RAIRO-Oper. Res. 51 (2017) 199–210
DOI: 10.1051/ro/2016014

RAIRO Operations Research
www.rairo-ro.org

FINDING A SOLUTION FOR MULTI-OBJECTIVE LINEAR FRACTIONAL PROGRAMMING PROBLEM BASED ON GOAL PROGRAMMING AND DATA ENVELOPMENT ANALYSIS

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Abstract. The multi-objective linear fractional programming is an interesting topic with many applications in different fields. Until now, various algorithms have been proposed in order to solve the multi-objective linear fractional programming (MOLFP) problem. An important point in most of them is the use of non-linear programming with a high computational complexity or the use of linear programming with preferences of the objective functions which are assigned by the decision maker. The current paper, through combining goal programming and data envelopment analysis (DEA), proposes an iterative method to solve MOLFP problems using only linear programming. Moreover, the proposed method provides an efficient solution which fairly optimizes each objective function when the decision maker has no information about the preferences of the objective functions. In fact, along with normalization of the objective functions, their relative preferences are fairly determined using the DEA. The implementation of the proposed method is demonstrated using numerical examples.

Mathematics Subject Classification. 90C29, 90C32, 90C05.

Received January 30, 2015. Accepted February 12, 2016.

1. Introduction

Multi-objective fractional programming is a programming problem with several ratio objective functions which should be optimized on a feasible region. If these objective functions are linear fractions (fractions with linear numerator and denominator) along with a feasible region obtained from linear constraints, the problem thus changes to a multi-objective linear fractional programming (MOLFP) problem. Until now, various methods have been proposed to solve the MOLFP problem. Steuer and Kornbluth [14] suggested an algorithm based on the simplex to find the weak efficient points of the MOLFP problem. They also utilized goal programming to solve the multi-objective linear fractional programming which resulted in a non-linear programming model (see [15]). Gueorguieva [12] used non-linear programming to find the weak efficient set. Iterative methods have also been proposed using problems with sum of ratio function (for instance see [7–9]), which again are considered as non-linear programming. There are some important difficulties in solving non-linear programming problems. One of them is that these problems often require more complicated calculations. On the other hand, available software

Keywords. Multi-Objective linear fractional programming, goal programming, data envelopment analysis, fair satisfaction.

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programs are faced with major limitations to solve these sorts of problems. To obviate high computational problems, many methods and algorithms have been proposed using the fuzzy set theory [5,11,17,21]. However, the algorithms obtained from the fuzzy set theory often result in just a satisfactory solution, not necessarily an efficient solution. For example, the linguistic variable approach of Luhandjula [7] and the modified linguistic approach proposed by Dutta *et al.* [11] with its certain shortcomings pointed out by Stancu–Minasian and Pop [21] can be mentioned.

In addition to the aforementioned attempts, Bogdana and Milan Stanojevic [22], using the efficiency test introduced by Lotfi *et al.* [16], proposed two procedures to derive weakly and strongly efficient solutions in MOLFP problems. Furthermore, Valipour *et al.* [23] extended the parametric approach of Dinkelbach [10] and proposed an iterative algorithm to solve MOLFP problems. They suggested an iterative parametric approach to solve MOLFP problems which always converges to an efficient solution. Although the two latter methods utilize linear programming and have a low computational complexity, they don't have much computational discretion for the objective functions. In other words, these methods optimize the objective functions corresponding to their preferences which often are assigned by the decision maker based on experience or the intrinsic knowledge of the problem.

In the present paper, through combining goal programming and data envelopment analysis (DEA), an iterative method is proposed to solve MOLFP problems which only uses linear programming and also has a managerial approach to find a fair solution when the decision maker has no information about the preferences of the objective functions. In fact, along with normalizing the objective functions, their relative preferences are fairly determined using the DEA.

The rest of the paper is organized as follows: Section 2 briefly discusses goal programming for the MOLFP problem. In Section 3, we propose an iterative method to solve the MOLFP problem which only uses linear programming and also converges to the efficient solution. The necessary background for DEA is presented in the Section 4. Moreover, in this section, the iterative method proposed in Section 3 is modified using the DEA technique to obtain a fair efficient solution. Finally, the conclusions of the paper are presented in the last section.

2. Goal programming problem to solve MOLFP problems

Consider a maximization linear fractional programming problem with $z_i(x) = \frac{N_i(x)}{D_i(x)} = \frac{p_i x+\alpha_i}{q_i x+\beta_i} (i = 1,\ldots,k)$ as the objective functions and $\mathbf{X} = \{x|Ax \le b, x \ge 0\}$ as the feasible region (where $q_i, p_i \in \mathbb{R}^n$ and α_i, β_i are scalars). This problem can be written as below:

$$\begin{aligned} &\text{Max } Z(x) = \big(z_1(x),\ldots,z_k(x)\big)\\ &s.t \quad x \in \mathbf{X}. \end{aligned} \tag{2.1}$$

where, it has been assumed that the feasible region $\mathbf{X}$ is bounded and the inequality $D_i(x) > 0$ holds for all $x \in \mathbf{X}$. Moreover, in the current paper, without loss of generality, we suppose that each component of the criterion vector $Z(x)$ is non-negative for all $x \in \mathbf{X}$. This is possible by adding a sufficiently positive constant to the objective functions (see [16]).

Definition 2.1. $\bar{x} \in \mathbf{X}$ is called an efficient solution of the problem (2.1) if there is no $x \in \mathbf{X}$ such that $z_i(x) \ge z_i(\bar{x})$ for all $i \in \{1,2,\ldots,k\}$ and $z_r(x) > z_r(\bar{x})$ at least for one index like r.

Definition 2.2. Suppose that $x^1, x^2 \in \mathbf{X}$. It is called that $Z(x^1)$ dominates $Z(x^2)$ if $z_i(x^1) \ge z_i(x^2)$ for all $i \in \{1,2,\ldots,k\}$ and $z_r(x^1) > z_r(x^2)$ at least for one index like r. In this case, x^1 is called better than x^2.

To achieve the linear fractional goal programming, first the model (2.2) should be solved for each $i \in \{1,2,\ldots,k\}$ as follows:

$$\begin{aligned} &\text{Max } z_i(x) = \frac{N_i(x)}{D_i(x)}\\ &s.t \quad x \in \mathbf{X}. \end{aligned} \tag{2.2}$$

Model (2.2) is a single objective linear fractional programming problem which can easily be linearized through the Charnes and Cooper transformation (see [3]). Suppose that $z_i^*(i = 1, \ldots, k)$ is the optimum value of the model (2.2) corresponding to the ith objective function of the model (2.1). Then, the goal programming model corresponding to the model (2.1) will be as follows:

$$\begin{aligned}
&\text{Min} \sum_{i=1}^{k} d_i \\
&s.t \quad \frac{N_i(x)}{D_i(x)} + d_i = z_i^* \quad i = 1, \ldots, k, \\
&\qquad d_i \geq 0 \quad i = 1, \ldots, k, \\
&\qquad x \in \mathbf{X}.
\end{aligned} \tag{2.3}$$

Model (2.3) provides an efficient solution for the MOLFP problem (2.1). However, this model is a non-linear programming problem which is difficult to solve. Since the model (2.3) cannot be always linearized, we create a little change in it. In fact, we consider the deviation variable $d_i(i = 1, \ldots, k)$ as $d_i = r_i/(q_i x + \beta_i)$. Then, through replacing the objective function $\sum_{i=1}^{k} d_i$ with $\sum_{i=1}^{k} r_i$, the linear goal programming model for the MOLFP problem (2.1) can be obtained as follows:

$$\begin{aligned}
&\text{Min} \sum_{i=1}^{k} r_i \\
&s.t \quad N_i(x) - z_i^* D_i(x) + r_i = 0 \quad i = 1, \ldots, k, \\
&\qquad r_i \geq 0 \quad i = 1, \ldots, k, \\
&\qquad x \in \mathbf{X}.
\end{aligned} \tag{2.4}$$

It should be mentioned that although the problem (2.4) is a linear programming problem, it does not guarantee to obtain the efficient solution of the MOLFP problem (2.1). In other words, model (2.4) may only provide a satisfactory solution of the MOLFP problem (2.1). In fact, the models (2.3) and (2.4) are not equivalent but one may use the latter model to obtain a satisfactory solution without dealing with the non-linearity of the problem to be solved. It can be said that the model (2.4) is similar to the methods which often lead to a satisfactory solution, however, not necessarily an efficient solution (*e.g.*, [5, 11]).

Example 1. Consider the MOLFP problem (2.5) with 4 objective functions and 4 linear constraints as follows:

$$\begin{aligned}
&\text{Max} \left(z_1 = \frac{x_0 - x_1 + 2x_2 + 3}{2x_0 + 3x_1 + x_2 + 2},\ z_2 = \frac{4x_2 + 9}{2x_0 - x_1 + x_2 + 5} \right) \\
&\text{Max} \left(z_3 = \frac{100x_0 - 100x_1 + 1000x_2 + 300}{x_0 + x_1 + x_2 + 3},\ z_4 = \frac{2000x_0 + 4000x_2 + 28000}{-x_0 + x_1 + x_2 + 10} \right) \\
&\quad s.t \quad x_0 + x_1 + x_2 = 1\,,\ x_0 + x_1 - x_2 \leq 2, \\
&\qquad\quad x_0 - x_1 + x_2 \leq 4,\ x_0 + 2x_2 \leq 4, \\
&\qquad\quad x_0, x_1, x_2 \geq 0.
\end{aligned} \tag{2.5}$$

All of the objective functions have non-negative values throughout the feasible region of the problem. Model (2.4) corresponding to the problem (2.5) can be written as follows (by separately maximizing the objective functions):

$$\begin{aligned}
&\text{Max} \ \sum_{i=1}^{4} r_i \\
&s.t \\
&-2.34x_0 - 6.01x_2 + 0.33x_3 + r_1 = 0.34, \\
&-4.5x_0 + 2.25x_2 + 1.75x_3 + r_2 = 2.25, \\
&-225x_0 - 425x_2 + 675x_3 + r_3 = 675, \\
&5333.34x_0 + 666.66x_2 - 3333.34x_3 + r_4 = 5333.34, \\
&x_0 + x_1 + x_2 = 1\,,\ x_0 + x_1 - x_2 \leq 2, \\
&x_0 - x_1 + x_2 \leq 4,\ x_0 + 2x_2 \leq 4, \\
&x_0, x_1, x_2 \geq 0.
\end{aligned} \tag{2.6}$$

Solving the model (2.6) leads to the optimal solution $(x_1, x_2, x_3) = (1, 0, 0)$ with the criterion vector $(z_1, z_2, z_3, z_4) = (1, 1.286, 100, 3333.34)$. In the next section, it will be shown that the obtained solution is an efficient solution of the MOLFP problem (2.5).

3. An iterative method to obtain an efficient solution for MOLFP problems

In the previous section, we proposed the linear goal programming model (2.4) to solve the MOLFP problem in which the provided solution might not be efficient while achieving the efficient solution is usually the main purpose of any method delivered to solve the MOLFP problem. Therefore, we should determine the efficiency status of the provided solution from the model (2.4) denoted by x^*. If x^* is an efficient solution, we are done; otherwise, it must be changed with an efficient solution.

There are several methods to determine the efficiency status of a given feasible solution of the MOLFP problem (*e.g.*, [16]). However, we propose a linear programming model based on goal programming similar to the model (2.4) which can easily determine the efficiency status of x^*. Moreover, if the solution x^* is not efficient, the proposed model provides a better feasible solution for the MOLFP problem. Finally, based on the proposed model, an iterative method is presented which certainly leads to an efficient solution of the MOLFP problem.

Suppose that $x^0 = x^*$. To determine the efficiency status of x^0, the following linear programming problem is suggested:

$$\begin{aligned}
&\text{Max} \ \sum_{i=1}^{k} r_i \\
&s.t \quad N_i(x) - z_i^0 D_i(x) - r_i = 0 \quad i = 1, \ldots, k, \\
&\qquad\ x \in \mathbf{X}, r = (r_1, \ldots, r_k) \geq 0.
\end{aligned} \tag{3.1}$$

where $z_i^0 = N_i(x^0)/D_i(x^0)$ $(i = 1, \ldots, k)$. Note that $\mathbf{X}$ is a compact set and $D_i(x) > 0$ for all $x \in \mathbf{X}$; therefore, each $z_i^0 (i = 1, \ldots, k)$ is a finite positive quantity. Details of the efficiency determining of the x^0 is presented in the Theorem 3.1.

Theorem 3.1. *Consider the problem (3.1). Then, x^0 is an efficient solution of the MOLFP problem (2.1) if and only if the optimum value of the problem (3.1) is zero. Otherwise, if (x^{1^*}, r^{1^*}) is an optimal solution of the problem (3.1), then x^{1^*} is a feasible solution of the MOLFP problem (2.1) which is better than x^0.*

Proof. First, suppose that x^0 is an efficient solution. We prove that the optimum value of the problem (3.1) is zero. Assume by contradiction that it is a positive quantity. Therefore, if (x^{1^*}, r^{1^*}) is an optimal solution of the problem (3.1) then the inequality $r^{1^*} \neq 0$ holds. Because $D_i(x^{1^*}) > 0$ $(i = 1, \dots, k)$, the first constraints of the problem (3.1) can be written at the optimal solution as follows:

$$z_i^1 = \frac{r_i^{1^*}}{D_i(x^{1^*})} + z_i^0 \quad i = 1, 2, \dots, k \tag{3.2}$$

where $z_i^1 = N_i(x^{1^*})/D_i(x^{1^*})$ $(i = 1, \dots, k)$. Since $x^{1^*} \in \mathbf{X}$ and $r^{1^*} \neq 0$, the relation (3.2) shows that x^0 is an inefficient solution of the MOLFP problem (2.1) and it is a contradiction. Thus, the optimum value of the problem (3.1) must be zero.

Conversely, suppose that the optimum value of the problem (3.1) is zero. Assume by contradiction that x^0 is an inefficient solution of the problem (2.1). Then, there should be a feasible solution of the problem (2.1) like $\bar{x}$ which is better than x^0. It means that:

$$\begin{aligned} \frac{N_i(\bar{x})}{D_i(\bar{x})} &\geq \frac{N_i(x^0)}{D_i(x^0)} \quad i = 1, 2, \dots, k, \\ \exists j \in \{1, 2, \dots, k\};\ \frac{N_j(\bar{x})}{D_j(\bar{x})} &> \frac{N_j(x^0)}{D_j(x^0)}. \end{aligned} \tag{3.3}$$

Since $\big(1/D_i(\bar{x})\big) > 0$ $(i = 1, \dots, k)$, then by adding the $\big(\bar{r}_i/D_i(\bar{x})\big)$s $(i = 1, \dots, k)$ as slack variables in the relation (3.3) such that $\bar{r} = (\bar{r}_1, \dots, \bar{r}_k) \neq 0$, the following equalities can be obtained:

$$\begin{aligned} \frac{N_i(\bar{x})}{D_i(\bar{x})} - \frac{\bar{r}_i}{D_i(\bar{x})} &= \frac{N_i(x^0)}{D_i(x^0)} \quad i = 1, 2, \dots, k \\ \bar{r} = (\bar{r}_1, \dots, \bar{r}_k) &\geq 0,\ \bar{r} \neq 0 \end{aligned} \tag{3.4}$$

by multiplying both sides of the ith equality to $D_i(\bar{x})(i = 1, 2, \dots, k)$ in the relation (3.4), it can be seen that $(\bar{x}, \bar{r})$ is a feasible solution of the problem (3.1). Now, $\bar{r} \neq 0$ shows that the optimum value of the objective function should be positive and it is a contradiction; thus, x^0 is efficient.

The second part of the theorem is proved in the same way as the first part. □

Now, the Theorem 3.2 shows that how the frequent use of the model (3.1) can lead to an efficient feasible solution of the MOLFP problem (2.1).

Theorem 3.2. *Suppose that x^0 is an arbitrary feasible solution of the problem (2.1). Let $h := 0$ and solve the following problem (where $z_i^h = z_i(x^h), i = 1, \dots, k$):*

$$\begin{aligned} & U_h^* = \max \sum_{i=1}^{k} r_i^h \\ & S.t. \quad N_i(x) - z_i^h D_i(x) - r_i^h = 0 \quad i = 1, 2, \dots, k, \\ & \qquad x \in \mathbf{X},\ r^h = (r_1^h, \dots, r_k^h) \geq 0. \end{aligned} \tag{3.5}$$

If $U_h^* > 0$, this means that x^h is an inefficient solution of the problem (2.1). In this situation, let $h := 1$ and $x^h := x^{h^*}$; where, (x^{h^*}, r^{h^*}) is an optimal solution of the model (3.5). Once again, by updating the z_i^h $(i = 1, \dots, k)$ let's solve the problem (3.5) and continue this process to obtain the sequence $\{x^h\}_{h=1}^{\infty}$. The obtained sequence certainly converges to an efficient feasible solution of the problem (2.1).

Proof. Before starting the proof, remember the non-negativity assumption of the objective functions. If h is an arbitrary iteration number, it is obvious that x^h is a feasible solution of the problem (2.1) such that

$Z(x^h) \geq Z(x^{h-1}) \geq 0$ and thus $||Z(x^h)||_1 \geq ||Z(x^{h-1})||_1$. If $U_h^* = 0$ then $x^{l+1} = x^h$ for all $l \geq h$ and there is nothing to prove. Then, suppose that $U_h^* > 0$. It can be seen that $||Z(x^{h+1}) - Z(x^h)||_1$ is obtained from the following relation:

$$||Z(x^{h+1}) - Z(x^h)||_1 = \sum_{i=1}^{k} \frac{r_i^h}{D_i(x^{h+1})} \geq 0 \tag{3.6}$$

As long as $U_h^* > 0$, relation (3.6) shows that $\{||Z(x^h)||_1\}_{h=1}^{\infty}$ is a strictly increasing sequence. On the other hand, $\mathbf{X}$ is a compact set and $||Z(.)||_1 : \mathbf{X} \to \mathbb{R}$ is a continuous function. This means that the function $||Z(.)||_1$, according to Weierstrass' extreme value theorem, attains its maximum on the $\mathbf{X}$. Therefore, since $x^h \in \mathbf{X}$ for all h, it is concluded that the sequence $\{x^h\}_{h=1}^{\infty}$ must converge to a feasible solution of the problem (2.1) like $\bar{x} = x^N$ such that $U_N^* = 0$. This means that x^N is an efficient feasible solution of the problem (2.1). □

It should be noted that the presented method can be considered as a generalization of the Isbell and Marlow's technique [13]. A comparison between some variations of this technique has been provided by Bhatt [2].

Remark 3.3. In practice, the sequence $\{x^h\}_{h=1}^{\infty}$ generated in the Theorem 3.2 converges in the first few elements. However, to avoid time wasting, one can use the stop condition as follows:

$$||Z(x^{h+1}) - Z(x^h)||_1 = \sum_{i=1}^{k} \frac{r_i^h}{D_i(x^{h+1})} \leq \varepsilon \tag{3.7}$$

where ε is a sufficiently small positive quantity. It should be mentioned that using the stop condition leads to a solution of the problem (2.1) which may not be efficient.

It is worth noting that in most cases it is necessary to obtain an efficient solution of the MOLFP problem (2.1), and so the presented condition in the relation (3.7) cannot be used. In such a situation, in order to reduce the volume of computations, problem (3.1) can get rid of its first k constraints by eliminating the variables r_is $(i = 1, \ldots, k)$ and putting them in the objective function. Now, each problem has fewer constraints and also it can be solved through fewer numbers of Simplex iterations which start by the last tableau of the precedent problem, because it is only the objective function that changes.

Now, using the model (3.1), it can be displayed that the obtained solution from the model (2.4) is efficient or not. Moreover, in the case of its inefficiency, the frequent use of the model (3.1), as was presented in the Theorem 3.2, can provide an efficient solution of the MOLFP problem (2.1). For instance, in the Example 1, the optimum value of the problem (3.1) corresponding to the solution $x^0 = (1, 0, 0)$ from the MOLFP problem (2.1) is equal to zero. This means that the mentioned solution is efficient for the problem (2.1) and there is no need to reuse the model (3.1).

4. Obtaining a fair solution for MOLFP problems by using the DEA technique

In the previous sections, it was shown that the MOLFP problem (2.1) can be solved first using the model (2.4) and then by frequent use of the model (3.1). However, in the proposed method, the objective functions were not normalized and also they had the same preferences. Since these may be determinant in the computations, we call the obtained solution in this situation as an "unfair solution". To overcome the mentioned problem and obtain a fair solution, the objective functions must be normalized and their preferences must be considered in the optimization method. Different techniques exist to normalize the objective functions. Therefore, we focus on the fair determination of the objective functions' preferences when the decision maker has no information about them. To this end, the concept of fairness related to preferences of the objective functions is defined based on the objective functions' scales using the DEA technique. The necessary background of DEA and the details of its use in the fair determination of the objective functions' preferences are presented in the following subsections.

4.1. Data envelopment analysis (DEA)

DEA is a technique based on the mathematical programming to evaluate the performance of the homogeneous decision making units (DMU) with multiple inputs and outputs. CCR and BCC models are two basic models of DEA which deal with constant returns to scale (CRS) and variable returns to scale (VRS) respectively (see [1,6]). Constant returns to scale simply means that the size of the inputs and outputs can be increased in the same proportion in the production possibility set (PPS).

In addition to the aforementioned DEA models which measure the efficiency of the DMUs, many other models have been developed with different applications and purposes. One of them is the additive model introduced by Charnes *et al.* [4]. This model was developed to determine the efficiency status of the DMUs. Moreover, the optimum value of the additive model is an inefficiency criterion for the corresponding DMUs. This means that the larger optimum value of the additive model results in the more inefficiency of the DMU under evaluation. In the traditional DEA literature, (x_0, y_0) is called an efficient (Pareto-efficient) DMU in the PPS if there is no $(\bar{x}, \bar{y}) \in PPS$ such that $(-\bar{x}, \bar{y}) \gneqq (-x_0, y_0)$. Naturally, in the additive model, a DMU will be more inefficient if its outputs are smaller and its inputs are larger.

In order to briefly present the additive model, consider the n DMUs where each $DMU_j (j = 1, \ldots, n)$ uses the input vector $x_j = (x_{1j}, \ldots, x_{mj})$ to produce the output vector $y_j = (y_{1j}, \ldots, y_{sj})$. The additive model to evaluate the DMU_o, $(o \in \{1, 2, \ldots, n\})$, assuming variable returns to scale, is as below:

$$\begin{aligned}
&\text{Max } 1_m s_o^- + 1_s s_o^+ \\
&s.t \quad \sum_{j=1}^{n} \lambda_j x_j + s_o^- = x_o, \\
&\qquad \sum_{j=1}^{n} \lambda_j y_j - s_o^+ = y_o, \\
&\qquad \sum_{j=1}^{n} \lambda_j = 1, \\
&\qquad \lambda_j \geq 0 \; j = 1, \ldots, n, \\
&\qquad s_o^- \geq 0, s_o^+ \geq 0,
\end{aligned} \tag{4.1}$$

where 1_m and 1_s are row vectors of ones of the appropriate sizes. If the optimal value of the model (4.1) equals to zero, DMU_o is an efficient DMU in the PPS. Otherwise, it is an inefficient DMU. In the last case, the larger the optimum value, the more inefficient is the DMU_o.

Sometimes it happens that the DMUs have no input and the evaluation of the DMUs must be conducted just based on their outputs. In this regard, Lovell and Pastor [18] proposed a pure output model in 1997. Nonetheless, two years later, they stated that from a production viewpoint it could be argued that each DMU is by itself 'the input', therefore, a single constant input is at hand (see [19]). On the other hand, Mahdiloo *et al.* [20] proved that the additive model with a single constant input is equivalent to the additive model without input. Accordingly, when the DMUs have no input, the additive model to evaluate the DMU_o can be used as follows:

$$\begin{aligned}
&\text{Max } 1 s_o^+ \\
&s.t \; \sum_{j=1}^{n} \lambda_j y_j - s_o^+ = y_o, \\
&\qquad \sum_{j=1}^{n} \lambda_j = 1, \\
&\qquad \lambda_j \geq 0 \; j = 1, \ldots, n, \\
&\qquad s_o^+ \geq 0.
\end{aligned} \tag{4.2}$$

4.2. Obtaining a fair solution for MOLFP problems

Consider the models (2.4) and (3.1) which were proposed in the previous sections. A major disadvantage of them becomes apparent when the value of an objective function is much larger than those of the other functions. This issue may be determinant in the goal programming computations. Therefore, before starting the related computations, the objective functions should be normalized. There are different normalization methods which can be used to this end. One of the most common normalization methods is as follows:

$$f_N(x) = \frac{f(x) - f_{\min}}{f_{\max} - f_{\min}}, \tag{4.3}$$

where, $f_N(x)$ is the normalized form of the objective function $f(x)$ and $f_{\max}$ along with $f_{\min}$ are the maximum and minimum values of the objective function on the feasible region of the problem respectively. Using the relation (4.3), each objective function ranges within the interval $[0, 1]$ and then the same magnitude among them is guaranteed.

Another disadvantage of the models (2.4) and (3.1), which could be determinant in the computations and thus must be resolved, is assigning the same preference to all of the objective functions. Most of the existing methods for solving the MOLFP problems use the preferences of the objective functions presented by the decision maker based on his/her experience. However, consider a situation in which decision maker has no information about the preferences of the objective functions. The purpose of the current subsection is assigning the relative preferences to the objective functions in the mentioned situation (using the additive DEA model).

In most of the practical MOLFP problems, an objective function is more significant for decision maker if it has smaller absolute values compared to other objective functions. Accordingly, considering the non-negativity of the objective functions' values, we suggest a method which assigns the higher preference to the objective function with the smaller scales. Of course, it should be noted that this idea may need to be changed in different cases. Therefore, before using the suggested preference assignment process, be careful about its compatibility with the given MOLFP problem.

Now, the question which arises is that how should the scale of the objective functions be determined? One may use the maximum (or minimum) value of the objective functions as their scales. Nonetheless, determination of the scale value using only one value of the function does not seem a reasonable idea. Therefore, in order to determine the scale of the objective function, we utilize its values (before normalization) at the solutions which individually maximize the objective functions. Accordingly, each of the objective functions is considered as a DMU without any input such that its outputs are the mentioned values of the associated objective function. Finally, the relative preferences will be assigned by the additive model (4.2) corresponding to these DMUs. The preference assignment process for the MOLFP problem (2.1), which must be done before the normalization of the objective functions, can be summarized as follows:

Step 0: Solve the model (4.4) for all $j \in \{1, \ldots, k\}$ and obtain its optimal solution denoted by $x_j^*(j = 1, \ldots, k)$. Then, go to Step 1.

$$\begin{aligned} \text{Max } & p_j x + \alpha_j t \\ S.t. \ & q_j x + \beta_j t = 1, \\ & Ax \leq bt, \\ & x \geq 0, t \geq 0. \end{aligned} \tag{4.4}$$

Step 1: Compute the cross-evaluation score $z_{ij} = z_i(x_j^*)$ for all $i, j \in \{1, \ldots, k\}$.

Step 2: Consider the DMUs with k outputs without any input as follows:

$$DMU_i = (z_{i1}, z_{i2}, \ldots, z_{ik}) \quad i = 1, \ldots, k. \tag{4.5}$$

In order to create a better distinction between the DMUs, add the super ideal DMU $(z_1^{**}, z_2^{**}, \dots, z_k^{**})$ to the mentioned DMUs and go to Step 3; where, z_j^{**} $(j = 1, \dots, k)$ is obtained from the relation (4.6) (ε is a sufficiently small positive value):

$$z_j^{**} = \varepsilon + \max\{z_{ij} : i = 1, \dots, k\} \quad j = 1, \dots, k. \tag{4.6}$$

Step 3: Evaluate the k DMUs corresponding to the objective functions, mentioned in the Step 2, through the additive model (4.2). Suppose that $R_i^*(i = 1, \dots, k)$ is the optimal value of the model (4.2) corresponding to the ith objective function. The value of the relative preference for the ith objective function $(i = 1, \dots, k)$, denoted by w_i, is obtained as follows:

$$w_i = \left(R_i^* / \sum_{j=1}^{k} R_j^* \right) \tag{4.7}$$

According to the relation (4.7), the relative preference has a larger value when R_i^* is larger. This is what we have been looking for because the larger R_i^* means that the ith objective function compared to others has a smaller scale.

Lemma 4.1. *Theorems 3.1 and 3.2 remain true when the objective function of the models (2.4) and (3.1) is replaced with $\sum_{i=1}^{k} w_i r_i$ such that $w_i > 0$ $(i = 1, \dots, k)$.*

Proof. This lemma can be proved exactly similar to the Theorems 3.1 and 3.2. □

Note that the super ideal DMU $(z_1^{**}, z_2^{**}, \dots, z_k^{**})$ dominates all other DMUs corresponding to the objective functions and this guarantees the positivity of the w_i for all $i \in \{1, \dots, k\}$. Therefore, as a main corollary of the lemma 4.1, the method to fairly solve the MOLFP problems can be presented as follows:

(1) Using the presented preference assignment process, determine the relative preferences of the objective functions *i.e.* w_is $(i = 1, \dots, k)$.
(2) Replace the original objective functions with the normalized ones in the models (2.4) and (3.1) by the use of the relation (4.3).
(3) Obtain a solution for the MOLFP problem through the model (2.4) and then model (3.1) with the modified objective function as $\sum_{i=1}^{k} w_i r_i$.

Example 2. In the Example 1, assigning the same preference to the objective functions of the MOLFP problem (2.5) without using any normalization process led to the efficient solution $\bar{x} = (1, 0, 0)$ with the criterion vector $\overline{Z} = (1, 1.286, 100, 3333.34)$. Now, according to the presented definition of the fairness, we want to obtain a fair solution for the problem (2.5).

To obtain a fair solution, first of all, the relative preferences of the objective functions must be determined. Table 1 represents the DMUs associated with the objective functions along with the super ideal DMU $(z_1^{**}, z_2^{**}, \dots, z_k^{**})$ in which ε is considered equal to 0.0001. Table 1 shows that the fourth objective function

TABLE 1. The DMUs related to the Example 2.

DMUs	Outputs			
1	1.66667	0.40000	1.66667	1.00000
2	2.16667	2.25000	2.16667	1.28571
3	325.00000	50.00000	325.00000	100.00000
4	2909.09091	2545.45455	2909.09091	3333.33333
Super Ideal	2909.09101	2545.45465	2909.09101	3333.33343

Table 2. The relative preferences.

	Preferences
w_1	0.34109725188
w_2	0.34100577379
w_3	0.31789696266
w_4	0.00000001167

Table 3. The maximum and minimum values of the objective functions.

Objective functions	Minimum	Maximum
1	0.40000	1.66667
2	1.28571	2.25000
3	50.00000	325.00000
4	2545.45455	3333.33333

compared to others has a larger scale; therefore, based on the presented discussion about the fairness, it is expected that a relative preference with smaller a value to be obtained for this objective function. The relative preferences according to the Table 1 using the additive model (4.2) and then the relative (4.7) are shown in the Table 2. It is clear that the obtained preferences are consistent with our desire.

Now, the objective functions must be normalized using the relation (4.3). To this end, the maximum and minimum values of the objective functions are obtained and represented in the Table 3. According to the Tables 2 and 3, the associated model (2.4) with the modified objective function will be as follows:

$$\begin{aligned}
\text{Min } z = {} & 0.341097r_1 + 0.3410058r_2 + 0.3178970r_3 + 0.00000001167r_4 \\
s.t \quad & x_0 + x_1 + x_2 = 1\,,\; x_0 + x_1 - x_2 \leq 2, \\
& x_0 - x_1 + x_2 \leq 4,\; x_0 + 2x_2 \leq 4, \\
& -2.34x_0 - 6.01x_1 + 0.33x_2 + r_1 = 0.34, \\
& -4.5x_0 + 2.25x_1 + 1.75x_2 + r_2 = 2.25, \\
& -225x_0 - 425x_1 + 675x_2 + r_3 = 675, \\
& 5333.34x_0 - 3333.34x_1 + 666.66x_3 + r_4 = 5333.4, \\
& x_0, x_1, x_2 \geq 0,\; r_i \geq 0 \; i = 1, \ldots, 4.
\end{aligned} \tag{4.8}$$

Solving the model (4.8) results in the solution $x^0 = (0, 0, 1)$ which may not be efficient. Therefore, to obtain an efficient solution of the problem (2.5), the associated model (3.1) with the modified objective function should be solved as follows:

$$\begin{aligned}
\text{Min } z = {} & 0.341097r_1 + 0.3410058r_2 + 0.3178970r_3 + 0.00000001167r_4 \\
s.t \quad & x_0 + x_1 + x_2 = 1\,,\; x_0 + x_1 - x_2 \leq 2, \\
& x_0 - x_1 + x_2 \leq 4,\; x_0 + 2x_2 \leq 4, \\
& -2.34x_0 - 6x_1 + 0.34x_3 - r_1 = 0.34, \\
& -4.34x_0 + 2.17x_1 + 1.84x_3 - r_2 = 1.84, \\
& -225x_0 - 425x_1 + 675x_2 - r_3 = 675, \\
& 4909.09x_0 - 2909.09x_1 + 1090.90x_2 - r_4 = 1090.90, \\
& x_0, x_1, x_2 \geq 0,\; r_i \geq 0 \; i = 1, \ldots, 4.
\end{aligned} \tag{4.9}$$

The Optimum value of the model (4.9) is equal to 0. This means that $x^0 = (0, 0, 1)$ is an efficient solution of the problem (2.5) and thus we are done.

The original criterion vector of x^0 in the Example 2 is equal to:

$$Z\big(x^0 = (0,0,1)\big) = \big(1.66667, 2.16667, 325.00000, 2909.09091\big).$$

The Comparison of the criterion vectors of $x^0 = (0,0,1)$ and $\bar{x} = (1,0,0)$ (obtained in the Example 1) shows the importance of selecting the relative preferences in obtaining a fair solution. Because all of the first to third objective functions, which have the smaller scales compared to the fourth objective function, are improved in the new solution (*i.e.* x^0); specifically, the first and third objective functions which attain their maximum values.

As a final point, it should be mentioned again that the proposed method in this section just obtained an efficient solution for the given MOLFP problem based on the presented definition of the fairness. When the decision maker is able to decide about the satisfaction level of the objective functions, model (3.1) can be used with a little change to obtain a satisfactory solution in an interactive process. In this case, starting from an arbitrary feasible solution of the given MOLFP problem like $\bar{x}$, the frequent use of the model (4.10) can lead to a satisfactory solution:

$$\begin{aligned} \text{Max} \ & \sum_{i \in \{1,\ldots,k\} - \mathbf{I}_1} r_i \\ S.t. \ & N_i(x) - r_i = (z_i(\bar{x}) - \Delta_i) D_i(x) \quad i \in \mathbf{I}_1, \\ & N_i(x) - r_i = z_i(\bar{x}) D_i(x) \quad i \in \{1,\ldots,k\} - \mathbf{I}_1, \\ & Ax \le b, \ x \ge 0, \ r \ge 0, \end{aligned} \tag{4.10}$$

where $\mathbf{I}_1$ is the index set of the objective functions to be relaxed and Δ_is ($i \in \mathbf{I}_1$) are the amounts by which they are to be relaxed. To reach the best compromise (satisfactory) solution, model (4.10) can be frequently solved by updating the $\bar{x}$ along with its associated index set $\mathbf{I}_1$ and the values of the Δ_is ($i \in \mathbf{I}_1$). At the end, the obtained satisfactory solution through frequent use of the model (3.1) (without any change) can be improved until the efficient solution is achieved.

5. Conclusions

The paper has pointed out that most of the existing methods for solving the MOLFP problems either use non-linear programming with a high computational complexity or they use linear programming with preferences of the objective functions which are assigned by the decision maker. Although in the latter case, the existing models have a low computational complexity, this paper discussed two points about them which may sometimes be important in optimization process. The first is the normalization of the objective functions and the second is the consideration of their relative preferences in the optimization process. Before addressing these points, an iterative method was proposed using the goal programming to solve the MOLFP problem which only uses linear programming and converges to the efficient solution. Then, the proposed method was developed using the additive DEA model to obtain a fair solution.

At the end, it should be noted that the concept of fairness to determine the relative preferences of the objective functions has been presented based on the objective functions' scales for when the decision maker has no information about the preferences of the objective functions. According to this definition, the objective function has a higher preference if it has a smaller scale. However, this definition may not always be consistent with the given MOLFP problem and may sometimes need to be changed. Furthermore, as has been mentioned in Section 4, if the decision maker is able to decide about the satisfaction level of the objective functions, the proposed method can be transformed to obtain a satisfactory efficient solution in an interactive process.

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