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MULTI-OBJECTIVE GEOMETRIC PROGRAMMING PROBLEM WITH KARUSH−KUHN−TUCKER CONDITION USING ϵ-CONSTRAINT METHOD

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Abstract. Optimization is an important tool widely used in formulation of the mathematical model and design of various decision making problems related to the science and engineering. Generally, the real world problems are occurring in the form of multi-criteria and multi-choice with certain constraints. There is no such single optimal solution exist which could optimize all the objective functions simultaneously. In this paper, ϵ-constraint method along with Karush−Kuhn−Tucker (KKT) condition has been used to solve multi-objective Geometric programming problems(MOGPP) for searching a compromise solution. To find the suitable compromise solution for multi-objective Geometric programming problems, a brief solution procedure using ϵ-constraint method has been presented. The basic concept and classical principle of multi-objective optimization problems with KKT condition has been discussed. The result obtained by ϵ-constraint method with help of KKT condition has been compared with the result so obtained by Fuzzy programming method. Illustrative examples are presented to demonstrate the correctness of proposed model.

Keywords. Geometric Programming, Karush−Kuhn−Tucker (KKT) condition, ϵ-constraint method, fuzzy programming, duality theorem, Pareto optimal solution.

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1. Introduction

Geometric programming(GP) is a powerful optimization technique developed to solve a class of non-linear optimization programming problems especially found in engineering design and manufacturing. In 1961, Duffin *et al.* [4] put a foundation stone to solve a wide range of engineering design problems developing the basic theories of Geometric programming and have shown its application in their text. In their work, they have considered many engineering design problems where an objective function consisting of component cost under certain constraints which are in the form of posynomials. Several methods have been proposed by the authors [1,2,5–7] to solve various non-linear programming problems subject to linear and non-linear constraints. A lot of work have been done in the field of production, transportation, distribution of products and risk management as well as inventory model in market planning [16–18,20,21,36]. The recent application of GP can be found in various fields including circuit design [22,23], information theory [24,25], queue proportional scheduling in fading broadcast channels [26] and also in certain convex optimization problems [31]. Since the recent real world problems are multi choice problems, optimizing a combination of objectives has the advantages of producing a single compromise solution require no further discussion with decision makers. If the optimal solution can not be accepted either due to function used or to an inappropriate setting of the coefficient of combining functions, a new runs of optimizer may be required until a suitable solution is found. Now-a-days, GP techniques have been used extensively to solve various engineering design problems which are in the form of multi-objective functions. These multi-objective functions with given constraints are usually incommensurate and conflict with one another. It indicates, multi-objective optimization problem does not have a single solution that could optimize all objective functions simultaneously. However, the decision makers are always in search of a most compromise solution that could optimize all objective functions, known as pareto optimal solution. In this paper, we have discussed a theory which has been developed for locating the points of maxima and minima of constrained and unconstrained nonlinear optimization problems, popularly known as Kuhn−Tucker theory. Though, it is not well suited for computational purposes, it provides a set of necessary and sufficient conditions for locating the point of optimality. Sinha *et al.* [42] used KKT transportation approach for solving multi-objective multi-level linear programming problems. Luptacik [45], in his paper presented optimization theory as an instrument for qualitative economic analysis using KKT theory. He has shown the relationship of Kuhn−Tucker conditions to their saddle points of the Lagrangian function in finding optimal solutions. It has been observed that there exist no unified approach to dualization in multi-objective optimization. In multi-objective optimization problem there exist a set of pareto optimal solution instead of unique solution. Luptacik also developed the parametric form of duality for a multi-objective optimization problem which can be used for solving multi-objective geometric programming problems and as an application he developed and analysed a nonlinear model of environmental control.

In an another paper, he has studied the optimal behaviour of a monopolistic firm and subsequently developed a model how to maximize revenue and profit based on neoclassical theory of monopolistic firm. In economics and engineering design, there are so many problems leading to geometric programming models involving several objectives or criteria that must be considered simultaneously. In a paper Pascual and Israel [46] have generated an efficient and properly efficient solutions of geometric programming problems using vector valued criteria and also proposed a new parametric problem with a nonlinear combination of objectives via power function. Soorpanth [32] has solved multi-objective analog circuit design problems using GP technique. Waiel *et al.* [33] have solved multi-objective transportation problems under fuzziness. In a recent paper, Ray *et al.* [34] developed multi-item inventory models of deteriorating items with space constraints under fuzzy environment. Cao.bing-yuan, first extended GPP in the study of fuzzy state problems and consider the situation where co-efficient are fuzzy [6,7,27,28]. Biswal [2] developed fuzzy programming with non-linear membership functions in the study of multi-objective Geometric programming problems. Some other classical methods such as Weighting mean and ϵ-constraint method are also used to solve multi-objective Geometric programming problems with fuzzy parameters. According to Hwang *et al.* [29], the methods for solving multi-objective optimization problems can be classified into three categories such as priori method, the interactive method and generation method. The priori methods are based on the goals or weights set by the decision maker before the solution process, being a difficult task for obtaining a compromise solution. In the interactive methods phases of dialogue with decision maker are changed with phases of calculation and the process usually converges towards the most preferred solution after a few iteration. The draw back in this method is decision maker can not see the whole pareto front or an approximation in it. The generation methods are less popular due to their computational effort in finding pareto optimal solutions. Various generation methods are found in literature which are presented by [12–14]. The ultimate aim of the multi-objective optimization is to achieve three important goals. First, the pareto front should be very close as possible as to the true pareto front. Secondly, the solution best known as pareto optimal should be uniformly distributed and finally the best pareto front should capture the whole spectrum of the pareto front.

In this paper we have applied ϵ-constraint method to solve a class of multi-objective Geometric programming problems. Using ϵ-constraint method, we can optimize one of the objective function at a time where other objectives are kept in the constraint part of the model as defined in Section 5. This method is found more suitable than other generating methods used for obtaining pareto optimal solutions. After obtaining lower and upper bounds of each objective function with the given constraints, we have generated a set of pareto optimal solution. The result so obtained has been compared with its corresponding solution obtained by fuzzy programming method. Berube *et al.* [43] in their recent paper have studied the multi-objective optimization problems by solving a series of single objective sub-problems, where all but one objectives are transformed into constraints. They have also shown, how the Pareto front of bi-objective problems can be efficiently generated with the [epsilon]-constraint method. In a recent paper, Laumanns *et al.* [44] have discussed for generating or approximating the Pareto set of multi-objective optimization problems by solving a sequence of constrained single-objective problems. The requirement of determining the constraint value *a priori* is shown to be a serious drawback of the original epsilon-constraint method. Therefore they have proposed a new adaptive scheme to generate appropriate constraint values during the run.

The organization of this paper is as follows: Following introduction, the concept of MOGPP has been discussed in Section 2. General mathematical formulation of KKT condition and solution of Geometric programming problem using KKT condition have been discussed in Sections 3 and 4 respectively, where as ϵ-constraint method has been discussed in Section 5. Standard GP with ϵ-constraint method and its corresponding dual GP have been discussed in Sections 6 and 7 to find most compromise solution of the multi-objective functions. Fuzzy programming method has been discussed in Section 8, to compare pareto optimal solution so obtained by ϵ-constraint method with its counterpart fuzzy solution and the convergence analysis of pareto set of solution has been discussed in Section 9. An illustrative example have been incorporated in Section 10 and finally some conclusions drawn from the results have been presented in Section 11.

2. Multi-objective geometric programming problem(MOGPP)

Optimization is an important activity in many fields of science and engineering. The classical framework for optimization is to find the optimum value of objective functions with respect to the given constraints. All the Conventional type optimization methods seek to find a single optimal solution based on a weighted sum of all objectives. If all objectives get better or worse together, then conventional approach can effectively find the optimal solution. In this case, a multi-objective optimization study should be performed which provides multiple tradeoff solutions among the objectives. A solution of multi-objective optimization problem is considered to be more a concept than a definition. In multi-objective optimization problems, what is optimal in terms of one of the objectives is usually non-optimal for the remaining objectives. Consequently, there is no single optimal solution exist for a multi-objective optimization problem. Hence we have to search for a solution which is acceptable to the decision maker. The method of optimizing systematically and simultaneously a collection of objective function is called multi-objective optimization or vector optimization. A multi-objective geometric programming problem can be stated as:

Find $x = (x_1, x_2, \ldots, x_n)^T$ so as to

$$\min : f_0^k(x) = \sum_{t=1}^{T_0^k} C_{0t}^k \prod_{j=1}^{n} x_j^{a_{0tj}^k}, \quad k = 1, 2, \ldots, p \tag{2.1}$$

subject to

$$g_i(x) = \sum_{t=1}^{T_i} C_{it} \prod_{j=1}^{n} x_j^{a_{itj}} \leq 1, \quad i = 1, 2, \ldots, m \tag{2.2}$$

$$x_j > 0, \quad j = 1, 2, \ldots, n \tag{2.3}$$

where:

$C_{0t}^k > 0$ for all k and t;
$C_{it} > 0$ for all i and t;
a_{0tj}^k and a_{itj} are real numbers for all i, j, k, t;
T_0^k = number of terms present in the kth objective function $f_0^k(x)$;
T_i= number of terms present in the ith constraint.

In the above multi-objective Geometric programming problem, there are p number of minimization type objective functions, m number of inequality type constraints and n number of strictly positive decision variables.

The multi-objective Geometric programming problem defined in (2.1)−(2.3) is considered as a Vector-minimization problem. It is assumed that the problem has an optimal compromise solution.

According to Pascual and Israel, a vector minimization problem is formulated in terms of a subset X of R^m and a vector valued function $F(x) = [f_1(x), f_2(x), \ldots, f_s(x)]$ which is defined on X, where X is the feasible set and F is the vector valued criterion function whose components are to be minimized in X.

A point $x_0 \in X$ is an efficient solution of vector minimization problem if $x \in X$, $F(x) \leq F(x_0) \Rightarrow x = x_0$. The point x_0 is an properly efficient solution of a vector minimization problem if it is efficient and if there is a scalar$M > 0$ such that $x_i \in X$, $f_i(x) < f_i(x_0) \Rightarrow [f_i(x) - f_i(x_0)]/[f_j(x) - f_j(x_0)] \leq M$ for some j with $f_j(x) > f_j(x_0)$. If x_0 is an optimal solution of a primal program, then x_0 will be an efficient solution of vector minimization problem.

3. Karush−Kuhn−Tucker (KKT) condition

The theory which has been developed for locating the points of maxima and minima of constrained and unconstrained optimization problems is known as Kuhn−Tucker theory. It provides a set of necessary and sufficient conditions for checking whether the given point is point of optimality or not. This section primarily deals with developing the necessary form of KKT condition for identifying stationary points of constrained non-linear optimization problems. The general form of KKT condition can be stated as follows.

In a general optimization problem, we have to find $x = (x_1, x_2, \ldots, x_n)$ so as to

$$\min: \ f(x) = f(x_1, x_2, \ldots, x_n) \tag{3.1}$$

subject to

$$g_i(x) \le b_i, i = 1, 2 \ldots, m$$
$$i.e. \quad b_i - g_i(x) \ge 0, i = 1, 2 \ldots, m \tag{3.2}$$

$$\text{where,} \quad x \ge 0. \tag{3.3}$$

Now, we can write the above optimization problem using KKT condition as follows:

$$\min: \; F(x) = f(x) - \sum_{i=1}^{m} \lambda_i \left(b_i - g_i(x) - \phi_i^2\right) \tag{3.4}$$

subject to the following conditions are satisfied.

$$\frac{\partial f(x)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial}{\partial x_j}(b_i - g_i(x) - \phi_i^2) = 0, j = 1, 2, 3, \ldots, n \tag{3.5}$$

$$\lambda_i(b_i - g_i(x)) = 0, \qquad i = 1, 2, 3, \ldots, m \tag{3.6}$$

$$(b_i - g_i(x)) \ge 0, \qquad i = 1, 2, 3, \ldots, m \tag{3.7}$$

$$\lambda_i \ge 0, \qquad i = 1, 2, 3 \ldots, m \quad and \quad x \ge 0 \tag{3.8}$$

where λ_i's are constant known as Lagrange's multiplier and ϕ_i is a complimentary surplus of ith constraint.

The KKT conditions are necessary conditions for a local maximum or minimum. They don't guarantee that a point satisfying them is actually a local maximum or minimum.

4. Geometric programming problem(GPP) with KKT condition

Minimising a posynomial subject to inequality constraints is known as geometric programming problem. A constrained posynomial GPP is represented as follows:

$$\min: \; f_0(x) = \sum_{t=1}^{T_0} c_{0t} \prod_{j=1}^{n} x_j^{a_{0tj}} \tag{4.1}$$

subject to

$$g_i(x) = \sum_{t=1}^{T_i} c_{it} \prod_{j=1}^{n} x_j^{a_{itj}} \le 1$$
$$i.e. \quad 1 - g_i(x) \ge 0 \tag{4.2}$$

where

$$i = 1, 2, 3, \dots, m, \quad j = 1, 2, 3, \dots, n. \quad \textit{and} \quad x_j > 0. \tag{4.3}$$

Here the posynomial $f_0(x)$ is an objective function containing T_0 number of terms where as posynomial $g_i(x)$ contains T_i, $(i = 1, 2, 3, \dots, m)$ number of inequality constraints.

The standard GPP using KKT condition can be represented as:

$$\min: \ F_0(x) = \sum_{t=1}^{T_0} c_{0t} \prod_{j=1}^{n} x_j^{a_{0tj}} - \sum_{t=1}^{T_i} \lambda_t \left(-c_{it} \prod_{j=1}^{n} x_j^{a_{itj}} + 1 - \psi_t^2 \right) \tag{4.4}$$

subject to

$$\frac{\partial}{\partial x_j} \left(\sum_{t=1}^{T_0} c_{0t} \prod_{j=1}^{n} x_j^{a_{0tj}} \right) - \frac{\partial}{\partial x_j} \left(\sum_{t=1}^{T_i} \lambda_t \left(-c_{it} \prod_{j=1}^{n} x_j^{a_{itj}} + 1 - \psi_t^2 \right) \right) = 0 \tag{4.5}$$

$$\sum_{t=1}^{T_i} \lambda_t \left(-c_{it} \prod_{j=1}^{n} x_j^{a_{itj}} + 1 \right) = 0 \tag{4.6}$$

$$\sum_{t=1}^{T_i} \left(-c_{it} \prod_{j=1}^{n} x_j^{a_{itj}} + 1 \right) \geq 0 \tag{4.7}$$

$$\lambda_t \geq 0 \quad \text{and} \quad x \geq 0. \tag{4.8}$$

where λ_t's are known as Lagrange multiplier and ψ_t is a complimentary surplus of tth constraint.

5. ϵ-Constraint method

A method which overcomes some of the convexity problems of the weighted sum technique is known as ϵ-constraint method. This method involves minimizing a primary objective and expressing the other objectives in the form of inequality constraints.

The ϵ-constraint method was proposed by Haimes *et al.* [30] for generating Pareto optimal solutions for the multi-objective optimization problem. This method generates the non inferior solutions of multi-objective optimization problems by considering one objective function at a time as primary one and converting the remaining objective functions as constraints. In other words, it minimizes one objective function and simultaneously maintain the maximum acceptability level for other objective function. The ϵ-constraint method is defined as:

$$\min: \ f_0^k(x), \quad \text{where } k \in \{1, 2, \dots, p\} \tag{5.1}$$

subject to

$$f_0^j(x) \leq \epsilon_j, \quad j = 1, 2, \dots, p, \quad j \neq k \tag{5.2}$$

$$g_i(x) \leq 1, \qquad i = 1, 2, \dots, m \tag{5.3}$$

we define

$$L_j \leq \epsilon_j \leq U_j, j = 1, 2, \dots, p, j \neq k$$

where

$$L_j = \min_{\forall x \in X} \quad f_0^j(x), j = 1, 2, \dots, p$$

and

$$U_j = \max_{\forall x \in X} \quad f_0^j(x), j = 1, 2, \dots, p$$

$$x \in X, \quad X \text{ being the feasible region.}$$

Changing the value of ϵ_j in the interval $[L_j, U_j]$, j=1,2,..., p, we generate the most preferred non-inferior solution of the MOGPP.

6. Standard MOGPP with ϵ-constraint method

For any non-linear multi-objective optimization problem, the solution obtained by ϵ-constraint method yields a weak pareto optimal solution [30]. A true pareto optimal solution can be obtained, either if the solution is unique or if the optimizations are done for all the objectives before reporting the solution [37]. As our proposed method is designed to deals with the real valued problems which are likely to have a continuous Pareto front, a systematic variation of ϵ_j will yields a set of non-dominated solution. However, the determination of the minimum level and assumption about the form of preference in finding the preferred decisions are often questionable in real world problems. The ϵ-constraint method is also applicable to a non-convex vector optimization problem.

The ϵ-constraint method has been incorporated into the surrogate worth trade-off method as an interactive decision making method. The bounded objective method is another variation of this approach.

Using ϵ-constraint method, the present multi-objective Geometric programming problem (2.1)−(2.3) can be redefined as a single-objective Geometric programming problem as:

$$\min: \ f_0^k(x) = \sum_{t=1}^{T_0^k} \prod_{j=1}^{n} x_j^{a_{0tj}^k}, \qquad k \in \{1, 2, \dots, p\} \tag{6.1}$$

subject to

$$f_0^r(x) = \sum_{t=1}^{T_0^r} C_{0t}^r \prod_{j=1}^{n} x_j^{a_{0tj}^r} \leq \epsilon_r, \quad r = 1, 2, \dots, \dots, p, r \neq k \tag{6.2}$$

and

$$g_i(x) = \sum_{t=1}^{T_i} C_{it} \prod_{j=1}^{n} x_j^{a_{itj}} \leq 1, \quad i = 1, 2, \ldots, m \tag{6.3}$$

$$x_j > 0, \quad j = 1, 2, \ldots, n \tag{6.4}$$

where $\epsilon_r > 0$, for all r.

Here we have p number of single-objective geometric programming problems.

These problems can be further simplified as:

$$\min : \; \widehat{f}_0^k(x) = \sum_{t=1}^{T_0^k} \widehat{C}_{0t}^k \prod_{j=1}^{n} x_j^{a_{0tj}^k}, \qquad k \in \{1, 2, \ldots, p\} \tag{6.5}$$

subject to

$$\widehat{f}_0^r(x) = \sum_{t=1}^{T_0^r} \frac{C_{0t}^r}{\epsilon_r} \prod_{j=1}^{n} x_j^{a_{0tj}^r} \leq 1, \quad r = 1, 2, \ldots, p, r \neq k$$

$$\widehat{f}_0^r(x) = \sum_{t=1}^{T_0^r} \widehat{C}_{0t}^r \prod_{j=1}^{n} x_j^{a_{0tj}^r} \leq 1, \quad r = 1, 2, \ldots, p, r \neq k \tag{6.6}$$

$$g_i(x) = \sum_{t=1}^{T_i} \widehat{C}_{it} \prod_{j=1}^{n} x_j^{a_{itj}} \leq 1, \quad i = 1, 2, \ldots, m \tag{6.7}$$

$$x_j > 0, \quad j = 1, 2, \ldots, n$$

where

$$\widehat{C}_{0t}^r = \frac{C_{0t}^r}{\epsilon_r} \tag{6.8}$$

for all $r, t, , r \neq k$

$$\widehat{C}_{it}^k = C_{it}^k, \quad i = 1, 2, \ldots, m \;\; \text{and} \;\; \widehat{f}_0^k(x) = f_0^k(x) \tag{6.9}$$

for $k = 1, 2, \ldots, p$.

The GPP has $p + m - 1$ constraints and n number of decision variables.

The degree of difficulty of this problem can be defined as:

$$d = \sum_{k=1}^{p} T_0^k + \sum_{i=1}^{m} T_i - n - 1 \tag{6.10}$$

7. Dual form of MOGPP with ϵ-Constraint method

The dual form of Geometric programming problems plays an important role in solving complex types of single and multi-objective optimization problems. It has been verified, if primal program of the problem is super consistent and attains its constrained minimum value, then its corresponding dual program is consistent and attains its corresponding maximum value where the optimal values due to primal and dual program are same. The model given by (6.5)−(6.9) is a conventional type Geometric programming problem. The solution procedure for a GP may be categorized as of two types. It is either primal based algorithms that directly solve non-linear primal problem, or dual based algorithms that solve the equivalent linear constraint dual program [38]. In view of Rajgopal and Bricker [39], the dual program has the desirable features of some linear constraints having an objective function with attractive structural properties, which enables for getting a solution. According to Beightler and Phillips [1] and Duffin *et al.* [4], one can obtain the corresponding dual program of (6.5)−(6.9) as follows:

Dual Program.

$$\max: \ V(w) = \prod_{k=1}^{p} \prod_{t=1}^{T_0^k} \left(\frac{C_{0t}^k}{w_{0t}^k} \right)^{w_{0t}^k} \prod_{i=1}^{m} \prod_{t=1}^{T_i} \left(\frac{C_{it}}{w_{it}} \right)^{w_{it}} \prod_{k=1}^{p} \left(\lambda^k\right)^{\lambda^k} \prod_{i=1}^{m} \left(\lambda^i\right)^{\lambda^i} \tag{7.1}$$

subject to

$$\sum_{t=1}^{T_0^r} w_{0t}^r = \lambda^r, \qquad r = 1, 2, \ldots, p, r \neq k \tag{7.2}$$

$$\sum_{t=1}^{T_0^k} w_{0t}^k = \lambda^k = 1 \tag{7.3}$$

(normality condition)

$$\sum_{t=1}^{T_i} w_{it} = \lambda_i, \qquad i = 1, 2, \ldots, m \tag{7.4}$$

$$\sum_{k=1}^{p} \sum_{t=1}^{T_0^k} a_{0tj}^k w_{0t}^k + \sum_{i=1}^{m} \sum_{t=1}^{T_i} a_{itj} w_{it} = 0, \quad j = 1, 2, \ldots, n \tag{7.5}$$

$$w_{it} \geq 0 \ \ \forall\ t, i$$

$$w_{0t}^k \geq 0 \ \ \forall\ k, t.$$

This dual problem can be solved by using duality theory of GPP.

8. Fuzzy programming method

Zadeh, introduced fuzzy set theory in the year 1965 which is a generalization of classical set theory to understand the uncertainty and vagueness in the complexity of the problems. Fuzzy programming Problem due to Zimmermann [41] based on the concept given by Bellman and Zadeh [40] has been successfully applied to solve various types of multi-objective decision making problems such as engineering design and maintenance, production planning and control, transportation, water resource management, managerial decision making and scheduling problems. A fuzzy set is associated with its membership function which is defined from its elements to the interval [0,1] plays an important role in solving multi-objective decision making problems. As there are several type of fuzzy membership functions, a suitable membership function is to be selected to solve the real world multi-objective mathematical programming problems. The following steps are used for solving a multi-objective optimization problem with a linear membership function by Geometric programming technique to find an optimal compromise solution.

Step 1. Choose one of the objective function $f_0^k(x), k = 1, 2, \ldots, p.$ and solve it as a single objective Geometric programming problem subject to the constraints (2.2) and (2.3) by using Geometric programming algorithms [22]. Let $X^{(1)}, X^{(2)}, \ldots, X^{(p)}$ be the respective optimal solution for p different ($k = 1, 2 \ldots, p$)) Geometric programming problems. It is assumed that at least two of these ideal solutions are different ($f_0^k(x), k = 1, 2, \ldots, p$) and has the different bound values. If all the optimal solutions $X^{(1)} = X^{(2)} = \ldots = X^{(p)} = X^*$ are same then stop and $X^{(*)}$ is the optimal compromise solution. Otherwise go to Step 2.

Step 2. Evaluate all these p objective functions $f_0^k(x), k = 1, 2, \ldots, p$, at all these p ideal solutions $X^{(1)}, X^{(2)}, \ldots, X^{(p)}$.

Step 3. Find the best value L_k (minimum value) and the worst value U_k (maximum value) of each objective function $f_0^k(x)$ such that

$$L_k \leq f_0^k(x) \leq U_k, \qquad k = 1, 2, \ldots, p.$$

Step 4. Define a fuzzy membership function $\mu_k(x)$ for the kth objective function $f_0^k(x)$ as:

$$\mu_k(x) = \begin{cases} 1 & \text{if} \quad f_0^k(x) \leq L_k; \\ \frac{U_k - f_0^{(k)}(x)}{U_k - L_k} & \text{if} \quad L_k \leq f_0^k(x) \leq U_k; \\ 0 & \text{if} \quad f_0^{(k)}(x) > U_k. \end{cases}$$

where $L_k \neq U_k, k = 1, 2, \ldots, p$

If $L_k = U_k$ then define $\mu_k(x) = 1$ for any value of k.

Now maximize the membership function $\mu_k(x), k = 1, 2, \ldots, p$ subject to the constraints (2.2) and (2.3) and then use max-min operator [23] to find a crisp model.

Step 5. Consider a Dummy variable θ and formulate a crisp model for fuzzy Geometric programming problem as :

$$\max : \theta \tag{8.1}$$

subject to

$$\frac{U_k - f_0^{(k)}(x)}{U_k - L_k} \geq \theta, \quad k = 1, 2, \ldots, p \tag{8.2}$$

$$g_i(x) \leq 1, \quad i = 1, 2, \ldots, m \tag{8.3}$$

$$\theta \geq 0, \quad x_j > 0, \quad j = 1, 2, \ldots, n \tag{8.4}$$

Further the inequality (8.2) can be represented as:

$$f_0^{(k)}(x) + (U_k - L_k)\,\theta \leq U_k, \quad k = 1, 2, \ldots, p. \tag{8.5}$$

Step 6. Solve the crisp Geometric programming problem defined in Step 5 by using Geometric programming algorithms to find x^* and evaluate all p number of objective functions (2.1) at this optimal solution x^*.

9. The ϵ-Constraint method and convergence of the optimal solution

There is no specific mathematical proof for the convergence of the pareto optimal solutions of the multi-objective mathematical programming problem available in the literature. However the decision maker try to find out the most compromise solutions by using some of the existing methods like fuzzy programming, goal programming and weighting methods. In the present work we have used ϵ-constraint method as defined in Section 5 to find the pareto optimal solution. Here, we have adopted the following steps to show the set of pareto optimal solutions are converging to certain point.

Step 1. First find the bounds of the objective functions $(f_0^{(k)}(x), k = 1, 2, \ldots, p)$ with the help of obtained ideal solutions $X^{(1)}, X^{(2)}, \ldots, X^{(p)}$ by using Geometric programming algorithms as discussed in the section 6, such that L_k and U_k are the best and worst values of $f_0^{(k)} i.e. L_k \leq f_0^{(k)}(x) \leq U_k, k = 1, 2 \ldots, p$

Step 2. Let ϵ_k, be a point in the interval such that $L_k \leq \epsilon_k \leq U_k, k = 1, 2 \ldots, p$

Step 3. Changing the value of ϵ_k in the interval $[L_k, U_k]$, it generate a set of pareto optimal solution.

Step 4. Compare the pareto optimal solution with the solution obtained by fuzzy programming method.

Step 5. If the pareto optimal solution obtained in Step 3 is equal to the optimal compromise solution obtained in Step 4, then stop and accept the pareto optimal solution of the problem. This indicate that the set of solution generated by ϵ-constraint method converges to this particular solution. However, the decision maker has the choice to choose his/her solution from the set of solution according to their satisfaction.

The following illustrative examples explain, How to find the pareto optimal solutions and convergence of the method.

10. Numerical examples

For illustration the following multi-objective Geometric programming problem can be considered.

Example 1. Find x_1, x_2, x_3 so as to

$$\min: \; f_1(x) = x_1^{-2} + \frac{1}{4}x_2^2x_3^{-1} \tag{10.1}$$

$$\min: \; f_2(x) = 2x_1^{-1}x_2^{-1}x_3^{-1} + 2x_1x_2 \tag{10.2}$$

subject to

$$\frac{3}{4}x_1^2x_2^{-2} + \frac{3}{8}x_2x_3^2 \leq 1 \tag{10.3}$$

where

$$x_1, x_2, x_3 > 0. \tag{10.4}$$

Case 1. Solution of Primal $f_1(x)$.
Find x_1, x_2, x_3 so as to

$$\min: \; f_1(x) = x_1^{-2} + \frac{1}{4}x_2^2x_3^{-1} \tag{10.5}$$

subject to

$$\frac{3}{4}x_1^2x_2^{-2} + \frac{3}{8}x_2x_3^2 \leq 1 \tag{10.6}$$

where

$$x_1, x_2, x_3 > 0. \tag{10.7}$$

Using KKT condition,the above problem can be written as follows:

$$\min: f_1(x) = x_1^{-2} + \frac{1}{4}x_2^2x_3^{-1} - \lambda\left(-\frac{3}{4}x_1^2x_2^{-2} - \frac{3}{8}x_2x_3^2 + 1\right) \tag{10.8}$$

subject to

$$\frac{\partial f_1(x)}{\partial x_1} = -2x_1^{-3} - \lambda\left(-\frac{3}{2}x_1x_2^{-2}\right) = 0 \tag{10.9}$$

$$\frac{\partial f_1(x)}{\partial x_2} = \frac{1}{2}x_2x_3^{-1} - \lambda\left(\frac{3}{2}x_1^2x_2^{-3} - \frac{3}{8}x_3^2\right) = 0 \tag{10.10}$$

$$\frac{\partial f_1(x)}{\partial x_3} = -\frac{1}{4}x_2^2x_3^{-2} - \lambda\left(-\frac{3}{4}x_2x_3\right) = 0 \tag{10.11}$$

$$\lambda\left(-\frac{3}{4}x_1^2x_2^{-2} - \frac{3}{8}x_2x_3^2 + 1\right) = 0 \tag{10.12}$$

$$\left(-\frac{3}{4}x_1^2x_2^{-2} - \frac{3}{8}x_2x_3^2 + 1\right) \geq 0 \tag{10.13}$$

where

$$x_1, x_2, x_3 \geq 0. \tag{10.14}$$

The corresponding Dual program using the condition given in Section 7 is given below:

$$\max_w : V(w) = \left(\frac{1}{w_{01}}\right)^{w_{01}} \left(\frac{1}{4w_{02}}\right)^{w_{02}} \left(\frac{3}{4w_{11}}\right)^{w_{11}} \left(\frac{3}{8w_{12}}\right)^{w_{12}} (w_{11} + w_{12})^{(w_{11}+w_{12})} \tag{10.15}$$

subject to

$$\begin{aligned} w_{01} + w_{02} &= 1 \\ -2w_{01} + 2w_{11} &= 0 \\ 2w_{02} - 2w_{11} + w_{12} &= 0 \\ -w_{02} + 2w_{12} &= 0 \\ w_{01}, w_{02}, w_{11}, w_{12} &\geq 0. \end{aligned}$$

The optimal primal solution of $f_1(x)$ using KKT condition is given by $f_1 = 1.171595$ for $x_1 = 1.239503,$ $x_2 = 1.270112,$ $x_3 = 0.7745138$ and $\lambda = 0.9112412$ where as same optimal value of objective function for dual problem is obtained for $w_{01} = 0.5555556, w_{02} = 0.444444, w_{11} = 0.5555556, w_{12} = 0.222222.$

Case 2. Solution of Primal $f_2(x)$.

Find x_1, x_2, x_3 so as to

$$\min : \ f_2(x) = 2x_1^{-1}x_2^{-1}x_3^{-1} + 2x_1x_2 \tag{10.16}$$

subject to

$$\frac{3}{4}x_1^2x_2^{-2} + \frac{3}{8}x_2x_3^2 \leq 1 \tag{10.17}$$

where

$$x_1, x_2, x_3 > 0. \tag{10.18}$$

Using KKT condition the above problem can be written as:

$$\min : \ f_2(x) = 2x_1^{-1}x_2^{-1}x_3^{-1} + 2x_1x_2 - \lambda\left(-\frac{3}{4}x_1^2x_2^{-2} - \frac{3}{8}x_2x_3^2 + 1\right) \tag{10.19}$$

$$\frac{\partial f_2(x)}{\partial x_1} = -2x_1^{-2}x_2^{-1}x_3^{-1} + 2x_2 - \lambda\left(-\frac{3}{2}x_1x_2^{-2}\right) = 0 \tag{10.20}$$

$$\frac{\partial f_2(x)}{\partial x_2} = -2x_1^{-1}x_2^{-2}x_3^{-1} + 2x_1 - \lambda\left(\frac{3}{2}x_1^2x_2^{-3} - \frac{3}{8}x_3^2\right) = 0 \tag{10.21}$$

$$\frac{\partial f_2(x)}{\partial x_3} = -2x_1^{-1}x_2^{-1}x_3^{-2} - \lambda\left(-\frac{3}{4}x_2x_3\right) = 0 \tag{10.22}$$

$$\lambda\left(-\frac{3}{4}x_1^2x_2^{-2} - \frac{3}{8}x_2x_3^2 + 1\right) = 0 \tag{10.23}$$

$$\left(-\frac{3}{4}x_1^2x_2^{-2} - \frac{3}{8}x_2x_3^2 + 1\right) \geq 0 \tag{10.24}$$

where

$$x_1, x_2, x_3 \geq 0. \tag{10.25}$$

The corresponding Dual program is given below

$$\max_w : V(w) = \left(\frac{2}{w_{01}}\right)^{w_{01}}\left(\frac{2}{w_{02}}\right)^{w_{02}}\left(\frac{3}{4w_{11}}\right)^{w_{11}}\left(\frac{3}{8w_{12}}\right)^{w_{12}}(w_{11}+w_{12})^{(w_{11}+w_{12})} \tag{10.26}$$

subject to

$$\begin{aligned} w_{01} + w_{02} &= 1 \\ -w_{01} + w_{02} + 2w_{11} &= 0 \\ -w_{01} + w_{02} - 2w_{11} + w_{12} &= 0 \\ -w_{01} + 2w_{12} &= 0 \\ w_{01}, w_{02}, w_{11}, w_{12} &\geq 0. \end{aligned}$$

The optimal primal solution of $f_2(x)$ using KKT condition is given by $f_2 = 3.504279\ for\ x_1 = 1.0.6227133,\ x_2 = 1.205879,\ x_3 = 1.330079\ and\ \lambda = 1.251528$ where as same optimal value of objective function for dual problem is obtained for $w_{01} = 0.5714286, w_{02} = 0.4285714, w_{11} = 0.07142857, w_{12} = 0.2857143$.

Using the solution of f_1 in f_2 and f_2 in f_1 as obtained above, we can find the lower bound L_i and upper bound U_i of the functions f_i for $i = 1, 2$ as:

$$L_1 = 1.171595 \leq f_1 \leq 2.852155 = U_1$$

$$L_2 = 3.504279 \leq f_2 \leq 4.788870 = U_2.$$

Considering ϵ_1 and ϵ_2 defined by

$$1.171595 \leq \epsilon_1 \leq 2.852155 \ \text{ and } \ 3.504279 \leq \epsilon_2 \leq 4.788870,$$

TABLE 1. Optimal solution of Primal (i).

ϵ_2	x_1	x_2	x_3	primal f_1
3.6	0.7761717	1.135443	1.235101	1.920866
3.8	0.9001469	1.136367	1.114596	1.523806
4.0	0.9880850	1.154432	1.020190	1.350848
4.2	1.061164	1.177817	0.9411259	1.256553
4.4	1.125801	1.204625	0.8738374	1.204157
4.6	1.185429	1.235360	0.8172388	1.178472
4.7	1.214197	1.252863	0.7931753	1.173041
4.75	1.228452	1.262344	0.7823372	1.171865
4.78	1.236983	1.268307	0.7762508	1.171609
4.788	1.239248	1.269929	0.7746880	1.171595
4.78887	1.239495	1.270107	0.7745190	1.171595

we can reformulate the above problem as two different problems using the ϵ-constraint method as follows.

Primal Problem (i) using ϵ-constraint method

$$\min: \ f_1(x) = x_1^{-2} + \frac{1}{4}x_2^2x_3^{-1} \tag{10.27}$$

subject to

$$\frac{3}{4}x_1^2x_2^{-2} + \frac{3}{8}x_2x_3^2 \leq 1 \tag{10.28}$$

$$2x_1^{-1}x_2^{-1}x_3^{-1} + 2x_1x_2 \leq \epsilon_2 \tag{10.29}$$

$$x_1, x_2, x_3 \geq 0. \tag{10.30}$$

Primal Problem (ii) using ϵ-constraint method

$$\min: \ f_2(x) = 2x_1^{-1}x_2^{-1}x_3^{-1} + 2x_1x_2 \tag{10.31}$$

subject to

$$\frac{3}{4}x_1^2x_2^{-2} + \frac{3}{8}x_2x_3^2 \leq 1 \tag{10.32}$$

$$x_1^{-2} + \frac{1}{4}x_2^2x_3^{-1} \leq \epsilon_1 \tag{10.33}$$

$$x_1, x_2, x_3 \geq 0. \tag{10.34}$$

Solution of Primal Problem (i)

Solution of the Primal (i) using general method is given in following Table 1.

Using KKT condition, we can write the same Primal Problem (i) as follows:

$$\min: \; f_1(x) = x_1^{-2} + \frac{1}{4}x_2^2x_3^{-1} - \lambda_1\left(-\frac{3}{4}x_1^2x_2^{-2} - \frac{3}{8}x_2x_3^2 + 1\right) - \lambda_2\left(-2x_1^{-1}x_2^{-1}x_3^{-1} - 2x_1x_2 + \epsilon_2\right) \tag{10.35}$$

subject to

$$\frac{\partial f_1(x)}{\partial x_1} = -2x_1^{-3} - \lambda_1\left(-\frac{3}{2}x_1x_2^{-2}\right) - \lambda_2\left(2x_1^{-2}x_2^{-1}x_3^{-1} - 2x_2\right) = 0 \tag{10.36}$$

$$\frac{\partial f_1(x)}{\partial x_2} = \frac{1}{2}x_2x_3^{-1} - \lambda_1\left(\frac{3}{2}x_1^2x_2^{-3} - \frac{3}{8}x_3^2\right) - \lambda_2(2x_1^{-1}x_2^{-2}x_3^{-1} - 2x_1) = 0 \tag{10.37}$$

$$\frac{\partial f_1(x)}{\partial x_3} = -\frac{1}{4}x_2^2x_3^{-2} - \lambda_1\left(-\frac{3}{4}x_2x_3\right) - \lambda_2(2x_1^{-1}x_2^{-1}x_3^{-2}) = 0 \tag{10.38}$$

$$\lambda_1\left(-\frac{3}{4}x_1^2x_2^{-2} - \frac{3}{8}x_2x_3^2 + 1\right) = 0 \tag{10.39}$$

$$\lambda_2\left(-2x_1^{-1}x_2^{-1}x_3^{-1} - 2x_1x_2 + \epsilon_2\right) = 0 \tag{10.40}$$

$$\left(-\frac{3}{4}x_1^2x_2^{-2} - \frac{3}{8}x_2x_3^2 + 1\right) \geq 0 \tag{10.41}$$

$$\left(-2x_1^{-1}x_2^{-1}x_3^{-1} - 2x_1x_2 + \epsilon_2\right) \geq 0 \tag{10.42}$$

where

$$x_1, x_2, x_3 \geq 0. \tag{10.43}$$

Solution of the above problem obtained by changing the value of ϵ_2 between 3.6 to 4.78887 is given by the following Table 2.

From the above table it is observed that objective values obtained by changing the value of ϵ_2 converging towards $f_1 = 1.171595$ for $x_1 = 1.239495$, $x_2 = 1.270107$, $x_3 = 0.7745190$, $\lambda_1 = 0.9112451$, $\lambda_2 = 0$.

The Dual Program of the Primal (i) is given below:

$$\max_w : V(w) = \left(\frac{1}{w_{01}}\right)^{w_{01}}\left(\frac{1}{4w_{02}}\right)^{w_{02}}\left(\frac{3}{4w_{11}}\right)^{w_{11}}\left(\frac{3}{8w_{12}}\right)^{w_{12}}(w_{11}+w_{12})^{(w_{11}+w_{12})} \left(\frac{2}{\epsilon_2 w_{21}}\right)^{w_{21}}\left(\frac{2}{\epsilon_2 w_{22}}\right)^{w_{22}}(w_{21}+w_{22})^{(w_{21}+w_{22})} \tag{10.44}$$

TABLE 2. Optimal solution of Primal (i) by ϵ-constraint method using KKT condition.

ϵ_2	x_1	x_2	x_3	λ_1	λ_2	primal f_1
3.6	0.7761717	1.135443	1.235101	5.106435	3.468290	1.920866
3.8	0.9001469	1.136367	1.114596	2.252495	1.194447	1.523806
4.0	0.9880850	1.154432	1.020190	1.546340	0.6207686	1.350848
4.2	1.061164	1.177817	0.9411259	1.229520	0.3490472	1.256553
4.4	1.125801	1.204625	0.8738374	1.058462	0.1866816	1.204157
4.6	1.185429	1.235360	0.8172388	0.9608445	0.0764288	1.178472
4.7	1.214197	1.252863	0.7931753	0.9302072	0.0332766	1.173041
4.75	1.228452	1.262344	0.7823372	0.9186274	0.014010	1.171865
4.78	1.236983	1.268307	0.7762508	0.9128042	0.0031247	1.171609
4.788	1.239248	1.269929	0.7746880	0.9113952	0.0003135	1.171595
4.78887	1.239495	1.270107	0.7745190	0.9112451	0.000009	1.171595

TABLE 3. Dual Solution.

ϵ_2	w_{01}	w_{02}	w_{11}	w_{12}	w_{21}	w_{22}	$Dual f_1$
3.6	0.864146	0.135853	0.931679	1.726716	3.317579	3.182513	1.920866
3.8	0.809922	0.190077	0.695640	0.782561	1.375045	1.603610	1.523806
4.0	0.758237	0.241762	0.628943	0.515774	0.789786	1.048373	1.350848
4.6	0.603850	0.396149	0.563066	0.252264	0.108379	0.189948	1.178472
4.7	0.578240	0.421759	0.558597	0.234390	0.047020	0.086306	1.173041
4.75	0.565465	0.434534	0.556780	0.227121	0.019708	0.037077	1.171865
4.78	0.557812	0.442187	0.555870	0.223281	0.004375	0.0083595	1.171609
4.788	0.5557488	0.444251	0.555577	0.222311	0.000372	0.000713	1.171595

subject to

$$
\begin{aligned}
w_{01} + w_{02} &= 1 \\
-2w_{01} + 2w_{11} - w_{21} + w_{22} &= 0 \\
2w_{02} - 2w_{11} + w_{12} - w_{21} + w_{22} &= 0 \\
-w_{02} + 2w_{12} - w_{21} &= 0 \\
w_{01}, w_{02}, w_{11}, w_{12}, w_{21}, w_{22} &\geq 0.
\end{aligned}
$$

Solution of the Dual problem obtained by changing the value of ϵ_2 between 3.6 to 4.78887 is given below in the following Table 3.

from the above table it is observed that the optimal objective value of dual problem is 1.171595 for $w_{01} = 0.5557488, w_{02} = 0.444281, w_{11} = 0.555577, w_{12} = 0.222311, w_{21} = 0.000372, w_{22} = 0.000713$.

TABLE 4. Optimal solution of Primal (ii).

ϵ_1	x_1	x_2	x_3	primal f_2
1.18	1.179935	1.232238	0.822119	4.581102
1.5	0.910294	1.137829	1.103910	3.820710
2.0	0.757922	1.138624	1.250490	3.579273
2.5	0.668465	1.172546	1.311444	3.513293
2.7	0.641207	1.190768	1.323795	3.505775
2.8	0.628857	1.200590	1.328192	3.504445
2.85	0.622963	1.205658	1.330006	3.504279

Solution of Primal Problem (ii)

Solution of the Primal (ii) using general method is given in following Table 4. Using KKT condition,we can write the Primal Problem (ii) as follows:

$$\begin{aligned}\min:\ f_2(x) = & 2x_1^{-1}x_2^{-1}x_3^{-1} + 2x_1x_2 - \lambda_1\left(-\frac{3}{4}x_1^2x_2^{-2} - \frac{3}{8}x_2x_3^2 + 1\right)\\ & - \lambda_2\left(-x_1^{-2} - \frac{1}{4}x_2^2x_3^{-1} + \epsilon_1\right)\end{aligned} \tag{10.45}$$

$$\frac{\partial f_2(x)}{\partial x_1} = -2x_1^{-2}x_2^{-1}x_3^{-1} + 2x_2 - \lambda_1\left(-\frac{3}{2}x_1x_2^{-2}\right) - \lambda_2(2x_1^{-3}) = 0 \tag{10.46}$$

$$\frac{\partial f_2(x)}{\partial x_2} = -2x_1^{-1}x_2^{-2}x_3^{-1} + 2x_1 - \lambda_1\left(\frac{3}{2}x_1^2x_2^{-3} - \frac{3}{8}x_3^2\right) + \lambda_2\left(\frac{1}{2}x_2x_3^{-1}\right) = 0 \tag{10.47}$$

$$\frac{\partial f_2(x)}{\partial x_3} = -2x_1^{-1}x_2^{-1}x_3^{-2} - \lambda_1\left(-\frac{3}{4}x_2x_3\right) - \lambda_2\left(\frac{1}{4}x_2^2x_3^{-2}\right) = 0 \tag{10.48}$$

$$\lambda_1\left(-\frac{3}{4}x_1^2x_2^{-2} - \frac{3}{8}x_2x_3^2 + 1\right) = 0 \tag{10.49}$$

$$\lambda_2\left(-x_1^{-2} - \frac{1}{4}x_2^2x_3^{-1} + \epsilon_1\right) = 0 \tag{10.50}$$

$$\left(-\frac{3}{4}x_1^2x_2^{-2} - \frac{3}{8}x_2x_3^2 + 1\right) \geq 0 \tag{10.51}$$

$$\left(-x_1^{-2} - \frac{1}{4}x_2^2x_3^{-1} + \epsilon_1\right) \geq 0 \tag{10.52}$$

where

$$x_1, x_2, x_3 \geq 0. \tag{10.53}$$

TABLE 5. Optimal solution of Primal (ii) by ϵ-constraint method using KKT condition.

ϵ_1	x_1	x_2	x_3	λ_1	λ_2	primalf_2
1.18	0.6227133	1.205879	1.330079	1.251528	0.0000	3.504279
1.5	0.6227134	1.205879	1.330073	1.251529	0.0000	3.504279
2.0	0.6227134	1.205879	1.330079	1.251528	0.0000	3.504279
2.5	0.6227133	1.205879	1.330079	1.251528	0.0000	3.504279
2.7	0.6227145	1.205878	1.330073	1.251538	0.0000	3.504279

Solution of the above problem obtained by changing the value of ϵ_1 between 1.17 to 2.852 is given by the following Table 5.

From the above table it is observed that objective values obtained by changing the value of ϵ_1 converging towards $f_2 = 3.504279$ for $x_1 = 0.6227145$, $x_2 = 1.205878$, $x_3 = 1.330073$, $\lambda_1 = 1.251538$, $\lambda_2 = 0$.

The Dual program of the primal (ii) is given below:

$$\max_{w} : V(w) = \left(\frac{2}{w_{01}}\right)^{w_{01}} \left(\frac{2}{w_{02}}\right)^{w_{02}} \left(\frac{3}{4w_{11}}\right)^{w_{11}} \left(\frac{3}{8w_{12}}\right)^{w_{12}} (w_{11}+w_{12})^{(w_{11}+w_{12})}$$
$$\left(\frac{1}{\epsilon_1 w_{21}}\right)^{w_{21}} \left(\frac{1}{4\epsilon_1 w_{22}}\right)^{w_{22}} (w_{21}+w_{22})^{(w_{21}+w_{22})} \tag{10.54}$$

subject to

$$\begin{aligned} w_{01} + w_{02} &= 1 \\ -w_{01} + w_{02} + 2w_{11} - 2w_{21} &= 0 \\ -w_{01} + w_{02} - 2w_{11} + w_{12} + 2w_{22} &= 0 \\ -w_{01} + 2w_{12} - w_{22} &= 0 \\ w_{01}, w_{02}, w_{11}, w_{12}, w_{21}, w_{22} &\geq 0. \end{aligned}$$

Solution of the above problem obtained by changing the value of ϵ_1 between 1.17 to 2.852 is given by the following Table 6.

from the above table it is observed that the optimal objective value of dual problem is 3.504279 for $w_{01} = 0.571353, w_{02} = 0.428646, w_{11} = 0.071505, w_{12} = 0.285684, w_{21} = 0.000151, w_{22} = 0.000016$

Convergence Analysis

For lower bound L_i and upper bound U_i of the functions f_i, for $i = 1, 2$ is given by.

$$L_1 = 1.171595 \leq f_1 \leq 2.852155 = U_1;$$
$$L_2 = 3.504279 \leq f_2 \leq 4.788870 = U_2$$

TABLE 6. Dual Solution.

ϵ_1	w_{01}	w_{02}	w_{11}	w_{12}	w_{21}	w_{22}	$Dual f_2$
1.18	0.365234	0.634765	1.702165	0.773053	1.83693	1.180873	4.581102
1.5	0.457818	0.542181	0.243350	0.263594	0.285531	0.069371	3.820710
2.0	0.517785	0.482214	0.133127	0.267479	0.115341	0.017173	3.579273
2.5	0.553805	0.446194	0.089935	0.279018	0.036129	0.004231	3.513293
2.7	0.564416	0.435583	0.078645	0.282991	0.014229	0.001566	3.505775
2.8	0.509118	0.430881	0.073786	0.284809	0.004667	0.000500	3.504445
2.85	0.571353	0.428646	0.071505	0.285684	0.000151	0.000016	3.504279

TABLE 7. Convergence test for Solution of Primal (i) by ϵ-constraint method.

ϵ_2	x_1	x_2	x_3	λ_1	λ_2	primal f_1
3.6	0.7761717	1.135443	1.235101	5.106435	3.468290	1.920866
3.7	0.8457904	1.131758	1.170157	3.054870	1.835243	1.671550
3.8	0.9001469	1.136367	1.114596	2.252495	1.19447	1.523806
3.9	0.9466340	1.144447	1.065098	1.818183	0.8440674	1.423355
4.0	0.9880850	1.154432	1.020190	1.546340	0.6207686	1.350848
4.1	1.025965	1.165656	0.9790380	1.361619	0.4648520	1.296986
4.2	1.061146	1.177817	0.9411259	1.229520	0.3490472	1.256553
4.3	1.094292	1.190804	0.9061239	1.131934	0.259060	1.226322
4.4	1.125806	1.204625	0.8738374	1.0584620	0.1866816	1.204157
4.5	1.156066	1.219398	0.8441914	1.002766	0.1268772	1.188569
4.6	1.185429	1.235360	0.8172388	0.9608445	0.0764288	1.178472
4.7	1.214197	1.252863	0.7931753	0.9302072	0.0332766	1.173041
4.75	1.228452	1.262344	0.7823372	0.9186274	0.014010	1.171865
4.76	1.23196	1.264312	0.780749	0.9165978	0.010327	1.171743
4.77	1.234140	1.266303	0.7782483	0.9146599	0.006698	1.171658
4.78	1.236983	1.268307	0.7762508	0.9128042	0.0031247	1.171609
4.788	1.239248	1.269929	0.7746880	0.9113952	0.0003135	1.171595
4.7888	1.239476	1.270093	0.7745326	0.9112571	0.000033	1.171595
4.78882	1.239481	1.270097	0.7745287	0.9112537	0.000026	1.171595
4.78884	1.239487	1.270101	0.7745248	0.9112502	0.000014	1.171595
4.78887	1.239495	1.270107	0.7745190	0.9112451	0.000009	1.171595

and changing ϵ_1 and ϵ_2 as

$$1.171595 < \epsilon_1 < 2.852155; \text{ and } 3.504279 < \epsilon_2 < 4.788870,$$

it has been observed that changing the value of ϵ_1 and ϵ_2 within the required range, the compromise solution of the objective functionf_1 and f_2 are obtained converging towards the suitable compromise values of the objective function within the range.

Convergence Test for Solution of Primal (i)

Changing the value of ϵ_2 in the interval [3.6,4.78887],the pareto optimal solution for primal problem (i)are presented in the Table 7.

From the above Table 5, it is observed that the set of solution generated are converging to the solution $f_1 = 1.171595$ for $x_1 = 1.239495$, $x_2 = 1.270107$,

TABLE 8. Convergence test for Solution of Primal (ii) by ϵ-constraint method.

ϵ_2	x_1	x_2	x_3	λ_1	λ_2	primal f_1
1.17	0.6227133	1.205879	1.330079	1.251528	0.0000	3.504279
1.4	0.6227133	1.205879	1.330079	1.251528	0.0000	3.504279
1.7	0.6227135	1.205878	1.330079	1.251530	0.0000	3.504279
2.0	0.6227134	1.205878	1.330079	1.251529	0.0000	3.504279
2.3	0.6227131	1.205880	1.330079	1.251529	0.0000	3.504279
2.6	0.6227146	1.205878	1.330072	1.251539	0.0000	3.504279
2.7	0.6227145	1.205878	1.330073	1.251538	0.0000	3.504279
2.8	0.6227145	1.205878	1.330073	1.251538	0.0000	3.504279
2.85	0.6227144	1.205879	1.330073	1.251538	0.0000	3.504279

$x_3 = 0.7745190$, $\lambda_1 = 0.9112451$ and $\lambda_2 = 0.000009$ which is same as the result obtained using KKT condition.

Convergence Test for Solution of Primal (ii)

Similarly changing the value of ϵ_1 in the interval [1.17,2.852],the pareto optimal solution obtained for Primal Problem (ii)are presented in the Table 8.

From above Table 6,it is observed that the set solutions so generated by changing the value of ϵ_2 converging to the value $f_2 = 3.504279$ for $x_1 = 0.6227144$, $x_2 = 1.205879$, $x_3 = 1.330073$, $\lambda_1 = 1.251538$ and $\lambda_2 = 0.0000$ which is same as the result obtained using KKT condition.

Comparison of above solution by Fuzzy Programming method

The solutions which are obtained by ϵ-constraint method along with KKT has been compared with the fuzzy programming method in the following two cases.

Case-1(Value of f_1 by Fuzzy programming method)

Using the steps of fuzzy programming method as defined in Section 8, the corresponding crisp model for f_1 is
$max : \theta$

subject to

$$\begin{aligned} x_1^{-2} + \frac{1}{4}x_2^2x_3^{-1} + (2.852155 - 1.171595)\theta \leq 2.852155 \\ \frac{3}{4}x_1^2x_2^{-2} + \frac{3}{8}x_2x_3^2 - 1 \leq 0 \\ \theta > 0, x_1 > 0, x_2 > 0, x_3 > 0. \end{aligned} \tag{10.55}$$

The optimal solution of $f_1 = 1.171595$ for $\theta = 1.0000$, $x_1 = 1.239502$, $x_2 = 1.270112$, $x_3 = 0.7745138$.

Case-2(Value of f_2 by Fuzzy programming method)

Similarly using the steps of fuzzy programming method, the corresponding crisp model for f_2 is defined as:

$max : \theta$

subject to

$$\begin{aligned} 2x_1^{-1}x_2^{-1}x_3^{-1} + 2x_1x_2 + (4.788870 - 3.504279)\theta \leq 4.788870 \\ \frac{3}{4}x_1^2x_2^{-2} + \frac{3}{8}x_2x_3^2 - 1 \leq 0 \\ \theta > 0, x_1 > 0, x_2 > 0, x_3 > 0. \end{aligned} \tag{10.56}$$

The optimal solution of $f_2 = 3.504279$ for $\theta = 1.0000$,$x_1 = 0.6227132$,$x_2 = 1.205879$ and $x_3 = 1.330079$.

The above discussion indicate that the solution obtained by ϵ-constrained method using KKT condition converging to the solution $f_1 = 1.171595$ for $x_1 = 1.239495$,$x_2 = 1.270107$,$x_3 = 0.7745190$,$\lambda_1 = 0.9112451$, $\lambda_2 = 0.000009$ and $f_2 = 3.504279$ for $x_1 = 0.6227144$, $x_2 = 1.205879$, $x_3 = 1.330073$, $\lambda_1 = 1.251538$ and $\lambda_2 = 0.0000$ is same as the solution obtained by fuzzy programming method. However the ϵ-constraint method gives a set of solution where the decision maker has a choice to change his/her solution according to their choice. But in fuzzy programming method it gives only one solution.

11. Conclusion

It is a challenging task for searching a suitable compromise solution corresponding to a given multi-objective optimization problem. In fact, the difficulty lies in conflict between our various objectives and goals. Because most decision and compromise made on basis of intuition. However, there are certain area where mathematical modeling and programming needed. In this paper we have used the ϵ-constraint method using KKT condition to find the set of primal and dual solutions of the multi-objective functions with given constraints. The corresponding fuzzy programming techniques have been applied to find the optimal values of the functions. From the computation it has been observed that the pareto optimal solution obtained by primal-dual techniques matches with their counterpart solution due to fuzzy programming method. The illustrative examples explain in getting the most allowable solution with suitable values of ϵ. The procedures adopted here for understanding the convergence analysis of multi-objective programming problem helps to incorporate the preferences of the decision maker, in order to focus on achieving the most approximate non-inferior solution instead of trying to generate the entire pareto front. Such a procedure may also alleviate the problem of many optimizations when dealing with many objective functions.

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