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# EQUILIBRIUM BALKING STRATEGIES IN THE OBSERVABLE GEO/GEO/1 QUEUE WITH DELAYED MULTIPLE VACATIONS

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**Abstract.** We consider the discrete-time Markovian single-server queue under delayed multiple vacations. Upon arriving, the customers observe the queue length and decide whether to join or balk. We derive equilibrium threshold balking strategies in two cases, according to the information for the server's state. We also illustrate the equilibrium thresholds and the social benefits for systems via numerical experiments.

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## 1. Introduction

Recently, economic analysis of queueing systems has been investigated extensively due to their widely applications for management in service system and electronic commerce, where the customers's strategic responses for maximizing their own benefit are considered. The customers are allowed to make decisions as to whether to join or balk, to wait or abandon, to buy priority or not, *etc.* The pioneering work is Naor [14] for an M/M/1 queue with a simple linear reward-cost structure, where each customer observes the queue length before his decision. Later, Edelson and Hildebrand [7] investigated Naor's model by assuming that there is no information on the queue length for an arriving customer. Moreover, several researchers have studied the same problem for various queuing systems considering diverse characteristics *e.g.* retrials, breakdown and repairs, priorities, reneging and jockeying, schedules, *etc.* The fundamental results in this area with extensive bibliographical references can be found in the comprehensive monograph by Hassin and Haviv [11]. Further studies can be referred to Aksin *et al.* [1], Boudali and Economou [2], Economou and Kanta [4,5], Hassin [10], Wang and Zhang [18] and references therein.

Queuing systems with vacations deal with situations where the servers may be unavailable for serving customers over some intervals of time. Such situations frequently occur in real applications; for example, a server may be deactivated for economic reasons (low traffic intensity and/or high stand-by costs), suffer random failures, go under preventive maintenance or attend a secondary system. A comprehensive study in vacation queueing models can be found in Takagi [16], Tian and Zhang [17]. In the past, the study of the queues with

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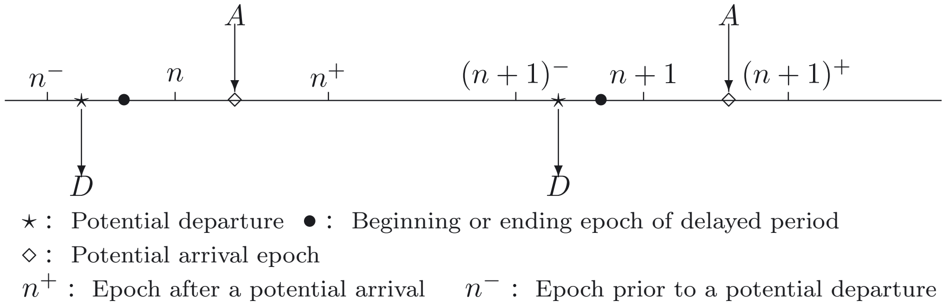



FIGURE 1. Various time epochs in an early arrival system.

vacations hasn't taken into customers's behavior. In fact, the customers's strategic behavior was first analyzed by Burnetas and Economou [3] in a single-server queue with exponential setup times. Subsequently, Guo and Hassin [8] considered other related models under various levels of information. Sun *et al.* [15] consider the strategic customer behavior in an M/M/1 system with closedown and setup. Guo and Hassin [9] studied the strategic behavior and social optimization in Markovian vacation queues with heterogeneous customers. Liu *et al.* [12] studied the equilibrium behavior of customers in continuous/discrete time observable queueing systems under single vacation policy. Ma *et al.* [13] considered the equilibrium balking behavior in the Geo/Geo/1 queueing system with multiple vacations under observable and unobservable cases. Zhang *et al.* [19] dealt with equilibrium balking strategies for an M/M/1 queue with working vacations. Different from above existing work, Economou *et al.* [6] study the joining-balking problem in the single-server queue with generally distributed service and vacation times. To the best of the authors's knowledge, so far, there exists no literature studying the discrete-time queueing system with delayed multiple vacations from an economic viewpoint. The aim of this paper is to study the equilibrium behavior of customers in the context of an observable discrete-time Geo/Geo/1 queue with delayed multiple vacations.

The paper is organized as follows. Descriptions of the model and price structure are given in Section 2. In Section 3, the equilibrium strategies for fully observable and partially observable queues are identified and the equilibrium social benefit for both models are derived. In Section 4, some numerical examples are presented to illustrate the effect of several parameters on the customers's behavior in the considered models.

## 2. Model formulation

In this paper, we denote $\bar{x} = 1 - x$ for any real number $x \in (0, 1)$. The Geo/Geo/1 queue with delayed multiple vacations we considered here is an early arrival system, that is, a potential arrival can only take place in $(n, n^+)$ and a potential departure can only take place in $(n^-, n)$. We assume that the beginning and ending of vacation occurs at division point $n$. Arriving customers are queued according to the first-come, first-served (FCFS) discipline. The server can serve only one customer at a time. Various stochastic processes involved in the system are independent of each other. The various time epochs at which events occur are depicted in Figure 1.

The inter-arrival times $\{T_n, n \in \mathbb{N}\}$ are independent and identically distributed (i.i.d) sequences with generic random variable $T$, where $T$ follows a geometric distribution with rate $p$ $(0 < p < 1)$,

$$P(T = k) = p\bar{p}^{k-1}, k \geq 1.$$

The service times $\{S_n, n \in \mathbb{N}\}$ are i.i.d sequences with generic random variable $S$, where $S$ follows a geometric distribution with rate $\mu$ $(0 < \mu < 1)$,

$$P(S = k) = \mu\bar{\mu}^{k-1}, k \geq 1.$$

After serving all the customers, the server will remain in the system for a period of time, referred as the delayed period. The delayed period $D$ follows a geometric distribution with parameter $\xi$ $(0 < \xi < 1)$,

$$P(D = k) = \xi\bar{\xi}^{k-1}, k \geq 1.$$

If a customer arrives during the delayed period, the server starts to serve the customer immediately. Otherwise, after the delayed period, the server will take a vacation immediately at the end of delayed period. If he finds the system still empty upon returning from the vacation, he will take another vacation, otherwise he begins to server the customers. The vacation times $\{V_n, n \in \mathbb{N}\}$ are i.i.d sequences with generic random variable $V$, where $V$ is geometrically distributed with rate $\theta$ $(0 < \theta < 1)$,

$$P(V = k) = \theta\bar{\theta}^{k-1}, k \geq 1.$$

We represent the state of the system at time $n$ by a pair $(N_n, J_n)$, where $N_n$ and $J_n$ denote the number of customers and the state of the server (0: the server is in vacation period; 1: the server is in busy period or delayed period) respectively. The process $\{(N_n, J_n), n \geq 0\}$ is a two-dimensional discrete-time Markov's chain with state space $\Omega = \{(k, j), k \geq 0, j = 0, 1\}$, where state $(k, 0), k \geq 0$, indicates that the server is in the vacation period and there are $k$ customers; state $(0, 1)$ indicates that the server is in the delayed period and there is no customer; state $(k, 1), k \geq 1$, indicates that the server is in the busy period and there are $k$ customers. Furthermore, we assume that the customers are allowed to decide whether to join or balk upon their arrival based on the information they have. After service, every customer receives a reward of $R$ units. This may reflect his satisfaction or the added value of being served. On the other hand, there exists a waiting cost of $C$ units per unit time when the customers remain in the system including the time of waiting in queue and being served. Customers are risk neutral and maximize their expected net benefit. Hereinafter, we assume the condition

$$R \geq \frac{C}{\mu} + \frac{C}{\theta}, \tag{2.1}$$

which ensures that a customer who joins the system when no customer is in the system and the server is in a vacation period expects a benefit $R - (\frac{C}{\mu} + \frac{C}{\theta})$, the customer then enters if this value is nonnegative. Otherwise, all individuals, even in the most ideal situation of observing an empty system in a vacation period, would balk. A customer who balks leaves the system and never returns. Their decisions are irrevocable that retrials of balking customers and reneging of entering customers are not allowed. Each customer can observe the number of customers ahead of him upon his arrival. In this paper, we consider two information cases regarding whether the customers observe also the state of the server or not, which are known as fully observable and partially observable cases in the literature.

## 3. Equilibrium threshold strategies

In this section, we shall show that there exist equilibrium balking strategies of thresholds type in two cases mentioned above. Intuitively, a strategy is an equilibrium if it is a best response against itself; a strategy is weakly dominant if it is a best response against any strategy, see Hassin and Haviv [11] for more details. Note that the terminology "balking strategy of threshold" means that arriving customers will join the system if the number of customer in the system does not exceed some threshold to be determined, and decide to balk otherwise. At this point, any such system is stable. In the fully observable queue, a pure threshold strategy is specified by a pair $(n_e(0), n_e(1))$ and has the form "observe $(N_n, J_n)$ at arrival epoch; enter if $N_n \leq n_e(J_n)$ and balk otherwise". In the almost observable queue, a pure threshold strategy is specified by a single number $n_e$ and has the form "observe $N_n$; enter if $N_n \leq n_e$ and balk otherwise".

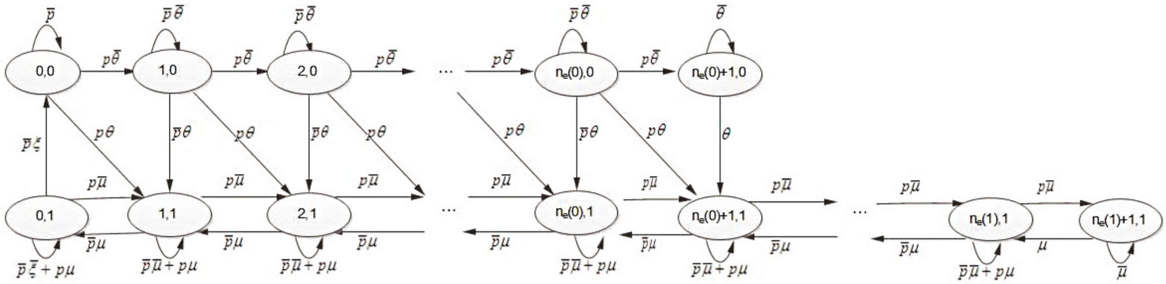



FIGURE 2. Transition rate diagram for the $(n_e(0), n_e(1))$ threshold strategy in the fully observable queue.

### 3.1. Fully observable case

We begin with the fully observable case in which arriving customers know both the number of present customers $N_n$ and the state of the server $J_n$. In equilibrium, a customer who joins the system when he observes state $(k, j)$ has mean sojourn time

$$(S_e(0), S_e(1)) = \left(\frac{k+1}{\mu} + \frac{1}{\theta}, \frac{k+1}{\mu}\right).$$

Thus his expected net benefit is

$$(B_e(0), B_e(1)) = \left(R - \frac{C(k+1)}{\mu} - \frac{C}{\theta}, R - \frac{C(k+1)}{\mu}\right).$$

The customer strictly prefers to enter if this value is positive and is indifferent between entering and balking if it equals zero. Thus we have the following theorem.

**Theorem 3.1.** *In the fully observable Geo/Geo/1 queue with delayed multiple vacations, there exists a pair of thresholds*

$$(n_e(0), n_e(1)) = \left(\left\lfloor \frac{R\mu}{C} - \frac{\mu}{\theta} \right\rfloor - 1, \left\lfloor \frac{R\mu}{C} \right\rfloor - 1\right), \tag{3.1}$$

*such that the strategy 'observe $(N_n, J_n)$, enter if $N_n \le n_e(J_n)$ and balk otherwise' is a unique equilibrium in the class of threshold strategies.*

Next, we focus on the stationary distribution of the system in the fully observable case. Note that if all customers follow the threshold strategy in (3.1), the system follows the Markov's chain $\{(N_n, J_n), n \ge 0\}$ with state space $\Omega_{fo} = \{(n, j) | 0 \le n \le n_e(j) + 1, j = 0, 1\}$. The transition rate diagram is depicted in Figure 2.

From Figure 2, we can see that the Markov's chain $\{(N_n, J_n), n \ge 0\}$ is an irreducible non-periodic Markov's chain with a finite number of states $\Omega_{fo}$, then the Markov's chain $\{(N_n, J_n), n \ge 0\}$ is ergodic. Define the corresponding stationary distribution as $\pi_{k,j} = \lim_{n\to\infty} P(N_n = k, J_n = j), (k, j) \in \Omega_{fo}$. Then from Figure 2, the stationary distribution $\{\pi_{k,j} : (k, j) \in \Omega_{fo}\}$ can be obtained as the unique positive normalized solution of the following system of balance equations:

$$\pi_{0,0} = \bar{p}\pi_{0,0} + \bar{p}\xi\pi_{0,1}, \tag{3.2}$$

$$\pi_{k,0} = \bar{p}\bar{\theta}\pi_{n,0} + p\bar{\theta}\pi_{k-1,0},\ k = 1, 2, \ldots, n_e(0), \tag{3.3}$$

$$\pi_{n_e(0)+1,0} = \bar{\theta}\pi_{n_e(0)+1,0} + p\bar{\theta}\pi_{n_e(0),0}, \tag{3.4}$$

$$\pi_{0,1} = (\bar{p}\bar{\xi} + p\mu)\pi_{0,1} + \bar{p}\mu\pi_{1,1}, \tag{3.5}$$

$$\pi_{k,1} = (\bar{p}\bar{\mu} + p\mu)\pi_{k,1} + p\bar{\mu}\pi_{k-1,1} + \bar{p}\mu\pi_{k+1,1} + p\theta\pi_{k-1,0} + \bar{p}\theta\pi_{k,0},$$
$$k = 1, 2, \ldots, n_e(0), \tag{3.6}$$

$$\pi_{n_e(0)+1,1} = (\bar{p}\bar{\mu} + p\mu)\pi_{n_e(0)+1,1} + p\bar{\mu}\pi_{n_e(0),1} + \bar{p}\mu\pi_{n_e(0)+2,1}$$
$$+ p\theta\pi_{n_e(0),0} + \theta\pi_{n_e(0)+1,0}, \tag{3.7}$$

$$\pi_{k,1} = (\bar{p}\bar{\mu} + p\mu)\pi_{k,1} + p\bar{\mu}\pi_{k-1,1} + \bar{p}\mu\pi_{k+1,1},\ k = n_e(0) + 2, \ldots, n_e(1) - 1, \tag{3.8}$$

$$\pi_{n_e(1),1} = (\bar{p}\bar{\mu} + p\mu)\pi_{n_e(1),1} + p\bar{\mu}\pi_{n_e(1)-1,1} + \mu\pi_{n_e(1)+1,1}, \tag{3.9}$$

$$\pi_{n_e(1)+1,1} = \bar{\mu}\pi_{n_e(1)+1,1} + p\bar{\mu}\pi_{n_e(1),1}. \tag{3.10}$$

Define $\rho = \frac{p\bar{\mu}}{\bar{p}\mu}, \sigma = \frac{p\bar{\theta}}{1-\bar{p}\bar{\theta}}$. Then we have the following theorem and its proof is given in Appendix.

**Theorem 3.2.** *Consider a fully observable Geo/Geo/1 queue with delayed multiple vacations and $\sigma \neq 1 \neq \rho \neq \sigma$, in which customers follow the threshold policy $(n_e(0), n_e(1))$ given in Theorem 3.1. The stationary probabilities $\{\pi_{k,j}, 0 \leq k \leq n_e(j) + 1, j = 0, 1\}$ are as follows:*

$$\pi_{k,0} = \frac{\bar{p}\xi}{p}\sigma^k\pi_{0,1},\ k = 0, 1, 2, \ldots, n_e(0), \tag{3.11}$$

$$\pi_{n_e(0)+1,0} = \frac{1}{1-\sigma}\frac{\bar{p}\xi}{p}\sigma^{n_e(0)+1}\pi_{0,1}, \tag{3.12}$$

$$\pi_{k,1} = \left(\rho^k + (\sigma^k - \rho^k)\frac{\xi}{\mu}\frac{1}{\sigma-\rho}\right)\pi_{0,1},\ k = 0, 1, 2, \ldots, n_e(0) + 1, \tag{3.13}$$

$$\pi_{k,1} = \rho^{k-n_e(0)-1}\left(\rho^{n_e(0)+1} + (\sigma^{n_e(0)+1} - \rho^{n_e(0)+1})\frac{\xi}{\mu}\frac{1}{\sigma-\rho}\right)\pi_{0,1},$$
$$k = n_e(0) + 2, \ldots, n_e(1), \tag{3.14}$$

$$\pi_{n_e(1)+1,1} = \bar{p}\rho^{n_e(1)-n_e(0)}\left(\rho^{n_e(0)+1} + (\sigma^{n_e(0)+1} - \rho^{n_e(0)+1})\frac{\xi}{\mu}\frac{1}{\sigma-\rho}\right)\pi_{0,1}, \tag{3.15}$$

*where $\pi_{0,1}$, can be found from the normalization equation*

$$\sum_{k=0}^{n_e(0)+1}\pi_{k,0} + \sum_{k=0}^{n_e(1)+1}\pi_{k,1} = 1.$$

It may be noted here that due to BASTA (Bernoulli arrivals see time averages) property, probabilities at pre-arrival epoch will be same as those of $\pi_{k,0}$ and $\pi_{k,1}$. Then the probability of balking is equal to $\pi_{n_e(0)+1,0} + \pi_{n_e(1)+1,1}$, the social benefit per time unit when all customers follow the threshold policy $(n_e(0), n_e(1))$ given in Theorem 3.1 equals:

$$SB_{fo} = pR(1 - \pi_{n_e(0)+1,0} - \pi_{n_e(1)+1,1}) - C\left(\sum_{k=0}^{n_e(0)+1}k\pi_{k,0} + \sum_{k=0}^{n_e(1)+1}k\pi_{k,1}\right).$$

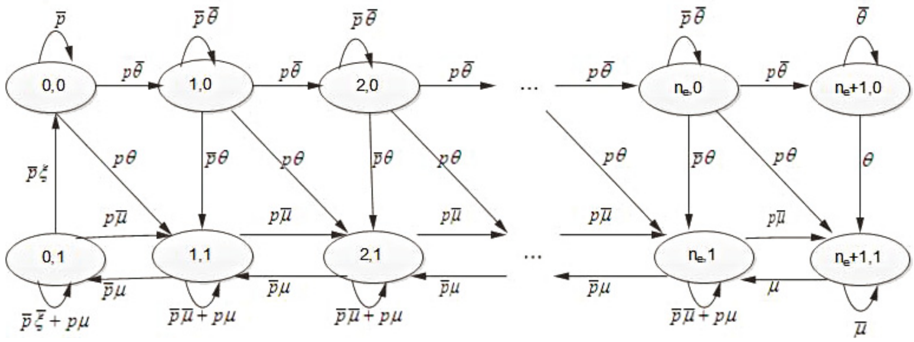



FIGURE 3. Transition rate diagram for the $n_e$ threshold strategy in the almost observable queue.

## 3.2. Almost observable case

In this subsection we proceed to the almost observable case where the arriving customers observe the number of customers upon arrival, but not the state of the server. Hence the stationary distribution of the corresponding Markov's chain is from Theorem 3.2 with $n_e(0) = n_e(1) = n_e$ and state space $\Omega_{ao} = \{(k, j)|0 \le k \le n_e + 1, j = 0, 1\}$. The transition diagram is depicted in Figure 3.

**Theorem 3.3.** *Consider an almost observable* Geo/Geo/1 *queue with delayed multiple vacations and* $\sigma \neq 1 \neq \rho \neq \sigma$, *in which customers follow the threshold policy* $n_e$. *The stationary probabilities of* $\{\pi_{k,j}, 0 \le k \le n_e+1, j = 0, 1\}$ *are as follows:*

$$\pi_{k,0} = \frac{\bar{p}\xi}{p}\sigma^k \pi_{0,1},\ k = 0, 1, 2, \dots, n_e, \tag{3.16}$$

$$\pi_{n_e+1,0} = \frac{1}{1-\sigma}\frac{\bar{p}\xi}{p}\sigma^{n_e+1}\pi_{0,1}, \tag{3.17}$$

$$\pi_{k,1} = \left(\rho^k + (\sigma^k - \rho^k)\frac{\xi}{\mu}\frac{1}{\sigma-\rho}\right)\pi_{0,1},\ k = 0, 1, 2, \dots, n_e, \tag{3.18}$$

$$\pi_{n_e+1,1} = \bar{p}\left(\rho^{n_e+1} + (\sigma^{n_e+1} - \rho^{n_e+1})\frac{\xi}{\mu}\frac{1}{\sigma-\rho}\right)\pi_{0,1}, \tag{3.19}$$

*where* $\pi_{0,1}$, *can be found from the normalization condition*

$$\sum_{k=0}^{n_e+1}(\pi_{k,0} + \pi_{k,1}) = 1.$$

Now, we proceed to find the expected net reward of a customer that observes $k$ customers ahead of him and decides to enter. We have the following lemma:

**Lemma 3.4.** *In the almost observable* Geo/Geo/1 *queue with delayed multiple vacations, where the customers enter to the system according to a threshold strategy "while arriving in* $(n, n^+)$, *observe* $N_n$; *enter if* $N_n \le n_e$

*and balk otherwise". The net benefit of a customer that observes $k$ customers and decides to enter is given by*

$$\begin{aligned} Be(k) &= R - \frac{C(k+1)}{\mu} - \frac{C}{\theta}\left(1 + \frac{\rho}{(\sigma-\rho)\bar{\mu}} + \left(\frac{\rho}{\sigma}\right)^k \left(\frac{p}{\bar{p}\xi} - \frac{\rho}{(\sigma-\rho)\bar{\mu}}\right)\right)^{-1}, \\ &\quad k = 0, 1, 2, \ldots, n_e, \end{aligned} \tag{3.20}$$

$$Be(n_e+1) = R - \frac{C(n_e+2)}{\mu} - \frac{C}{\theta}\left(1 + (1-\sigma)\bar{p}\left(\frac{\rho}{(\sigma-\rho)\bar{\mu}} + \left(\frac{\rho}{\sigma}\right)^k \left(\frac{p}{\bar{p}\xi} - \frac{\rho}{(\sigma-\rho)\bar{\mu}}\right)\right)\right)^{-1}. \tag{3.21}$$

*Proof.* Denote by $(N^-, J^-)$ the steady-state of the system just prior to an arrival of a customer, due to BASTA property, we have $P(N^- = k, J^- = j) = \pi_{k,j}$. Because the expected net benefit of a customer who finds $k, k \geq 0$ customers in the system, if he decides to enter, is equal to

$$Be(k) = R - \frac{C(k+1)}{\mu} - \frac{C}{\theta} P(J^- = 0 | N^- = k), \tag{3.22}$$

where $P(J^- = 0|N^- = k)$ is the probability that an arriving customer finds the server in a vacation time, given that there are $k$ customers. Using the various forms of $\pi_{k,j}$ from (3.16)–(3.19), we get

$$\begin{cases} P(J^- = 0|N^- = k) = \dfrac{1}{1 + \frac{\rho}{(\sigma-\rho)\bar{\mu}} + \left(\frac{\rho}{\sigma}\right)^k \left(\frac{p}{\bar{p}\xi} - \frac{\rho}{(\sigma-\rho)\bar{\mu}}\right)}, \ k = 0, 1, 2, \ldots, n_e, \\ P(J^- = 0|N^- = n_e+1) = \dfrac{1}{1 + (1-\sigma)\bar{p}\left[\frac{\rho}{(\sigma-\rho)\bar{\mu}} + \left(\frac{\rho}{\sigma}\right)^{n_e+1} \left(\frac{p}{\bar{p}\xi} - \frac{\rho}{(\sigma-\rho)\bar{\mu}}\right)\right]}. \end{cases} \tag{3.23}$$

Then, equations (3.20) and (3.21) can be obtained by plugging equation (3.23) in (3.22) for every $k$.

It should be noted that a customer does not enter the system even if he finds no customers in front of him if $Be(0) < 0$, otherwise, he enters the queue. □

In view of the characteristics of equations (3.20) and (3.21), we introduce the function

$$\begin{aligned} f(x,k) =& R - \frac{C(k+1)}{\mu} - \frac{C}{\theta}\left(1 + x\left(\frac{\rho}{(\sigma-\rho)\bar{\mu}} + \left(\frac{\rho}{\sigma}\right)^k \left(\frac{p}{\bar{p}\xi} - \frac{\rho}{(\sigma-\rho)\bar{\mu}}\right)\right)\right)^{-1}, \\ & x \in [(1-\sigma)\bar{p}, 1], \ k = 0, 1, 2, \ldots \end{aligned} \tag{3.24}$$

which will permit us to prove the existence of equilibrium threshold strategies and derive the corresponding thresholds. Let

$$\begin{aligned} f_U(k) &= f(1,k) \\ &= R - \frac{C(k+1)}{\mu} - \frac{C}{\theta}\left(1 + \frac{\rho}{(\sigma-\rho)\bar{\mu}} + \left(\frac{\rho}{\sigma}\right)^k \left(\frac{p}{\bar{p}\xi} - \frac{\rho}{(\sigma-\rho)\bar{\mu}}\right)\right)^{-1}, \end{aligned} \tag{3.25}$$

$$\begin{aligned} f_L(k) &= f(1-\sigma,k) \\ &= R - \frac{C(k+1)}{\mu} - \frac{C}{\theta}\left(1 + (1-\sigma)\bar{p}\left(\frac{\rho}{(\sigma-\rho)\bar{\mu}} + \left(\frac{\rho}{\sigma}\right)^k \left(\frac{p}{\bar{p}\xi} - \frac{\rho}{(\sigma-\rho)\bar{\mu}}\right)\right)\right)^{-1}. \end{aligned} \tag{3.26}$$

Easily it can be seen that

$$f_U(0) = R - \frac{C}{\mu} - \frac{C}{\theta}\frac{1}{1 + \frac{p}{\bar{p}\xi}} > R - \frac{C}{\mu} - \frac{C}{\theta} \geq 0,$$

and

$$f_L(0) = R - \frac{C}{\mu} - \frac{C}{\theta}\frac{1}{1+\frac{p}{\xi}(1-\sigma)} > R - \frac{C}{\mu} - \frac{C}{\theta} \geq 0.$$

(because of (2.1)). In addition,

$$\lim_{k\to\infty} f_U(k) = \lim_{k\to\infty} f_L(k) = -\infty.$$

Hence, there exists $k_U$ such that

$$f_U(0), f_U(1), \ldots, f_U(k_U) > 0, \text{ and } f_U(k_U+1) \leq 0. \tag{3.27}$$

Because the function $f(x,k)$ is increasing with respect to $x$ for every fixed $k$, we get the relation $f_L(k) \leq f_U(k), k = 0,1,2,\ldots$. Specially, $f_L(k_U+1) \leq f_U(k_U+1) \leq 0$, while $f_L(0) > 0$. Hence, there exists $k_L \leq k_U$ such that

$$f_L(k_L) > 0, \text{ and } f_L(k_L+1), \ldots, f_L(k_U), f_L(k_U+1) \leq 0. \tag{3.28}$$

We can now establish the existence of the equilibrium threshold policies in the almost observable case and give the following theorem.

**Theorem 3.5.** *In the almost observable* Geo/Geo/1 *queue with delayed multiple vacations, all pure threshold strategies "observe $N_n$, enter if $N_n \leq n_e$ and balk otherwise" for $n_e = k_L, k_L+1, \ldots, k_U$ are equilibrium balking strategies.*

*Proof.* Consider a tagged customer at his arrival instant and assume all other customers follow the same threshold strategy "observe $N_n$, enter if $N_n \leq n_e$ and balk otherwise" for some fixed $n_e \in \{k_L, k_L+1, \ldots, k_U\}$.

If the tagged customer finds $k \leq n_e$ customers in front of him and decides to enter, his expected benefit is equal to $f_U(k) > 0$ because of (3.20), (3.25) and (3.27). So in this case the customer prefers to enter.

If the tagged customer finds $k \leq n_e + 1$ customers in front of him and decides to enter, his expected benefit is equal to $f_L(n_e+1) \leq 0$ because of (3.21), (3.26) and (3.28). So in this case the customer prefers to balk. □

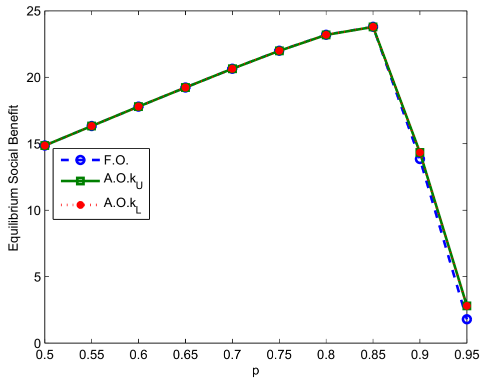
Because the probability of balking equals $\pi_{n_e+1,0} + \pi_{n_e+1,1}$, the social benefit per time unit when all customers follow the threshold policy given in Theorem 3.5 equals:

$$SB_{ao} = pR(1 - \pi_{n_e+1,0} - \pi_{n_e+1,1}) - C\sum_{k=0}^{n_e+1} k(\pi_{k,0} + \pi_{k,1}).$$

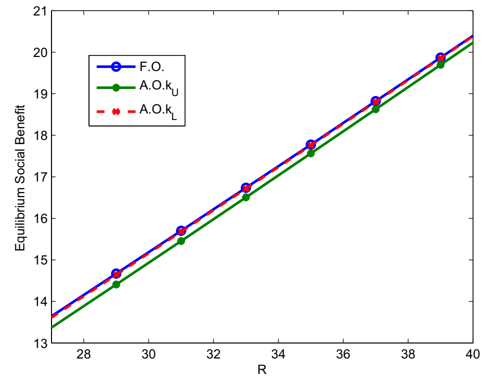
## 4. Numerical examples

In this section, based on the results obtained, we present some numerical examples to show the effect of the information level and several parameters on the behavior of customers. Here we mainly focus on the social benefits per unit time when customers follow equilibrium strategies for both the fully and almost observable models.

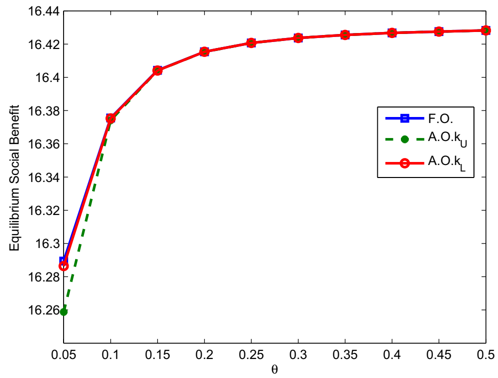
In Figures 4a–4d, we pay attention to the curves of the social benefit under the equilibrium threshold strategies for the fully and almost observable cases with the change of the parameters $p, \theta, R$ and $\xi$, respectively, and give a quantitative comparison between the two cases. For the almost observable case, we only present the social benefit under the two extreme thresholds $k_L$ and $k_U$. In Figures 4a–4c, we observe that the difference in equilibrium social benefit is not obvious between the fully and almost observable cases, the three pieces of curves nearly overlap each other. Moreover, Figures 4b–4d show that the social benefit of all customers in fully



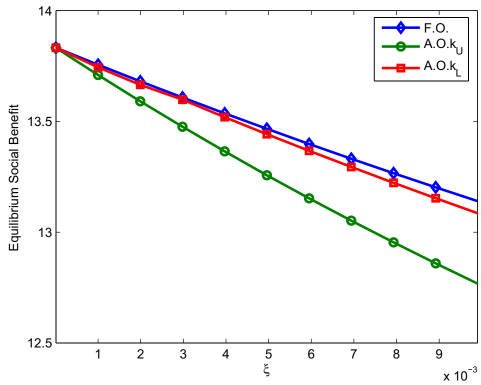

(a) $\mu = 0.9, \theta = 0.08, \xi = 0.0001, R = 30, C = 1$




(c) $\mu = 0.95, p = 0.55, \xi = 0.01, \theta = 0.05, C = 1$




(b) $\mu = 0.95, p = 0.55, \xi = 0.001, R = 30, C = 1$




(d) $\mu = 0.65, p = 0.5, \theta = 0.05, R = 30, C = 1$

FIGURE 4. Social benefit for the fully observable and almost observable system W.R.T. different sensitivities.

observable case is more than (or not less than) that in the almost observable extreme cases. However, it is not always tenable in Figure 4a.

As to the sensitivity of social benefit in equilibrium, in Figure 4a, we observe that the social benefit achieves a maximum for intermediate values regarding the arrival rate $p$. The reason is that customers can be served soon and the social benefit improves when the arrival rate is relatively small, whereas negative effect will be brought to social benefit if $p$ continues to increase. In Figures 4b and 4c, we find that it increases with respect to the vacation rate $\theta$ and the service reward $R$, which agree with the intuitive expectations. However, Figure 4d shows that the equilibrium social benefit decreases with respect to the delayed rate $\xi$. The reason is that the probability that an arriving customer finds (or happens to encounter) the server in a delayed period reduces so that the expected delay from server activation maybe increases with the delayed rate $\xi$.

## APPENDIX A. PROOF OF THEOREM 3.2

From (3.2)–(3.4), we can easily obtain (3.11) and (3.12). By iterating (3.8) and taking into account (3.9) and (3.10), we obtain

$$\pi_{k,1} = \rho^{n-n_e(0)-1}\pi_{n_e(0)+1,1}, k = n_e(0)+1, \ldots, n_e(1) \tag{A.1}$$

$$\pi_{n_e(1)+1,1} = \bar{p}\rho^{n_e(1)-n_e(0)}\pi_{n_e(0)+1,1}. \tag{A.2}$$

Equation (3.6) and (3.7) indicates that $\{\pi_{k,1}, k = 0, 1, \ldots, n_e(0)+1\}$ is a solution of the nonhomogeneous linear difference equation with constant coefficients

$$\begin{aligned}\bar{p}\mu x_{k+1} - (\bar{p}\mu + p\bar{\mu})x_k + p\bar{\mu}x_{k-1} &= -p\theta\pi_{k-1,0} - \bar{p}\theta\pi_{k,0} \\ &= -\frac{\theta}{\bar{\theta}}\frac{\bar{p}\xi}{p}\sigma^k\pi_{0,1}, k = 1, 2, \ldots, n_e(0). \end{aligned} \tag{A.3}$$

Corresponding characteristic equation is as follows

$$\bar{p}\mu x^2 - (\bar{p}\mu + p\bar{\mu})x + p\bar{\mu} = 0,$$

which has two roots at 1 and $\rho$. Then the general solution of the homogeneous version of (A.3) is $x_k^{\text{hom}} = A + B\rho^k$ (assume that $\rho \neq 1$). The general solution $x_k^{\text{gen}}$ of (A.3) is given as

$$x_k^{\text{gen}} = A + B\rho^k + x_k^{\text{spec}},$$

where $x_k^{\text{spec}}$ is a specific solution of (A.3). Because the nonhomogeneous part of (A.3) is geometric with parameter $\sigma$, we can find a specific solution $C\sigma^k$ (assume that $\sigma \neq 1, \sigma \neq \rho$). Substituting $x_k^{\text{spec}} = C\sigma^k$ into (A.3), we have that

$$C = \frac{\xi}{(\sigma-\rho)\mu}\pi_{0,1}. \tag{A.4}$$

Hence the general solution of (A.3) is given as

$$x_k^{\text{gen}} = A + B\rho^k + C\sigma^k,\ k = 0, 1, 2, \ldots, n_e(0)+1, \tag{A.5}$$

where $C$ is given by (A.4) and $A, B$ are to be determined.

From (A.5) for $k = 0, 1$ and taking into account (3.3), we obtain

$$A + B + C = \pi_{0,1}, \tag{A.6}$$

$$A + B\rho + C\sigma = \left(\rho + \frac{\xi}{\mu}\right)\pi_{0,1}. \tag{A.7}$$

Solving the system of (A.6) and (A.7), we obtain

$$\begin{cases} A = 0, \\ B = \left(1 - \dfrac{\xi}{\mu}\dfrac{1}{\sigma-\rho}\right)\pi_{0,1}. \end{cases}$$

Thus, from (A.5),

$$\pi_{k,1} = \left(\rho^k + (\sigma^k - \rho^k)\frac{\xi}{\mu}\frac{1}{\sigma-\rho}\right)\pi_{0,1},\ k = 0, 1, 2, \ldots, n_e(0)+1. \tag{A.8}$$

From equations (3.16), (3.17), (A.1), (A.2) and (A.8), we know that all stationary probabilities are functions of $\pi_{0,1}$. The remaining probability, $\pi_{0,1}$, can be found from the normalization equation:

$$\sum_{k=0}^{n_e(0)+1}\pi_{k,0} + \sum_{k=0}^{n_e(1)+1}\pi_{k,1} = 1.$$

After some algebraic simplifications, we can express all stationary probabilities in terms of $\rho$ and $\sigma$ as given in Theorem 3.2. $\square$

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