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ANALYSIS OF A GI/M/1 QUEUE IN A MULTI-PHASE SERVICE ENVIRONMENT WITH DISASTERS

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Abstract. In this paper, we study a single server GI/M/1 queue in a multi-phase service environment with disasters, where the disasters occur only when the server is busy serving customers. Whenever a disaster occurs in an operative service phase, all present customers are forced to leave the system simultaneously, the server abandons the service and an exponential repair time is set on. After the system is repaired, the server resumes his service and moves to service phase i immediately with probability $q_i, i = 1, 2, \dots, N$. Using the matrix analytic approach and semi-Markov process, we obtain the stationary queue length distribution at both arrival and arbitrary epochs. After introducing tagged customers and the concept of a cycle, we also derive the sojourn time distribution, the duration of a cycle, and the length of the server's working time in a service cycle. In addition, numerical examples are presented to illustrate the impact of some critical model parameters on performance measures.

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1. Introduction

Due to important applications in complex modern communication systems, networks and manufacturing systems, there has been a notable interest in the study of queues with disasters. Towsley and Tripathi [22] studied the M/M/1 queue with disasters for the purpose of studying distributed database systems with site failures. Jain and Sigman [9], Yang and Chae [23] respectively extended the idea in [22] to M/G/1 queue and GI/M/1 queue. Since the introduction of disasters, a rapid increase of researches on this topic has appeared, such as the work on a queue with system disasters and impatient customers (see *e.g.* [5, 21, 24]). In [24], he first assumed that the system suffers occasionally a disastrous breakdown. Then, he analyzed the model and derived some performance measures. Next, Chakravarthy [5] generalized the model in [24] to Markovian arrivals and obtained the steady state probabilities by the matrix analytic approach. Sudhesh [21] obtained an explicit transient solution for the state probabilities of the same model studied in [24]. Recently, Dimou and Economou [7] gave a complementing study of [24], where the customers become impatient and leave the system according to a geometric distribution while the server is in repair. Baumann and Sandmann [2] studied a state dependent M/M/c queue with disasters in random environment, and provided a matrix analytic algorithm to obtain the stationary distribution of the queue length. Kim and Lee [11] dealt with an M/G/1 queue with disasters and working breakdown services, in which the system continues to providing services for arriving customers during

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the repair period, instead of stopping serving customers completely. Mytalas and Zazanis [16] considered a queueing system with batch Poisson arrival with catastrophes and repairs under a multiple adapted vacation policy, and provided a quite complete analysis of the system. Jiang *et al.* [10] investigated an M/G/1 queue in a multi-phase random environment, in which the system is subjected to disastrous breakdowns, and they obtained various important performance measures, such as the sojourn time distribution and the length of the server's working time in a service cycle. Due to the discrete-time queues are more suitable for describing the telecommunication network, digital communication systems and other related areas, there is a growing interest in the analysis of the discrete-time queues with disasters. For the discrete-time queues with disasters, there exists a large volume of references as well (see *e.g.* [1, 18, 19], *etc.*). Furthermore, Lee *et al.* [13] discussed a discrete-time single server Geo/G/1 queue with disasters and general repair time. Lee and Yang [12] studied a N-policy of a discrete-time Geo/G/1 queue with disasters in order to analyze the power saving scheme in wireless sensor networks under unreliable network connections. There are also a number of researchers studied the topic from an economic viewpoint. Some excellent papers can be seen in ([3, 4, 8]). In these papers, all of the authors studied the queueing systems with the assumption that the arriving customers choose whether to join the system or balk, relying on a natural reward-cost structure.

Motivated by applications of queueing models on computer networks and telecommunication systems, and the excellent work of Paz and Yechiali [20], we generalize the queueing model presented by [20] in which the M/M/1 queue was studied, and extend this model to a GI/M/1 queue operating in a multi-phase service environment with disasters. The difference between [20] and this paper is that we assume the interarrival times are independent and identically distributed (iid), follow a general distribution. More importantly, we assume that the disasters have no effect on the system whenever the server is idle in operative service phase or under repair. Following the idea provided by Li *et al.* [14] and Li and Tian [15], which used matrix analytic method and semi-Markov process in analyzing GI/M/1 queue with working vacations, we obtain the stationary queue size distribution at both arrival and arbitrary epochs. We also give some important performance measures such as the cycle analysis, the sojourn time of a customer and the length of the server's working time in a service cycle.

The rest of this paper is organized as follows: In Section 2, we give the system description. In Sections 3 and 4, using the matrix analytic method and constructing a semi-Markov process, we derive the stationary queue length distribution at arrival epochs and arbitrary epochs. Sections 5, 6 and 7 are devoted to the sojourn time distribution, cycle analysis and results for the length of the server's working time in a service cycle. In Section 8, some numerical examples are provided to illustrate the impact of some parameters on some performance measures. Section 9 is the conclusion.

2. Model description

In this paper, we consider a GI/M/1 queue in a multi-phase service environment with disasters. The queueing model is described in detail below.

(1) Interarrival times $\{A_k, k \geq 1\}$ are iid with a general distribution denoted by $A(\nu)$ with a mean $\frac{1}{\lambda}$ and a Laplace Stieltjes transform (LST) denoted by $A^*(s)$.
(2) Under operative service phase i, the service times are exponentially distributed with parameter $\mu_i, i = 1, 2, \ldots, N$.
(3) Time to a disaster in service phase i also follows an exponential distribution with parameter $\eta_i, i = 1, 2, \ldots, N$.
(4) Whenever a disaster occurs, all customers are forced to leave the system simultaneously, and the server stops working completely. After a repair time, the system resumes service and moves to operative service phase i immediately with probability q_i, where $\sum_{i=1}^{N} q_i = 1$. The repair times follow an exponential distribution with parameter η_0.

We further assume that the disasters have no effect on the system whenever the server is idle in operative service phase or in repair phase, that is, the disasters occur only when the server is in operation.

3. The stationary queue length distribution at arrival epochs

In this section, we construct an embedded Markov chain to derive the stationary queue length distribution at arrival epochs. It is worth noting that as long as $\eta_i > 0, i = 0, 1, 2, \ldots, N$, the system in consideration can be analyzed in steady state. Actually, whenever a disaster occurs, all customers are forced to abandon the system, which means that the number of customers never goes to infinity.

Suppose τ_k be the arrival epoch of kth customer with $\tau_0 = 0$. Let $L(t)$ denote the number of customers in the system at time t, and let $L_k = L(\tau_k - 0)$ be the number of customers seen by the kth arrival instant. Define

$$J_k = \begin{cases} 0, & \text{the } k\text{th arrival occurs during a repair period,} \\ i, & \text{the } k\text{th arrival occurs during operative service phase } i, \end{cases}$$

for $i = 1, 2, \ldots, N$.

Then the system can be described by the process $\{(L_k, J_k), k \geq 1\}$, which is an embedded Markov chain with state space

$$\{(n, i), n \geq 0, i = 0, 1, 2, \ldots, N\}.$$

Next, we introduce the following transition probabilities of $\{(L_k, J_k)\}$ to express the transition matrix $\{(L_k, J_k), k \geq 1\}$. Let

$$P_{(h,l),(m,j)} = P(L_{k+1} = m, J_{k+1} = j | L_k = h, J_k = l), 0 \leq m \leq h + 1, l, j = 0, 1, 2, \ldots, N.$$

Now, we consider various cases according to the ordering of various times (remaining repair time or time to a disaster, next interarrival time, overall service time for the present customers, *etc.*).

Case 1: The system is in an operating mode i, not all present customers are served before the arrival of the next customer, and the next interarrival time is less than the time to a disaster. Transitions of type $(n, i) \to (m, i)$ with $n \geq 0$, $1 \leq m \leq n + 1$, $i = 1, 2, \ldots, N$.

Case 2: In this case, we consider three different ways. First, the system is in an operating mode i, all present customers are served before the arrival of the next customer, and the next interarrival time is less than the time to a disaster. Second, the system is in an operating mode i, all present customers are served before the occurrence of the next catastrophe, and the next interarrival time is greater than the time to a disaster. Third, the system is in an operating mode i, not all present customers are served before the occurrence of the next catastrophe, the server enters into a repair period and the repair process has ended before the next interarrival time, meanwhile, the server resumes his service and moves to service phase i once again. Transitions of type $(n, i) \to (0, i)$ with $n \geq 0$, $i = 1, 2, \ldots, N$.

Case 3: The system is in an operating mode i, not all present customers are served before the occurrence of the next catastrophe, the server enters into a repair period and the repair process has ended before the arrival of the next customer, meanwhile, the server resumes his service and moves to service phase $j, j \neq i$. Transitions of type $(n, i) \to (0, j)$ with $n \geq 0$, $i, j = 1, 2, \ldots, N, i \neq j$.

Case 4: The system is in an operating mode i, the overall service time for the present customers is greater than the time to a disaster, the server enters into a repair period and the repair process has not ended as the arrival of the next customer. Transitions of type $(n, i) \to (0, 0)$ with $n \geq 0$, $i = 1, 2, \ldots, N$.

Case 5: The system is in repairing mode, the next interarrival time is less than the remaining repair time. Transitions of type $(n, 0) \to (n + 1, 0)$ with $n \geq 0$.

Case 6: The system is in repairing mode, the remaining repair time is less than the next interarrival time. After the server is repaired, the system moves to operative service phase i. In operating mode i, not all present customers are served before the arrival of the next customer, and the arrival of the next customer occurs before the occurrence of the next catastrophe. Transitions of type $(n,0) \to (m,i)$ with $n \geq 0, 1 \leq m \leq n+1, i = 1,2,\dots,N$.

Case 7: The system is in repairing mode, the remaining repair time is less than the next interarrival time. After the server is repaired, the system moves to operative service phase i. In operating mode i, not all present customers are served before the occurrence of the next catastrophe, the server enters into a new repair period, and the arrival of the next customer occurs before the end of the repair process. Transitions of type $(n,0) \to (0,0)$ with $n \geq 0$.

Case 8: The system is in repairing mode, the remaining repair time is less than the next interarrival time. After the server is repaired, the system moves to operative service phase i. Then, we consider three possible events. First, in operating mode i, all present customers are served before the arrival of the next customer, and the arrival of the next customer occurs before the occurrence of the next catastrophe. Second, in operating mode i, the overall service time for the present customers is less than the time to a disaster, the arrival of the next customer occurs after the occurrence of the next catastrophe. Third, in operating mode i, not all present customers are served before the occurrence of the next catastrophe, the server enters into a new repair period, and the repair period has ended before the arrival of the next customer. After the server is repaired again, the system resumes its service and moves to service phase i. Transitions of type $(n,0) \to (0,i)$ with $n \geq 0, i = 1,2,\dots,N$.

In order to give the formulas for each case, we define various random variables as follows:

A stands for the limit of A_k as $k \to \infty$;

D_i denotes the interarrival times of disaster in service phase $i, i = 1,2,\dots,N$;

$S_{h,i}$ denotes the service time of the hth customer in service phase i with $S_{0,i} \equiv 0, i = 1,2,\dots,N$;

T_0 denotes the repair time;

$\{T_0^{(k)}, k \geq 0\}$ is an iid sequence of the repair time with $T_0 = T_0^{(0)}$.

Then, we will give the explicit expressions for various cases. For Case 1, the probability of the transition from (n,i) to (m,i) can be obtained by

$$\begin{aligned} P_{(n,i),(m,i)} &= P(A < D_i, n+1-m \text{ customers are served in } A) \\ &= \int_0^\infty \mathrm{e}^{-\eta_i t} \frac{(\mu_i t)^{n+1-m}}{(n+1-m)!} \mathrm{e}^{-\mu_i t} \mathrm{d}A(t) = b_{n+1-m,i}, 1 \leq m \leq n+1. \end{aligned}$$

For case 2, we have the transition probability from (n,i) to $(0,i)$

$$\begin{aligned} P_{(n,i),(0,i)} =& P\left(A < D_i, \sum_{h=0}^{n+1} S_{h,i} < A\right) + P\left(A > D_i, \sum_{h=0}^{n+1} S_{h,i} < D_i\right) \\ &+ q_i P\left(A > D_i + T_0, \sum_{h=0}^{n+1} S_{h,i} > D_i\right) \\ =& \int_0^\infty \mathrm{e}^{-\eta_i t} \left[1 - \sum_{h=0}^{n} \frac{(\mu_i t)^h}{h!} \mathrm{e}^{-\mu_i t}\right] \mathrm{d}A(t) \\ &+ \int_0^\infty \int_0^t \left[1 - \sum_{h=0}^{n} \frac{(\mu_i x)^h}{h!} \mathrm{e}^{-\mu_i x}\right] \eta_i \mathrm{e}^{-\eta_i x} \mathrm{d}x \mathrm{d}A(t) \\ &+ q_i \int_0^\infty \int_0^t \left[1 - \mathrm{e}^{-\eta_0 (t-x)}\right] \left[\sum_{h=0}^{n} \frac{(\mu_i x)^h}{h!} \mathrm{e}^{-\mu_i x}\right] \eta_i \mathrm{e}^{-\eta_i x} \mathrm{d}x \mathrm{d}A(t) \\ =& 1 - \sum_{h=0}^{n} b_{h,i} - \sum_{h=0}^{n} c_{h,i} + q_i \left(\sum_{h=0}^{n} c_{h,i} - \sum_{h=0}^{n} d_{h,i}\right). \end{aligned}$$

For Case 3, we have the transition probability from (n,i) to $(0,j)$

$$\begin{aligned}
P_{(n,i),(0,j)} &= q_j P\left(A > D_i + T_0, \sum_{h=0}^{n+1} S_{h,i} > D_i\right) \\
&= q_j \int_0^\infty \int_0^t \left[1 - \mathrm{e}^{-\eta_0(t-x)}\right]\left[\sum_{h=0}^{n} \frac{(\mu_i x)^h}{h!}\mathrm{e}^{-\mu_i x}\right]\eta_i \mathrm{e}^{-\eta_i x}\mathrm{d}x\mathrm{d}A(t) \\
&= q_j \left(\sum_{h=0}^{n} c_{h,i} - \sum_{h=0}^{n} d_{h,i}\right), j \neq i, j = 1, 2, \ldots, N.
\end{aligned}$$

For Case 4, the probability of the transition from (n,i) to $(0,0)$ can be obtained by

$$\begin{aligned}
P_{(n,i),(0,0)} &= P\left(D_i < A < D_i + T_0, \sum_{h=0}^{n+1} S_{h,i} > D_i\right) \\
&= \int_0^\infty \int_0^t \mathrm{e}^{-\eta_0(t-x)}\left[\sum_{h=0}^{n} \frac{(\mu_i x)^h}{h!}\mathrm{e}^{-\mu_i x}\right]\eta_i \mathrm{e}^{-\eta_i x}\mathrm{d}x\mathrm{d}A(t) \\
&= \sum_{h=0}^{n} d_{h,i}.
\end{aligned}$$

For Case 5, we have the transition probability from (n,i) to $(n+1,0)$

$$P_{(n,0),(n+1,0)} = P(A < T_0) = \int_0^\infty \mathrm{e}^{-\eta_0 t}\mathrm{d}A(t) = A^*(\eta_0).$$

For Case 6, we have the transition probability from $(n,0)$ to (m,i)

$$\begin{aligned}
P_{(n,0),(m,i)} &= q_i P(T_0 < A < T_0 + D_i, n+1-m \text{ customers are served in} A - T_0) \\
&= q_i \int_0^\infty \int_0^t \mathrm{e}^{-\eta_i(t-x)}\frac{(\mu_i(t-x))^{(n+1-m)}}{(n+1-m)!}\mathrm{e}^{-\mu_i(t-x)}\eta_0 \mathrm{e}^{-\eta_0 x}\mathrm{d}x\mathrm{d}A(t) \\
&= q_i v_{n+1-m,i}^{(0)}, 1 \leq m \leq n+1.
\end{aligned}$$

For Case 7, we have the transition probability from $(n,0)$ to $(0,0)$

$$\begin{aligned}
P_{(n,0),(0,0)} &= \sum_{i=1}^{N} q_i P\left(T_0 + D_i < A < T_0 + D_i + T_0^{(1)}, \sum_{h=0}^{n+1} S_{h,i} > D_i\right) \\
&= \sum_{i=1}^{N} q_i \int_0^\infty \int_0^t \left[\sum_{h=0}^{n} \frac{(\mu_i x)^h}{h!}\mathrm{e}^{-\mu_i x}\right]\eta_i \mathrm{e}^{-\eta_i x}((t-x)\eta_0 \mathrm{e}^{-\eta_0(t-x)})\mathrm{d}x\mathrm{d}A(t) \\
&= \sum_{i=1}^{N} q_i m_{n,i}^{(0)}.
\end{aligned}$$

Similarly, for Case 8, the probability of the transition from $(n,0)$ to $(0,i)$ can be obtained by

$$
\begin{aligned}
P_{(n,0),(0,i)} &= q_i P\left(T_0 < A < T_0 + D_i, \sum_{h=0}^{n+1} S_{h,i} < A - T_0\right) \\
&\quad + q_i P\left(A > T_0 + D_i, \sum_{h=0}^{n+1} S_{h,i} < D_i\right) \\
&\quad + q_i \sum_{j=1}^{N} q_j P\left(A > T_0 + D_j + T_0^{(1)}, \sum_{h=0}^{n+1} S_{h,j} > D_j\right) \\
&= q_i[1 - A^*(\eta_0)] - q_i \sum_{h=0}^{n} v_{h,i}^{(0)} - q_i E_{n,i} + q_i \sum_{j=1}^{N} q_j \left(E_{n,j} - m_{n,j}^{(0)}\right).
\end{aligned}
$$

The definition of the various quantities are given by

$$
\begin{aligned}
b_{k,i} &= \int_0^\infty \mathrm{e}^{-\eta_i t} \frac{(\mu_i t)^k}{k!} \mathrm{e}^{-\mu_i t} \mathrm{d}A(t), k \geq 0, \\
c_{k,i} &= \int_0^\infty \int_0^t \frac{(\mu_i x)^k}{k!} \mathrm{e}^{-\mu_i x} \eta_i \mathrm{e}^{-\eta_i x} \mathrm{d}x \mathrm{d}A(t), k \geq 0, \\
d_{k,i} &= \int_0^\infty \int_0^t \mathrm{e}^{-\eta_0(t-x)} \frac{(\mu_i x)^k}{k!} \mathrm{e}^{-\mu_i x} \eta_i \mathrm{e}^{-\eta_i x} \mathrm{d}x \mathrm{d}A(t), k \geq 0, \\
v_{k,i}^{(0)} &= \int_0^\infty \int_0^t \mathrm{e}^{-\eta_i(t-x)} \frac{(\mu_i(t-x))^k}{k!} \mathrm{e}^{-\mu_i(t-x)} \eta_0 \mathrm{e}^{-\eta_0 x} \mathrm{d}x \mathrm{d}A(t), k \geq 0, \\
m_{k,i}^{(0)} &= \int_0^\infty \int_0^t \left[\sum_{h=0}^{k} \frac{(\mu_i x)^h}{h!} \mathrm{e}^{-\mu_i x}\right] \eta_i \mathrm{e}^{-\eta_i x} ((t-x)\eta_0 \mathrm{e}^{-\eta_0(t-x)}) \mathrm{d}x \mathrm{d}A(t), k \geq 0, \\
E_{k,i} &= \int_0^\infty \int_0^t \left[\sum_{h=0}^{k} \frac{(\mu_i x)^h}{h!} \mathrm{e}^{-\mu_i x}\right] \eta_i \mathrm{e}^{-\eta_i x} (1 - \mathrm{e}^{-\eta_0(t-x)}) \mathrm{d}x \mathrm{d}A(t), k \geq 0.
\end{aligned}
$$

Once the transition probabilities are obtained, by the lexicographic sequence for the states, the transition probability matrix of $\{(L_k, J_k), k \geq 1\}$ can be written as the following block form

$$
P = \begin{bmatrix}
B_0 & A_0 & & & & \\
B_1 & A_1 & A_0 & & & \\
B_2 & A_2 & A_1 & A_0 & & \\
B_3 & A_3 & A_2 & A_1 & A_0 & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},
$$

where

$$A_0 = \begin{bmatrix} A^*(\eta_0) & q_1 v_{0,1}^{(0)} & q_2 v_{0,2}^{(0)} & \cdots & q_N v_{0,N}^{(0)} \\ & b_{0,1} & & & \\ & & b_{0,2} & & \\ & & & \ddots & \\ & & & & b_{0,N} \end{bmatrix},$$

$$A_n = \begin{bmatrix} 0 & q_1 v_{n,1}^{(0)} & q_2 v_{n,2}^{(0)} & \cdots & q_N v_{n,N}^{(0)} \\ & b_{n,1} & & & \\ & & b_{n,2} & & \\ & & & \ddots & \\ & & & & b_{n,N} \end{bmatrix}, n \geq 1,$$

$$B_n = \begin{bmatrix} \sum\limits_{i=1}^{N} q_i m_{n,i}^{(0)} & q_1 \beta_{n,1} & q_2 \beta_{n,2} & \cdots & q_N \beta_{n,N} \\ \sum\limits_{h=0}^{n} d_{h,1} & \gamma_{n,1,1} & \gamma_{n,1,2} & \cdots & \gamma_{n,1,N} \\ \sum\limits_{h=0}^{n} d_{h,2} & \gamma_{n,2,1} & \gamma_{n,2,2} & \cdots & \gamma_{n,2,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum\limits_{h=0}^{n} d_{h,N} & \gamma_{n,N,1} & \gamma_{n,N,2} & \cdots & \gamma_{n,N,N} \end{bmatrix},$$

with

$$\begin{aligned} \beta_{n,i} &= [1 - A^*(\eta_0)] - \sum_{h=0}^{n} v_{h,i}^{(0)} - E_{n,i} + \sum_{j=1}^{N} q_j \left(E_{n,j} - m_{n,j}^{(0)} \right), \\ \gamma_{n,i,i} &= 1 - \sum_{h=0}^{n} (b_{h,i} + c_{h,i}) + q_i \sum_{h=0}^{n} (c_{h,i} - d_{h,i}), \\ \gamma_{n,i,j} &= q_j \sum_{h=0}^{n} (c_{h,i} - d_{h,i}), i, j = 1, 2, \ldots, N, i \neq j. \end{aligned}$$

Because of the special upper triangular form of the matrices A_0 and $A_n, n \geq 1$, it follows from the matrix equation $R = \sum\limits_{n=0}^{\infty} R^n A_n$, which can be found in [17], that R is also a special upper triangular matrix having the following form

$$R = \begin{bmatrix} r_{0,0} & r_{0,1} & r_{0,2} & \cdots & r_{0,N} \\ 0 & r_{1,1} & 0 & \cdots & 0 \\ 0 & 0 & r_{2,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r_{N,N} \end{bmatrix}.$$

Lemma 3.1. *If $\eta_i > 0, i = 0, 1, 2, \ldots, N$, then the matrix equation $R = \sum_{n=0}^{\infty} R^n A_n$ has the minimal nonnegative solution*

$$R = \begin{bmatrix} r_{0,0} & r_{0,1} & r_{0,2} & \cdots & r_{0,N} \\ 0 & r_{1,1} & 0 & \cdots & 0 \\ 0 & 0 & r_{2,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r_{N,N} \end{bmatrix},$$

where $r_{0,0} = A^*(\eta_0)$, $r_{i,i}$ is the unique root in the range $0 < z < 1$ of the equations $z = A^*(\eta_i + \mu_i(1 - z)), i = 1, 2, \ldots, N$, and $r_{0,i} = q_i\alpha_i(r_{i,i} - r_{0,0})$ with $\alpha_i = \dfrac{\eta_0}{(\eta_0 - \eta_i) - \mu_i(1 - r_{0,0})}, i = 1, 2, \ldots, N$.

Proof. It is easy to see that $r_{0,0} = A^*(\eta_0)$, and $r_{i,i}$ satisfies the equation

$$\sum_{n=0}^{\infty} r_{i,i}^n b_{n,i} = r_{i,i}, i = 1, 2, \ldots, N,$$

that is, $r_{i,i}$ is the unique root in the range $0 < z < 1$ of the equation $z = A^*(\eta_i + \mu_i(1 - z))$. For the expression of $r_{0,i}$, by the matrix equation $R = \sum_{n=0}^{\infty} R^n A_n$, we find that $r_{0,i}$ satisfies the following equation

$$\sum_{n=0}^{\infty} q_i r_{0,0}^n v_{n,i}^{(0)} + \sum_{n=1}^{\infty} m_i(n) b_{n,i} = r_{0,i}, i = 1, 2, \ldots, N,$$

where $m_i(n) = r_{0,i}(r_{0,0}^{n-1} + r_{0,0}^{n-2} r_{i,i} + \ldots + r_{0,0} r_{i,i}^{n-2} + r_{i,i}^{n-1})$.

According to the result of $v_{n,i}^{(0)}$, we have

$$\sum_{n=0}^{\infty} q_i r_{0,0}^n v_{n,i}^{(0)} = \frac{q_i \eta_0}{(\eta_0 - \eta_i) - \mu_i(1 - r_{0,0})}(A^*(\eta_i + \mu_i(1 - r_{0,0})) - r_{0,0}).$$

Since

$$\begin{aligned} \sum_{n=1}^{\infty} m_i(n) b_{n,i} &= r_{0,i} \frac{1}{r_{0,0} - r_{i,i}} \sum_{n=1}^{\infty} (r_{0,0}^n - r_{i,i}^n) b_{n,i} \\ &= r_{0,i} \frac{1}{r_{0,0} - r_{i,i}} (A^*(\eta_i + \mu_i(1 - r_{0,0})) - r_{i,i}), \end{aligned}$$

we have

$$r_{0,i} = q_i \alpha_i (r_{i,i} - r_{0,0}),$$

where $\alpha_i = \dfrac{\eta_0}{(\eta_0 - \eta_i) - \mu_i(1 - r_{0,0})}$. Then we can gain the conclusion. □

If $\eta_i > 0, i = 0, 1, 2, \ldots, N$, the system under consideration can be analyzed in steady state. Let (L, J) be the stationary limit of the Markov chain (L_k, J_k). Denote

$$\begin{aligned} \pi_n &= (\pi_{n,0}, \pi_{n,1}, \pi_{n,2}, \ldots, \pi_{n,N}), \quad n \geq 0, \\ \pi_{n,i} &= \lim_{k \to \infty} P(L_k = n, J_k = i), \quad n \geq 0, \quad i = 0, 1, 2, \ldots, N. \end{aligned}$$

Theorem 3.2. *If $\eta_i > 0, i = 0, 1, 2 \ldots, N$, the steady state probability*

$$\pi_0 = (\pi_{0,0}, \pi_{0,1}, \pi_{0,2}, \ldots, \pi_{0,N})$$

satisfies the following set of equations

$$(B[R] - I)^T (\pi_{0,0}, \pi_{0,1}, \pi_{0,2}, \ldots, \pi_{0,N})^T = (0, 0, 0, \ldots, 0)^T, \tag{3.1}$$

$$\pi_0 (I - R)^{-1} \mathrm{e} = 1, \tag{3.2}$$

and π_n satisfies

$$\pi_n = \pi_0 R^n, n \geq 1,$$

where

$$B[R] = \begin{bmatrix} 1 - \sum_{i=1}^{N} \delta_i & \delta_1 & \delta_2 & \cdots & \delta_N \\ \theta_{1,0} & 1 - \sum_{j=0, j\neq 1}^{N} \theta_{1,j} & \theta_{1,2} & \cdots & \theta_{1,N} \\ \theta_{2,0} & \theta_{2,1} & 1 - \sum_{j=0, j\neq 2}^{N} \theta_{2,j} & \cdots & \theta_{2,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \theta_{N,0} & \theta_{N,1} & \theta_{N,1} & \cdots & 1 - \sum_{j=0, j\neq N}^{N} \theta_{N,j} \end{bmatrix},$$

and

$$\begin{aligned} \theta_{i,0} &= \frac{1}{1 - r_{i,i}} \frac{\eta_i [A^*(\eta_0) - r_{i,i}]}{\eta_i - \eta_0 + \mu_i (1 - r_{i,i})}, i = 1, 2, \ldots, N, \\ \theta_{i,j} &= q_j \xi_i - q_j \sigma_i, j = 1, 2, \ldots, N, j \neq i, \\ \delta_i &= \sum_{n=0}^{\infty} r_{0,0}^n q_i \left([1 - A^*(\eta_0)] - \sum_{h=0}^{n} v_{h,i}^{(0)} - E_{n,i} + \sum_{j=1}^{N} q_j (E_{n,j} - m_{n,j}^{(0)}) \right) \\ &\quad + \sum_{n=1}^{\infty} \sum_{j=1}^{N} m_j(n) \gamma_{n,j,i}, \end{aligned}$$

with

$$\begin{aligned} \xi_i &= \frac{\eta_i}{\mu_i + \eta_i - \mu_i r_{i,i}}, \\ \sigma_i &= \theta_{i,0} = \frac{1}{1 - r_{i,i}} \frac{\eta_i [A^*(\eta_0) - r_{i,i}]}{\eta_i - \eta_0 + \mu_i (1 - r_{i,i})}, \\ \sum_{n=0}^{\infty} r_{0,0}^n \sum_{h=0}^{n} v_{h,i}^{(0)} &= \frac{1}{1 - r_{0,0}} \alpha_i [A^*(\eta_i + \mu_i - \mu_i r_{0,0}) - A^*(\eta_0)], \\ \sum_{n=0}^{\infty} r_{0,0}^n E_{n,i} &= \frac{1}{1 - r_{0,0}} \frac{\eta_i}{\eta_i + \mu_i - \mu_i r_{0,0}} [1 - A^*(\eta_i + \mu_i - \mu_i r_{0,0})] \\ &\quad + \frac{1}{1 - r_{0,0}} \frac{\eta_i}{\eta_i - \eta_0 + \mu_i - \mu_i r_{0,0}} [A^*(\eta_i + \mu_i - \mu_i r_{0,0}) - A^*(\eta_0)], \\ \sum_{n=0}^{\infty} r_{0,0}^n m_{n,i}^{(0)} &= \frac{1}{1 - r_{0,0}} \frac{\eta_i \eta_0}{\eta_i - \eta_0 + \mu_i (1 - r_{0,0})} \int_0^{\infty} t e^{-\eta_0 t} \mathrm{d}A(t) \\ &\quad - \frac{1}{1 - r_{0,0}} \frac{\eta_i \eta_0 [A^*(\eta_0) - A^*(\eta_i + \mu_i - \mu_i r_{0,0})]}{[\eta_i - \eta_0 + \mu_i (1 - r_{0,0})]^2}, \end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{j=1}^{N} m_j(n)\gamma_{n,j,i} = {} & q_i\alpha_i - q_i\alpha_i\frac{\eta_i}{\mu_i+\eta_i-\mu_i r_{i,i}} + q_i\sum_{j=1}^{N} q_j\alpha_j\frac{\eta_j}{\mu_j+\eta_j-\mu_j r_{j,j}}\\ & - q_i\sum_{j=1}^{N} q_j\frac{\alpha_j}{1-r_{j,j}}\frac{\eta_j[A^*(\eta_0)-r_{j,j}]}{\eta_j-\eta_0+\mu_j(1-r_{j,j})}\\ & - q_i\frac{\alpha_i}{1-r_{0,0}}[1-A^*(\mu_i+\eta_i-\mu_i r_{0,0})]\\ & + q_i\frac{\alpha_i}{1-r_{0,0}}\frac{\eta_i[1-A^*(\mu_i+\eta_i-\mu_i r_{0,0})]}{\mu_i+\eta_i-\mu_i r_{0,0}}\\ & - q_i\sum_{j=1}^{N} q_j\frac{\alpha_j}{1-r_{0,0}}\frac{\eta_j[1-A^*(\mu_j+\eta_j-\mu_j r_{0,0})]}{\mu_j+\eta_j-\mu_j r_{0,0}}\\ & + q_i\sum_{j=1}^{N} q_j\frac{\alpha_j}{1-r_{0,0}}\frac{\eta_j[A^*(\eta_0)-A^*(\mu_j+\eta_j-\mu_j r_{0,0})]}{\eta_j-\eta_0+\mu_j(1-r_{0,0})}.\end{aligned}$$

Proof. The proof of this theorem can be seen in Appendix. □

4. The stationary queue length distribution at arbitrary epochs

In this section, using the semi-Markov process (SMP), we derive the limit distribution of $L(t)$.

Let L be the number of customers in the system at an arbitrary epoch, and let

$$P_n = P\{L=n\} = \lim_{t\to\infty} P\{L(t)=n\}.$$

We will construct a semi-Markov process to find P_n.

First, we define a new process $\{(Z(t),K(t)),t\geq 0\}$, where $Z(t)$ is the number of customers in the system just before the most recent arrival and $K(t)$ equals to 0 or i, if the most recent arrival sees the system in repair period or in operative service phase $i, i=1,2,\ldots,N$. In fact, $Z(t)=L_k, \tau_k\leq t<\tau_k+1$, and $K(t)=J_k, \tau_k\leq t<\tau_k+1$. Obviously, the Markov chain (L_k,J_k) is irreducible, aperiodic and positive recurrent, so, from the theory of semi-Markov process, $\{(Z(t),K(t)),t\geq 0\}$ is a SMP having $\{(L_k,J_k),k\geq 1\}$ for its embedded Markov chain. Let $w_{n,i}$ be the expected time that the semi-Markov process resides in the state (n,i), where

$$(n,i)\in\{(m,j)|m\geq 0, j=0,1,2,\ldots,N\}.$$

According to the definition of semi-Markov process, we have $w_{m,i}=\frac{1}{\lambda}$ for all (m,i). Let $f_{n,i}$ denote the limiting probability that the SMP is in state (n,i). Note that

$$f_{n,i} = \lim_{t\to\infty} P\{(Z(t),K(t))=(n,i)\} = \lim_{k\to\infty} P\{(L_k,J_k)=(n,i)\} = \pi_{n,i}.$$

From the theory of semi-Markov process, we also have

$$f_{n,i} = \frac{\pi_{n,i}w_{n,i}}{\sum\limits_{j=0}^{N}\sum\limits_{h=0}^{\infty}\pi_{h,j}w_{h,j}} = \pi_{n,i},$$

for all

$$(n,i)\in\{(m,j)|m\geq 0, j=0,1,2,\ldots,N\}.$$

Define A_E as the elapsed interarrival time at an arbitrary epoch in steady state. The density function of A_E is known as $\lambda P(A>t)$, where A is the interarrival time.

Then, for $n \geq 1$,

$$\begin{aligned} P_n &= f_{n-1,0} + \sum_{i=1}^{N} \sum_{j=n-1}^{\infty} f_{j,i} \int_0^\infty \mathrm{e}^{-\eta_i t} \frac{(\mu_i t)^{j+1-n}}{(j+1-n)!} \mathrm{e}^{-\mu_i t} \lambda(1-A(t))\mathrm{d}t \\ &= \pi_{n-1,0} + \sum_{i=1}^{N} \sum_{j=n-1}^{\infty} \pi_{j,i} \int_0^\infty \mathrm{e}^{-\eta_i t} \frac{(\mu_i t)^{j+1-n}}{(j+1-n)!} \mathrm{e}^{-\mu_i t} \lambda(1-A(t))\mathrm{d}t, \end{aligned} \tag{4.1}$$

and $P_0 = 1 - \sum\limits_{n=1}^{\infty} P_n$.

The expectation of L is given by

$$E[L] = \sum_{n=0}^{\infty} n P_n.$$

5. The stationary sojourn time distribution

In this section, considering a tagged customer, we will derive the LST of the stationary sojourn time of an arbitrary customer under the first-come-first-served (FCFS) discipline, where the sojourn time is defined as the period from the epoch at which he enters into the system to the epoch at which he leaves the system by either the occurrence of a disaster or the service completion. Let W and $W^*(s)$ denote the stationary sojourn time of an arbitrary customer and its LST, respectively.

First, we give a lemma below.

Lemma 5.1. *Let X, Y and Z be the random variables, and $Z = \min(X,Y)$, where X follows a general (arbitrary) distribution with mean $\frac{1}{\mu}$. The distribution function and its LST are $F_X(\nu)$ and $F_X^*(s)$. If Y follows an exponential distribution with parameter η, and X and Y are assumed to be independent, then*

$$E[\mathrm{e}^{-sZ}] = E[\mathrm{e}^{-s\min(X,Y)}] = \frac{\eta + sF_X^*(s+\eta)}{s+\eta}\cdot$$

Proof. First, we have

$$P(Z>t) = P(\min(X,Y)>t) = P(X>t)P(Y>t) = \mathrm{e}^{-\eta t}(1-F_X(t)), t>0,$$

and

$$P(Z<t) = 1 - \mathrm{e}^{-\eta t}(1-F_X(t)).$$

Then

$$\begin{aligned} E[\mathrm{e}^{-sZ}] &= \int_0^\infty \mathrm{e}^{-st}\mathrm{d}P(Z<t) \\ &= -\int_0^\infty \mathrm{e}^{-(s+\eta)t} d[1-F_X(t)] + \int_0^\infty \eta \mathrm{e}^{-(s+\eta)t}[1-F_X(t)]\mathrm{d}t \\ &= F_X^*(s+\eta) + \frac{\eta[1-F_X^*(s+\eta)]}{s+\eta} \\ &= \frac{\eta + sF_X^*(s+\eta)}{s+\eta}\cdot \end{aligned} \tag{5.1}$$

We can also prove the lemma with the following method.

$$E[\mathrm{e}^{-sZ}] = P(X<Y)E[\mathrm{e}^{-sX}|X<Y] + P(X>Y)E[\mathrm{e}^{-sY}|X>Y],$$

where $E[\mathrm{e}^{-sX}|X<Y]$ can be derived by conditional expectation, *i.e.*,

$$\begin{aligned} E[\mathrm{e}^{-sX}|X<Y] &= \frac{1}{P(X<Y)}\int_0^\infty \left[\int_x^\infty E[\mathrm{e}^{-sX}|X=x]\mathrm{d}P(Y<t)\right]\mathrm{d}P(X<x) \\ &= \frac{1}{P(X<Y)}\int_0^\infty \mathrm{e}^{-(s+\eta)x}\mathrm{d}P(X<x) \\ &= \frac{1}{P(X<Y)}F_X^*(s+\eta). \end{aligned} \tag{5.2}$$

Similarly, $E[\mathrm{e}^{-sY}|X>Y]$ can be obtained by

$$E[\mathrm{e}^{-sY}|X>Y] = \frac{\eta}{s+\eta}\frac{1-F_X^*(s+\eta)}{P(X>Y)}\cdot$$

So, the result is obtained. □

Let $B_{h,i}$ and $B_{h,i}^*(s)$ denote the total service time of h customers in service phase i and its LST, respectively. In order to derive the results, we consider two cases.

Case 1: The tagged customer arrives in state $(n,i), n\geq 0, i=1,2,\ldots,N$. Based on the result of Lemma 5.1, we have

$$\begin{aligned} \sum_{n=0}^\infty \pi_{n,i}E\left[\mathrm{e}^{-s\min(B_{n+1,i},D_i)}\right] &= \sum_{n=0}^\infty \pi_{n,i}\frac{\eta_i+sB_{n+1,i}^*(s+\eta_i)}{\eta_i+s} \\ &= \sum_{n=0}^\infty \pi_{n,i}\left(\frac{\eta_i}{\eta_i+s}+\frac{s}{\eta_i+s}\left(\frac{\mu_i}{s+\eta_i+\mu_i}\right)^{n+1}\right). \end{aligned} \tag{5.3}$$

Case 2: The tagged customer arrives in state $(n,0), n\geq 0$. In this case, we have

$$\sum_{n=0}^\infty \pi_{n,0}E\left[\mathrm{e}^{-sT_0}\right]\sum_{i=1}^N q_iE\left[\mathrm{e}^{-s\min(B_{n+1,i},D_i)}\right] = \sum_{i=1}^N\sum_{n=0}^\infty \pi_{n,0}\frac{\eta_0q_i}{\eta_0+s}\left(\frac{\eta_i}{\eta_i+s}+\frac{s}{\eta_i+s}\left(\frac{\mu_i}{s+\eta_i+\mu_i}\right)^{n+1}\right). \tag{5.4}$$

Then the LST of the stationary sojourn time W can be obtained by

$$W^*(s) = \sum_{i=1}^N\sum_{n=0}^\infty \pi_{n,i}E\left[\mathrm{e}^{-s\min(B_{n+1,i},D_i)}\right] + \sum_{i=1}^N\sum_{n=0}^\infty \pi_{n,0}q_iE\left[\mathrm{e}^{-sT_0}\right]E[\mathrm{e}^{-s\min(B_{n+1,i},D_i)}\Big].$$

After some manipulations, the explicit expression of $W^*(s)$ is derived by the following theorem.

Theorem 5.2. *If $\eta_i>0, i=0,1,2\ldots,N$, the LST of the stationary sojourn time W is*

$$\begin{aligned} W^*(s) = &\sum_{i=1}^N\frac{\eta_i}{\eta_i+s}\left(\frac{\pi_{0,0}r_{0,i}}{(1-r_{0,0})(1-r_{i,i})}+\frac{\pi_{0,i}}{1-r_{i,i}}\right) \\ &+\sum_{i=1}^N\frac{s}{\eta_i+s}\frac{\pi_{0,0}r_{0,i}\mu_i^2}{(s+\eta_i+\mu_i-\mu_ir_{0,0})(s+\eta_i+\mu_i-\mu_ir_{i,i})} \\ &+\sum_{i=1}^N\frac{s}{\eta_i+s}\frac{\pi_{0,i}\mu_i}{s+\eta_i+\mu_i-\mu_ir_{i,i}} \\ &+\sum_{i=1}^N\frac{\eta_0\pi_{0,0}q_i}{\eta_0+s}\left(\frac{\eta_i}{\eta_i+s}\frac{1}{1-r_{0,0}}+\frac{s}{\eta_i+s}\frac{\mu_i}{s+\eta_i+\mu_i-\mu_ir_{0,0}}\right). \end{aligned} \tag{5.5}$$

From Theorem 5.2, we obtain the expected sojourn time of an arbitrary customer

$$
\begin{aligned}
E[W] = -\frac{dW^*(s)}{ds}|_{s=0} &= \sum_{i=1}^{N} \frac{1}{\eta_i} \left(\frac{\pi_{0,0} r_{0,i}}{(1-r_{0,0})(1-r_{i,i})} + \frac{\pi_{0,i}}{1-r_{i,i}} \right) \\
&- \sum_{i=1}^{N} \frac{1}{\eta_i} \left(\frac{\pi_{0,0} r_{0,i} \mu_i^2}{(\eta_i+\mu_i-\mu_i r_{0,0})(\eta_i+\mu_i-\mu_i r_{i,i})} + \frac{\pi_{0,i}\mu_i}{\eta_i+\mu_i-\mu_i r_{i,i}} \right) \\
&+ \frac{\pi_{0,0}}{\eta_0} \left(\frac{1}{1-r_{0,0}} \right) + \sum_{i=1}^{N} \frac{\pi_{0,0} q_i}{\eta_i} \left(\frac{1}{1-r_{0,0}} - \frac{\mu_i}{\eta_i+\mu_i-\mu_i r_{0,0}} \right).
\end{aligned}
$$

6. Cycle analysis

This section mainly gives the cycle analysis. To avoid confusion, a cycle considered here is defined as the time interval between two consecutive instants at which a repair process commences. In our system, We define two types of cycle.

Type-1: there are no customers arrive during repair period,
Type-2: there are $k, k \geq 1$ customers arrive during repair period.

Let C denote the length of a cycle, let C_1 denote the length of Type-1 cycle, let $C_{2,k}$ denote the length of Type-2 cycle, and let $H_{i,h}$ denote the length of busy period caused by $h, h \geq 1$ customers in service phase i, $i = 1, 2, \ldots, N$. We also define U_i as the time duration from the end of the repair period to the next occurrence point of a disaster during the Type-1 cycle in service phase i. Since the interarrival times don't have the memoryless property, we further define A_R as the residual lifetime of an interarrival time. It is known that A_E and A_R have the same limiting distribution, so the density function of A_R is $\lambda P(A > t)$. According to our assumptions, the expression of U_i can be easily obtained by

$$
U_i = \begin{cases} U_{1,i}, D_i < H_{i,1} \\ U_{2,i} + U_i, D_i > H_{i,1} \end{cases},
$$

where

$$
U_{1,i} = A_R + (D_i | D_i < H_{i,1}), U_{2,i} = A_R + (H_{i,1} | D_i > H_{i,1}).
$$

First, we have

$$
A_R^*(s) = \int_0^\infty \mathrm{e}^{-st} \mathrm{d}P(A_R < t) = \int_0^\infty \mathrm{e}^{-st} \lambda [1 - P(A < t)] \mathrm{d}t = \frac{\lambda}{s}[1 - A^*(s)].
$$

Then, based on Lemma 5.1, $U_{1,i}^*(s)$ and $U_{2,i}^*(s)$ are obtained as follows:

$$
U_{1,i}^*(s) = A_R^*(s) E(\mathrm{e}^{-sD_i} | D_i < H_{i,1}) = \frac{\lambda[1 - A^*(s)]}{s} \frac{\eta_i}{s+\eta_i} \frac{1 - H_{i,1}^*(s+\eta_i)}{P(D_i < H_{i,1})},
$$

$$
U_{2,i}^*(s) = A_R^*(s) E(\mathrm{e}^{-sH_{i,1}} | D_i > H_{i,1}) = \frac{\lambda[1 - A^*(s)]}{s} \frac{H_{i,1}^*(s+\eta_i)}{P(D_i > H_{i,1})}.
$$

From the above expressions, $U_i^*(s)$ is obtained as

$$
U_i^*(s) = P(D_i < H_{i,1}) U_{1,i}^*(s) + P(D_i > H_{i,1}) U_{2,i}^*(s) U_i^*(s),
$$

i.e.,

$$
U_i^*(s) = \frac{P(D_i < H_{i,1}) U_{1,i}^*(s)}{1 - P(D_i > H_{i,1}) U_{2,i}^*(s)} = \frac{\eta_i}{s+\eta_i} \frac{\lambda[1 - A^*(s)][1 - H_{i,1}^*(s+\eta_i)]}{s - \lambda[1 - A^*(s)] H_{i,1}^*(s+\eta_i)}.
$$

From the proof of Lemma 5.1, we have

$$
\begin{aligned}
E[\mathrm{e}^{-sT_0}|A_R > T_0] &= \frac{\eta_0}{s+\eta_0}\frac{1-A_R^*(s+\eta_0)}{P(A_R>T_0)} \\
&= \frac{\eta_0}{(s+\eta_0)^2}\frac{s+\eta_0-\lambda[1-A^*(s+\eta_0)]}{P(A_R>T_0)}.
\end{aligned}
$$

Expressing C_1 in terms of U_i, we have $C_1 = (T_0|A_R > T_0) + \sum_{i=1}^{N} q_i U_i$. Therefore, $C_1^*(s)$ can be obtained by

$$
\begin{aligned}
C_1^*(s) &= E[\mathrm{e}^{-sT_0}|A_R > T_0]\sum_{i=1}^{N} q_i U_i^*(s) \\
&= \frac{\eta_0}{(s+\eta_0)^2}\frac{s+\eta_0-\lambda[1-A^*(s+\eta_0)]}{P(A_R>T_0)} \\
&\quad \times \left[\sum_{i=1}^{N} q_i \frac{\eta_i}{s+\eta_i}\frac{\lambda[1-A^*(s)][1-H_{i,1}^*(s+\eta_i)]}{s-\lambda[1-A^*(s)]H_{i,1}^*(s+\eta_i)}\right],
\end{aligned}
\tag{6.1}
$$

where $P(A_R > T_0) = 1 - A_R^*(\eta_0)$.

Next, we consider Type-2 cycle. In this case, we first define $p_k, k \geq 1$ as the probability that there are k customers in the system at the instant of a repair completion, *i.e.*, k customers arrive during a repair period. Then, p_k can be given by

$$
\begin{aligned}
p_k &= P(A_R + \sum_{h=1}^{k-1} A_h < T_0 < A_R + \sum_{h=1}^{k} A_h) \\
&= P(A_R + \sum_{h=1}^{k-1} A_h < T_0) - P(A_R + \sum_{h=1}^{k} A_h < T_0).
\end{aligned}
$$

Let $A^{(k)}(t)$ denote the distribution function of $\sum_{h=1}^{k} A_h$, it is easy to find that

$$
\begin{aligned}
A^{(1)}(t) &= A(t), \\
A^{(k)}(t) &= \int_0^t A^{(k-1)}(t-x)\mathrm{d}A(x), \quad k \geq 2.
\end{aligned}
$$

That is $A^{(k)}(t)$ is the k-fold convolution of $A(t)$ (denoted as $A^{(k)} = A^{(k-1)} * A$ for convenience). Then the probability of $P(A_R + \sum_{h=1}^{k} A_h < T_0)$ is given by

$$
\begin{aligned}
P\left(A_R + \sum_{h=1}^{k} A_h < T_0\right) &= \int_0^\infty P\left(A_R + \sum_{h=1}^{k} A_h < t\right)\eta_0\mathrm{e}^{-\eta_0 t}\mathrm{d}t \\
&= \int_0^\infty \int_0^t P\left(\sum_{h=1}^{k} A_h < t-x\right)\eta_0\mathrm{e}^{-\eta_0 t} dA_R(x)\mathrm{d}t \\
&= \int_0^\infty \int_0^t A^{(k)}(t-x)\eta_0\mathrm{e}^{-\eta_0 t}\lambda(1-A(x))\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

and

$$p_k = \int_0^\infty \int_0^t [A^{(k-1)}(t-x) - A^{(k)}(t-x)]\eta_0 \mathrm{e}^{-\eta_0 t}\lambda(1-A(x))\mathrm{d}x\mathrm{d}t.$$

Similarly to U_i in Type-1 cycle, we define the random variable $V_{k,i}$ as the remaining Type-2 cycle after the repair completion in service phase i. The subscript k in $V_{k,i}$ denotes the number of customers at the instant of the repair completion. Then, we have

$$V_{k,i} = \begin{cases} V_{1,k,i}, D_i < H_{i,k} \\ V_{2,k,i} + U_i, D_i > H_{i,k} \end{cases}, i = 1, 2, \ldots, N,$$

where $V_{1,k,i} = (D_i | D_i < H_{i,k})$, $V_{2,k,i} = (H_{i,k} | D_i > H_{i,k})$, and $V^*_{1,k,i}(s)$ and $V^*_{2,k,i}(s)$ can be obtained by

$$V^*_{1,k,i}(s) = E(\mathrm{e}^{-sD_i} | D_i < H_{i,k}) = \frac{\eta_i}{s+\eta_i} \frac{1 - H^*_{i,k}(s+\eta_i)}{P(D_i < H_{i,k})},$$

$$V^*_{2,k,i}(s) = E(\mathrm{e}^{-sH_{i,k}} | D_i > H_{i,k}) = \frac{H^*_{i,k}(s+\eta_i)}{P(D_i > H_{i,k})}\cdot$$

From the above expressions, $V^*_{k,i}(s)$ is obtained as

$$V^*_{k,i}(s) = P(D_i < H_{i,k})V^*_{1,k,i}(s) + P(D_i > H_{i,k})V^*_{2,k,i}(s)U^*_i(s).$$

Expressing $C_{2,k}$ in terms of $V_{k,i}$, we have

$$C_{2,k} = \left(T_0 | A_R + \sum_{h=1}^{k-1} A_h < T_0 < A_R + \sum_{h=1}^{k} A_h \right) + \sum_{i=1}^{N} q_i V_{k,i}.$$

Therefore, $C^*_{2,k}(s)$ is given by

$$C^*_{2,k}(s) = E\left[\mathrm{e}^{-sT_0} | A_R + \sum_{h=1}^{k-1} A_h < T_0 < A_R + \sum_{h=1}^{k} A_h \right] \sum_{i=1}^{N} q_i V^*_{k,i}(s), \tag{6.2}$$

where

$$E\left[\mathrm{e}^{-sT_0} | A_R + \sum_{h=1}^{k-1} A_h < T_0 < A_R + \sum_{h=1}^{k} A_h \right] = \frac{\eta_0 A^*_R(s+\eta_0)(A^*(s+\eta_0))^{k-1}(1 - A^*(s+\eta_0))}{(s+\eta_0)p_k}\cdot$$

In fact, $H^*_{i,k}(s), k \geq 1$ have very complex expressions, and the results can be seen in Cohen [6] (see p. 227). From [6], we have the distribution function of the busy period by

$$H(t) = \sum_{n=1}^{\infty} \mathrm{e}^{-\mu_i t} \frac{(\mu_i t)^{n-1}}{n!} \int_0^t [1 - A^{(n)}(\nu)]\mu_i \mathrm{d}\nu,$$

and its LST by

$$H^*_{i,1}(s) = \int_0^\infty \mathrm{e}^{-st}\mathrm{d}H(t), H^*_{i,k}(s) = (H^*_{i,1}(s))^k.$$

Once $C^*_1(s)$ and $C^*_{2,k}(s)$ are derived, the expression of C is given by

$$C^*(s) = P(A_R > T_0)C^*_1(s) + \sum_{k=1}^{\infty} p_k C^*_{2,k}(s).$$

7. The length of working time in a cycle

In this section, we mainly concentrate on the LST of the length of the server's working time in a cycle. To avoid terminological confusion, the working time is the duration that the server is busy in service phase (it doesn't contain the time that the server is idle). According to the section of cycle analysis, this section gets much easier. Let Y denote the length of working time in a cycle. In Type-1 cycle, we define $Y_{0,i}$ as the length of working time in service phase i of Type-1 cycle. Then, we have

$$Y_{0,i} = \begin{cases} Y^{(1)}_{1,0,i}, D_i < H_{i,1} \\ Y^{(1)}_{2,0,i} + Y_{0,i}, D_i > H_{i,1} \end{cases},$$

where

$$Y^{(1)}_{1,0,i} = (D_i|D_i < H_{i,1}), Y^{(1)}_{2,0,i} = (H_{i,1}|D_i > H_{i,1}).$$

Similarly to the analysis of the above section, $Y^*_{0,i}(s)$ can be obtained by

$$Y^*_{0,i}(s) = \frac{\eta_i}{s+\eta_i}[1 - H^*_{i,1}(s+\eta_i)] + H^*_{i,1}(s+\eta_i)Y^*_{0,i}(s),$$

i.e.,

$$Y^*_{0,i}(s) = \frac{\eta_i}{s+\eta_i}\cdot$$

In Type-2 cycle, we define $Y_{k,i}$ as the length of working time in service phase i of Type-2 cycle. Then, we have

$$Y_{k,i} = \begin{cases} Y^{(2)}_{1,k,i}, D_i < H_{i,k} \\ Y^{(2)}_{2,k,i} + Y_{0,i}, D_i > H_{i,k} \end{cases}, k \geq 1,$$

where $Y^{(2)}_{1,k,i} = (D_i|D_i < H_{i,k})$, $Y^{(2)}_{2,k,i} = (H_{i,k}|D_i > H_{i,k})$. Then

$$Y^*_{k,i}(s) = \frac{\eta_i}{s+\eta_i}[1 - H^*_{i,k}(s+\eta_i)] + H^*_{i,k}(s+\eta_i)Y^*_{0,i}(s) = \frac{\eta_i}{s+\eta_i}\cdot$$

So the expression of $Y^*(s)$ can be obtained by

$$Y^*(s) = P(A_R > T_0)\sum_{i=1}^{N} q_i Y^*_{0,i}(s) + \sum_{k=1}^{\infty} p_k \sum_{i=1}^{N} q_i Y^*_{k,i}(s) = \sum_{i=1}^{N} q_i \frac{\eta_i}{s+\eta_i}\cdot$$

8. Numerical examples

In this section, we provide a set of numerical examples on the basis of the results obtained by this paper. In particular, we assume that the interarrival times are exponentially distributed with parameter λ, then, the system in consideration translates into an M/M/1 queue in a multi-phase service environment with disasters. Hence, $r_{i,i}$ is the unique root of $\mu_i(1-z)z + (\lambda+\eta_i)z - \lambda = 0$ in $(0,1)$, and an immediately result of $r_{i,i}$ can be directly calculated by

$$r_{i,i} = \frac{(\lambda+\mu_i+\eta_i) - \sqrt{(\lambda+\mu_i+\eta_i)^2 - 4\lambda\mu_i}}{2\mu_i}, \quad i = 1, 2, \ldots, N.$$

The nonzero entries of matrix R can be easily obtained:

$$r_{0,0} = \frac{\lambda}{\lambda+\eta_0},$$

$$r_{0,i} = \frac{q_i\eta_0(r_{i,i} - r_{0,0})}{(\eta_0 - \eta_i) - \mu_i(1 - r_{0,0})} = \frac{q_i\eta_0 r_{0,0}}{(\lambda+\mu_i+\eta_i) - (r_{i,i} + r_{0,0})\mu_i}, \quad i = 1, 2, \ldots, N.$$

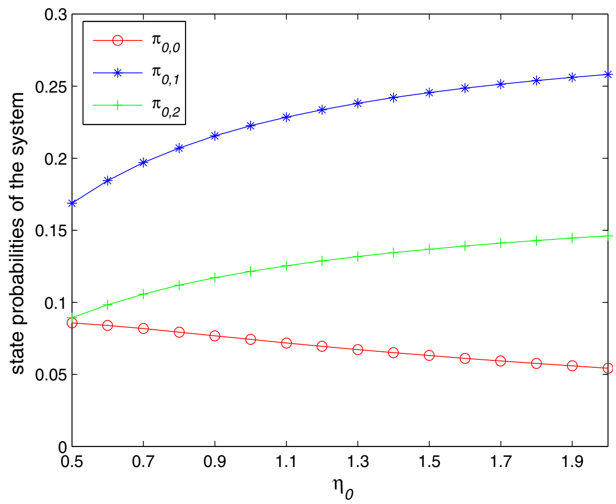

FIGURE 1. $\pi_{0,0}$, $\pi_{0,1}$, $\pi_{0,2}$ *versus* η_0 ($\lambda = 1.1$, M/M/1).

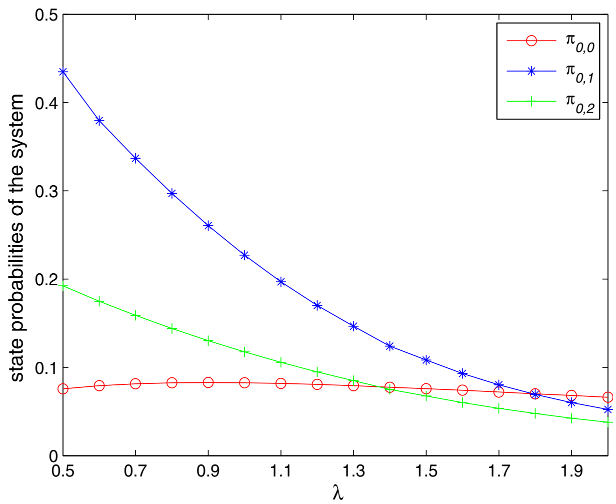

FIGURE 2. $\pi_{0,0}$, $\pi_{0,1}$, $\pi_{0,2}$ *versus* $\lambda(\eta_0 = 0.7$, M/M/1).

In [20], if $\lambda_0 = \lambda_1 = \ldots = \lambda_N$, all entries of R obtained by Paz and Yechiali [20] are in accordance with the result in our paper.

Next, based on Theorem 3.2, we have the steady state probabilities

$$\pi_k = (\pi_{k,0}, \pi_{k,1}, \pi_{k,2}, \ldots, \pi_{k,N}), k \geq 0.$$

Once the steady state probabilities are obtained, the stationary queue length distribution at arbitrary epochs, the sojourn time distribution of an arbitrary customer and the duration of a cycle can be respectively obtained.

In the following part, we give some figures to show the effect of system parameters on the steady state queue length distributions $\pi_{0,0}$, $\pi_{0,1}$, $\pi_{0,2}$ and the expected sojourn time $E[W]$. Without loss of generality, we assume $N = 2$, *i.e.*, the system has two operative phase and a repair phase.

In Figures 1 and 2, we illustrate the steady state queue length distributions $\pi_{0,0}$, $\pi_{0,1}$, $\pi_{0,2}$ in the M/M/1 queue with parameters $q_1 = 0.6$, $\mu_1 = 1.5$, $\mu_2 = 2$, $\eta_1 = 0.2$, $\eta_2 = 0.4$. Obviously, from Figure 1, with the increase

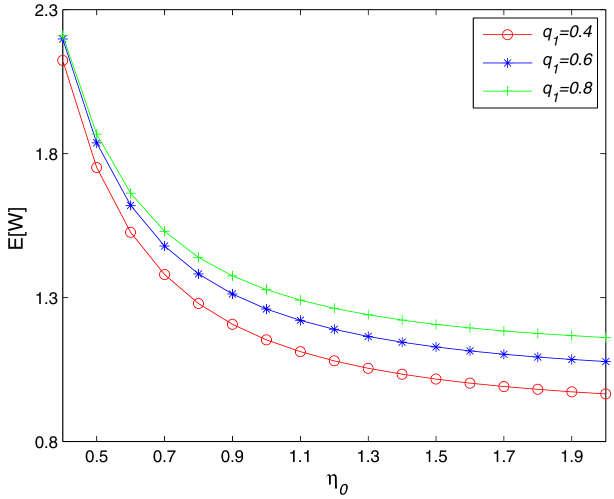

FIGURE 3. The mean sojourn time $E[W]$ *versus* η_0 ($\lambda = 1.1$, M/M/1).

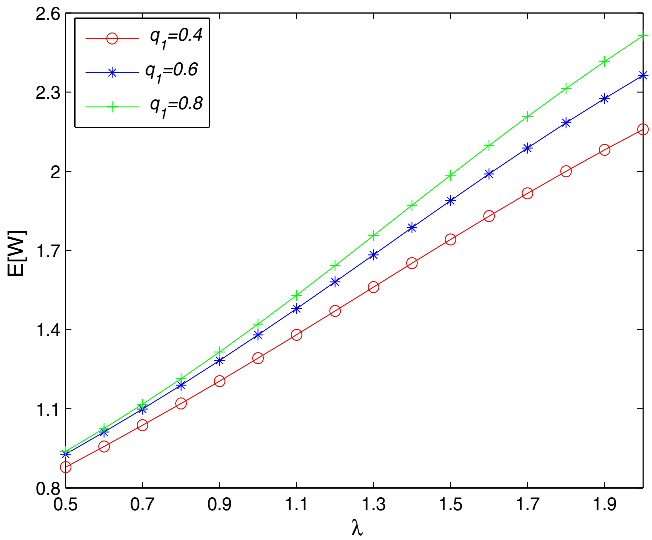

FIGURE 4. The mean sojourn time $E[W]$ *versus* λ ($\eta_0 = 0.7$, M/M/1).

of η_0, the steady state probabilities $\pi_{0,1}$ and $\pi_{0,2}$ become larger and $\pi_{0,0}$ declines. From Figure 2, we find that $\pi_{0,1}$ and $\pi_{0,2}$ decrease with the increase of λ, $\pi_{0,0}$ firstly increases and then descends with the increasing of λ. Furthermore, Figures 1 and 2 indicate that the effect of η_0 and λ on $\pi_{0,0}$ is less than the effect on $\pi_{0,1}$ and $\pi_{0,2}$.

In Figures 3 and 4, we plot the trend of the change for the expected sojourn time $E[W]$ as η_0 increases from 0.4 to 2 and λ increases from 0.5 to 2 for different values of q_1, respectively. Figures 3 and 4 confirm that $E[W]$ decreases with the increase of η_0 and increases with the increase of λ, which are consistent with the intuitive expectations. It is noteworthy that, in Figures 3 and 4, if η_0 and λ are fixed, the smaller q_1 is, the smaller $E[W]$ becomes. Actually, it is in accordance with the fact. This is because that after the server is repaired, as q_1 decreases, the system moves to operative phase 2 with a higher probability. Based on our assumption $\mu_1 < \mu_2$, the server has a higher service rate in operative phase 2, which leads to the smaller values of $E[W]$. Although we only focus on showing the effect of the parameters on the expected sojourn time for M/M/1 queue, we believe that the similar change trends exist for other queues (D/M/1 queue and E_2/M/1 queue *etc.*).

9. Conclusion

In this paper, we investigated a GI/M/1 queue in a multi-phase service environment with disasters in order to establish the theoretical foundations for applications and obtain the explicit computation formulas for the performance measures. By the matrix analytic method and semi-Markov process, we obtained the stationary queue length distribution both at arrival and arbitrary epochs. Furthermore, we derived the LST of the stationary sojourn time distribution of an arbitrary customer. More importantly, we presented an elaborate analysis of the duration of a cycle and the server's working time in a cycle, respectively. Beside, we performed some numerical examples to show the effect of parameters on the steady state probabilities and the expected sojourn time. We expect that the results and the method can be applied to more queueing systems.

Appendix.

In this Appendix, we will give the proof of Theorem 3.2.

First, let $B[R]=\sum\limits_{n=0}^{\infty}R^nB_n$. Substituting R and B_n into $B[R]$,we have

$$B[R]=\sum_{n=0}^{\infty}R^nB_n=\begin{bmatrix} F_{0,0} & F_{0,1} & F_{0,2} & \cdots & F_{0,N} \\ F_{1,0} & F_{1,1} & F_{1,2} & \cdots & F_{1,N} \\ F_{2,0} & F_{2,1} & F_{2,2} & \cdots & F_{2,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{N,0} & F_{N,1} & F_{N,2} & \cdots & F_{N,N} \end{bmatrix},$$

where

$$\begin{aligned} F_{0,0} &= \sum_{n=0}^{\infty} r_{0,0}^n \sum_{i=1}^{N} q_i m_{n,i}^{(0)} + \sum_{n=1}^{\infty}\sum_{i=1}^{N} m_i(n)\sum_{h=0}^{n} d_{h,i} \\ &= \sum_{n=0}^{\infty} r_{0,0}^n \sum_{i=1}^{N} q_i m_{n,i}^{(0)} + \sum_{n=1}^{\infty}\sum_{i=1}^{N} q_i\alpha_i(r_{i,i}^n - r_{0,0}^n)\sum_{h=0}^{n} d_{h,i}, \\ F_{0,i} &= \sum_{n=0}^{\infty} r_{0,0}^n q_i\beta_{n,i} + \sum_{n=1}^{\infty}\sum_{j=1}^{N} m_j(n)\gamma_{n,j,i}, \\ F_{i,0} &= \sum_{n=0}^{\infty} r_{i,i}^n \sum_{h=0}^{n} d_{h,i}, i=1,2,\ldots,N, \\ F_{i,i} &= \sum_{n=0}^{\infty} r_{i,i}^n\gamma_{n,i,i}, i=1,2,\ldots,N, \\ F_{i,j} &= \sum_{n=0}^{\infty} r_{i,i}^n\gamma_{n,i,j}, i\neq j, i=1,2,\ldots,N, j=1,2,\ldots,N. \end{aligned}$$

Next, we will give the explicit results of the above expressions. For $F_{i,0}$, it is easy to obtain

$$F_{i,0}=\sum_{n=0}^{\infty} r_{i,i}^n \sum_{h=0}^{n} d_{h,i} = \frac{1}{1-r_{i,i}}\frac{\eta_i[A^*(\eta_0)-r_{i,i}]}{\eta_i-\eta_0+\mu_i(1-r_{i,i})}=\sigma_i=\theta_{i,0}. \tag{A.1}$$

For $F_{i,i}$, we first have

$$\sum_{n=0}^{\infty} r_{i,i}^n \gamma_{n,i,i} = \sum_{n=0}^{\infty} r_{i,i}^n \left(1 - \sum_{h=0}^{n} (b_{h,i} + c_{h,i}) + q_i \sum_{h=0}^{n} (c_{h,i} - d_{h,i})\right),$$

then $F_{i,i}$ can be obtained by

$$\begin{aligned} F_{i,i} &= \frac{1}{1-r_{i,i}} \left(1 - r_{i,i} - q_i \frac{\eta_i [A^*(\eta_0) - r_{i,i}]}{\eta_i - \eta_0 + \mu_i(1-r_{i,i})}\right) - (1-q_i) \frac{\eta_i}{\mu_i + \eta_i - \mu_i r_{i,i}} \\ &= 1 - q_i\sigma_i - (1-q_i)\xi_i = 1 - \sum_{j=0, j\neq i}^{N} \theta_{i,j}, \quad i = 1, 2, \ldots, N. \end{aligned} \tag{A.2}$$

and $F_{i,j}$ can be easily obtained by

$$\begin{aligned} F_{i,j} &= q_j \frac{\eta_i}{\mu_i + \eta_i - \mu_i r_{i,i}} - q_j \frac{1}{1-r_{i,i}} \frac{\eta_i [A^*(\eta_0) - r_{i,i}]}{\eta_i - \eta_0 + \mu_i(1-r_{i,i})} \\ &= q_j\xi_i - q_j\sigma_i = \theta_{i,j}, \quad i = 1, 2, \ldots N, \quad j = 1, 2, \ldots N, j \neq i. \end{aligned} \tag{A.3}$$

Finally, we will give the expressions of $F_{0,0}$ and $F_{0,i}$. First, we have

$$\sum_{n=0}^{\infty} r_{0,0}^n q_i \beta_{n,i} = \sum_{n=0}^{\infty} r_{0,0}^n q_i \left([1 - A^*(\eta_0)] - \sum_{h=0}^{n} v_{h,i}^{(0)} - E_{n,i} + \sum_{j=1}^{N} q_j \left(E_{n,j} - m_{n,j}^{(0)}\right)\right),$$

where

$$\begin{aligned} \sum_{n=0}^{\infty} r_{0,0}^n \sum_{h=0}^{n} v_{h,i}^{(0)} &= \sum_{n=0}^{\infty} r_{0,0}^n \sum_{h=0}^{n} \int_0^{\infty} \int_0^t \mathrm{e}^{-\eta_i(t-x)} \frac{(\mu_i(t-x))^h}{h!} \mathrm{e}^{-\mu_i(t-x)} \eta_0 \mathrm{e}^{-\eta_0 x} \mathrm{d}x \mathrm{d}A(t) \\ &= \frac{1}{1-r_{0,0}} \alpha_i [A^*(\eta_i + \mu_i - \mu_i r_{0,0}) - A^*(\eta_0)], \end{aligned}$$

$$\sum_{n=0}^{\infty} r_{0,0}^n E_{n,i} = \frac{[1 - A^*(\eta_i + \mu_i - \mu_i r_{0,0})]}{1 - r_{0,0}} - \frac{\alpha_i [A^*(\eta_i + \mu_i - \mu_i r_{0,0}) - A^*(\eta_0)]}{1 - r_{0,0}},$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{j=1}^{N} m_j(n) \gamma_{n,j,i} &= \sum_{n=1}^{\infty} \sum_{j=1}^{N} q_j \alpha_j (r_{j,j}^n - r_{0,0}^n) \gamma_{n,j,i} \\ &= \sum_{n=1}^{\infty} q_i \alpha_i (r_{i,i}^n - r_{0,0}^n) \gamma_{n,i,i} + \sum_{n=1}^{\infty} \sum_{j=1, j\neq i}^{N} q_j \alpha_j (r_{j,j}^n - r_{0,0}^n) \gamma_{n,j,i}. \end{aligned}$$

After some calculations, we have the explicit expression of $\sum\limits_{n=1}^{\infty} \sum\limits_{j=1}^{N} m_j(n)\gamma_{n,j,i}$ as present in Theorem 3.2, and $F_{0,i}$ can be also obtained by substituting the above equations, which is denoted by δ_i.

For $F_{0,0}$, we have

$$\sum_{i=1}^{N} q_i \sum_{n=0}^{\infty} r_{0,0}^n m_{n,i}^{(0)} = \sum_{i=1}^{N} q_i \frac{1}{1-r_{0,0}} \frac{\eta_i \eta_0}{\eta_i - \eta_0 + \mu_i(1-r_{0,0})} \int_0^{\infty} t\mathrm{e}^{-\eta_0 t} \mathrm{d}A(t)$$
$$- \sum_{i=1}^{N} q_i \frac{1}{1-r_{0,0}} \frac{\eta_i \eta_0 [A^*(\eta_0) - A^*(\eta_i + \mu_i - \mu_i r_{0,0})]}{[\eta_i - \eta_0 + \mu_i(1-r_{0,0})]^2},$$

and

$$\sum_{n=1}^{\infty} \sum_{i=1}^{N} q_i \alpha_i (r_{i,i}^n - r_{0,0}^n) \sum_{h=0}^{n} d_{h,i} = \sum_{i=1}^{N} q_i \alpha_i \left(\frac{1}{1-r_{i,i}} \frac{\eta_i [A^*(\eta_0) - r_{i,i}]}{\eta_i - \eta_0 + \mu_i(1-r_{i,i})} \right)$$
$$- \sum_{i=1}^{N} q_i \alpha_i \left(\frac{1}{1-r_{0,0}} \frac{\eta_i [A^*(\eta_0) - A^*(\eta_i + \mu_i - \mu_i r_{0,0})]}{\eta_i - \eta_0 + \mu_i(1-r_{0,0})} \right),$$

then, the expression of $F_{0,0}$ can be obtained, and it is equal to $F_{0,0} = 1 - \sum_{i=1}^{N} \delta_i$. Once all elements of the matrix $B[R]$ are given, we can derive the result of stationary probability $\pi_0 = (\pi_{0,0}, \pi_{0,1}, \pi_{0,2}, \ldots, \pi_{0,N})$.

According to the equation $\pi_0 B[R] = \pi_0$, which is equal to

$$(B[R] - I)^T (\pi_{0,0}, \pi_{0,1}, \pi_{0,2}, \ldots, \pi_{0,N})^T = (0, 0, 0, \ldots, 0)^T, \tag{A.4}$$

where

$$(B[R] - I)^T = \begin{bmatrix} -\sum_{i=1}^{N} \delta_i & \theta_{1,0} & \theta_{2,0} & \cdots & \theta_{N,0} \\ \delta_1 & -\sum_{j=0,j\neq 1}^{N} \theta_{1,j} & \theta_{2,1} & \cdots & \theta_{N,1} \\ \delta_2 & \theta_{1,2} & -\sum_{j=0,j\neq 2}^{N} \theta_{2,j} & \cdots & \theta_{N,1} \\ \vdots & \vdots & \vdots & & \vdots \\ \delta_N & \theta_{1,N} & \theta_{2,N} & \cdots & -\sum_{j=0,j\neq N}^{N} \theta_{N,j} \end{bmatrix}.$$

It is easy to find that $\det((B[R] - I)^T) = 0$, that is equation (A.4) has non-zero solutions. Using the normalizing condition

$$\pi_0 (I - R)^{-1} \mathrm{e} = 1, \tag{A.5}$$

the unique solution of π_0 can be derived by simultaneously solving (A.4) and (A.5) with Cramer's Rule. Then, $\pi_n, n \geq 1$ can be obtained by

$$\pi_n = \pi_0 R^n, n \geq 1.$$

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