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EXPLORING THE DISJUNCTIVE RANK OF SOME FACET-INDUCING INEQUALITIES OF THE ACYCLIC COLORING POLYTOPE

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Abstract. In a previous work we presented six facet-inducing families of valid inequalities for the polytope associated to an integer programming formulation of the acyclic coloring problem. In this work we study their disjunctive rank, as defined by [E. Balas, S. Ceria and G. Cornuéjols, *Math. Program.* **58** (1993) 295–324]. We also propose to study a dual concept, which we call the disjunctive anti-rank of a valid inequality.

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1. Introduction

A *coloring* of a graph G is an assignment of colors to the vertices of G such that any two vertices receive distinct colors whenever they are adjacent. An *acyclic coloring* of a graph G is a coloring such that no cycle of G receives exactly two colors, *i.e.*, such that the subgraph of G induced by any two color classes is acyclic. The *acyclic chromatic number* $\chi_A(G)$ of a graph G is the minimum number of colors in any such coloring of G. Given a graph G, the *acyclic coloring problem* consists in finding $\chi_A(G)$, and this problem has been shown to be NP-hard [12].

The acyclic coloring problem arises in the context of matrix partitioning for the estimation of the Hessian matrix associated to numerical optimization problems [13,15], although it was introduced by Grünbaum in [18] in a different context. Many previous research efforts on this problem consisted in finding bounds on $\chi_A(G)$ for particular classes of graphs [2,6–8,11,14]. Efficient heuristic algorithms for the acyclic coloring problem were developed in [16,17]. However, not too many approaches in order to solve this problem in practice exist.

These considerations and the interest of this problem as a combinatorial model, motivate us to approach this problem from the perspective of integer linear programming. We presented in a previous work [10] an integer programming model for the acyclic coloring problem, based on existing formulations for the classical vertex coloring problem. We studied the structure of the polyhedron associated with this formulation with the objective of finding valid inequalities that can contribute to an algorithm based on cutting-plane methods. In particular, we introduced six families of facet-inducing inequalities.

Lift-and-project methods provide a systematic way to generate a sequence of convex relaxations of a polytope, converging to the convex hull of the feasible solutions. These methods usually start with a linear relaxation, and

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construct a sequence of polytopes – each included in the previous one – that ends with the convex hull. Many of these methods use higher dimensional formulations during the construction of this sequence of polytopes, because these formulations sometimes allow compact representations of exponentially many facets. Many lift-and-project operators have been proposed, the most prominent being the Balas–Ceria–Cornuéjols operator [5], the Sherali-Adams operator [27], the Lovász–Schriver operator [23], and the Lasserre operator [19]. For a thorough analysis of these procedures, we refer the reader to [4].

A concept derived from the existence of such operators is the *rank* of a valid inequality, defined as the minimum number of applications of the operator needed to get a polytope for which the inequality is valid (this concept is well defined since the last polytope in the sequence is the convex hull of feasible solutions, so it satisfies the valid inequality). This value has been proposed as a measure of theoretical interest of a valid inequality, in contrast to computational measurements assesing the contribution of the inequality within cutting plane environments.

In this work we are interested in the rank of the families of valid inequalities presented in [10]. In particular, we study the rank associated to the Balas–Ceria–Cornuéjols (BCC) operator, usually called the *disjunctive rank*. We propose to also study a dual concept, which we call the *disjunctive anti-rank* of a valid inequality, defined as the maximum number of applications of the BCC operator ensuring a polytope that satisfies the inequality. In [10] a preliminary branch-and-cut procedure was implemented, experimentally showing that two of the families of valid inequalities considered in this work allowed to achieve the best perfomance. An additional motivation for the present work is to verify whether these computational results correlate with the theoretical strength of these inequalities, as measured by their disjunctive rank and anti-rank. Previous analyses of the disjunctive rank of valid inequalities for particular problems can be found in [1, 22, 25], and further studies of lift-and-project applications to particular problems are carried out in [3, 20, 21, 24, 26], among others.

This paper is organized as follows. In Section 2 we recall the definition of the BCC operator and the disjunctive rank of a valid inequality, and introduce the definition of the anti-rank of a valid inequality. In Section 3 we introduce the integer linear programming model for the acyclic coloring problem and give some definitions. In Section 4 we study the disjunctive rank and anti-rank of six families of facet-inducing inequalities. Finally, in Section 5 we provide some concluding remarks and directions for future work. A preliminary version of these results appeared without proofs in the conference paper [9].

2. The BCC operator

We now formally define the BCC operator introduced by Balas *et al.* [5]. Let $P = \operatorname{conv}\{x \in \{0,1\}^n : Ax \leq b\}$ be the convex hull of the integer points within $L = \{x \in [0,1]^n : Ax \leq b\}$. The BCC operator takes the polytope L and a variable x_i for $i \in \{1, \ldots, n\}$ and generates a new polytope $P_{x_i}(L) \subseteq L$, in the following way:

(1) Multiply the system $Ax \leq b$ by x_i and $1 - x_i$, getting the systems $x_i(b - Ax) \geq 0$ and $(1 - x_i)(b - Ax) \geq 0$.
(2) Identify $x_i := x_i^2$ and $y_k := x_i x_k$ for $k \neq i$, thus getting a lifted polytope $L^i \subseteq \mathbb{R}^{2n-1}$.
(3) Project L^i back to the space of the original x-variables, and call $P_{x_i}(L)$ the resulting polytope.

We refer to the procedure applied to the variable x_i as BCC_{x_i}. We can now repeat the procedure with some other variable x_j, for $j \neq i$, thus getting the polytope $P_{x_j}(P_{x_i}(L))$. It can be seen that the order of the lifted variables does not change the resulting polytope [5], *i.e.*, $P_{x_j}(P_{x_i}(L)) = P_{x_i}(P_{x_j}(L))$, so we simply denote this polytope by $P_{\mathbb{A}}(L)$, where $\mathbb{A} = \{x_i, x_j\}$. If $\mathbb{A} \subset \mathbb{A}'$ then $P_{\mathbb{A}'}(L) \subseteq P_{\mathbb{A}}(L)$ and, crucially, $P_{\mathbb{V}}(L) = P$ for $\mathbb{V} = \{x_1, \ldots, x_n\}$.

Figure 1 provides an example for the linear relaxation $L = \{x \in \mathbb{R}^3_+ : x_1 + x_2 + x_3 \leq 1 + \varepsilon\}$ for $\varepsilon \in (0, 1)$, hence the convex hull of the integer solutions is $P = \{x \in \mathbb{R}^3_+ : x_1 + x_2 + x_3 \leq 1\}$. Starting from L, each path generates a sequence of polytopes ending at P after 3 steps. Consider the inequality $x_1 \leq 1$, which is valid for P (although not for L). The polytopes marked with (*) in the figure are those satisfying $x_1 \leq 1$. For every path from L to P, at some point the inequality $x_1 \leq 1$ is satisfied, and the *disjunctive rank* is the minimum height k such that some polytope at height k satisfies the inequality.

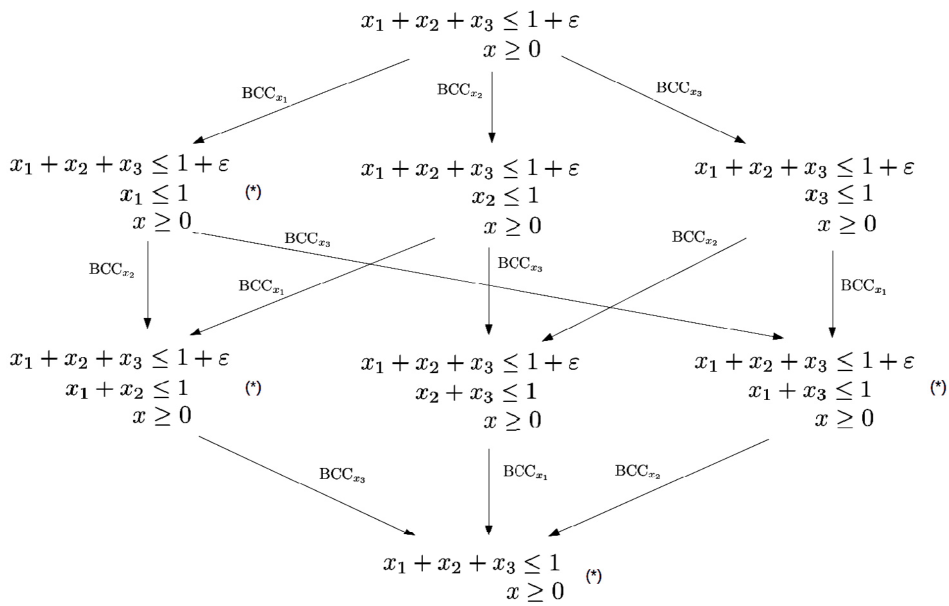

FIGURE 1. The sketch shows the polyhedra obtained after successively applying the BCC operator from the linear relaxation $L = \{x \in \mathbb{R}^3_+ : x_1 + x_2 + x_3 \leq 1 + \varepsilon\}$. The polytopes marked with (*) satisfy the valid inequality $x_1 \leq 1$. One polytope in the first level (*i.e.*, obtained by exactly one application of the BCC operator) safisfies the inequality $x_1 \leq 1$, hence this inequality has disjunctive rank 1. All polytopes in the third level satisfy $x_1 \leq 1$ whereas there is a polytope in the second level not satisfying this inequality, hence $x_1 \leq 1$ has disjunctive anti-rank 2.

Definition 2.1. [5] Let $\pi x \leq \pi_0$ be a valid inequality for P. The inequality $\pi x \leq \pi_0$ has *disjunctive rank* k if and only if there exists a set $\mathbb{A}$ of variables such that $|\mathbb{A}| = k$ and $\pi x \leq \pi_0$ is valid for $P_{\mathbb{A}}(L)$, and $\pi x \leq \pi_0$ is not valid for $P_{\mathbb{B}}(L)$ for any set $\mathbb{B}$ of variables with $|\mathbb{B}| = k - 1$.

The disjunctive rank is a theoretical measure associated with a valid inequality, given by the minimum number of applications of the *BCC* operator [5] needed to obtain the inequality. Note that if the disjunctive rank of a valid inequality for P is 0 then it is also valid for the linear relaxation L. In this work we propose to also study the *maximum* number of such applications, which we call the *disjunctive anti-rank* of a valid inequality and is, in some sense, the dual concept of the disjunctive rank. In Figure 1, the disjunctive rank corresponds to the maximum height t such that there exists some polytope at height t *not* satisfying the valid inequality.

Definition 2.2. Let $\pi x \leq \pi_0$ be a valid inequality for P with nonzero disjunctive rank. The inequality $\pi x \leq \pi_0$ has *disjunctive anti-rank* t if and only if there exists a set $\mathbb{B}$ of variables with $|\mathbb{B}| = t$ such that $\pi x \leq \pi_0$ is not valid for $P_{\mathbb{B}}(L)$, and $\pi x \leq \pi_0$ is valid for $P_{\mathbb{A}}(L)$ for any set $\mathbb{A}$ of variables with $|\mathbb{A}| = t + 1$.

The disjunctive rank of a valid inequality is less than or equal to the anti-rank. Therefore if the disjunctive anti-rank of a valid inequality is 0 then the rank is also 0. Moreover, if the disjunctive rank is 0, then the linear relaxation satisfies the inequality and the anti-rank is 0 too. The disjunctive anti-rank is a natural measure associated with a valid inequality, in this case providing a lower bound on the number of BCC iterations needed to obtain a polytope $P_{\mathbb{A}}(L)$ satisfying the valid inequality without regard of the choice of the set $\mathbb{A}$ of variables, see Figure 1.

The following result provides a useful property of the BCC operator.

Theorem 2.3. [5] *If $\mathbb{A}$ is a subset of variables, then* $P_{\mathbb{A}}(L) = \text{conv}\{x \in L : x_i \in \{0,1\}$ *for every* $x_i \in \mathbb{A}\}$.

This theorem is the basis for the analysis of the disjunctive rank of valid inequalities, since it provides a straightforward way of checking whether a (possibly fractional) solution in L belongs to $P_{\mathbb{A}}(L)$ or not, for a given subset $\mathbb{A}$ of variables. This direct check enables the approaches followed in the proofs in Section 4.

To the best of our knowledge, the other lift-and-project operators mentioned in the introduction do not admit a similar result characterizing feasible solutions of the resulting convex bodies, and this makes the exploration of the corresponding disjunctive ranks a more difficult issue. Lacking such a simple characterization, the strategy for finding bounds on the disjunctive rank followed in this work might not be applied for these other operators in a direct way. Due to these facts, we concentrate in this work on the BCC operator as a first approach on the lift-and-project rank of the known inequalities for the standard formulation of the acyclic coloring problem.

3. Integer programming formulation for acyclic coloring

Let $G = (V,E)$ be a simple connected graph, and denote by $\mathcal{T}$ the set of available colors. For $v \in V$ and $c \in \mathcal{T}$, we define the *assignment variable* x_{vc} to be $x_{vc} = 1$ if the vertex v is assigned the color c, and $x_{vc} = 0$ otherwise. For every $c \in \mathcal{T}$ we define the *color variable* w_c to be $w_c = 1$ if some vertex uses the color c, and $w_c = 0$ otherwise.

Denote by $\mathbf{C}(G) \subseteq 2^V$ the set of all cycles of G. The acyclic coloring problem can be formulated in terms of the assignment variables and the color variables in the following way:

$$\min \sum_{c\in\mathcal{T}} w_c$$

$$\text{s.t.} \quad \sum_{c\in\mathcal{T}} x_{vc} = 1 \qquad \forall v \in V, \tag{3.1}$$

$$x_{uc} + x_{vc} \le w_c \qquad \forall uv \in E \quad \forall c \in \mathcal{T}, \tag{3.2}$$

$$\sum_{v\in A} x_{vc} + x_{vc'} \le |A| - 1 \quad \forall A \in \mathbf{C}(G),\ \forall c, c' \in \mathcal{T}, \tag{3.3}$$

$$x_{vc} \in \{0,1\} \qquad \forall v \in V, c \in \mathcal{T}, \tag{3.4}$$

$$w_c \in \{0,1\} \qquad \forall c \in \mathcal{T}. \tag{3.5}$$

Let $p = |V||\mathcal{T}| + |\mathcal{T}|$. We define $P(G,\mathcal{T}) \subset \mathbb{R}^p$ to be the convex hull of the vectors $(x,w) \in \{0,1\}^p$ satisfying constraints (1)–(5). Let $L(G,\mathcal{T}) \subset [0,1]^p$ be the linear relaxation of $P(G,\mathcal{T})$, *i.e.*, the points $(x,w) \in [0,1]^p$ satisfying the constraints (1)–(3). When the graph G and the color set $\mathcal{T}$ are clear from the context, we denote the linear relaxation $L(G,\mathcal{T})$ by L. Finally, let $\mathbb{V}$ be the set of variables from models (1)–(5), *i.e.*, $\mathbb{V} = \{x_{vj} : v \in V, c \in \mathcal{T}\} \cup \{w_c : c \in \mathcal{T}\}$.

In the following lemma we collect some straightforward facts that will arise frequently in Section 4.

Lemma 3.1. *Let $(x,w) \in L(G,\mathcal{T})$, let $\mathbf{C} \subseteq V$ be an even cycle, and let $c \in \mathcal{T}$.*

(i) *The point (x,w) satisfies the inequality* $\sum_{v\in\mathbf{C}} x_{vc} \le \frac{|\mathbf{C}|}{2} w_c$.
(ii) *If at most $|\mathbf{C}| - 1$ variables from $\{x_{vd}\}_{v\in\mathbf{C},d\in\mathcal{T}}$ are allowed to take fractional values, then there exist at most $|\mathbf{C}|/2 - 1$ variables from $\{x_{vc}\}_{v\in\mathbf{C}}$ at fractional values.*

Proof. For part (i), since $(x,w) \in L(G,\mathcal{T})$ then (3.2) implies $x_{uc} + x_{vc} \le w_c$ for every edge uv in the cycle (considered as an edge set). By summing these inequalities over all the edges from the cycle, part (i) follows.

Now for part (ii). If at least $|\mathbf{C}|/2$ variables in $\{x_{vc}\}_{v\in\mathbf{C}}$ were fractional then by (3.1), there would exist at least $|\mathbf{C}|/2$ variables in $\{x_{vd}\}_{v\in\mathbf{C},d\in\mathcal{T}\setminus\{c\}}$ that are fractional. This would therefore lead to a number of fractional variables in $\{x_{vd}\}_{v\in\mathbf{C},d\in\mathcal{T}}$ being at least $|\mathcal{T}|$. □

	v_1	v_2	v_3	v_4	....	$v_{\lvert\mathbf{C}\rvert}$	w_c
c_0	$\mathbf{\frac{1}{2}}$	$\mathbf{\frac{1}{2}}$	$\mathbf{\frac{1}{2}}$	$\mathbf{\frac{1}{2}}$	....	$\mathbf{\frac{1}{2}}$	$\mathbf{1}$
c_1	$\mathbf{\frac{1}{2}}$	0	$\mathbf{\frac{1}{2}}$	0	....	0	$\mathbf{\frac{1}{2}}$
c_2	0	$\mathbf{\frac{1}{2}}$	0	$\mathbf{\frac{1}{2}}$	....	$\mathbf{\frac{1}{2}}$	1
c_3	0	0	0	0	....	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
$c_{\lvert\mathcal{T}\rvert}$	0	0	0	0	....	0	0

FIGURE 2. Construction for the proof of Theorem 4.2. The values in boldface correspond to the variables in $\mathbb{V}\backslash\mathbb{B}$.

4. Disjunctive rank and anti-rank of known inequalities

In this section we study the disjunctive rank and anti-rank of the families of facet-inducing inequalities presented in [10] for the case where the graph G is a cycle. We assume throughout this section that $G = \mathbf{C}$ is an even cycle, and all the inequalities presented in this section involve such a cycle. Under technical hypotheses, these inequalities are facet-defining for $P(G, \mathcal{T})$ for any graph G [10]. For $v \in \mathbf{C}$, we define $\mathbf{C}_v \subset \mathbf{C}$ to be the set of all vertices at even distance in $\mathbf{C}$ to the vertex v. We define $\mathbf{C} = \{v_1, v_2, \ldots, v_{|\mathbf{C}|}\}$ to be the set of vertices where $v_i v_{i+1} \in E$ for $1 \le i \le |\mathbf{C}| - 1$ and $v_{|\mathbf{C}|} v_1 \in E$.

4.1. The two-color inequalities

We first study the so-called two-color inequalities, which was the best-performing familiy of valid inequalities in the branch-and-cut procedure reported in [10].

Definition 4.1. Let $c_0, c_1 \in \mathcal{T}$ with $c_0 \neq c_1$. The *two-color inequality* associated with $\mathbf{C}$, c_0, and c_1 is

$$\sum_{v \in \mathbf{C}} (x_{vc_0} + x_{vc_1}) \le 1 + \left(\frac{|\mathbf{C}|}{2} - 1\right) w_{c_0} + \left(\frac{|\mathbf{C}|}{2} - 1\right) w_{c_1}. \tag{4.1}$$

In order to efficiently describe the constructions of feasible solutions given in this section, we introduce the graphical representation depicted in Figure 2. This graphical representation specifies the value that each variable takes in a solution. For example, the value 0 in column v_2 and row c_1 in Figure 2 asserts that the variable $x_{v_2 c_1}$ takes value 0. The last column represents the values of the w-variables.

Theorem 4.2. *The disjunctive rank of the two-color inequality* (4.1) *is* 2.

Proof. We shall first prove that the disjunctive rank of (4.1) is less than or equal to 2. Let $\mathbb{A} \subseteq \mathbb{V}$ be the set $\{w_{c_0}, w_{c_1}\}$. We must prove that (4.1) is valid for $P_{\mathbb{A}}(L)$. Let $z = (x, w)$ be a (possibly fractional) solution such that $w_{c_0}, w_{c_1} \in \{0, 1\}$ (so Thm. 2.3 implies $z \in P_{\mathbb{A}}(L)$) and consider the following cases:

(1) If the variables in $\mathbb{A}$ take value 1 in z, *i.e.*, $w_{c_0} = w_{c_1} = 1$, then the left-hand side (LHS) of (4.1) is less than or equal to $|\mathbf{C}| - 1$, as z satisfies (3.3), with $A = \mathbf{C}$, $c = c_0$ and $c' = c_1$. Note that the right-hand side (RHS) of inequality (4.1) equals $|\mathbf{C}| - 1$ in this case.
(2) If exactly one variable in $\mathbb{A}$ takes value 1 in z, say, $w_{c_0} = 1$ and $w_{c_1} = 0$, then the LHS of (4.1) is less than or equal to $\frac{|\mathbf{C}|}{2}$ by Lemma 3.1(i). Since the RHS of inequality (4.1) equals $\frac{|\mathbf{C}|}{2}$, then z satisfies (4.1).
(3) If both variables in $\mathbb{A}$ take value 0 in z, *i.e.*, $w_{c_0} = w_{c_1} = 0$, then by (3.2), the LHS of (4.1) takes value 0. The RHS of inequality (4.1) equals value 1 hence it is satisfied by z.

Since in the three cases the two-color inequality is satisfied and z is an arbitrary solution of $P_{\mathbb{A}}(L)$, we conclude that (4.1) is valid for $P_{\mathbb{A}}(L)$. Then, the disjunctive rank of (4.1) is less than or equal to 2.

We now prove that the disjunctive rank of (4.1) is greater than or equal to 2. Let $\mathbb{B} \subseteq \mathbb{V}$ be an arbitrary set of variables with cardinality 1. In order to prove that the two-color inequality is not valid for $P_{\mathbb{B}}(L)$, we show a solution $z \in P_{\mathbb{B}}(L)$ that violates (4.1). To this end, consider the solution depicted in Figure 2, which violates (4.1) but satisfies (3.1)-(3.3), and consider the following cases:

- If $\mathbb{B} = \{w_{c_i}\}$ for some $i \neq 1$, then the solution in Figure 2 belongs to $P_{\mathbb{B}}(L)$ by Theorem 2.3 and violates (4.1).
- If $\mathbb{B} = \{w_{c_1}\}$, then the solution obtained from Figure 2 by swapping the colors c_0 and c_1 violates (4.1) and belongs to $P_{\mathbb{B}}(L)$ by Theorem 2.3.
- If $\mathbb{B} = \{x_{vc_1}\}$, then assume w.l.o.g. that $v = v_2$. Again, the solution specified by Figure 2 violates (4.1).
- If $\mathbb{B} = \{x_{vc_0}\}$, again assume w.l.o.g. that $v = v_2$. The solution obtained from Figure 2 by swapping the colors c_0 and c_1 violates (4.1).
- If $\mathbb{B} = \{x_{vc}\}$ with $c \notin \{c_0, c_1\}$, then assume w.l.o.g. that $v = v_1$. The solution obtained from Figure 2 by swapping the colors c and c_2 violates (4.1).

We conclude that for any singleton $\mathbb{B}$ the inequality (4.1) is not valid for $P_{\mathbb{B}}(L)$, hence the disjunctive rank of (4.1) equals 2. □

In order to establish the disjunctive anti-rank of (4.1) we first prove the following lemma.

Lemma 4.3. *If $z = (x, w) \in L$ violates the two-color inequality, then $1 < w_{c_0} + w_{c_1} < 2$.*

Proof. Let $z = (x, w) \in L$ be a solution violating (4.1). Then,

$$1 + \left(\frac{|\mathbf{C}|}{2} - 1 \right) (w_{c_0} + w_{c_1}) < \sum_{v \in \mathbf{C}} (x_{vc_0} + x_{vc_1}).$$

By Lemma 3.1(i), we have $\sum_{v \in \mathbf{C}} x_{vc_0} \leq \frac{|\mathbf{C}|}{2} w_{c_0}$ and $\sum_{v \in \mathbf{C}} x_{vc_1} \leq \frac{|\mathbf{C}|}{2} w_{c_1}$. Therefore, $\sum_{v \in \mathbf{C}} (x_{vc_0} + x_{vc_1}) \leq \frac{|\mathbf{C}|}{2}(w_{c_0} + w_{c_1})$ and

$$1 + \left(\frac{|\mathbf{C}|}{2} - 1 \right) (w_{c_0} + w_{c_1}) < \frac{|\mathbf{C}|}{2}(w_{c_0} + w_{c_1}).$$

So we conclude $1 < w_{c_0} + w_{c_1}$.

On the other hand, the constraint (3.3) asserts $\sum_{v \in \mathbf{C}} (x_{vc_0} + x_{vc_1}) \leq |\mathbf{C}| - 1$. Then,

$$1 + \left(\frac{|\mathbf{C}|}{2} - 1 \right) (w_{c_0} + w_{c_1}) < |\mathbf{C}| - 1.$$

So we conclude $w_{c_0} + w_{c_1} < 2$. □

We can now prove the following result. Recall that $p = |V||\mathcal{T}| + |\mathcal{T}|$.

Theorem 4.4. *The disjunctive anti-rank of the two-color inequality* (4.1) *is* $p - (|\mathbf{C}| + 1)$.

Proof. We first prove that for any set $\mathbb{A} \subseteq \mathbb{V}$ with $p - |\mathbf{C}|$ variables, the inequality (4.1) is valid for $P_{\mathbb{A}}(L)$. Let $(x, w) \in P_{\mathbb{A}}(L)$. If $w_{c_0} + w_{c_1} \leq 1$ or $w_{c_0} + w_{c_1} = 2$ then Lemma 4.3 implies that (4.1) is satisfied, so assume $1 < w_{c_0} + w_{c_1} < 2$ (hence $w_{c_0} > 0$ and $w_{c_1} > 0$) and consider the following cases:

(1) If $\boldsymbol{w_{c_0} = 1}$ then $0 < w_{c_1} < 1$, hence none of the variables in $\{x_{vc_1}\}_{v \in \mathbf{C}}$ can take value 1. Since $|\mathbb{V} \setminus (\mathbb{A} \cup \{w_{c_1}\})| = |\mathbf{C}| - 1$, by Lemma 3.1(ii) at most $\frac{|\mathbf{C}|}{2} - 1$ variables from $\{x_{vc_1}\}_{v \in V}$ can take fractional values and the remaining ones take value 0. The model constraint (3.2) implies $x_{vc_1} \leq w_{c_1}$, so $\sum_{v \in \mathbf{C}} x_{vc_1} \leq \left(\frac{|\mathbf{C}|}{2} - 1 \right) w_{c_1}$. As $\sum_{v \in \mathbf{C}} x_{vc_0} \leq \frac{|\mathbf{C}|}{2}$, the LHS of (4.1) is less than or equal to $\left(\frac{|\mathbf{C}|}{2} - 1 \right) w_{c_1} + \frac{|\mathbf{C}|}{2}$. Note that the RHS of inequality (4.1) equals this value, so (4.1) is satisfied. A symmetrical argument settles the case $w_{c_1} = 1$.

	v_1	v_2	v_3	v_4	....	$v_{\|\mathbf{C}\|}$	w_c
c_0	0	1	0	1	....	1	1
c_1	$\mathbf{\frac{1}{2}}$	0	$\mathbf{\frac{1}{2}}$	0	....	0	$\mathbf{\frac{1}{2}}$
c_2	$\mathbf{\frac{1}{2}}$	0	$\mathbf{\frac{1}{2}}$	0	....	0	1
c_3	0	0	0	0	....	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
$c_{\|\mathcal{T}\|}$	0	0	0	0	....	0	0

FIGURE 3. The values in boldface correspond to the variables in $\mathbb{V}\backslash\mathbb{B}$.

(2) If $\mathbf{0 < w_{c_0} < 1}$, then we are left with the case $0 < w_{c_1} < 1$. Therefore, at most $|\mathbf{C}| - 2$ variables different from w_{c_0} and w_{c_1} can take fractional values. Again, by Lemma 3.1(ii) at most $\frac{|\mathbf{C}|}{2} - 1$ variables from $\{x_{vc_0}\}_{v\in\mathbf{C}}$ (respectively $\{x_{vc_1}\}_{v\in\mathbf{C}}$) can take fractional values and the remaining ones take value 0, implying that the LHS of (4.1) is less than or equal to $\left(\frac{|\mathbf{C}|}{2} - 1\right)(w_{c_0} + w_{c_1})$. Note that the RHS is greater than this value, so (4.1) is satisfied.

We conclude that the disjunctive anti-rank of (4.1) is less than or equal to $p - (|\mathbf{C}| + 1)$.

In order to prove the opposite inequality, let $\mathbb{B} \subseteq \mathbb{V}$ be the set $\{x_{vc} : v \in \mathbf{C},\, c \in \mathcal{T}, c \neq c_1, c_2\} \cup \{x_{vc} : v \in \mathbf{C}_{v_2}, c = c_1, c_2\} \cup \{w_c : c \in \mathcal{T}, c \neq c_1\}$. The cardinality of $\mathbb{B}$ is $p - (|\mathbf{C}| + 1)$. Let $z = (x, w) \in P_{\mathbb{B}}(L)$ be the feasible solution depicted in Figure 3. This solution satisfies the model constraints, violates the inequality (4.1), and the variables in $\mathbb{B}$ take 0-1 values. Then, the anti-rank of (4.1) is greater than or equal to $p-(|\mathbf{C}|+1)$, and the theorem follows. □

4.2. The distinguished colors inequalities

We now introduce the distinguished colors inequalities, which include in their definition an arbitrary subset of colors, and study their disjunctive rank and anti-rank.

Definition 4.5. Let $\mathcal{D} \subset \mathcal{T}$ with $\mathcal{D} \neq \emptyset$. The *distinguished colors inequality* associated with $\mathbf{C}$ and $\mathcal{D}$ is

$$\sum_{v\in\mathbf{C}} \sum_{c\in\mathcal{D}} x_{vc} \leq |\mathbf{C}| - 3 + \sum_{c\in\mathcal{D}} w_c. \tag{4.2}$$

Note that we need not consider $\mathcal{D} = \emptyset$, since (4.2) becomes $0 \leq 1$ in this case.

Theorem 4.6. *The disjunctive rank of the distinguished colors inequality is*

(1) $|\mathcal{D}|$ *if* $|\mathcal{D}| \geq 2$;
(2) 0 *if* $|\mathcal{D}| \leq 1$.

Proof. Consider the following cases.

Case I: $|\mathcal{D}| = 1$. Let $\mathcal{D} = \{d\}$. The LHS of (4.2) equals $\sum_{v\in\mathbf{C}} x_{vd}$ and by Lemma 3.1(i) $\sum_{v\in\mathbf{C}} x_{vd} \leq \frac{|\mathbf{C}|}{2} w_d$. Since $|\mathbf{C}| \geq 4$ then $2\frac{|\mathbf{C}|-3}{|\mathbf{C}|-2} \geq 1$, hence $\frac{|\mathbf{C}|}{2} w_d \leq |\mathbf{C}| - 3 + w_d$. The inequality (4.2) is, therefore, satisfied and has disjunctive rank 0.

Case II: $|\mathcal{D}| \geq 2$. We first prove that the disjunctive rank of (4.2) is less than or equal to $|\mathcal{D}|$.

Let $\mathbb{A} \subseteq \mathbb{V}$ be the set $\{w_c : c \in \mathcal{D}\}$. We shall prove that (4.2) is valid for $P_{\mathbb{A}}(L)$. Let $z = (x, w) \in P_{\mathbb{A}}(L)$ be an arbitrary solution, so by Theorem 2.3 we have $w_c \in \{0, 1\}$, and consider the following cases:

(1) If the number of w-variables in $\mathbb{A}$ that take value 1 is greater than or equal to 3, then the RHS of inequality (4.2) is no less than $|\mathbf{C}|$, hence (4.2) is satisfied.

	v_i	v_{i+1}	v_{i+2}	v_{i+3}	v_{i+4}	....	v_j	v_{j+1}	v_{j+2}	v_{j+3}	v_{j+4}	....	$v_{\lvert\mathbf{C}\rvert}$	w_c
d_1	0	1	0	1	0	....	0	1	0	1	0	....	1	1
d_2	$\mathbf{\frac{1}{2}}$	0	1	0	1	....	$\mathbf{\frac{1}{2}}$	0	1	0	1	....	0	1
d_3	$\mathbf{\frac{1}{2}}$	0	0	0	0	....	$\mathbf{\frac{1}{2}}$	0	0	0	0	....	0	$\frac{1}{2}$
d_4	0	0	0	0	0	....	0	0	0	0	0	....	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
$d_{\lvert\mathcal{D}\rvert}$	0	0	0	0	0	....	0	0	0	0	0	....	0	0
c_1	0	0	0	0	0	....	0	0	0	0	0	....	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
$c_{\lvert\mathcal{T}\setminus\mathcal{D}\rvert}$	0	0	0	0	0	....	0	0	0	0	0	....	0	0

FIGURE 4. Solution for the proof of Theorem 4.6. The values in boldface specify some variables guaranteed not to belong to $\mathbb{B}$.

(2) If exactly two variables in $\mathbb{A}$ take value 1, say w_d and $w_{d'}$, then the model constraints (3.2) imply that the LHS of (4.2) is equal to $\sum_{v\in\mathbf{C}}(x_{vd}+x_{vd'})$, and the RHS of (4.2) equals $|\mathbf{C}|-1$. Since z satisfies (3.3), the inequality (4.2) is satisfied.
(3) If exactly one variable in $\mathbb{A}$ takes value 1, then by Lemma 3.1(i) the LHS of (4.2) is less than or equal to $\frac{|\mathbf{C}|}{2}$. As $\frac{|\mathbf{C}|}{2} \leq |\mathbf{C}|-2$ if and only if $|\mathbf{C}| \geq 4$, the inequality (4.2) is satisfied.
(4) If all the variables in $\mathbb{A}$ take value 0, then the LHS is equal to 0 and the inequality (4.2) is satisfied.

Since in the four cases the distinguished colors inequality is satisfied and z is an arbitrary solution of $P_{\mathbb{A}}(L)$, we conclude that (4.2) is a valid inequality for $P_{\mathbb{A}}(L)$. Then, the disjunctive rank of (4.2) is less than or equal to $|\mathcal{D}|$.

We now prove that the disjunctive rank of (4.2) is greater than or equal to $|\mathcal{D}|$. Let $\mathbb{B} \subseteq \mathbb{V}$ be an arbitrary set of $|\mathcal{D}|-1$ variables. We must prove that the distinguished colors inequality is not valid for $P_{\mathbb{B}}(L)$.

Let $\mathcal{D} = \{d_1,\ldots,d_{|\mathcal{D}|}\}$ and $\mathcal{T}\setminus\mathcal{D} = \{c_1,\ldots,c_{|\mathcal{T}\setminus\mathcal{D}|}\}$. Since $|\mathbb{B}| = |\mathcal{D}|-1$, there exists some color in $\mathcal{D}$, say d_3, such that $w_{d_3} \notin \mathbb{B}$ and $x_{vd_3} \notin \mathbb{B}$ for every $v \in \mathbf{C}$ (if every $d \in \mathcal{D}$ had $w_d \in \mathbb{B}$ or $x_{vd} \in \mathbb{B}$ for some $v \in \mathbf{C}$ then we would have at least one variable in $\mathbb{B}$ for each color in $\mathcal{D}$, implying $|\mathbb{B}| \geq |\mathcal{D}|$, a contradiction). The set $\mathbb{D} = \{x_{vc}\}_{v\in\mathbf{C},c\in\mathcal{D}\setminus\{d_3\}}$ contains $|\mathbf{C}|(|\mathcal{D}|-1)$ variables. Since $\mathbb{B}$ contains just $|\mathcal{D}|-1$ variables, then there exists a color in $\mathcal{D}\setminus\{d_3\}$, say d_2, and two vertices in $\mathbf{C}$ located at even distance in $\mathbf{C}$, say v_i and v_j, such that $x_{v_id_2}, x_{v_jd_2} \notin \mathbb{B}$. The solution depicted in Figure 4 satisfies (3.1)–(3.3), belongs to $P_{\mathbb{B}}(L)$ by Theorem 2.3, but violates the inequality (4.2).

We, therefore, conclude that the disjunctive rank of (4.2) is less than or equal to $|\mathcal{D}|$. □

Theorem 4.7. *The disjunctive anti-rank of the distinguished colors inequality is*

(1) 0, *if* $|\mathcal{D}| \leq 1$;
(2) $p-(|\mathbf{C}|+1)$, *if* $|\mathcal{D}| = 2$;
(3) $p-5$, *if* $|\mathcal{D}| \geq 3$.

Proof. Consider the following cases.

Case I: $|\mathcal{D}| \leq 1$. The disjunctive rank is 0, which implies that the linear relaxation satisfies inequality (4.2). Hence, the disjunctive anti-rank of (4.2) is also 0.

Case II: $|\mathcal{D}| = 2$. We first verify that for any set $\mathbb{A} \subseteq \mathbb{V}$ with $p - |\mathbf{C}|$ variables, the inequality (4.2) is valid for $P_{\mathbb{A}}(L)$. Let $(x, w) \in P_{\mathbb{A}}(L)$ be an arbitrary point, and consider the following cases:

(1) If $\boldsymbol{w_{d_1}, w_{d_2}} \in \mathbb{Z}$ then we split the analysis into the following cases:
   (a) If $w_{d_1} = w_{d_2} = 1$, then the inequality (4.2) states that $\sum_{v\in\mathbf{C}}(x_{vd_1} + x_{vd_2}) \le |\mathbf{C}| - 1$. This inequality is satisfied as it is one of the model constraints (3.3).
   (b) If $w_{d_1} \ne w_{d_2}$, say $w_{d_1} = 1$ and $w_{d_2} = 0$, then the inequality (4.2) states that $\sum_{v\in\mathbf{C}} x_{vd_1} \le |\mathbf{C}| - 2$. This inequality is satisfied as $\sum_{v\in\mathbf{C}} x_{vd_1} \le \frac{|\mathbf{C}|}{2}$ by Lemma 3.1(i).
   (c) If $w_{d_1} = w_{d_2} = 0$, then (4.2) is trivially satisfied.

(2) If $\boldsymbol{w_{d_1} = 0}$ **and** $\boldsymbol{0 < w_{d_2} < 1}$, then $x_{vd_1} = 0$ for $v \in \mathbf{C}$. Constraints (3.2) imply $x_{ud_2} + x_{vd_2} \le w_{d_2}$ for every edge uv in the cycle, and summing over all such edges yields $\sum_{v\in\mathbf{C}} x_{vd_2} \le w_{d_2}|\mathbf{C}|/2$. The RHS of inequality (4.2) equals $|\mathbf{C}| + w_{d_2} - 2$ and, since $w_{d_2}|\mathbf{C}|/2 \le |\mathbf{C}| + w_{d_2} - 2$, the inequality (4.2) is satisfied.

(3) If $\boldsymbol{w_{d_1} = 1}$ **and** $\boldsymbol{0 < w_{d_2} < 1}$, then the inequality (4.2) states that $\sum_{v\in\mathbf{C}}(x_{vd_1} + x_{vd_2}) \le |\mathbf{C}| - 2 + w_{d_2}$. By contradiction, suppose that $\sum_{v\in\mathbf{C}}(x_{vd_1} + x_{vd_2}) > |\mathbf{C}| - 2 + w_{d_2}$ for a solution $(x, w) \in P_{\mathbb{A}}(L)$. Since $|\mathbb{V} \setminus (\mathbb{A} \cup \{w_{d_2}\})| = |\mathbf{C}| - 1$, Lemma 3.1(ii) implies that at most $\frac{|\mathbf{C}|}{2} - 1$ variables from $\{x_{vd_2}\}_{v\in\mathbf{C}}$ take fractional values and the remaining ones are 0. As $w_{d_1} = 1$, then $\sum_{v\in\mathbf{C}}(x_{vd_1} + x_{vd_2}) \le \frac{|\mathbf{C}|}{2} + \left(\frac{|\mathbf{C}|}{2} - 1\right) w_{d_2}$. Hence,

$$\frac{|\mathbf{C}|}{2} + \left(\frac{|\mathbf{C}|}{2} - 1\right) w_{d_2} > |\mathbf{C}| - 2 + w_{d_2}. \tag{4.3}$$

   From (4.3) we obtain that $w_{d_2} > 1$ if $|\mathbf{C}| \ne 4$, and $0 > 0$ if $|\mathbf{C}| = 4$; a contradiction, hence (4.2) is satisfied.

(4) If $\boldsymbol{0 < w_{d_1}, w_{d_2} < 1}$ (hence $w_{d_1}, w_{d_2} \notin \mathbb{A}$), Lemma 3.1(ii) ensures that at most $\frac{|\mathbf{C}|}{2} - 1$ variables from $\{x_{vd_1}\}_{v\in\mathbf{C}}$ (respectively $\{x_{vd_2}\}_{v\in\mathbf{C}}$) can take fractional values. Therefore,

$$\sum_{v\in\mathbf{C}}(x_{vd_1} + x_{vd_2}) \le \left(\frac{|\mathbf{C}|}{2} - 1\right)(w_{d_1} + w_{d_2}).$$

   We now prove that

$$\left(\frac{|\mathbf{C}|}{2} - 1\right)(w_{d_1} + w_{d_2}) \le |\mathbf{C}| - 3 + w_{d_1} + w_{d_2}. \tag{4.4}$$

   If (4.4) does not hold and $|\mathbf{C}| \ne 4$, then $w_{d_1} + w_{d_2} > 2\frac{|\mathbf{C}|-3}{|\mathbf{C}|-4} \ge 1$, implying $w_{d_1} + w_{d_2} > 2$, a contradiction. If (4.4) does not hold and $|\mathbf{C}| = 4$, we get $0 > 1$. Thus, (4.4) holds and the inequality (4.2) is satisfied.

We conclude that the disjunctive anti-rank of (4.2) is less than or equal to $p - (|\mathbf{C}| + 1)$.

For the converse direction, note that $|\mathcal{T} \setminus \mathcal{D}| \ge 1$, since otherwise the polytope is empty. Let $\mathbb{B} \subseteq \mathbb{V}$ be the set $\{x_{vc} : v \in \mathbf{C}, c \in \mathcal{T}, c \ne c_0, d_2\} \cup \{x_{vc} : v \in \mathbf{C}_{v_2}, c = c_0, d_2\} \cup \{w_c : c \in \mathcal{T}, c \ne d_2\}$. The cardinality of $\mathbb{B}$ is $p - (|\mathbf{C}| + 1)$. Let $z = (x, w) \in P_{\mathbb{B}}(L)$ be the solution depicted in Figure 5, where $\mathcal{D} = \{d_1, d_2\}$ and $\mathcal{T} \setminus \mathcal{D} = \{c_1, \ldots, c_n\}$.

This solution satisfies the model constraints, violates the inequality (4.2) and the variables in $\mathbb{B}$ take values in $\{0, 1\}$. Therefore, the disjunctive anti-rank is greater than or equal to $p - (|\mathbf{C}| + 1)$.

Case III: $|\mathcal{D}| \ge 3$. In order to prove that the disjunctive anti-rank is less than or equal to $p - 5$, we shall verify that for any set $\mathbb{A} \subseteq \mathbb{V}$ of $p - 4$ variables, the inequality (4.2) is valid for $P_{\mathbb{A}}(L)$. Consider the following cases:

(1) If $\sum_{d\in\mathcal{D}} w_d \ge 3$, then the inequality (4.2) is trivially satisfied.

(2) If $\{w_d\}_{d\in\mathcal{D}} \subseteq \mathbb{A}$ and at most two variables from this set take value 1, say w_{d_1} and w_{d_2}, then the inequality (4.2) states that $\sum_{v\in\mathbf{C}}(x_{vd_1} + x_{vd_2}) \le |\mathbf{C}| - 1$, which is equivalent to the model constraint (3.3) and is therefore satisfied.

	v_1	v_2	v_3	v_4	$\ldots$	$v_{\lvert\mathbf{C}\rvert}$	w_c
d_1	0	1	0	1	$\ldots$	1	1
d_2	$\mathbf{1-\frac{2}{\lvert C\rvert}}$	0	$\mathbf{1-\frac{2}{\lvert C\rvert}}$	0	$\ldots$	0	$\mathbf{1-\frac{2}{\lvert C\rvert}}$
c_0	$\mathbf{\frac{2}{\lvert C\rvert}}$	0	$\mathbf{\frac{2}{\lvert C\rvert}}$	0	$\ldots$	0	1
c_1	0	0	0	0	$\ldots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
$c_{\lvert\mathcal{T}\rvert-2}$	0	0	0	0	$\ldots$	0	0

FIGURE 5. The values in boldface correspond to variables guaranteed to belong to $\mathbb{V}\backslash\mathbb{B}$.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	....	$v_{\lvert\mathbf{C}\rvert}$	w_c
d_1	0	1	0	1	0	1	0	....	1	1
d_2	$\mathbf{\frac{1}{2}}$	0	$\mathbf{\frac{1}{2}}$	0	1	0	1	....	0	1
d_3	$\mathbf{\frac{1}{2}}$	0	$\mathbf{\frac{1}{2}}$	0	0	0	0	....	0	$\mathbf{\frac{1}{2}}$
d_4	0	0	0	0	0	0	0	....	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
$d_{\lvert\mathcal{D}\rvert}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
c_1	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
$c_{\lvert\mathcal{T}\setminus\mathcal{D}\rvert}$	0	0	0	0	0	0	0	....	0	0

FIGURE 6. Solution for the proof of Theorem 4.7. The values in boldface correspond to variables guaranteed not to belong to $\mathbb{B}$.

(3) If at least one variable from $\{w_d\}_{d\in\mathcal{D}}$ is not in $\mathbb{A}$, and adding the facts that at most four variables take fractional values, that $\sum_{d\in\mathcal{D}} w_d < 3$ and that the model constraint (3.1) must be satisfied, then at most one vertex has the corresponding x_{vd}-variable with a fractional value and the rest of them take value 0. Let v be such a vertex. Furthermore, at most two w-variables in $\mathbb{A}$ take value 1. Consider the following cases.
   (a) Two variables from $\{w_d\}_{d\in\mathcal{D}}$ in $\mathbb{A}$, say w_{d_1} and w_{d_2}, take value 1. The LHS is less than or equal to $|\mathbf{C}|-1+\sum_{\substack{d\in\mathcal{D}\\ d\neq d_1,d_2}} x_{vd}$. As $\sum_{\substack{d\in\mathcal{D}\\ d\neq d_1,d_2}} x_{vd} \leq \sum_{d\in\mathcal{D}} w_d - 2$, the inequality (4.2) is satisfied.
   (b) Exactly one variable from $\{w_d\}_{d\in\mathcal{D}}$ in $\mathbb{A}$, say w_{d_1}, takes value 1. The LHS is less than or equal to $\frac{|\mathbf{C}|}{2}+\sum_{\substack{d\in\mathcal{D}\\ d\neq d_1}} x_{vd}$. As $\sum_{\substack{d\in\mathcal{D}\\ d\neq d_1}} x_{vd} \leq \frac{|\mathbf{C}|}{2} - 3 + \sum_{d\in\mathcal{D}} w_d$, the inequality (4.2) is satisfied.
   (c) None of the variables from $\{w_d\}_{d\in\mathcal{D}}$ in $\mathbb{A}$ take value 1. The LHS is equal to $\sum_{d\in D} x_{vd}$ and this is less than or equal to the RHS of (4.2). Then the inequality (4.2) is satisfied.

For the converse direction, let $\mathbb{B} = \mathbb{V}\setminus\{x_{v_1d_2}, x_{v_1d_3}, x_{v_3d_2}, x_{v_3d_3}, w_{c_3}\}$, which has $p-5$ elements. Let $z = (x,w) \in P_{\mathbb{B}}(L)$ be the feasible solution depicted in Figure 6. This solution satisfies the model constraints, violates the inequality (4.2) and the variables in $\mathbb{B}$ take values in $\{0,1\}$. Therefore, the disjunctive anti-rank is greater than or equal to $p-5$. We conclude that the disjunctive anti-rank of (4.2) is $p-5$ when $|\mathcal{D}|\geq 3$. □

4.3. The prominent vertex inequalities

We now turn to the so-called prominent vertex inequalities, whose rank we, unfortunately, were not able to fully characterize. We first present the (partial) characterization and then discuss the open case. Recall that $\mathbf{C}$ is an even cycle.

Definition 4.8. Let $c_0, c_1 \in \mathcal{T}$ and let $\mathcal{D} \subset \mathcal{T} \setminus \{c_0, c_1\}$. The *prominent vertex inequality* associated with the cycle $\mathbf{C}$, the colors c_0, c_1, and the color set $\mathcal{D}$ is

$$\sum_{u \in \mathbf{C} \setminus \{v_2\}} x_{uc_0} + \sum_{u \in \mathbf{C}_{v_2}} \sum_{c \in \mathcal{D} \cup \{c_0\}} x_{uc} \leq \frac{|\mathbf{C}|}{2} w_{c_0} + \sum_{c \in \mathcal{D}} w_c + \sum_{c \in \mathcal{T} \setminus \mathcal{D} \cup \{c_0, c_1\}} \sum_{u \in \mathbf{C}_{v_1}} x_{uc} + \left(\frac{|\mathbf{C}|}{2} - 2 \right). \quad (4.5)$$

We first study the disjunctive rank of the prominent vertex inequalities. In the following result, the hypothesis states that the set of colors $\mathcal{D}$ is not empty and excludes the case in which $|\mathbf{C}| = 4$ and $|\mathcal{D}| = 1$.

Theorem 4.9. *The disjunctive rank of* (4.5) *is* $|\mathcal{D}| + 1 + \lfloor \frac{|\mathbf{C}|}{4} \rfloor$ *whenever **(I)** $\mathcal{D} \neq \emptyset$ and **(II)** $|\mathbf{C}| \geq 6$ or $|\mathcal{D}| > 1$.*

Proof. We shall first prove that the disjunctive rank of (4.5) is less than or equal to $|\mathcal{D}| + 1 + \lfloor \frac{|\mathbf{C}|}{4} \rfloor$ whenever $\mathcal{D} \neq \emptyset$. Let $\mathbb{A} \subseteq \mathbb{V}$ be the set $\{w_{c_0}\} \cup \{w_d : d \in \mathcal{D}\} \cup \{x_{v_{4k-1}c_0} : k = 1, \ldots, \lfloor \frac{|\mathbf{C}|}{4} \rfloor\}$. We must prove that (4.5) is valid for $P_{\mathbb{A}}(L)$. Let $z = (x, w) \in P_{\mathbb{A}}(L)$ be a (possibly fractional) solution (so Thm. 2.3 implies that all the variables in $\mathbb{A}$ take values in $\{0, 1\}$) and consider the following cases:

(1) If the number of w-variables in $\mathbb{A}$ that take value 1 in z is greater than or equal to 2, then Lemma 3.1(i) and constraint (3.1) imply that the LHS of (4.5) is less or equal than $\frac{|\mathbf{C}|}{2} w_{c_0} + \frac{|\mathbf{C}|}{2}$. Note that the RHS of (4.5) is greater than or equal to $\frac{|\mathbf{C}|}{2} w_{c_0} + \frac{|\mathbf{C}|}{2}$, as $\sum_{c \in \mathcal{D}} w_c \geq 2$.

(2) If the number of w-variables in $\mathbb{A}$ that take value 1 in z is less than or equal to 1, then consider the following cases:

 (a) If all the variables in $\mathbb{A}$ associated with the color c_0 take value 1 in z, then $x_{vc_0} = 0$ for all $v \in \mathbf{C}_{v_2}$, as z satisfies (3.2) and $w_{c_0} = 1$.

  (i) If there exists $d \in \mathcal{D}$ such that $w_d = 1$, then the assumption of Case 2 implies that $w_{d'} = 0$ for every $d' \in D$. Then, the LHS of (4.5) is equal to $\sum_{v \in \mathbf{C}_{v_1}} x_{vc_0} + \sum_{v \in \mathbf{C}_{v_2}} x_{vd}$, implying

$$\sum_{v \in \mathbf{C}_{v_1}} x_{vc_0} + \sum_{v \in \mathbf{C}_{v_2}} x_{vd} \leq \sum_{v \in \mathbf{C}} (x_{vc_0} + x_{vd}) \leq |\mathbf{C}| - 1,$$

  as z represents an acyclic coloring. As the right-hand-side of (4.5) is greater than or equal to $|\mathbf{C}| - 1$, the inequality is satisfied.

  (ii) If $w_d = 0$ for all $d \in \mathcal{D}$, then the LHS of (4.5) is equal to $\sum_{v \in \mathbf{C}_{v_1}} x_{vc_0}$, which is less than or equal to $\frac{|\mathbf{C}|}{2}$. On the other side, the RHS of (4.5) is greater than or equal to $|\mathbf{C}| - 2$, so the inequality (4.5) is satisfied.

 (b) If there exist $x_{v_i c_0}, x_{v_j c_0} \in \mathbb{A}$ such that $x_{v_i c_0} = 0$ and $x_{v_j c_0} = 1$, then $w_{c_0} = 1$. Let $\mathbf{C}' = \mathbf{C} \setminus \{v_2, v_j\}$. Since $x_{v_j c_0} = 0$, Lemma 3.1(i) implies that

$$\sum_{v \in \mathbf{C} \setminus \{v_2\}} x_{vc_0} = \sum_{v \in \mathbf{C}'} x_{vc_0} \leq \left(\frac{|\mathbf{C}|}{2} - 1 \right) w_{c_0} = \left(\frac{|\mathbf{C}|}{2} - 1 \right). \quad (4.6)$$

 Consider the following cases:

  (i) If there exists $d \in \mathcal{D}$ such that $w_d = 1$, then the RHS of (4.5) is greater then or equal to $|\mathbf{C}| - 1$. By combining (4.6) and the fact that $\sum_{v \in \mathbf{C}_{v_2}} (x_{vd_1} + v_{vc_0}) \leq \frac{|\mathbf{C}|}{2}$, we get that the LHS of (4.5) is less than or equal to $|\mathbf{C}| - 1$. Then, the inequality (4.5) is satisfied.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	$\cdots$	$v_{\vert\mathbf{C}\vert}$	w_c
c_0	0	1	0	1	0	1	0	....	1	1
c_1	1	0	$\mathbf{\frac{1}{2}}$	0	$\mathbf{\frac{1}{2}}$	0	1	....	0	1
d_1	0	0	$\mathbf{\frac{1}{2}}$	0	$\mathbf{\frac{1}{2}}$	0	0	....	0	$\mathbf{\frac{1}{2}}$
d_2	0	0	0	0	0	0	0	....	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
$d_{\vert\mathcal{D}\vert}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
t_1	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
$t_{\vert\mathcal{T}\setminus\mathcal{D}\vert-2}$	0	0	0	0	0	0	0	....	0	0

FIGURE 7. Solution for the proof of Theorem 4.9, for $i = 3$. The variables in boldface do not belong to $\mathbb{B}$.

(ii) If $w_d = 0$ for all $d \in \mathcal{D}$, then the LHS of inequality (4.5) is equal to $\sum_{v\in\mathbf{C}\setminus\{v_2\}} x_{vc_0} + \sum_{v\in\mathbf{C}_{v_2}} v_{vc_0}$. Since $x_{v_ic_0} = 1$, then $x_{v_{i-1}c_0} = x_{v_{i+1}c_0} = 0$, where the subindices are taken modulo n. This implies $\sum_{v\in\mathbf{C}_{v_2}} v_{vc_0} \le \frac{|\mathbf{C}|}{2} - 2$. So, the LHS of (4.5) is less then or equal to $|\mathbf{C}| - 3$. As the RHS of (4.5) is greater than or equal to this value, the inequality (4.5) is satisfied.

(c) If $x_{v_ic_0} = 0$ for all $x_{v_ic_0} \in \mathbb{A}$, then consider the following cases:

(i) If $w_d = 1$ for some $d \in \mathcal{D}$ and $w_{c_0} = 1$, then a similar argument as in Case 2(b)(i) shows that (4.5) is satisfied.

(ii) If $w_d = 1$ for some $d \in D$ and $w_{c_0} = 0$, then $w_{d'} = 0$ for every $d' \in \mathcal{D}$, $d' \neq d$. This, together with the constraints (3.1), implies $\sum_{v\in\mathbf{C}_{v_1}} \sum_{c\in\mathcal{T}\setminus\mathcal{D}\cup\{c_0,c_1\}} x_{vc} = \frac{|\mathbf{C}|}{2} - \sum_{v\in\mathbf{C}_{v_1}} (x_{vc_1} + x_{vd})$, hence we can rewrite the inequality (4.5) in the following way:

$$\sum_{v\in\mathbf{C}_{v_2}} x_{vd} + \sum_{v\in\mathbf{C}_{v_1}} (x_{vc_1} + v_{vd}) \le |\mathbf{C}| - 1,$$

and this inequality is satisfied since z represents an acyclic coloring.

(iii) If $w_d = 0$ for every $d \in \mathcal{D}$, then (4.5) is trivially satisfied.

Since in all the cases the prominent vertex inequality is satisfied and z is an arbitrary solution of $P_{\mathbb{A}}(L)$, we conclude that (4.5) is valid for $P_{\mathbb{A}}(L)$. Then the disjunctive rank of (4.5) is less than or equal to $|\mathcal{D}| + 1 + \lfloor\frac{|\mathbf{C}|}{4}\rfloor$.

We now prove that the disjunctive rank of (4.5) is greater than or equal to $|\mathcal{D}| + 1 + \lfloor\frac{|\mathbf{C}|}{4}\rfloor$. Let $\mathbb{B} \subseteq \mathbb{V}$ be an arbitrary set of variables with cardinality $|\mathcal{D}| + \lfloor\frac{|\mathbf{C}|}{4}\rfloor$. In order to prove that the prominent vertex inequality is not valid for $P_{\mathbb{B}}(L)$, we show a solution $z \in P_{\mathbb{B}}(L)$ that violates (4.5). To this end, consider the following cases:

(1) If $w_d \notin \mathbb{B}$ for some $d \in \mathcal{D}$, since $|C| \ge 6$ or $|\mathcal{D}| > 1$, a straightforward counting argument shows that $|\mathbb{B}| = |\mathcal{D}| + \lfloor\frac{|\mathbf{C}|}{4}\rfloor$, implying that

(a) there exists $d_1 \in \mathcal{D}$ and $i \in \{1, \dots, n\}$ such that $w_{d_1} \notin \mathbb{B}$ and $x_{v_id_1}, x_{v_{i+2}d_1}, x_{v_ic_1}, x_{v_{i+2}c_1} \notin \mathbb{B}$ (see Fig. 7 for an example with $i = 3$), or

(b) there exist $d_1, d_2 \in \mathcal{D}$ and $i \in \{1, \dots, n\}$ such that $w_{d_2} \notin \mathbb{B}$ and $x_{v_id_1}, x_{v_{i+2}d_1}, x_{v_id_2}, x_{v_{i+2}d_2} \notin \mathbb{B}$ (see Fig. 8 for an example with $i = 4$).

If (a) holds then the solution depicted in Figure 7 settles this case, and if (b) holds then the solution depicted in Figure 8 settles this case.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	$\cdots$	$v_{\vert\mathbf{C}\vert}$	w_c
c_0	0	0	0	0	0	0	0	....	0	0
c_1	1	0	1	0	1	0	1	....	0	1
d_1	0	$\mathbf{\frac{1}{2}}$	0	$\mathbf{\frac{1}{2}}$	0	1	0	....	1	1
d_2	0	$\mathbf{\frac{1}{2}}$	0	$\mathbf{\frac{1}{2}}$	0	0	0	....	0	$\mathbf{\frac{1}{2}}$
d_3	0	0	0	0	0	0	0	....	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
$d_{\vert\mathcal{D}\vert}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
t_1	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
$t_{\vert\mathcal{T}\setminus\mathcal{D}\vert-2}$	0	0	0	0	0	0	0	....	0	0

FIGURE 8. Solution for the proof of Theorem 4.9, for $i = 2$. The variables in boldface do not belong to $\mathbb{B}$.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	$\cdots$	$v_{\vert\mathbf{C}\vert}$	w_c
c_0	1	0	$\mathbf{\frac{1}{2}}$	$\mathbf{\frac{1}{2}}$	$\mathbf{\frac{1}{2}}$	0	1	....	0	1
c_1	0	0	$\mathbf{\frac{1}{2}}$	0	$\mathbf{\frac{1}{2}}$	0	0	....	0	1
d_1	0	1	0	$\mathbf{\frac{1}{2}}$	0	1	0	....	1	1
d_2	0	0	0	0	0	0	0	....	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
$d_{\vert\mathcal{D}\vert}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
t_1	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
$t_{\vert\mathcal{T}\setminus\mathcal{D}\vert-2}$	0	0	0	0	0	0	0	....	0	0

FIGURE 9. Solution for the proof of Theorem 4.9. The variables in boldface do not belong to $\mathbb{B}$.

(2) If $w_d \in \mathbb{B}$ for every $d \in \mathcal{D}$, then there must exist some $d_1 \in \mathcal{D}$ and $i \in \{1, \dots, n\}$ such that $x_{v_i c_0}, x_{v_{i+1} c_0}, x_{v_{i+2} c_0}, x_{v_i c_1}, x_{v_{i+2} c_1}, x_{v_{i+1} d_1} \not\in \mathbb{B}$. If such a structure is not present, then $\mathbb{B}$ contains at least $|\mathbf{C}|/3$ variables from $\{x_{vc_0}, x_{vc_1}\}_{v\in\mathbf{C}}$. The solution specified by Figure 9 only has fractional values for these variables and violates (4.5), thus establishing this case.

Therefore, for any set $\mathbb{B}$ we can construct a solution that violates the inequality (4.5) and satisfies (3.1)–(3.3). Then, we conclude that for any set $\mathbb{B}$ the inequality (4.5) is not valid for $P_{\mathbb{B}}(L)$. $\square$

Note that the disjunctive rank of (4.5) is not fully characterized by Theorem 4.9. If the hypothesis (b) does not hold, *i.e.*, if $|\mathbf{C}| = 4$ and $|\mathcal{D}| = 1$, then the disjuctive rank depends on the existence of colors in $\mathcal{T} \setminus \mathcal{D} \cup \{c_0, c_1\}$, as the following proposition shows. We ommit the proof since it is based on similar arguments as the proof of Theorem 4.9.

Proposition 4.10. *Assume* $|\mathbf{C}| = 4$ *and* $|\mathcal{D}| = 1$. *If* $\mathcal{T} \setminus \mathcal{D} \cup \{c_0, c_1\} \neq \emptyset$ *then the disjunctive rank of* (4.5) *is* 3, *and the disjunctive rank of* (4.5) *is* 2 *otherwise.*

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	$\cdots$	$v_{\lvert\mathbf{C}\rvert}$	w_c
c_0	0	$\frac{\lvert\mathbf{C}\rvert-2}{\lvert\mathbf{C}\rvert}$	0	$\frac{\lvert\mathbf{C}\rvert-2}{\lvert\mathbf{C}\rvert}$	0	$\frac{\lvert\mathbf{C}\rvert-2}{\lvert\mathbf{C}\rvert}$	0	....	$\frac{\lvert\mathbf{C}\rvert-2}{\lvert\mathbf{C}\rvert}$	$\frac{\lvert\mathbf{C}\rvert-2}{\lvert\mathbf{C}\rvert}$
c_1	1	0	1	0	1	0	1	....	0	1
t_1	0	$\frac{2}{\lvert\mathbf{C}\rvert}$	0	$\frac{2}{\lvert\mathbf{C}\rvert}$	0	$\frac{2}{\lvert\mathbf{C}\rvert}$	0	....	$\frac{2}{\lvert\mathbf{C}\rvert}$	1
t_2	0	0	0	0	0	0	0	....	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
$t_{\lvert\mathcal{T}\rvert-2}$	0	0	0	0	0	0	0	....	0	0

FIGURE 10. A solution violating (4.5).

Theorem 4.9 and Proposition 4.10 leave the case $\mathcal{D} = \emptyset$ open. We conjecture the rank to be 1 in this case. The solution depicted in Figure 10 violates the inequality (4.5) but satisfies (3.1)–(3.3). In order to prove that the rank of (4.5) is at most 1, we must take a singleton $\mathbb{A}$ and show that (4.5) holds for $P_{\mathbb{A}}(L)$. Unfortunately, we were not able to settle this case. We conjecture that $\mathbb{A} = \{x_{v_3 c_0}\}$ may allow to complete such a proof, but showing that a solution $(x, w) \in P_{\mathbb{A}}(L)$ with $x_{v_3 c_0} = 0$ satisfies the inequality does not seem to be a straightforward task.

Theorem 4.11. *The disjunctive anti-rank of* (4.5) *is* $p - 5$ *if* $\mathcal{D} \neq \emptyset$.

Proof. We first prove that for any set $\mathbb{A} \subseteq \mathbb{V}$ with $p - 4$ variables, the inequality (4.5) is valid for $P_{\mathbb{A}}(L)$. Let (x, w) be an arbitrary extreme point in $P_{\mathbb{A}}(L)$, which has, by Theorem 2.3, at most four variables with fractional values.

If $\sum_{\mathbf{d}\in\mathcal{D}} \mathbf{w_d} \geq \mathbf{2}$, then the RHS of (4.5) is greater than or equal to $(1 + w_{c_0})\frac{|\mathbf{C}|}{2}$. Constraint (3.1) ensures $\sum_{u\in\mathbf{C}_{v_2}} \sum_{c\in\mathcal{D}\cup\{c_0\}} x_{uc} \leq \frac{|\mathbf{C}|}{2}$ which, together with Lemma 3.1(i), implies that (4.5) is satisfied. We, therefore, restrict ourselves to the case $\sum_{\mathbf{d}\in\mathcal{D}} \mathbf{w_d} \leq \mathbf{1}$. Assume that this holds and consider the following cases:

(1) If $\sum_{\boldsymbol{v}\in\mathbf{C}_{\boldsymbol{v_2}}} \boldsymbol{x_{vc_0}} = \frac{\mathbf{|C|}}{\mathbf{2}}$, then $\sum_{v\in\mathbf{C}\setminus\{v_2\}} x_{vc_0} = \frac{|\mathbf{C}|}{2} - 1$. Also, the second term in the LHS of (4.5) is less than or equal to $|\mathbf{C}|/2$. Since (x, w) satisfies the model constraint (3.3), then $\sum_{c\in\mathcal{D}} w_c + \sum_{c\in\mathcal{T}\setminus\mathcal{D}\cup\{c_0,c_1\}} \sum_{u\in\mathbf{C}_{v_1}} x_{uc} \geq 1$. Thus, the inequality (4.5) is satisfied.

(2) If $\frac{\mathbf{|C|}}{\mathbf{2}} - \mathbf{1} < \sum_{\boldsymbol{v}\in\mathbf{C}_{\boldsymbol{v_2}}} \boldsymbol{x_{vc_0}} < \frac{\mathbf{|C|}}{\mathbf{2}}$, then $w_{c_0} = 1$, at least one variable from $\{x_{vc_0}\}_{v\in\mathbf{C}_{v_2}}$ takes a fractional value, and $x_{vc_0} \neq 0$ for all $v \in \mathbf{C}_{v_2}$. Since at most four variables can take fractional values, then at most two variables from $\{x_{vc_0}\}_{v\in\mathbf{C}_{v_2}}$ can take fractional values, and this implies that $x_{vc_0} = 0$ for every $v \in \mathbf{C}_{v_1}$. Therefore, $\sum_{v\in\mathbf{C}\setminus\{v_2\}} x_{vc_0} \leq \frac{|\mathbf{C}|}{2} - 1$ and the LHS of (4.5) is less than or equal to $|\mathbf{C}| - 1$. As $\sum_{v\in\mathbf{C}} (x_{vc_0} + x_{vc_1}) \leq |\mathbf{C}| - 1$ by constraint (3.3), then $\sum_{c\in\mathcal{T}\setminus\{c_0,c_1\}} \sum_{u\in\mathbf{C}_{v_1}} x_{uc} \geq 1$. If $w_{d_1} < 1$ for some $d_1 \in \mathcal{D}$, then $x_{vd_1} = 0$ for every $v \in \mathbf{C}_{v_1}$. So, $\sum_{c\in\mathcal{T}\setminus\mathcal{D}\cup\{c_0,c_1\}} \sum_{u\in\mathbf{C}_{v_1}} x_{uc} \geq 1$ and the inequality (4.5) is satisfied. If $w_{d_1} = 1$ for some $d_1 \in \mathcal{D}$ (hence $w_d = 0$ for every $d \in \mathcal{D}, d \neq d_1$ since $\sum_{\mathbf{d}\in\mathcal{D}} \mathbf{w_d} \leq \mathbf{1}$), then the inequality (4.5) is trivially satisfied.

(3) If $\mathbf{0} < \sum_{\boldsymbol{v}\in\mathbf{C}_{\boldsymbol{v_2}}} \boldsymbol{x_{vc_0}} \leq \frac{\mathbf{|C|}}{\mathbf{2}} - \mathbf{1}$ and $\boldsymbol{w_{c_0}} = \mathbf{1}$, we claim that $\sum_{v\in\mathbf{C}\setminus\{v_2\}} x_{vc_0} \leq \frac{|\mathbf{C}|}{2} - 1$. To this end, consider the following cases:
   (a) If $x_{vc_0} \in \mathbb{Z}$ for all $v \in \mathbf{C}_{v_1}$ or for all $v \in \mathbf{C}_{v_2}$, then $\sum_{v\in\mathbf{C}\setminus\{v_2\}} x_{vc_0} \leq \frac{|\mathbf{C}|}{2} - 1$.
   (b) If there exist $x_{v_t c_0} \in \mathbf{C}_{v_2}$ and $x_{v_{t'} c_0} \in \mathbf{C}_{v_1}$ with $0 < x_{v_t c_0}, x_{v_{t'} c_0} < 1$ and such that $v_t v_{t'} \notin E$, then there exist colors $c_2, c_3 \in \mathcal{T}\setminus\{c_0\}$ such that $x_{v_t c_2}$ and $x_{v_{t'} c_3}$ take fractional values. Since (x, w) has at most four variables with fractional values, then $x_{v_{t-1} c_0} = x_{v_{t+1} c_0} = x_{v_{t'-1} c_0} = x_{v_{t'+1} c_0} = 0$, where indices are taken modulo n. This implies that at least three variables from $\{x_{vc_0}\}_{v\in\mathbf{C}\setminus\{v_2\}}$ take null values, hence $\sum_{v\in\mathbf{C}\setminus\{v_2\}} x_{vc_0} \leq \frac{|\mathbf{C}|}{2} - 1$.

(c) If there exist $x_{v_t c_0} \in \mathbf{C}_{v_2}$ and $x_{v_{t'} c_0} \in \mathbf{C}_{v_1}$ with $0 < x_{v_t c_0}, x_{v_{t'} c_0} < 1$ and such that $v_t v_{t'} \in E$, then a similar argument shows that $x_{v_{t-1} c_0} = x_{v_{t'+1} c_0} = 0$. Therefore, $\sum_{v \in \mathbf{C} \setminus \{v_2\}} x_{vc_0} \leq \frac{|\mathbf{C}|}{2} - 1$.

In all three cases the claim holds, hence the LHS of (4.5) is less than or equal to $|\mathbf{C}| - 2$. As the RHS of (4.5) is greater than or equal to this value, the inequality (4.5) is satisfied.

(4) If $\sum_{v \in \mathbf{C}_{v_2}} \boldsymbol{x_{vc_0} = 0}$ and $\boldsymbol{w_{c_0} = 1}$, then $\sum_{v \in \mathbf{C} \setminus \{v_2\}} x_{vc_0} \leq \frac{|\mathbf{C}|}{2}$. On the one hand, if $w_{d_1} = 1$ for some $d_1 \in \mathcal{D}$, the assumption $\sum_{d \in \mathcal{D}} w_d \leq 1$ implies $w_d = 0$ for every $d \in \mathcal{D}, d \neq d_1$. Then, the LHS of (4.5) is equal to $\sum_{v \in \mathbf{C} \setminus \{v_2\}} x_{vc_0} + \sum_{v \in \mathbf{C}_{v_2}} x_{vd_1}$ which, by the model constraint (3.3), is less than or equal to $|\mathbf{C}| - 1$. We conclude that the inequality (4.5) is satisfied. On the other hand, if $\sum_{d \in \mathcal{D}} w_d < 1$, then at most one variable from $\{x_{vd}\}_{v \in \mathbf{C}_{v_2}, d \in \mathcal{D}}$ can take a fractional value, since at most four variables take fractional values in (x, w). This implies $\sum_{v \in \mathbf{C}_{v_2}} \sum_{d \in \mathcal{D}} x_{vd} \leq \sum_{d \in \mathcal{D}} w_d$, and the inequality (4.5) is satisfied.

(5) If $\boldsymbol{0 < w_{c_0} < 1}$, then at most three variables from $\{x_{vc}\}_{v \in \mathbf{C}, c \in \mathcal{T}}$ are allowed to take fractional values. Therefore, at most one variable from $\{x_{vc_0}\}_{v \in \mathbf{C}}$ can take a fractional value. Furthermore, if $x_{v_t c_0}$ takes a fractional value, then at most one variable from $\{w_d\}_{d \in \mathcal{D}}$ can take a value different from 0 (since otherwise we would have more than four variables with fractional values). Let w_{d_1} be such variable, and consider the following cases:

(a) If $w_{d_1} = 1$ and $\frac{|\mathbf{C}|}{2} - 1 < \sum_{v \in \mathbf{C}_{v_2}} x_{vd_1} \leq \frac{|\mathbf{C}|}{2}$, then, as $\sum_{v \in \mathbf{C}} (x_{vc_1} + x_{vd_1}) \leq |\mathbf{C}| - 1$, the sum $\sum_{c \in \mathcal{T} \setminus \mathcal{D} \cup \{c_0, c_1\}} \sum_{u \in \mathbf{C}_{v_1}} x_{uc}$ is greater than or equal to 1. Then the inequality (4.5) is satisfied.

(b) If $w_{d_1} = 1$ and $\sum_{v \in \mathbf{C}_{v_2}} x_{vd_1} \leq \frac{|\mathbf{C}|}{2} - 1$, then the LHS of (4.5) is less than or equal to $2w_{c_0} + \frac{|\mathbf{C}|}{2} - 1$. Therefore, the inequality (4.5) is satisfied.

(c) If $w_{d_1} < 1$, then $x_{v_t d_1}$ is the only variable in $\{x_{vd_1}\}_{v \in \mathbf{C}}$ allowed to take a non-null value (if $x_{ud_1} > 0$ for some $u \neq v_t$, then $x_{ud_1} \leq w_{d_1} < 1$, thus generating at least five variables with fractional values). So, the LHS of the inequality is less than or equal to $2x_{v_t c_0} + x_{v_t d_1}$. As this value is less than or equal to $2w_{c_0} + w_{d_1}$, the inequality (4.5) is satisfied.

(6) If $\boldsymbol{w_{c_0} = 0}$, then the LHS of (4.5) is equal to $\sum_{v \in \mathbf{C}_{v_2}} \sum_{d \in \mathcal{D}} x_{vd}$. Consider the following cases:

(a) If $\sum_{d \in \mathcal{D}} w_d < 1$, then a similar argument as in Case 4 shows that (4.5) holds.

(b) If $w_{d_1} = 1$ for some $d_1 \in \mathcal{D}$, then the LHS of (4.5) is equal to $\sum_{v \in \mathbf{C}_{v_2}} x_{vd_1}$. If $\sum_{v \in \mathbf{C}_{v_2}} x_{vd_1} \leq \frac{|\mathbf{C}|}{2} - 1$, then the inequality (4.5) is satisfied. If $\frac{|\mathbf{C}|}{2} - 1 < \sum_{v \in \mathbf{C}_{v_2}} x_{vd_1} \leq \frac{|\mathbf{C}|}{2}$, then $\sum_{v \in \mathbf{C}_{v_1}} x_{vd_1} = 0$. Therefore, $\sum_{c \in \mathcal{T} \setminus \mathcal{D} \cup \{c_0, c_1\}} \sum_{u \in \mathbf{C}_{v_1}} x_{uc} \geq 1$ and again the inequality (4.5) is satisfied.

We conclude that the disjunctive anti-rank of (4.5) is less than or equal to $p - 5$.

We now prove that the disjunctive anti-rank is greater than or equal to $p - 5$. Let $d_1 \in \mathcal{D}$ and define $\mathbb{B} \subseteq \mathbb{V}$ to be the set $\mathbb{V} \setminus \{x_{v_1 c_1}, x_{v_1 d_1}, x_{v_3 c_1}, x_{v_3 d_1}, w_{d_1}\}$, which has $|\mathbb{B}| = p - 5$. Let $z = (x, w) \in P_{\mathbb{B}}(L)$ be the feasible solution depicted in Figure 11. This solution satisfies the model constraints, violates the inequality (4.5) and the variables in $\mathbb{B}$ take 0-1 values. Hence, the disjunctive anti-rank is greater than or equal to $p - 5$ and the theorem follows. $\square$

4.4. Further families of valid inequalities

We now present results for the rest of the families presented in [10]. We do not include the proofs of the following theorems as they involve similar arguments to the previous ones.

Definition 4.12. Let $v_1 \in \mathbf{C}$ and let $c_0, c_1 \in \mathcal{T}$, $c_0 \neq c_1$. The *reinforced two-color inequality* associated with $\mathbf{C}$, v_1, c_0 and c_1 is

$$\sum_{u \in \mathbf{C} \setminus \{v_1\}} x_{uc_0} + \sum_{u \in \mathbf{C}_{v_1}} (x_{uc_0} + x_{uc_1}) \leq \frac{|\mathbf{C}|}{2} + \left(\frac{|\mathbf{C}|}{2} - 1 \right) w_{c_0}. \tag{4.7}$$

Theorem 4.13. *The disjunctive rank of* (4.7) *is* $\lfloor \frac{|\mathbf{C}|}{4} \rfloor$ *and the disjunctive anti-rank of* (4.7) *is* $p - (|\mathbf{C}| + 1)$.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	$\cdots$	$v_{\lvert\mathbf{C}\rvert}$	w_c
c_0	0	1	0	1	0	1	0	....	1	1
c_1	$\mathbf{\frac{1}{2}}$	0	$\mathbf{\frac{1}{2}}$	0	1	0	1	....	0	1
d_1	$\mathbf{\frac{1}{2}}$	0	$\mathbf{\frac{1}{2}}$	0	0	0	0	....	0	$\mathbf{\frac{1}{2}}$
d_2	0	0	0	0	0	0	0	....	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
$d_{\lvert\mathcal{D}\rvert}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
t_1	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	....	$\vdots$	$\vdots$
$t_{\lvert\mathcal{T}\setminus\mathcal{D}\rvert-2}$	0	0	0	0	0	0	0	....	0	0

FIGURE 11. Solution for the proof of Theorem 4.11. The values in boldface correspond to the variables in $\mathbb{V}\backslash\mathbb{B}$

Definition 4.14. Let $v_1, v_2, v_3 \in \mathbf{C}$ be three consecutive vertices and let $c_0 \in \mathcal{T}$. Let $\mathcal{D}, \mathcal{D}' \subset \mathcal{T} \setminus \{c_0\}$ such that $\mathcal{D} \cap \mathcal{D}' = \emptyset$. The *three-consecutive vertices inequality* associated with $\mathbf{C}$, v_1, v_2, v_3, c_0, $\mathcal{D}$, and $\mathcal{D}'$ is

$$\sum_{u\in\mathbf{C}\setminus\{v_2\}} x_{uc_0} + \sum_{c\in\mathcal{D}} x_{v_3c} + \sum_{c\in\mathcal{D}'} x_{v_1c} + \sum_{c\in\mathcal{D}\cup\mathcal{D}'} x_{v_2c} \le \left(\frac{|\mathbf{C}|}{2} - 1\right) w_{c_0} + \sum_{c\in\mathcal{D}\cup\mathcal{D}'} w_c + \sum_{u\in\mathbf{C}_{v_2}\setminus\{v_2\}} \sum_{c\in\mathcal{T}\setminus(\{c_0\}\cup\mathcal{D}\cup\mathcal{D}')} x_{uc}. \tag{4.8}$$

Theorem 4.15. *If $\mathcal{T} \setminus (\mathcal{D} \cup \mathcal{D}' \cup \{c_0\}) = \emptyset$, then the disjunctive rank of (4.8) is*

(1) $|\mathcal{D}| + |\mathcal{D}'| + 1$, *if* $\mathcal{D} \neq \emptyset$, $\mathcal{D}' \neq \emptyset$ *and* $|\mathbf{C}| \geq \frac{|\mathcal{D}|+|\mathcal{D}'|+2}{|\mathcal{D}|+|\mathcal{D}'|-1}2$*;*
(2) $\frac{|\mathbf{C}|}{2} - 1$, *if* $\mathcal{D} \neq \emptyset$, $\mathcal{D}' \neq \emptyset$, $|\mathbf{C}| \leq 6$ *and* $|\mathcal{D}| + |\mathcal{D}'| = 2$*;*
(3) 2, *if* $|\mathbf{C}| = 4$, $\mathcal{D} \neq \emptyset$, $\mathcal{D}' \neq \emptyset$ *and* $|\mathcal{D}| + |\mathcal{D}'| = 3$*;*
(4) $|\mathcal{D}'|$, *if* $\mathcal{D} = \emptyset$*;*
(5) $|\mathcal{D}|$, *if* $\mathcal{D}' = \emptyset$*.*

If $|\mathcal{T} \setminus (\mathcal{D} \cup \mathcal{D}' \cup \{c_0\})| \geq 1$, then the disjunctive rank of (4.8) is

(1) $|\mathcal{D}| + |\mathcal{D}'| + \lfloor\frac{|\mathbf{C}|}{4}\rfloor$ *if* $|\mathcal{D} \cup \mathcal{D}'| \geq 2$*;*
(2) $\lfloor\frac{|\mathbf{C}|}{4}\rfloor$ *if* $|\mathcal{D} \cup \mathcal{D}'| = 1$*;*
(3) 0, *if* $|\mathcal{D} \cup \mathcal{D}'| = 0$*.*

The disjunctive anti-rank of (4.8) is

(1) $p - 5$, *if* $|\mathcal{D} \cup \mathcal{D}'| \geq 2$*;*
(2) $p - 6$, *if* $|\mathcal{D} \cup \mathcal{D}'| = 1$*;*
(3) 0, *if* $\mathcal{D} \cup \mathcal{D}' = \emptyset$*.*

Definition 4.16. Let $v_1, v_2, v_3, v_4 \in \mathbf{C}$ be four consecutive vertices and let $c_0, c_1, c_2 \in \mathcal{T}$. The *four-consecutive vertices inequality* associated with $\mathbf{C}$, v_1, v_2, v_3, v_4, c_0, c_1, and c_2 is

$$\sum_{u\in\mathbf{C}\setminus\{v_3\}} x_{uc_0} + \sum_{u\in\mathbf{C}\setminus\{v_2\}} x_{uc_1} + x_{v_1c_0} + x_{v_4c_1} \le \left(\frac{|\mathbf{C}|}{2} - 1\right)(w_{c_0} + w_{c_1}) + \sum_{c\in\mathcal{T}\setminus\{c_0,c_2\}} x_{v_2c} + \sum_{c\in\mathcal{T}\setminus\{c_1,c_2\}} x_{v_3c} + 1. \tag{4.9}$$

Theorem 4.17. *The disjunctive rank of (4.9) is 4. The disjunctive anti-rank of (4.9) is $p - (|\mathbf{C}| + 1)$.*

5. Conclusions and open problems

In this work we studied the disjunctive rank of six families of valid inequalities presented in a previous work. We introduced a dual concept, the disjunctive anti-rank of a valid inequality, which gives the number of BCC iterations needed in order to always obtain a polytope that satisfies the inequality. It is interesting to note that the disjunctive rank of the families of inequalities studied in this work does not seem to be correlated with the practical contribution of each family to a branch-and-cut procedure for acyclic coloring. In the preliminary branch-and-cut procedure implemented in [10], the two-color inequalities (4.1) and the distinguished colors inequalities (4.2) allowed to achieve the best performances. This observation does not seem to correlate to the disjunctive ranks presented in Section 4.

This work leaves open many issues, among which we can mention the following ones.

- The rank and anti-rank for the prominent vertex inequalities is not fully characterized by the results in this paper. It would be interesting to further explore this issue.
- In this work we settled ourselves to exploring the rank of the known valid inequalities for the standard formulation of acyclic coloring associated with the BCC operator, since this operator provides a neat environment for such study. Anyway, it would be interesting to study the rank of these inequalities under the other known lift-and-project operators. In particular, it would be interesting to assess whether the ranks are similar across the different operators.
- Finally, exploring the Chvátal rank of these inequalities may also be an interesting task. This could not be an easy task for some of the more involved valid inequalities.

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