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OPTIMAL MANUFACTURING BATCH SIZE WITH REWORK FOR A FINITE-HORIZON AND TIME-VARYING DEMAND RATES INVENTORY MODEL *

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Abstract. This paper proposes a finite-horizon and time-varying demand rate function formulations for the optimal manufacturing batch size model with rework. The basic model is found in [Jamal *et al.* *Comput. Ind. Eng.* **47** (2004) 77–89.]. Two policies 1 and 2 are considered. In Policy 1 defective items produced in a given period are remanufactured within the same period while Policy 2 accumulates the defective items until the last period. The search for the optimal manufacturing batch size for policies 1 and 2 is shown to reduce to the problem of determining the number of manufacturing-rework periods as well as their starting and finishing times. This leads to the examination of two mixed integer non-linear programming problem which are completely solved by appealing to some established techniques proposed in [Al-Khamis *et al.* *Int. J. Syst. Sci.* **45** (2014) 2196–2202]. Numerical results are also presented for illustration.

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1. Introduction

Producing imperfect quality items in a manufacturing process is inevitable due controllable and uncontrollable factors. Driven by economic considerations, environmental awareness, and (or) or governmental legislations imperfect quality items may be reworked to serviceable condition. The term rework refers to the repetition of the production process to bring a product or service into conformance with its original requirements (see [8,9]). Sarker *et al.* [21] cite the example of filing cabinet in the metal industries where shelves and defective filing cabinets are reworked. Other examples may be found in semiconductor, automobile, pharmaceutical, and food industries.

The work of Schrady [22] appears to be one the earliest work to focus on remanufacturing process with constant demand and return rates (see also [16] for related work). Teunter [23] examined a more general problem of Schrady by considering the numbers n_1 of recovery lots (that are of equal size) and n_2 of manufacturing lots

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of equal sizes, together with their respective quantities. In [23] only heuristics were proposed. Konstantaras and Papachristos [14] proposed an exact procedure for finding the optimal values of n_1 and n_2 together with their corresponding optimal quantities.

Jamal *et al.* [12] examined an (EPQ) model with imperfect items, where demand is constant. Optimal batch-quantity were developed for a multi-stage manufacturing system for two operational policies: policies 1 and 2. Policy 1 deals with the rework in the same cycle. The second policy considers rework after N cycles. Refinement and extensions of Jamal *et al.* [12] are found in [5–7, 21, 26]. Related work may also be found in [4, 24, 25].

In all the above cited papers the demand rate is assumed constant over time. Although, this assumption may be considered to be reasonable for some products over a period of time, it cannot be seen to be a true reflection of reality for a large number of product-demands. Particularly, for newly launched products, and seasonal products. Therefore, the need to include time-varying demand in modeling arises.

This paper is concerned with the determination of optimal batch quantities in a manufacturing system with an imperfect production process where the demand is time-varying and the planning horizon is finite for the model of Jamal *et al.* [12]. Examination of time-varying demand with remanufactured items has been examined in [18,19], where the planning horizon is made-up of regular production and remanufacturing runs. The objective is the determination of the sequence as well as the timings of the production runs and remanufacturing runs that minimizes some inventory costs. Modeling and numerical results are presented. Structural results for model [18] are found in [2]. The extension of the work of Jamal *et al.* [12] to time-varying demand rate call for the examination of classical optimal batching models for time-varying demand. The work of Hill [10], Hill *et al.* [11], and Omar and Smith [17] are possibly one of the earliest model on the subject. Of particular relevance to the present work is a general form of the inventory cost function within a cycle found in Omar and Smith [17]. These papers examined the optimal batching problem for linear demand rate function where numerical results are presented. Rau and Ou Yang [20] proposed a complete solution procedure for the linear demand rate case. Recently, Al-Khamis *et al.* [1] suggested a methodology for tackling finite-horizon batching inventory model for general demand rate function. Crucial in the success of the methodology are a number of properties of the form of the inventory cost function within a cycle found in [17]. This form, as we shall see, reappears in the extension of the model of Jamal *et al.* [12] to time-varying demand rate models.

The main contributions of this paper are two fold:

(i) extend the model in [12] to finite planning horizon with time-varying demand rate,
(ii) to show that the optimal inventory policies within classes of policies 1 and 2 defined in [12] exist and are unique under some technical requirements. A complete characterization of the policies will also be given. This will be done by solving two mixed integer non-linear programming problems.

The remainder of the paper is organized as follows. The next section contains the notations and the assumptions. Section 3 is concerned with Policy 1 where the model is introduced along with the derivation of the total inventory costs and the optimal batching policy. Section 4 is related to Policy 2. Numerical examples as well as sensitivity analysis are found in Section 5. Section 6 concludes the paper with some general remarks and a list of possible extensions of the paper.

2. Assumptions and notation

The models considered in the following notation which are similar to those found in [12] with obvious modification when required.

2.1. Assumptions

(1) A single product inventory system is considered over a known and finite planning horizon.
(2) The demand rate is known and is given by a continuously differentiable function.
(3) All demands must be satisfied from good items.
(4) The proportion of defective items is known and is constant in each cycle.

(5) The production rate is known and is constant in each cycle.
(6) No defective items are produced during the rework.
(7) The inspection cost is ignored.

Note that Assumptions (2)–(7) are reproduced from [21]. We shall comment briefly on the implication of relaxing Assumption (7) later on.

2.2. Notations

Below are some of the main notations used in this paper:

- H: the total planning horizon, $H > 0$,
- $D(t)$: demand rate at time t, where $D : [0.H] \to \mathbb{R}_+$, and belongs to the space of differentiable functions in $[0, H]$,
- p : production rate, with $p > 0$,
- β : proportion of defective items in a stage (cycle), where $0 < \beta < 1$, and $(1 - \beta)p > D(t)$, for all $t \in \mathbb{R}_+$,
- c_c : the unit production cost, $c_c > 0$,
- c_h : holding cost per item per unit time,
- c_p : penalty cost per item per unit time, $c_p > 0$,
- K : the set-up cost for a regular production run, $K > 0$,
- S : the set-up cost for a rework production run, $S > 0$.

Note that in this paper defective items are held in stock until they are reworked. The unit holding cost for such item is c_p. Also, the time is taken in unit of time (day or otherwise) and money in unit of money (dollars or otherwise).

3. Policy 1: Rework within the same cycle

This model assumes that for a given planning horizon H, production is undertaken in stages and good items produced are continuously shipped to a buyer. Each stage consists of a regular production run followed by a rework production run. In the latter run defective items produced in the former run are reworked to quality as good as new. For a typical stage (cycle) i which begins at time t_{i-1} and finishes at time t_i (say): see Figure 1, a regular production lasts from time t_{i-1} up to some time τ_i, $t_{i-1} < \tau_i < t_i$. During this run a number of defective items are produced which are reworked from time τ_i to t_i^p, with $\tau_i < t_i^p < t_i$. On the interval $[t_i^p, t_i)$ production is stopped and the inventory accumulated during the production runs is left to deplete until it reaches level zero at t_i, at which time a new production stage is triggered if $t_i < H$. Otherwise, production is stopped. Let $I(t)$ be the inventory level of good items at time t, and set $\alpha := 1 - \beta$. On the interval $[t_{i-1}, \tau_i)$ the changes in the inventory level is described by the differential equation:

$$I'(t) = \alpha p - D(t), \tag{3.1}$$

with initial condition $I(t_{i-1}) = 0$.

The solution to (3.1) is given by:

$$I(t) = \int_{t_{i-1}}^{t} \{\alpha p - D(u)\}\mathrm{d}u. \tag{3.2}$$

Using integration by parts, the amount of inventory A_1 on $[t_{i-1}, \tau_i)$ is then:

$$A_1 = \int_{t_{i-1}}^{\tau_i} (\tau_i - t)\{\alpha p - D(t)\}\mathrm{d}t. \tag{3.3}$$

On the interval $[\tau_i, t_i^p)$ the changes in the level of inventory is described by the equation:

$$I'(t) = p - D(t), \tag{3.4}$$

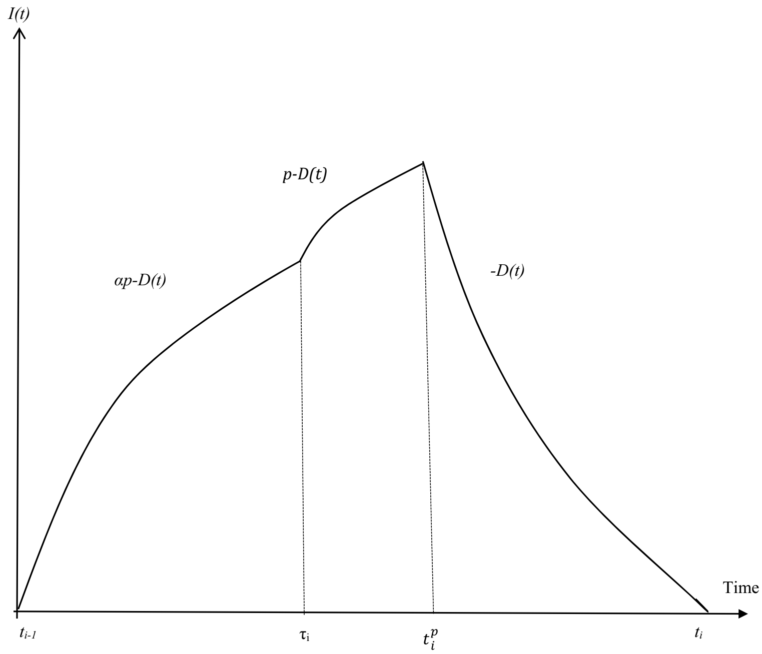

FIGURE 1. The level of inventory of good items for a typical period i for Policy 1.

with initial condition

$$I(\tau_i) = \int_{t_{i-1}}^{\tau_i} \{\alpha p - D(t)\}\mathrm{d}t. \tag{3.5}$$

It follows that for $\tau_i \leq t < t_i^p$, $I(t)$ is given by:

$$I(t) = \int_{\tau_i}^{t} \{p - D(u)\}\mathrm{d}u + \int_{t_{i-1}}^{\tau_i} \{\alpha p - D(u)\}\mathrm{d}u. \tag{3.6}$$

The amount of inventory on the interval $[\tau_i, t_i^p)$ can be shown to be:

$$A_2 := \int_{\tau_i}^{t_i^p} (t_i^p - t)\{p - D(t)\}\mathrm{d}t + (t_i^p - \tau_i)\int_{t_{i-1}}^{\tau_i} \{\alpha p - D(t)\}\mathrm{d}t. \tag{3.7}$$

For $t \in [t_i^p, t_i]$, the changes in $I(t)$ is described by the differential equation

$$I'(t) = -D(t), \tag{3.8}$$

with boundary condition $I(t_i) = 0$.

Therefore,

$$I(t) = \int_{t}^{t_i} D(u)\mathrm{d}u, \qquad t_i^p \leq t < t_i. \tag{3.9}$$

The amount of inventory on the interval $[t_i^p, t_i]$ is then

$$A_3 := \int_{t_i^p}^{t_i} (t - t_i^p)D(t)\mathrm{d}t. \tag{3.10}$$

Our objective next is to express the total amount of the inventory $A_1 + A_2 + A_3$ as a function of t_{i-1} and t_i. This as we shall see is key in the determination of an optimal inventory policy (to be defined below). Before we proceed further note that

$$\alpha p(\tau_i - t_{i-1}) + p(t_i^p - \tau_i) = \int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t. \tag{3.11}$$

Relation (3.11) means that amount of good items produced in stage i is consumed in that stage. Also, we have

$$p(\tau_i - t_{i-1}) = \int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t. \tag{3.12}$$

It follows from (3.11) and (3.12) that

$$p(t_i^p - \tau_i) = (1-\alpha)\int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t, \tag{3.13}$$

and

$$\tau_i = t_{i-1} + \frac{1}{p}\int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t. \tag{3.14}$$

Expressions (3.13) and (3.14) lead after some simple computation to:

$$t_i^p = t_{i-1} + \frac{2-\alpha}{p}\int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t. \tag{3.15}$$

Lemma 3.1. *The total amount of inventory of good items $A_1 + A_2 + A_3$ is given by*

$$\int_{t_{i-1}}^{t_i} (t - t_{i-1})D(t)\mathrm{d}t - \frac{1}{2p'}\left\{\int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t\right\}^2, \tag{3.16}$$

where

$$p' = \frac{p}{1 + (1-\alpha)(2-\alpha)}\cdot \tag{3.17}$$

The proof of the lemma may be found in the appendix.

Remark 3.2. Note that (3.16) is a special case of the form

$$c_1\left\{\int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t\right\}^2 + c_2\left[\int_{t_{i-1}}^{t_i} (t - t_{i-1})D(t)\mathrm{d}t - \frac{1}{2z}\left\{\int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t\right\}^2\right], \tag{3.18}$$

with $c_1 = 0$, $c_2 = 1$, and $z = p'$. This form is found in [1] and will be key in the determination of the optimal inventory policy. Also, for $t \in [0, H]$

$$p' > D(t). \tag{3.19}$$

It is easy to see that since $\alpha p > D(t)$ by assumption and $0 < \alpha < 1$, (3.19) is true if $(1-\alpha)^2 > 0$, which certainly holds.

The amount of inventory of defective items, in stage i, needing rework can be shown (see Fig. 2) to be:

$$\frac{1}{2}(1-\alpha)(\tau_i - t_{i-1})(t_i^p - t_{i-1})p = \frac{(1-\alpha)(2-\alpha)}{2p}\left\{\int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t\right\}^2. \tag{3.20}$$

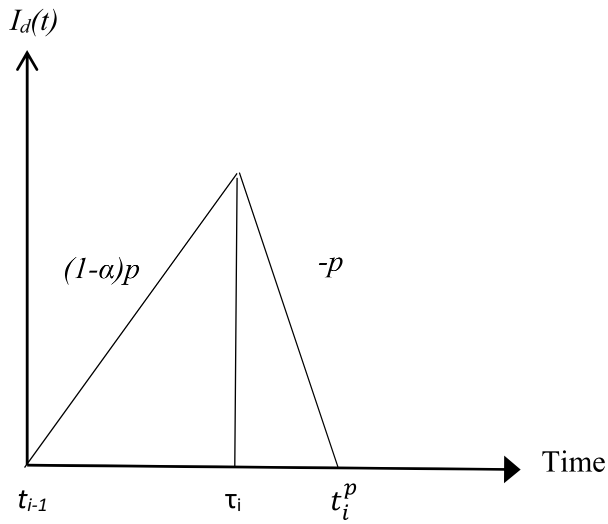

FIGURE 2. The level of inventory of defective items for Policy 1.

The last equality follows from (3.12) and (3.15). Moreover, the amount of defective items in a given cycle i can be shown to be $(1-\alpha)\int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t$. Therefore, the total amount of defective items on the the whole planning horizon is

$$(1-\alpha)\sum_{i=1}^{H}\int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t,$$

which is equal to $(1-\alpha)\int_0^H D(t)\mathrm{d}t$ and is constant. Hence, this term is dropped from any computations of the overall inventory costs.

The total cost incurred in stage i is equal to:

set-up costs + holding costs + penalty costs + purchasing costs.

It follows from (3.16) and (3.20) that the total cost in cycle i is given by:

$$K + S + c_c\int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t + c_p L(t_{i-1}, t_i) + c_h Q(t_{i-1}, t_i), \tag{3.21}$$

where

$$L(x,y) := \frac{(1-\alpha)(2-\alpha)}{2p}\left\{\int_x^y D(t)\mathrm{d}t\right\}^2, \tag{3.22}$$

$$Q(x,y) := \int_x^x (t-x)D(t)\mathrm{d}t - \frac{1}{2p'}\left\{\int_x^y D(t)\mathrm{d}t\right\}^2, \tag{3.23}$$

with p' given by (3.17).

An optimal inventory policy consists of determining the number of production stages and at each stage the schedules of the regular production and the rework production. This reduces to considering the optimization problem

$$\begin{aligned}&\mathbf{P}:\ \min\ n(K+S) + c_c\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t + \sum_{i=1}^{n}\left\{c_p L(t_{i-1}, t_i) + c_h Q(t_{i-1}, t_i)\right\},\\ &\text{subject to}:\quad t_0 = 0 < t_1 < \ldots < t_n = H,\end{aligned} \tag{3.24}$$

where the decision variables are n and the vector $(t_1,\ldots,t_n)$, and $L(t_{i-1},t_i)$, and $Q(t_{i-1},t_i)$ are given respectively by (3.22) and (3.23).

Remark 3.3. Note that once the starting and the finishing times of cycle i, say, are known, then so are τ_i (the finishing time of a regular production and the starting time of the rework run) and t_i^p (the finishing time of the rework run). These can be obtained from (3.14) and (3.15).

The next subsection contains the solution to Problem **P**.

3.1. Optimal inventory policy

This subsection contains the solution to Problem **P**. The mathematical foundations for solving **P** are found in Benkherouf and Gilding [3] and Al-Khamis *et al.* [1].

Note first that objective function term

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t=\int_0^H D(t)\mathrm{d}t, \tag{3.25}$$

is fixed and therefore will be dropped since it has no effect on the optimal solution. In this case **P** reduces to

$$\mathbf{P}':\ \min\ n(K+S)+\sum_{i=1}^{n}R(t_{i-1},t_i), \tag{3.26}$$

$$\text{subject to}:\ t_0=\ \ 0<t_1<\ldots<t_n=H, \tag{3.27}$$

with

$$R(x,y)=c_1\left\{\int_x^y D(t)\mathrm{d}t\right\}^2+c_2Q(x,y), \tag{3.28}$$

where $c_1=\dfrac{(1-\alpha)(2-\alpha)}{2p}c_p$, $c_2=c_h$, and $Q(x,y)$ is given by (3.23).

Problem $\mathbf{P}'$ belongs to a class of optimization problem of finite-horizon batching problem which was initiated in [10] and examined by Hill, Omar and Smith [11] and Omar and Smith [17]. This problem was solved in [1]. We shall state the results pertaining to $\mathbf{P}'$ without proof. Interested readers may consult [1].

Fix n and consider the problem of finding the minimum of the function

$$V_n(t_1,\ldots,t_n)=\sum_{i=1}^{n}R(t_{i-1},t_i), \tag{3.29}$$

under constraints (3.27).

Let

$$r:=2c_1-\frac{c_2}{p}, \tag{3.30}$$

$$F(x):=\frac{D'(x)}{rD(x)+c_2}\cdot \tag{3.31}$$

Theorem 3.4. *If the demand rate is log-concave, and*

(i) *$(r=0)$, or*
(ii) *$(r>0)$, and F is non-increasing or*
(iii) *$(r<0)$, and F is non-decreasing, then the minimum of the function $V_n(t_1,\ldots,t_n)$ under constraint (3.27) exist and is unique. Moreover, this minimum is the stationary point of the function V_n.*

Remark 3.5. Note that Theorem 3.4 implies that the solution of $\nabla V_n(t_1,\dots,t_n) = 0$ gives the unique optimal solution of $\mathbf{P}'$. In practice one can use any the available off the shelve software or use the univariate line search method alluded to in [3] to find the stationary point.

Theorem 3.6. *If $\boldsymbol{v}_n$ refers to the optimal value function of the function V_n under the assumptions of the Theorem 3.4, then $\boldsymbol{v}_n$ is convex in n.*

Theorem 3.7. *If $\boldsymbol{v}_n$ refers to the optimal value function of the function V_n under the assumptions of the Theorem 3.4, then the optimal number of stages n^* is prescribed by:*

(i) *if $K + S > \boldsymbol{v}_1 - \boldsymbol{v}_2$, then $n^* = 1$,*
(ii) *if there exists $N \geq 2$ such that $\boldsymbol{v}_{N-1} - \boldsymbol{v}_N > K + S > \boldsymbol{v}_N - \boldsymbol{v}_{N+1}$, then $n^* = N$,*
(iii) *if there exists $N \geq 2$ such that $K + S = \boldsymbol{v}_N - \boldsymbol{v}_{N+1}$, then $n^* = N$, and $n^* = N + 1$.*

Remark 3.8. Note under the assumptions of Theorem 3.4 and if the demand rate D is monotonic then so are the lengths of the stages.

4. Policy 2: N-cycle rework model

For an n-normal manufacturing run defective items produced are accumulated until the end of nth run. Stage $n+1$ is dedicated to the rework of these defective items: (see Figs. 3a and 3b). As a result Policy 2 may require large storage space to implement and thus higher penalty costs. Here, a typical cycle i begins at time t_{i-1} and ends at time t_i. If $i \leq n$, regular production lasts from time t_{i-1} to time t_i^p, $t_i^p < t_i$. Production is then stopped from time t_i^p to time t_i. Defective items produced at this stage are accumulated until the rework stage, which begins at time t_n and finishes at time H.

Key relations for this model are:

$$t_i^p - t_{i-1} = \frac{1}{\alpha p}\int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t, \tag{4.1}$$

and

$$t_i^p = t_{i-1} + \frac{1}{\alpha p}\int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t. \tag{4.2}$$

Similar computations to those undertaken in Section 2 show that the amount of inventory in stage $i \leq n$ is equal to:

$$W(t_i, t_{i-1}) := \int_{t_{i-1}}^{t_i} (t - t_{i-1})D(t)\mathrm{d}t - \frac{1}{2\alpha p}\left\{\int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t\right\}^2. \tag{4.3}$$

In this model defective items are stored in a warehouse until the start of the rework stage. Let h refers to the starting time of the rework stage. The next result shows that if the planning horizon is known, then so is h.

Lemma 4.1. *The starting time of the rework period is uniquely determined as a function of the planning horizon.*

Proof. For a given time span x of regular production, the amount of defective items produced is given by

$$(1-\alpha)\int_0^x D(t)\mathrm{d}t. \tag{4.4}$$

The fact that all defective items produced are reworked in the last period implies that

$$(1-\alpha)\int_0^h D(t)\mathrm{d}t = \int_h^H D(t)\mathrm{d}t. \tag{4.5}$$

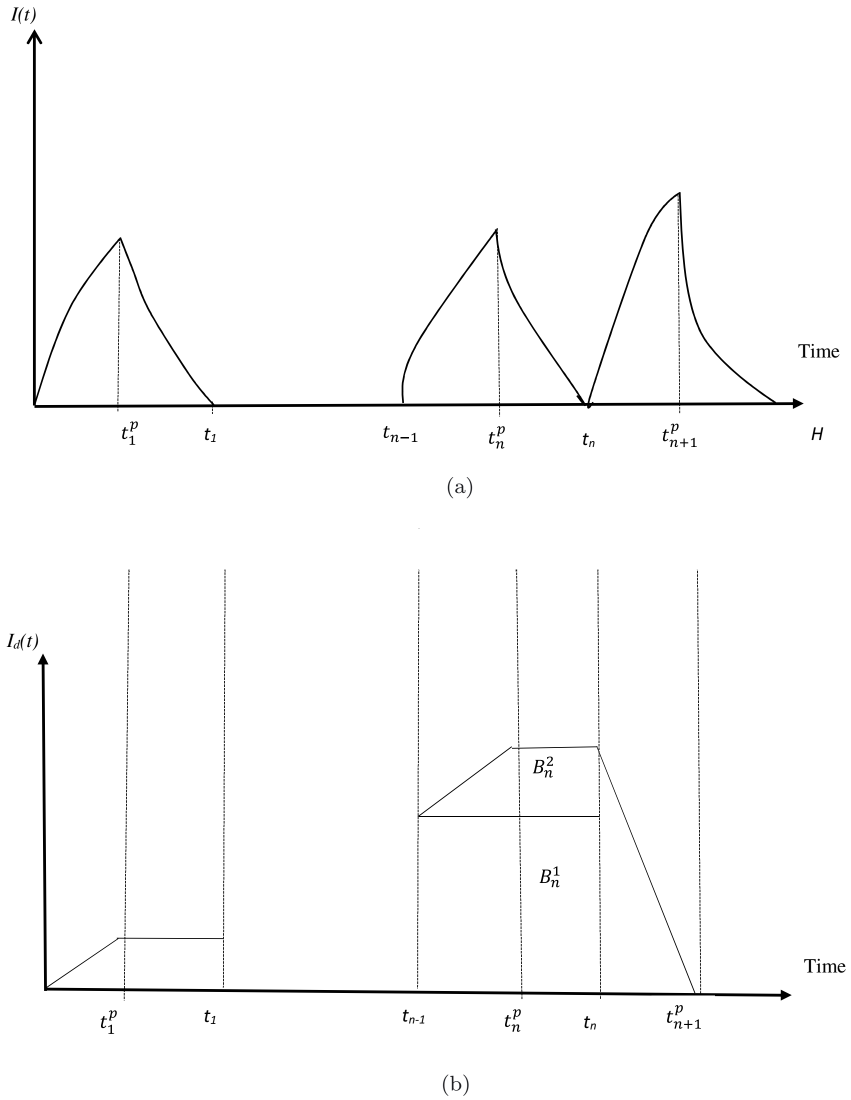

FIGURE 3. The level of inventory for a typical period i for Policy 1. (a) The level of stock of good items. (b) The level of stock of defective items.

This suggests examining the function

$$G(x) := (1-\alpha)\int_0^x D(t)\mathrm{d}t - \int_x^H D(t)\mathrm{d}t. \tag{4.6}$$

The function G is strictly increasing in x with $G(0) < 0$, and $G(H) > 0$. Therefore, $G(x) = 0$ has a unique solution on $[0, H]$. This leads to the required result. □

We shall next turn our attention to the computation of the amount of inventory of defective items in a typical stage $i \leq n$ (see Fig. 3b). This is equal to

$$B_i^1 + B_i^2. \tag{4.7}$$

B_i^1 refers to the area of a rectangle with length $(t_i - t_{i-1})$ and width $(1-\alpha)\sum_{j=1}^{i-1} p(t_j^p - t_{j-1})$. But by (4.1)

$$(1-\alpha)\sum_{j=1}^{i-1} p(t_j^p - t_{j-1}) = \frac{1-\alpha}{\alpha}\sum_{j=1}^{i-1}\int_{t_{j-1}}^{t_j} D(t)\mathrm{d}t = \frac{1-\alpha}{\alpha}\int_0^{t_{i-1}} D(t)\mathrm{d}t. \tag{4.8}$$

It follows that

$$B_i^1 = \frac{1-\alpha}{\alpha}(t_i - t_{i-1})\int_0^{t_{i-1}} D(t)\mathrm{d}t. \tag{4.9}$$

Now (see Fig. 3b),

$$B_i^2 = \frac{1}{2}(1-\alpha)p(t_i^p - t_{i-1})^2 + (1-\alpha)p(t_i^p - t_{i-1})(t_i - t_i^p). \tag{4.10}$$

By virtue of (4.1) and (4.2)

$$B_i^2 = -\left(\frac{1-\alpha}{2\alpha^2 p}\right)\left\{\int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t\right\}^2 + \left(\frac{1-\alpha}{\alpha}\right)(t_i - t_{i-1})\int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t. \tag{4.11}$$

The total area (pertaining to the defective items) in stage i is then given by:

$$\begin{aligned}&\left(\frac{1-\alpha}{\alpha}\right)\left\{t_i\int_0^{t_i} D(t)\mathrm{d}t - t_{i-1}\int_0^{t_{i-1}} D(t)\mathrm{d}t\right\}\\ &\quad + \left(\frac{1-\alpha}{\alpha}\right)\left[\int_{t_{i-1}}^{t_i}(t - t_{i-1})D(t)\mathrm{d}t - \frac{1}{2\alpha p}\left\{\int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t\right\}^2\right] - \left(\frac{1-\alpha}{\alpha}\right)\int_{t_{i-1}}^{t_i} tD(t)\mathrm{d}t.\end{aligned} \tag{4.12}$$

Remark 4.2. The sum of the first line of expression (4.12) from stage 1 up to the beginning of the rework stage is

$$\left(\frac{1-\alpha}{\alpha}\right)h\int_0^h D(t)\mathrm{d}t, \tag{4.13}$$

which is fixed by Lemma 4.1. Similarly, the sums for the last term in (4.12) is

$$-\left(\frac{1-\alpha}{\alpha}\right)\int_0^h tD(t)\mathrm{d}t, \tag{4.14}$$

which is again fixed.

The cost in the last stage can be shown to be:

$$\begin{aligned}&c_h\left\{\int_h^H (t-h)D(t)\mathrm{d}t - \frac{1}{2p}\left\{\int_h^H D(t)\mathrm{d}t\right\}^2\right\} + \frac{1}{2}c_p(1-\alpha)(t_{n+1}^p - h)\int_0^h D(t)\mathrm{d}t\\ &= c_h\left\{\int_h^H (t-h)D(t)\mathrm{d}t - \frac{1}{2p}\left\{\int_h^H D(t)\mathrm{d}t\right\}^2\right\} + \frac{1}{2p}c_p\left\{\int_h^H D(t)\mathrm{d}t\right\}^2.\end{aligned} \tag{4.15}$$

Thus, the total inventory cost when $n-$regular production cycles are initiated:

set-up costs + holding costs + penalty costs + purchasing costs.

$$nK + S + \sum_{i=1}^{n} \left\{ c_h + \left(\frac{1-\alpha}{\alpha}\right) c_p \right\} W(t_{i-1}, t_i) + c_h \left\{ \int_h^H (t-h)D(t)\mathrm{d}t - \frac{1}{2p}\left\{ \int_h^H D(t)\mathrm{d}t \right\}^2 \right\}$$
$$+ \frac{1}{2p}c_p \left\{ \int_h^H D(t)\mathrm{d}t \right\}^2 + \left(\frac{1-\alpha}{\alpha}\right) c_p \left\{ h\int_0^h D(t)\mathrm{d}t - \int_0^h tD(t)\mathrm{d}t \right\} + c_c \int_0^H D(t)\mathrm{d}t, \tag{4.16}$$

where $W(t_{i-1}, t_i)$ is given by (4.3). This is justified by (4.5) and

$$p(t_{n+1}^p - h) = \int_0^H D(t)\mathrm{d}t. \tag{4.17}$$

As in Section 3 we drop the terms that are fixed (see Rem. 4.2). It follows that in order to determine the optimal number of regular production stages as well as their starting times and finishing times, we need to solve the following mixed integer non-linear programming problem:

$$\mathbf{D}: \ \min \ nK + \sum_{i=1}^{n} \tilde{R}(t_{i-1}, t_i),$$
$$\text{subject to:} \quad t_0 = 0 < t_1 < \ldots < t_n = h, \tag{4.18}$$

where

$$\tilde{R}(x,y) = \left\{ c_h + \left(\frac{1-\alpha}{\alpha}\right) c_p \right\} W(x,y), \tag{4.19}$$

and the decision variables are n and $(t_1, \ldots, t_{n-1})$.

4.1. Optimal inventory policy

Problem **D** is similar to Problem **P** of Section 3, and belongs to the class of problems treated in [1].

For fixed n, Problem $\mathbf{D'}$ reduces to the consideration of the following nonlinear programming problem of which the solution is given in Theorem 4.3 below.

$$\mathbf{D'}: \ \min \ \sum_{i=1}^{n} \tilde{R}(t_{i-1}, t_i),$$
$$\text{subject to:} \quad t_0 = 0 < t_1 < \ldots < t_n = h. \tag{4.20}$$

Let

$$\phi(x) = D(x) - \frac{D'(x)}{\alpha p D(x)}\cdot \tag{4.21}$$

The next result follows from [1].

Theorem 4.3. *If the demand rate is log-concave and ϕ is a non-decreasing function on the interval $[0,h]$, then $\mathbf{D'}$ has a unique solution. This solution is the stationary point of the objective function.*

Note that if D is linear or exponential, then they are log-concave and ϕ is non-decreasing.

Write

$$\tilde{V}_n(t_1, \ldots, t_n) = \sum_{i=1}^{n} \tilde{R}(t_{i-1}, t_i). \tag{4.22}$$

Theorems 3.6, 3.7 and Remark 3.8 apply verbatim by taking V_n to be $\tilde{V}_n$ and $(S+K)$ to be K.

5. Numerical experiments

This section is devoted to two sets of example: the linear and the exponential demand rates function. Both demand rate function satisfy the hypotheses of Theorems 3.4 and 4.3, where r defined in (3.30) is strictly negative. Also, c_c is set to zero since it has no effect on the selection of the optimal inventory policies. The numerical results below represent a sample of a much larger experiment.

Example 5.1. Let $D(t) = a + bt$, with $a = 10$, $b = 10$, $H = 5$, $\beta = 0.05$ ($\alpha = 0.95$), $c_h = 10$, $c_p = 12$, $K = 25$, $S = 30$, $p = 100$. The optimal times, number of stages, and costs for policies 1 and 2 are given in Table 1 below: Note that in Policy 2, the last run is devoted to rework. Also, the sign($-$) in the table means that no value is assigned to the corresponding times in the row.

Table 1. The optimal inventory policy for the linear demand rate function.

Policy	n^*	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}	t_{11}	TC_n
1	7	0.874	1.639	2.347	3.025	3.685	4.341	5	–	–	–	–	774.345
2	12	0.562	1.064	1.530	1.973	2.401	2.817	3.227	3.634	4.039	4.447	5	739.351

Policy 1 recommends 7 runs with a total cost of 774.345, while Policy 2 recommends 11 normal runs and 1 rework run with a total cost of 739.351. It is not surprising that Policy 2 does better than Policy 1 for some parameters values of problem. However, it seems reasonable to expect that as the penalty cost c_p increases, Policy 1 will outperform Policy 2. In fact, there exists a threshold value for c_p (c_p^*) after which Policy 1 does always better than Policy 2 (see Tab. 2):

Computations show that $c_p^* \approx 14.32165$.

Table 2. Sensitivity analysis with respect to c_p.

c_p	12	13	14	15	16	17	18
TC_n : Policy 1	774.345	779.672	780.269	781.377	782.484	783.591	784.698
TC_n : Policy 2	739.351	757.129	772.952	792.685	814.991	828.242	846.02

Table 2 reveals, that by taking the model with $c_p = 12$ as a base model, the value of the optimal cost function for Policy 2 is highly sensitive to changes in the penalty cost as compared to Policy 1. For example an increase of around 33% in c_p has an effect of 1% increase on the cost of Policy 1 and around 10% on Policy 2.

Table 3 contains the results sensitivity experiment analysis with respect to the set-up cost, S, for rework. It is noticeable when S is small Policy 1 does better than Policy 2 but as S increases the balance shifts towards Policy 2. Again, there is a critical value of the set-up cost $S^* \approx 25$ after which Policy 2 is the recommended policy. It can also be deduced from the table that the optimal cost for Policy 1 is sensitive to changes in S, whereas the optimal inventory policy for Policy 2 remains unchanged.

Table 3. Sensitivity analysis with respect to S.

S	0	10	20	30	40	50	60	70
TC_n : Policy 1	477.388	601.093	685.31	774.345	865.31	917.21	968.941	1028.94
TC_n : Policy 2	709.351	719.351	729.351	739.351	740.351	750.351	769.351	779.351

Tables 4 and 5 below present the results of sensitivity analysis with respect to β (the proportion of defective items produced) and c_h (the holding cost). The base model taken is that found in Table 1. Table 4 shows that Policy 1 is not sensitive to changes in β. In fact an increase from 5% to 9% (a change of 80%) lead to an increase

TABLE 4. Sensitivity analysis with respect to β.

β	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
TC_n : Policy 1	772.295	772.793	773.300	773.818	774.345	774.881	775.427	775.983	776.548
TC_n : Policy 2	538.990	621.210	661.273	698.921	739.351	778.700	818.265	858.053	898.07

TABLE 5. Sensitivity analysis with respect to c_h.

c_h	6	7	8	9	10	11	12	13	14
TC_n : Policy 1	624.685	662.103	713.102	745.577	774.345	811.756	851.057	875.470	908.426
TC_n : Policy 2	634.740	665.464	689.847	717.249	739.351	764.489	784.702	808.266	826.891

of less than 0.5% in the overall inventory costs. However, Policy 2 is sensitive to changes in β. For example an increase of β from 5% to 9% lead to an increase of costs of around 21%. Moreover, Table 6 indicates that Policies 1 and 2 are sensitive to changes in c_h. For a change from 10% to 40% in c_h, Policy 1's costs varied from approximately 4% to approximately 19 % while that of Policy 2 from 3% to 12%.

Example 5.2. Let $D(t) = a\exp(bt)$, with $a = 1$, $b = 0.85$, $H = 5$, $\beta = 0.05$ ($\alpha = 0.95$), $c_h = 10$, $c_p = 12$, $K = 25$, $S = 30$, $p = 100$. The optimal times, number of stages, and costs for policies 1 and 2 are given in Table 6.

TABLE 6. The optimal inventory policy for the exponential demand rate function.

Policy	n^*	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	TC_n
1	6	1.599	2.621	3.375	3.982	4.508	5	–	–	541.395
2	8	1.268	2.139	2.796	3.322	3.790	4.197	4.574	5	437.446

Policy 1 recommends 6 runs with a total cost of 541.395, while Policy 2 recommends 8 normal runs and 1 rework run with a total cost of 697.766, and hence the presence of the sign $(-)$ for Policy 1. Here Policy 2 does better for this set of parameters. Also, similar computations to those undertaken for Example 5.1 can be repeated here to get a value c_p^* of c_p where $c_p^* > 12$ for which Policy 1 outperforms Policy 2 when $c_p < c_p^*$. Likewise, there exists a value of $S < 30$ where Policy 1 again overtakes Policy 2. This is left for interested readers to check. Finally, sensitivity analysis with respect to β and c_h revealed similar behavior, as in Example 5.1, of the total inventory costs.

6. CONCLUSIONS

This paper proposes a method for finding the optimal batch size for an inventory model with rework, time-varying demand and finite planning horizon. Optimal inventory policies were developed under two operational policies: in Policy 1 defective items are reworked within cycle while in Policy 2 defective items are reworked after n cycles. The key in the derivation of the optimal policies is form (3.18) of the cost functions in policies 1 and 2. Direct application of an earlier methodology in [1] leads to a complete characterization of the optimal policies. Also, numerical experiments show that Policy 2 outperforms Policy 1 when the penalty cost is relatively small or the set-up for rework is relatively large. Moreover, Policy 2 is sensitive to the set-up cost of rework unlike Policy 1 which remains unchanged. Also, changes in the defect factor appears to have very little influence on the overall costs for Policy 1, unlike that of Policy 2. However, both costs of policies 1 and 2 are slightly sensitive to changes in the holding costs.

The model examined in this paper, although a simplification of reality, can provide inventory managers with a mean of quantifying costs for time-varying demand inventory models with imperfect production. Insight can also be gained from the optimal Policy suggested in this paper, which state that careful timing and schedule of regular and rework runs could result in significant cost reduction.

Possible extensions of the present paper may include:

(1) the possibility of variation of the defective rate β from cycle to cycle, throughout the planning horizon, as in [21]. The form (3.18) will be a function of the a defective items which is cycle dependent. This will not change fundamentally the problem as the general theory developed in [3] may be applied for this case. Technical details remains to be worked out.
(2) the possibility of extension of the EPQ with production capacity limitation and breakdown as in [24].
(3) the possibility of variation within Policy 2 as discussed in Liu *et al.* [15]. Here, given n regular production runs and m rework runs, it is desired to determine the sequence in which these runs are operated. Policy 2 is denoted as $(n,1)$ policy since it allows for n normal production run and one rework.
(4) deterioration of the product as in [27]. This will allow for extra flexibility in modeling items that may lose quality over time. The form (3.18) for the present study will no longer be valid.
(5) inspection as in Ullah and Kang [25] or Konstantaras *et al.* [13]. In this case, form (3.18) of the total costs is lost. It remains an open problem how the results of [1,3] apply to this case.
(6) different detection scenarios of defective items. This is discussed in [4].

Appendix A. Proof of Lemma 1

Proof. Expressions (3.3), (3.7), and (3.10) show that $A_1 + A_2 + A_3$ is equal to:

$$\int_{t_{i-1}}^{\tau_i} (\tau_i - t)\{\alpha p - D(t)\}\mathrm{d}t + \int_{\tau_i}^{t_i^p} (t_i^p - t)\{p - D(t)\}\mathrm{d}t + (t_i^p - \tau_i)\int_{t_{i-1}}^{\tau_i}\{\alpha p - D(t)\}\mathrm{d}t + \int_{t_i^p}^{t_i}(t - t_i^p)D(t)\mathrm{d}t$$

$$\begin{aligned} &= \tau_i \int_{t_{i-1}}^{\tau_i}\{\alpha p - D(t)\}\mathrm{d}t - \int_{t_{i-1}}^{\tau_i} t\{\alpha p - D(t)\}\mathrm{d}t \\ &\quad + t_i^p\int_{\tau_i}^{t_i^p}\{p - D(t)\}\mathrm{d}t - \int_{\tau_i}^{t_i^p} t\{p - D(t)\}\mathrm{d}t \\ &\quad + t_i^p\int_{t_{i-1}}^{\tau_i}\{\alpha p - D(t)\}\mathrm{d}t - \tau_i\int_{tt_{i-1}}^{\tau_i}\{\alpha p - D(t)\}\mathrm{d}t \\ &\quad + \int_{t_i^p}^{t_i} tD(t)\mathrm{d}t - \int_{t_i^p}^{t_i} t_i^p D(t)\mathrm{d}t. \end{aligned} \tag{A.1}$$

Some algebra, using (A.1), lead to:

$$-\alpha p\int_{t_{i-1}}^{\tau_i} t\mathrm{d}t + t_i^p\left\{p(t_i^p - \tau_i) + \alpha p(\tau_i - t_{i-1}) - \int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t\right\} + \int_{t_{i-1}}^{t_i} tD(t)\mathrm{d}t - p\int_{\tau_i}^{t_i^p} t\mathrm{d}t.$$

Now (3.11) gives that $A_1 + A_2 + A_3$ is equal to:

$$-\alpha p\int_{t_{i-1}}^{\tau_i} t\mathrm{d}t + \int_{t_{i-1}}^{t_i} tD(t)\mathrm{d}t - p\int_{\tau_i}^{t_i^p} t\mathrm{d}t = -\frac{1}{2}\alpha p(\tau_i - t_{i-1})(\tau_i + t_{i-1}) - \frac{1}{2}p(t_i^p - \tau_i)(t_i^p + \tau_i) + \int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t. \tag{A.2}$$

Expression (A.2) reduces, by (3.12), (3.14), and (3.15), to:

$$-\frac{1}{2}\alpha\left\{\int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t\right\}\left\{2t_{i-1} + \frac{1}{p}\int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t\right\} - \frac{1}{2}(1-\alpha)\left\{\int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t\right\}\left\{2\tau_i + \frac{1-\alpha}{p}\int_{t_{i-1}}^{t_i} D(t)\mathrm{d}t\right\} + \int_{t_{i-1}}^{t_i} tD(t)\mathrm{d}t. \tag{A.3}$$

Again, after some algebra the above simplifies to:

$$\int_{t_{i-1}}^{t_i} (t - t_{i-1}) D(t) \mathrm{d}t - \frac{1}{2p} \{1 + (1-\alpha)(2-\alpha)\} \left\{ \int_{t_{i-1}}^{t_i} D(t) \mathrm{d}t \right\}^2, \tag{A.4}$$

which completes the proof. □

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