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ANALYSIS OF A CLEARING QUEUEING SYSTEM WITH SETUP TIMES *

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Abstract. We consider a Markovian clearing queueing system with setup times. When the system is empty, the server gets into the state of vacation. Once a new customer arrives the system, an exponential setup time is required before the server renders the service again. The customers are accumulated according to Poisson arrival process and the service times are exponentially distributed. Upon their arrivals, customers decide whether to join or balk the queue based on a natural linear reward-cost structure which reflects their desire for service and their unwillingness to wait. According to the state of server under some condition, we obtain the balking strategies of customers, the stationary distribution of system state, the expected queue length and the social optimal benefit. Finally, some numerical experiments describe how the expected queue length and the social optimal benefit depend on the arrival rate, the service time and the setup time.

Keywords. Clearing system, setup times, dominant strategy, Nash equilibrium.

Mathematics Subject Classification. 90B22, 60K25.

1. Introduction

A clearing system is a usual queueing system with batch services and there are many realistic situations in our true-life about it. For example, the development of modern transport not only provides convenience for urban daily traveling, what is more, it also remits the pressure of urban traffic. The other actual model is

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the process of eliminating the viruses simultaneously in computer. In view of this property, this system allows a certain extent of congestion, and unless otherwise stated, the capacity of the system is assumed to be infinite.

During the last decade, there is an emerging tendency to study vacation queueing models. In these models, the server is turned off when the system becomes empty. As soon as a new customer arrives, a random time is required for setup before the server renders the service again. Bischof [3] computed the first two moments of the waiting time, the queue length, and the cycle time in a M/G/1 queue with setup times and vacations. Under six service disciplines, he used these measures to analyze the performance of system. In [1] Artalejo et al. got limiting distribution of the system state, waiting time analysis, busy period and maximum queue length in a multiserver queue with setup time. Other models, such as single vacation, multiple vacation and working vacation can be seen in references [5, 12].

From the work [11] initiated by Naor, more and more scholars focused their attention on the economic analysis from the point of game theory. In [11], the economic analysis was based on some reward-cost structure which reflected the customers' desire for service and their unwillingness to wait, *i.e.*, the customers must bear some cost for holding in the queueing system and only after the service the customers can get some kind of reward. In order to maximize their individual benefit, customers are permitted to make their decisions whether to join or balk the system. Afterward his study was complemented by Edelson and Hilderbrand [8] who considered the same system, but assumed that customers did not observe the state of system. These results were further extended in [9, 14]. Besides, some scholars analyzed the behaviors of customers in system to find the Nash equilibrium strategy. Economou and Kanta [6] considered a Markovian single-server queue that alternated between on and off periods. A newly arriving customer observed the queue length and decided whether to join or balk. According to the information of server's state, they derived corresponding equilibrium threshold balking strategies. Wang and Zhang [13] studied the same queueing system but assumed that once the server failed, it would experience an exponential delay time to begin the repair process. During the delay process, the server stopped providing service to arriving customers and waited to be repaired by repair facility. In observable case, they obtained the equilibrium threshold balking strategies for customers.

Recently, Baumann and Sandmann [2] presented a detailed steady state analysis by computing the stationary distribution under a continuous-time level dependent quasi-birth-and-death processes with catastrophes. In [4], whenever a catastrophe occurred, all customers were forced to abandon the system, the server was rendered inoperative and an exponential repair time was set on. Under a natural reward-cost function, Nash equilibrium and social optimal strategies were obtained. Economou and Manou [7] studied the balking behavior of the customers and derived the corresponding Nash equilibrium strategies in an alternating environment for a clearing queueing system.

In this paper we study a Markovian clearing queueing system with setup time. Using a natural linear reward-cost structure, we will determine the balking

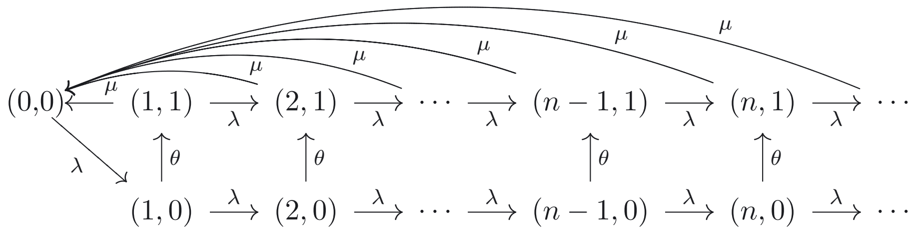

FIGURE 1. Transition rate diagram of the original model.

strategies of customers under two different cases, before making their decisions about whether to join or not. Furthermore, we get the steady state distribution of system, the expected queue length and the social optimal benefit.

The organization of this paper is as follows. The model under consideration is described in Section 2. And the following two sections give the customers balking strategies. In Section 5, some numerical experiments are presented to demonstrate the effect of information level to the expected queue length and the social optimal benefit. Finally, some conclusions are given in Section 6.

2. Model description

We consider a transportation system, such as the subway, with infinite waiting space and the facility can serve all the customers present in system simultaneously. In this paper, we assume that customers arrive the system according to Poisson process with rate λ and the service times are exponentially distributed with rate μ. Once the system is empty, the server will get into the state of vacation and will be reactivated when a new customer arrives the empty system. The time that required to set up is also exponentially distributed with rate θ. During the setup time, customers continue to arrive the system, but they can not be served until the vacation is over. We assume the service discipline is FIFO and the interarrival times, services times and setup times are mutually independent.

The state of system is specified at time t by a pair $(N(t), I(t))$, where $N(t)$ records the number of customers in system and $I(t) = \{0, 1\}$ denotes the state of server ($0 \leftrightarrow$ in vacation, $1 \leftrightarrow$ in service) respectively. The process $\{(N(t), I(t)),\ t \geq 0\}$ is a continuous time Markov chain with state space $\Omega = \{(n, i)|\ n \geq 1,\ i = 0, 1\} \bigcup \{(0, 0)\}$ and transition rate diagram is in Figure 1.

We are interested in the behaviors of customers under the observation of the length of queue and the state of server. Upon their arrivals, they have options to decide whether to join or balk the system. Generally, every customer would like to enter the system because he will receive a reward of R units after completing the service. Moreover, there exists a waiting cost of C units per time unit and the total cost that he affords is directly proportional to the holding time in system. Similar to the papers above, the decisions made by the customers are assumed irrevocable,

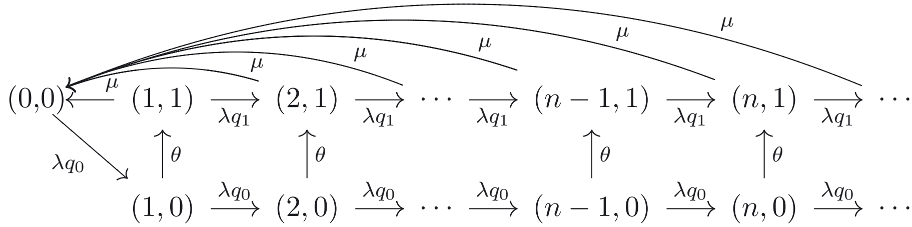

FIGURE 2. Transition rate diagram in the observable case.

in the sense that retrials of balking customers and reneging of entering customers are not allowed.

Since all customers are considered to be homogeneous, the situation can be modeled as a symmetric game among them. Intuitively, a strategy is weakly dominant if it is a best response against any strategy. A strategy is an equilibrium if it is a best response against itself (see [10] for details).

In the next two sections, we obtain customers equilibrium strategies for joining/balking. We distinguish two cases depending on the state of server available to the customers at their arrivals instants, before the decisions are made:

– The observable case: Customers observe the state of server $I(t)$.

– The unobservable case: Customers do not observe the state of server $I(t)$.

3. THE OBSERVABLE CASE

In this case, we will show that the balking strategies are independent of the number of customers in system. Let S_i denotes the sojourn time of the customer in system if the server is in state $i(i = 0, 1)$. Since the facility serve all the customers simultaneously, S_i is independent of the length of customers in system. So the mean sojourn time $E[S_i]$ does not depend on $N(t)$. From Figure 1 the following two conclusions

$$E[S_i] = \frac{1}{\mu} + \frac{1-i}{\theta}, \quad i = 0, 1, \tag{3.1}$$

$$E[S_1] < E[S_0], \tag{3.2}$$

are obviously. Then, the weakly dominate strategies in the observable case are specified in Theorem 3.1 by a pair of number (q_0, q_1), where q_i is the joining probability of a customer if the server state upon arrival is $i,\ i = 0, 1$.

Theorem 3.1. *In the observable case of a clearing queueing system with setup times, there always exist weakly dominant strategies, which depend on the comparison of the value between* $\frac{R}{C}$ *and* $E[S_i]$ *that given in equations* (3.1)*. The specific results are summarized in Table* 1*.*

TABLE 1. Weakly Dominant strategies in the observable case.

Comparison relation	$\frac{R}{C} < E[S_1]$	$\frac{R}{C} = E[S_1]$	$E[S_1] < \frac{R}{C} < E[S_0]$	$\frac{R}{C} = E[S_0]$	$E[S_0] < \frac{R}{C}$
Dominant strategies	(0,0)	$(0, q)\ \ q \in [0, 1]$	(0,1)	$(q, 1)\ \ q \in [0, 1]$	(1,1)

Proof. Consider a tagged customer that observes the state of server upon arrival. Then the expected benefit of customer who finds the server in state i and decides to join the system is

$$S_o(i) = R - CE[S_i]. \tag{3.3}$$

Therefore the tagged customer is willing to join the system if $S_o(i) > 0$, chooses to balk if $S_o(i) < 0$ and he is indifferent between joining and balking if $S_o(i) = 0$. Through simple computations we can get the results in Table 1. □

4. The unobservable case

In this section, the customers, upon their arrivals and before making their decisions about whether to join or balk, do not observe the state of server. Similar to the conclusion in Section 3, we will show that the balking strategies are also independent of the number of customers present in the system. Firstly, we suppose that a tagged customer observes n customers in the system upon arrival. Although his mean sojourn time does not depend on n, the information about n influences the joining probabilities that the system is found in state 0 or 1. If the number n is below a certain threshold n_0, we expect that the tagged customer will benefit from joining the system. In order to ensure the customer who finds the system empty prefers to join the queue, we assume $R - \frac{C}{\mu} - \frac{C}{\theta} > 0$.

Theorem 4.1. *In the unobservable case of a clearing queueing system with setup times, where the customers enter the system according to a pure threshold strategy 'while arriving at time t, observe $N(t)$; enter if $N(t) \leq n_0 - 1$ and balk otherwise', the net benefit of a customer that observes n customers and decides to enter is always positive,* i.e., *all the customers choose to join the queue regardless of n_0.*

Proof. The expected net benefit of a customer, who observes n customer present in system upon his arrival and decides to enter the system, is

$$S_u(n) = R - CE(n), \tag{4.1}$$

where $E(n)$ means his expected sojourn time. We denote $P(I^- = i|N^- = n),\ i = 0, 1$ is the probability that the state of server is i when he observes n customers in the system upon his arrival. Conditioning on the state of server we have

$$E(n) = P(I^- = 0|N^- = n)E[S_0] + P(I^- = 1|N^- = n)E[S_1], \tag{4.2}$$

where

$$P(I^- = i|N^- = n) = \frac{\lambda P(n,i)}{\lambda P(n,0) + \lambda P(n,1) I\{n \geq 1\}}, \quad i = 0, 1, \tag{4.3}$$

$I\{n \geq 1\}$ is the indicator function of the set $\{1, 2, \ldots\}$ and the stationary distribution $P(n, i)$ is to be determined. Making use of the equations (3.1) and (4.1)–(4.3), we have

$$\begin{aligned} S_u(n) &= R - \frac{C}{\mu} - \frac{CP(I^- = 0|N^- = n)}{\theta} \\ &= R - \frac{C}{\mu} - \frac{C}{\theta} \frac{P(n,0)}{P(n,0) + P(n,1) I\{n \geq 1\}} \qquad (4.4) \\ &\geq R - \frac{C}{\mu} - \frac{C}{\theta} > 0. \qquad (4.5) \end{aligned}$$

So the expected benefit of the customer is positive and they always choose to join the queue. For this reason, the customer balking strategies are independent of the threshold n_0 and the number $N(t)$. We prove the Theorem 4.1. □

In the following theorem, on the basis of Theorem 4.1, we study the stationary distribution if all the customers join the system.

Theorem 4.2. *In the unobservable case of a clearing queueing system with setup times, when all the customers enter the system, the stationary distribution* $(P(n, i) : (n, i) \in \Omega_u = \Omega = \{(n, i)|n \geq 1,\ i = 0, 1\} \bigcup \{(0, 0)\})$ *is given as follows:*

$$P(0,0) = \left\{1 + \frac{\lambda(\theta + \mu)}{\mu\theta}\right\}^{-1}, \tag{4.6}$$

$$P(n,0) = \sigma^n P(0,0), \quad n \geq 1, \tag{4.7}$$

$$P(n,1) = \frac{\theta}{\theta - \mu}(\omega^n - \sigma^n) P(0,0), \quad n \geq 1. \tag{4.8}$$

where

$$\rho = \frac{\lambda}{\mu}, \quad \sigma = \frac{\lambda}{\lambda + \theta}, \quad \omega = \frac{\lambda}{\lambda + \mu},$$

Proof. To get the unique positive normalized solution $P(n, i)$, we firstly get the following system of balance equations from Figure 1:

$$\lambda P(0,0) = \sum_{n=1}^{\infty} \mu P(n,1), \tag{4.9}$$

$$(\lambda + \theta) P(n,0) = \lambda P(n-1,0), \quad n \geq 1, \tag{4.10}$$

$$(\lambda + \mu) P(1,1) = \theta P(1,0), \tag{4.11}$$

$$(\lambda + \mu) P(n,1) = \theta P(n,0) + \lambda P(n-1,1), \quad n \geq 2. \tag{4.12}$$

By iterating (4.10) we have

$$P(n,0) = \sigma^n P(0,0), \quad n \geq 1. \tag{4.13}$$

From (4.12) we observe that $(P(n,1) : n \geq 1)$ are the solutions of the nonhomogeneous first order linear difference equation with constants coefficients

$$\begin{aligned}(\lambda + \mu)P_n - \lambda P_{n-1} &= \theta P(n,0) \\ &= \theta \sigma^n P(0,0), \quad n \geq 2,\end{aligned} \tag{4.14}$$

where the last equation is due to (4.13). Using the standard approach for solving such equations, we solve the corresponding characteristic equation

$$(\lambda + \mu)x - \lambda = 0$$

with solution ω. Then the general solution of the homogeneous version of (4.14) is $P_n^{hom} = A\omega^n P(0,0)$ and A is to be determined. The general solution of (4.14) is given as $P_n^{gen} = P_n^{hom} + P_n^{spec}$, where P_n^{spec} is a specific solution of (4.14). Because the nonhomogeneous part $\theta\sigma^n P(0,0)$ of (4.14) is geometric with parameter σ, we can again use the standard approach to find a specific solution, which has the form $B\sigma^n$ (if $\sigma \neq \omega$ or $\theta \neq \mu$). Through putting it in (4.14) we obtain $B = \frac{\theta}{\mu-\theta} P(0,0)$. Hence, the general solution of (4.14) is given as

$$P(n,1) = P_n^{gen} = (A\omega^n + \frac{\theta}{\mu - \theta}\sigma^n)P(0,0), \quad n \geq 1. \tag{4.15}$$

Substituting (4.15) in (4.11), taking into account (4.13), we have

$$A = \frac{\theta}{\theta - \mu}\cdot \tag{4.16}$$

So

$$P(n,1) = \frac{\theta}{\theta - \mu}(\omega^n - \sigma^n)P(0,0), \quad n \geq 1. \tag{4.17}$$

We have thus expressed all stationary probabilities by means of $P(0,0)$ in formulas (4.13) and (4.17). Finally, the remaining probability, $P(0,0)$, can be easily found from the normalization equation

$$P(0,0) + \sum_{n=1}^{\infty} P(n,1) + \sum_{n=1}^{\infty} P(n,0) = 1,$$

as well as (4.13) and (4.17). The proof is completed. □

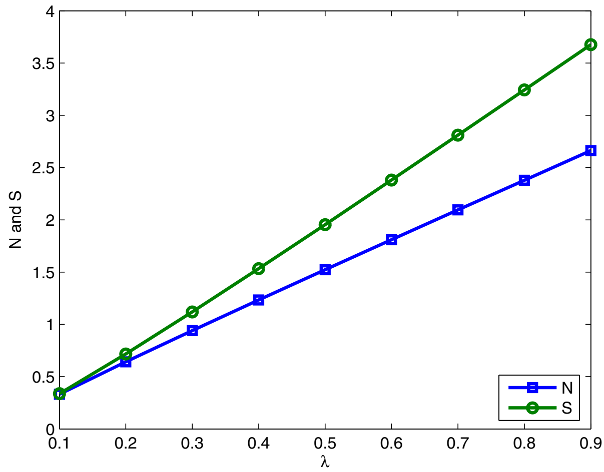

FIGURE 3. N and S *vs.* λ (where $\mu = 1, \theta = 0.4,\ R = 10,\ C = 2$).

In addition, from the Theorem 4.2 we can get the expected queue length N and the social optimal benefit S per time

$$\begin{aligned} N &= \sum_{n=1}^{\infty} n(P(n,0) + P(n,1)) \\ &= \left(\frac{\lambda^2(\mu^2 + \mu\theta + \theta^2)}{\mu\theta} + \lambda(\theta + \mu) \right) (\lambda\theta + \mu\theta + \lambda\mu)^{-1}, \qquad (4.18) \\ S &= R\lambda - CN \\ &= R\lambda - C \left(\frac{\lambda^2(\mu^2 + \mu\theta + \theta^2)}{\mu\theta} + \lambda(\theta + \mu) \right) (\lambda\theta + \mu\theta + \lambda\mu)^{-1}. \qquad (4.19) \end{aligned}$$

From (4.18) we know the expected queue length N is a function related to $\lambda,\ \mu,\ \theta$ and is independent of R and C. The reason is that all the customers choose to join the queue if $R - \frac{C}{\mu} - \frac{C}{\theta} > 0$. Meanwhile the social optimal benefit S is explicit expression by the parameters $\lambda,\ \mu,\ \theta,\ C$ and R. These results will help us determine the actual capacity and benefit in reality for the facility operator.

5. NUMERICAL EXPERIMENTS

In this section, we give some figures to illustrate the tendency of variables N and S in the unobservable case. From Figure 3, we can find the expected queue length N will increase if the arrival rate λ increases. This is coincidence with the case in reality. Meanwhile the social optimal benefit S will also increase. And from Figures 4 and 5, the expected queue length N will decrease because the bigger service rate μ or the bigger repair rate θ may enhance the service efficiency and

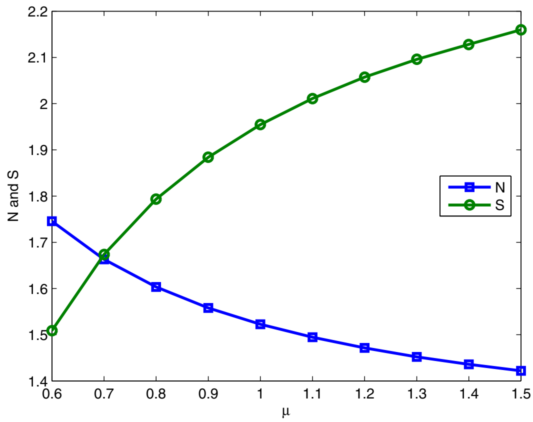

FIGURE 4. N and S *vs.* μ (where $\lambda = 0.5, \theta = 0.4,\ R = 10,\ C = 2$).

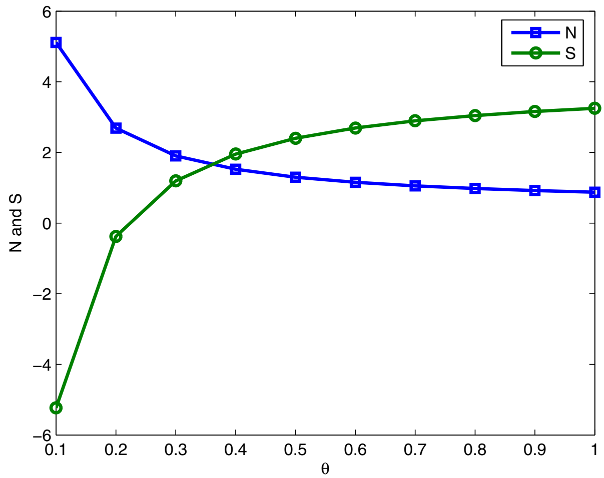

FIGURE 5. N and S *vs.* θ (where $\lambda = 0.5, \mu = 1,\ R = 10,\ C = 2$).

reduce the waiting time. Naturally, S is bigger than before which is contrast to the tendency of N.

6. CONCLUSIONS

In this article, we studied balking strategies of the clearing system with setup times under two different cases. In the unobservable case, we obtained the stationary distribution of system state, the expected queue length and the social optimal benefit. On the other hand, every customer who decides to join the queue does not impose any externalities to other customers because the server removes all the customers in the system simultaneously. So in the observable and the unobservable cases the equilibrium strategies are also the strategies that maximize the social benefit. On account of complexity of the model, there is still some work to do, such as investigating the balking strategies for other vacation models.

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