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ITERATIVE METHODS WITH ANALYTICAL PRECONDITIONING TECHNIQUE TO LINEAR COMPLEMENTARITY PROBLEMS: APPLICATION TO OBSTACLE PROBLEMS

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Abstract. For solving linear complementarity problems *LCP* more attention has recently been paid on a class of iterative methods called the matrix-splitting. But up to now, no paper has discussed the effect of preconditioning technique for matrix-splitting methods in *LCP*. So, this paper is planning to fill in this gap and we use a class of preconditioners with generalized Accelerated Overrelaxation (*GAOR*) methods and analyze the convergence of these methods for *LCP*. Furthermore, Comparison between our methods and other non-preconditioned methods for the studied problem shows a remarkable agreement and reveals that our models are superior in point of view of convergence rate and computing efficiency. Besides, by choosing the appropriate parameters of these methods, we derive same results as the other iterative methods such as *AOR*, *JOR*, *SOR* etc. Finally the method is tested by some numerical experiments.

Keywords. Linear complementarity problems, preconditioning, iterative methods, H-matrix, obstacle problems.

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1. Introduction

For a given real vector $q \in R^n$ and a given matrix $M \in R^{n\times n}$ the linear complementarity problem abbreviated as *LCP (M, q)*, consists in finding vectors $z \in R^n$ such that,

$$W = Mz + q, z^T = 0, z \geq 0, W \geq 0. \tag{1}$$

Where, z^T denotes the transpose of the vector z. Many problems in various scientific computing and engineering areas can lead to the solution of an *LCP* of the form (1). For more details (see [1–5] and the references therein).

In several decades, many methods for solving the *LCP (M, q)* have been introduced. Most of these methods originate from those for the system of the linear equations where may be classified into two principal kinds, direct (see [3–6]) and iterative methods (see [2, 3, 7–12]). Iterative methods often fall into *splitting* and *multisplitting* methods. For example Cottle *et al.* [5] studied the convergence of the splitting and two–stage methods when matrix M is symmetric or nonsymmetric. Bai and Evans in [13–15] studied the multisplitting techniques for solving (1.1)which are useful in parallel computing. Based on the models in [13], Yuan and Song [16] proposed a class of modified AOR ($MAOR$) methods to solve (1.1), when M is a $2-cyclic$ matrix. Furthermore, under certain conditions, Li and Dai in [17] studied Generalized Accelerated Overrelaxation ($GAOR$) methods, for LCP based on [13]. $GAOR$ algorithm was first proposed for solving systems of linear equations by James [18] in 1973 and has been extensively studied by some authors (see [19, 20]). All these methods are in a class of iterative methods called the matrix-splitting. There are many solution methods available for solving linear systems and also, some of these methods apply $Jacobi$ and $Gauss-Seidel$ iterations as preconditioners. But up to now, no paper has discussed the effective of preconditioning technique for above matrix-splitting methods in *LCP (M, q)*.

This paper is devoted to the preconditioning technique for *LCP (M, q)*. The development of efficient and authentic preconditioning strategy is the key for the successful application of scientific computation to the solution of many large scale Problems. The convergence rate of iterative methods depends on spectral properties of the coefficient matrix, so in preconditioning schemes the attempt is, to transform the original system into another one, that has the same solution but more desirable properties for iterative solution. In this paper, $GAOR$ methods are adopted and the effect of preconditioning is investigated. Here we extend $(I + S) - type$ preconditioners for linear equations to LCP and show that the preconditioned $GAOR$ methods are superior to the basic $GAOR$ methods.

2. Prerequisite

We begin with some basic notation and preliminary results which we refer to later. First of all, the matrix $A = (a_{ij})$ is nonnegative (positive) if for any $i, j; a_{i,j} \geq 0$ $(a_{i,j} > 0)$. In this case we write $A \geq 0 (A > 0)$. Similarly, for *n-dimensional* vectors x, by identifying them with $n \times 1$ matrices, we can also define $x \geq 0 (x > 0)$.

Definition 2.1 [21, 22]**.** A real $n \times n$ matrix $A = (a_{ij})$ is called;

(i) Z-matrix if for any $i \neq j; a_{i,j} \leq 0$,
(ii) M–matrix, if A is nonsingular, and $A^{-1} \geq 0$,
(iii) H–matrix if and only if $< A >= (m_{i,j}) \in R^{n \times n}$ is an M–matrix, where;

$$m_{i,i} = |a_{i,i}|;\ m_{i,j} = -|a_{i,j}|,\ i \neq j.$$

Definition 2.2 [16, 17]**.** For $x \in R^n$, vector x_+ is defined such that $(x_+)_j = \max\{0, x_j\}$,Then, for any $x, y \in R^n$, the following facts hold:

(i) $(x + y)_+ \leq x_+ + y_+$.
(ii) $x_+ - y_+ \leq (x - y)_+$.
(iii) $|x| = x_+ - (-x)_+$.
(iv) $x \leq y$ implies $x_+ \leq y_+$.

Definition 2.3 [21, 22]**.** Let A be a real matrix. The splitting $A = M - N$ is called;

(i) convergent if $\rho(M^{-1}N) < 1$,
(ii) regular if $M^{-1} \geq 0$ and$N \geq 0$,
(iii) weak regular if $M^{-1}N \geq 0$ $N \geq 0$.

Clearly, a regular splitting is weak regular.

Lemma 2.4 [21, 23]**.** *Let A be a Z-matrix. Then A is M–matrix if and only if there is a positive vector x such that $Ax > 0$. Lemma* 2.2 [21, 23]*. Let $A = M - N$ be an M–splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if A is M–matrix. Lemma* 2.3 [22]*. Let A, B are Z-matrices and A is an M–matrix, if $A \leq B$ then B is also an M–matrix. Lemma* 2.4 [22]*. If $A \geq 0$, then;*

(i) A has a nonnegative real eigenvalue equal to its spectral radius,
(ii) For $\rho(A) > 0$, there corresponds an eigenvector $x \geq 0$,
(iii) $\rho(A)$ does not decrease when any entry of A is increased.

Lemma 2.5 [23]**.** *Let $T \geq 0$. If there exist $x \geq 0$ and a scalar α such that;*

(i) $Tx \geq \alpha x$, then $\rho(T) \geq \alpha$. Moreover, if $Tx < \alpha x$, then $\rho(T) < \alpha$.
(i) $Tx \leq \alpha x$, then $\rho(T) \leq \alpha$. Moreover, if $Tx > \alpha x$, then $\rho(T) > \alpha$.

Lemma 2.6 [8, 16]**.** *$LCP(M, q)$ can be equivalently transformed to a fixed-point system of equations;*

$$z = (z - \alpha E(Mz + q))_+, \tag{2}$$

where α is some positive constant and E is a diagonal matrix with positive diagonal elements.

Lemma 2.7 [13]**.** *Let $M \in R^{n\times n}$ be an H–matrix with positive diagonal elements. Then the $LCP(M,q)$ has a unique solution $Z^* \in R^n$.*

Let the matrix M be as;

$$M = D + L + U, \tag{3}$$

Where, D diagonal, L and U are strictly lower and upper triangular matrices of M, respectively. Then by choice of $\alpha E = D^{-1}$ and Lemma 2.6 we have,

$$z = (z - D^{-1}(Mz + q))_+. \tag{4}$$

So, in order to solve $LCP(M,q)$, $GAOR$ iterative methods defined in [17] is;

$$z^{k+1} = (z^k - D^{-1}[\alpha\Omega L Z^{k+1} + (\Omega M - \alpha\Omega L)z^k + \Omega q)_+, \tag{5}$$

Where, α is a real parameter and $\Omega = (w_1,\ldots,w_n)$ is a real diagonal relaxation matrix.

The operator $f : R^n \to R^n$, is defined such that $f(z) = \xi$, where ξ is the fixed point of the system;

$$\xi = (z - D^{-1}[\alpha\Omega L\xi + (\Omega M - \alpha\Omega L)z + \Omega q)_+, \tag{6}$$

In next lemma, we have the convergence theorem, proposed in [17] for the $GAOR$ methods.

Lemma 2.8 [11]**.** *Let $M \in R^{n\times n}$ be an H–matrix with positive diagonal elements. Moreover, let*

$$G = I - \alpha\Omega D^{-1}|L|, \quad F = |I - D^{-1}(\Omega M - \alpha L)|, \tag{7}$$

then, for any initial vector $z^0 \in R^n$, the iterative sequence z^k generated by the $GAOR$ method converges to the unique solution z^* of the $LCP(M,q)$ and;

$$\rho(G^{-1}F) \le \mathrm{Max}\{|1 - w_i| + w_i\rho(|J|)\} < 1,$$

if

$$0 < w_i < 2/(1 + \rho(|J|)), \quad 0 \le \alpha \le 1,$$

where $\rho(|J|)$ is the spectral radius Jacobi iteration matrix ($J = D^{-1}(L + U)$).

3. Preconditioning technique in $GAOR$ methods for $LCP(M,q)$

In this section, $GAOR$ methods for LCP and the effect of preconditioning for these methods are investigated. In these iterative methods, for increasing the convergence rate, an acceleration parameter has been used. However, it is impossible to estimate an optimal parameter in actual problems. Moreover, it does not provide an essential methodology. In other words, this strategy has a high cost. From trade off cost, efficiency and also numerical techniques point of view, Preconditioning is effective to change the convergence rate. A *preconditioner* is defined as an auxiliary approximate solver, which will be combined with an iterative method. According to critical importance of spectral radius, in preconditioning; we find a more desired spectral radius. In the literature, various authors have suggested different model of $(I+S)-type$ preconditioner for linear systems $A=I-L-U$; where I is the identity matrix and L, U are strictly lower and strictly upper triangular matrices of A, respectively. (see [24–31] and the references therein). These preconditioners have reasonable effectiveness and low construction cost. For example In 1987 Milaszewicz [24] presented the preconditioner of $(I+S)-type$, where the elements of the first column below the diagonal of A eliminate. Gunawardena, Jain and Snyder in [25] considered the alternative preconditioner, which eliminates the elements of the first upper diagonal. In [26], Usui *et al.* proposed to adopt, as the preconditioned matrix, where $P=I+L$ is strictly lower triangular of matrix A. They obtained excellent convergence rate compared with that by other iterative methods. In [27], we presented some preconditioners for solving linear systems $Ax=b$. In these preconditioners, we let, $(I+S)$ be one model of above preconditioners. Then, our preconditioners are given by the following;

$$P_1=(I+S)\{(I-S)+(L+U)(I+S)\}. \tag{8}$$

$$P_2=(I+S)\{3I-A(I+S)(3I-A(I+S))\}. \tag{9}$$

In the present section, same preconditioners as above for solving linear complementarity problem are used. Consider M in (3) is nonsingular. Then preconditioning in M is;

$$\bar{M}=D+L+U+SD+SL+SU=\bar{D}+\bar{L}+\bar{U}, \tag{10}$$

where, $\bar{D},\bar{L},\bar{U}$ are diagonal, strictly lower and strictly upper triangular parts of $\bar{M}$ and;

$$\bar{q}=(I+S)q.$$

Therefore, Milaszewicz's preconditioner is as follow;

$$(I+S_1),$$

where,

$$s_1=\frac{1}{m_{11}}\begin{pmatrix} 0 & 0\dots 0\\ -m_{21} & 0\dots 0\\ -m_{31} & 0\dots 0\\ \vdots & \vdots\dots\vdots\\ -m_{n1} & 0\dots 0\end{pmatrix}. \tag{11}$$

Gunawardena *et al.*'s preconditioner is as follow;

$$(I+S_2),$$

where,

$$s_2 = \begin{pmatrix} 0 & \frac{-m_{12}}{m_{22}} & 0 & 0 \\ 0 & 0 & \frac{-m_{23}}{m_{33}} & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{-m_{n-1,n}}{m_{n,n}} \\ 0 & 0 & \dots & 0 \end{pmatrix}. \tag{12}$$

Usui *et al.*'s preconditioner is as follow;

$$(I + S_3),$$

where,

$$s_3 = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ \frac{-m_{21}}{m_{11}} & 0 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \frac{-m_{n-1,1}}{m_{11}} & \dots & \frac{-m_{n-1,n-2}}{m_{n-2,n-2}} & 0 & 0 \\ \frac{-m_{n,1}}{m_{11}} & \dots & \frac{-m_{n,n-2}}{m_{n-2,n-2}} & \frac{-m_{n,n-1}}{m_{n-1,n-1}} & 0 \end{pmatrix}. \tag{13}$$

And our preconditioners for LCP are;

$$P_1 = (I + K_1) = (I + S_i)\{(I - S_i) + (l + u)(I + S_i)\}. \tag{14}$$

$$P_2 = (I + K_2) = (I + S_i)\{3I - M(I + S_i)(3I - M(I + S_i))\}. \tag{15}$$

where, $l = -D^{-1}L$ and $u = -D^{-1}U$ are strictly lower and strictly upper triangular matrices of $M = D + L + U = D(I - l - u)$. Furthermore, for i=0,1,2 and 3,we have,

$$\bar{M} = (I + K_1)M,$$

$$\bar{q} = (I + K_i)q,$$

$$K_0 = (I + S_i), i = 1, 2, 3.$$

Thus the preconditioned $GAOR$ methods for LCP are:

$$z^{k+1} = (z^k - \bar{D}^{-1}[\alpha\Omega\bar{L}Z^{k+1} + (\Omega\bar{M} - \alpha\Omega\bar{L})z^k + \Omega\bar{q})_+, \tag{16}$$

Lemma 3.1. *Let M be an $H-matrix$, then the preconditioned $\bar{M} = (I + K_i)M$ also is $H-matrix$.*

Proof. Let M be an $H-matrix$, then $< M >$ is $M-matrix$ and by Lemma 2.1;

$$\exists x > 0, s.t; < M > x > 0.$$

Since $< \bar{M} >= (I + |K_i|) < M >)$, then

$$< \bar{M} > x = (I + |K_i|) < M > x > 0.$$

Therefore $\bar{M}$ is $M-matrix$ and the proof is completed. □

Theorem 3.2. *Let M with positive diagonal elements be an $H-matrix$ and $\bar{M} = (I + K_i)M$ is preconditioned form of M with preconditioners* (11)–(15)*. Then if the conditions of Lemma* 2.8 *are satisfied we have;*

$$\rho(\bar{G}^{-1}\bar{F}) \leq \rho(G^{-1}F) < 1.$$

Proof. By Lemma 3.1 $\bar{M}$ is an $H-matrix$. Hence $< \bar{M} = \bar{G} - \bar{F}$ is $M-matrix$, and by Lemma 2.2 $\rho(G^{-1}F) < 1$. Since $(< \bar{M} >) < \bar{G}$, by Lemma 2.3 $\bar{G}$ is $M-matrix$. As same demonstration G is also $M-matrix$. Thus

$$\bar{G}^{-1} \geq 0, \bar{G}^{-1}\bar{F} \geq 0.$$

$$G^{-1} \geq 0, G^{-1}F \geq 0.$$

Then by Lemma 2.4, there exist a positive vector x such that $(G^{-1}F)x = \rho(G^{-1}F)x$. Therefore,

$$(G - F)x =< M > x = G(I - G^{-1}F)x = \frac{1 - \rho(G^{-1}F)}{\rho(G^{1}F)}Fx \geq 0.$$

Furthermore, for $(I + K_i)$;say $(I + K_0)$ and $(S_i = S_2)$ we have;

$$\begin{aligned} < \bar{M} > &= (I + |S_2|) < M > \\ &= (I + |S_2|)(D - |L| - |U|) \\ &= D - |L| - |U| + |S| D - |S| |L| - |S| |U| \qquad = (\bar{D} - \left|\bar{L}\right| - \left|\bar{U}\right|), \end{aligned}$$

where,

$$|S| |L| = D_1 + L_1 + U_1,$$

$$\bar{D} = D - D_1 \leq D,$$

$$\left|\bar{L}\right| = ||L| + L_1| \geq |L| ,$$

$$\left|\bar{U}\right| = ||U| + U_1 + |S| |U| - |S| D| .$$

Thus, $\bar{G} \leq G$ and in view of the fact that both $\bar{G}$ and G are $M - matrices$ we have;

$$\bar{G}^{-1}(I + S_2) \geq \bar{G}^{-1} \geq G^{-1}.$$

Therefore,

$$0 \leq [\bar{G}^{-1}(I + S_2) - G^{-1}](G - F)x =$$

$$(I - \bar{G}^{-1}\bar{F}x - (I - G^{-1}G)x) =$$

$$G^{-1}Fx - \bar{G}^{-1}\bar{F}x = \rho(G^{-1}F)x - \bar{G}^{-1}\bar{F}x.$$

And by Lemma 2.5 we have;

$$\rho(\bar{G}^{-1}\bar{F}) \leq \rho(G^{-1}F).$$

Therefore by Lemma 2.8 the proof is completed. □

Now, following [13, 16, 17], we show that in LCP, the convergence rate of *preconditioned* $GAOR$ methods are faster than of the $GAOR$ methods.

Theorem 3.3. *Let M with positive diagonal elements be an $H-matrix$ and $\bar{M} = (I + K_i)M$ is preconditioned form of M with preconditioners (11)–(15). Then, convergence rate of preconditioned $GAOR$ methods are faster than of the $GAOR$ methods.*

Proof. Let iterative sequence $\{z^i\}$, $i = 0, 1,$ generated by (16). From the assumption that M is an $H-matrix$, it follows by Lemma 3.1, $\bar{M}$ is an $H-matrix$ and therefore by Lemma 2.7, the vector sequence $\{z^i\}$ is uniquely defined and the $LCP(M, q)$ has a unique solution Z^*. Similar to (6), we define the operator $v : R^n \to R^n$, such that $v(z) = \bar{\xi}$, where $\bar{\xi}$ is the fixed point of the following system;

$$\bar{\xi} = (z - \bar{D}^{-1}[\alpha\Omega\bar{L}\bar{\xi} + (\Omega\bar{M} - \alpha\Omega\bar{L})z + \Omega\bar{q})_+. \tag{17}$$

Let;

$$\bar{\psi} = v(x) = (x - \bar{D}^{-1}[\alpha\Omega\bar{L}\bar{\psi} + (\Omega\bar{M} - \alpha\Omega\bar{L})x + \Omega\bar{q})_+. \tag{18}$$

By subtracting (17)and (18), we get;

$$\bar{\xi} - \bar{\psi} = ((z - x) - \bar{D}^{-1}[\alpha\Omega\bar{L}(\bar{\xi} - \bar{\psi}) + (\Omega\bar{M} - \alpha\Omega\bar{L})(z - x))_+.$$

$$\bar{\psi} - \bar{\xi} = ((x - z) - \bar{D}^{-1}[\alpha\Omega\bar{L}(\bar{\psi} - \bar{\xi}) + (\Omega\bar{M} - \alpha\Omega\bar{L})(x - z))_+.$$

Therefore, by above relations we have;

$$\left|\bar{\xi} - \bar{\psi}\right| = (\bar{\xi} - \bar{\psi})_+ + (\bar{\psi} - \bar{\xi})_+ \leq \bar{G}^{-1}\bar{F}(z - x).$$

Thus from the definition of the preconditioned $GAOR$ methods and above relation we can write;

$$\left|z^{k+1} - z^*\right| = \left|v(z^k) - v(z^*)\right| \leq \bar{G}^{-1}\bar{F}\left|z^k - z^*\right|.$$

Hence, the iterative sequence $\{z^k\}$, $k = 0, 1,$ converges to z^* if $\rho(\bar{G}^{-1}\bar{F}) < 1$. Furthermore, since by Theorem 3.2, $\rho(\bar{G}^{-1}\bar{F}) \leq \rho(G^{-1}F)$ then, we conclude that for solving LCP, the preconditioned $GAOR$ iterative methods are better than of the $GAOR$ methods form point of view of the convergence speed and the proof is completed. □

Corollary 3.4. *By choosing special parameters in $GAOR$ methods, it can be obtained the similar results for other well known iterative methods. For example,*

1) $GSOR$ (generalized SOR) methods [17] *for $\alpha = 1$.*
2) AOR (accelerated Overrelaxation) methods [32] *for $\alpha = r/w$, $\Omega = wI$.*
3) $MSOR$ (modified SOR) methods [16] *for $\alpha = 1$, $\Omega = (w_1 I, w_2 I)$.*
4) $EAOR$ (extrapolated AOR) methods [33] *for $\alpha = r^2/w^2$, $\Omega = (w^2/r)I$.*
5) SOR methods [21,23] *for $\alpha = 1$, $\Omega = wI$.*

6) JOR (Jacobi Overrelaxation) methods [34] *for* $\alpha = 0$, $\Omega = wI$.
7) Gauss – Seidel method [21,23] *for* $\alpha = 1$, $\Omega = I$.
8) Jacobi method for [21,23] *for* $\alpha = 0$, $\Omega = I$.
These preconditioning techniques and their results are also applicable for parallel computing such as multisplitting [13–15], *SIMD and MIMD systems* [10,12].

4. Numerical Examples

Here we give some examples, to illustrate the results obtained in previous section. In these experiments, the initial approximation of z^0 is $z^0 = (1, 1, \ldots, 1)^T$ and as a stopping criterion we choose;

$$\|\min(Mz^l + q, z^k)\|_\infty \leq 10^{-6}.$$

$$\|\min(\bar{M}z^l + \bar{q}, z^k)\|_\infty \leq 10^{-6}.$$

Furthermore, we report the CPU time and number of iterations for the corresponding $GAOR$ and preconditioned $GAOR$ methods by CPU and $Iter$, respectively. All the numerical experiments presented in this section were computed with MATLAB 7 on a PC with a 1.86GHz 32-bit processor and 1GB memory.

Example 4.1. Consider $LCP(M, q)$ as;

$$M = I \otimes B + R \otimes I \in R^{N \times N}.$$

$$q = (-1, 1, \ldots, (-1)^{n^2})^T \in R^N.$$

where $I \in R^N \times N$ and $\otimes$ denotes the Kronecker product. Furthermore, B and R are $n \times n$ tridiagonal matrices given by;

$$B = \text{tridiagonal}\left[-\frac{2+h}{8}, 1, -\frac{2-h}{8}\right]$$

$$R = \text{tridiagonal}\left[-\frac{1+h}{4}, 0, -\frac{1-h}{4}\right]$$

$$h = 1/n, \ N = n^2.$$

Evidently, M is an $H-matrix$ with positive diagonal elements so, $LCP(M, q)$ has a unique solution. Then, we solved the $n^2 \times n^2$ $H-matrix$ yielded by the iterative methods, and Preconditioned forms. In this experiment, we choose *Gunawardena et al.'s model* and our models (P_1, P_2) as preconditioners. In Table 1, we report the CPU time and number of iterations for the corresponding $GAOR$ and preconditioned $GAOR$ methods. Moreover, the N parameters w_i, are taken from the N equal–partitioned points of the interval [0.91.1] and *alpha* is *one*. Here, the preconditioned $GAOR$ methods with *Gunawardena et al.'s preconditioner* is denoted by $PREC(Guna)$, while $PREC(P_1)$, $PREC(P_2)$ corresponds to our preconditioners $(P_i); i = 1, 2$.

TABLE 1. Show the result of example 4.1 for GAOR.

Method	$GAOR$		$PREC(Guna)$		$PREC(P_1)$		$PREC(P_2)$	
n	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
7	53	0.070	34	0.020	21	0.010	14	0.000
14	147	0.700	92	0.470	56	0.310	36	0.210
25	315	19.520	199	16.314	120	11.060	78	7.200

TABLE 2. Show the result of example 4.1 for AOR.

Method	AOR		$PREC(Guna)$		$PREC(P_1)$		$PREC(P_2)$	
n	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
9	94	0.130	63	0.060	35	0.030	24	0.020
18	248	3.800	119	0.761	66	0.411	44	0.280
25	375	27.689	256	23.999	142	13.339	95	8.933

TABLE 3. Show the result of example 4.1 for SOR.

Method	SOR		$PREC(Guna)$		$PREC(P_1)$		$PREC(P_2)$	
n	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
10	111	0.270	72	0.140	42	0.060	28	0.030
20	288	5.538	188	3.696	109	2.203	72	1.402
30	460	72.554	370	58.594	178	34.740	117	22.693

In Table 2, we report the CPU time and number of iterations by different n for the corresponding AOR and *preconditioned* AOR methods with $w = 1$ and $r = 0.8$.

In Table 3, we report the CPU time and number of iterations by different n for the corresponding SOR and *preconditioned* SOR methods with $w = 0.9$.

Example 4.2 Application to the obstacle problems**.**

The test problem comes from the finite difference discretization of the one side obstacle problem [35],

$$< -\Delta u - b, v - u >\geq 0, \ \forall v \in K,$$

where, $K = \{v \in H_0(\Omega) : v \geq 0\}$, $b = 4sin(4xy)$, $\Omega = (0,1) \times (0,1)$. By discretization, we obtain the problem as $LCP(M,q)$,where, $h = 1/n$, $n = m^2$ and $q = (4h^2 sin(4ij/m^2))_{i,j}$, $i,j = 1,\ldots,m$,

$$M = \begin{pmatrix} A & -I & & & \\ -I & A & -I & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -I \\ & & & -I & A \end{pmatrix} \in R^{n\times n}.$$

TABLE 4. Show the result of example 4.2.

Method	GAOR		$PREC(Usui)$		$PREC(P_1)$		$PREC(P_2)$	
n	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
100	102	0.004054	82	0.003262	43	0.001657	28	0.001055
225	198	0.064214	172	0.063753	87	0.037722	57	0.014490
400	345	0.289446	307	0.248204	157	0.119898	105	0.099793
625	472	0.839852	436	0.726877	225	0.389484	153	0.258722
900	608	1.960319	600	1.628224	313	0.881270	212	0.513456
1225	744	3.688457	740	3.625643	403	1.979734	278	1.425091
1600	907	8.138793	859	7.415670	484	4.126375	337	2.689793
2025	1063	25.073047	972	22.962480	579	8.990856	409	5.712374
2500	1249	27.860959	1069	21.915278	697	14.568068	510	8.092789

I is the identity matrix of m -dimension, and

$$A = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 4 \end{pmatrix} \in R^{m\times m}.$$

In this experiment, we choose *Usui et al.'s model* and our models (P_1, P_2) as preconditioners. In Table 4, we report the CPU time and number of iterations for the corresponding $GAOR$ and *preconditioned* $GAOR$ methods. Furthermore, the N parameters w_i, are taken from the N equal–partitioned points of the interval $[1, 1.02]$ and *alpha* is *one*. Here, the preconditioned $GAOR$ methods with *Usui et al.'s* preconditioner is denoted by $PREC(Usui)$, while $PREC(P_1)$, $PREC(P_2)$ corresponds to our preconditioners (P_i); $i = 1, 2$.

From the tables, we can see that the preconditioned iterative methods are superior to the basic iterative methods and our preconditioners are better than other preconditioners. The tables have also shown that the preconditioned iterative methods associated with (P_2) are the best.

5. CONCLUSIONS

In this paper we have proposed the preconditioned $GAOR$ methods for linear complementarity problem and analyzed the convergence for these methods under certain conditions. We have also studied how the iterative methods for LCP are affected, if the system is preconditioned by our model. Besides, convergence areas and comparison results can be extended to the other well known iterative methods such as $MAOR$, AOR, SOR, etc. numerical results show the influence of our theorems.

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