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A RELAXED LOGARITHMIC BARRIER METHOD FOR SEMIDEFINITE PROGRAMMING

Kettab Samia[1] and Benterki Djamel[1]

Abstract. Interior point methods applied to optimization problems have known a remarkable evolution in the last decades. They are used with success in linear, quadratic and semidefinite programming. Among these methods, primal-dual central trajectory methods have a polynomial convergence and are credited of a good numerical behavior. In this paper, we propose a new central trajectory method where a relaxation parameter is introduced in order to give more flexibility to the theoretical and numerical aspects of the perturbed problems and accelerate the convergence of the algorithm. This claim is confirmed by numerical tests showing the good behavior of the algorithm which is proposed in this paper.

Keywords. Linear programming, semidefinite programming, central trajectory Methods.

Mathematics Subject Classification. 90C22, 90C51.

1. Introduction

The problem of semidefinite programming (SDP) is of paramount importance for its involvement in various mathematical and practical problems of great interest, namely, control theory, combinatorial optimization, nonlinear programming, the maximum cut problem in graph theory and the problem of min-max eigenvalue.

Recently, SDP has known a remarkable evolution in theory as that of the practice, after the revival of interior point methods from the investigations made by Karmarkar [10] for linear programming problems. Then several algorithms

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[1] Laboratory of Fundamental and Numerical Mathematics, Setif-1 University, 19000 Setif, Algeria. dj_benterki@yahoo.fr

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including potential reduction methods and primal-dual methods of central trajectory have been developed for linear and quadratic programming problems, complementarity problems. The extensions of these methods to SDP have been pioneered by Alizadeh [1] and Monteiro [13]. The polynomial convergence of these algorithms has been studied and proved by several researchers [6–8, 14, 15, 18].

In this work, we are interested in one extension of the central trajectory type method to semidefinite programming [13]. We relax the perturbed problem by a new parameter in order to give more flexibility to the theoretical and numerical study of the obtained perturbed problem and accelerate the convergence of the algorithm.

The paper is organized as follows. In Section 2, we present the problem SDP and we give weak and strong results of duality. In Section 3, we study and prove the existence and uniqueness of the optimal solution of the perturbed problem $(SDP)_\mu$ and we give the corresponding algorithm. Section 4, is reserved to the relaxed problem SDP_W, where we relax the parameter μ by a diagonal semidefinite matrix W and we discuss also the effective computation of direction and stepsize of the obtained algorithm. In Section 5, we present some promising numerical tests in a comparative framework. And lastly a conclusion is given.

1.1. Notation

- M^n is the set of real $(n \times n)$ matrices.
- $S^n = \{A \in M^n /\ A \text{ is symmetric}\}$.
- $S^n_+ = \{A \in S^n /\ A \text{ is positive semidefinite}\}$.
- $S^n_{++} = \{A \in S^n /\ A \text{ is positive definite}\}$.
- Given $X \in M^n$, $\operatorname{tr}(X) = \sum\limits_{i=1}^{n} X_{ii}$ is the trace of the matrix X.
- Given $X \in M^n$, $\operatorname{diag}(X)$ is the $n \times n$ diagonal matrix with diagonal entries X_{ii}.
- Given $x \in \mathbb{R}^n$, $\operatorname{diag}(x)$ is the $n \times n$ diagonal matrix with diagonal entries x_i.
- Given $X, Y \in M^n$, the scalar product of X and Y is defined by

$$\langle X, Y \rangle = \operatorname{tr}\left(X^t Y\right) = \sum_{i,j=1}^{n} X_{ij} Y_{ij}$$

  and the norm of X is defined by $\|X\| = \sqrt{\langle X, X \rangle}$.
- Given $f : \mathbb{R}^n \to \mathbb{R}$, ∇f denotes the gradient of f and $\nabla^2 f$ denotes its Hessian matrix.

2. Statement of the problem

The standard form of a semidefinite program is:

$$\min_X \left[\operatorname{tr}(CX)\ :\ X \in S^n_+,\ \operatorname{tr}(A_i X) = b_i \ \text{ for } i = 1, \ldots, m\right], \tag{SDP}$$

where C, A_i , $i = 1, \ldots, m$ belong to S^n and $b \in \mathbb{R}^m$.

Its dual problem is:

$$\max_{y}\left[b^t y\ :\ C-\sum_{i=1}^{m} y_i A_i = S,\ S\in S_+^n,\ y\in \mathbb{R}^m\right]. \tag{SDD}$$

We denote by:

$$R(P)=\left\{X\in S_+^n : \langle A_i, X\rangle = b_i,\ i=1,\dots,m\right\},$$

the set of feasible primal solutions of (SDP), and

$$R^\circ(P)=\left\{X\in R(P) : X\in S_{++}^n\right\},$$

the set of strictly feasible primal solutions of (SDP).

Similarly,

$$R(D)=\left\{(y,S)\in \mathbb{R}^m\times S_+^n\ :\ C-\sum_{i=1}^{m} y_i A_i = S\right\},$$

the set of feasible dual solutions of (SDD), and

$$R^\circ(D)=\left\{(y,S)\in R(D) : S\in S_{++}^n\right\},$$

the set of strictly feasible dual solutions of (SDD).

We denote by $val(P)$ and $val(D)$ the optimal values of (SDP) and (SDD) respectively.

Theorem 2.1 (Weak duality [17])**.** *If $X\in R(P)$ and $(y,S)\in R(D)$ then:*

$$\langle C, X\rangle - b^t y = \langle S, X\rangle \geq 0.$$

And, in case where $\langle S, X\rangle = 0$, X is one optimal solution of (SDP) and (y,S) is one optimal solution of (SDD).

Theorem 2.2 (Strong duality [17])**.**

1. *If $val(P) > -\infty$ and $R^\circ(P)\neq\phi$, then the set of optimal solutions of (SDD) is nonempty and bounded and we have:*

$$val(P)=val(D).$$

2. *If $val(D) < +\infty$ and $R^\circ(D)\neq\phi$, then the set of optimal solutions of (SDP) is nonempty and bounded and we have:*

$$val(P)=val(D).$$

3. *If $R^\circ(P)\neq\phi$ and $R^\circ(D)\neq\phi$, then the sets of optimal solutions of (SDP) and (SDD) are nonempty and bounded and we have:*

$$val(P)=val(D).$$

Moreover, for all X optimal solution of (SDP) and (y,S) optimal solution of (SDD) one has:

$$\begin{cases} C - \sum_{i=1}^{m} y_i A_i = S \\ \langle A_i, X\rangle = b_i, i = 1,\dots,m \\ XS = 0 \end{cases} \tag{2.1}$$

Throughout the remaining of the paper, we assume that both $R^\circ(P)$ and $R^\circ(D)$ are nonempty.

3. The logarithmic penalization

Given $\mu > 0$, we consider the problem

$$\min_{X\in S^n} [\, f_\mu(X) : \langle A_i, X\rangle = b_i,\ i = 1,\dots,m] \tag{$(SDP)_\mu$}$$

where f_μ is defined by

$$f_\mu(X) = \begin{cases} \langle C, X\rangle - \mu \ln\det X & \text{if } X \in S^n_{++} \\ +\infty & \text{otherwise.} \end{cases}$$

3.1. Study of the problem $(SDP)_\mu$

We start with a lemma.

Lemma 3.1. $\{Y \in S^n_+ : \langle C, Y\rangle \le 0,\ \langle A_i, Y\rangle = 0,\ \ i = 1,\dots,m\} = \{0\}$.

Proof. We know that the set of optimal solutions of (SDP) $(Sol(SDP))$ is convex closed bounded and nonempty. Therefore its recession cone $Sol(SDP)^\infty$ is reduced to the singleton $\{0\}$. But

$$Sol(SDP)^\infty = \{Y \in S^n_+ : \langle C, Y\rangle \le 0,\ \langle A_i, Y\rangle = 0,\ \ i = 1,\dots,m\}. \qquad \square$$

Theorem 3.2. *The problem $(SDP)_\mu$ admits one unique optimal solution.*

Proof. **Existence:** It is sufficient to prove that the recession cone of the closed convex set

$$S_\lambda = \{\, X : \ \langle A_i, X\rangle = b_i\ \forall i,\ f_\mu(X) \le \lambda\}$$

is reduced to $\{0\}$. One has

$$S_\lambda^\infty = \{Y : \langle A_i, Y\rangle = 0,\ \ i = 1,\dots,m\} \cap \{Y : f_\mu^\infty(Y) \le 0\,\}.$$

Let us compute the recession function f_μ^∞. Let some $X \in S^n_{++}$. Then, for $Y \in S^n$,

$$f_\mu^\infty(Y) = \lim_{t\to\infty} \frac{f_\mu(X+tY) - f_\mu(X)}{t}.$$

It is clear that if $Y \notin S^n_+$, for t large enough, $X + tY$ does not belong to S^n_{++} and therefore $f_\mu(Y) = +\infty$. Assume that $Y \in S^n_+$. Then,

$$f_\mu^\infty(Y) = \langle C, Y\rangle - \mu \lim_{t\to\infty} \frac{\ln\det(X+tY) - \ln\det(X)}{t}.$$

Recall that X is positive definite, there is a symmetric positive definite matrix Z such that $X = Z^2$. Next, there exists a matrix P and a positive semidefinite diagonal matrix D such that $Z^{-1}YZ^{-1} = P^tDP$ and $P^tP = I$. It follows that

$$\det(X+tY) = \det(ZP^t(I+tD)PZ) = \det(X)\prod_{i=1}^{n}(1+td_i)$$

where the d_is are the diagonal entries of D. Therefore,

$$f_\mu^\infty(Y) = \langle C, Y\rangle - \mu \sum_{i=1}^{n} \lim_{t\to\infty} \frac{\ln(1+td_i)}{t} = \langle C, Y\rangle.$$

Apply lemma 1, we obtain

$$S_\lambda^\infty = \{0\}.$$

Uniqueness: Since f_μ is strictly convex, it follows that the optimal solution of $(SDP)_\mu$ is unique. □

3.1.1. Optimality conditions for (SDP)$_\mu$

The problem is convex. Applying the KKT conditions, we have that $X \in S^n_{++}$ is an optimal solution of problem $(SDP)_\mu$ if and only if there exists $y \in \mathbb{R}^m$ such that:

$$\begin{cases} C - \mu X^{-1} - \sum\limits_{i=1}^{m} y_i A_i = 0 \\ \langle A_i, X\rangle = b_i, \ i = 1, \ldots, m. \end{cases} \tag{3.1}$$

Set $S = \mu X^{-1}$, the system (3.1) can be written as:

$$\begin{cases} \sum\limits_{i=1}^{m} y_i A_i + S = C \quad , \quad S \in S^n_{++} \\ \langle A_i, X\rangle = b_i, \ i = 1, \ldots, m \quad , \quad X \in S^n_{++} \\ XS = \mu I \quad , \quad \mu > 0. \end{cases} \tag{3.2}$$

The system (3.2) is the parameterized system of (2.1). We denote by $(X(\mu), y(\mu), S(\mu))$ one solution of the system. We already know that $X(\mu)$ (and thereby also $S(\mu)$) is uniquely defined. From to now, we add the additional assumption: the matrices $A_i, i = 1, \ldots, m$ are linearly independent. Then, $y(\mu)$ is also uniquely defined.

Definition 3.3. The set

$$T_c(\mu) = \{(X(\mu), y(\mu), S(\mu)) \ / \ \mu > 0\}$$

is called the central trajectory associated to our penalization.

3.2. Central trajectory method

If $(X(\mu), y(\mu), S(\mu))$ is an optimal solution of $(SDP)_\mu$ with $\mu > 0$ then

$$\lim_{\mu\to 0}(X(\mu), y(\mu), S(\mu)) = (X, y, S)$$

is an optimal solution of (SDP), see [12].

To facilitate the study, we consider in the following the notation (X, y, S) instead of $(X(\mu), y(\mu), S(\mu))$.

To solve the problem $(SDP)_\mu$, we use the primal-dual central trajectory method. The strategy of this method is to find at each iteration an approximate solution for the nonlinear system (3.2), in the neighborhood of the central trajectory, *i.e.*, by obtaining a decreasing sequence of the matrix duality gap X^kS^k (or in an equivalent manner, the convergence of the parameter μ_k to 0).

Newton's method is considered as one of the best methods for solving the system (3.2). At the iteration k, assume $U^k = (X^k, y^k, S^k) \in R^\circ(P) \times R^\circ(D)$, we search a new iterate $U^{k+1} = (X^{k+1}, y^{k+1}, S^{k+1})$ defined by:

$$\begin{cases} X^{k+1} = X^k + \Delta X^k \\ y^{k+1} = y^k + \Delta y^k \\ S^{k+1} = S^k + \Delta S^k. \end{cases}$$

Then, $(X^{k+1}, y^{k+1}, S^{k+1})$ satisfies the nonlinear system (3.2) and we have:

$$\begin{cases} \sum_{i=1}^m (y_i^k + \Delta y_i^k)A_i + (S^k + \Delta S^k) = C, \quad S^k \in S^n_{++} \\ \left\langle A_i, X^k + \Delta X^k \right\rangle = b_i,\ i = 1, \ldots, m, \quad X^k \in S^n_{++} \\ (X^k + \Delta X^k)(S^k + \Delta S^k) = \mu^k I, \quad \mu^k > 0, \end{cases} \tag{3.3}$$

where $(\Delta X^k, \Delta y^k, \Delta S^k)$ is the solution of the linear system:

$$\begin{cases} \sum_{i=1}^m \Delta y_i^k A_i + \Delta S^k = 0, \quad S^k \in S^n_{++} \\ \left\langle A_i, \Delta X^k \right\rangle = 0,\ i = 1, \ldots, m, \quad X^k \in S^n_{++} \\ X^k \triangle S^k + S^k \triangle X^k = \mu^k I - X^k S^k, \quad \mu^k > 0. \end{cases} \tag{3.4}$$

A point is said near the central trajectory if it belongs to the following set:

$$S_\sigma(\mu) = \left\{ (X, y, S) \in R^\circ(P) \times R^\circ(D) \ / \ \left\| X^{1/2} S X^{1/2} - \mu I \right\| \le \sigma\mu \right\},$$

with $0 < \sigma < 1$.

3.2.1. Algorithm of central trajectory

Begin of the algorithm

Initialization:

Choose $\varepsilon > 0$ and $0 < \sigma < 1$, such that $(X^0, y^0, S^0) \in S_\sigma(\mu^0)$, with $\mu^0 = \frac{\langle X^0, S^0\rangle}{n}$, $k = 0$.

- While $\mu^k > \varepsilon$ do
  - Compute $(\triangle X^k, \triangle y^k, \triangle S^k)$ by solving the linear system (3.4).
  - Take $(X^{k+1}, y^{k+1}, S^{k+1}) = (X^k, y^k, S^k) + (\triangle X^k, \triangle y^k, \triangle S^k)$.
  - Take $\mu^{k+1} = \frac{\langle X^{k+1}, S^{k+1}\rangle}{n}$ and $k = k + 1$.
- End While.

End of the algorithm.

Remark 3.4. The major difficult in central trajectory methods is the obtention of one feasible solution. Once this point obtained, the convergence of the algorithm is guaranteed as soon as the point is in some neighborhood of the central trajectory [4, 9, 11, 16, 20, 21]. In this sense, several studies have been conducted to answer this question. Besides, introducing a weight factor in barrier function, many researchers have proved the convergence results of the algorithm. See for instance [5, 25].

In the other hand, the iterates are infeasible because the found direction is not in general symmetric. Several families of symmetrized direction are proposed, namely HAO direction [1], NT direction [21], H.K.M direction [8, 12, 13] and the Monteiro family of direction [13–15].

Remark 3.5. The parameterized nonlinear equation $XS = \mu I$ of system (3.2), limited the theoretical and numerical study of the algorithm. In order to give more flexibility to the equation $XS = \mu I$, we relax the parameter μ by the spectral radius of a diagonal positive definite matrix.

4. Relaxation of parameter μ

We propose in this section a generalization of the primal-dual central trajectory method by replacing the parameter μ in the system (3.2) by the spectral radius $\rho(W)$ of the diagonal positive definite matrix W.

The parameterized problem associated to SDP becomes:

$$\min_{X \in S^n} [\, f_W(X) : \langle A_i, X\rangle = b_i,\ i = 1, \ldots, m]. \qquad (SDP)_W$$

Where f_W is defined by

$$f_W(X) = \begin{cases} \langle C, X\rangle - \rho(W) \ln \det X & \text{if } X \in S^n_{++} \\ +\infty & \text{otherwise.} \end{cases}$$

For the same reasons than in $(SDP)_\mu$, the problem $(SDP)_W$ also admits a unique optimal solution. Consequently, the optimality conditions are necessary and sufficient. It follows that X is an optimal solution of $(SDP)_W$ if there exists $y \in \mathbb{R}^m$ such that:

$$\begin{cases} C - \rho(W)X^{-1} - \sum_{i=1}^{m} y_i A_i = 0 \\ \langle A_i, X\rangle = b_i,\ i = 1, \dots, m. \end{cases}$$

Let $S = \rho(W)X^{-1}$, then we obtain:

$$\begin{cases} \sum_{i=1}^{m} y_i A_i + S = C, \qquad S \in S^n_{++} \\ \langle A_i, X\rangle = b_i,\ i = 1, \dots, m, \quad X \in S^n_{++} \\ XS = \rho(W)I, \qquad W \in S^n_{++}. \end{cases} \tag{4.1}$$

We denote by $(X(W), y(W), S(W))$ one solution of the system (4.1). We already know that $X(W)$ (and thereby also $S(W)$) is uniquely defined. Recall that the matrices $A_i, i = 1, \dots, m$ are linearly independent. Then, $y(W)$ is also uniquely defined.

Definition 4.1. The set

$$T_c(W) = \left\{(X(W), y(W), S(W)) \ \ / \ \ W \in S^n_{++}\right\}$$

is called the central trajectory associated to our penalization.

Here, we consider also (X, y, S) instead of $(X(W), y(W), S(W))$ to facilitate the study.

The system (4.1) is nonlinear which can be written as

$$F_W(X, y, S) = 0, \text{ such that } X, S, W \in S^n_{++},$$

where

$$F_W(X, y, S) = \begin{bmatrix} \sum_{i=1}^{m} y_i A_i + S - C & \\ \langle A_i, X\rangle - b_i, & i = 1, \dots, m \\ XS - \rho(W)I & \end{bmatrix}$$

By applying Newton's method and linearization, the resolution of the nonlinear system

$$F_W(X + \triangle X, y + \triangle y, S + \triangle S) = 0, \text{ such that } X, S, W \in S^n_{++}$$

returns to the following linear system:

$$F_W(X, y, S) + (\triangle X, \triangle y, \triangle S)^T \nabla F_W(X, y, S) = 0, \text{ such that } X, S, W \in S^n_{++}$$

which is equivalent to

$$\begin{cases} \sum_{i=1}^{m} \triangle y_i A_i + \Delta S = 0, \quad S \in S_{++}^n \\ \langle A_i, \triangle X \rangle = 0\ ,\ i = 1, \ldots, m, \quad X \in S_{++}^n \\ X \triangle S + S \triangle X = \rho(W)I\ - XS, \qquad W \in S_{++}^n. \end{cases} \tag{4.2}$$

Hence, the new iterate is given by:

$$(X^+, y^+, S^+) = (X, y, S) + (\triangle X, \triangle y, \triangle S).$$

A point is said near the central trajectory if it belongs to the following set:

$$S_\sigma(W) = \left\{ (X, y, S) \in R^\circ(P) \times R^\circ(D)\ /\ \left\| X^{1/2} S X^{1/2} - \rho(W) I \right\| \leq \sigma \rho(W) \right\},$$

with $0 < \sigma < 1$.

4.1. Computing of the direction

Recall that the computation of the direction $(\triangle X, \triangle y, \triangle S)$ requires at each iteration, the resolution of system (4.2). Then, the new iterate

$$(X^+, y^+, S^+) = (X, y, S) + (\triangle X, \triangle y, \triangle S)$$

must be strictly feasible *i.e.*, $X^+ \in R^\circ(P)$ and $(y^+, S^+) \in R^\circ(D)$.

Unfortunately, X^+ and S^+ are not always symmetrical. To remedy these difficulties, Zhang [24] proposed an alternative to guarantee the symmetrization of the iterates as follows:

We consider the linear transformation:

$$H_P(M) = \frac{1}{2} \left[P M P^{-1} + P^{-T} M^T P^T \right],$$

where P is an invertible matrix and P^{-T} is the inverse matrix of P^T.

Zhang showed that if M is a positive definite matrix, then:

$$H_P(M) = \mu I \Leftrightarrow M = \mu I.$$

Therefore, we can replace the matrix XS in the systems (3.2) and (4.1) by $H_P(XS)$.

Remark 4.2. If $P = I$, the direction obtained coincides with (HAO) direction of Alizadeh, Haeberly and Overton [1].

If $P = [X^{\frac{1}{2}}(X^{\frac{1}{2}} S X^{\frac{1}{2}})^{-\frac{1}{2}} X^{\frac{1}{2}}]^{\frac{1}{2}}$, we obtain the NT direction of Nesterov and Todd [21].

If $P = X^{-\frac{1}{2}}$ or $P = S^{\frac{1}{2}}$, we obtain the H.K.M direction of Helmberg *et al.*, Kojima *et al.* and Monteiro [8, 12, 13].

In the practice, the HAO direction is the most used thanks to its simplicity.

To solve the system (4.2), we apply the HAO alternative. For this, we define the operator:

$$A : S^n \to \mathbb{R}^m$$
$$A(X) = (\langle A_i, X\rangle)_{i=1}^m$$

The associate adjoint operator of A is defined by:

$$A^* : \mathbb{R}^m \to S^n$$
$$A^*(y) = \sum_{i=1}^m y_i A_i$$

Replacing the matrix XS by $\dfrac{XS+SX}{2}$, the system (4.2) becomes:

$$\begin{cases} A^*(\Delta y) + \Delta S = 0, \quad S \in S^n_{++} \\ A(\Delta X) = 0, \qquad X \in S^n_{++} \\ \hbar(\Delta X) + \pounds(\Delta S) = 2\rho(W)I - (XS+SX), \quad W \in S^n_{++} \end{cases} \tag{4.3}$$

where:

$$\hbar, \pounds : S^n \to S^n$$
$$\hbar P = \frac{1}{2}(SP + PS)$$
$$\pounds P = \frac{1}{2}(XP + PX)$$

Theorem 4.3 [25]**.** *Let $\hbar, \pounds$: $S^n \to S^n$ two operators, such as $\hbar$ invertible and $\hbar^{-1}\pounds$ positive definite operator, then the solution of system (4.3) is unique and given by:*

$$\begin{cases} \Delta y = (A\hbar^{-1}\pounds A^*)^{-1}(-A\hbar^{-1}(2\rho(W)I - (XS+SX))) \\ \Delta S = -A^*(\Delta y), \qquad X, S \in S^n_{++} \\ \Delta X = \hbar^{-1}(2\rho(W)I - (XS+SX) - \pounds(\Delta S)), \quad W \in S^n_{++}. \end{cases}$$

Remark 4.4. The convergence speed of the algorithm depends largely on the manner of computing the direction $(\triangle X, \triangle y, \triangle S)$ and the stepsize along the direction. Thus, the current iterate is defined by:

$$\begin{cases} X^+ = X + \alpha\triangle X \\ y^+ = y + \beta\triangle y \\ S^+ = S + \beta\triangle S \end{cases}$$

4.2. Computation of the stepsize

It means to search (α,β) which guaranteed the positivity definiteness of the matrices X^+, S^+ and improves the convergence speed of the algorithm. For this, we set:

$$\alpha_X = \begin{cases} \min\left(\dfrac{\sum_{i\neq j=1}^{n}|X_{ij}|-X_{ii}}{\Delta X_{ii}-\sum_{i\neq j=1}^{n}|\Delta X_{ij}|}\right) & \text{if } \ I_0 \neq \phi \\ 1 & \text{if } \ I_0 = \phi \end{cases}$$

$$\beta_S = \begin{cases} \min\left(\dfrac{\sum_{i\neq j=1}^{n}|S_{ij}|-S_{ii}}{\Delta S_{ii}-\sum_{i\neq j=1}^{n}|\Delta S_{ij}|}\right) & \text{if } \ I_1 \neq \phi \\ 1 & \text{if } \ I_1 = \phi, \end{cases}$$

where

$$I_0 = \left\{ i \in \{1,\dots,n\} \ / \ \Delta X_{ii} - \sum_{i\neq j=1}^{n} |\Delta X_{ij}| < 0 \right\}$$

$$I_1 = \left\{ i \in \{1,\dots,n\} \ / \ \Delta S_{ii} - \sum_{i\neq j=1}^{n} |\Delta S_{ij}| < 0 \right\}.$$

Lemma 4.5 [23]. *If $\alpha^* = min(\alpha_X, \beta_S)$, then*

$$X^+ = X + \alpha^* \triangle X \in S^n_{++} \text{ and } S^+ = S + \alpha^* \triangle S \in S^n_{++}.$$

4.3. Description of the algorithm

Begin of the algorithm

Initialization:

Choose $\varepsilon > 0$ and $0 < \sigma < 1$, such that $(X^0, y^0, S^0) \in S_\sigma(W^0)$, with $W^0 = \dfrac{\operatorname{diag}(X^0S^0)}{n}$, $k = 0$.

- While $\rho(W^k) > \varepsilon$ do
  - Compute $(\triangle X^k, \triangle y^k, \triangle S^k)$ by solving the linear system (4.3).
  - Take $(X^{k+1}, y^{k+1}, S^{k+1}) = (X^k, y^k, S^k) + \alpha^k(\triangle X^k, \triangle y^k, \triangle S^k)$, where $\alpha^k = \min(\alpha_{X^k}, \beta_{S^k})$.
  - Take $W^{k+1} = \dfrac{\operatorname{diag}(X^{k+1}S^{k+1})}{n}$ and $k = k + 1$.
- End While.

End of the algorithm.

5. Numerical tests

The following examples are taken from the literature see for instance [2,23]. The tests were done using Matlab 10. We have taken $\varepsilon = 10^{-7}$ and $\sigma = 0.1$.

In the table of results, (ex (m, n)) represents the size of the example, (Itr) represents the number of iterations necessary to obtain the optimal solution and (time (s)) represents the time of computation.

- **Examples with fixed size**

	parameter μ		parameter $\rho(W)$	
ex (m,n)	Itr	time (s)	Itr	time (s)
ex $(2,3)$	4	0.017895	3	0.011789
ex $(3,5)$	7	0.018817	5	0.012493
ex $(3,6)$	7	0.022879	6	0.018049

- **Examples with variable size**

Recall that the considered problem is

$$\min_X \left[\, \mathrm{tr}(CX) \ : \ X \in S_+^n, \ \mathrm{tr}(A_i X) = b_i \ \text{ for } i = 1, \ldots, m\right], \qquad (SDP)$$

where

$C = -I$, $b\,(i) = 2$, $i = 1, \ldots, m, n = 2m$, and the matrices A_k, $k = 1, \ldots, m$ are defined as follows

$$A_k\,(i,j) = \begin{cases} 1 \text{ if } i = j = k \\ 1 \text{ if } i = j \text{ and } i = m + k \\ 0 \quad \text{Otherwise.} \end{cases}$$

	parameter μ		parameter $\rho(W)$	
ex (m,n)	Itr	time (s)	Itr	time (s)
$(10,20)$	7	0.031858	3	0.029764
$(30,60)$	8	0.163532	3	0.090465
$(50,100)$	8	0.458340	3	0.181330
$(100,200)$	8	1.579016	3	0.640606
(200,400)	8	7.010987	3	2.837008

5.1. Comments

Through the numerical tests and for different dimension of the examples, we remark that the results show the importance of the introduced modification, expressed by significant reduction in the number of iteration and computation time. Generally, the choice of the parameter $\rho(W)$ is much better than the parameter μ.

Conclusion

In this paper, we have introduced a relaxation in the perturbed problem of SDP. This modification led to improving the behavior of algorithm, and provided a flexibility in the theoretical study. Furthermore, the numerical tests confirm the theoretical propositions and open the way to other modifications which aim to lead a more effective algorithm.

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