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EQUILIBRIUM JOINING STRATEGIES IN M/M/1 QUEUES WITH WORKING VACATION AND VACATION INTERRUPTIONS *

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Abstract. We study the equilibrium joining strategies for customers in an M/M/1 queue with working vacations and vacation interruptions. The service rate switches between a low and a high value depending on system dynamics. The server will take a multiple working vacation when the system is empty, during which a low service rate is provided to the arriving customers if any. Upon completion of the first customer's service, given that the system is not empty, the working vacation will be terminated which means the server comes back and serves the following customers with a higher service rate. Otherwise, if the system is found empty upon completion of the first service, the server will continue his working vacation. Arriving customers may or may not know the state of the server and/or the number of the customers upon arrival, but they have to decide whether to enter the system or balk based on a linear reward-cost structure. We investigate customer behavior according to different levels of information regarding the system state. The equilibrium strategies for the customers are derived and the stationary behavior of the system under these strategies are analyzed. Finally, the effect of different levels of information on equilibrium thresholds and equilibrium entrance probabilities is illustrated by several numerical examples.

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1. Introduction

In many practical queueing scenarios, servers may become unavailable for a random amount of time when there are no jobs in the waiting line at a service completion instant. This period of absence time called server vacation time and generally for some economic reasons or routine system maintenance, queueing systems with vacation are frequently used in telecommunication and manufacturing fields and extensively studied in the queueing literature. Detailed surveys about classical vacation queueing systems are contained in the monographs of Takagi [17] and Tian and Zhang [18], among others.

During the last decade, considerable attentions have been devoted to the economic analysis of queueing systems with strategic customers' behavior. For the general theory on this topic, interested readers are referred to Hassin and Haviv [9] and Stidham [13] for extensive surveys. In particular for the game-theoretic analysis of vacation queueing systems, Burnetas and Economou [2] first presented a Markovian queue with setup times

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and customer strategic behavior under four different levels of system information, *i.e.*, fully observable, almost observable, almost unobservable and fully unobservable were studied. Economou and Kanta [4] discussed an observable M/M/1 queue with server breakdowns and repairs. For the fully and partially observable queues, Liu *et al.* [10] introduced classical single vacation policy, whereas Wang and Zhang [19] focused on server breakdowns and delayed repairs. Sun *et al.* [15, 16] considered fully observable and unobservable queues with several types of setup/closedown policies: interruptible, skippable and insusceptible policies, respectively. Moreover, Guo and Hassin [7, 8] elaborately studied fully observable and unobservable queues with homogeneous and heterogeneous customers under N-policy, respectively. Recently, Economou *et al.* [5] further discussed the unobservable and partially observable queues with general service and vacation times. On the other hand, for discrete-time queueing systems, Ma *et al.* [11] investigated some Geo/Geo/1 queues with multiple vacations.

All the previous literature above considers customers' behavior in queueing systems with various classical vacation policies. That is, the server stops serving customers during a vacation period. Motivated by the analysis of a WDM optical access network using multiple wavelengths which can be reconfigured, Servi and Finn [12] introduced a half-vacation policy called working vacation (WV) in which the server serves customers at a low rate rather than stop working during the vacation time. This vacation policy is different from the classical vacation queueing models. They studied the M/M/1/WV system and obtained the total numbers and expected mean sojourn time of the customers in the queue. Wu and Takagi [20] generalized their study to an M/G/1 queue with working vacation. Baba [1] discussed a GI/M/1 queue with multiple working vacations. Do [3] studied an M/M/1 retrial queue with working vacations which is motivated by the performance analysis of a Media Access Control function in wireless systems. However, studies on customers' equilibrium balking strategies in queues with working vacation appeared very few. Sun and Li [14] and Zhang *et al.* [21] both studied the single-server Markovian queues with multiple working vacations in which customers maximize their benefit, and they derived the customers' equilibrium and socially optimal behavior under different levels of the system information.

In some practical situations, it is noted that the server may terminate his working vacation to improve the quality of service and provide more flexible service scheme. A practical example of the proposed model arises from health care scenario. We consider a telephone consultation service system staffed with a chief physician (called main server) and a physician assistant (called substitute server). The physician assistant only provides service to the patients when the chief physician is on vacation (called working vacation) and the service rate of the physician assistant is usually lower than that of the chief physician. When the chief physician finds no patient call, he will need to rest from his work, *i.e.*, go on a vacation. During the chief physician is on vacation, the physician assistant will serve the patients, if any, and after his service completion if there are patients in the system, the chief physician will come back from his vacation no matter his vacation ends or not, *i.e.*, vacation interruption happens. Meanwhile, if there is no patient when a vacation ends, the chief physician begins another vacation, otherwise, the chief physician takes over the physician assistant. To understand the patient's condition, the chief physician will restart his service no matter how long the physician assistant has served the patient.

To the best of authors' knowledge, there is no work concerning customers' equilibrium balking strategies in queue with working vacation and vacation interruptions. The main objective of our work is to study the customers' equilibrium balking strategies in this queueing system. When customers arrive at the system, in the light of their acquired different precision levels of system information, they need to make decisions of whether to join the queue or not. Here we study four types of queueing systems: the observable queues, the almost observable queues, the almost unobservable queues and the unobservable queues. Customer equilibrium strategies are obtained in each case, along with the stationary behavior of the corresponding system and the social benefit for all customers. The effect of the information level on the equilibrium behavior and the social benefit are investigated *via* analytical and numerical comparisons.

This paper is organized as follows. In Section 2 we describe the model and the reward-cost structure. In Section 3, the equilibrium threshold strategies are derived in fully observable and almost observable queues. In Section 4, we study the almost unobservable and fully unobservable queues and the equilibrium mixed joining probabilities are derived. In Section 5, some numerical examples are presented to investigate the effects of various

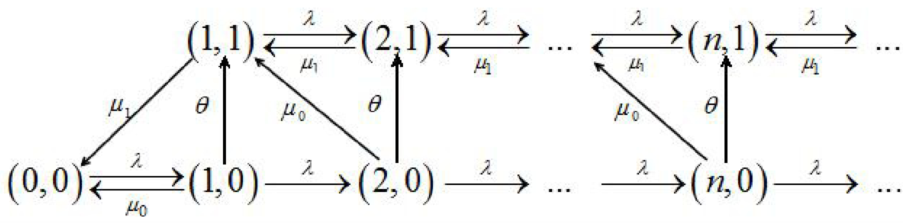

Figure 1. Transition rate diagram of the original model.

values of parameters on customers' behavior in the considered cases. Finally, in Section 6, some conclusions are given.

2. Description of the model

We consider a single server Markovian queue with infinite waiting room under the FCFS discipline, where potential customers arrive according to a Poisson's process with rate λ. The server works with service rate μ_1 and it goes to a vacation period once there is no customer upon completion of a service. If customers arrive during the vacation time, the system provides a low service rate μ_0 ($\mu_0 < \mu_1$). We assume that the working vacation time is exponentially distributed with of mean $1/\theta$. When a working vacation ends, if there is no customer in the queue, the server takes another working vacation. That is, the server adopts a multiple working vacation policy. However, the server will terminates its vacation if there are more follow-up customers waiting in the queue when the first customer completes the service. The service rate switches from μ_0 to μ_1 after the vacation interruption. It is assumed that arrival times, service times, working vacation times are mutually independent.

The state of the system can be represented by a pair $(N(t), I(t))$ at time t, where $N(t)$ denotes the number of customers in the system and $I(t)$ stands for the state of the server (0: on working vacation; 1: normal working state) respectively. It is easy to see that the process $\{N(t), I(t), t \geq 0\}$ is a two-dimensional continuous time Markov chain with state space $\mathbb{S} = \{(n,i)|n = 0,1,2,\ldots; i = 0,1\}$ and non-zero transition rates: $q_{(n,i)(n+1,i)} = \lambda, n = i, i+1, i+2, \ldots; i = 0,1; q_{(1,1)(0,0)} = \mu_1; q_{(1,0)(0,0)} = \mu_0; q_{(n+1,1)(n,1)} = \mu_1; n = 1,2,3\ldots q_{(n+1,0)(n,1)} = \mu_0; n = 1,2,3\ldots q_{(n,0)(n,1)} = \theta; n = 1,2,3\ldots$ The corresponding transition rate diagram is illustrated in Figure 1. We are interested in customer strategic behavior when they can decide whether to join the system or balk based on available information upon their arrival. More specifically, we assume that every customer gets a reward of R units for completing service, but there exists a waiting cost of C units per time unit when they are waiting in the queue or in service. Customers are risk neutral and they want to maximize their expected net benefit. Throughout the paper, it is assumed that

$$R > C\left(\frac{1}{\mu_0 + \theta} + \frac{\theta}{\mu_0 + \theta} \cdot \frac{1}{\mu_1}\right),$$

which enables that when a customer finds the system empty then he will join in the queue because the profit for service absolutely surpasses the expected cost. Finally, the decisions are irrevocable, that is, reneging of entering customers and retrials of balking customers are not allowed.

In the next sections we investigate equilibrium threshold strategies for joining or balking the queueing system. Depending on the information available to customers at their arrival epoch, there are four different levels of information, before their decisions are made (see, for example Burnetas and Economou [2]).

(1) Fully observable case: Customers are informed about the queue length and the server state.
(2) Almost observable case: Customers are informed only about the queue length.
(3) Almost unobservable case: Customers are informed only about the server state.

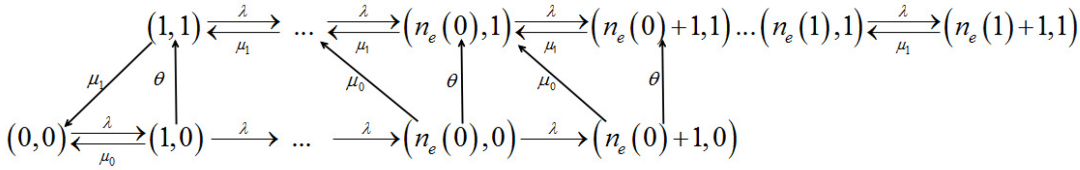

FIGURE 2. Transition rate diagram for the $(n_e(0), n_e(1))$ threshold strategy in the fully observable model.

(4) Fully unobservable case: Customers are not informed about the queue length nor the server state.

For the terminology, we adopt throughout this paper that S_{fo} stands for the individual benefit per time unit in fully observable case, and S_{fu} the individual benefit per time unit in fully unobservable case.

3. Equilibrium threshold strategies for the observable cases

In this section, we consider equilibrium customer strategies of threshold type in the fully observable and almost observable cases. In the fully observable case where customers can observe both the state of the server $I(t)$ and the number of waiting customers $N(t)$ at the arrival time t, a pure threshold strategy is given by a pair $(n_e(0), n_e(1))$, and the balking strategy is 'While arriving at time t, observe $(N(t), I(t))$; enter if $N(t) \leq n_e(I(t))$ and balk otherwise'. In the almost observable case where customers can observe only the number of waiting customers $N(t)$ at arrival time t, a pure threshold strategy is specified by a single number n_e and the strategy is 'While arriving at time t, observe $N(t)$; enter if $N(t) \leq n_e$ and balk otherwise'.

3.1. Fully observable case

In the dynamics of the queueing system showed in Figure 2, where customers can observe both the state of the server $I(t)$ and the number of waiting customers $N(t)$ at arrival time t in the fully observable case. And we have the following results.

Theorem 3.1. *In the fully observable M/M/1 queue with working vacations and vacation interruptions there exist thresholds which are given by*

$$n_e(0) = \lfloor x_e \rfloor, \tag{3.1}$$

$$n_e(1) = \left\lfloor \frac{R\mu_1}{C} \right\rfloor - 1, \tag{3.2}$$

where

$$x_e = \frac{R\mu_1(\mu_0 + \theta) - C(\mu_1 + \theta)}{C(\mu_0 + \theta)}.$$

That is, when a customer arrives and finds the state of the system is $(N(t), I(t))$, if $N(t) \leq n_e(I(t))$, he will join in the system, otherwise he will balk.

Proof. Based on the reward-cost structure which is defined on the system. For an arriving customer, his expected profit is

$$S_{fo}(n, i) = R - CT(n, i), \tag{3.3}$$

where $T(n,i)$ represents his expected mean sojourn time when the system is at state $(N(t), I(t))$ upon his arrival. Then we can get balance equations as follows:

$$T(0,0) = \frac{1}{\mu_0+\theta} + \frac{\theta}{\mu_0+\theta}\cdot\frac{1}{\mu_1}, \tag{3.4}$$

$$T(n,0) = \frac{1}{\mu_0+\theta} + \frac{\theta}{\mu_0+\theta}T(n,1) + \frac{\mu_0}{\mu_0+\theta}T(n-1,1),\ n=1,2,\ldots, \tag{3.5}$$

$$T(n,1) = \frac{n+1}{\mu_1},\ n=1,2,\ldots \tag{3.6}$$

By iterating (3.5) and taking (3.6) into account , we obtain

$$T(n,0) = \frac{(u_1+\theta)+(\mu_0+\theta)n}{\mu_1(\mu_0+\theta)},\ n=1,2,\ldots \tag{3.7}$$

It is obvious that $T(n,0)$ is strictly increasing for n form (3.7). A customer decides to enter if the reward for service surpasses the spend of waiting. We can solve $S_{fo}(n,i)=0$ to find thresholds, *i.e.*,

$$T(n,0) = \frac{(\mu_1+\theta)+(\mu_0+\theta)n}{\mu_1(\mu_0+\theta)} = \frac{R}{C},$$
$$T(n,1) = \frac{n+1}{\mu_1} = \frac{R}{C}.$$

By solving these two equations, we can get results (3.1) and (3.2).

As mentioned above, $R > CT(0,0)$, that is $R > C(\frac{1}{\mu_0+\theta} + \frac{\theta}{\mu_0+\theta}\cdot\frac{1}{\mu_1})$, enables that when a customer finds the system empty then he will join in the queue because the profit for service absolutely surpasses the expected cost. By solving $S_{fo}(n,i) \geq 0$ for n, using (3.3), (3.6) and (3.7), we obtain that the customer arriving at time t decides to enter if and only if $n \leq n_e(I(t))$ where $(n_e(0), n_e(1))$ are given by (3.1) and (3.2). This strategy is preferable, independent of what the other customers do, *i.e.* balks or join in. Furthermore, it is a weakly dominant strategy. □

Remark 3.2. The individual optimal strategies are irrelevant to arrival rate λ. This is because when an arriving customer decides to join a first-come first-served queue, future customers are not influenced by his decision. And we consider the limiting case. When $\mu_0 \to 0$, we can check that $(n_e(0), n_e(1))$ is consistent with the thresholds derived by Burnetas and Economou [2] for the M/M/1 queue with setup times.

Remark 3.3. By simple computation, one can concludes that the threshold $n_e(1)$ of our model is greater than that in the model studied in [2]. It is quite intuitive that in our model when an arriving customer finds the server is in working vacation, due to the fact of interruption policy, the waiting time should be shorter than the model without vacation interruption. It is also noted that the expression of $n_e(0)$ keeps the same in both models.

3.2. Almost observable case

In this section we study the almost observable case. In this case, the arriving customers can observe the number of the present customers in the system, but can not observe the state of the server. Figure 3 depicts the transition diagram of the dynamic of this model. We seek equilibrium strategies within the class of pure threshold strategies. To this end, it is imperative to obtain the stationary distribution of the system when the customers follow a given pure threshold strategy. We have the following results.

Lemma 3.4. *In the almost observable M/M/1 queue with working vacations and vacation interruptions where the customers enter the system according to a threshold strategy 'While arriving at time t, observe $N(t)$; enter*

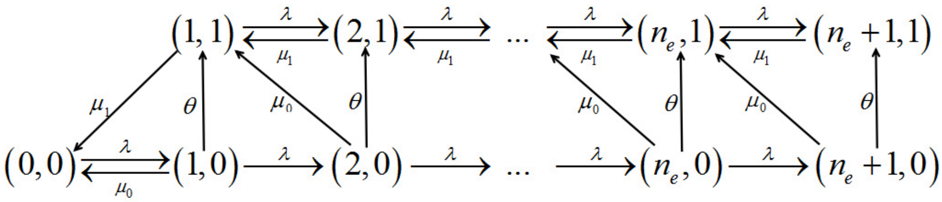

FIGURE 3. Transition rate diagram for the n_e threshold strategy in the almost observable model.

if $N(t) \le n_e$ and balk otherwise', the stationary distribution is given as follows:

$$p_{ao}(n,0) = \sigma^n p(0,0),\ n = 0,1,2,\ldots,n_e, \tag{3.8}$$

$$p_{ao}(n_e+1,0) = \frac{\sigma^{n_e+1}}{1-\sigma}p(0,0), \tag{3.9}$$

$$p_{ao}(n,1) = \frac{z\sigma-\rho}{\sigma-\rho}(\rho^n-\sigma^n)p(0,0), n = 1,2,\ldots,n_e, \tag{3.10}$$

$$p_{ao}(n_e+1,1) = (\frac{z\sigma-\rho}{\sigma-\rho}\rho^{n_e+1} + \frac{\rho(1-\sigma)-z\sigma(1-\rho)}{(\sigma-\rho)(1-\sigma)}\sigma^{n_e+1})p(0,0), \tag{3.11}$$

$$p\,(0,0) = \left[\frac{z\sigma-\rho}{\sigma-\rho}\left(\frac{\rho}{1-\rho}-\frac{\sigma}{1-\sigma}\right) - \frac{1}{\sigma-\rho}\left(\frac{z\sigma-\rho}{1-\rho}\rho^{n_e+2} - \frac{\rho\,(1-z)}{1-\sigma}\sigma^{n_e+2}\right) + \frac{\sigma}{1-\sigma}\right]^{-1}, \tag{3.12}$$

where

$$\rho = \frac{\lambda}{\mu_1}, \tag{3.13}$$

$$\sigma = \frac{\lambda}{\lambda+\mu_0+\theta}, \tag{3.14}$$

$$z = \frac{\mu_0}{\mu_1}\cdot \tag{3.15}$$

Proof. By routine procedure, we get the system balance equations as follows:

$$\lambda p\,(0,0) = \mu_1 p\,(1,1) + \mu_0 p\,(1,0)\,, \tag{3.16}$$

$$(\lambda+\mu_0+\theta)\,p\,(n,0) = \lambda p\,(n-1,0)\,, n = 1,\ldots,n_e, \tag{3.17}$$

$$(\mu_0+\theta)\,p\,(n_e+1,0) = \lambda p\,(n_e,0)\,, \tag{3.18}$$

$$(\lambda+\mu_1)\,p\,(1,1) = \theta p\,(1,0) + \mu_0 p\,(2,0) + \mu_1 p\,(2,1)\,, \tag{3.19}$$

$$(\lambda+\mu_1)\,p\,(n,1) = \lambda p\,(n-1,1) + \mu_1 p\,(n+1,1) + \theta p\,(n,0) + \mu_0 p\,(n+1,0)\ , \quad n = 2,\ldots,n_e, \tag{3.20}$$

$$\mu_1 p\,(n_e+1,1) = \lambda p\,(n_e,1) + \theta p\,(n_e+1,0). \tag{3.21}$$

By iterating (3.17), we can get

$$p(n,0) = \left(\frac{\lambda_0}{\lambda_0+\mu_0+\theta}\right)^n p(0,0),\ n = 0,1,2,\ldots,n_e. \tag{3.22}$$

By (3.18) and (3.22) we can get $p(n_e+1,0)$ as follow,

$$p(n_e+1,0)=\left(\frac{\lambda}{\lambda+\mu_0+\theta}\right)^{n_e}\left(\frac{\lambda}{\lambda+\theta}\right)p(0,0).$$

Letting $\sigma=\frac{\lambda}{\lambda+\mu_0+\theta}$, $\rho=\frac{\lambda}{u_1}$, we can get (3.8) and (3.9).

On the other hand, by (3.20), we can get

$$\mu_1 p(n+1,1)-(\lambda+\mu_1)p(n,1)+\lambda p(n-1,1)=-\left(\frac{\theta(\lambda+\mu_0+\theta)+\mu_0\lambda}{\lambda+\mu_0+\theta}\right)\sigma^n p(0,0). \tag{3.23}$$

In the following, we use a rather standard method to solve this type of equation by solving a linear difference equation with constant coefficients as $\mu_1 x^2-(\lambda+\mu_1)x+\lambda=0$, see *e.g.* Elaydi [6] and Burnetas *et al.* [2].

It is readily seen that the above equation has two roots 1 and ρ and the common root of the homogeneous transformation equation (3.23) is

$$\begin{cases} x_n^{\text{hom}}=A1^n+B\rho^n, & \rho\neq 1;\\ x_n^{\text{hom}}=A1^n+Bn1^n, & \rho=1.\end{cases} \tag{3.24}$$

So the general solution of equation (3.23) is $x_n^{gen}=x_n^{\text{hom}}+x_n^{spec}$, where x_n^{spec} is a special root of the equation (3.23).

Next we want to find a special root of equation (3.23) to replace x_n^{spec}, and find the special root is like $D\sigma^n$ (when $\sigma\neq 1$ and $\sigma\neq\rho$), or like $Dn\sigma^n$(when $\sigma=1$ or $\sigma=\rho$), or like $Dn^2\sigma^n$(when $\sigma=1=\rho$). According to the discussion on the root solution given by [2], we need only consider the common situation. That is, find the special root is like $D\sigma^n$ for the regular case $\sigma\neq 1$ and $\sigma\neq\rho$. Therefore, by simple computation, the solution of the equation (3.23) is given by

$$x_n^{gen}=A1^n+B\rho^n+D\sigma^n,\ \ n=1,2,3,\dots,n_e-1. \tag{3.25}$$

Letting $x_n=D\sigma^n$ and take (3.23) into account, we can get the value of D as follows.

$$D=\frac{\lambda+\theta}{\mu_1-\mu_0-\theta-\lambda}p\left(0,0\right)=-\frac{z\sigma-\rho}{\sigma-\rho}p(0,0).$$

Now, we need to know the values of A and B for the purpose of getting the expression of x_n^{gen}. Letting $n=1$ and $n=2$, using (3.25), we can get

$$\begin{cases} A+B\rho+C\sigma=p(1,1);\\ A+B\rho^2+C\sigma^2=p(2,1).\end{cases} \tag{3.26}$$

We get $p(1,1)$ and $p(2,1)$ from (3.16) and (3.19)

$$\begin{cases} p(1,1)=\frac{\lambda(\lambda+\theta)}{\mu_1(\lambda+\mu_0+\theta)}p(0,0);\\ p(2,1)=\frac{\lambda(\lambda+\mu_1)(\lambda+\theta)(\lambda+\mu_0+\theta)-\theta\lambda\mu_1(\lambda+\mu_0+\theta)-\mu_0\mu_1\lambda^2}{\mu_1^2(\lambda+\mu_0+\theta)^2}p(0,0).\end{cases} \tag{3.27}$$

Solving the equation (3.26), we can get A and B.

$$\begin{cases} A=\frac{p(2,1)-\rho p(1,1)-D\sigma^2+D\sigma\rho}{1-\rho}p(0,0);\\ B=\frac{p(1,1)-D\sigma}{\rho}=\frac{z\sigma-\rho}{\sigma-\rho}p(0,0).\end{cases} \tag{3.28}$$

With the help of known values of A, B, D and equation (3.25) we can obtain (3.10). Consequently, we can get the expression of $p(n_e+1,1)$ by taking (3.9) (3.10) into (3.21). Based on the above results, we can conclude that all probabilities involved can be expressed *via* $p(0,0)$. Finally, we can get the expression of $p(0,0)$ by normalization equation

$$\sum_{n=0}^{n_e+1} p(n,0) + \sum_{n=1}^{n_e+1} p(n,1) = 1,$$

which reaches the result (3.22). This completes the proof of this theorem. □

Next, we will proceed to study the profit net of the almost observable case. In this case, the arriving customers can only observe the number of customers. For an arriving customer, if he finds n customers in front of him and decided to enter in this system, the sojourn time of an arriving customer is $\frac{n+1}{u_1} + \frac{\mu_1-\mu_0}{\mu_1(\mu_0+\theta)} \Pr(I^- = 0|N^- = n)$, where $\Pr(I^- = 0|N^- = n)$ is the probability that the server is off when the system have n customers waiting. So the profit for this customer is

$$R - C\left[\frac{n+1}{\mu_1} + \frac{\mu_1-\mu_0}{\mu_1(\mu_0+\theta)} \Pr\left(I^- = 0|N^- = n\right)\right]. \tag{3.29}$$

To find the equilibrium strategies of threshold type, we should compute $\Pr(I^- = 0|N^- = n)$ as follow.

$$\Pr(I^- = 0\,\big|N^- = n) = \frac{\lambda p_{ao}(n,0)}{\lambda p_{au}(n,1) + \lambda p_{au}(n,0)\, I\{n \geq 1\}},\ n = 1,2,\ldots,n_e+1. \tag{3.30}$$

where

$$I\{n \geq 1\} = \begin{cases} 0, & n = 0; \\ 1, & n \geq 1. \end{cases}$$

Taking (3.8)–(3.10) into (3.30), we can get

$$Pr(I^- = 0\,\big|N^- = n) = \left[1 + \frac{\lambda+\theta}{\mu_1-\mu_0-\lambda-\theta}\left(1 - \left(\frac{\rho}{\sigma}\right)^n\right)\right]^{-1}, \tag{3.31}$$

$$Pr(I^- = 0\big|N^- = n_e+1) = \left[1 + \frac{\lambda+\theta}{\mu_1-\mu_0-\lambda-\theta}\left(\frac{\theta+\frac{\mu_0}{\nu_1}\lambda}{\lambda+\theta} - \frac{\mu_0+\theta}{\lambda+\mu_0+\theta}(\frac{\rho}{\sigma})^{n_e+1}\right)\right]^{-1}. \tag{3.32}$$

Following the method used in Burnetas *et al.* [2], and according to (3.29), (3.31)and (3.32), we can define a function as

$$f(x,n) = R - \tfrac{C(n+1)}{\mu_1} - \tfrac{C(\mu_1-\mu_0)}{(\mu_0+\theta)\mu_1}\left\{1 + \tfrac{1}{\mu_1-\mu_0-\lambda-\theta}\left[(\lambda+\theta-\tfrac{\mu_1-\mu_0}{\mu_1}\lambda x) - (\lambda+\theta-\tfrac{\lambda(\lambda+\theta)}{\lambda+\mu_0+\theta}x)(\tfrac{\rho}{\sigma})^n\right]\right\}^{-1}, \quad x \in [0,1], n = 0,1,2,\ldots \tag{3.33}$$

Next we will prove the existence of equilibrium threshold strategies and give the corresponding thresholds by using $f(x,n)$. Define

$$f_L(n) = f(1,n) = R - \tfrac{C(n+1)}{\mu_1} - \tfrac{C(\mu_1-\mu_0)}{(\mu_0+\theta)\mu_1}\left\{1 + \tfrac{1}{\mu_1-\mu_0-\lambda-\theta}[(\lambda+\theta-\tfrac{\mu_1-\mu_0}{\mu_1}\lambda) - (\lambda+\theta-\tfrac{\lambda(\lambda+\theta)}{\lambda+\mu_0+\theta})(\tfrac{\rho}{\sigma})^n]\right\}^{-1}, \quad x \in [0,1], n = 0,1,2,\ldots, \tag{3.34}$$

$$f_U(n) = f(0,n) = R - \tfrac{C(n+1)}{\mu_1} - \tfrac{C(\mu_1-\mu_0)}{(\mu_0+\theta)\mu_1}\left\{1 + \tfrac{1}{\mu_1-\mu_0-\lambda-\theta}[(\lambda+\theta) - (\lambda+\theta)(\tfrac{\rho}{\sigma})^n]\right\}^{-1}, \quad x \in [0,1], n = 0,1,2,\ldots, \tag{3.35}$$

we can obtain the following results.

Theorem 3.5. *For $f_L(n)$ and $f_U(n)$ defined by (3.34) and (3.35), there are finite non-negative integers $n_L \leq n_U$, such as*

$$f_U(0), f_U(1), \ldots f_U(n_U) \geq 0, f_U(n_U+1) \leq 0, \tag{3.36}$$

and

$$f_L(n_U+1), f_L(n_U) \ldots, f_L(n_L+1) \leq 0, f_L(n_L) \geq 0, \tag{3.37}$$

or

$$f_L(n_U+1), \ldots, f_L(2), f_L(1), f_L(0) \leq 0. \tag{3.38}$$

In the almost observable M/M/1 queue with working vacations and vacation interruptions, equilibrium strategies are 'While arriving at time t, observe: enter if $n \leq n_e$ and balk otherwise' for $n_e \in \{n_L, n_L+1, \ldots n_U\}$.

Proof. Firstly, we prove that $n_e \in \{n_L, n_L+1, \ldots n_U\}$.

(1) It is readily seen that $f_U(0) > 0$ as $S(0) = R - CT\,(0,0) > 0$, and $\lim\limits_{n\to\infty} f_U(n) = -\infty$, so there exist integers n_U that satisfy $f_U(n_U) \geq 0, f_U(n_U+1) \leq 0$.
(2) It can be easily proved that $f_U(n) > f_L(n)$ for $n = 1, 2, \ldots$ So we can get that $f_L(n_U+1) \leq f_U(n_U+1) \leq 0$. Then we have the conclusion above.
Next, we will prove that n_e is the threshold. On one hand, for an arriving customer, and he decides to enter the system, in which there are $n (n \leq n_e)$ customers ahead of him, then his expected benefit is $S(n) = f_U(n) > 0$ by virtue of equation (3.36). So he prefers to enter the system.
On the other hand, for an arriving customer, and he decides to enter the system, in which there are $n = n_e+1$ customers ahead of him, then his expected benefit is $S(n_e+1) = f_L(n_e+1) < 0$ by equations (3.37) or (3.38). So the customer prefers to balk the system.
So, we can conclude that any $n_e \in \{n_L, n_L+1, \ldots, n_U\}$ is the threshold. □

Remark 3.6. In the present model, there is a "Follow-The-Crowd" (FTC) situation, in which ones optimal response to a strategy x adopted by all others is increasing for x. Suppose a customer, there are n customers ahead of him and he decides to enter while others follow the threshold n_e, so we define his net benefit is $S_{n_e}(n)$. Next, we suppose that others follow the threshold n_e+1. In this case, for an arriving customer, his expected benefit is $S_{n_e+1}(n_e+1) = f_U(n_e+1) \geq f_L(n_e+1) = S_{n_e}(n_e+1)$ when there are n_e+1 customers ahead of him and he decides to enter. So we can easily know that a threshold best response of the tagged customer while others follow the threshold n_e+1 is greater than his best threshold response while others follow the threshold n_e. This means an arriving customer prefers to enter the system while others have a higher threshold. In conclusion, we have an FTC situation.

4. Equilibrium threshold strategies for the unobservable cases

In this section we will focus on almost unobservable case and fully unobservable case in which equilibrium mixed strategies exist.

4.1. Almost unobservable case

In this case, customers can observe the state of server $I(t)$ only. Since all customers are assumed indistinguishable, we can consider the situation as a symmetric game among them. In this model, the arriving customers have two pure strategies, to join or to balk. A mixed strategy is studied by the entrance probability of an arriving customer who informed the state of server. Our main point is to find the Nash equilibrium mixed balking strategies.

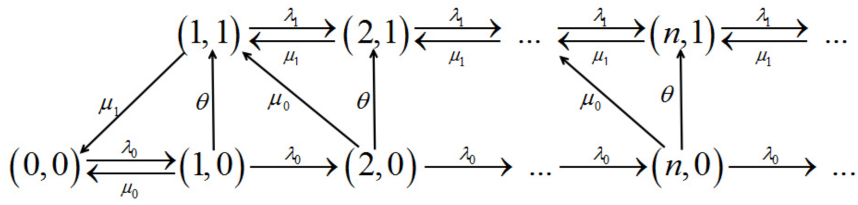

Figure 4. Transition rate diagram for the $(q(0), q(1))$ mixed strategy in the almost unobservable model.

Assume that all customers follow a mixed strategy $(q(0), q(1))$ where $q(i)$ is the probability of joining in queue when sever is in state i. In this case, the system is similar to that described in Figure 1, the only difference is the arrival rate $\lambda_i = \lambda q(i)$ for states where sever is in state i. The transition diagram is showed in Figure 4.

The system is stable if and only if $\lambda_1 < \mu_1$. Let $(p_{au}(n,i) : (n,i) \in \{(0,0)\} \cup \{1,2,\ldots\} \times \{0,1\})$ be the stationary distribution of the corresponding system. We have following results.

Lemma 4.1. *In the almost unobservable queue with working vacations and vacation interruptions in which all customers follow a mixed balking strategy $(q(0), q(1))$, where $q(i)$ is the entrance probability when the server is in state i, the stationary distribution is given as follows:*

$$p_{au}(n,0) = \frac{z(1-\sigma(0))(1-\rho(1))}{z(1-\rho(1)) + \rho(0) - \sigma(0)} \times \sigma^n(0), n = 0,1,2\ldots, \tag{4.1}$$

$$p_{au}(n,1) = \frac{z(1-\sigma(0))(\rho(0)-\sigma(0))(1-\rho(1))}{z(\rho(1)-\sigma(0))[z(1-\rho(1)) + (\rho(0)-\sigma(0))]} \times [\rho^n(1) - \sigma^n(0)], \quad n = 0,1,2,\ldots, \tag{4.2}$$

where

$$\sigma(0) = \frac{\lambda_0}{\lambda_0 + \mu_0 + \theta}, \tag{4.3}$$

$$\rho(1) = \frac{\lambda_1}{\mu_1}, \tag{4.4}$$

$$\rho(0) = \frac{\lambda_0}{\mu_0}, \tag{4.5}$$

$$z = \frac{\mu_1}{\mu_0}\cdot \tag{4.6}$$

Proof. The balance equations are given as follows.

$$\lambda_0 p(0,0) = \mu_1 p(1,1) + \mu_0 p(1,0), \tag{4.7}$$

$$(\lambda_0 + \mu_0 + \theta)p(n,0) = \lambda_0 p(n-1,0), \quad n \geq 1, \tag{4.8}$$

$$(\lambda_1 + \mu_1)p(1,1) = \theta p(1,0) + \mu_0 p(2,0) + \mu_1 p(2,1), \tag{4.9}$$

$$(\lambda_1 + \mu_1)p(n,1) = \lambda p(n-1,1) + \mu_1 p(n+1,1) + \theta p(n,0) + \mu_0 p(n+1,0), \quad n \geq 2. \tag{4.10}$$

By iterating (4.8) we can obtain

$$p(n,0) = \left(\frac{\lambda_0}{\lambda_0 + \mu_0 + \theta}\right)^n p(0,0).$$

From (4.10) it follows that $p\,(n,1)$ $(n = 1,2,\ldots)$ is a solution of the nonhomogeneous linear difference equation with constant coefficients.

$$\mu_1 x_{n+1} - (\lambda_1 + \mu_1)\, x_n + \lambda_1 x_{n-1} = -\left(\theta + \frac{\lambda_0 \mu_0}{\mu_0 + \lambda_0 + \theta}\right)\left(\frac{\lambda_0}{\mu_0 + \lambda_0 + \theta}\right)^n p\,(0,0)\,.$$

Letting $\sigma\,(0) = \frac{\lambda_0}{\lambda_0+\mu_0+\theta}$, $\rho(1) = \frac{\lambda_1}{\mu_1}$. By using the same approach as used in the proof of Lemma 1, we can get (4.2).

Next, we consider an arriving customer who finds the server is at state i and we will give the expected sojourn time of a customer that decides to enter given that the others follow the same mixed strategy $(q\,(0)\,, q\,(1))$.

Case 1. When the server is at state 1, the expected sojourn time is

$$T_{au}\,(1) = \frac{E\,[N|1] + 1}{\mu_1}\cdot \tag{4.11}$$

Case 2. When the server is at state 0, the expected sojourn time is

$$T_{au}\,(0) = \frac{1}{\mu_0 + \theta} + \frac{\mu_0}{\mu_0 + \theta}\frac{E\,[N|0]}{\mu_1} + \frac{\theta}{\mu_0 + \theta}\frac{E\,[N|0] + 1}{\mu_1}, \tag{4.12}$$

where

$$E\,[N|i] = \sum_{n=i}^{\infty} \frac{np\,(n,i)}{\sum\limits_{k=i}^{\infty} p\,(k,i)}\cdot \tag{4.13}$$

Computing $E\,[N|i]$ by taking (4.1) and (4.2) into (4.13), we can get

$$E\,[N|0] = \sum_{n=0}^{\infty} \frac{np\,(n,0)}{\sum\limits_{k=0}^{\infty} p\,(k,0)} = \frac{\sigma\,(0)}{1 - \sigma\,(0)}, \tag{4.14}$$

$$E\,[N|1] = \sum_{n=1}^{\infty} \frac{np\,(n,1)}{\sum\limits_{k=1}^{\infty} p\,(k,1)} = \frac{1}{1 - \sigma\,(0)} + \frac{\rho\,(1)}{1 - \rho\,(1)}\cdot \tag{4.15}$$

Taking (4.14), (4.15), (4.3) and (4.4) into (4.11), (4.12), we can deprive the $T_{au}\,(0)$ and $T_{au}\,(1)$ as follows:

$$T_{au}\,(0) = \frac{\mu_0 + \lambda_0 + \theta}{\mu_1\,(\mu_0 + \theta)},$$
$$T_{au}\,(1) = \frac{\mu_0 + \lambda_0 + \theta}{\mu_1\,(\mu_0 + \theta)} + \frac{\mu_1}{\mu_1 - \lambda}\cdot$$

Based on the reward-cost structure, the expected benefit for an arriving customer who is informed the server is at state i is given as follows.

$$S_{au}\,(0) = R - C\frac{\mu_1 + \lambda_0 + \theta}{\mu_1\,(\mu_0 + \theta)}, \tag{4.16}$$

$$S_{au}\,(1) = R - \frac{C}{\mu_1 - \lambda_1} - \frac{C}{\mu_1}\frac{\lambda_0 + \mu_0 + \theta}{\mu_0 + \theta}\cdot \tag{4.17}$$

Now we will give the mixed equilibrium strategies of an arriving customer in the almost unobservable case.

Theorem 4.2. *In the almost unobservable model of the M/M/1 queue with working vacations and vacation interruptions and $\lambda < \mu_1$, we can get a unique Nash equilibrium mixed strategy $(q(0), q(1))$, 'observe I(t) and enter with probability $q_e(I(t))$', where the $(q(0), q(1))$ is given as follows:*

Case 1. $\frac{C(\mu_1+\theta)}{\mu_1(\mu_0+\theta)} < R < \frac{C(\mu_1+\lambda+\theta)}{\mu_1(\mu_0+\theta)}$.

Case 1a. $\frac{C(\mu_1-\mu_0)}{\mu_1(\mu_0+\theta)} < \frac{C}{\mu_1}$.

$$(q_e(0), q(1)) = \left(\frac{1}{\lambda}\left(\frac{R\mu_1}{C}(\mu_0+\theta) - \mu_1 - \theta\right), 0\right).$$

Case 1b. $\frac{C}{\mu_1} \le \frac{C(\mu_1-\mu_0)}{\mu_1(\mu_0+\theta)} \le \frac{C}{\mu_1-\lambda}$.

$$(q_e(0), q(1)) = \left(\frac{1}{\lambda}\left(\frac{R\mu_1}{C}(\mu_0+\theta) - \mu_1 - \theta\right), \frac{\mu_1(\mu_1-\theta-2\mu_0)}{\lambda(\mu_1-\mu_0)}\right).$$

Case 1c. $\frac{C(\mu_1-\mu_0)}{\mu_1(\mu_0+\theta)} > \frac{C}{\mu_1-\lambda}$.

$$(q_e(0), q(1)) = \left(\frac{1}{\lambda}\left(\frac{R\mu_1}{C}(\mu_0+\theta) - \mu_1 - \theta\right), 1\right),$$

Case 2. $R \ge \frac{C(\mu_1+\lambda+\theta)}{\mu_1(\mu_0+\theta)}$.

Case 2a. $R < \frac{C}{\mu_1} + \frac{C}{\mu_1}\frac{\lambda+\theta+\mu_0}{\mu_0+\theta}$.

$$(q_e(0), q(1)) = (1, 0)\cdot$$

Case 2b. $\frac{C}{\mu_1} + \frac{C}{\mu_1}\frac{\lambda+\theta+\mu_0}{\mu_0+\theta} \le R \le \frac{C}{\mu_1-\lambda} + \frac{C}{\mu_1}\frac{\lambda+\theta+\mu_0}{\mu_0+\theta}$.

$$(q_e(0), q(1)) = \left(1, \frac{1}{\lambda}\left(\mu_1 - \frac{C}{R - \frac{C(\lambda+\mu_0+\theta)}{\mu_1(\mu_0+\theta)}}\right)\right)\cdot$$

Case 2c. $R > \frac{C}{\mu_1-\lambda} + \frac{C}{\mu_1}\frac{\lambda+\theta+\mu_0}{\mu_0+\theta}$.

$$(q_e(0), q(1)) = (1, 1).$$

Proof. Now we consider a customer's expected net benefit when the server is at state 0, and he decides to enter.

Firstly, we consider $q_e(0)$. There are two cases:

Case 1. $\frac{C(\mu_1+\theta)}{\mu_1(\mu_0+\theta)} < R < \frac{C(\mu_1+\lambda+\theta)}{\mu_1(\mu_0+\theta)}$. In this case, we can get $S_{au}(0) = R - C\frac{u_1+\lambda_0+\theta}{u_1(u_0+\theta)} < 0$. When the system is empty, and all customers decide to enter with probability 1, then the tagged customer will get a negative benefit if he decides to enter. And if all customers decide to enter with probability 0, then the tagged customer will get a positive benefit if he decides to enter. So there is a mixed equilibrium strategy. We can get this mixed equilibrium strategy by solving the following equation.

$$S_{au}(0) = R - C\frac{\mu_1 + \lambda q_e(0) + \theta}{\mu_1(\mu_0+\theta)} = 0, \tag{4.18}$$

which gives the the solution of $q_e(0)$ as

$$q_e(0) = \frac{1}{\lambda}\left(\frac{R\mu_1}{C}(\mu_0+\theta) - \mu_1 - \theta\right)\cdot$$

Case 2. $R \ge \frac{C(\mu_1+\lambda+\theta)}{\mu_1(\mu_0+\theta)}$. In this case, we can get $S_{au}(0) = R - C\frac{\mu_1+\lambda_0+\theta}{\mu_1(\mu_0+\theta)} \ge 0$, and no matter what strategy the other customers adopt, the tagged customer will get a positive benefit if he decides to enter. Therefore $q_e(0) = 1$.

Next we consider $q_e(1)$. We tag an arriving customer when the server is at state 1. His expected benefit is as follows:

$$S_{au}(1) = R - \frac{C}{\mu_1 - \lambda_1} - \frac{C}{\mu_1}\frac{\lambda_0 + \mu_0 + \theta}{\mu_0 + \theta} = \begin{cases} \frac{C(\mu_1-\mu_0)}{\mu_1(\mu_0+\theta)} - \frac{C}{\mu_1 - \lambda q_e(1)}, & case1; \\ R - \frac{C}{\mu_1 - \lambda q_e(1)} - \frac{C}{\mu_1}\frac{\lambda+\mu_0+\theta}{\mu_0+\theta}, & case2. \end{cases}$$

To find $q_e(1)$ we must examine Cases 1 and 2, and consider some possible subcases. By using a similar way, we can get the mixed equilibrium strategy as follows:

Case 1. $\frac{C(\mu_1+\theta)}{\mu_1(\mu_0+\theta)} < R < \frac{C(\mu_1+\lambda+\theta)}{\mu_1(\mu_0+\theta)}$.

Case 1a. $\frac{C(\mu_1-\mu_0)}{\mu_1(\mu_0+\theta)} < \frac{C}{\mu_1}$.

$$(q_e(0), q(1)) = \left(\frac{1}{\lambda}\left(\frac{R\mu_1}{C}(\mu_0+\theta) - \mu_1 - \theta\right), 0\right).$$

Case 1b. $\frac{C}{\mu_1} \le \frac{C(\mu_1-\mu_0)}{\mu_1(\mu_0+\theta)} \le \frac{C}{\mu_1-\lambda}$.

$$(q_e(0), q(1)) = \left(\frac{1}{\lambda}\left(\frac{R\mu_1}{C}(\mu_0+\theta) - \mu_1 - \theta\right), \frac{\mu_1(\mu_1-\theta-2\mu_0)}{\lambda(\mu_1-\mu_0)}\right).$$

Case 1c. $\frac{C(\mu_1-\mu_0)}{\mu_1(\mu_0+\theta)} > \frac{C}{\mu_1-\lambda}$.

$$(q_e(0), q(1)) = \left(\frac{1}{\lambda}\left(\frac{R\mu_1}{C}(\mu_0+\theta) - \mu_1 - \theta\right), 1\right).$$

Case 2. $R \ge \frac{C(\mu_1+\lambda+\theta)}{\mu_1(\mu_0+\theta)}$.

Case 2a. $R < \frac{C}{\mu_1} + \frac{C}{\mu_1}\frac{\lambda+\theta+\mu_0}{\mu_0+\theta}$.

$$(q_e(0), q(1)) = (1, 0).$$

Case 2b. $\frac{C}{\mu_1} + \frac{C}{\mu_1}\frac{\lambda+\theta+\mu_0}{\mu_0+\theta} \le R \le \frac{C}{\mu_1-\lambda} + \frac{C}{\mu_1}\frac{\lambda+\theta+\mu_0}{\mu_0+\theta}$.

$$(q_e(0), q(1)) = \left(1, \frac{1}{\lambda}\left(\mu_1 - \frac{C}{R - \frac{C(\lambda+\mu_0+\theta)}{\mu_1(\mu_0+\theta)}}\right)\right).$$

Case 2c. $R > \frac{C}{\mu_1-\lambda} + \frac{C}{\mu_1}\frac{\lambda+\theta+\mu_0}{\mu_0+\theta}$.

$$(q_e(0), q(1)) = (1, 1).$$

This completes the proof. □

Remark 4.3. In Theorem 3, we assumed that $\lambda < \mu_1$, now we consider the opposite case. Similarly, we can get the mixed strategy. In the almost unobservable model of the M/M/1 queue with working vacations and vacation interruptions and $\lambda \ge \mu_1$, we can get a unique Nash equilibrium mixed strategy $(q(0), q(1))$ 'observe $I(t)$ and enter with probability $q_e(I(t))$' where $(q(0), q(1))$ is given as follows:

Case 1. $\frac{C(\mu_1+\theta)}{\mu_1(\mu_0+\theta)} < R < \frac{C(\mu_1+\lambda+\theta)}{\mu_1(\mu_0+\theta)}$.

Case 1a: $\frac{C(\mu_1-\mu_0)}{\mu_1(\mu_0+\theta)} < \frac{C}{\mu_1}$.

$$(q_e(0), q(1)) = \left(\frac{1}{\lambda}\left(\frac{R\mu_1}{C}(\mu_0+\theta) - \mu_1 - \theta\right), 0\right).$$

Case 1b: $\frac{C(\mu_1-\mu_0)}{\mu_1(\mu_0+\theta)} \ge \frac{C}{\mu_1}$.

$$(q_e(0), q(1)) = \left(\frac{1}{\lambda}\left(\frac{R\mu_1}{C}(\mu_0+\theta) - \mu_1 - \theta\right), \frac{\mu_1(\mu_1-\theta-2\mu_0)}{\lambda(\mu_1-\mu_0)}\right).$$

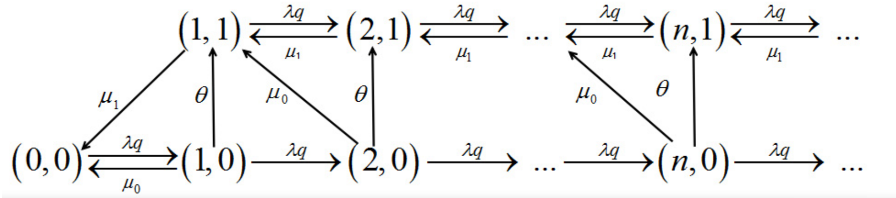

FIGURE 5. Transition rate diagram for the q_e mixed strategy in the fully unobservable model.

Case 2. $R \geq \frac{C(\mu_1+\lambda+\theta)}{\mu_1(\mu_0+\theta)}$.

Case 2a: $R < \frac{C}{\mu_1} + \frac{C}{\mu_1}\frac{\lambda+\theta+\mu_0}{\mu_0+\theta}$.

$$(q_e(0), q(1)) = (1, 0).$$

Case 2b: $R \geq \frac{C}{\mu_1} + \frac{C}{\mu_1}\frac{\lambda+\theta+\mu_0}{\mu_0+\theta}$.

$$(q_e(0), q(1)) = \left(1, \frac{1}{\lambda}\left(\mu_1 - \frac{C}{R - \frac{C(\lambda+\mu_0+\theta)}{\mu_1(\mu_0+\theta)}}\right)\right).$$

4.2. Fully unobservable case

Finally, we consider the fully unobservable case. In this case, the customer do not observe the server state at all. In this model, there are also two strategies, to join or to balk. Here is a mixed strategy for an arriving customer is specified by the probability q of entering and the actual arrival rate is $\lambda q(1) = \lambda q(0) = \lambda q$. Figure 5 gives the corresponding transition diagram.

Using known expressions of $p(n,0)$ and $p(n,1)$ in Lemma 3, letting $\lambda q(1) = \lambda q(0) = \lambda q$, the mean number of the customers in the system is

$$E[N] = \sum_{n=1}^{\infty} n(p(n,0) + p(n,1)) = \frac{z\sigma(0)(1-p(1))^2 + (p(0)-\sigma(0))(1-\sigma(0)p(1))}{(1-p(1))(1-\sigma(0))(z-\sigma(0))}. \tag{4.19}$$

By using the Little's Law, we can get the sojourn time of a customer who decides to enter,

$$E[T(N)] = \frac{E[N]}{\lambda q} = \frac{z\sigma(0)(1-p(1))^2 + (p(0)-\sigma(0))(1-\sigma(0)p(1))}{\lambda q(1-p(1))(1-\sigma(0))(z-\sigma(0))}. \tag{4.20}$$

Theorem 4.4. *In the fully unobservable model of the M/M/1 queue with working vacations and vacation interruptions and $\lambda < \mu_1$, we can get a unique Nash equilibrium mixed strategy 'enter with probability q_e', where q_e is given by*

$$q_e = \begin{cases} q_e^*, \text{ if } R \in \left(\frac{C}{\mu_0+\theta} + \frac{C\theta}{\mu_1(\mu_0+\theta)}, \frac{C}{\mu_0+\theta} + \frac{C\mu_1(\lambda+\theta)}{[\lambda(\mu_1+\mu_0)+\mu_1(\mu_0+\theta)](\mu_1-\lambda)}\right); \\ 1, \quad \text{if } R \in \left(\frac{C}{\mu_0+\theta} + \frac{C\mu_1(\lambda+\theta)}{[\lambda(\mu_1+\mu_0)+\mu_1(\mu_0+\theta)](\mu_1-\lambda)}, \infty\right), \end{cases}$$

where q_e^* is the unique root of equation

$$R - C\left(\frac{1}{\mu_0+\theta} + \frac{\mu_1(\lambda q+\theta)}{[\lambda q(\mu_1-\mu_0) + \mu_1(\mu_0+\theta)](\mu_1-\lambda q)}\right) = 0.$$

Proof. The expected benefit of an arriving customer is

$$S_{fu}\left(q\right)=R-CE[T(N)]=R-C\left(\frac{1}{\mu_0+\theta}+\frac{\mu_1\left(\lambda q+\theta\right)}{\left[\lambda q\left(\mu_1-\mu_0\right)+\mu_1\left(\mu_0+\theta\right)\right]\left(\mu_1-\lambda q\right)}\right).$$

Firstly, we investigate the monotonicity of S_{fu}. Define

$$g\left(q\right)=\frac{\mu_1\left(\lambda q+\theta\right)}{\left[\lambda q\left(\mu_1-\mu_0\right)+\mu_1\left(\mu_0+\theta\right)\right]\left(u_1-\lambda q\right)},$$
$$\frac{dg\left(q\right)}{dq}=\frac{\mu_1\lambda\left(\mu_1-\mu_0\right)\left(\lambda q+\theta\right)^2+\mu_1\lambda\mu_0(\mu_1+\theta)^2}{\left\{\left[\lambda q\left(\mu_1-\mu_0\right)+\mu_1\left(\mu_0+\theta\right)\right]\left(\mu_1-\lambda q\right)\right\}^2}.$$

Since $\mu_1>\mu_0$, $\frac{dg(q)}{dq}=\frac{u_1\lambda(\mu_1-\mu_0)(\lambda q+\theta)^2+\mu_1\lambda\mu_0(\mu_1+\theta)^2}{\{[\lambda q(\mu_1-\mu_0)+\mu_1(\mu_0+\theta)](\mu_1-\lambda q)\}^2}>0, q\in[0,1]$, we can conclude that $g\left(q\right)$ is strictly increasing, and therefore $S_{fu}\left(q\right)=R-Cg\left(q\right)$ is strictly decreasing. So, there exists a unique solution of equation $S_{fu}\left(q\right)=0$, denoted by q_e^*.

When $R\in\left(\frac{C}{\mu_0+\theta}+\frac{C\theta}{\mu_1(\mu_0+\theta)},\frac{C}{\mu_0+\theta}+\frac{C\mu_1(\lambda+\theta)}{[\lambda(\mu_1+\mu_0)+\mu_1(\mu_0+\theta)](\mu_1-\lambda)}\right)$, we find a unique root q_e^* in (0,1). On the other hand, for the case of $R\in\left(\frac{C}{\mu_0+\theta}+\frac{C\mu_1(\lambda+\theta)}{[\lambda(\mu_1+\mu_0)+\mu_1(\mu_0+\theta)](\mu_1-\lambda)},\infty\right)$, for an arriving customer the expected benefit is positive for every q. In this case $q_e=1$. □

5. Numerical examples

In this section, we give some numerical examples to study the sensitivity with varies of parameters. Specifically, we discuss the equilibrium thresholds in observable cases and the equilibrium entrance probability in unobservable cases.

In Figures 6−10, we investigate the varies of equilibrium thresholds in observable cases with the varies of setup rate θ, arrival rate λ, service reward R, normal service rate μ_1, and vacation service rate μ_0, along with the assumption that $C=1$. In Figures 6−10, we can observe that the equilibrium thresholds for the almost observable cases $n_e\in\{n_L,n_L+1,\ldots,n_U\}$ is contained in the range from $n_e\left(0\right)$ to $n_e\left(1\right)$ for the fully observable cases. This situation shows that when the customers do not know the server state, their entrance probability is always in the middle of the two separated thresholds.

Next, we study the Figures 6−10 one by one. In Figure 6, we observe that thresholds are increasing functions to the parameter θ except for $n_e\left(1\right)$ which is fixed. The reason is that the customers will reduce their sojourn time when the server is more faster to change state from working vacation to the normal working level. The customers are willing to enter. And $n_e\left(1\right)$ is always fixed since θ is irrelevant to customers decision, which is concluded in Theorem 3.1. Under different arrival rate λ, which can be studied in Figure 7. We can know $n_e\left(0\right)$ and $n_e\left(1\right)$ are fixed, since λ is irrelevant to customers decision when the customers know the fully information from Theorem 3.1. And the thresholds for the almost observable case are increasing with the λ, which indicates that an arriving customer prefers to enter the queue, when he only knows the number of waiting customers upon arrival. Figure 8 shows that the equilibrium thresholds are increasing with the increase of reward R. Finally, in Figures 9−10, with the increasing of normal service rate μ_1 and vacation service rate μ_0, we can conclude that customers are willing to enter the system.

Similarly, in Figures 11−15, the equilibrium entrance probability q_e is always between $q_e\left(0\right)$ and $q_e\left(1\right)$. At the same time, one can see that there are some differences among the Figures 11−15, such as the order of $q_e\left(0\right)$ and $q_e\left(1\right)$. So we can conclude that an arriving customer enters when they have no information about the system with a probability between those in almost unobservable cases that they are informed the server state.

Finally, we study the sensitivity to the setup rate θ, arrival rate λ, server reward R, work service rate μ_1, and vacation service rate μ_0. The entrance probabilities are increasing with the reward R. The reason is customer will get more reward, they prefer to enter. With the increasing of θ, $q_e\left(0\right)$ and q_e are increasing, $q_e\left(1\right)$ is decreasing

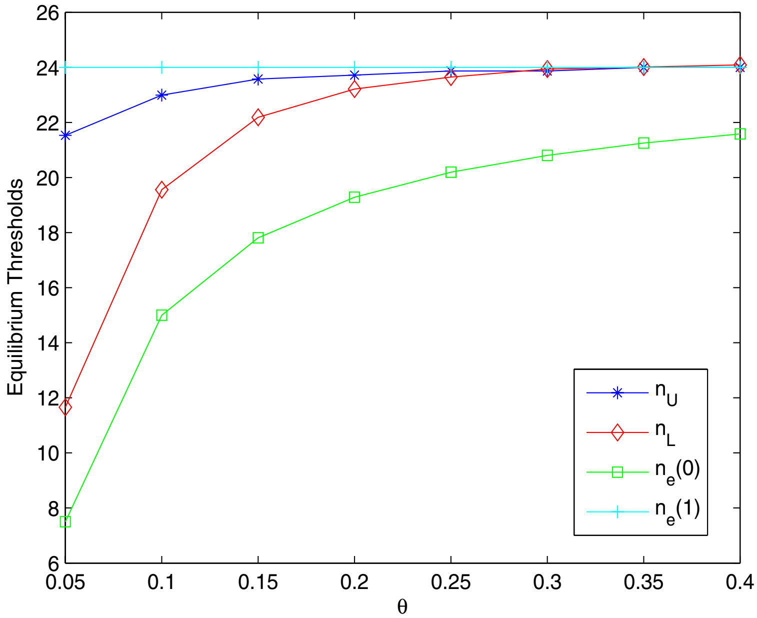

FIGURE 6. Equilibrium thresholds for observable models *vs.* θ for $\lambda = 0.8, \mu_1 = 1, R = 25, C = 1, \mu_0 = 0.01$.

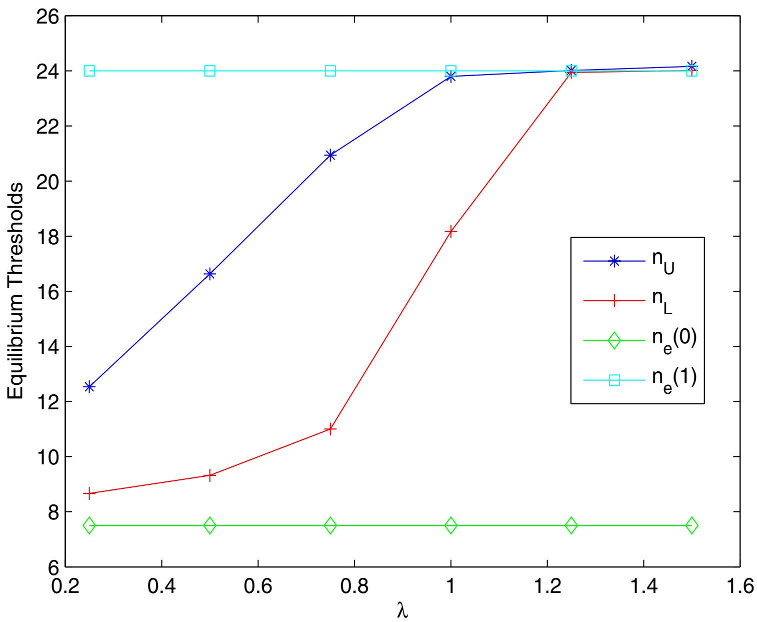

FIGURE 7. Equilibrium thresholds for observable models *vs.* λ for $\theta = 0.05, \mu_1 = 1, R = 25, C = 1, \mu_0 = 0.01$.

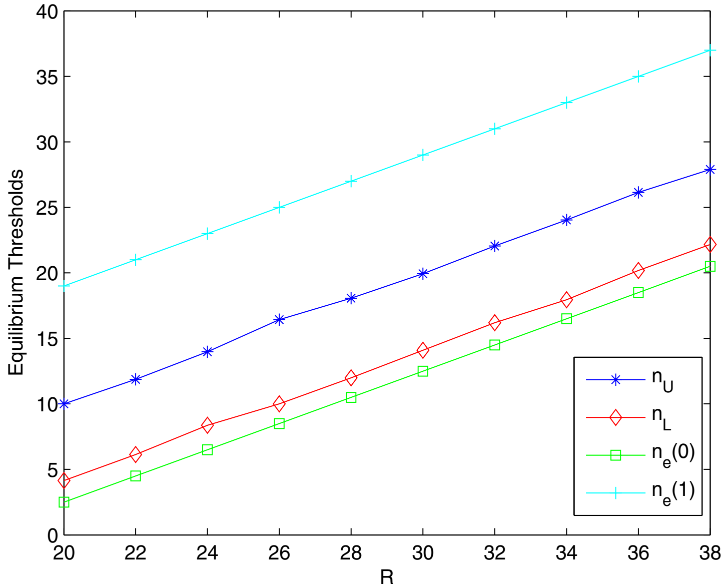

FIGURE 8. Equilibrium thresholds for observable models *vs.* R for $\lambda = 0.4, \mu_1 = 1, \theta = 0.05, C = 1, \mu_0 = 0.01$.

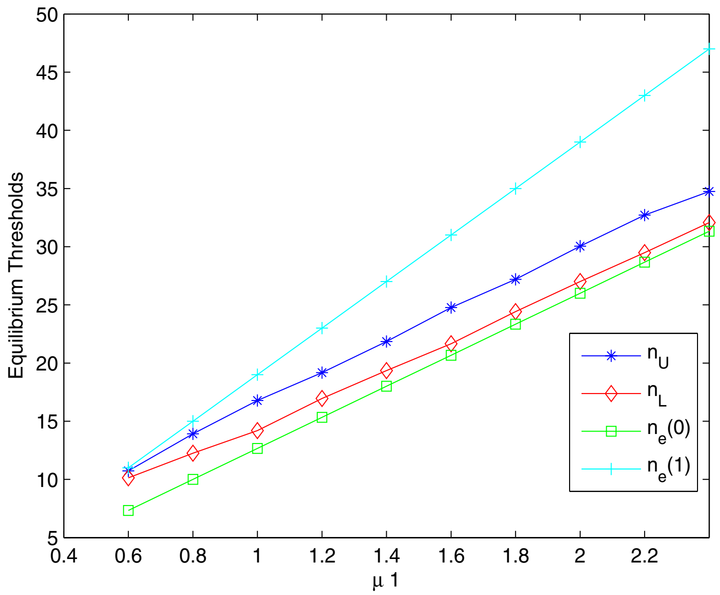

FIGURE 9. Equilibrium thresholds for observable models *vs.* μ_1. for $\lambda = 0.5, R = 20, \theta = 0.1, C = 1, \mu_0 = 0.05$.

first to a certain level, and then increasing. That is, $q_e(1)$ has a minimum. To the arrival rate λ, the entrance probability is decreasing in the almost unobservable case. The reason is that customers expect that the system is more loaded and prefer to balk with the increasing of λ. This is different from the observable cases, since the customer is not informed the number of waiting customers. So, for an arriving customer, he will think the

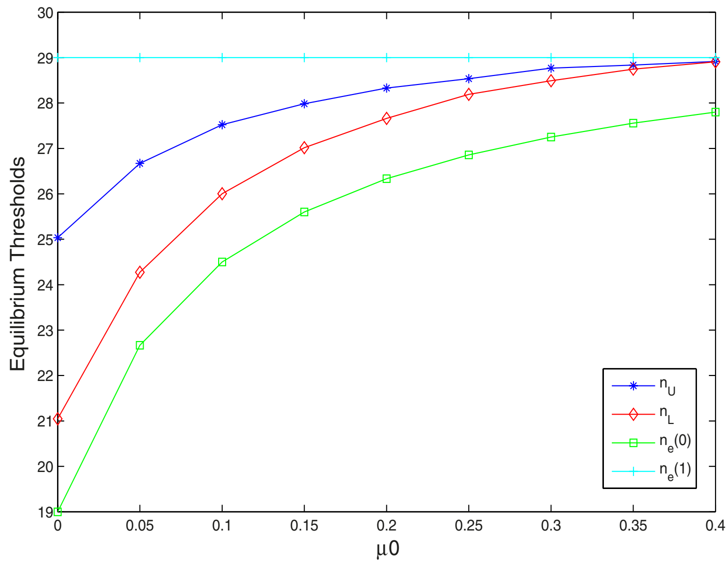

FIGURE 10. Equilibrium thresholds for observable models *vs.* μ_0. for $\lambda = 0.5, R = 30, \theta = 0.1, C = 1, \mu_1 = 1$.

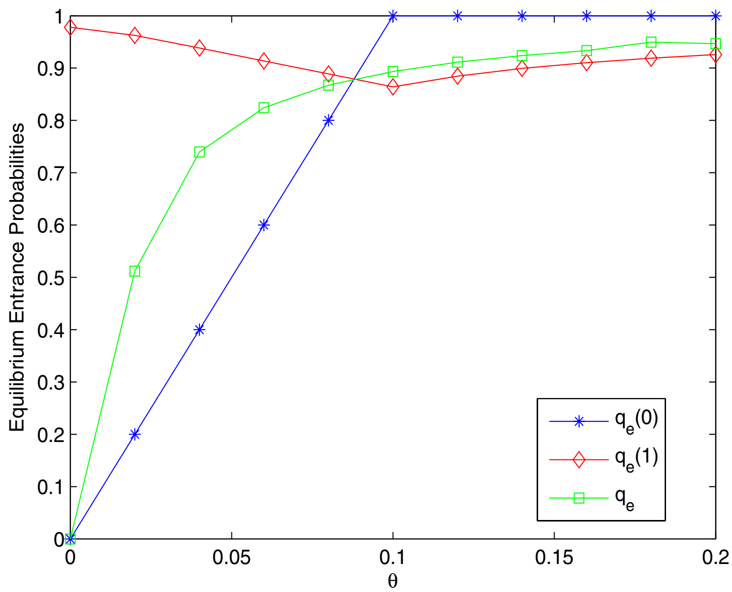

FIGURE 11. Equilibrium entrance probabilities for unobservable models *vs.* θ for $\lambda = 0.9, \mu_1 = 1, R = 10, C = 1, \mu_0 = 0.1$.

system is crowded and decides to balk with the high arrival rate. With respect to μ_1, entrance probability is increasing with the increasing of normal service rate μ_1, since the customers expected waiting time will decrease no matter what state the server is. So the customers prefer to enter. For vacation service rate μ_0, similarly to θ,

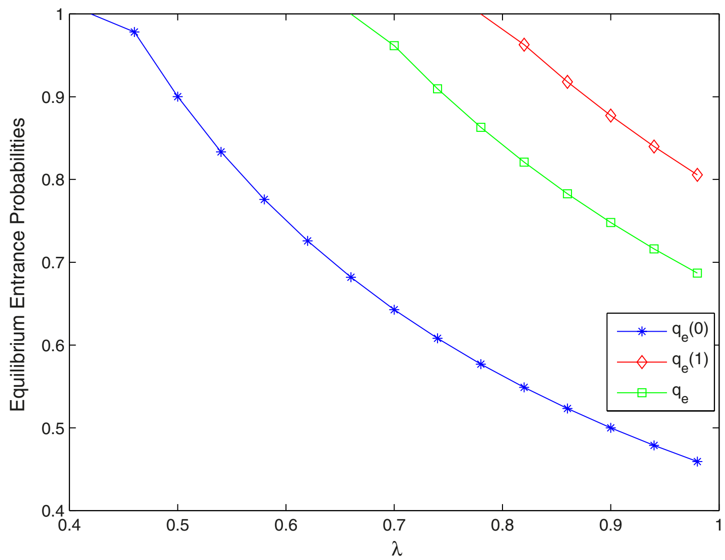

FIGURE 12. Equilibrium entrance probabilities for unobservable models *vs.* λ for $\mu_1 = 1, R = 20, C = 1, \theta = 0.05, \mu_0 = 0.05$.

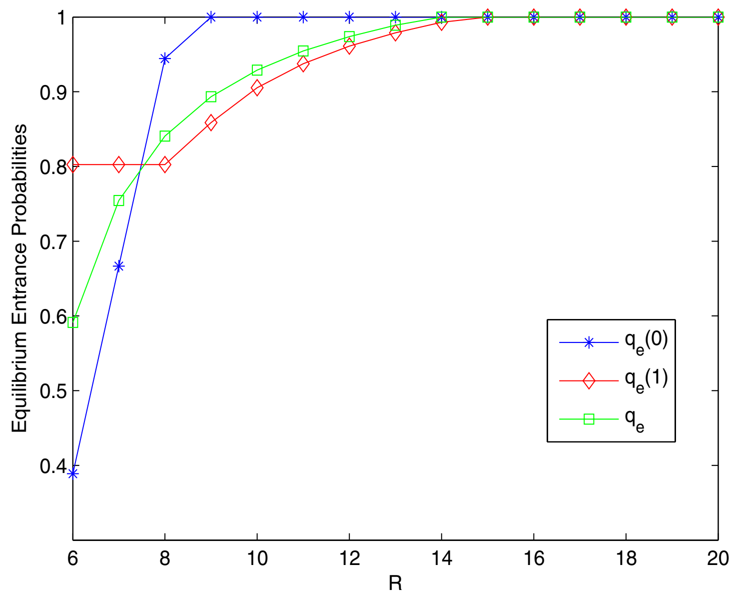

FIGURE 13. Equilibrium entrance probabilities for unobservable models *vs.* R for $\lambda = 0.9, \mu_1 = 1, \theta = 0.15, C = 1, \mu_0 = 0.1$.

$q_e(0)$ and q_e are increasing, and $q_e(1)$ is decreasing to a certain level, and then increasing. The reason is that the customers who observe that the server is on working vacations or do not have any information about the state of the server tend to enter the queue when more customers are served during working vacations. And when the vacation service rate μ_0 starts to increase, the customers who observe that the system is on the working

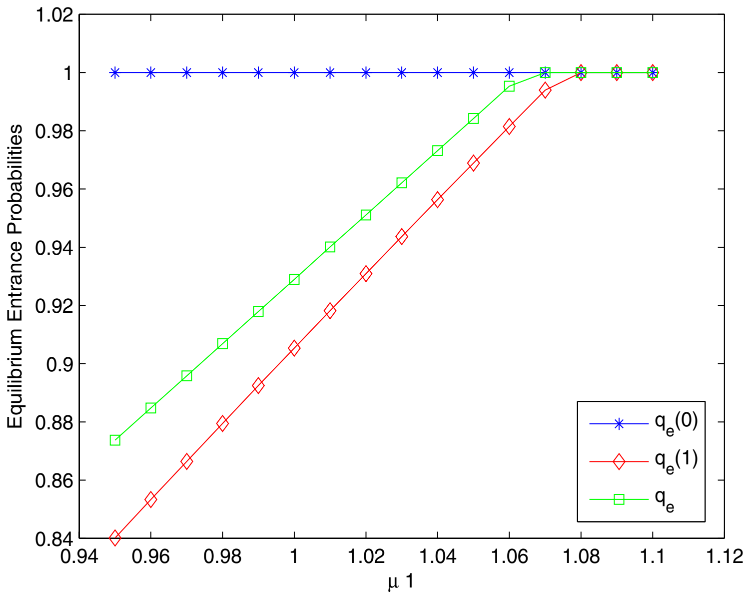

FIGURE 14. Equilibrium entrance probabilities for unobservable models *vs.* μ_1. for $\lambda = 0.9, \theta = 0.15, R = 10, C = 1, \mu_0 = 0.1$.

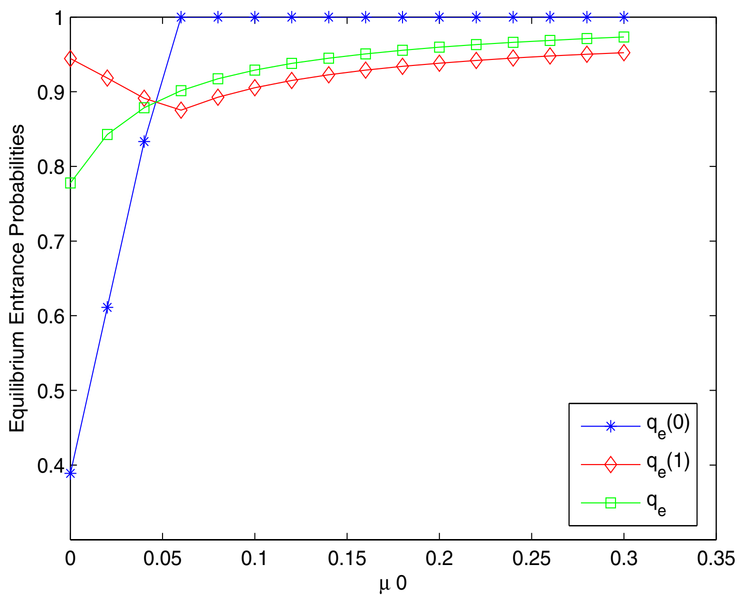

FIGURE 15. Equilibrium entrance probabilities for unobservable models *vs.* μ_0 for $\lambda = 0.9, \mu_1 = 1, R = 10, C = 1, \theta = 0.15$.

vacation are less incline to enter the system. However, as μ_0 continues to increase, the customers who observe that the server is not on working vacations expect the system may be not much loaded and is more likely to enter.

6. Conclusions

In this paper, we studied customers' strategies behavior in an M/M/1 queue with working vacation and vacation interruptions. We considered four cases under different levels of information. For each case, we gave the corresponding equilibrium strategies for the customer. Specifically, we gave the equilibrium threshold strategies in the observable cases and the equilibrium mixed joining probabilities for arriving customers in the unobservable cases. Furthermore, we also investigated the effects of various values of parameters on the equilibrium threshold and equilibrium entrance probabilities through numerical examples. This is the first time that queueing systems with working vacations and vacation interruptions are studied from an economic viewpoint with game-theoretic method. For further research, one can further consider the corresponding social optimization problems arising from each information level studied in this paper.

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