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A SIMPLE ANALYSIS OF SYSTEM CHARACTERISTICS IN THE BATCH SERVICE QUEUE WITH INFINITE-BUFFER AND MARKOVIAN SERVICE PROCESS USING THE ROOTS METHOD: $GI/C\text{-}MSP^{(a,b)}/1/\infty$

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Abstract. We consider an infinite-buffer single-server queue with renewal input and Markovian service process where customers are served in batches according to a general bulk service rule. Queue-length distributions at epochs of pre-arrival, arbitrary and post-departure have been obtained along with some important performance measures such as mean queue lengths and mean waiting times in both the system as well as the queue. We also obtain the steady-state service batch size distributions as well as system-length distributions. The proposed analysis is based on roots of the associated characteristic equation of the vector-generating function of queue-length distribution at a pre-arrival epoch. Also, we provide analytical and numerical comparison between the roots method used in this paper and the matrix geometric method in terms of computational complexities and required computation time to evaluate pre-arrival epoch probabilities for both the methods. Later, we have established heavy- and light-traffic approximations as well as an approximation for the tail probabilities at pre-arrival epoch based on one root of the characteristic equation. Numerical results for some cases have been presented to show the effect of model parameters on the performance measures.

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1. Introduction

Bulk service queues have received considerable attention due to their wide applications in several areas such as computer-communications, telecommunications, transportation systems and automatic manufacturing systems. Chaudhry and Templeton [15] and Dshalalow [21] provide an extensive discussion of bulk-service queueing systems. In such queues, customers are served by a single server in batches of maximum size 'b' with a minimum threshold size 'a'. Such type of service rule is referred to as the general bulk service rule. In recent years some

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authors have analyzed models that involve this rule, see *e.g.* Gold and Tran-Gia [25] as well as Chaudhry and Gupta [14]. For the more general model $GI/G^{(1,b)}/1/N$, Hébuterne and Rosenberg [30] have obtained relations among queue-length distributions at various epochs. Chakravarthy [10, 11] analyzes $GI/PH^{(1,b)}/1/b$ and $MAP/G^{(1,b)}/1/b$ queueing systems. Later, Gupta and Vijaya Laxmi [29] and Laxmi and Gupta [44] carried out the analysis of $MAP/G^{(a,b)}/1/N$ and $GI/M^{(1,b)}/1/N$ queues, respectively. Recently, Yu and Alfa [45] have analyzed the $D\text{-}MAP/G^{(1,a,b)}/1/N$ queue with batch-size-dependent service time.

Queueing models with non-renewal arrivals and Markov service processes are often used to model networks of complex computer and communication systems. In such systems, both the arrival and service processes may exhibit correlations which have significant impact on queueing behaviour. Markovian arrival process ($C\text{-}MAP$) is used to capture the correlation among the inter-arrival times. Similarly, batch Markovian arrival process ($C\text{-}BMAP$) is used to capture correlation among inter-batch arrival times. $C\text{-}BMAP$ is a convenient representation of the versatile Markovian point process (N-process) which was introduced and discussed by Neuts [38]. Later on it was formalized by Lucantoni *et al.* [33] and Lucantoni [31]; see Pacheco *et al.* [40] for an historical account on the $C\text{-}BMAP$ and related Markov additive processes of arrivals. Considerable amount of literature is available on the queueing systems with $C\text{-}MAP$ or $C\text{-}BMAP$ arrivals. Like these non-renewal arrival processes, Markovian service process $C\text{-}MSP$ is a versatile service process which can capture the correlation among successive service times. Several other service processes, *e.g.*, Poisson process, Markov modulated Poisson process ($C\text{-}MMPP$), PH-type renewal process, *etc.* can be considered as special cases of MSP. For details of $C\text{-}MSP$ readers are referred to Bocharov [5], Albores and Tajonar [1] and Gupta and Banik [28]. In Bocharov [5], the analysis of finite and infinite-buffer $G/C\text{-}MSP/1/r$ ($r \leq \infty$) queue has been performed; in [1], $GI/C\text{-}MSP/n/r$ queue has been analyzed; and in [28], $GI/C\text{-}MSP/1$ queue with finite- and infinite-buffer is discussed along with its computational procedure.

In this paper, we carry out the analytic analysis of the $GI/C\text{-}MSP^{(a,b)}/1/\infty$ queue through the calculation of roots of the denominator of the underlying generating function for the steady state probabilities of the embedded Markov chain after a particular threshold level for infinite state space. In this connection, readers are referred to Chaudhry *et al.* [17, 18], Tijms [43] and Chaudhry *et al.* [16], who have used the roots method. The roots can be easily found using one of the several commercially available packages such as Maple and Mathematica. Historically, when MAPLE and Mathmatica could not find a large number of roots, a software package called QROOT developed by Chaudhry [13] was used by him and his collaborators to find a large number of roots and use them in solving several queueing models. The algorithm for finding such roots is available in some papers, *e.g.*, see Chaudhry *et al.* [16]. The purpose of studying this queueing model using roots is that we obtain computationally simple and analytically closed form solution to the infinite-buffer $GI/C\text{-}MSP^{(a,b)}/1$ queue. It may be remarked here that the matrix-geometric method (MGM) uses iterative procedure to get steady-state probabilities at the pre-arrival epochs. Further, it is well known that while for the case of the MGM it is required to solve the non-linear matrix equation and the dimension of each matrix in this equation is the maximum number of service phases involved in a $GI/C\text{-}MSP^{(a,b)}/1$ queue. However, in roots method we do not have to investigate the structure of the transition probability matrices (TPM) at the embedded pre-arrival epochs. It may be worthy to mention here that Mitrani and Chakka [35] have made a comparison of matrix-geometric and spectral expansion method used to evaluate steady-state probabilities for a class of similar stochastic models. Also, Mitrani and Chakka [35] have concluded that the spectral expansion method has certain advantages over the matrix-geometric method. Further, it may be stated that it is always interesting and instructive to have an alternative solution to a complex problem such as the one being considered here. It may be mentioned here that the basic idea of correlated service was first introduced by Chaudhry [12]. It may be remarked that the analysis of the finite-buffer queues with renewal input and batch Markovian service process ($GI/C\text{-}MSP^{(a,b)}/1/N$) has been carried out by Banik *et al.* [4]. The present queueing model is an infinite-buffer queue which is more difficult to handle than the corresponding finite-buffer counterpart, *e.g.*, see Banik *et al.* [4]. Secondly, we obtain several other quantitative measures such as queue-length distribution at post-departure epoch, service batch size distributions, system-length distributions, evaluation of expected busy and idle periods which are not derived in [4]. Further, we provide a comparison between the computational complexities of the roots method used in this paper and matrix-geometric method. It may worth mentioning that matrix-geometric method due to Neuts [39] may be used to obtain embedded pre-arrival epoch probabilities for this particular queueing model under consideration. In fact, the roots method is simple analytically, notationally and computationally. Besides, whereas the roots method is insensitive to the traffic intensity (ρ), the matrix-geometric is not. Also, we have established heavy- and light-traffic approximations as well as approximation for the tail probabilities at pre-arrival epoch based on one root of the characteristic equation. Finally, some numerical results have been presented which may help researchers/practitioners to tally their results with those of ours.

It may be added here that the queueing model studied in this paper finds applications in the design of semi-conductor manufacturing workstations and cloud computing centers. Recently, Banik [3] mentions such applications in the analysis of similar bulk-service queueing model. Machines involved in semiconductor manufacturing process can be characterized into discrete processing machines (DPMs) and batch processing machines (BPMs). The batch processing machines can process a subset of queued lots at the same time without interruption. As presented in Cha *et al.* [9], there is a limit to the number of lots, called maximum batch size, which may be processed in a particular batch due to the limitation of the machine and/or the process constraints. The determination of the optimal minimum and maximum batch size in different manufacturing environments is generally a topic of research. In cloud computing, transmissions of user requests are of various types (*e.g.*, data, voice, video, images *etc.*). There are many applications which need bulk data to be transferred in the web which are accessed by a large client at the same time. As per the service level agreement (SLA) all the requests received by the clients may be processed in bulk. The instances of the target web application running into virtual machines (VMs) act as service centers to process the requests in the queue. When one by one requests are submitted in the queue, the requests are forwarded to the VM which is currently idle because of lack of sufficient number of requests. The requests are served in batches with a minimum of a and a maximum of b requests for this VM, where $1 \leq a \leq b$. The system under consideration contains multiple homogeneous VMs which render service in order of task request arrivals that is, first-come, first served (FCFS). For a recent survey of system model of bulk service on cloud, the readers are referred to Goswami *et al.* [26]. Following above examples, we propose to study $GI/C\text{-}MSP^{(a,b)}/1/\infty$ queue for semiconductor wafer fabrication and cloud computing centers.

2. Description of the model

Let us consider a single-server infinite-buffer queueing system wherein inter-arrival times are independent and identically distributed (i.i.d.) random variables (r.vs.) with distribution function (DF) $A(x)$, probability density function $a(x)$, Laplace–Stieltjes transform (LST) $A^*(\theta)$ and mean inter-arrival time $1/\lambda$. The service process is $C\text{-}MSP^{(a,b)}$ and is governed by an underlying m-state Markov chain of $C\text{-}MSP$. The inter-arrival times are independent of the service process.

The customers are served in batches according to $C\text{-}MSP$ with matrix representation $(\mathbf{L}_0, \mathbf{L}_1)$, wherein the services are governed by an underlying m-state recurrent Markov chain. Let us assume that the Markov process is in state i $(1 \leq i \leq m)$. The sojourn time in this state is exponentially distributed with parameter μ_i. Now, if at some instant a batch is under service and the service is in phase i $(1 \leq i \leq m)$, then in a "small" time Δt with probability $[L_0]_{i,j}\Delta t + o(\Delta t)$ the service phase changes to j, $j = 1, \ldots, m$, $j \neq i$, and the service of the batch continues; and, with probability $[L_1]_{i,j}\Delta t + o(\Delta t)$ the service phase changes to j, $j = 1, \ldots, m$, with the batch service being completed and the batch in service leaving from the system. If we let $[L_0]_{i,i} = -\mu_i = -\Big(\sum\limits_{j\neq i}[L_0]_{i,j} + \sum\limits_{j}[L_1]_{i,j}\Big)$, then the matrix $\mathbf{L}_0 = [[L_0]_{i,j}]$ has nonnegative off-diagonal and negative diagonal elements, and the matrix $\mathbf{L}_1 = [[L_1]_{i,j}]$ has nonnegative elements. We also assume that the service phase does not change in an idle period, *i.e.*, the service process is interrupted during idle periods. Let $N(t)$ denote the number of batches served in t units of time and $J(t)$ the state of the underlying Markov chain at time t with its state space $\{i : 1 \leq i \leq m\}$. Then $\{N(t), J(t)\}$ is a two-dimensional Markov process with state space $\{(\ell, i) : \ell \geq 0, 1 \leq i \leq m\}$. The infinitesimal generator for this Markov process is given by Chaudhry *et al.* [17]. Let us denote $\mathbf{L}(z) = \mathbf{L}_0 + \mathbf{L}_1 z$ with $\mathbf{L} \equiv \mathbf{L}(1) = \mathbf{L}_0 + \mathbf{L}_1$ being an irreducible infinitesimal generator of the underlying Markov chain $\{J(t)\}$. Therefore from Chaudhry *et al.* [17], we have $\mathbf{Le} = \mathbf{0}$ (throughout $\mathbf{e}$ denotes the column vector of ones with the appropriate dimension depending on the context). Average batch service rate μ^* (the so called fundamental service rate) of the stationary $C\text{-}MSP$ is given by $\mu^* = \overline{\boldsymbol{\pi}}\mathbf{L}_1\mathbf{e}$, where $\overline{\boldsymbol{\pi}} = [\overline{\pi}_1, \overline{\pi}_2, \ldots, \overline{\pi}_m]$ with $\overline{\pi}_j$ denoting the steady-state probability of server being in phase j $(1 \leq j \leq m)$. The stationary probability row-vector $\overline{\boldsymbol{\pi}}$ can be calculated from $\overline{\boldsymbol{\pi}}\mathbf{L} = \mathbf{0}$ with $\overline{\boldsymbol{\pi}}\mathbf{e} = 1$.

The customers are served in batches according to a Markovian service process. The service takes place in batches of maximum size b with a minimum threshold equal to a $(1 \leq a \leq b)$. However, if fewer than a (≥ 1) customers are present in the queue, the server waits till the number of customers in the queue reaches a and then initiates service for that group of customers. The offered load ρ is defined as $\rho = \lambda/b\mu^*$. The customers are served in batches according to a $C\text{-}MSP^{(a,b)}$ rule with mean batch service time $1/\mu^*$.

Further, let us define $\{\mathbf{P}(n,t) : n \geq 0, t \geq 0\}$ as the $m \times m$ matrix whose (i,j)th element is the conditional probability defined as

$$P_{i,j}(n,t) = Pr\{N(t) = n, J(t) = j | N(0) = 0, J(0) = i\}, \quad 1 \leq i,j \leq m.$$

This system may be written in matrix notation as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{P}(0,t) = \mathbf{P}(0,t)\mathbf{L}_0 \tag{2.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{P}(n,t) = \mathbf{P}(n,t)\mathbf{L}_0 + \mathbf{P}(n-1,t)\mathbf{L}_1,\ n \geq 1, \tag{2.2}$$

with $\mathbf{P}(0,0) = \mathbf{I}_m$ and $\mathbf{P}(n,0) = \mathbf{0}$, $n \geq 1$, where $\mathbf{I}_m$ is the identity matrix of order $m \times m$. Let us define the matrix-generating function $\mathbf{P}^*(z,t)$ as

$$\mathbf{P}^*(z,t) = \sum_{n=0}^{\infty} \mathbf{P}(n,t) z^n, \quad |z| \leq 1. \tag{2.3}$$

From Chaudhry *et al.* [17], we get

$$\mathbf{P}^*(z,t) = \mathrm{e}^{\mathbf{L}(z)t}, \quad |z| \leq 1,\ t \geq 0. \tag{2.4}$$

Recall that the arrival times are assumed to be i.i.d. random variables and they are independent of the service process.

Let $\mathbf{S}_n$ $(n \geq 0)$ denote the matrix of order $m \times m$ whose (i,j)th element represents the conditional probability that during an inter-arrival period n batches are served and the service process passes to phase j, provided at the initial instant of the previous arrival epoch there were at least n batches (*i.e.*, at least $(n-1)b+a$ customers) in the system and the service process was in phase i. Then

$$\mathbf{S}_n = \int_0^{\infty} \mathbf{P}(n,t)\mathrm{d}A(t),\ n \geq 0. \tag{2.5}$$

If we let $\mathbf{S}(z)$ denote the matrix-generating function of $\mathbf{S}_n$, where $S_{i,j}(z)$ $(1 \leq i,j \leq m)$ are the elements of $\mathbf{S}(z)$, then, using (2.5) and (2.4), we get

$$\begin{aligned}\mathbf{S}(z) &= \sum_{n=0}^{\infty} \mathbf{S}_n z^n = \int_0^{\infty} \sum_{n=0}^{\infty} \mathbf{P}(n,t) z^n \mathrm{d}A(t) \\ &= \int_0^{\infty} \mathbf{P}^*(z,t)\mathrm{d}A(t) = \int_0^{\infty} \mathrm{e}^{\mathbf{L}(z)t} a(t)\mathrm{d}t. \end{aligned} \tag{2.6}$$

The evaluation of the matrices $\mathbf{S}_n$ can be carried out along the lines proposed by Lucantoni [31]. Chaudhry *et al.* [18] have given the procedure of obtaining $\mathbf{S}(z)$.

As above, we introduce the matrices $\mathbf{\Omega}_n$ $(n \geq 0)$ of order $m \times m$ whose (i, j)th element represents the limiting probability that n batches are served during an elapsed inter-arrival time of the arrival process with the service process being in phase j, given that there were at least $(n + 1)$ batches in the system with the service process being in phase i at the beginning of the inter-arrival period. Then, from Markov renewal theory as given by (Chaudhry and Templeton [15], p. 80), equation (2.3.14), we have

$$\mathbf{\Omega}_n = \lambda \int_0^\infty \mathbf{P}(n, x)(1 - A(x))\, \mathrm{d}x, \quad n \geq 0. \tag{2.7}$$

The matrices $\mathbf{\Omega}_n$ can be expressed in terms of the matrices $\mathbf{S}_n$. This is shown as follows:

$$\begin{aligned}
\mathbf{S}_n &= \int_0^\infty \mathbf{P}(n, t)\mathrm{d}A(t) = -\int_0^\infty \mathbf{P}(n, t)\mathrm{d}[1 - A(t)],\\
&= \delta_{n,0}\mathbf{I}_m + \int_0^\infty \mathbf{P}^{(1)}(n, t)[1 - A(t)]\mathrm{d}t, \quad \text{where } \mathbf{P}^{(1)}(n, t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{P}(n, t) \text{ and } \delta_{n,0} \text{ is the}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{Kronecker's delta,}\\
&= \delta_{n,0}\mathbf{I}_m + \frac{1}{\lambda}\mathbf{\Omega}_n\mathbf{L}_0 + \frac{1}{\lambda}\mathbf{\Omega}_{n-1}\mathbf{L}_1.\mathit{1}_{\{n\geq 1\}}, \quad \text{where } \mathit{1}_{\{n\geq 1\}} \text{ is the indicator function.}
\end{aligned} \tag{2.8}$$

To obtain the last equality, we have replaced $\mathbf{P}^{(1)}(n, t)$ from equation (2.2). Therefore,

$$\mathbf{\Omega}_0 = \lambda\Big(\mathbf{I}_m - \mathbf{S}_0\Big)(-\mathbf{L}_0)^{-1}, \tag{2.9}$$

$$\mathbf{\Omega}_n = \Big(\mathbf{\Omega}_{n-1}\mathbf{L}_1 - \lambda\mathbf{S}_n\Big)(-\mathbf{L}_0)^{-1}, \quad n \geq 1. \tag{2.10}$$

Further, let $\widetilde{P}_{ij}(n, t)$ be the conditional probability that at least n batches are served in $(0, t]$ and the service process is in phase j at the end of the nth batch service completion, given that there were n batches in the system and the service process was in phase i at time $t = 0$. The probabilities $\widetilde{P}_{ij}(n, t),\ n \geq 1,\ t \geq 0$, then satisfy the equations

$$\widetilde{P}_{ij}(n, t + \varDelta t) = \widetilde{P}_{ij}(n, t) + \sum_{k=1}^{m} P_{ik}(n - 1, t)[L_1]_{kj}\varDelta t + o(\varDelta t),$$

with the initial condition $\widetilde{P}_{ij}(n, 0) = 0,\ n \geq 1$. Rearranging the terms and taking the limit as $\varDelta t \to 0$, it reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{P}_{ij}(n, t) = \sum_{k=1}^{m} P_{ik}(n - 1, t)[L_1]_{kj}, \quad n \geq 1,$$

for $t \geq 0,\ 1 \leq i, j \leq m$, with the initial conditions $\widetilde{P}_{ij}(n, 0) = 0$. This system may be written in matrix notation as

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathbf{P}}(n, t) = \mathbf{P}(n - 1, t)\mathbf{L}_1, \quad n \geq 1, \tag{2.11}$$

with $\widetilde{\mathbf{P}}(n, 0) = \mathbf{0}$.

Further, let $\mathbf{S}_n^*$ denote the matrix of order $m \times m$ whose (i, j)th element represents the probability that at least n batches are served during an inter-arrival period and the service process is in phase j at the end of the nth batch service completion, provided at the initial instant of previous arrival epoch there were exactly n batches in the system and the service process was in phase i. Then

$$\begin{aligned}\mathbf{S}_n^* &= \int_0^\infty \widetilde{\mathbf{P}}(n,t)\mathrm{d}A(t)\\ &= -\int_0^\infty \widetilde{\mathbf{P}}(n,t)\mathrm{d}[1-A(t)]\\ &= -\Big[\widetilde{\mathbf{P}}(n,t)[1-A(t)]\Big]_0^\infty + \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathbf{P}}(n,t)[1-A(t)]\mathrm{d}t\\ &= \int_0^\infty \mathbf{P}(n-1,t)\mathbf{L}_1[1-A(t)]\mathrm{d}t\\ &= \frac{1}{\lambda}\mathbf{\Omega}_{n-1}\mathbf{L}_1,\;\; n\ge 1. \end{aligned} \tag{2.12}$$

Further, we introduce the matrix $\mathbf{\Omega}_n^*$ ($n \ge 1$) as the $m \times m$ matrix whose (i,j)th element represents the limiting probability that n or more batches have been served during an elapsed inter-arrival time period of the arrival process with the service process being in phase j at the end of nth batch service completion in the period, provided at the previous arrival epoch the service process was in phase i and the arrival lead the system to stay with n batches of customers. Then, from Markov renewal theory, we have

$$\mathbf{\Omega}_n^* = \lambda\int_0^\infty \widetilde{\mathbf{P}}(n,x)(1-A(x))\;\mathrm{d}x,\quad n\ge 1. \tag{2.13}$$

Using (2.11) in the above equation, we have

$$\mathbf{\Omega}_n^* = \lambda\int_0^\infty \left(\int_0^t \mathbf{P}(n-1,x)\mathbf{L}_1\mathrm{d}x\right)[1-A(t)]\mathrm{d}t,\quad n\ge 1. \tag{2.14}$$

Using (2.1) in (2.14), for $n=1$, we obtain

$$\begin{aligned}\mathbf{\Omega}_1^* &= \lambda\int_0^\infty \left(\int_0^t \mathrm{e}^{\mathbf{L}_0 x}\mathbf{L}_1\mathrm{d}x\right)[1-A(t)]\mathrm{d}t\\ &= \lambda\int_0^\infty \left((\mathbf{I}_m - \mathrm{e}^{\mathbf{L}_0 t}).(-\mathbf{L}_0)^{-1}\right)[1-A(t)]\mathrm{d}t\mathbf{L}_1\\ &= \lambda\int_0^\infty \left(\mathbf{I}_m - \mathrm{e}^{\mathbf{L}_0 t}\right)[1-A(t)]\mathrm{d}t\,(-\mathbf{L}_0)^{-1}\,\mathbf{L}_1. \end{aligned} \tag{2.15}$$

To evaluate $\int_0^\infty \left(\mathbf{I}_m - \mathrm{e}^{\mathbf{L}_0 t}\right)[1-A(t)]\mathrm{d}t$ we assume $\mathbf{X} = \int_0^\infty \left(\mathbf{I}_m - \mathrm{e}^{\mathbf{L}_0 t}\right)[1-A(t)]\mathrm{d}t$ and consider the following

$$\begin{aligned}\int_0^\infty \left(\mathbf{I}_m - \mathrm{e}^{\mathbf{L}_0 t}\right)\mathrm{d}A(t) &= -\int_0^\infty \left(\mathbf{I}_m - \mathrm{e}^{\mathbf{L}_0 t}\right)\mathrm{d}[1-A(t)]\\ &= -\int_0^\infty \mathrm{e}^{\mathbf{L}_0 t}[1-A(t)]\mathrm{d}t\cdot\mathbf{L}_0\\ &= \int_0^\infty \left(\mathbf{I}_m - \mathrm{e}^{\mathbf{L}_0 t}\right)[1-A(t)]\mathrm{d}t\cdot\mathbf{L}_0 - \int_0^\infty [1-A(t)]\mathrm{d}t\cdot\mathbf{L}_0\\ &= \mathbf{X}\cdot\mathbf{L}_0 - \frac{1}{\lambda}\cdot\mathbf{L}_0. \end{aligned} \tag{2.16}$$

The left-hand side of the above equation (2.16) can be written as

$$\int_0^\infty \left(\mathbf{I}_m - \mathrm{e}^{\mathbf{L}_0 t}\right)\mathrm{d}A(t) = \mathbf{I}_m - \mathbf{S}_0. \tag{2.17}$$

Comparing equations (2.16) and (2.17), we obtain

$$\mathbf{X} = \left(\mathbf{I}_m - \mathbf{S}_0 + \frac{1}{\lambda} \cdot \mathbf{L}_0\right) \cdot (\mathbf{L}_0)^{-1}. \tag{2.18}$$

Therefore, from equation (2.15), we get, using (2.18),

$$\begin{aligned} \mathbf{\Omega}_1^* &= \lambda \left(\mathbf{I}_m - \mathbf{S}_0 + \frac{1}{\lambda} \cdot \mathbf{L}_0\right) . (\mathbf{L}_0)^{-1} \cdot (-\mathbf{L}_0)^{-1} \mathbf{L}_1, \\ &= (\mathbf{I}_m - \mathbf{\Omega}_0) \cdot (-\mathbf{L}_0)^{-1} \mathbf{L}_1. \end{aligned} \tag{2.19}$$

Using (2.2) in (2.14), we have

$$\mathbf{\Omega}_{n+1}^* = \lambda \int_0^\infty \int_0^t \left(\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{P}(n,x) - \mathbf{P}(n-1,x)\mathbf{L}_1\right) (\mathbf{L}_0)^{-1} \mathbf{L}_1 \mathrm{d}x[1-A(t)]\mathrm{d}t, \quad n \geq 1,$$

which leads to

$$\mathbf{\Omega}_{n+1}^* = (\mathbf{\Omega}_n^* - \mathbf{\Omega}_n) (-\mathbf{L}_0)^{-1} \mathbf{L}_1, \quad n \geq 1. \tag{2.20}$$

For more information on $C\text{-}MSP$, readers are referred to Bocharov [5], Albores and Tajonar [1] and Gupta and Banik [28].

3. Analysis of the model

We consider a $GI/C\text{-}MSP^{(a,b)}/1/\infty$ queueing system as described above. Since we study the system in steady-state, we assume $\rho < 1$.

3.1. Queue-length distribution at pre-arrival epoch

We now consider the queue-length distribution just before an arrival epoch of a customer. Let the kth customer arrive at time instant τ_k, $k = 0, 1, \ldots$, with $\tau_0 = 0$, and let τ_k^- denote the pre-arrival epoch of a customer, *i.e.*, the time epoch just before the arrival instant τ_k. Then the state of the system at τ_k^- defined as $\zeta_k = \{N_{\tau_k^-}, J_{\tau_k^-}, \xi_{\tau_k^-}\}$ is a Markov chain, where $N_{\tau_k^-}$ represents the number of customers in the queue excluding the batch in service, $J_{\tau_k^-}$ the phase of the service process at the embedded point τ_k^- and $\xi_{\tau_k^-}$ the state of the server, *i.e.*, either busy ($\xi_{\tau_k^-} = 1$) or idle ($\xi_{\tau_k^-} = 0$). In the limiting case, let us define $\pi_{j,0}^-(n) = \lim_{k\to\infty} P(N_{\tau_k^-} = n, J_{\tau_k^-} = j, \xi_{\tau_k^-} = 0)$, $0 \leq n \leq a-1, 1 \leq j \leq m$, and $\pi_{j,1}^-(n) = \lim_{k\to\infty} P(N_{\tau_k^-} = n, J_{\tau_k^-} = j, \xi_{\tau_k^-} = 1)$, $n \geq 0, 1 \leq j \leq m$, where $\pi_{j,0}^-(n)$ represents the pre-arrival epoch probability of a customer when there are n $(< a)$ customers in the system and the server is idle in phase j. On the other hand, $\pi_{j,1}^-(n)$ denotes the pre-arrival epoch probability of an arrival when there are n (≥ 0) customers in the queue and the server is busy in phase j. Let $\boldsymbol{\pi}_l^-(n)$ $(l = 0, 1)$ be the $1 \times m$ vector whose jth component is $\pi_{j,l}^-(n)$. Observing the state of the system at consecutive imbedded points, we have an imbedded Markov chain whose state space is equivalent to $\Omega = \{(i, r, 0),\ 0 \leq i \leq a-1,\ 1 \leq r \leq m\} \bigcup \{(j, s, 1),\ j \geq 0,\ 1 \leq s \leq m\}$. The one step transition probability matrix (TPM), $\mathcal{P}$ of the above Markov chain can be constructed similar to the corresponding finite-buffer $GI/C\text{-}MSP^{(a,b)}/1$ queue; for details, see Banik *et al.* [4]. To obtain the vector-generating function (VGF) of the distribution of the number of customers in the system at pre-arrival epochs, we write $\boldsymbol{\pi}^- = \boldsymbol{\pi}^-\mathcal{P}$, where $\boldsymbol{\pi}^- = [\boldsymbol{\pi}_0^-(0), \boldsymbol{\pi}_0^-(1), \ldots, \boldsymbol{\pi}_0^-(a-1), \boldsymbol{\pi}_1^-(0),\ \boldsymbol{\pi}_1^-(1), \boldsymbol{\pi}_1^-(2), \ldots]$. Therefore, we have the following system of vector difference equations

$$\boldsymbol{\pi}_0^-(0) = \boldsymbol{\pi}_0^-(a-1)\mathbf{S}_1^* + \sum_{i=0}^{\infty}\sum_{j=i.b+a-1}^{(i+1)b-1}\boldsymbol{\pi}_1^-(j)\mathbf{S}_{i+2}^*, \tag{3.1}$$

$$\boldsymbol{\pi}_0^-(n) = \boldsymbol{\pi}_0^-(n-1) + \sum_{i=0}^{\infty}\boldsymbol{\pi}_1^-(i.b+n-1)\mathbf{S}_{i+1}^*, \quad 1 \le n \le a-1, \tag{3.2}$$

$$\boldsymbol{\pi}_1^-(0) = \boldsymbol{\pi}_0^-(a-1)\mathbf{S}_0 + \sum_{i=0}^{\infty}\sum_{j=i.b+a-1}^{(i+1)b-1}\boldsymbol{\pi}_1^-(j)\mathbf{S}_{i+1}, \tag{3.3}$$

$$\boldsymbol{\pi}_1^-(n) = \sum_{i=0}^{\infty}\boldsymbol{\pi}_1^-(i.b+n-1)\mathbf{S}_i, \quad n \ge 1. \tag{3.4}$$

Multiplying (3.4) by z^n and summing from $n=1$ to ∞, adding equation (3.3) and using the vector-generating function $\boldsymbol{\pi}_1^{-*}(z) = \sum_{n=0}^{\infty}\boldsymbol{\pi}_1^-(n)z^n$, we get

$$\boldsymbol{\pi}_1^{-*}(z)[\mathbf{I}_m - z\mathbf{S}(z^{-b})] = \boldsymbol{\pi}_0^-(a-1)\mathbf{S}_0 - \sum_{i=1}^{\infty}\sum_{r=0}^{i.b-1}\boldsymbol{\pi}_1^-(r)\mathbf{S}_i z^{-i.b+1+r}, \tag{3.5}$$

leading to

$$\boldsymbol{\pi}_1^{-*}(z) = \frac{\left(\boldsymbol{\pi}_0^-(a-1)\mathbf{S}_0 - \sum_{i=1}^{\infty}\sum_{r=0}^{i.b-1}\boldsymbol{\pi}_1^-(r)\mathbf{S}_i z^{-i.b+1+r}\right)\mathrm{Adj}[\mathbf{I}_m - z\mathbf{S}(z^{-b})]}{\det[\mathbf{I}_m - z\mathbf{S}(z^{-b})]}\cdot \tag{3.6}$$

For further analysis, we first determine an analytic expression for each component of $\boldsymbol{\pi}_1^{-*}(z)$. Each component $\pi_{j,1}^{-*}(z)$ defined as $\pi_{j,1}^{-*}(z) = \sum_{n=0}^{\infty}\pi_{j,1}^-(n)z^n$ of the VGF $\boldsymbol{\pi}_1^{-*}(z)$ given in (3.6) being convergent in $|z| \le 1$ implies that $\boldsymbol{\pi}_1^{-*}(z)$ is convergent in $|z| \le 1$. Since rational Laplace transforms cover a large class of distributions (see Botta *et al.* [8]), for further analysis we assume that the inter-arrival time distributions have a rational Laplace transforms. In view of this, each element of $\mathbf{S}(z^{-b})$ will be a rational function (for details, see Chaudhry *et al.* [18]) implying each element of $\det[\mathbf{I}_m - z\mathbf{S}(z^{-b})]$ will also be a rational function and thus we assume that

$$\det[\mathbf{I}_m - z\mathbf{S}(z^{-b})] = \frac{\mathrm{d}(z)}{\varphi(z)}\cdot$$

Equation (3.6) can be rewritten element-wise as

$$\pi_{j,1}^{-*}(z) = \frac{\xi_j(z)}{\mathrm{d}(z)}, \quad 1 \le j \le m, \tag{3.7}$$

where the jth component of $\left(\boldsymbol{\pi}_0^-(a-1)\mathbf{S}_0 - \sum_{i=1}^{\infty}\sum_{r=0}^{i.b-1}\boldsymbol{\pi}_1^-(r)\mathbf{S}_i z^{-i.b+1+r}\right)\mathrm{Adj}[\mathbf{I}_m - z\mathbf{S}(z^{-b})]\varphi(z)$ is represented by $\xi_j(z)$. To evaluate the vector in the numerator of equation (3.6), we show that the equation $\det[\mathbf{I}_m z - \mathbf{S}(z^b)] = 0$ has exactly m roots inside the unit circle $|z| = 1$, see Appendix A. Let these roots be γ_i $(1 \le i \le m)$. Now, consider the zeros of the function $\mathrm{d}(z)$. Since the equation $\det[\mathbf{I}_m z - \mathbf{S}(z^b)] = 0$ has m roots γ_i inside the unit circle, the function $\det[\mathbf{I}_m - z\mathbf{S}(z^{-b})]$ has m zeros $1/\gamma_i$ outside the unit circle. As $\pi_{j,1}^{-*}(z)$ is an analytic function of z for $|z| \le 1$, applying the partial-fraction method, we have

$$\pi_{j,1}^{-*}(z) = \sum_{i=1}^{m}\frac{k_{ij}}{1-\gamma_i z}, \; 1 \le j \le m, \tag{3.8}$$

where k_{ij} are constants to be determined. Now, collecting the coefficient of z^n from both sides of (3.8), we have

$$\pi^-_{j,1}(n) = \sum_{i=1}^{m} k_{ij}\gamma_i^n,\ 1 \le j \le m,\ n \ge 0. \tag{3.9}$$

Using (3.9) in (3.4) for $r = 1, 2, 3, \ldots, m-1$, we have

$$\left[\sum_{i=1}^{m} k_{i1}\gamma_i^r, \sum_{i=1}^{m} k_{i2}\gamma_i^r, \ldots, \sum_{i=1}^{m} k_{im}\gamma_i^r\right] = \sum_{n=0}^{\infty}\left[\sum_{i=1}^{m} k_{i1}\gamma_i^{nb+r-1}, \sum_{i=1}^{m} k_{i2}\gamma_i^{nb+r-1}, \ldots,\right.$$
$$\left.\times \sum_{i=1}^{m} k_{im}\gamma_i^{nb+r-1}\right]\mathbf{S}_n. \tag{3.10}$$

Equations (3.10) gives $m(m-1)$ simultaneous equations in m^2 unknowns, k_{ij}'s $(1 \le i \le m,\ 1 \le j \le m)$. We must use other m equations which can be obtained in the following way. Substituting the values of $\pi^-_{j,1}(n)$ $(n \ge 0)$ from equation (3.9) into equation (3.3), we obtain

$$\boldsymbol{\pi}_0^-(a-1) = \left(\left[\sum_{i=1}^{m} k_{i1}, \sum_{i=1}^{m} k_{i2}, \ldots, \sum_{i=1}^{m} k_{im}\right]\right.$$
$$\left. - \sum_{n=0}^{\infty}\sum_{j=n.b+a-1}^{(n+1)b-1}\left[\sum_{i=1}^{m} k_{i1}\gamma_i^j, \sum_{i=1}^{m} k_{i2}\gamma_i^j, \ldots, \sum_{i=1}^{m} k_{im}\gamma_i^j\right]\mathbf{S}_{n+1}\right)(\mathbf{S}_0)^{-1}, \tag{3.11}$$

After getting the elements of the vector $\boldsymbol{\pi}_0^-(a-1)$ in terms of the roots γ_i $(1 \le i \le m)$ and k_{ij}'s $(1 \le i \le m,\ 1 \le j \le m)$ from the above equation (3.11), we substitute $\pi^-_{j,1}(n)$ $(n \ge 0)$ from (3.9) and $\boldsymbol{\pi}_0^-(a-1)$ from (3.11) into equations (3.1) to (3.2). As a result of these substitutions we finally obtain each element of the vectors $\boldsymbol{\pi}_0^-(i)$ $(0 \le i \le a-1)$ in terms of the roots γ_i $(1 \le i \le m)$ and k_{ij}'s $(1 \le i \le m,\ 1 \le j \le m)$ as follows.

$$\boldsymbol{\pi}_0^-(0) = \boldsymbol{\pi}_0^-(a-1)\mathbf{S}_1^* + \sum_{l=0}^{\infty}\sum_{j=l.b+a-1}^{(l+1)b-1}\left[\sum_{i=1}^{m} k_{i1}\gamma_i^j, \sum_{i=1}^{m} k_{i2}\gamma_i^j, \ldots, \sum_{i=1}^{m} k_{im}\gamma_i^j\right]\mathbf{S}_{l+2}^*, \tag{3.12}$$

$$\boldsymbol{\pi}_0^-(n) = \boldsymbol{\pi}_0^-(n-1) + \sum_{l=0}^{\infty}\left[\sum_{i=1}^{m} k_{i1}\gamma_i^{l.b+n-1}, \sum_{i=1}^{m} k_{i2}\gamma_i^{l.b+n-1}, \ldots, \sum_{i=1}^{m} k_{im}\gamma_i^{l.b+n-1}\right]\mathbf{S}_{l+1}^*,$$
$$1 \le n \le a-1. \tag{3.13}$$

Finally, we equate the elements of the vector $\boldsymbol{\pi}_0^-(a-1)$ obtained from (3.13) for $n = a-1$ and (3.11), as a result of which we have another m equations. Therefore, we have a total of $m(m-1) + m = m^2$ equations with m^2 unknowns k_{ij}'s $(1 \le i \le m,\ 1 \le j \le m)$. One may note here that we ignore one component of equation (3.10) for $r = m-1$ which is a redundant equation and instead use the normalization condition given by

$$\sum_{j=1}^{m}\pi^{-*}_{j,1}(1) + \sum_{n=0}^{a-1}\sum_{j=1}^{m}\pi^-_{j,0}(n) = 1. \tag{3.14}$$

Above normalization condition can be simplified by putting $z = 1$ in equation (3.8) leading to

$$\sum_{j=1}^{m}\pi^{-*}_{j,1}(1) + \sum_{n=0}^{a-1}\sum_{j=1}^{m}\pi^-_{j,0}(n) = \sum_{j=1}^{m}\sum_{i=1}^{m}\frac{k_{ij}}{1-\gamma_i} + \sum_{n=0}^{a-1}\sum_{j=1}^{m}\pi^-_{j,0}(n) = 1. \tag{3.15}$$

Thus solving these m^2 equations, we get m^2 unknowns.

Remark 3.1. If $m = 1$, the above results (3.1)–(3.4) are similar to those for $GI/M^{(a,b)}/1$ queueing system which is discussed by Madill and Chaudhry [34].

3.1.1. Comparison of roots method with the matrix-geometric method

In the previous subsection we have used roots method to obtain pre-arrival epoch probabilities $\boldsymbol{\pi}_0^-(i)$ $(0 \leq i \leq a-1)$ and $\boldsymbol{\pi}_1^-(n)$ $(n \geq 0)$. One may be interested to use matrix-geometric method (MGM) to compute pre-arrival epoch probability vectors $\boldsymbol{\pi}_0^-(i)$ $(0 \leq i \leq a-1)$ and $\boldsymbol{\pi}_1^-(n)$ $(n \geq 0)$. In this subsection we discuss the procedure to calculate probability vectors at pre-arrival epoch using the MGM and also give computational complexities involved in both methods.

3.1.1.1. ***Determination of pre-arrival probability vectors using matrix-geometric method***

MGM is a popular method to solve a continuous-time Markov chain (CTMC) with a special block structure, see Neuts [39]. Steady-state Probability vectors at pre-arrival epochs for $GI/C\text{-}MSP^{(a,b)}/1/\infty$ queue can be obtained by the MGM. In case of the MGM, one has to determine the rate matrix $\mathbf{R}$ of the Markov chain described by the state space Ω in the previous subsection. As given by Neuts [39] and Banik [2] $\mathbf{R}$ is the minimal non-negative solution of the matrix equation as follows:

$$\mathbf{R} = \sum_{r=0}^{\infty} \mathbf{R}^{br} \mathbf{S}_r. \tag{3.16}$$

Pre-arrival probability vectors at server's busy state can be given by

$$\boldsymbol{\pi}_1^-(\eta + k) = \boldsymbol{\pi}_1^-(\eta)\mathbf{R}^k, \quad k \geq 0, \quad \eta > 0. \tag{3.17}$$

Pre-arrival probability vectors at server's idle state, *i.e.*, $\boldsymbol{\pi}_0^-(0)$, $\boldsymbol{\pi}_0^-(1)$, $\ldots$, $\boldsymbol{\pi}_0^-(a-1)$ and busy state, *i.e.*, $\boldsymbol{\pi}_1^-(0)$, $\boldsymbol{\pi}_1^-(0)$, $\cdots$, $\boldsymbol{\pi}_1^-(\eta)$ can be obtained from first $(a+\eta+1)$ components of the left-stationary vector of the stochastic matrix $\mathbf{B}[\mathbf{R}]$ of order $(a+\eta+1)m \times (a+\eta+1)m$. The $\mathbf{B}[\mathbf{R}]$ matrix can be obtained by truncating the TPM $\mathcal{P}$ at any of the server's busy state. Let $\mathbf{k}^* = \{(k,j,0), 0 \leq k \leq a-1, 1 \leq j \leq m\}$ be the idle state of the server and $\mathbf{n} = \{(n,j,1), n \geq 0, 1 \leq j \leq m\}$ be the busy state of the server. Now if we truncate $\mathcal{P}$ at server's busy state $\boldsymbol{\eta}$ then the $\mathbf{B}[\mathbf{R}]$ matrix can be given by

States	$\mathbf{0}^*$	$\mathbf{1}^*$	$\cdots$	$(\mathbf{a}-\mathbf{1})^*$	$\mathbf{0}$	$\mathbf{1}$	$\cdots$	$\mathbf{a}$	$\cdots$	$\mathbf{b}$	$\cdots$	$\mathbf{a}+\mathbf{b}$	$\cdots$	$\boldsymbol{\eta}$
$\mathbf{0}^*$	$\mathbf{0}$	$\mathbf{I}_m$	$\cdots$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$\mathbf{1}^*$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$\mathbf{2}^*$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(\mathbf{a}-\mathbf{2})^*$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{I}_m$	$\mathbf{0}$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$(\mathbf{a}-\mathbf{1})^*$	$\mathbf{S}_1^*$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\mathbf{S}_0$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{S}_1^*$	$\cdots$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{S}_0$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{a}-\mathbf{1}$	$\mathbf{S}_2^*$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\mathbf{S}_1$	$\mathbf{0}$	$\cdots$	$\mathbf{S}_0$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{b}-\mathbf{1}$	$\mathbf{S}_2^*$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\mathbf{S}_1$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{S}_0$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$\mathbf{b}$	$\mathbf{0}$	$\mathbf{S}_2^*$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{S}_1$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$		$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{a}+\mathbf{b}-\mathbf{1}$	$\mathbf{S}_3^*$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\mathbf{S}_2$	$\mathbf{0}$	$\cdots$	$\mathbf{S}_1$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{S}_0$	$\cdots$	$\mathbf{0}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathbf{2b}-\mathbf{1}$	$\mathbf{S}_3^*$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\mathbf{S}_2$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{S}_1$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$\mathbf{2b}$	$\mathbf{0}$	$\mathbf{S}_3^*$	$\cdots$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{S}_2$	$\cdots$	$\mathbf{0}$	$\cdots$	$\mathbf{0}$		$\mathbf{0}$	$\cdots$	$\mathbf{0}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\boldsymbol{\eta}$	$\mathbf{B}_0^{(\eta)}[\mathbf{R}]$	$\mathbf{B}_1^{(\eta)}[\mathbf{R}]$	$\cdots$	$\mathbf{B}_{a-1}^{(\eta)}[\mathbf{R}]$	$\mathbf{B}_a^{(\eta)}[\mathbf{R}]$	$\mathbf{B}_{a+1}^{(\eta)}[\mathbf{R}]$	$\cdots$	$\mathbf{B}_{2a}^{(\eta)}[\mathbf{R}]$	$\cdots$	$\mathbf{B}_{a+b}^{(\eta)}[\mathbf{R}]$	$\cdots$	$\mathbf{B}_{2a+b}^{(\eta)}[\mathbf{R}]$	$\cdots$	$\mathbf{B}_\eta^{(\eta)}[\mathbf{R}]$

.

We can express η as $b\nu + \beta$, where ν (≥ 0) is a non-negative integer and $1 \leq \beta \leq b$, then $\mathbf{B}_\iota^{(\eta)}[\mathbf{R}]$

$(\iota = 0, 1, \cdots \eta)$ are given by

$$\mathbf{B}_0^{(\eta)}[\mathbf{R}] = \begin{cases} \sum\limits_{i=\nu}^{\infty} \sum\limits_{j=bi+a-\beta}^{b(i+1)-\beta} \mathbf{R}^j \mathbf{S}^*_{i+2}, & 1 \le \beta \le a-1, \\ \sum\limits_{j=0}^{b-\beta} \mathbf{R}^j \mathbf{S}^*_{\nu+2} + \sum\limits_{i=\nu+1}^{\infty} \sum\limits_{j=bi+a-\beta}^{b(i+1)-\beta} \mathbf{R}^j \mathbf{S}^*_{i+2}, & a \le \beta \le b, \end{cases} \tag{3.18}$$

$$\mathbf{B}_k^{(\eta)}[\mathbf{R}] = \begin{cases} \sum\limits_{i=0}^{\infty} \mathbf{R}^{b(i+1)+k-\beta} \mathbf{S}^*_{i+\nu+2}, & 1 \le k \le \beta-1, \\ \sum\limits_{i=0}^{\infty} \mathbf{R}^{bi+k-\beta} \mathbf{S}^*_{i+\nu+1}, & \beta \le k \le a-1, \end{cases} \tag{3.19}$$

$$\mathbf{B}_a^{(\eta)}[\mathbf{R}] = \begin{cases} \sum\limits_{i=\nu}^{\infty} \sum\limits_{j=bi+a-\beta}^{b(i+1)-\beta} \mathbf{R}^j \mathbf{S}_{i+1}, & 1 \le \beta \le a-1, \\ \sum\limits_{j=0}^{b-\beta} \mathbf{R}^j \mathbf{S}_{\nu+1} + \sum\limits_{i=\nu+1}^{\infty} \sum\limits_{j=bi+a-\beta}^{b(i+1)-\beta} \mathbf{R}^j \mathbf{S}_{i+1}, & a \le \beta \le b, \end{cases} \tag{3.20}$$

$$\mathbf{B}_{a+k}^{(\eta)}[\mathbf{R}] = \begin{cases} \sum\limits_{i=0}^{\infty} \mathbf{R}^{b(i+1)+k-\beta} \mathbf{S}_{i+\nu+1}, & 1 \le k \le \beta-1, \\ \sum\limits_{i=0}^{\infty} \mathbf{R}^{bi+k-\beta} \mathbf{S}_{i+\nu}, & k \ge \beta. \end{cases} \tag{3.21}$$

It should be mentioned here that pre-arrival epoch probabilities calculated through the above procedure must be normalized using the normalizing condition $\sum\limits_{n=0}^{a-1} \boldsymbol{\pi}_0^-(n)\mathbf{e} + \sum\limits_{n=0}^{\infty} \boldsymbol{\pi}_1^-(n)\mathbf{e} = 1$.

3.1.1.2. ***Computational complexities of the roots method and the matrix-geometric method***

The main difference between the two methods is described below. The roots method requires the extraction of m roots (inside the unit circle $|z| = 1$) of the associated characteristic equation of the vector-generating function followed by the determination of m^2 unknowns by solving m^2 simultaneous linear equations. On the other hand MGM requires the derivation of the rate matrix $\mathbf{R}$ and then evaluation of the left-stationary vector of the stochastic matrix $\mathbf{B}[\mathbf{R}]$ of order $Nm \times Nm$ $(N > a)$.

In the roots method, the main goal is to calculate m roots inside the unit circle of the associated characteristic equation of the vector-generating function. That is, we require the zeros of the determinant of the matrix $[\mathbf{I}_m z - \mathbf{S}(z^b)]$ of order $m \times m$ inside the unit circle $|z| = 1$. It may be noted that if the entries of an $m \times m$ matrix are constant then the complexity of computing the determinant by Laplace expansion is $O(m!)$, whereas LU decomposition has complexity $O(m^3)$. Without loss of generality we can assume that m is the rank of the associated polynomial-matrix with polynomials entries of at most degree d. Mulders and Storjohann [36] proposed an algorithm to evaluate the determinant of a polynomial-matrix with complexity $O(m^3 d^2)$. After successful determination of the characteristic equation our next job is to extract m roots inside the unit circle $|z| = 1$. It should be mentioned here that we can numerically locate roots of a polynomial using Newton's method. Root extraction using Newton's method and multiplication of two numbers has equal complexity, see Borwein and Borwein [7]. That is, if we want to calculate roots of the characteristic equation up to d_1 digits accuracy, then the computational complexity of per root extraction will be at most $O(d_1^2)$. The last part is to evaluate m^2 unknowns by solving m^2 simultaneous linear equations. There are several methods to solve simultaneous equations, *e.g.*, Cramer's rule, Gaussian elimination, LU decomposition etc. One of the most efficient method to determine the m^2 unknowns is LU decomposition with complexity $O(m^6)$.

Essentially matrix-geometric method is based on the determination of the rate matrix. Evaluation of the rate matrix $\mathbf{R}$ follows an iterative procedure that is applied to equation (3.16). There are several methods to compute $\mathbf{R}$ iteratively such as successive substitution (SS) method, see Neuts [39] and Newton's iteration (NI) method, see Neuts [37] and Ramaswami [42]. Let Λ be the smallest index n such that any element of $\mathbf{S}_n$ $(n > \Lambda)$

is zero (numerically very small number). The SS algorithm converges linearly (that means it requires large number of iterations) with complexity $O(\Lambda m^3)$ per iteration, while the NI has complexity $O(m^6 + \Lambda m^4)$ per iteration but converges faster (that is to say it requires few iterations to converge). Pérez *et al.* [41] used Schur-decomposition method to the Newton's iteration to reduce the per iteration complexity from $O(m^6 + \Lambda m^4)$ to $O(\Lambda m^4)$. The left-stationary vector of the stochastic matrix $\mathbf{B}[\mathbf{R}]$ of order $Nm \times Nm$ $(N > a)$ can be computed by GTH (Grassmann, Taksar, Heyman) algorithm, see Grassmann *et al.* [27]. The GTH algorithm has complexity $O(m^4N^4)$, see Gelenbe and Lent [24].

3.2. Stationary distribution at arbitrary epoch

We now derive explicit expressions for the steady-state queue-length distribution $\boldsymbol{\pi}_l(n) = [\pi_{1,l}(n), \pi_{2,l}(n), \ldots, \pi_{m,l}(n)]$, $n \geq 0; l = 1$ or $0 \leq n \leq a-1; l = 0$, at an arbitrary epoch using the classical argument based on renewal theory which relates the steady-state queue-length distribution at an arbitrary epoch to that at the corresponding pre-arrival epoch. Using the results of Markov renewal theory and semi-Markov processes, see, *e.g.*, Çinlar [20] or Lucantoni and Neuts [32], we obtain

$$\boldsymbol{\pi}_0(0) = \boldsymbol{\pi}_0^-(a-1)\boldsymbol{\Omega}_1^* + \sum_{i=0}^{\infty} \sum_{j=i.b+a-1}^{(i+1).b-1} \boldsymbol{\pi}_1^-(j)\boldsymbol{\Omega}_{i+2}^*, \tag{3.22}$$

$$\boldsymbol{\pi}_0(n) = \boldsymbol{\pi}_0^-(n-1) + \sum_{i=0}^{\infty} \boldsymbol{\pi}_1^-(i.b+n-1)\boldsymbol{\Omega}_{i+1}^*, \quad 1 \leq n \leq a-1, \tag{3.23}$$

$$\boldsymbol{\pi}_1(0) = \boldsymbol{\pi}_0^-(a-1)\boldsymbol{\Omega}_0 + \sum_{i=0}^{\infty} \sum_{j=i.b+a-1}^{(i+1).b-1} \boldsymbol{\pi}_1^-(j)\boldsymbol{\Omega}_{i+1}, \tag{3.24}$$

$$\boldsymbol{\pi}_1(n) = \sum_{i=0}^{\infty} \boldsymbol{\pi}^-(i.b+n-1)\boldsymbol{\Omega}_i, \quad n \geq 1. \tag{3.25}$$

Note that since the service process is interrupted during periods in which the system is empty or system has less than or equal to $a-1$ customers, it follows that

$$\sum_{n=0}^{\infty} \boldsymbol{\pi}_1(n)\,(\mathbf{L}_0 + \mathbf{L}_1) = \mathbf{0}, \tag{3.26}$$

which in turn implies that

$$\sum_{n=0}^{\infty} \boldsymbol{\pi}_1(n) = C\overline{\boldsymbol{\pi}}, \tag{3.27}$$

for some positive constant C, since $\overline{\boldsymbol{\pi}}(\mathbf{L}_0 + \mathbf{L}_1) = \mathbf{0}$. Thus, by post multiplying the members of the previous equation by $\mathbf{e}$, we conclude that

$$\sum_{n=0}^{\infty} \boldsymbol{\pi}_1(n)\mathbf{e} = C. \tag{3.28}$$

Therefore, using (3.28) in (3.27) we obtain

$$\frac{1}{(\sum_{n=0}^{\infty} \boldsymbol{\pi}_1(n)\mathbf{e})} \sum_{n=0}^{\infty} \boldsymbol{\pi}_1(n) = \overline{\boldsymbol{\pi}}, \tag{3.29}$$

$$i.e., \ \frac{1}{\rho'} \sum_{n=0}^{\infty} \boldsymbol{\pi}_1(n) = \overline{\boldsymbol{\pi}}, \tag{3.30}$$

where $\rho' = \sum_{n=0}^{\infty} \boldsymbol{\pi}_1(n)\mathbf{e}$ represents the probability that the server is busy. The above result (3.29) is useful while performing numerical calculations.

3.3. Queue-length distribution at post-departure epoch of a batch and its relation with pre-service epoch

In this subsection, we derive the probabilities for the states of the system immediately after a service completion takes place. Let $\boldsymbol{\pi}_l^{+}(n) = [\pi_{1,l}^{+}(n), \pi_{2,l}^{+}(n), \ldots, \pi_{m,l}^{+}(n)],\ n \geq 0; l = 1$ or $0 \leq n \leq a-1; l = 0$, be the $1 \times m$ vector whose ith component $\pi_{i,l}^{+}(n)$ represents the post-departure epoch probability that there are n customers in the queue immediately after a service completion of a batch and the server is busy $(l = 1)$ in phase i or the server becomes idle $(l = 0)$ in phase i. The post-departure epoch thus occurs immediately after the server has either reduced the queue or become idle. Hence, using level-crossing arguments given in ([15], p. 299), we have

$$\boldsymbol{\pi}_0^{+}(n) = \frac{1}{\mu^* \rho'} \boldsymbol{\pi}_1(n) \mathbf{L}_1, \quad 0 \leq n \leq a-1, \tag{3.31}$$

$$\boldsymbol{\pi}_1^{+}(0) = \frac{1}{\mu^* \rho'} \sum_{n=a}^{b} \boldsymbol{\pi}_1(n) \mathbf{L}_1, \tag{3.32}$$

$$\boldsymbol{\pi}_1^{+}(n) = \frac{1}{\mu^* \rho'} \boldsymbol{\pi}_1(n+b) \mathbf{L}_1, \quad n \geq 1. \tag{3.33}$$

It may be noted that $\sum_{n=0}^{\infty} \boldsymbol{\pi}_1(n) \mathbf{L}_1 \mathbf{e} = \mu^* \rho'$, which represents the steady-state batch departure rate.

Let $\boldsymbol{\pi}_l^{s-}(n) = [\pi_{1,l}^{s-}(n), \pi_{2,l}^{s-}(n), \ldots, \pi_{m,l}^{s-}(n)],\ n \geq 0; l = 1$ or $0 \leq n \leq a-1; l = 0$, be the $1 \times m$ vector whose ith component $\pi_{i,l}^{s-}(n)$ represents the pre-service epoch probability that there are n customers in the queue immediately before a service of a batch takes place and the server is busy $(l = 1)$ or idle $(l = 0)$ in phase i. It is apparent that $\pi_{i,0}^{s-}(n) = 0$ for any i $(1 \leq i \leq m)$ and $0 \leq n \leq a-1$. The argument to find post-departure epoch probabilities may be based on the distribution of the probabilities for the system at pre-service epoch of a batch, the instant in time immediately before a real service of a batch starts. Using the above arguments, we obtain the following results.

$$\boldsymbol{\pi}_1^{s-}(n) = \boldsymbol{\pi}_0^{+}(n), \quad 0 \leq n \leq a-1, \tag{3.34}$$

$$\sum_{n=a}^{b} \boldsymbol{\pi}_1^{s-}(n) = \boldsymbol{\pi}_1^{+}(0), \tag{3.35}$$

$$\boldsymbol{\pi}_1^{s-}(n+b) = \boldsymbol{\pi}_1^{+}(n), \quad n \geq 1. \tag{3.36}$$

4. Service batch size distribution

In case of general bulk service rule, we can determine the distribution of batch size in service. Let the batch or group size in service at a given epoch be denoted by G_i^c, where c represents the epoch under consideration and i $(1 \leq i \leq m)$ is the phase of the service process. Further, let the probability that the server is busy with a group of size j and the phase of the service process is i be denoted by $g_i^c(j) = P(G_i^c = j)$. It may be noted that $g_i^c(0)$ is the probability that the server will be serving a group of size zero or a potential batch. Let $\mathbf{g}^c(j)$ denote $1 \times m$ vector whose ith component is $g_i^c(j)$. To be consistent $\mathbf{g}^c(0)$ must be equal to $\sum_{i=0}^{a-1} \boldsymbol{\pi}_0^c(i)$. This can be validated for each of the epochs considered above. The other trivial result is that $\mathbf{g}^c(a) = \sum_{n=0}^{\infty} \boldsymbol{\pi}_1^c(n)$ if $a = b$. The other cases for $\mathbf{g}^c(j)$ need to be solved and they must obey $\sum_{j=a}^{b} \mathbf{g}^c(j) = \sum_{n=0}^{\infty} \boldsymbol{\pi}_1^c(n)$. Let us further define P_B^c and P_I^c as being the probability that the server is busy and idle, respectively, in the corresponding epoch "c". Then, we obtain from the above argument $\sum_{j=a}^{b} \mathbf{g}^c(j)\mathbf{e} = \sum_{n=0}^{\infty} \boldsymbol{\pi}_1^c(n)\mathbf{e} = P_B^c$ and $\mathbf{g}^c(0)\mathbf{e} = \sum_{i=0}^{a-1} \boldsymbol{\pi}_0^c(i)\mathbf{e} = P_I^c$.

The approach adopted to solve for $\mathbf{g}^c(j)$ $(a \leq j \leq b)$ is to enumerate the mutually exclusive cases which can result in a group of size j in service at the corresponding epoch and then weigh these by the probability of their occurrence. Once this distribution has been found, the mean group size in service at a given epoch, M_1^c, and the second moment of the group size in service, M_2^c, can be calculated using

$$M_1^c = \sum_{j=a}^{b} j\mathbf{g}^c(j)\mathbf{e},\ M_2^c = \sum_{j=a}^{b} j^2\mathbf{g}^c(j)\mathbf{e}. \tag{4.1}$$

4.1. Service batch size distribution at pre-arrival epoch

The pre-arrival epoch results must be found before the arbitrary epoch results as they occur as part of the latter. Considering, in steady state, all possible mutually exclusive events which can occur between the "current" pre-arrival epoch and the subsequent which can result in a batch of size j being in service at the subsequent pre-arrival epoch, it is found that

$$\mathbf{g}^-(a) = \boldsymbol{\pi}_0^-(a-1)\mathbf{S}_0 + \sum_{n=0}^{\infty}\boldsymbol{\pi}_1^-(n.b+a-1)\mathbf{S}_{n+1} + \mathbf{g}^-(a)\mathbf{S}_0, \quad a \neq b, \tag{4.2}$$

$$\mathbf{g}^-(j) = \sum_{n=0}^{\infty}\boldsymbol{\pi}_1^-(n.b+j-1)\mathbf{S}_{n+1} + \mathbf{g}^-(j)\mathbf{S}_0, \quad a < j < b, \tag{4.3}$$

$$\mathbf{g}^-(b) = \sum_{n=1}^{\infty}\sum_{i=0}^{b-1}\boldsymbol{\pi}_1^-(n.b+i-1)\left(\sum_{j=1}^{n}\mathbf{S}_j\right) + \mathbf{g}^-(b)\mathbf{S}_0, \quad b \neq a. \tag{4.4}$$

It may be noted here that $\sum_{j=a}^{b}\mathbf{g}^-(j)\mathbf{e} = \sum_{n=0}^{\infty}\boldsymbol{\pi}_1^-(n)\mathbf{e} = P_B^-$ and $\mathbf{g}^-(0)\mathbf{e} = \sum_{i=0}^{a-1}\boldsymbol{\pi}_0^-(i)\mathbf{e} = P_I^-$.

4.2. Service batch size distribution at arbitrary epoch

To derive the distribution of the batch size in service at a random epoch, the same argument which was used to obtain the arbitrary epoch probabilities from pre-arrival epoch probabilities is applied here. That is, to find the probabilities for the group size in service at the end of an elapsed inter-arrival time since the last arrival, it is sufficient to change the time frame being considered by using the probabilities appropriate to the elapsed inter-arrival time, rather than the inter-arrival time period. The expressions of these probabilities are given below.

$$\mathbf{g}(a) = \boldsymbol{\pi}_0^-(a-1)\boldsymbol{\Omega}_0 + \sum_{n=0}^{\infty}\boldsymbol{\pi}_1^-(n.b+a-1)\boldsymbol{\Omega}_{n+1} + \mathbf{g}^-(a)\boldsymbol{\Omega}_0, \quad a \neq b, \tag{4.5}$$

$$\mathbf{g}(j) = \sum_{n=0}^{\infty}\boldsymbol{\pi}_1^-(n.b+j-1)\boldsymbol{\Omega}_{n+1} + \mathbf{g}^-(j)\boldsymbol{\Omega}_0, \quad a < j < b, \tag{4.6}$$

$$\mathbf{g}(b) = \sum_{n=1}^{\infty}\sum_{i=0}^{b-1}\boldsymbol{\pi}_1^-(n.b+i-1)\left(\sum_{j=1}^{n}\boldsymbol{\Omega}_j\right) + \mathbf{g}^-(b)\boldsymbol{\Omega}_0, \quad b \neq a. \tag{4.7}$$

One may note here that the mean group size in service at a random epoch, $M_1 = \sum_{j=a}^{b} j\mathbf{g}(j)\mathbf{e}$, should be equal to λ/μ^*. This serves as a good check while performing numerical computations. Also one may note here that $\sum_{j=a}^{b}\mathbf{g}(j)\mathbf{e} = \sum_{n=0}^{\infty}\boldsymbol{\pi}_1(n)\mathbf{e} = P_B = \rho'$ and $\mathbf{g}(0)\mathbf{e} = \sum_{j=0}^{a-1}\boldsymbol{\pi}_0(j)\mathbf{e} = P_I = 1-\rho'$.

4.3. Service batch size distribution at post-departure epoch

The derivation of the probability distribution for the group size in service following a departure, $\mathbf{g}^+(j)$, is straightforward compared to either the pre-arrival epoch or the random epoch case. In fact, it is evident that the following relationships between this distribution and the pre-service epoch probabilities ($\boldsymbol{\pi}_1^{s-}(j)$) must hold

$$\mathbf{g}^{+}(0) = \sum_{n=0}^{a-1} \boldsymbol{\pi}_1^{s-}(n), \tag{4.8}$$

$$\mathbf{g}^{+}(j) = \boldsymbol{\pi}_1^{s-}(j), \quad a \le j < b, \tag{4.9}$$

$$\mathbf{g}^{+}(b) = \sum_{n=b}^{\infty} \boldsymbol{\pi}_1^{s-}(n), \quad b \ge a. \tag{4.10}$$

4.4. Another conditional distribution and conservation of rate considerations

As the setting of the quorum is a management decision to promote the efficiency of the queueing system under consideration, one may be interested in the distribution of the numbers in the batch in service. It is apparent that the probability that an individual joining the queue is the rth $(1 \le r \le b)$ member of his batch $(\mathbf{p}^{-}(r))$, is identical to the probability of arrival finding $(h.b + r - 1),\ h \ge 0$, customers in queue. Thus, using vector notations as above we obtain this distribution as

$$\mathbf{p}^{-}(r) = \begin{cases} \boldsymbol{\pi}_0^{-}(r-1) + \sum_{n=0}^{\infty} \boldsymbol{\pi}_1^{-}(n.b + r - 1), & 1 \le r \le a, \\ \sum_{n=0}^{\infty} \boldsymbol{\pi}_1^{-}(n.b + r - 1), & a \le r \le b. \end{cases} \tag{4.11}$$

The rate conservation principle states that in stationary or steady-state the frequency of entry into a service channel equals the frequency of departure from this channel. This result must hold true for rth customer of a batch as derived above. The stationary departure rate for the rth customer is given by $\sum_{j=(a,r)^+}^{b} \mathbf{g}(j)\mathbf{L}_1\mathbf{e}$ and stationary arrival rate of rth customer equals $\lambda \mathbf{p}^{-}(r)\mathbf{e}$, where $(a,r)^+$ denotes the maximum of a or r. The above argument implies

$$\sum_{j=(a,r)^+}^{b} \mathbf{g}(j)\mathbf{L}_1\mathbf{e} = \lambda \mathbf{p}^{-}(r)\mathbf{e}. \tag{4.12}$$

5. Performance measures

As state probabilities at various epochs are known, performance measures can be easily obtained. The average number of customers in queue at an arbitrary epoch is given by

$$L_q = \sum_{i=1}^{a-1} i\boldsymbol{\pi}_0(i)\mathbf{e} + \sum_{n=1}^{\infty} n\boldsymbol{\pi}_1(n)\mathbf{e}. \tag{5.1}$$

The second moment of queue length, M_q, which is required if the variance of queue-length is to be calculated, is expressed similarly as

$$M_q = \sum_{i=1}^{a-1} i^2\boldsymbol{\pi}_0(i)\mathbf{e} + \sum_{n=1}^{\infty} n^2\boldsymbol{\pi}_1(n)\mathbf{e}. \tag{5.2}$$

Since it is difficult to directly find the system-length distribution that includes the size of a batch in service at a given epoch, it is derived later (see below) using the queue-length distributions. However, one can immediately obtain average system-length at an arbitrary epoch as follows. Let $\mathbf{y}_l(n) = [y_{1,l}(n), y_{2,l}(n), \ldots, y_{m,l}(n)],\ n \ge a; l = 1$ or $0 \le n \le a-1; l = 0$, be the $1 \times m$ vector whose ith component $y_{i,l}(n)$ represents the arbitrary epoch probability that there are n customers in the system (including the size of a batch in service) and the Markov service process is in phase i with server busy ($l = 1$) or idle ($l = 0$). Then, the vector generating function of $\mathbf{y}_l(n)$ may be defined by $\mathbf{y}(z) = \sum_{n=0}^{a-1} \mathbf{y}_0(n)z^n + \sum_{n=a}^{\infty} \mathbf{y}_1(n)z^n$. The expected value of the system-length distribution L_s may be obtained by

$$L_s = \frac{\mathrm{d}}{\mathrm{d}z}\mathbf{y}(z) \mid_{z=1} \mathbf{e} = \sum_{i=1}^{m} \frac{\mathrm{d}}{\mathrm{d}z} y_i(z) \mid_{z=1} = L_q + M_1, \tag{5.3}$$

where M_1 is obtained from (4.5)–(4.7), see the line below (4.7). Using the Little's law, we can also get mean waiting time of a customer in the queue W_q as $W_q = \frac{L_q}{\lambda}$. The expected virtual sojourn time or the waiting time in the system will be given by $W_s = W_q + \frac{1}{\mu^*}$. One may also use the Little's law to obtain W_s which is given by $W_s = \frac{L_s}{\lambda}$. In the following we present a procedure to derive system-length distribution in a slightly different way than above.

5.1. System-length distributions at various epochs

We need to derive system-length distribution when the server is busy as queue-length and system-length distributions become the same when the server is idle. Let $\Gamma_{i,j,s}^c$ denote the probability that the server is busy with a group of size j $(a \le j \le b)$ and the phase of the service process is s $(1 \le s \le m)$ with i $(i \ge 0)$ customers waiting in the queue at any given epoch "c". Also, let $\mathbf{\Gamma}_{i,j}^c$ denote the corresponding row vector whose sth component is $\Gamma_{i,j,s}^c$. Now, at any given epoch, the following relationships hold.

$$\sum_{j=a}^{b} \mathbf{\Gamma}_{i,j}^c = \boldsymbol{\pi}_1^c(i), \quad i \ge 0, \tag{5.4}$$

$$\sum_{i=0}^{\infty} \mathbf{\Gamma}_{i,j}^c = \mathbf{g}^c(j), \quad a \le j \le b. \tag{5.5}$$

After this, inspecting the number of customers in system at two consecutive pre-arrival epochs, we have the following relationships:

$$\begin{aligned}
\mathbf{\Gamma}_{0,a}^- &= \boldsymbol{\pi}_0^-(a-1)\mathbf{S}_0 + \sum_{i=0}^{\infty}\sum_{j=a}^{b} \mathbf{\Gamma}_{i.b+a-1,j}^- \mathbf{S}_{i+1} \\
&= \boldsymbol{\pi}_0^-(a-1)\mathbf{S}_0 + \sum_{i=0}^{\infty} \boldsymbol{\pi}_1^-(i.b+a-1)\mathbf{S}_{i+1}, &(5.6)\\
\mathbf{\Gamma}_{0,j}^- &= \sum_{i=0}^{\infty} \boldsymbol{\pi}_1^-(i.b+j-1)\mathbf{S}_{i+1}, \quad a+1 \le j \le b, &(5.7)\\
\mathbf{\Gamma}_{i,j}^- &= \mathbf{\Gamma}_{i-1,j}^-\mathbf{S}_0, \quad a \le j \le b-1,\ i \ge 1, &(5.8)\\
\mathbf{\Gamma}_{n,b}^- &= \mathbf{\Gamma}_{n-1,b}^-\mathbf{S}_0 + \sum_{i=1}^{\infty} \boldsymbol{\pi}_1^-(i.b+n-1)\mathbf{S}_i, \quad n \ge 1. &(5.9)
\end{aligned}$$

Similarly, we can derive system-length distribution at an arbitrary epoch along the lines proposed in Section 3.2, and we have the following relationships:

$$\begin{aligned}\mathbf{\Gamma}_{0,a} &= \boldsymbol{\pi}_0^-(a-1)\mathbf{\Omega}_0 + \sum_{i=0}^{\infty}\sum_{j=a}^{b}\mathbf{\Gamma}^-_{i.b+a-1,j}\mathbf{\Omega}_{i+1}\\ &= \boldsymbol{\pi}_0^-(a-1)\mathbf{\Omega}_0 + \sum_{i=0}^{\infty}\boldsymbol{\pi}_1^-(i.b+a-1)\mathbf{\Omega}_{i+1},\end{aligned} \tag{5.10}$$

$$\mathbf{\Gamma}_{0,j} = \sum_{i=0}^{\infty}\boldsymbol{\pi}_1^-(i.b+j-1)\mathbf{\Omega}_{i+1}, \quad a+1 \le j \le b, \tag{5.11}$$

$$\mathbf{\Gamma}_{i,j} = \mathbf{\Gamma}^-_{i-1,j}\mathbf{\Omega}_0, \quad a \le j \le b-1,\ i \ge 1, \tag{5.12}$$

$$\mathbf{\Gamma}_{n,b} = \mathbf{\Gamma}^-_{n-1,b}\mathbf{\Omega}_0 + \sum_{i=1}^{\infty}\boldsymbol{\pi}_1^-(i.b+n-1)\mathbf{\Omega}_i, \quad n \ge 1. \tag{5.13}$$

One may verify that equations (5.4) and (5.5) are satisfied. This can be checked by using the above equations (5.6)–(5.13) along with the previous equations (3.3)–(3.4), (3.24)–(3.25) and (4.2)–(4.7).

5.2. Expected length of busy and idle periods

Since for this system, in the limiting case, the fractions of time the server is busy and idle are $\rho^{'}$ and $1-\rho^{'}$, respectively, we have

$$\frac{E(B)}{E(I)} = \frac{\rho^{'}}{1-\rho^{'}}, \tag{5.14}$$

where B and I are random variables denoting the lengths of busy and idle periods, respectively. We first discuss the mean busy period $E(B)$, which is comparatively easy to evaluate. Let at time t, $N_q(t)$ denote the number of customers in queue (excluding the size of a batch in service in the case of busy server) and $\xi_q(t)$ be the state of the server, *i.e.*, busy $(=1)$ or idle $(=0)$. The state $\{N_q(t), \xi_q(t)\}$ enters the set of idle states, $\Upsilon \equiv \{(0,0),(1,0),(2,0),\ldots,(a-1,0)\}$ at the termination of a busy period. Let us denote the busy states by $\Delta \equiv \{(0,1),(1,1),(2,1),\ldots\}$. The conditional probability that $\{N_q(t), \xi_q(t)\}$ enters state $(i,0)$ given that $\{N_q(t), \xi_q(t)\}$ enters Υ, is therefore $D\boldsymbol{\pi}_0^+(i)\mathbf{e}$, $0 \le i \le a-1$. Then one can evaluate $D = \frac{1}{\sum_{i=0}^{a-1}\boldsymbol{\pi}_0^+(i)\mathbf{e}}\cdot$ Now $\{N_q(t), \xi_q(t)\}$ enters busy states or idle states irrespective of customers' arrival during a service time of a batch, which may happen in expected time $E(S)$, where $E(S) = 1/\mu^*$. Thus

$$E(B) = \frac{\left(\sum_{i=0}^{\infty}\boldsymbol{\pi}_1^+(i)\mathbf{e} + \sum_{j=0}^{a-1}\boldsymbol{\pi}_0^+(j)\mathbf{e}\right).E(S)}{\sum_{i=0}^{a-1}\boldsymbol{\pi}_0^+(i)\mathbf{e}} = \frac{E(S)}{\sum_{i=0}^{a-1}\boldsymbol{\pi}_0^+(i)\mathbf{e}}\cdot \tag{5.15}$$

Substituting $E(B)$ from equation (5.15) in equation (5.14), we obtain

$$E(I) = \frac{1-\rho^{'}}{\rho^{'}}.\frac{E(S)}{\sum_{i=0}^{a-1}\boldsymbol{\pi}_0^+(i)\mathbf{e}}\cdot \tag{5.16}$$

6. Approximation for queue-length distributions based on one root

One can get several approximate results such as the tail probabilities of the queue-length distribution at pre-arrival epoch, heavy- or light-traffic behaviours of the queue-length distributions based on the real root of $\det[\mathbf{I}_m - z\mathbf{S}(z^{-b})] = 0$ which is closest to 1 and outside $|z| \le 1$. The existence of such a root has been discovered since long time in the literature, *e.g.*, Feller ([22], pp. 276–277) calculated the tail probabilities using a single root of the denominator which is the smallest root in absolute value. Also Chaudhry *et al.* [16] have given a formal proof of the existence of such a root. One may note that it is not difficult to calculate this root numerically. In this context it is worth to mention that sometimes an approximate value of this root may be used to get desired queue-length distribution and this approximate value may be obtained in the following way. We investigate the approximate root inside $|z| \leq 1$ by expanding the matrix of the left-hand side of the characteristic equation $\det[\mathbf{I}_m z - \mathbf{S}(z^b)] = 0$ in powers of ρ as

$$\begin{aligned}\mathbf{I}_m z - \mathbf{S}(z^b) &= \mathbf{I}_m z - \mathbf{S}(\rho + z^b - \rho)\\ &= \mathbf{I}_m z - \mathbf{S}(z^b - \rho) - \rho\mathbf{S}^{'}(z^b - \rho) - \frac{\rho^2}{2!}\mathbf{S}^{''}(z^b - \rho) + o(\rho^2)\overline{\mathbf{H}},\end{aligned} \tag{6.1}$$

where $\overline{\mathbf{H}}$ is some unknown matrix and $\mathbf{S}^{'}(.)$, $\mathbf{S}^{''}(.)$ are the successive differentiation of $\mathbf{S}(.)$ of order one and two, respectively. Also one may note that in equation (6.1), as usual, $o(x)$ represents a function of x with the property that $\frac{o(x)}{x} \to 0$ as $x \to 0$. Multiplying the right-hand side of the above equation (6.1) by the vector $\overline{\boldsymbol{\pi}}$ from left and the vector $\mathbf{e}$ from right, we may write the characteristic equation as follows

$$z - \overline{\boldsymbol{\pi}}\mathbf{S}(z^b - \rho)\mathbf{e} - \rho\overline{\boldsymbol{\pi}}\mathbf{S}^{'}(z^b - \rho)\mathbf{e} - \frac{\rho^2}{2!}\overline{\boldsymbol{\pi}}\mathbf{S}^{''}(z^b - \rho)\mathbf{e} + o(\rho^2)\overline{c} = 0. \tag{6.2}$$

where $\overline{c} = \overline{\boldsymbol{\pi}}\overline{\mathbf{H}}\mathbf{e}$ and is a finite constant. Now an approximate value of the root for the above described three cases is obtained as follows.

- Light-traffic case: Applying $\rho \downarrow 0$ in equation (6.2) gives

$$z - \overline{\boldsymbol{\pi}}\mathbf{S}(z^b - \rho)\mathbf{e} - \rho\overline{\boldsymbol{\pi}}\mathbf{S}^{'}(z^b - \rho)\mathbf{e} - \frac{\rho^2}{2!}\overline{\boldsymbol{\pi}}\mathbf{S}^{''}(z^b - \rho)\mathbf{e} = 0, \tag{6.3}$$

  which gives an approximate value of this one root.
- Heavy-traffic case: Replacing ρ by $(1 - \rho)$ in equation (6.2) and applying the condition $\rho \uparrow 1$, we obtain

$$z - \overline{\boldsymbol{\pi}}\mathbf{S}(z^b - 1 + \rho)\mathbf{e} - (1 - \rho)\overline{\boldsymbol{\pi}}\mathbf{S}^{'}(z^b - 1 + \rho)\mathbf{e} - \frac{(1 - \rho)^2}{2!}\overline{\boldsymbol{\pi}}\mathbf{S}^{''}(z^b - 1 + \rho)\mathbf{e} = 0. \tag{6.4}$$

  Solving (6.4) we get the desired value.
- Tail probabilities at a pre-arrival epoch: If ρ_1 denotes any arbitrary offered load numerically very close to ρ then we replace ρ by $(\rho - \rho_1)$ in equation (6.2) and applying the condition $\rho \to \rho_1$ to get

$$z - \overline{\boldsymbol{\pi}}\mathbf{S}(z^b - \rho + \rho_1)\mathbf{e} - (\rho - \rho_1)\overline{\boldsymbol{\pi}}\mathbf{S}^{'}(z^b - \rho + \rho_1)\mathbf{e} - \frac{(\rho - \rho_1)^2}{2!}\overline{\boldsymbol{\pi}}\mathbf{S}^{''}(z^b - \rho + \rho_1)\mathbf{e} = 0. \tag{6.5}$$

  Hence, we can obtain the desired root, say z_1, by solving equation (6.5) for z.

Finally, it may be noted that once we obtain an approximate value of a root, we can obtain the exact root through various numerical methods. The equation $\det[\mathbf{I}_m z - \mathbf{S}(z^b)] = 0$ may be used to find the original root which is closest to 1 in all the above described cases. For the sake of completeness we present below the procedure to calculate tail probabilities at pre-arrival epoch based on this one root. To get the tail probabilities, assume

$$\pi^{-}_{j,1}(n) \simeq k_{1,j} z_1^n = p^{a1}_{j,1}(n), \quad n > n_{\overline{\epsilon}},\ 1 \leq j \leq m, \tag{6.6}$$

where $n_{\overline{\epsilon}}$ is chosen as the smallest integer such that $|(\pi^{-}_{j,1}(n) - p^{a1}_{j,1}(n))/\pi^{-}_{j,1}(n)| < \overline{\epsilon}$, *i.e.*, $|1 - \frac{p^{a1}_{j,1}(n)}{\pi^{-}_{j,1}(n)}| < \overline{\epsilon}$, $\overline{\epsilon} > 0$. But, since the probability $p^{a1}_{j,1}(n)$ follows a geometric distribution with common ratio z_1, it is better to choose $n_{\overline{\epsilon}}$ such that $|\frac{\pi^{-}_{j,1}(n)}{z_1\pi^{-}_{j,1}(n-1)} - 1| < \overline{\epsilon}$. The approximation gets better if more than one root, in ascending order of magnitude, is used. It should however be mentioned that those roots that occur in complex-conjugate pairs should be used in pairs. Thus, the tail probabilities using three roots can be approximated by

$$\pi^{-}_{j,1}(n) \simeq \sum_{i=1}^{3} k_{i,j} z_i^n = p^{a3}_{j,1}(n), \quad n > n^1_{\epsilon},\ 1 \le j \le m, \tag{6.7}$$

where z_i $(i = 1, 2, 3)$ are the roots in ascending order of magnitude and n^1_{ϵ} may be chosen by $|1 - \frac{p^{a3}_{j,1}(n)}{\pi^{-}_{j,1}(n)}| < \overline{\epsilon},\ n > n^1_{\epsilon}$. Similar procedure may be adopted to calculate queue-length distributions for the cases of light and heavy traffics.

Remark 6.1. It may be remarked that this root can also be obtained accurately by simply using high precision of the software packages mentioned earlier.

7. Numerical results and discussion

To demonstrate the applicability of the results obtained in the previous sections, some numerical results have been presented in two self explanatory tables. At the bottom of the tables, several performance measures are given. Since various distributions can be either represented or approximated by PH-distribution, we take inter-arrival time distribution to be of PH-type having the representation $(\boldsymbol{\alpha}, \mathbf{T})$, where $\boldsymbol{\alpha}$ and $\mathbf{T}$ are of dimension ν. Then $\mathbf{S}(z)$ can be derived as follows using the procedure adopted in Chaudhry *et al.* [18].

$$\mathbf{S}(z) = (\mathbf{I}_m \otimes \alpha)(\mathbf{L}(z) \oplus \mathbf{T})^{-1}(\mathbf{I}_m \otimes \mathbf{T}\mathbf{e}_{\nu})$$

with $\mathbf{L}(z) \oplus \mathbf{T} = (\mathbf{L}(z) \otimes \mathbf{I}_{\nu}) + (\mathbf{I}_m \otimes \mathbf{T})$, where $\oplus$ and $\otimes$ are used for Kronecker product and sum, respectively. For the derivation of $\mathbf{S}(z)$, see Chaudhry *et al.* [18]. Knowing that each element of $\mathbf{L}(z)$ is a polynomial in z, each element of $\mathbf{L}(z) \oplus \mathbf{T}$ is also a polynomial in z and hence the determinant of $(\mathbf{L}(z) \oplus \mathbf{T})^{-1}$ is a rational function in z. Thus, from the above expression for $\mathbf{S}(z)$, we can immediately say that each element of $\mathbf{S}(z)$ is a rational function in z with the same denominator.

We have carried out extensive numerical work based on the procedure discussed in this paper by considering different service matrices $C\text{-}MSP$ $(\mathbf{L}_0, \mathbf{L}_1)$ and phase-type inter-arrival time distribution $PH(\boldsymbol{\alpha}, \mathbf{T})$. Further, we tested our procedure for different values of $\mathbf{L}_0$ and $\mathbf{L}_1$ with reasonable positive lag-1 correlation coefficient, *i.e.*, correlation coefficient between two consecutive service times of batches. Several outputs were generated and the results were matched against some existing results, *e.g.*, we matched our results with Gagandeep *et al.* [23] in the special case of $PH/PH^{(a,b)}/1/\infty$ queue. In all the cases matching was found almost perfect. The calculations were performed on a PC having Intel(R) Core 2 Duo processor @3.00 GHz with 8 GB DDR2 RAM using MAPLE 18 until unless specified further. Besides these all the numerical results were carried out in high precision, they are reported here in 6 decimal places due to lack of space.

In Tables 1 and 2, we have presented various epoch probabilities for a $PH/C\text{-}MSP^{(4,10)}/1/\infty$ queue using our method described in this paper. Inter-arrival time is PH-type and its representation is given by $\boldsymbol{\alpha} = \begin{bmatrix}0.7\ 0.3\end{bmatrix}$, $\mathbf{T} = \begin{bmatrix} -3.23 & 0.60 \\ 1.7 & -7.36 \end{bmatrix}$ with $\lambda = 3.2268898$. We choose service batch sizes $a = 4,\ b = 10$ and the $C\text{-}MSP$ matrices as $\mathbf{L}_0 = \begin{bmatrix} -2.119 & 0.761 \\ 1.985 & -4.541 \end{bmatrix}$, $\mathbf{L}_1 = \begin{bmatrix} 1.119\ 0.239 \\ 2.015\ 0.541 \end{bmatrix}$ with stationary mean service rate $\mu^* = 1.5976$ and $\overline{\boldsymbol{\pi}} = \begin{bmatrix}0.8\ 0.2\end{bmatrix}$ so that $\rho = \lambda/(b.\mu^*) = 0.201983$. To calculate queue-length distribution we need to calculate the roots of

$$\det[\mathbf{I}_m z - \mathbf{S}(z^b)] = 0, \tag{7.1}$$

where $m = 2$ and $b = 10$ are given above. Here $\mathbf{S}(z)$ may be obtained by equation (B.3), see Appendix B. The $m = 2$ roots of (7.1) inside $|z| < 1$ are evaluated. The corresponding k_{ij} $(1 \le i \le 2,\ 1 \le j \le 2)$ values are calculated using the procedure described in Section 3.1, see Appendix C. Now using equations (3.9), (3.12)

TABLE 1. Queue-length distributions at pre-arrival and arbitrary epoch.

	Pre-arrival $\pi^-_{j,0}(n)$			Arbitrary $\pi_{j,0}(n)$		
n	$j=1$	$j=2$	$\sum_{j=1}^2$	$j=1$	$j=2$	$\sum_{j=1}^2$
0	0.059886	0.013790	0.073677	0.062294	0.014345	0.076639
1	0.100224	0.023091	0.123315	0.101686	0.023429	0.125115
2	0.127573	0.029403	0.156976	0.128563	0.029632	0.158195
3	0.146095	0.033679	0.179775	0.146766	0.033834	0.180600
	Pre-arrival $\pi^-_{j,1}(n)$			Arbitrary $\pi_{j,1}(n)$		
n	$j=1$	$j=2$	$\sum_{j=1}^2$	$j=1$	$j=2$	$\sum_{j=1}^2$
0	0.121325	0.029379	0.150704	0.120208	0.029149	0.149357
1	0.081719	0.020346	0.102065	0.080293	0.020008	0.100301
2	0.055131	0.013949	0.069080	0.054172	0.013713	0.067885
3	0.037229	0.009509	0.046738	0.036582	0.009347	0.045929
4	0.025154	0.006461	0.031615	0.024718	0.006350	0.031068
5	0.017002	0.004381	0.021383	0.016707	0.004305	0.021012
⋮	⋮	⋮	⋮	⋮	⋮	⋮
10	0.002402	0.000622	0.003024	0.002360	0.000611	0.002971
⋮	⋮	⋮	⋮	⋮	⋮	⋮
20	0.000048	0.000012	0.000060	0.000047	0.000012	0.000059
⋮	⋮	⋮	⋮	⋮	⋮	⋮
100	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
⋮	⋮	⋮	⋮	⋮	⋮	⋮
sum	0.806834	0.193165	1.00000	0.806870	0.193130	1.000000
L_q	=1.941546,	W_q (Little's law)	=0.601677			

TABLE 2. Batch-size distributions at pre-arrival and arbitrary epoch.

	Pre-arrival $g^-_j(n)$			Arbitrary $g_j(n)$		
n	$j=1$	$j=2$	$\sum_{j=1}^2$	$j=1$	$j=2$	$\sum_{j=1}^2$
4	0.320762	0.080123	0.400885	0.315437	0.078853	0.394290
5	0.016925	0.004232	0.021157	0.016869	0.004219	0.021088
6	0.011448	0.002863	0.014311	0.011411	0.002854	0.014265
7	0.007743	0.001936	0.009679	0.007718	0.001930	0.009648
8	0.005236	0.001310	0.006546	0.005220	0.001305	0.006525
9	0.003541	0.000886	0.004427	0.003529	0.000883	0.004412
10	0.007397	0.001850	0.009247	0.007373	0.001844	0.009217
sum	0.373055	0.093201	0.466256	0.367558	0.091889	0.459448
		L_s	=3.961371,	$\sum_{n=a}^{b} \mathbf{g}(n)\mathbf{L}_1\mathbf{e}$	=0.734014	$W_s = 1.227613$

and (3.13), one can obtain system-length distribution at pre-arrival epoch and after that using relations (3.22)–(3.25) the arbitrary epoch probabilities may be derived (see Tab. 1). Also, in Table 2, we present service batch size distribution as presented in Section 4 of the paper, see equations (4.2)–(4.7). One may note the required matrices $\mathbf{S}_n$ $(n \geq 0)$ may be calculated using Theorem B.1 in Appendix B.

It may be noted that in the above numerical experiment we find the probability that a customer joins as a first member in a servicing batch as $\mathbf{p}^-(1) = \left[0.183664\ 0.043803\right]$, see equation (4.11). Therefore, the stationary arrival rate of this first customer is $\lambda.\mathbf{p}^-(1)\mathbf{e} = 0.734014$ which matches with the stationary departure rate of this first customer as shown in the bottom of the above table. Also we have calculated system-length distribution

TABLE 3. Queue-length distributions at pre-arrival and arbitrary epoch.

	$\pi^-_{j,0}(n)$				$\pi_{j,0}(n)$			
n	$j=1$	$j=2$	$j=3$	$\sum_{j=1}^3$	$j=1$	$j=2$	$j=3$	$\sum_{j=1}^3$
0	0.116146	0.000293	0.072953	0.189391	0.116276	0.000293	0.073119	0.189688
1	0.137953	0.000390	0.107475	.245818	0.137974	0.000390	0.107553	0.245917
2	0.142465	0.000454	0.123866	.266785	0.142470	0.000455	0.123903	.266828
	$\pi^-_{j,1}(n)$				$\pi_{j,1}(n)$			
n	$j=1$	$j=2$	$j=3$	$\sum_{j=1}^3$	$j=1$	$j=2$	$j=3$	$\sum_{j=1}^3$
0	0.068337	0.001214	0.024440	0.093991	0.068169	0.001216	0.024350	0.093735
1	0.032397	0.001460	0.004412	0.038269	0.032317	0.001461	0.004394	0.038172
2	0.015391	0.001562	0.000834	0.017787	0.015353	0.001562	0.000831	0.017746
3	0.007344	0.001598	0.000183	0.009125	0.007326	0.001598	0.000183	0.009107
4	0.003535	0.001603	0.000060	0.005199	0.003527	0.001603	0.000060	0.005190
5	0.001733	0.001594	0.000035	0.003362	0.001729	0.001594	0.000035	0.003358
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10	0.000146	0.001501	0.000024	0.001671	0.000146	0.001501	0.000024	0.001670
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
20	0.000094	0.001312	0.000021	0.001427	0.000094	0.001312	0.000021	0.001427
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
100	0.000032	0.000447	0.000007	0.000486	0.000032	0.000447	0.000007	0.000486
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
sum	0.535248	0.128627	0.336124	0.999999	0.535083	0.128623	0.336293	0.999999
$L_q=$	11.204321,	$W_q=$	23.070422					

as discussed in Section 5 (see Eqs. (5.6)–(5.13)) of performance measures and checked the validity of some of equations (5.4) and (5.5). We found that numerical matchings are almost perfect.

In Tables 3 and 4, we have presented various epoch probabilities for a $HE_2/C\text{-}MSP^{(3,6)}/1/\infty$ queue using our method described in this paper. Inter-arrival time is 2-phase hyper-exponential distribution and its PH-type representation is given by $\boldsymbol{\alpha} = \begin{bmatrix}0.4 & 0.6\end{bmatrix}$, $\mathbf{T} = \begin{bmatrix}-0.53 & 0.0\\ 0.0 & -0.46\end{bmatrix}$ with $\lambda = 0.485657$. We choose service batch sizes $a=3,\ b=6$ and the $C\text{-}MSP$ matrices as

$$\mathbf{L}_0 = \begin{bmatrix}-0.542409519 & 0.0037279 & 0.0\\ 0.004349217 & -0.02298872 & 0.000621317\\ 0.0 & 0.001242633 & -2.269670072\end{bmatrix}, \quad \mathbf{L}_1 = \begin{bmatrix}0.020503453 & 0.0 & 0.518178166\\ 0.0 & 0.017396869 & 0.000621317\\ 2.259107688 & 0.004970534 & 0.004349217\end{bmatrix}$$

with stationary mean service rate $\mu^* = 0.5000000$ and $\boldsymbol{\overline{\pi}} = \begin{bmatrix}0.464978 & 0.428426 & 0.106596\end{bmatrix}$ so that $\rho = \lambda/(b.\mu^*) = 0.161886$. Similar procedure as described in Tables 1 and 2, we calculate the roots of

$$\det[\mathbf{I}_m z - \mathbf{S}(z^b)] = 0, \tag{7.2}$$

where $m=3$ and $b=6$ are given above. The roots as well as corresponding k_{ij} $(1 \le i \le 2,\ 1 \le j \le 2)$ values are calculated similar to Tables 1 and 2, see Appendix C. In Tables 3 and 4, system-length distributions and service batch size distributions are presented similar to Tables 1 and 2.

One may note that for Tables 3 and 4, almost all numerical matchings as discussed in Tables 1 and 2 are almost perfect. As above one can check that $\lambda.\mathbf{p}^-(1)\mathbf{e} = 0.148780$ which checks the validity of our numerical as well as analytical results, see equation (4.12).

TABLE 4. Batch-size distributions at pre-arrival and arbitrary epoch.

	$g_j^-(n)$				$g_j(n)$			
n	$j=1$	$j=2$	$j=3$	$\sum_{j=1}^3$	$j=1$	$j=2$	$j=3$	$\sum_{j=1}^3$
3	0.129187	0.033276	0.028295	0.190758	0.128866	0.033272	0.028187	0.190325
4	0.000397	0.001334	0.000808	0.002538	0.000396	0.001334	0.000806	0.002536
5	0.000221	0.001271	0.000396	0.001888	0.000220	0.001271	0.000396	0.001887
6	0.008876	0.091575	0.002331	0.102782	0.008875	0.091575	0.002330	0.102780
		$L_s =$	12.411557,	$\sum_{n=a}^{b} \mathbf{g}(n)\mathbf{L}_1\mathbf{e}$	=0.148764	$W_s =$	25.070422	

TABLE 5. Queue-length distributions at pre-arrival and arbitrary epoch.

	Pre-arrival $\pi_{j,0}^-(n)$			Arbitrary $\pi_{j,0}(n)$		
n	$j=1$	$j=2$	$\sum_{j=1}^2$	$j=1$	$j=2$	$\sum_{j=1}^2$
0	0.042760	0.078846	0.121606	0.035165	0.064477	0.099642
1	0.069069	0.129879	0.198948	0.064777	0.121476	0.186253
2	0.084400	0.160067	.244467	0.081904	0.155139	0.237043
	Pre-arrival $\pi_{j,1}^-(n)$			Arbitrary $\pi_{j,1}(n)$		
n	$j=1$	$j=2$	$\sum_{j=1}^2$	$j=1$	$j=2$	$\sum_{j=1}^2$
0	0.075262	0.106575	0.181837	0.078892	0.115312	0.194204
1	0.046286	0.059947	0.106233	0.051368	0.067277	0.118645
2	0.027286	0.034435	0.061721	0.030440	0.038541	0.068981
3	0.015895	0.019907	0.035802	0.017759	0.022262	0.040021
4	0.009227	0.011530	0.020757	0.010313	0.012891	0.023204
5	0.005351	0.006682	0.012033	0.005982	0.007470	0.013452
⋮	⋮	⋮	⋮	⋮	⋮	⋮
10	0.000350	0.000437	0.000787	0.000391	0.000489	0.000880
⋮	⋮	⋮	⋮	⋮	⋮	⋮
20	0.000001	0.000002	0.000003	0.000002	0.000002	0.000004
⋮	⋮	⋮	⋮	⋮	⋮	⋮
100	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
⋮	⋮	⋮	⋮	⋮	⋮	⋮
sum	0.382916	0.617083	0.999999	0.384850	0.615149	0.999999
L_q	=1.333983,	W_q (Little's law)	=2.094461			

In Tables 5 and 6, we have presented various-epoch probabilities of a $WB/C\text{-}MSP^{(3,7)}/1/\infty$ queue, where WB stands for Weibull distribution whose probability density and distribution functions are given by $a(x) = \left(\frac{\alpha}{\beta}\right)\left(\frac{x}{\beta}\right)^{(\alpha-1)}\mathrm{e}^{-(x/\beta)^{\alpha}}$, $\alpha = 1.3$, $\beta = 1.7$, $x > 0$, and $A(x) = 1 - \mathrm{e}^{-0.501668x^{1.3}}$, $x > 0$, with $\lambda = 0.636910$, respectively. We choose service batch sizes $a = 3$ and $b = 7$, and the service matrices as

$$\mathbf{L}_0 = \begin{bmatrix} -1.5125 & 0.750 \\ 0.875 & -1.025 \end{bmatrix}, \ \mathbf{L}_1 = \begin{bmatrix} 0.1625 & 0.600 \\ 0.125 & 0.025 \end{bmatrix}$$

with stationary mean service rate $\mu^* = 0.410638$ and $\boldsymbol{\pi} = \left[0.425532 \ 0.574468\right]$ so that $\rho = \lambda/(b.\mu^*) = 0.221575$. To calculate system-length distribution we need to calculate the roots of

$$\det[\mathbf{I}_m z - \mathbf{S}(z^b)] = 0, \tag{7.3}$$

TABLE 6. Batch-size distributions at pre-arrival and arbitrary epoch.

	Pre-arrival $g_j^-(n)$			Arbitrary $g_j(n)$		
n	$j=1$	$j=2$	$\sum_{j=1}^{2}$	$j=1$	$j=2$	$\sum_{j=1}^{2}$
3	0.162979	0.216193	0.379172	0.178874	0.241283	0.420157
4	0.009963	0.013488	0.023451	0.010140	0.013771	0.023911
5	0.005777	0.007822	0.013599	0.005880	0.007986	0.0138664
6	0.003349	0.004534	0.007883	0.003409	0.004630	0.008039
7	0.004619	0.006253	0.010872	0.004701	0.006385	0.011086
sum	0.186688	0.248291	0.434979	0.203005	0.274056	0.477061
		L_s	=2.885271,	$\sum_{n=a}^{b} \mathbf{g}(n)\mathbf{L}_1\mathbf{e}$	=0.195899	$W_s = 4.529694$

where $m = 2$ and $b = 7$. Here $\mathbf{S}(z)$ may be obtained by equation (2.6) as $\mathbf{S}(z) = A^*(-\mathbf{L}(z))$ and it is not possible to compute the matrix $\mathbf{S}(z)$ as the LST of Weibull distribution is in non-rational form. Therefore, we calculate 20 moments from $a(x)$ to construct an approximate $A^*(\theta)$ through the Padé approximation [4/5] as:

$$A^*(\theta) \simeq \frac{1.0 + 1.4186658\theta + 0.7474740\theta^2 + 0.1384240\theta^3 + 0.4616040\theta^4}{1.0 + 2.9887380\theta + 3.4657670\theta^2 + 1.8942610\theta^3 + 0.4543150\theta^4 + 0.2875370\theta^5}, \tag{7.4}$$

where [4/5] stands for a rational function with degree of numerator polynomial 4 and degree of denominator polynomial 5. Taking these $\mathbf{S}(z)$ (using Eq. (7.4)) into account, we obtain $m = 2$ roots of (7.3) inside $|z| < 1$, see Appendix C. The corresponding k_{ij} $(1 \leq i \leq 9,\ 1 \leq j \leq 3)$ values are calculated similarly as above. Now using the procedure discussed in Tables 1 and 2, we obtain system-length distribution at pre-arrival epoch and the arbitrary epoch, see Tables 3 and 4. One may note the required matrices $\mathbf{S}_n$ $(n \geq 0)$ may be calculated using the procedure discussed in an appendix, see the first few lines of Appendix B.

Almost all numerical matchings are found almost perfect as discussed for above tables. Here $\lambda.\mathbf{p}^-(1)\mathbf{e} = 0.195899$ which also checks the validity of our numerical as well as analytical results.

Rest of this section presents numerical results corresponding to the comparison of roots method and the MGM as described in Section 3.1.1. The numerical values and tables presented so far were based on results obtained by the roots method. Further, in Section 3.1.1 we have given complexities of the steps involved in both the roots method and the MGM. In Tables 7 and 8, some numerical data have been provided to compare the computational aspects of these methods. Before giving the numerical results it should be mentioned that theoretically both methods give the exact result, but in practice results obtained by one method slightly differ from the other. The variation of results due to the dependency of accuracy on several factors. Rest of the numerical results presented in this section were generated by MAPLE 18 software in a 64-bit windows 7 professional OS having Intel(R) core i5-0330 processor @ 3.00 GHz with 4GB DDR3 RAM. Also, the rest of the numerical calculations were done by 15 decimal digit numbers.

The critical step of the roots method is determination of roots of the associated characteristic equation, whereas the MGM includes the determination of $\mathbf{R}$. We have used "fsolve" command in MAPLE 17 to efficiently evaluate the roots of a polynomial. Determination of $\mathbf{R}$ is an iterative procedure. We have used the matrix-geometric method *via* successive substitution (MGM-SS) method and the matrix-geometric method *via* Newton's iteration (MGM-NI) method proposed by Neuts [37, 39], respectively to evaluate the $\mathbf{R}$ matrix. The iteration scheme for MGM-SS method can be given by

$$\mathbf{R}(n+1) = \sum_{r=0}^{\infty} \mathbf{R}(n)^{br} \mathbf{S}_r, \tag{7.5}$$

TABLE 7. Iteration count, time (in seconds) to get π_0^- by using roots, MGM-SS and MGM-NI.

$\rho = 0.161885779144375$						
Time (roots) = 41.636, π_0^-(roots) = 0.701994982046078						
ϵ	Iterations (MGM-SS)	Iterations (MGM-NI)	Time (MGM-SS)	Time (MGM-NI)	π_0^- (MGM-SS)	π_0^- (MGM-NI)
10^{-3}	12	5	15.771	177.887	0.695488528079496	0.702149662941305
10^{-4}	23	5	16.520	177.029	0.701385965895218	0.702149662941307
10^{-5}	34	5	18.610	177.154	0.701931974338031	0.702149662941309
10^{-6}	45	5	18.782	177.544	0.701988505020825	0.702149662941307
10^{-7}	57	10	20.155	363.092	0.701994552031505	0.701995296934400
10^{-8}	68	10	22.089	351.610	0.701995057393527	0.701995296934399
10^{-9}	79	10	22.978	360.533	0.701995110371120	0.701995296934402
10^{-10}	90	14	23.462	490.623	0.701995115924871	0.701995116758121
10^{-11}	101	14	24.772	491.153	0.701995116507084	0.701995116758122
10^{-12}	113	14	26.036	493.041	0.701995116569443	0.701995116758124
10^{-13}	124	14	26.800	490.186	0.701995116574655	0.701995116758123
10^{-14}	135	18	28.048	640.945	0.701995116575201	0.701995116575311
10^{-15}	146	18	29.031	642.240	0.701995116575258	0.701995116575312
10^{-16}	157	22	30.388	772.875	0.701995116575265	0.701995116575266
10^{-17}	164	22	31.075	765.200	0.701995116575266	0.701995116575266
10^{-18}	168	22	31.527	751.425	0.701995116575266	0.701995116575266

TABLE 8. Time (in seconds) taken by the roots method and the MGM-SS method to get π_0^- for different ρ.

ρ	Time (roots)	Time (MGM)
0.10	26.722	25.084
0.15	30.747	29.936
0.20	28.672	35.349
0.25	27.970	42.447
0.30	27.643	48.251
0.35	27.752	54.537
0.40	29.234	62.712
0.45	27.081	69.295
0.50	28.111	77.563
0.55	30.076	91.276
0.60	29.094	105.004
0.65	26.582	120.822
0.70	27.892	144.784
0.75	29.577	170.930
0.80	26.457	204.220
0.85	29.062	272.143

where $\mathbf{R}(n)$ is the computed value of $\mathbf{R}$ after nth iteration, starting with $\mathbf{R}(n) = \mathbf{0}$. Proceeding in a similar fashion discussed by Neuts [37], one can have the iteration scheme for MGM-NI method as

$$\mathbf{r}(n+1) = \mathbf{r}(n) + \mathcal{F}[\mathbf{R}(n)] \left\{ \mathbf{I} - \sum_{r=1}^{\infty} \sum_{\nu=0}^{br-1} [\mathbf{R}(n)^{br-1-\nu} \otimes \mathbf{R}(n)_T^{\nu}](\mathbf{I} \otimes \mathbf{S}_r) \right\}^{-1}, \tag{7.6}$$

where $\mathbf{r}(n)$ is the row vector obtained from $\mathbf{R}(n)$ by taking the direct sum of the rows of $\mathbf{R}(n)$, $\mathcal{F}$ is an operator defined by $\mathcal{F}[\mathbf{R}(n)] = \sum_{r=0}^{\infty} \mathbf{R}(n)^{br}\mathbf{S}_r - \mathbf{R}(n)$ and $\mathbf{X}_T$ is the transpose of the matrix $\mathbf{X}$. Convergence of the

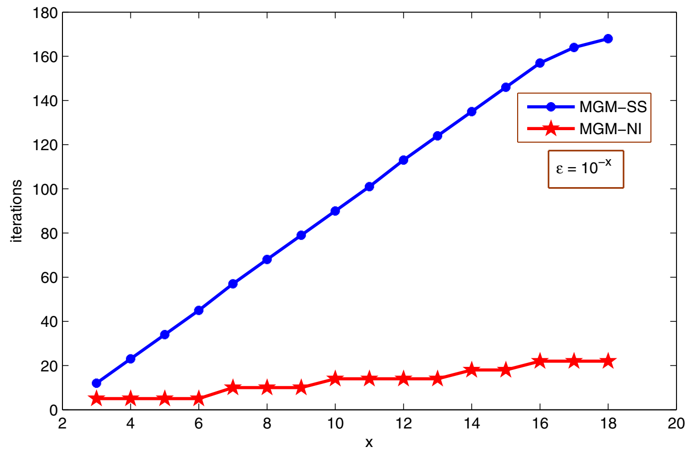

FIGURE 1. Iteration count *versus* x.

iteration depends on the desired accuracy and different model parameters, such as ρ and Λ. We can terminate the iterations for SS and NI methods when

$$\max_{i,j} |\mathbf{R}(n+1)(i,j) - \mathbf{R}(n)(i,j)| < \epsilon \quad \text{and} \quad \max_{j} |\mathbf{r}(n+1)(j) - \mathbf{r}(n)(j)| < \epsilon,$$

respectively, where $\epsilon(> 0)$ is the desired accuracy, $\mathbf{X}(i,j)$ is the (i,j)th element of the matrix $\mathbf{X}$ and $\mathbf{x}(j)$ is the jth element of the row vector $\mathbf{x}$. Let π_0^- be the probability of having an idle server at pre-arrival epoch, *i.e.*, $\pi_0^- = \sum_{n=0}^{a-1} \boldsymbol{\pi}_0^-(n)\boldsymbol{e}$. In order to compare the two methods we have calculated π_0^- for the $HE_2/C\text{-}MSP^{(3,6)}/1/\infty$ queue by roots method and the MGM. The parameters of the HE_2 and the $C\text{-}MSP$ are given while describing Tables 3 and 4. Repeated Results have been generated by the MGM-SS and MGM-NI methods for different accuracy level of the matrix $\mathbf{R}$ and have been presented in Table 7. In Figure 1, we have plotted the number of iterations required to determine the $\mathbf{R}$ matrix in both MGM-SS and MGM-NI methods against x, where the assigned accuracy is $\epsilon = 10^{-x}$, $x = 3, 4, \ldots, 18$. We can see from Figure 1 and Table 7 that with increasing accuracy, the iteration count for the MGM-SS method increases almost linearly and the value of π_0^- approaches to a constant, while in case of the MGM-NI method the iteration count is much lesser than the MGM-SS method. Although the iteration count for the MGM-NI method is less than MGM-SS method, we experienced that the MGM-NI method took much more time than the other method and Table 7 reflects this fact. Pérez *et al.* [41] proposed a new NI method to reduce the time per iteration to compute $\mathbf{R}$. It should be mentioned here that for constructing the $\mathbf{B}[\mathbf{R}]$ matrix we have considered $\eta = a + 1$.

We also evaluate π_0^- for the $HE_2/C\text{-}MSP^{(3,6)}/1/\infty$ queue with different ρ by changing the the diagonal elements of the matrix $\mathbf{T}$ of the corresponding PH-type representation used in the HE_2 distribution. The time taken by the roots method and the MGM-SS method to calculate π_0^- has been listed in Table 8. Results given in Table 8 have been plotted in Figure 2. From Figure 2 one can see that the time required by roots for different ρ remains almost same, while MGM-SS method required more time for higher ρ and grows significantly. Further, it should be mentioned that variability of the time in roots method is due to the initial approximation in Newton's iteration to determine the roots of a polynomial. It may further be mentioned that the computation time for the MGM-NI method is larger than that of the MGM-SS method. Due to this reason in Figure 2 and Table 8, we have made a comparison of the computation time between the roots method and the MGM-SS method.

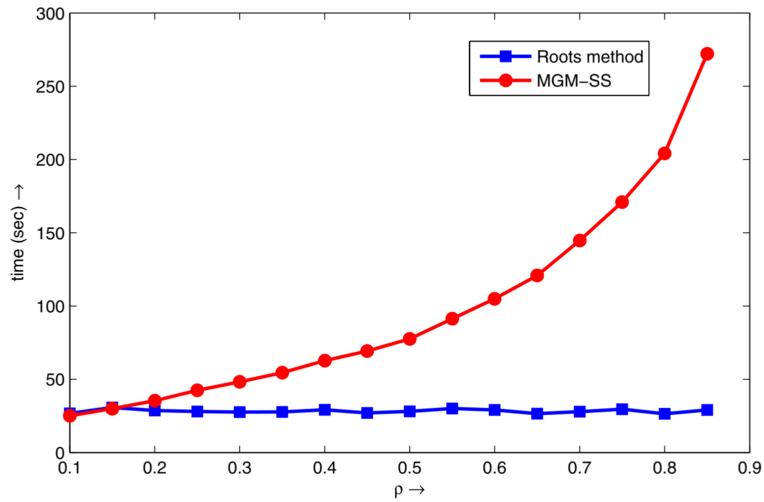

FIGURE 2. Computation time (in seconds) *versus* ρ.

8. Conclusions and future scope

In this paper, we have successfully analyzed $GI/C\text{-}MSP^{(a,b)}/1$ queue with infinite-buffer. We have suggested a procedure to obtain steady-state distributions of the number of customers in the queue as well as system at various epochs. Also we obtain service batch size probability distributions. Several other batch service/arrival queueing models under Markovian service process can be analyzed similarly, *e.g.*, analysis of the corresponding batch arrival queue, *i.e.*, $GI^{[X]}/C\text{-}MSP^{(a,b)}/1/\infty$ system, or the analysis of $GI/C\text{-}MSP^{[Y]}/1/\infty$ queue. Analysis of this batch service queue with single and multiple vacation policies are other possible extensions of this work. We leave these problems as a future extension of the current research.

Appendix A.

Theorem A.1. *Every function $z - S_{i,i}(z^b)$, $1 \le i \le m$ has exactly one zero inside the unit circle.*

Proof. Consider absolute values of $f(z) = z$ and $\bar{F}(z) = -S_{i,i}(z^b)$ on the circle $|z| = 1 - \delta$, where δ is positive and sufficiently small. First, note that $\mathbf{S} \equiv \mathbf{S}(1)$ is the imbedded transition probability matrix of $J(t)$ in a random amount of time with the distribution of an inter-arrival time. Since the state space of $J(t)$ is finite, $\mathbf{S}$ is an irreducible and aperiodic discrete-time Markov chain, which is necessarily ergodic. This implies, in particular, that $S_{i,i}(1) \le 1$, for $1 \le i \le m$. As equation (2.6) yields $\mathbf{S}'(1) = \sum_{n=1}^{\infty} n\mathbf{S}_n$ which represents the mean number of customers served during an inter-arrival time period and the phase changes of the underlying Markov chain, we have $S'_{i,i}(1) \ge 0$. Now let us consider the following inequality for $|\bar{F}(z)|$, with $|z| = 1 - \delta$,

$$\begin{aligned} |\bar{F}(z)| \le |\bar{F}(|z|)| &= S_{i,i}\Big((1-\delta)^b\Big) \\ &= S_{i,i}(1) - b\delta S'_{i,i}(1) + o(\delta). \end{aligned} \tag{A.1}$$

Also

$$|f(z)| = |z| = 1 - \delta. \tag{A.2}$$

We note that $\mathbf{S} = \mathbf{S}(1)$ is a stochastic matrix which represents the number of batches of customers served during an inter-arrival time and the phase changes of the underlying Markov chain during a busy period. Thus, we have

$\mathbf{S}(1)\mathbf{e} = \mathbf{e}$. Hence, (A.1) yields

$$\begin{aligned}
|\bar{F}(z)| &\le S_{i,i}(1) - b\delta S'_{i,i}(1) + o(\delta) \\
&= S_{i,i}(1 - b\delta + o(\delta)) + o(\delta), \quad \text{using Taylor's series expansion} \\
&= S_{i,i}\Big((1-\delta)^b\Big) + o(\delta), \quad \text{using Taylor's series expansion} \\
&= 1 - \sum_{j=1,\ j\ne i}^{m} S_{i,j}\Big((1-\delta)^b\Big) + o(\delta) \qquad \text{(A.3)} \\
&< 1 - \delta = |f(z)|, \quad \text{for sufficiently small } \delta \\
&\text{and } \lim_{\delta\downarrow 0} \sum_{j=1,\ j\ne i}^{m} S_{i,j}\Big((1-\delta)^b\Big) > 0. \qquad \text{(A.4)}
\end{aligned}$$

Hence, using the well-known Rouché's theorem, $f(z)$ and $f(z) + \bar{F}(z)$ have the same number of zeros inside the unit circle. It is obvious that $f(z) = z$ has exactly one zero inside the unit circle. Thus, $f(z)+\bar{F}(z) = z - S_{i,i}(z^b)$ has exactly one zero inside the unit circle. $\square$

Theorem A.2. *The following inequalities hold on the circle $|z| = 1 - \delta$:*

$$|z - S_{i,i}(z^b)| > \sum_{j=1,\ j\ne i}^{m} |S_{i,j}(z^b)|, \ 1 \le i \le m. \qquad \text{(A.5)}$$

Proof. On the circle $|z| = 1 - \delta$, using the Taylor series expansion we have

$$\begin{aligned}
|z - S_{i,i}(z^b)| &\ge |z| - |S_{i,i}(z^b)| \\
&\ge 1 - \delta - (S_{i,i}(1) - b\delta S'_{i,i}(1) + o(\delta)) \\
&\ge \sum_{j=1,\ j\ne i}^{m} S_{i,j}(1) - \delta(1 - bS'_{i,i}(1)) + o(\delta),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=1,\ j\ne i}^{m} |S_{i,j}(z^b)| &\le \sum_{j=1,\ j\ne i}^{m} S_{i,j}(|z|^b) \\
&\le \sum_{j=1,\ j\ne i}^{m} S_{i,j}(1) - b\delta \sum_{j=1,\ j\ne i}^{m} S'_{i,j}(1) + o(\delta).
\end{aligned}$$

Now, assume that the following inequality holds:

$$|z - S_{i,i}(z^b)| \le \sum_{j=1,\ j\ne i}^{m} |S_{i,j}(z^b)|, \ 1 \le i \le m.$$

This implies that

$$\begin{aligned}
\sum_{j=1,\ j\neq i}^{m} S_{i,j}(1) - \delta(1 - bS'_{i,i}(1)) + o(\delta) &\leq \sum_{j=1,\ j\neq i}^{m} S_{i,j}(1) - b\delta \sum_{j=1,\ j\neq i}^{m} S'_{i,j}(1) + o(\delta)\\
&\Rightarrow (1 - bS'_{i,i}(1)) \geq b \sum_{j=1,\ j\neq i}^{m} S'_{i,j}(1)\\
&\Rightarrow 1 \geq b\sum_{j=1}^{m} S'_{i,j}(1)\\
&\Rightarrow \overline{\pi}_i \geq b.\overline{\pi}_i \sum_{j=1}^{m} S'_{i,j}(1)\\
&\Rightarrow \sum_{i=1}^{m} \overline{\pi}_i \geq b. \sum_{i=1}^{m} \overline{\pi}_i \sum_{j=1}^{m} S'_{i,j}(1) = b.\overline{\boldsymbol{\pi}}\mathbf{S}'(1)\mathbf{e}\\
&\Rightarrow 1 \geq \frac{1}{\rho} \Rightarrow \rho \geq 1.
\end{aligned}$$

This contradicts the system stability condition $\rho < 1$, and hence equation (A.5) is satisfied. □

Theorem A.3. *The determinant* $\det[z\mathbf{I}_m - \mathbf{S}(z^b)]$ *has exactly* m *zeros inside the unit circle.*

Proof. Mathematical induction is used to prove this theorem. Let us denote

$$\mathbf{D}_n(z) = \det[z\mathbf{I}_n - \mathbf{S}_n(z^b)], \quad 1 \leq n \leq m, \tag{A.6}$$

where $\mathbf{S}_n(z^b)$ is the principal minor of order n of the matrix $\mathbf{S}(z^b)$ starting from the element $S_{1,1}(z^b)$. First, we show that the statement is true for $n = 1$. For $n = 1$, equation (A.6) becomes $\mathbf{D}_1(z) = z - S_{1,1}(z^b)$. This is the characteristic function for the classical $GI/M^{(a,b)}/1$ queue. It has been proved at several places that $z = S(z^b)$ has exactly one root inside the unit circle as $\rho < 1$, see, *e.g.*, Madill and Chaudhry [34]. Next for $n = 2$, equation (A.6) becomes

$$\begin{aligned}
\mathbf{D}_2(z) &= \begin{vmatrix} z - S_{1,1}(z^b) & -S_{1,2}(z^b) \\ -S_{2,1}(z^b) & z - S_{2,2}(z^b) \end{vmatrix}\\
&= -S_{2,1}(z^b)C_{2,1}(z) + [z - S_{2,2}(z^b)]\mathbf{D}_1(z),
\end{aligned} \tag{A.7}$$

where $C_{2,1}(z) = -S_{1,2}(z^b)$ is the cofactor of $-S_{2,1}(z^b)$. Again, we can write (A.7) as

$$\begin{aligned}
\left|\frac{\mathbf{D}_2(z) - [z - S_{2,2}(z^b)]\mathbf{D}_1(z)}{[z - S_{2,2}(z^b)]\mathbf{D}_1(z)}\right| &= \left|\frac{-S_{2,1}(z^b)C_{2,1}(z)}{[z - S_{2,2}(z^b)]\mathbf{D}_1(z)}\right|\\
&= \frac{\left|S_{2,1}(z^b)\right| |y_{2,1}(z)|}{|z - S_{2,2}(z^b)|} < 1,
\end{aligned} \tag{A.8}$$

where

$$|y_{2,1}(z)| = \frac{|C_{2,1}(z)|}{|\mathbf{D}_1(z)|} = \frac{|S_{1,2}(z^b)|}{|z - S_{1,1}(z^b)|} < 1 \quad \text{and} \quad \frac{\left|S_{2,1}(z^b)\right|}{|z - S_{2,2}(z^b)|} < 1,$$

by Theorem A2 for $i = 1$ and $i = 2$, respectively. Hence, by Rouché's theorem $\mathbf{D}_2(z)$ has 2 roots inside the unit circle, $|z| = 1$, since $(z - S_{2,2}(z^b))\mathbf{D}_1(z)$ has 2 roots. Finally, we assume that the statement is true for $n = m - 1$.

It must then be shown that the statement holds for $n = m$. The determinant $\mathbf{D}_n(z)$ for $n = m$ is given by

$$\mathbf{D}_m(z) = \begin{vmatrix} z - S_{1,1}(z^b) & -S_{1,2}(z^b) & \cdots & -S_{1,m-1}(z^b) & -S_{1,m}(z^b) \\ -S_{2,1}(z^b) & z - S_{2,2}(z^b) & \cdots & -S_{2,m-1}(z^b) & -S_{2,m}(z^b) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -S_{m-1,1}(z^b) & -S_{m-1,2}(z^b) & \cdots & z - S_{m-1,m-1}(z^b) & -S_{m-1,m}(z^b) \\ -S_{m,1}(z^b) & -S_{m,2}(z^b) & \cdots & -S_{m,m-1}(z^b) & z - S_{m,m}(z^b) \end{vmatrix}. \tag{A.9}$$

Now, we can rewrite (A.9) in the following way

$$\mathbf{D}_m(z) = -\sum_{j=1}^{m-1} S_{m,j}(z^b)C_{m,j}(z) + [z - S_{m,m}(z^b)]\mathbf{D}_{m-1}(z), \tag{A.10}$$

where $C_{m,j}(z)$ is the cofactor of $-S_{m,j}(z^b)$. Again, we can write (A.10) as

$$\begin{aligned} \left|\frac{\mathbf{D}_m(z) - [z - S_{m,m}(z^b)]\mathbf{D}_{m-1}(z)}{[z - S_{m,m}(z^b)]\mathbf{D}_{m-1}(z)}\right| &= \left|\frac{-\sum_{j=1}^{m-1} S_{m,j}(z^b)C_{m,j}(z)}{[z - S_{m,m}(z^b)]\mathbf{D}_{m-1}(z)}\right| \\ &\leq \frac{\sum_{j=1}^{m-1} \left|S_{m,j}(z^b)\right| |y_{m,j}(z)|}{|z - S_{m,m}(z^b)|}, \end{aligned} \tag{A.11}$$

where $|y_{m,j}(z)| = \frac{|C_{m,j}(z)|}{|\mathbf{D}_{m-1}(z)|}$ is the unique solution (by Cramer's rule, provided $|\mathbf{D}_{m-1}(z)| \neq 0$) of the system of equations

$$\begin{aligned} &\begin{pmatrix} z - S_{1,1}(z^b) & -S_{1,2}(z^b) & \cdots & -S_{1,m-1}(z^b) \\ -S_{2,1}(z^b) & z - S_{2,2}(z^b) & \cdots & -S_{2,m-1}(z^b) \\ \vdots & \vdots & \vdots & \vdots \\ -S_{m-1,1}(z^b) & -S_{m-1,2}(z^b) & \cdots & z - S_{m-1,m-1}(z^b) \end{pmatrix} \cdot \begin{pmatrix} y_{m,1}(z) \\ y_{m,2}(z) \\ \vdots \\ y_{m,m-1}(z) \end{pmatrix} \\ &= \begin{pmatrix} -S_{1,m}(z^b) \\ -S_{2,m}(z^b) \\ \vdots \\ -S_{m-1,m}(z^b) \end{pmatrix}. \end{aligned} \tag{A.12}$$

The kth equation of (A.12) is given by

$$[z - S_{k,k}(z^b)]y_{m,k}(z) - \sum_{j=1, j\neq k}^{m-1} S_{k,j}(z^b)y_{m,j}(z) = -S_{k,m}(z^b), \quad 1 \leq k \leq m-1. \tag{A.13}$$

Now, the entries of the matrix $\mathbf{D}_n(z)$ $(1 \leq n \leq m)$ satisfy Hadamard's condition, *i.e.*, the modulus of each entry of the matrix is less than or equal to one and the modulus of the diagonal element is greater than the sum of the moduli of all other elements in that row on the circle $|z| = 1 - \delta$. It implies that the matrix $\mathbf{D}_{m-1}(z)$ is nonsingular and the system (A.12) has a unique solution $y_{m,j}(z)$ with $|y_{m,j}(z)| < 1,\ 1 \leq j \leq m-1$ on the circle $|z| = 1 - \delta$. Let us assume the contrary that

$$\mathrm{Max}_j |y_{m,j}(z)| = |y_{m,k}(z)| \geq 1. \tag{A.14}$$

Because of our assumption $\left|\frac{y_{m,j}(z)}{y_{m,k}(z)}\right| \leq 1$ and $\left|\frac{1}{y_{m,k}(z)}\right| \leq 1$, we can rewrite (A.13) in the form

$$\begin{aligned}|z - S_{k,k}(z^b)| &\leq \sum_{j=1,j\neq k}^{m-1} \left|S_{k,j}(z^b)\right| \left|\frac{y_{m,j}(z)}{y_{m,k}(z)}\right| + \left|S_{k,m}(z^b)\right| \left|\frac{1}{y_{m,k}(z)}\right| \\ &\leq \sum_{j=1,j\neq k}^{m} \left|S_{k,j}(z^b)\right|.\end{aligned}$$

This contradicts Theorem A.2. Thus we have $|y_{m,j}(z)| < 1$. Hence, using Theorem A.2 and $|y_{m,j}(z)| < 1$, the right-hand side expression of equation (A.11) is smaller than one, and therefore $|\bar{G}(z)| < |f(z)|$, where $f(z) = [z - S_{m,m}(z^b)]\mathbf{D}_{m-1}(z)$ and $\bar{G}(z) = \mathbf{D}_m(z) - [z - S_{m,m}(z^b)]\mathbf{D}_{m-1}(z)$. By Rouché's theorem $f(z)$ and $f(z) + \bar{G}(z)$ have the same number of zeros inside the unit circle, $|z| = 1$. Since by our assumption, $\mathbf{D}_{m-1}(z)$ has $(m-1)$ zeros, and $[z - S_{m,m}(z^b)]$ has one zero by Theorem A1, $f(z)$ has m zeros inside the unit circle. This implies that $f(z) + \bar{G}(z) = \mathbf{D}_m(z)$ has exactly m zeros inside the unit circle, $|z| = 1$, if we let $\delta \to 0$. □

Appendix B.

For the sake of completeness, we have given the procedure of obtaining $\mathbf{S}_n$ as follows:
As presented in [31], applying the uniformization argument, $\mathbf{P}(n,t)$ is of the form:

$$\mathbf{P}(n,t) = \sum_{l=n}^{\infty} \mathrm{e}^{-\theta t} \frac{(\theta t)^l}{l!} \mathbf{U}_n^{(l)} \quad n \geq 0, \tag{B.1}$$

where $\theta = \max_i\{-[L_0]_{ii}\}$ $(1 \leq i \leq m)$ and $\mathbf{U}_n^{(l)}$ is given by

$$\begin{aligned}\mathbf{U}_0^{(0)} &= \mathbf{I}_m, \quad \mathbf{U}_n^{(0)} = \mathbf{0}, \qquad n \geq 1, \\ \mathbf{U}_0^{(l+1)} &= \mathbf{U}_0^{(l)}(\mathbf{I}_m + \theta^{-1}\mathbf{L}_0), \quad l \geq 0, \\ \mathbf{U}_n^{(l+1)} &= \mathbf{U}_n^{(l)}(\mathbf{I}_m + \theta^{-1}\mathbf{L}_0) + \theta^{-1}\mathbf{U}_{n-1}^{(l)}\mathbf{L}_1, \quad n \geq 1,\ l \geq 0.\end{aligned}$$

By substituting the values of $\mathbf{P}(n,t)$ in (B.1), we obtain

$$\mathbf{S}_n = \sum_{l=0}^{\infty} v_l \mathbf{U}_n^{(l)}, \qquad n \geq 0, \tag{B.2}$$

where $v_l = \int_0^\infty \mathrm{e}^{-\theta t} \frac{(\theta t)^l}{l!} \mathrm{d}A(t)$.

However, when the inter-arrival time distributions are of phase type (PH-distribution), these matrices can be evaluated using a procedure given by Neuts [39]. The following theorem gives a procedure for the computation of the matrices $\mathbf{S}_n$.

Theorem B.1. *If $A(t)$ follows a PH-distribution with irreducible representation $(\boldsymbol{\alpha}, \mathbf{T})$, where $\boldsymbol{\alpha}$ and $\mathbf{T}$ are of dimension ν, then the matrices $\mathbf{S}_n$ are given by*

$$\mathbf{S}_n = \mathbf{U}_n(\mathbf{I}_m \otimes \mathbf{T}^0), \quad n \geq 0,$$

where

$$\begin{aligned}\mathbf{U}_0 &= -(\mathbf{I}_m \otimes \boldsymbol{\alpha})[\mathbf{L}_0 \otimes \mathbf{I}_\nu + \mathbf{I}_m \otimes \mathbf{T}]^{-1}, \\ \mathbf{U}_n &= -\mathbf{U}_{n-1}(\mathbf{L}_1 \otimes \mathbf{I}_\nu)[\mathbf{L}_0 \otimes \mathbf{I}_\nu + \mathbf{I}_m \otimes \mathbf{T}]^{-1}, \quad n \geq 1,\end{aligned}$$

with $\mathbf{T}^0 = -\mathbf{T}\mathbf{e}_\nu$ and the symbol $\otimes$ denotes the Kronecker product of two matrices.

Proof. The proof is given by Neuts [39] for PH-type service. □

Theorem B.2. *If inter-arrival time distribution is of phase type having the parameters* $(\boldsymbol{\alpha}, \mathbf{T})$, *then the matrix-generating function* $\mathbf{S}(z)$ *of* $\mathbf{S}_n$*'s is given by*

$$\begin{aligned} \mathbf{S}(z) &= \sum_{n=0}^{\infty} \mathbf{S}_n z^n = \int_0^{\infty} \mathrm{e}^{\mathbf{L}(z)t} \otimes a(t)\mathrm{d}t \\ &= (\mathbf{I}_m \otimes \alpha)(\mathbf{L}(z) \oplus \mathbf{T})^{-1}(\mathbf{I}_m \otimes \mathbf{T}\mathbf{e}_\nu), \end{aligned} \tag{B.3}$$

where $\mathbf{L}(z) \oplus \mathbf{T} = (\mathbf{L}(z) \otimes \mathbf{I}_\nu) + (\mathbf{I}_m \otimes \mathbf{T})$.

Proof. Similar proof was presented by Chaudhry *et al.* [18] for C-MAP arrival and service time distribution with rational Laplace–Stieltjes transform and is applicable in our case with some modification. We present it as follows. As inter-arrival time distribution is of phase type with representation $(\boldsymbol{\alpha}, \mathbf{T})$, we can write its density function as $a(t) = \boldsymbol{\alpha}\mathrm{e}^{\mathbf{T}t}\mathbf{T}^0$ and $\mathbf{T}^0 = -\mathbf{T}\mathbf{e}_\nu$. Now write (2.6) as

$$\mathbf{S}(z) = \int_0^{\infty} \mathrm{e}^{\mathbf{L}(z)t} \otimes a(t)\mathrm{d}t, \tag{B.4}$$

where $\otimes$ is a Kronecker product of two square matrices of order m and 1, respectively. We write (B.4) in a form such that each square matrix is the product of three different matrices (not necessarily square matrices) as

$$\mathbf{S}(z) = \int_0^{\infty} \left(\mathbf{I}_m \mathrm{e}^{\mathbf{L}(z)t}\mathbf{I}_m\right) \otimes \left(\boldsymbol{\alpha}\mathrm{e}^{\mathbf{T}t}\mathbf{T}^0\right) \mathrm{d}t.$$

Using the well-known rule of Kronecker product $(\mathbf{E}_1\mathbf{E}_2\mathbf{E}_3) \otimes (\mathbf{F}_1\mathbf{F}_2\mathbf{F}_3) = (\mathbf{E}_1 \otimes \mathbf{F}_1)(\mathbf{E}_2 \otimes \mathbf{F}_2)(\mathbf{E}_3 \otimes \mathbf{F}_3)$, Now write

$$\begin{aligned} \mathbf{S}(z) &= \int_0^{\infty} (\mathbf{I}_m \otimes \boldsymbol{\alpha}) \left(\mathrm{e}^{\mathbf{L}(z)t} \otimes \mathrm{e}^{\mathbf{T}t}\right) \left(\mathbf{I}_m \otimes \mathbf{T}^0\right) \mathrm{d}t \\ &= (\mathbf{I}_m \otimes \boldsymbol{\alpha}) \left(\int_0^{\infty} \left(\mathrm{e}^{\mathbf{L}(z)t} \otimes \mathrm{e}^{\mathbf{T}t}\right) \mathrm{d}t\right) \left(\mathbf{I}_m \otimes \mathbf{T}^0\right). \end{aligned}$$

We use another important rule of Kronecker product, *i.e.*, $\mathrm{e}^{\mathbf{E}_1} \otimes \mathrm{e}^{\mathbf{F}_1} = \mathrm{e}^{\mathbf{E}_1 \oplus \mathbf{F}_1}$, where $\oplus$ is the Kronecker sum defined as $\mathbf{E}_1 \oplus \mathbf{F}_1 = \mathbf{E}_1 \otimes \mathbf{I}_\nu + \mathbf{I}_m \otimes \mathbf{F}_1$. Here, it is assumed that $\mathbf{E}_1$ and $\mathbf{F}_1$ are square matrices of order m and ν, respectively. We let $\mathbf{E} = (\mathbf{L}(z) \oplus \mathbf{T})$ and assume that $\mathbf{E}^{-1}$ exists and, in fact, this is the case here. From the fact that $\int_0^{\infty} \mathrm{e}^{\mathbf{E}t}\mathrm{d}t = -\mathbf{E}^{-1}$, we finally get

$$\mathbf{S}(z) = (\mathbf{I}_m \otimes \boldsymbol{\alpha}) \left(-(\mathbf{L}(z) \oplus \mathbf{T})^{-1}\right) (\mathbf{I}_m \otimes \mathbf{T}^0),$$

where $\mathbf{L}(z) \oplus \mathbf{T} = (\mathbf{L}(z) \otimes \mathbf{I}_\nu) + (\mathbf{I}_m \otimes \mathbf{T})$. □

Using $\mathbf{T}^0 = -\mathbf{T}\mathbf{e}_\nu$, we can finally write

$$\mathbf{S}(z) = (\mathbf{I}_m \otimes \boldsymbol{\alpha})(\mathbf{L}(z) \oplus \mathbf{T})^{-1}(\mathbf{I}_m \otimes \mathbf{T}\mathbf{e}_\nu).$$

Appendix C.

The roots used in Tables 1 and 2 are given as follows: $\gamma_1 = 0.3990684$, $\gamma_2 = 0.676222$. The corresponding k_{ij} $(1 \le i \le 2,\ 1 \le j \le 2)$ values are as follows: $k_{1,1} = 0.001169$, $k_{1,2} = -0.001729$, $k_{2,1} = 0.120157$, $k_{2,2} = 0.031108$.

The roots used in Tables 3 and 4 are given below: $\gamma_1 = 0.176834$, $\gamma_2 = 0.473153$ and $\gamma_3 = 0.986627$. The corresponding k_{ij} $(1 \le i \le 3,\ 1 \le j \le 3)$ values are as follows: $k_{1,1} = 0.000000$, $k_{1,2} = -0.000013$,

$k_{1,3} = 0.024184,\ k_{2,1} = 0.068215,\ k_{2,2} = -0.000490,\ k_{2,3} = 0.000229,\ k_{3,1} = 0.000123,\ k_{3,2} = 0.001717,\ k_{3,3} = 0.000027$.

The roots used in Tables 5 and 6 are as follows: $\gamma_1 = 0.171335,\ \gamma_2 = .579662$. The corresponding k_{ij} $(1 \leq i \leq 2,\ 1 \leq j \leq 2)$ values are as follows: $k_{1,1} = -0.006514,\ k_{1,2} = 0.004483,\ k_{2,1} = 0.081776,\ k_{2,2} = 0.102092$.

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