RAIRO-Oper. Res. 50 (2016) 935–949
DOI: 10.1051/ro/2015056

RAIRO Operations Research
www.rairo-ro.org

COMPLEXITY ANALYSIS OF INTERIOR POINT METHODS FOR LINEAR PROGRAMMING BASED ON A PARAMETERIZED KERNEL FUNCTION

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Abstract. Kernel function plays an important role in defining new search directions for primal-dual interior point algorithm for solving linear optimization problems. This problem has attracted the attention of many researchers for some years. The goal of their works is to find kernel functions that improve algorithmic complexity of this problem. In this paper, we introduce a real parameter $p > 0$ to generalize the kernel function (5) given by Bai *et al.* in [Y.Q. Bai, M El Ghami and C. Roos, *SIAM J. Optim.* **15** (2004) 101–128.], and give the corresponding primal-dual interior point methods for linear optimization. This parameterized kernel function yields the similar complexity bound given in [Y.Q. Bai, M El Ghami and C. Roos, *SIAM J. Optim.* **15** (2004) 101–128.] for both large-update and small-update methods.

Mathematics Subject Classification. 90C05, 90C31, 90C51.

Received October 26, 2014. Accepted November 12, 2015.

1. Introduction

We consider the standard linear optimization

$$(P)\ \min\{c^T x : Ax = b, x \geq 0\},$$

where $A \in \mathbb{R}^{m\times n}$, $rank(A) = m$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$, and its dual problem

$$(D)\ \max\{b^T y : A^T y + s = c, s \geq 0\}.$$

In 1984, Karmarkar [14] proposed a new polynomial-time method for solving linear programs. This method and its variants that were developed subsequently are now called interior point methods **IPMs**. For a survey, we refer to recent books on the subject [22]. The primal-dual interior point algorithm which is the most efficient for a computational point of view [1]. It is generally agreed that the iteration complexity of the algorithm is an appropriate measure for its efficiency [10]. At present, the best known theoretical iteration bound for small-update **IPMs** is better than the one for large-update **IPMs**. However, in practice, large-update **IPMs**

Keywords. Linear optimization, Kernel function, interior point methods, complexity bound.

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Article published by EDP Sciences

are much more efficient than small-update **IPMs** [18, 21, 22]. Many researchers proposed and analyzed various primal-dual interior point methods for linear optimization **LO** based on the logarithmic barrier function. In particular, Andersen *et al.* [1], den Hertog [13] and Todd [20] proposed for different logarithmic barrier functions a primal-dual interior point methods with complexity $\mathbf{O}\left(n\log\frac{n}{\epsilon}\right)$ for large-update methods and $\mathbf{O}\left(\sqrt{n}\log\frac{n}{\epsilon}\right)$ for small-update methods. Peng *et al.* [16, 17] introduced self-regular barrier functions for primal-dual **IPMs** for **LO** and obtained the best complexity result so far $\mathbf{O}(\sqrt{n}\log n\log\frac{n}{\epsilon})$ for large-update primal-dual **IPMs** with some specific self-regular barrier functions. Recently, Bai *et al.* [2–7], Ghami *et al.* [10, 11] and Cho [8] proposed new primal-dual **IPMs** for **LO** problems based on various kernel functions to improve the iteration bound for large-update methods from $\mathbf{O}\left(n\log\frac{n}{\epsilon}\right)$ to $\mathbf{O}\left(\sqrt{n}\log n\log\frac{n}{\epsilon}\right)$. For its part, EL Ghami *et al.* [9] used a new kernel function with a trigonometric barrier term and proposed a new primal-dual **IPMs** and proved that the iteration bound of large-update methods is $\mathbf{O}\left(n^{\frac{3}{4}}\log\frac{n}{\epsilon}\right)$. Motivated by the above works, in this paper we present a primal-dual interior-point algorithm for **LO** based on the generalization of the kernel function (5) given by Bai *et al.* in [5]. This function is defined

$$\psi(t) = p\left(\frac{t^2-1}{2}\right) + \mathrm{e}^{p\left(\frac{1}{t}-1\right)} - 1,$$

where the parameter p is assumed to be a positive real number. If $p=1$, we obtain the kernel function (5) given by Bai *et al.* in [5]. We show that the iteration bounds are $\mathbf{O}(\sqrt{n}(\log n)^2\log\frac{n}{\epsilon})$ for large-update methods and $\mathbf{O}(\sqrt{n}\log\frac{n}{\epsilon})$ for small-update methods.

Without loss of generality, we assume that (P) and (D) satisfy the interior point condition **IPC**, *i.e.*, there exist (x^0, y^0, s^0) such that

$$Ax^0 = b,\ x^0 > 0,\ A^T y^0 + s^0 = c,\ s^0 > 0 \tag{1.1}$$

It is well-known that finding an optimal solution of (P) and (D) is equivalent to solve the following system:

$$\begin{aligned} &Ax = b,\ x \geq 0,\\ &A^T y + s = c,\ s \geq 0,\\ &xs = 0. \end{aligned} \tag{1.2}$$

The paper is organized as follows. In Section 2, we recall how a given kernel function defines a primal-dual corresponding **IPMs**, and we present the generic form of this algorithm. In Section 3, we define a parameterized kernel function and give its properties which are essential for the complexity analysis. In Section 4, we derive decrease of the barrier function during an inner iteration result for both large-update and small-update methods. Finally, concluding remarks are given in Section 5.

We use the following notations throughout the paper. $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ denote the set of n-dimensional nonnegative vectors and positive vectors respectively. For $x, s \in \mathbb{R}^n$, $x_{\min}$ and xs denote the smallest component of the vector x and the vector componentwise product of the vector x and s, respectively. We denotes by $X = \mathrm{diag}(x)$ the $n\times n$ diagonal matrix with the components of the vector $x \in \mathbb{R}^n$ are the diagonal entries. e denotes the n-dimensional vector of ones. For functions $f, g : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$, $f(x) = \mathbf{O}(g(x))$ if $f(x) \leq C_1 g(x)$ for some positive constant C_1 and $f(x) = \mathbf{\Theta}(g(x))$ if $C_2 g(x) \leq f(x) \leq C_3 g(x)$ for some positive constant C_2 and C_3 and finally, $\|\|$ denotes the 2-norm of a vector.

2. The prototype algorithm

The basic idea of primal-dual **IPMs** is to replace the equation of complementarity condition for (P) and (D) define in (1.2), by the parameterized equation $xs = \mu e$, with $\mu > 0$. Thus we consider the system

$$\begin{aligned} &Ax = b,\ x \geq 0,\\ &A^T y + s = c,\ s \geq 0,\\ &xs = \mu e. \end{aligned} \tag{2.1}$$

If the **IPC** is satisfied, then there exists a solution, for each $\mu > 0$, and this solution is unique. It is denoted as $(x(\mu), y(\mu), s(\mu))$, and we call $x(\mu)$ the μ-center of (P) and $(y(\mu), s(\mu))$ the μ-center of (D). The set of μ-centers (with μ running through all positive real numbers) gives a path, which is called the central path of (P) and (D). The relevance of the central path for **LO** was recognized firstly by Sonnevend [19] and Megiddo [15]. If $\mu \to 0$, then the limit of the central path exists, and since the limit points satisfy the complementarity condition, the limit yields optimal solutions for (P) and (D).

From a theoretical point of view, the **IPC** can be assumed without loss of generality. In fact, we may and will assume that $x^0 = s^0 = e$. In practice, this can be realized by embedding the given problems (P) and (D) into a homogeneous self-dual problem which has two additional variables and two additional constraints. For this and the other properties mentioned above, see [18].

Without loss of generality, we assume that $(x(\mu), y(\mu), s(\mu))$ is known for some positive μ. For example, due to the above assumption, we assume that for $\mu = 1$, $x(1) = s(1) = e$. We then decrease μ to $\mu = (1-\theta)\mu$ for some fixed $\theta \in]0, 1[$, and we solve the following Newton system:

$$\begin{aligned} &A\varDelta x = 0, \\ &A^T\varDelta y + \varDelta s = 0, \\ &s\varDelta x + x\varDelta s = \mu e - xs. \end{aligned} \tag{2.2}$$

This system uniquely defines a search direction $(\varDelta x, \varDelta y, \varDelta s)$. By taking a step along the search direction with the stepsize defined by some line search rules, we construct a new triple (x, y, s). If necessary, we repeat the procedure until we find iterates that are close to $(x(\mu), y(\mu), s(\mu))$. Then μ is again reduced by the factor $1-\theta$, and we apply Newton's method targeting the new μ-centers, and so on. This process is repeated until μ is small enough, *i.e.*, until $n\mu \le \epsilon$; at this stage, we have found an $\epsilon-$optimal solution of problems (P) and (D). The result of a Newton step with step size α is denoted as

$$x^+ = x + \alpha\varDelta x,\ y^+ = y + \alpha\varDelta y,\ s^+ = s + \alpha\varDelta s, \tag{2.3}$$

where the step size α satisfies $(0 < \alpha \le 1)$.

Now, we introduce the scaled vector v and the scaled search directions d_x and d_s as follows:

$$v = \sqrt{\tfrac{xs}{\mu}},\ d_x = \tfrac{v\varDelta x}{x},\ d_s = \tfrac{v\varDelta s}{s}. \tag{2.4}$$

The system (2.2) can be rewritten as follows:

$$\begin{aligned} &\overline{A}d_x = 0, \\ &\overline{A}^T\varDelta y + d_s = 0, \\ &d_x + d_s = v^{-1} - v, \end{aligned} \tag{2.5}$$

where $\overline{A} = \frac{1}{\mu}AV^{-1}X$, $V = \mathrm{diag}(v)$, $X = \mathrm{diag}(x)$. Note that the right-hand side of the third equation in (2.5) equals to the negative gradient of the logarithmic barrier function Ψ, that is

$$d_x + d_s = -\nabla\Psi\left(v\right), \tag{2.6}$$

where the barrier function $\Psi : \mathbb{R}^n_{++} \to_+$ is defined as follows:

$$\Psi\left(v\right) = \Psi\left(x, s; \mu\right) = \sum_{i=1}^{n}\psi\left(v_i\right), \tag{2.7}$$

where n is the dimension of the problem (number of variables) and ψ is an univariate function called kernel. So, each barrier function Ψ is determined by its kernel ψ. And each kernel function gives rise to a primal-dual interior point algorithm.

$$\psi\left(v_i\right) = \frac{v_i^2 - 1}{2} - \log v_i. \tag{2.8}$$

ψ is called the kernel function of the logarithmic barrier function Ψ. In this paper, we introduce a parameterized kernel function, which will be defined in Section 2. Note that the pair (x, s) coincides with the μ-center $(x(\mu), s(\mu))$ if and only if $v = e$. One can easily verify that the kernel function ψ as defined by (2.8) is a strictly convex function which is defined for any $t \in \mathbb{R}_{++}$ and which is minimal at $t = 1$, where as the minimal value equals 0.

It is clear from the above description that the closeness of (x, s) to $(x(\mu), s(\mu))$ is measured by the value of $\Psi(v)$, with $\tau > 0$ as a threshold value. If $\Psi(v) \leq \tau$, then we start a new outer iteration by performing a $\mu-$update; otherwise we enter an inner iteration by computing the search directions at the current iterates with respect to the current value of μ and apply (2.4) to get new iterates. If necessary, we repeat the procedure until we find iterates that are in the neighborhood of $(x(\mu), s(\mu))$. Then μ is again reduced by the factor $1 - \theta$ with $0 < \theta < 1$, and we apply Newton's method targeting the new μ-centers, and so on. This process is repeated until μ is small enough, *i.e.*, until $n\mu < \epsilon$. At this stage, we have found an ϵ-approximate solution of **LO**.

The parameters τ, θ and the step size α should be chosen in so that the algorithm is optimized in the sense that the number of iterations required by the algorithm is as small as possible. The choice of the so-called barrier update parameter θ plays an important role both in theory and in practice of **IPMs**. Usually, if θ is a constant independent of the dimension n of the problem, for instance $\theta = \frac{1}{2}$, then we call the algorithm a large-update (or long-step) method. If θ depends on the dimension of the problem, such as $\theta = \frac{1}{\sqrt{n}}$, then the algorithm is named a small-update (or short-step) method.

The choice of the step size α, $(0 < \alpha \leq 1)$ is another crucial issue in the analysis of the algorithm. It has to be made such that the closeness of the iterates to the current μ-center improves by a sufficient amount. In the theoretical analysis, the step size α is usually given a value that depends on the closeness of the current iterates to the μ-center.

Prototype algorithm for **LO**

Begin algorithm
 A threshold parameter $\tau > 0$;
 an accuracy parameter $\epsilon > 0$;
 a fixed barrier update parameter $\theta, 0 < \theta < 1$;
begin
 $x = e$; $s = e$; $\mu = 1$; $v = e$.
 while $n\mu \geq \epsilon$ **do**
 begin (outer iteration)
   $\mu = (1 - \theta)\,\mu$;
   while $\Psi(x, s; \mu) > \tau$ **do**
   begin (inner iteration)
      solve the system (2.5) via (2.4) to obtain $(\Delta x, \Delta y, \Delta s)$;
      chose a suitable a step size α;
     $x = x + \alpha\Delta x$;
     $y = y + \alpha\Delta y$;
     $s = s + \alpha\Delta s$;
     $v = \sqrt{\frac{xs}{\mu}}$;
  end (inner iteration)
 end (outer iteration)
 End algorithm.

FIGURE 1. Algorithm.

3. The parameterized Kernel function and its properties

In this section, we present a parameterized Kernel function and give its properties which are essential to our complexity analysis.

We call univariate function $\psi : \mathbb{R}_{++} \to \mathbb{R}_{+}$ a Kernel function if ψ is twice differentiable and satisfies the following conditions:

$$\psi'(1) = \psi(1) = 0,$$

$$\psi''(t) > 0,$$

$$\lim_{t \to 0^+} \psi(t) = \lim_{t \to +\infty} \psi(t) = +\infty. \tag{3.1}$$

Now, we recall that our parameterized univariate function ψ is defined by:

$$\psi(t) = p\left(\frac{t^2-1}{2}\right) + \mathrm{e}^{p\left(\frac{1}{t}-1\right)} - 1, \tag{3.2}$$

where, the parameter p is assumed to be a positive real number. For this purpose, we give the first three derivatives with respect to t as follows:

$$\psi'(t) = pt - \frac{p}{t^2}\mathrm{e}^{p\left(\frac{1}{t}-1\right)},$$

$$\psi''(t) = p + \left(\frac{2p}{t^3} + \frac{p^2}{t^4}\right)\mathrm{e}^{p\left(\frac{1}{t}-1\right)},$$

$$\psi'''(t) = -\left(\frac{6p}{t^4} + \frac{6p^2}{t^5} + \frac{p^3}{t^6}\right)\mathrm{e}^{p\left(\frac{1}{t}-1\right)}. \tag{3.3}$$

Obviously, ψ is a Kernel function and

$$\psi''(t) > p. \tag{3.4}$$

In this paper, the barrier function Ψ is defined by the parameterized kernel function, then (2.6) becomes

$$d_x + d_s = -\nabla\Psi(v), \tag{3.5}$$

where

$$\Psi(v) = \sum_{i=1}^{n} \psi(v_i), \tag{3.6}$$

ψ is defined in (3.2). Hence, the new search direction $(\Delta x, \Delta y, \Delta s)$ is obtained by solving the following modified Newton system:

$$A\Delta x = 0,$$

$$A^T\Delta y + \Delta s = 0,$$

$$s\Delta x + x\Delta s = -\mu v \nabla\Psi(v). \tag{3.7}$$

Note that d_x and d_s are orthogonal because vector d_x belongs to null space and vector d_s to the row space of the matrix $\overline{A}$.

Since d_x and d_s are orthogonal, we have

$$d_x = d_s = 0 \Longleftrightarrow \nabla\Psi(v) = 0 \Longleftrightarrow v = e \Longleftrightarrow \Psi(v) = 0 \Longleftrightarrow \begin{cases} x = x(\mu) \\ s = s(\mu). \end{cases}$$

We use Ψ as the proximity function to measure the distance between the current iterate and the μ-center for given $\mu > 0$. We also define the norm-based proximity measure, $\delta : \mathbb{R}^n_{++} \to \mathbb{R}_+$, as follows:

$$\delta(v) = \tfrac{1}{2} \left\|\nabla\Psi\left(v\right)\right\| = \tfrac{1}{2}\left\|d_x + d_s\right\|. \tag{3.8}$$

Lemma 3.1. *For ψ, we have the following results.*

(i) *ψ is exponentially convex for all $t > 0$; that is*

$$\psi\left(\sqrt{t_1 t_2}\right) \leq \frac{1}{2}\left(\psi\left(t_1\right) + \psi\left(t_2\right)\right).$$

(ii) *ψ'' is monotonically decreasing for all $t > 0$.*
(iii) *$t\psi''\left(t\right) - \psi'\left(t\right) > 0$ for all $t > 0$.*
(iv) *$\psi''\left(t\right)\psi'\left(\beta t\right) - \beta\psi'\left(t\right)\psi''\left(\beta t\right) > 0$, $t > 1$, $\beta > 1$.*

Proof. For (i), using (3.3), we have

$$t\psi''\left(t\right) + \psi'\left(t\right) = 2pt + \left(\frac{p}{t^2} + \frac{p^2}{t^3}\right)\mathrm{e}^{p\left(\frac{1}{t}-1\right)} > 0 \text{ for all } t > 0,$$

and by Lemma 2.1.2 in [17], we have the result.

For (ii), using (3.3), we have $\psi'''(t) < 0$, so we have the result.

For (iii), using (3.3), we have

$$t\psi''\left(t\right) - \psi'\left(t\right) = \left(\frac{3p}{t^2} + \frac{p^2}{t^3}\right)\mathrm{e}^{p\left(\frac{1}{t}-1\right)} > 0 \text{ for all } t > 0.$$

For (iv), using Lemma 2.4 in [5], (ii) and (iii), we have the result. This completes the proof. □

Lemma 3.2. *For ψ, we have*

$$\frac{p}{2}\left(t-1\right)^2 \leq \psi\left(t\right) \leq \frac{1}{2p}\left[\psi'\left(t\right)\right]^2, \quad t > 0 \tag{3.9}$$

$$\psi\left(t\right) \leq \frac{p^2+3p}{2}\left(t-1\right)^2, \quad t > 1. \tag{3.10}$$

Proof. For (3.9), using (3.1) and (3.4), we have

$$\psi\left(t\right) = \int_1^t\int_1^x \psi''\left(y\right)\mathrm{d}y\mathrm{d}x \geq \int_1^t\int_1^x p\mathrm{d}y\mathrm{d}x = \frac{p}{2}\left(t-1\right)^2$$

$$\begin{aligned}
\psi\left(t\right) &= \int_1^t\int_1^x \psi''\left(y\right)\mathrm{d}y\mathrm{d}x \\
&\leq \frac{1}{p}\int_1^t\int_1^x \psi''\left(y\right)\psi''\left(x\right)\mathrm{d}y\mathrm{d}x \\
&= \frac{1}{p}\int_1^t \psi''\left(x\right)\psi'\left(x\right)\mathrm{d}x \\
&= \frac{1}{p}\int_1^t \psi'\left(x\right)\mathrm{d}\psi'\left(x\right) \\
&= \frac{1}{2p}\left[\psi'\left(t\right)\right]^2.
\end{aligned}$$

For (3.10), since $\psi(1) = \psi'(1) = 0$, $\psi'''(t) < 0$, $\psi''(1) = p^2 + 3p$, and by using Taylor's theorem, we have

$$\begin{aligned}
\psi(t) &= \psi(1) + \psi'(1)(t-1) + \frac{1}{2}\psi''(1)(t-1)^2 + \frac{1}{6}\psi'''(\xi)(\xi-1)^3 \\
&= \frac{1}{2}\psi''(1)(t-1)^2 + \frac{1}{6}\psi'''(\xi)(\xi-1)^3 \\
&\le \frac{1}{2}\psi''(1)(t-1)^2 \\
&= \frac{p^2+3p}{2}(t-1)^2
\end{aligned}$$

for some ξ, $1 \le \xi \le t$. This completes the proof. □

Let $\varrho : [0, +\infty[\to [1, +\infty[$ be the inverse function of ψ for $t \ge 1$ and $\rho : [0, +\infty[\to]0, 1]$ be the inverse function of $\frac{-1}{2}\psi'$ for all $t \in]0, 1]$. Then we have the following lemma.

Lemma 3.3. *For ψ, we have*

$$1 + \sqrt{\frac{2}{p^2+3p}s} \le \varrho(s) \le 1 + \sqrt{\frac{2}{p}s}, \quad s \ge 0, \tag{3.11}$$

$$\rho(z) > \frac{1}{\sqrt{\frac{2}{p}z + 1}}, \quad z \ge 0. \tag{3.12}$$

Proof. For (3.11), let $s = \psi(t)$, $t \ge 1$, *i.e.*, $\varrho(s) = t$, $t \ge 1$. By the definition of $\psi(t)$, we have

$$s = \frac{p}{2}t^2 + \mathrm{e}^{p\left(\frac{1}{t}-1\right)} - \left(1 + \frac{p}{2}\right), \ p > 0.$$

By (3.4) and $t \ge 1$, we have

$$\begin{aligned}
\psi''(t) > p &\Leftrightarrow \int_1^t \int_1^x \psi''(y)\,\mathrm{d}y\mathrm{d}x > \int_1^t \int_1^x p\mathrm{d}y\mathrm{d}x \\
&\Leftrightarrow \psi(t) > \frac{p}{2}(t-1)^2 \\
&\Longrightarrow s > \frac{p}{2}(t-1)^2
\end{aligned}$$

which implies that

$$t = \varrho(s) \le 1 + \sqrt{\frac{2}{p}s}.$$

By (3.10), we have $s = \psi(t) \le \frac{p^2+3p}{2}(t-1)^2$, so

$$t = \varrho(s) \ge 1 + \sqrt{\frac{2}{p^2+3p}s}.$$

For (3.12), let $z=\frac{-1}{2}\psi'\left(t\right),\, t\in\left]0,1\right]$. By the definition of $\rho:\rho\left(z\right)=t,\, t\in\left]0,1\right]$, we have $\mathrm{e}^{p\left(\frac{1}{t}-1\right)}>1$. By the definition of $\psi\left(t\right)$, we have

$$\begin{aligned} z &= -\frac{1}{2}\left(pt-\frac{p}{t^2}\mathrm{e}^{p\left(\frac{1}{t}-1\right)}\right) \\ &> -\frac{1}{2}\left(pt-\frac{p}{t^2}\right)=\frac{p}{2}\left(\frac{1}{t^2}-t\right) \\ &= \frac{p}{2}\left(\left(\frac{1}{t^2}-1\right)+(1-t)\right) \\ &\geq \frac{p}{2}\left(\frac{1}{t^2}-1\right) \end{aligned}$$

which implies that

$$t=\rho\left(z\right)>\frac{1}{\sqrt{\frac{2}{p}z+1}}\cdot$$

This completes the proof. $\square$

Let $\psi_b(t)=\frac{p}{t^2}\mathrm{e}^{p\left(\frac{1}{t}-1\right)},\ p>0$, for all $t\in]0,1]$ and $\underline{\rho}:[0,+\infty[\to]0,1]$ be the inverse function of ψ_b. Then, we have the following lemma.

Lemma 3.4. *For ψ_b, we have*

$$\underline{\rho}\left(z\right)>\frac{1}{1+\log\left(\frac{z}{p}\right)^{\frac{1}{p}}},\quad z\geq 0 \tag{3.13}$$

$$\rho\left(z\right)\geq\underline{\rho}\left(p+2z\right),\quad z\geq 0. \tag{3.14}$$

Proof. For (3.13), let $z=\psi_b\left(t\right)=\frac{p}{t^2}\mathrm{e}^{p\left(\frac{1}{t}-1\right)},\ p>0$, for all $t\in\left]0,1\right]$.

We have

$$\mathrm{e}^{p\left(\frac{1}{t}-1\right)}=\frac{t^2z}{p}\leq\frac{z}{p},$$

which implies that

$$t=\underline{\rho}\left(z\right)\geq\frac{1}{1+\log\left(\frac{z}{p}\right)^{\frac{1}{p}}},\quad z\geq 0.$$

For (3.14), we have

$$z=\frac{-1}{2}\psi'\left(t\right)=\frac{-1}{2}\left(pt-\psi_b\left(t\right)\right),\ z\ \geq 0,$$

which implies that

$$\psi_b\left(t\right)=pt+2z$$

since $t\leq 1$, we have

$$\psi_b\left(t\right)\leq p+2z$$

and since $\underline{\rho}$ is monotone decreasing with respect to $z\geq p$, we have

$$t=\rho\left(z\right)\geq\underline{\rho}\left(p+2z\right).$$

This completes the proof. $\square$

Lemma 3.5. *Let* $\varrho : [0, +\infty[\to [1, +\infty[$ *be the inverse function of* ψ *for* $t \geq 1$*. Then, we have*

$$\Psi(\beta v) \leq n\psi\left(\beta\varrho\left(\frac{\Psi(v)}{n}\right)\right), \qquad v \in \mathbb{R}_{++},\ \beta \geq 1.$$

Proof. Using Lemma 3.1(iv), and Theorem 3.2 in [5], we can get the result. This completes the proof. □

Lemma 3.6. *Let* $0 \leq \theta < 1$*,* $v_+ = \frac{v}{\sqrt{1-\theta}}$*, If* $\Psi(v) \leq \tau$ *then, we have*

$$\Psi(v_+) \leq \frac{(p^2+3p)}{2(1-\theta)}\left(\theta\sqrt{n} + \sqrt{\frac{2}{p}\tau}\right)^2.$$

Proof. Since $\frac{1}{\sqrt{1-\theta}} \geq 1$ and $\varrho(\frac{\Psi(v)}{n}) \geq 1$, we get $\frac{\varrho(\frac{\Psi(v)}{n})}{\sqrt{1-\theta}} \geq 1$.

By (3.10), with

$$\psi(t) \leq \frac{p^2+3p}{2}(t-1)^2, \qquad t > 1,$$

Using Lemma 3.5 with $\beta = \frac{1}{\sqrt{1-\theta}}$, (3.11) and $\Psi(v) \leq \tau$, we have

$$\begin{aligned}
\Psi(v_+) &\leq n\psi\left(\frac{1}{\sqrt{1-\theta}}\varrho\left(\frac{\Psi(v)}{n}\right)\right) \\
&\leq n\frac{p^2+3p}{2}\left(\frac{1}{\sqrt{1-\theta}}\varrho\left(\frac{\Psi(v)}{n}\right) - 1\right)^2 \\
&= \frac{n}{2(1-\theta)}(p^2+3p)\left(\varrho\left(\frac{\Psi(v)}{n}\right) - \sqrt{1-\theta}\right)^2 \\
&\leq \frac{n}{2(1-\theta)}(p^2+3p)\left(1 + \sqrt{\frac{2}{p}\left(\frac{\Psi(v)}{n}\right)} - \sqrt{1-\theta}\right)^2 \\
&\leq \frac{n}{2(1-\theta)}(p^2+3p)\left(\theta + \sqrt{\frac{2}{p}\left(\frac{\tau}{n}\right)}\right)^2 \\
&= \frac{(p^2+3p)}{2(1-\theta)}\left(\theta\sqrt{n} + \sqrt{\frac{2}{p}\tau}\right)^2 \\
&= \frac{(p^2+3p)}{2(1-\theta)}\left(\theta\sqrt{n} + \sqrt{\frac{2}{p}\tau}\right)^2 = (\Psi)_0,
\end{aligned}$$

where the last inequality holds from $1 - \sqrt{1-\theta} = \frac{\theta}{1+\sqrt{1-\theta}} \leq \theta$. This completes the proof. □

Denote

$$(\Psi)_0 = \frac{(p^2+3p)}{2(1-\theta)}\left(\theta\sqrt{n} + \sqrt{\frac{2}{p}\tau}\right)^2 = L(n, \theta, \tau), \tag{3.15}$$

then, $(\Psi)_0$ is an upper bound for $\Psi(v_+)$ during the process of the algorithm.

4. Decrease of the barrier function during an inner iteration

In this section, we compute a default, step size α and the resulting decrease of the barrier function. After a damped step we have

$$\begin{aligned}
x^+ &= x + \alpha\Delta x; \\
y^+ &= y + \alpha\Delta y; \\
s^+ &= s + \alpha\Delta s.
\end{aligned}$$

Using (2.4), we have

$$x^{+} = x\left(e + \alpha\frac{\Delta x}{x}\right) = x\left(e + \alpha\frac{d_x}{v}\right) = \frac{x}{v}\left(v + \alpha d_x\right),$$
$$s^{+} = s\left(e + \alpha\frac{\Delta s}{s}\right) = s\left(e + \alpha\frac{d_s}{v}\right) = \frac{s}{v}\left(v + \alpha d_s\right).$$

So, we have

$$v_{+} = \sqrt{\frac{x^{+}s^{+}}{\mu}} = \sqrt{\left(v + \alpha d_x\right)\left(v + \alpha d_s\right)}.$$

Define, for $\alpha > 0$,

$$f\left(\alpha\right) = \Psi\left(v_{+}\right) - \Psi\left(v\right).$$

Then $f(\alpha)$ is the difference of proximities between a new iterate and a current iterate for fixed μ. By Lemma 3.1(i), we have

$$\Psi\left(v_{+}\right) = \Psi\left(\sqrt{\left(v + \alpha d_x\right)\left(v + \alpha d_s\right)}\right) \leq \frac{1}{2}\left(\Psi\left(v + \alpha d_x\right) + \Psi\left(v + \alpha d_s\right)\right).$$

Therefore, $f\left(\alpha\right) \leq f_1\left(\alpha\right)$, where

$$f_1\left(\alpha\right) = \frac{1}{2}\left(\Psi\left(v + \alpha d_x\right) + \Psi\left(v + \alpha d_s\right)\right) - \Psi\left(v\right). \tag{4.1}$$

Obviously, $f(0) = f_1(0) = 0$. Taking the first two derivatives of $f_1\left(\alpha\right)$ with respect to α, we have

$$f_1'\left(\alpha\right) = \frac{1}{2}\sum_{i=1}^{n}\left(\psi'\left(v_i + \alpha d_{x_i}\right)d_{x_i} + \psi'\left(v_i + \alpha d_{s_i}\right)d_{si}\right),$$
$$f_1''\left(\alpha\right) = \frac{1}{2}\sum_{i=1}^{n}\left(\psi''\left(v_i + \alpha d_{x_i}\right)d_{x_i}^2 + \psi''\left(v_i + \alpha d_{s_i}\right)d_{s_i}^2\right).$$

Using (3.5) and (3.8), we have

$$f_1'\left(0\right) = \frac{1}{2}\nabla\Psi\left(v\right)^t\left(d_x + d_s\right) = -\frac{1}{2}\nabla\Psi\left(v\right)^t\nabla\Psi\left(v\right) = -2\delta(v)^2.$$

For convenience, we denote

$$v_1 = \min\left(v\right),\ \ \delta = \delta(v),\ \ \Psi = \Psi\left(v\right).$$

Lemma 4.1. *Let $\delta(v)$ be as defined in* (3.8). *Then, we have*

$$\delta(v) \geq \sqrt{\frac{p}{2}\Psi\left(v\right)} = \sqrt{\frac{p}{2}\Psi}.$$

Proof. Using (3.9), we have

$$\Psi\left(v\right) = \sum_{i=1}^{n}\psi\left(v_i\right) \leq \sum_{i=1}^{n}\frac{1}{2p}\left(\psi'\left(v_i\right)\right)^2 = \frac{1}{2p}\left\|\nabla\Psi\left(v\right)\right\|^2 = \frac{4}{2p}\delta(v)^2,$$

so

$$\delta(v) \geq \sqrt{\frac{p}{2}\Psi\left(v\right)} = \sqrt{\frac{p}{2}\Psi}.$$

This completes the proof. □

Remark 4.2. Throughout the paper, we assume that $\tau \geq 1$. Using Lemma 4.1 and the assumption that $\Psi(v) \geq \tau$, we have

$$\delta(v) \geq \sqrt{\frac{p}{2}}.$$

From Lemmas 4.1–4.4 in [5], we have the following Lemmas 4.3–4.6.

Lemma 4.3. *Let $f_1(\alpha)$ be as defined in* (3.14) *and $\delta(v)$ be as defined in* (3.8)*. Then, we have*

$$f_1''(\alpha) \leq 2\delta^2 \psi''(v_{\min} - 2\alpha\delta).$$

Lemma 4.4. *If the step size α satisfies the inequality*

$$\psi'(v_{\min}) - \psi'(v_{\min} - 2\alpha\delta) \leq 2\delta, \tag{4.2}$$

then, we have

$$f_1'(\alpha) \leq 0.$$

Lemma 4.5. *let $\rho : [0, +\infty[\to]0, 1]$ be the inverse function of $\frac{-1}{2}\psi'$ for all $t \in]0, 1]$. Then, the largest step size $\bar{\alpha}$ satisfying* (4.2) *is given by*

$$\bar{\alpha} = \frac{1}{2\delta}\left(\rho(\delta) - \rho(2\delta)\right).$$

Lemma 4.6. *Let $\bar{\alpha}$ be as defined in Lemma 4.5. Then*

$$\bar{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))}.$$

Lemma 4.7. *Let ρ and $\bar{\alpha}$ be as defined in Lemma 4.6. If*

$$\Psi = \Psi(v) \geq \tau \geq 1,$$

then we have

$$\bar{\alpha} \geq \frac{1}{p + \left[(2+p)(p+4)\sqrt{\frac{p}{2}\Psi}\right]\left[1 + \log\left(1 + \frac{4}{p}\sqrt{\frac{p}{2}\Psi}\right)^{\frac{1}{p}}\right]^2}.$$

Proof. Using Lemma 4.6, the definition of $\psi''(t)$ and (3.14), we have

$$\bar{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))} \geq \frac{1}{\psi''\left(\underline{\rho}(p + 2(2\delta))\right)}.$$

By Lemma 4.1, we have $\delta \geq \sqrt{\frac{p}{2}\Psi}$, and of the increasing functions $\psi''(t)$ and $\rho(\delta)$, we have

$$\begin{aligned}
\delta \geq \sqrt{\frac{p}{2}\Psi} &\Leftrightarrow \rho(2\delta) \geq \rho\left(2\sqrt{\frac{p}{2}\Psi}\right) \\
&\Leftrightarrow \psi''(\rho(2\delta)) \leq \psi''\left(\rho\left(2\sqrt{\frac{p}{2}\Psi}\right)\right) \\
&\Leftrightarrow \frac{1}{\psi''(\rho(2\delta))} \geq \frac{1}{\psi''\left(\rho\left(2\sqrt{\frac{p}{2}\Psi}\right)\right)}.
\end{aligned}$$

Then

$$\begin{aligned}\bar{\alpha} &\geq \frac{1}{\psi''\left(\rho\left(2\delta\right)\right)}\\ &\geq \frac{1}{\psi''\left(\rho\left(2\sqrt{\frac{p}{2}\Psi}\right)\right)}\\ &\geq \frac{1}{\psi''\left(\underline{\rho}\left(p+2\left(2\sqrt{\frac{p}{2}\Psi}\right)\right)\right)}.\end{aligned}$$

Now putting $t=\underline{\rho}\left(p+2\left(2\sqrt{\frac{p}{2}\Psi}\right)\right)=\underline{\rho}\left(p+4\sqrt{\frac{p}{2}\Psi}\right)$ then we obtain $t\leq 1$ and

$$\begin{aligned}\bar{\alpha} &\geq \frac{1}{\psi''\left(t\right)}\\ &= \frac{1}{p+\left(\frac{2p}{t^3}+\frac{p^2}{t^4}\right)\mathrm{e}^{p\left(\frac{1}{t}-1\right)}}\\ &= \frac{1}{p+\left(\frac{2t+p}{t^2}\right)\psi_b\left(t\right)}\\ &\geq \frac{1}{p+\left(\frac{2+p}{t^2}\right)\psi_b\left(t\right)} \geq \frac{1}{p+\left(\frac{2+p}{t^2}\right)\psi_b\left(t\right)} \geq \frac{1}{p+\left(2+p\right)\frac{1}{t^2}\psi_b\left(t\right)}.\end{aligned}$$

We have also

$$\begin{aligned}\frac{1}{t^2} &= \frac{1}{\left[\underline{\rho}\left(p+4\sqrt{\frac{p}{2}\Psi}\right)\right]^2}\\ &\leq \left[1+\log\left(\frac{p+4\sqrt{\frac{p}{2}\Psi}}{p}\right)^{\frac{1}{p}}\right]^2\\ &= \left[1+\log\left(1+\frac{4}{p}\sqrt{\frac{p}{2}\Psi}\right)^{\frac{1}{p}}\right]^2\end{aligned}$$

and

$$\psi_b\left(t\right)=p+4\sqrt{\frac{p}{2}\Psi}=p\left(1+\frac{4}{p}\sqrt{\frac{p}{2}\Psi}\right).$$

Finally we get

$$\begin{aligned}\bar{\alpha} &\geq \frac{1}{p+\left(2+p\right)\left(p+4\sqrt{\frac{p}{2}\Psi}\right)\left[1+\log\left(1+\frac{4}{p}\sqrt{\frac{p}{2}\Psi}\right)^{\frac{1}{p}}\right]^2}\\ &\geq \frac{1}{p+\left[\left(2+p\right)\left(p+4\right)\sqrt{\frac{p}{2}\Psi}\right]\left[1+\log\left(1+\frac{4}{p}\sqrt{\frac{p}{2}\Psi}\right)^{\frac{1}{p}}\right]^2}.\end{aligned}$$

This completes the proof □

Denoting

$$\tilde{\tilde{\alpha}}=\frac{1}{p+\left[\left(2+p\right)\left(p+4\right)\sqrt{\frac{p}{2}\Psi}\right]\left[1+\log\left(1+\frac{4}{p}\sqrt{\frac{p}{2}\Psi}\right)^{\frac{1}{p}}\right]^2}. \tag{4.3}$$

$\tilde{\tilde{\alpha}}$ becomes a default step size and that $\tilde{\tilde{\alpha}}\leq\bar{\alpha}$.

Lemma 4.8 (4.5 in [5]). *If the step size α satisfies $\alpha \leq \bar{\alpha}$, then*

$$f(\alpha) \leq -\alpha\delta^2.$$

Remark 4.9. The decrease of the barrier function Ψ depends of the sign of the function f. In fact, in Lemma 4.8, we have $f(\alpha) < 0$ for some α and in Lemma 4.11, we obtain the total number of inner iteration K which guaranteed $\Psi(v) < \tau$.

Lemma 4.10. *Let $\tilde{\tilde{\alpha}}$ be the default step size as defined in* (4.3) *and let*

$$(\Psi)_0 \geq \Psi(v) \geq 1.$$

Then

$$f\left(\tilde{\tilde{\alpha}}\right) \leq \frac{-\sqrt{\frac{p}{2}}\,[(\Psi)_0]^{\frac{1}{2}}}{\left[\sqrt{\frac{2}{p}}p + (2+p)(p+4)\right]\left[1 + \log\left(1 + \frac{4}{p}\sqrt{\frac{p}{2}(\Psi)_0}\right)^{\frac{1}{p}}\right]^2}. \tag{4.4}$$

Proof. Using Lemma 4.8 (4.5 in [5]) with $\alpha = \tilde{\tilde{\alpha}}$ and (4.3), we have

$$\begin{aligned} f\left(\tilde{\tilde{\alpha}}\right) &\leq -\tilde{\tilde{\alpha}}\delta^2 \\ &= -\frac{\delta^2}{p + \left[(2+p)(p+4)\sqrt{\frac{p}{2}\Psi}\right]\left[1 + \log\left(1 + \frac{4}{p}\sqrt{\frac{p}{2}\Psi}\right)^{\frac{1}{p}}\right]^2}. \end{aligned}$$

By Lemma 4.1, we have $\delta \geq \sqrt{\frac{p}{2}\Psi}$, and $\sqrt{\frac{p}{2}\Psi} \geq \sqrt{\frac{p}{2}}$ so

$$\begin{aligned} f\left(\tilde{\tilde{\alpha}}\right) &\leq -\frac{\left(\sqrt{\frac{p}{2}\Psi}\right)^2}{\sqrt{\frac{p}{2}\Psi}\left[\sqrt{\frac{2}{p}}p + (2+p)(p+4)\right]\left[1 + \log\left(1 + \frac{4}{p}\sqrt{\frac{p}{2}\Psi}\right)^{\frac{1}{p}}\right]^2} \\ &= -\frac{\sqrt{\frac{p}{2}\Psi}}{\left[\sqrt{\frac{2}{p}}p + (2+p)(p+4)\right]\left[1 + \log\left(1 + \frac{4}{p}\sqrt{\frac{p}{2}\Psi}\right)^{\frac{1}{p}}\right]^2} \\ &\leq -\frac{\sqrt{\frac{p}{2}(\Psi)_0}}{\left[\sqrt{\frac{2}{p}}p + (2+p)(p+4)\right]\left[1 + \log\left(1 + \frac{4}{p}\sqrt{\frac{p}{2}(\Psi)_0}\right)^{\frac{1}{p}}\right]^2}. \end{aligned}$$

This completes the proof. □

After the update of μ to $(1-\theta)\mu$, we have

$$\Psi(v_+) \leq \frac{(p^2+3p)}{2(1-\theta)}\left(\theta\sqrt{n} + \sqrt{\frac{2}{p}\tau}\right)^2 = L(n,\theta,\tau).$$

We need to count how many inner iterations are required to return to the situation where $\Psi(v) \leq \tau$. We denote the value of $\Psi(v)$ after the μ update as $(\Psi)_0$; the subsequent values in the same outer iteration are denoted as $(\Psi)_k$, $k = 1, 2, \ldots, K$, where K denotes the total number of inner iterations in the outer iteration. The decrease in each inner iteration is given by (4.4). In [5], we can find the appropriate values of κ and $\gamma \in]0,1]$:

$$\kappa = \frac{\sqrt{\frac{p}{2}}}{\left[\sqrt{\frac{2}{p}}p + (2+p)(p+4)\right]\left[1 + \log\left(1 + \frac{4}{p}\sqrt{\frac{p}{2}(\Psi)_0}\right)^{\frac{1}{p}}\right]^2}, \qquad \text{with } \gamma = \frac{1}{2}.$$

Lemma 4.11. *Let K be the total number of inner iterations in the outer iteration. Then we have*

$$K \leq \frac{4\sqrt{p}+2\sqrt{2}\,(2+p)\,(p+4)}{\sqrt{p}}\left[1+\log\left(1+\frac{4}{p}\sqrt{\frac{p}{2}\,(\Psi)_0}\right)^{\frac{1}{p}}\right]^2 [(\Psi)_0]^{\frac{1}{2}}.$$

Proof. By Lemma 1.3.2 in [16], we have

$$\begin{aligned} K &\leq \frac{[(\Psi)_0]^{\gamma}}{\kappa\gamma} \\ &= \frac{4\sqrt{p}+2\sqrt{2}\,(2+p)\,(p+4)}{\sqrt{p}}\left[1+\log\left(1+\frac{4}{p}\sqrt{\frac{p}{2}\,(\Psi)_0}\right)^{\frac{1}{p}}\right]^2 [(\Psi)_0]^{\frac{1}{2}}. \end{aligned}$$

This completes the proof. □

Theorem 4.12. *Let an* **LO** *problem be given, let $(\Psi)_0$ be as defined in* (3.15) *and let $\tau \geq 1$. Then, the total number of iterations to have an approximate solution with $n\mu < \epsilon$ is bounded by*

$$\frac{4\sqrt{p}+2\sqrt{2}\,(2+p)\,(p+4)}{\sqrt{p}}\left[1+\log\left(1+\frac{4}{p}\sqrt{\frac{p}{2}\,(\Psi)_0}\right)^{\frac{1}{p}}\right]^2 [(\Psi)_0]^{\frac{1}{2}}\,\frac{\log\frac{n}{\epsilon}}{\theta}.$$

Proof. Recall that $(\Psi)_0$ is the upper bound according to (3.15). The number of outer iterations is bounded above by $\frac{\log\frac{n}{\epsilon}}{\theta}$ (see [18], Lem. II.17, page 116). Through multiplying the number of outer iterations by the number of inner iterations, we get an upper bound for the total number of iterations, namely,

$$\frac{4\sqrt{p}+2\sqrt{2}\,(2+p)\,(p+4)}{\sqrt{p}}\left[1+\log\left(1+\frac{4}{p}\sqrt{\frac{p}{2}\,(\Psi)_0}\right)^{\frac{1}{p}}\right]^2 [(\Psi)_0]^{\frac{1}{2}}\,\frac{\log\frac{n}{\epsilon}}{\theta}.$$

This completes the proof. □

For large-update methods with $\tau = \mathbf{O}(n)$ and $\theta = \mathbf{\Theta}(1)$, we distinguish the two cases:
The first case if $p \in [1,+\infty[$, we get for large-update methods $(\Psi)_0 = \mathbf{O}\left(p^2 n\right)$ and $\mathbf{O}\left(\sqrt{np^5}\,(\log pn)^2 \log\frac{n}{\epsilon}\right)$ iterations.
The second case if $p \in]0,1[$, we get for large-update methods $(\Psi)_0 = \mathbf{O}(n)$ and $\mathbf{O}\left(\sqrt{\frac{n}{p^5}}(\log\frac{n}{p})^2\log\frac{n}{\epsilon}\right)$ iterations.

For small-update methods, we have $\tau = \mathbf{O}(1)$ and $\theta = \mathbf{\Theta}(\frac{1}{\sqrt{n}})$, we distinguish the two cases:
The first case if $p \in [1,+\infty[$, we get for small-update methods $(\Psi)_0 = \mathbf{O}(p^2)$ and $\mathbf{O}(\sqrt{np^5}\log\frac{n}{\epsilon})$ iterations.
The second case if $p \in]0,1[$, we get for small-update methods $(\Psi)_0 = \mathbf{O}(1)$ and $\mathbf{O}(\sqrt{\frac{n}{p^5}}\log\frac{n}{\epsilon})$ iterations.

5. Concluding remarks

In this paper, we have analyzed large-update and small-update versions of the primal-dual interior point algorithm described in Figure 1 that are based on the parameterized kernel function (3.2). The proposed function is not logarithmic and not self-regular. We proved that the iteration bound of a large-update interior point method based on the kernel function considered in this paper is $\mathbf{O}(\sqrt{n}(\log n)^2\log\frac{n}{\epsilon})$ and for small-update methods, we obtain the best know iteration bound, namely $\mathbf{O}(\sqrt{n}\log\frac{n}{\epsilon})$, just take $p = \mathbf{\Theta}(1)$.

Acknowledgements. The authors are very grateful and would like to thank the Editor-in-Chief and the anonymous referees for their suggestions and helpful comments which significantly improved the the presentation of this paper.

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