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A DISCRETE-TIME GEO$^{[X]}$/G/1 RETRIAL QUEUE WITH GENERAL RETRIAL TIME AND M-ADDITIONAL OPTIONS FOR SERVICE*

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Abstract. This paper concerns a discrete time Geo$^{[X]}$/G/1 retrial queue with general retrial time in which all the arriving customers require first essential service with probability α_0 while only some of them demand one of other optional services: $type-r$ $(r = 1, 2, 3, \dots M)$ service with probability α_r. The system state distribution, the orbit size and the system size distributions are obtained in terms of generating functions. The stochastic decomposition law holds for the proposed model. Performance measures of the system in steady state are obtained. Finally, some numerical illustrations are presented to justify the influence of parameters on several performance characteristics.

Keywords. Discrete-time queue, first essential service (FES), multi-optional service, retrial queue.

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1. Introduction

There has been rapid growth in the literature on the discrete-time queue due to their applications in communication systems and other related areas, see for

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instance [4,11,12,17,19,27,31] and the references therein. Many computer and communication systems operate on a discrete-time basis where events can only happen at regularly spaced epochs. In fact, discrete-time queues are more appropriate than their continuous-time counterparts for modelling computer and telecommunication systems because they work in slotted time basis that resemble packet transmitting times and machine cycles [11,31]. Moreover, discrete-time models can be used to derive the results for continuous-time models but not *vice versa* [27].

In recent years, the interest in analysis of retrial queueing systems has grown and these systems have been widely used in the field of computers and communications. Retrial queueing system is characterized by the feature that the arriving customers who find the server busy join the retrial queue (orbit) to try again for their requests after some random time. With the recent advancements in mobile communications the issue of retrials is becoming more important. In the retrial queueing literature, survey papers such as the paper of Falin [14] and the paper of Kulkurni and Liang [13] provide classified collections of work done in this research area. Moreover, the monographs of Artalejo and Gomez [6] and Falin and Templeton [15] have presented analytical and computational techniques that are commonly used to analyze retrial queues.

The study of discrete-time queues was initiated by Meisling [23], and Powell *et al.* [25]. More detailed applications on discrete-time queueing theory are included in the two monographs [11,27]. In the past, the study of the retrial queues has been focused on the continuous case. In fact, the paper of Yang and Li [32] is the first one that considered discrete-time retrial queues. They have analyzed a Geo/G/1 retrial using a generating function approach. Choi and Kim [13] and Li and Yang [20] have analysed the models with two types of customers. The batch arrival retrial queue has been addressed by Takahashi *et al.* [28], Atencia and Moreno [8] and Artalejo *et al.* [5]. Retrial queueing systems with an unreliable server have been analyzed by Atencia and Moreno [9,10], Moreno [24] and Wang and Zhao [29,30]. The balking behavior of the customers has been studied by Aboul-Hassan *et al.* [1,2].

Recently, there have been several contributions considering retrial queueing systems in which the server may provide a second phase of service. Such situations occur in day-to-day life, where all arriving customers require the main service and only some may require the subsidiary service provided by the same server. Madan [21] has studied an M/G/1 queue with second optional service in which the first essential service time follows a general distribution but the second optional service is assumed to be exponentially distributed. Medhi [22] has generalized the model by considering that the second optional service is also governed by a general distribution. Jau-Chuan Ke [18] has studied the model with J-additional options for service. Senthil Kumar and Arumuganathan [26] have analysed single server batch arrival retrial queue with two phases of service and Bernoulli schedule vacation. Wang and Zhao [29] have extended the study to discrete-time retrial queues with second optional service but without considering the general retrial time.

Most of the existing works focus on continuous-time models. Moreover the models studied in [18,21,22] have assumed a single arrival stream. The proposed work is the generalization of the model of Atencia and Moreno [8] and Aboul-Hassan *et al.* [3] by considering the additional M-options for service. The performance characteristics of the system are obtained.

The rest of this paper is organized as follows. In the next section, the mathematical description of the model is introduced. In Section 3, the Markov chain associated with the system is analyzed. The orbit and system sizes distribution are obtained together with several performance measures of the system. Section 4 gives two different stochastic decomposition laws regarding the probability generating function of the system size. Finally, in Section 5, some numerical results are illustrated to justify the impact of the service rate, retrial rate and so on.

2. Model description

In a discrete-time single server retrial queue, the time axis is segmented into slots and the time axis is marked by 0, 1, 2,... It is assumed that all queueing activities (arrivals, departures and retrials) occur at the slot boundaries. Different from continuous-time queues, the probability of an arrival and a departure and other queueing activities occurring concurrently may not be zero in discrete-time queues any more. So, it is necessary to specify the order in which the arrivals and departures take place in case of simultaneity. Thus, early arrival scheme (known also as departure first rule) [27] is applied in this paper. According to this scheme, the departures occur in the interval (m^-, m), while arrivals and retrials occur in the interval (m, m^+), where m^- is the instant immediately before time point m and m^+ is the instant immediately after time point m.

New customers arrive in batches according to a geometric arrival process with probability p where p, is the probability that a batch of customers arrives in the interval (m, m^+). Batch sizes are independent and identically distributed with probability distribution function $\{c_l\}_{l=1}^{\infty}$, generating function $C(z) = \sum\limits_{l=1}^{\infty} c_l z^l$ and nth factorial moments ζ_n. If, upon arrival, the server is busy, then the arriving customers join the orbit, whereas if the server free, then one of the arriving customers (selected at random) begins the first essential service (FES) immediately and the others join the orbit. It is always supposed that retrials and services can be started only at slot boundaries and their durations are integral multiples of slot duration.

The service times of FES are independent and identically distributed with general distribution $\{s_{0,i}\}_{i=1}^{\infty}$, generating function $S_0(x) = \sum\limits_{i=1}^{\infty} s_{0,i} x^i$ and nth factorial moments $\mu_{0,n}$. On completion of the FES, a customer decides with probability α_r to receive a rth type of optional service and with complementary probability α_0 to abandon the system forever $\left(i.e., \sum\limits_{r=0}^{M} \alpha_r = 1\right)$. The service times of rth type of multi-optional service are independent with general distribution $\{s_{r,i}\}_{i=1}^{\infty}$, generating function $S_r(x) = \sum_{i=1}^{\infty} s_{r,i}x^i$ and nth factorial moments $\mu_{r,n}$.

Customers in orbit are assumed to form a FCFS queue. A customer waits in this queue until he moves to the head of the queue. At this time, the customer begins to retry joining the server. The time between retrials (the retrial time) is assumed to follow a general distribution $\{a_i\}_{i=0}^{\infty}$, generating function $A(x) = \sum_{i=0}^{\infty} a_i x^i$ and nth factorial moments υ_n.

Finally, various stochastic processes involved in the system are assumed to be independent of each other. It is denoted that $\bar{p} = 1 - p$ where $0 < p < 1$. Further, the traffic intensity is denoted as $\rho = \rho_1 + \rho_2$ where $\rho_1 = p\zeta_1\mu_{0,1}$ and $\rho_2 = p\zeta_1 \sum_{r=1}^{M} \alpha_r \mu_{r,1}$.

3. The Markov chain

At time m^+, the system can be described by the process,

$$\{X_m = (C_m, \xi_m, \eta_{r,m}, N_m), r = 0, 1, 2, 3, \dots M; m = 0, 1, 2, \dots\}$$

where C_m denotes the state of the server, (0, 1, or $r+1$ according to whether the server is free, busy providing a FES, busy providing a rth type of optional service) and N_m, the number of customers in the retrial group. If $C_m = 0$, then ξ_m represents the remaining retrial time, and if $C_m = 1$, then $\eta_{0,m}$ corresponds the remaining service time of FES and if $C_m = r+1$, then $\eta_{r,m}$ represents the remaining service time of rth $(r = 1, 2, 3, \dots M)$ type of optional service.

The future dynamics of X_m depends only on the current state after introducing the above supplementary variables. Given the current state, the next state and the evolution of the system prior to the current state are independent. So, it can be shown that $\{X_m, m \in N\}$ is the Markov chain of the proposed queueing system, whose state space is

$$\{(0,0); (0,i,k) : i \geq 1, k \geq 1; (r+1,i,k) : r = 0, 1, 2, \cdots M, i \geq 1, \text{k} \geq 0\}.$$

Let

$$\pi_{0,0} = \lim_{m\to\infty} \Pr(C_m = 0, N_m = 0);$$

$$\pi_{0,i,k} = \lim_{m\to\infty} \Pr(C_m = 0, \xi_m = i, N_m = k); i \geq 1, k \geq 1;$$

$$P_{r,i,k} = \lim_{m\to\infty} \Pr(C_m = r+1, \eta_{r,m} = i, N_m = k); r = 0, 1, 2, \dots M, i \geq 1, k \geq 0$$

be the stationary distributions of the Markov chain $\{X_m, m \in N\}$. In this case, Kolmogorov equations are also referred to the global balance equations. The

Kolmogorov equations for the stationary distribution of our system are:

$$\pi_{0,0} = \bar{p}\pi_{0,0} + \bar{p}\alpha_0 P_{0,1,0} + \bar{p}\sum_{r=1}^{M} P_{r,1,0} \tag{3.1}$$

$$\pi_{0,i,k} = \bar{p}\pi_{0,i,k} + \alpha_0\bar{p}a_i P_{0,1,k} + \bar{p}a_i \sum_{r=1}^{M} P_{r,1,k}; \quad i \geq 1, k \geq 1 \tag{3.2}$$

$$\begin{aligned}P_{0,i,k} &= pc_{k+1}s_{0,i}\pi_{0,0} + \bar{p}s_{0,i}\pi_{0,1,k+1} + (1-\delta_{0k})ps_{0,i}\sum_{l=0}^{k-1}\sum_{j=1}^{\infty}\pi_{0,j,k-1}c_{l+1}\\ &\quad + ps_{0,i}\alpha_0\sum_{l=0}^{k} c_{l+1}P_{0,1,k-l} + \bar{p}P_{0,i+1,k} + (1-\delta_{0k})p\sum_{l=1}^{k} c_l P_{0,i+1,k-l}\\ &\quad + \alpha_0\bar{p}a_0 s_{0,i}P_{0,1,k+1} + ps_{0,i}\sum_{r=1}^{M}\sum_{l=0}^{k} c_{l+1}P_{r,1,k-l}\\ &\quad + \bar{p}a_0 s_{0,i}\sum_{r=1}^{M} P_{r,1,k+l},\ i\geq 1,\ k\geq 0 \end{aligned}\tag{3.3}$$

$$\begin{aligned}P_{r,i,k} &= (1-\delta_{0k})ps_{r,i}\sum_{l=1}^{k} c_l\alpha_r P_{0,1,k-l} + \bar{p}s_{r,i}\alpha_r P_{0,1,k}\\ &\quad + (1-\delta_{0k})p\sum_{l=1}^{k} c_l P_{r,i+1,k-l} + \bar{p}P_{r,i+1,k}, \quad r = 1,2,3,\ldots M, i\geq 1, k\geq 0\end{aligned}\tag{3.4}$$

where $\delta_{a,b}$ denotes Kronecker's delta, and the normalizing condition is

$$\pi_{0,0} + \sum_{l=1}^{\infty}\sum_{k=1}^{\infty}\pi_{0,i,k} + \sum_{r=0}^{M}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} P_{r,i,k} = 1. \tag{3.5}$$

In order to solve (3.1)–(3.4), the following generating functions are introduced

$$\phi_0(x,z) = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\pi_{0,i,k}x^i z^k; \Omega_r(x,z) = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} P_{r,i,k}x^i z^k;\ r = 0,1,2\ldots M,$$

and the auxiliary generating functions are,

$$\phi_{0,i}(z) = \sum_{k=1}^{\infty}\pi_{0,i,k}z^k; \Omega_{r,i}(z) = \sum_{k=0}^{\infty} P_{r,i,k}z^k;\ r = 0,1,2,\ldots M, i\geq 1.$$

The following lemmas will be used while deriving the main result.

Lemma 3.1. *If* $\zeta_1 p\left(\mu_{0,1}+\sum\limits_{r=1}^{M}\alpha_r\mu_{r,1}\right)\leq 1-\zeta_1\bar{p}(1-A(\bar{p}))$, *then*

$$\left\{(C(z)+\bar{p}A(\bar{p})(1-C(z))\left[\alpha_0+\sum_{r=1}^{M}\alpha_r S_r(\gamma)\right]S_0(\gamma)-z\gamma\right\}\geq 0$$

for $0\leq z\leq 1$, *where* $\gamma=\bar{p}+pC(z)$.

Proof. Let

$$g(z)=\frac{\left\{\left[C(z)+\bar{p}A(\bar{p})(1-C(z))\right]\left[\alpha_0+\sum\limits_{r=1}^{M}\alpha_r S_r(\gamma)\right]S_0(\gamma)\right\}}{\gamma}.$$

This function satisfies the following properties:

(i) $g(0)=A(\bar{p})\left[\alpha_0+\sum\limits_{r=1}^{M}\alpha_r S_r(\bar{p})\right]S_0(\bar{p})$;

(ii) $g(1)=1$;

(iii) $g'(1)=\zeta_1 p\left(\mu_{0,1}+\sum\limits_{r=1}^{M}\alpha_r\mu_{r,1}\right)+\zeta\bar{p}(1-A(\bar{p})<1$;

(iv) Assuming $0\leq z<1$ and so $g''(z)>0$. This implies $g(z)$ is convex.

Based on above properties of $g(z)$, $g(z)-z>0$ for $0\leq z<1$. So,

$$\left\{[C(z)+\bar{p}A(\bar{p})(1-C(z))]\left[\alpha_0+\sum_{r=1}^{M}\alpha_r S_r(\gamma)\right]S_0(\gamma)-z\gamma\right\}>0.$$

Hence Lemma 3.1 is proved. The following theorem gives an explicit expression for the generating function of the stationary distribution of the system state. □

Theorem 3.2. *If* $\zeta_1 p\left(\mu_{0,1}+\sum\limits_{r=1}^{M}\alpha_r\mu_{r,1}\right)\leq 1-\zeta_1\bar{p}(1-A(\bar{p})$ *then*

$$\phi_0(x,z)=\frac{A(x)-A(\bar{p})}{x-\bar{p}}\frac{px\left(z\gamma-C(z)S_0(\gamma)\left[\alpha_0+\sum\limits_{r=1}^{M}\alpha_r S_r(\gamma)\right]\right)\pi_{0,0}}{[C(z)+\bar{p}A(\bar{p})(1-C(z))]\left[\alpha_0+\sum\limits_{r=1}^{M}\alpha_r S_r(\gamma)\right]S_0(\gamma)-z\gamma}$$

$$\Omega_0(x,z)=\frac{S_0(x)-S_0(\gamma)}{x-\gamma}\frac{px\gamma A(\bar{p})(1-C(z))\pi_{0,0}}{[C(z)+\bar{p}A(\bar{p})(1-C(z))]\left[\alpha_0+\sum\limits_{r=1}^{M}\alpha_r S_r(\gamma)\right]S_0(\gamma)-z\gamma}$$

$$\Omega_r(x,z)=\frac{S_r(x)-S_r(\gamma)}{x-\gamma}\frac{\alpha_r S_0(\gamma)px\gamma A(\bar{p})(1-C(z))\pi_{0,0}}{[C(z)+\bar{p}A(\bar{p})(1-C(z))]\left[\alpha_0+\sum\limits_{r=1}^{M}\alpha_r S_r(\gamma)\right]S_0(\gamma)-z\gamma}$$

where

$$\pi_{0,0} = \frac{1 - \zeta_1 p\left[\mu_{0,1} + \sum\limits_{r=1}^{M} \alpha_r \mu_{r,1}\right] - \zeta_1 \bar{p}(1 - A(\bar{p}))}{A(\bar{p})}.$$

Proof. Multiplying (3.2)–(3.4) by z^k and summing over k and using boundary condition (3.1), these equations become,

$$\phi_{0,i}(z) = \bar{p}\phi_{0,i+1}(z) + \bar{p}a_i\alpha_0\Omega_{0,1}(z) - pa_i\pi_{0,0} \tag{3.6}$$

$$\begin{aligned}\Omega_{0,i}(z) = {} & \left(\frac{C(z) - a_0}{z}\right) ps_{0,i}\pi_{0,0} + \frac{\bar{p}}{z}s_{0,i}\phi_{0,1}(z) + \frac{C(z)}{z}ps_{0,i}\phi_0(1,z) \\ & + \left(\frac{\bar{p}a_0 + pC(z)}{z}\right)\alpha_0 s_{0,i}\Omega_{0,1}(z) + [\bar{p} + pC(z)]\Omega_{0,i+1}(z) \\ & + \left(\frac{\bar{p}a_0 + pC(z)}{z}\right) s_{0,i}\sum_{r=1}^{M}\Omega_{r,1}(z)\end{aligned} \tag{3.7}$$

$$\Omega_{r,i}(z) = (pC(z) + \bar{p})s_{r,i}\Omega_{0,1}(z) + (pC(z) + \bar{p})\Omega_{r,i+1}(z). \tag{3.8}$$

Multiplying (3.6)–(3.8) by x^i and summing over i, then

$$\begin{aligned}\left(\frac{x - \bar{p}}{x}\right)\phi_0(x,z) = {} & \bar{p}[A(x) - a_0]\alpha_0\Omega_{0,1}(z) - p[A(x) - a_0]\pi_{0,0} \\ & + \bar{p}[A(x) - a_0]\sum_{r=1}^{M}\Omega_{r,1}(z) - \bar{p}\phi_{0,1}(z)\end{aligned} \tag{3.9}$$

$$\begin{aligned}\left(\frac{x - (pC(z) + \bar{p})}{x}\right)\Omega_0(x,z) = {} & \left(\frac{C(z) - a_0}{z}\right) pS_0(x)\pi_{0,0} + \frac{\bar{p}}{z}S_0(x)\phi_{0,1}(z) \\ & + \frac{C(z)}{z}pS_0(x)\phi_0(1,z) + \left(\frac{\bar{p}a_0 + pC(z)}{z}\right)\alpha_0 S_0(x)\Omega_{0,1}(z) \\ & + \left(\frac{\bar{p}a_0 + pC(z)}{z}\right) S_0(x)\sum_{r=1}^{M}\Omega_{r,1}(z) - (pC(z) + \bar{p})\Omega_{0,1}(z)\end{aligned} \tag{3.10}$$

$$\begin{aligned}\left(\frac{x - (pC(z) + \bar{p})}{x}\right)\Omega_r(x,z) = {} & (pC(z) + \bar{p})\alpha_r S_r(x)\Omega_{0,1}(z) \\ & - (pC(z) + \bar{p})\Omega_{r,1}(z) \qquad r = 1, 2, 3, \ldots M.\end{aligned} \tag{3.11}$$

To obtain $\phi_0(1,z)$, put $x=1$ in (3.9). Hence,

$$\begin{aligned} p\phi_0(1,z) =& \bar{p}[1-a_0]\alpha_0\Omega_{0,1}(z) - p[1-a_0]\pi_{0,0} \\ &+ \bar{p}[1-a_0]\sum_{r=1}^{M}\Omega_{r,1}(z) - \bar{p}\phi_{0,1}(z). \end{aligned} \tag{3.12}$$

Substitute (3.12) in equation (3.10), then

$$\begin{aligned} \left(\frac{x-(pC(z)+\bar{p})}{x}\right)\Omega_0(x,z) =& \frac{\bar{p}(1-C(z))}{z}S_0(x)\phi_{0,1}(z) - \frac{pa_0(1-C(z))}{z}S_0(x)\pi_{0,0} \\ &+ \left[\left(\frac{C(z)+(1-C(z))\bar{p}a_0}{z}\right)\alpha_0 S_0(x) - (pC(z)+\bar{p})\right]\Omega_{0,1}(z) \\ &+ \left(\frac{C(z)+(1-C(z))\bar{p}a_0}{z}\right)S_0(x)\sum_{r=1}^{M}\Omega_{r,1}(z). \end{aligned} \tag{3.13}$$

To find $\phi_{0,1}(z)$ and $\Omega_{0,1}(z)$, first put $x=\bar{p}$ in (3.9) and $x=\bar{p}+pC(z)$ in (3.13) and (3.11). Hence

$$\begin{aligned} p[A(\bar{p})-a_0]\pi_{0,0} =& \bar{p}[A(\bar{p})-a_0]\alpha_0\Omega_{0,1}(z) \\ &+ \bar{p}[A(\bar{p})-a_0]\sum_{r=1}^{M}\Omega_{r,1}(z) - \bar{p}\phi_{0,1}(z) \end{aligned} \tag{3.14}$$

$$\begin{aligned} \left(\frac{pa_0(1-C(z))}{z}\right)S_0(\gamma)\pi_{0,0} =& \frac{\bar{p}(1-C(z))}{z}S_0(\gamma)\phi_{0,1}(z) \\ &+ \left[\left(\frac{C(z)+(1-C(z))\bar{p}a_0}{z}\right)\alpha_0 S_0(\gamma) - \gamma\right]\Omega_{0,1}(z) \\ &+ \left(\frac{C(z)+(1-C(z))\bar{p}a_0}{z}\right)S_0(\gamma)\sum_{r=1}^{M}\Omega_{r,1}(z) \end{aligned} \tag{3.15}$$

$$\gamma\,\Omega_{r,1}(z) = \gamma\alpha_r S_r(\gamma)\Omega_{0,1}(z) \;\; r=1,2,3,\ldots M \tag{3.16}$$

where $\gamma = pC(z)+\bar{p}$

Substitute $\Omega_{r,1}(z)$ in (3.14) and (3.15), we have,

$$p[A[(\bar{p})-a_0]\pi_{0,0} = \bar{p}[A(\bar{p})-a_0]\left(\alpha_0+\sum_{r=1}^{M}\alpha_r S_r(\gamma)\right)\Omega_{0,1}(z) - \bar{p}\phi_{0,1}(z) \tag{3.17}$$

$$\begin{aligned} \left(\frac{pa_0(1-C(z))}{z}\right)S_0(\gamma)\pi_{0,0} =& \frac{\bar{p}(1-C(z))}{z}S_0(\gamma)\phi_{0,1}(z) \\ &+ \left[\left(\frac{C(z)+(1-C(z))\bar{p}a_0}{z}\right)\left(\alpha_0+\sum_{r=1}^{M}\alpha_r S_r(\gamma)\right)S_0(\gamma) - \gamma\right]\Omega_{0,1}(z). \end{aligned} \tag{3.18}$$

Then, solving (3.17) and (3.18), $\phi_{0,1}(z), \Omega_{0,1}(z)$, and $\Omega_{r,1}(z)$ are given by,

$$\phi_{0,1}(z) = \frac{p(A(\bar{p}) - a_0)\left(z\gamma - C(z)S_0(\gamma)\left[\alpha_0 + \sum_{r=1}^{M} \alpha_r S_r(\gamma)\right]\right)\pi_{0,0}}{\left([C(z) + \bar{p}A(\bar{p})(1 - C(z))]\left[\alpha_0 + \sum_{r=1}^{M} \alpha_r S_r(\gamma)\right] S_0(\gamma) - z\gamma\right)\bar{p}}. \quad (3.19)$$

$$\Omega_{0,1}(z) = \frac{pA(\bar{p})(1 - C(z))S_0(\gamma)\pi_{0,0}}{\left([C(z) + \bar{p}A(\bar{p})(1 - C(z))]\left[\alpha_0 + \sum_{r=1}^{M} \alpha_r S_r(\gamma)\right] S_0(\gamma) - z\gamma\right)}. \quad (3.20)$$

$$\Omega_{r,1}(z) = \frac{\alpha_r S_r(\gamma) pA(\bar{p})(1 - C(z))S_0(\gamma)\pi_{0,0}}{\left([C(z) + \bar{p}A(\bar{p})(1 - C(z))]\left[\alpha_0 + \sum_{r=1}^{M} \alpha_r S_r(\gamma)\right] S_0(\gamma) - z\gamma\right)};$$
$$r = 1, 2, 3, \ldots M. \quad (3.21)$$

Based on Lemma 3.1

$$\phi_0(x, z) = \frac{A(x) - A(\bar{p})}{x - \bar{p}} \frac{px\left(z\gamma - C(z)S_0(\gamma)\left[\alpha_0 + \sum_{r=1}^{M} \alpha_r S_r(\gamma)\right]\right)\pi_{0,0}}{\left([C(z) + \bar{p}A(\bar{p})(1 - C(z))]\left[\alpha_0 + \sum_{r=1}^{M} \alpha_r S_r(\gamma)\right] S_0(\gamma) - z\gamma\right)}$$

$$\Omega_0(x, z) = \frac{S_0(x) S_0(\gamma)}{x - \gamma} \frac{px\gamma A(\bar{p})(1 - C(z))\pi_{0,0}}{\left([C(z) + \bar{p}A(\bar{p})(1 - C(z))]\left[\alpha_0 + \sum_{r=1}^{M} \alpha_r S_r(\gamma)\right] S_0(\gamma) - z\gamma\right)}$$

$$\Omega_r(x, z) = \frac{S_r(x) - S_r(\gamma)}{x - \gamma} \frac{\alpha_r S_0(\gamma) px\gamma A(\bar{p})(1 - C(z))\pi_{0,0}}{\left([C(z) + \bar{p}A(\bar{p})(1 - C(z))]\left[\alpha_0 + \sum_{r=1}^{M} \alpha_r S_r(\gamma)\right] S_0(\gamma) - z\gamma\right)}$$
$$r = 1, 2, 3, \ldots M.$$

$$\pi_{0,0} + \phi_0(1, 1) + \sum_{r=0}^{M} \Omega_r(1, 1) = 1$$

then,
$$\pi_{0,0} = \frac{1 - \zeta_1 p\left(\mu_{0,1} + \sum_{r=1}^{M} \alpha_r \mu_{r,1}\right) - \zeta_1 \bar{p}(1 - A(\bar{p}))}{A(\bar{p})}.$$

This completes the proof of the theorem. □

Corollary 3.3.
1. *The marginal generating function of the number of customers in the orbit when the server is idle is given by*

$$\pi_{0,0}+\phi_0(1,z)=\frac{A(\bar{p})\gamma\left(S_0(\gamma)\left[\alpha_0+\sum_{r=1}^{M}\alpha_rS_r(\gamma)\right]-z\right)\pi_{0,0}}{\left([C(z)+\bar{p}A(\bar{p})(1-C(z))]\left[\alpha_0+\sum_{r=1}^{M}\alpha_rS_r(\gamma)\right]S_0(\gamma)-z\gamma\right)}.$$

2. *The marginal generating function of the number of customers in the orbit when the server is busy is given by,*

$$\sum_{r=0}^{M}\Omega_r(1,z)=\frac{\gamma A(\bar{p})\left[(1-S_0(\gamma))+\sum_{r=1}^{M}S_0(\gamma)\alpha_r(1-S_r(\gamma))\right]\pi_{0,0}}{\left([C(z)+\bar{p}A(\bar{p})(1-C(z))]\left[\alpha_0+\sum_{r=1}^{M}\alpha_rS_r(\gamma)\right]S_0(\gamma)-z\gamma\right)}.$$

3. *The probability generating function of the orbit size is given by*

$$\begin{aligned}\Psi(z)&=\pi_{0,0}+\phi_0(1,z)+\sum_{r=0}^{M}\Omega_r(1,z)\\&=\frac{\gamma A(\bar{p})(1-z)\pi_{0,0}}{\left([C(z)+\bar{p}A(\bar{p})(1-C(z))]\left[\alpha_0+\sum_{r=1}^{M}\alpha_rS_r(\gamma)\right]S_0(\gamma)-z\gamma\right)}.\end{aligned}$$

4. *The probability generating function of the number of customers in the system is given by,*

$$\begin{aligned}\Phi(z)&=\pi_{0,0}+\phi_0(1,z)+z\sum_{r=0}^{M}\Omega_r(1,z)\\&=\frac{\gamma A(\bar{p})(1-z)\left[\alpha_0+\sum_{r=1}^{M}\alpha_rS_r(\gamma)\right]S_0(\gamma)\pi_{0,0}}{\left([C(z)+\bar{p}A(\bar{p})(1-C(z))]\left[\alpha_0+\sum_{r=1}^{M}\alpha_rS_r(\gamma)\right]S_0(\gamma)-z\gamma\right)}.\end{aligned}$$

Next, some performance measures for the system at the stationary regime are presented.

Corollary 3.4.
1. *The probability that the system is idle is given by,*

$$\pi_{0,0}=\frac{1-\zeta_1p\left(\mu_{0,1}+\sum_{r=1}^{M}\alpha_r\mu_{r,1}\right)-\zeta_1\bar{p}(1-A(\bar{p}))}{A(\bar{p})}.$$

2. *The probability that the system is busy is given by,*

$$1-\pi_{0,0}=\frac{\zeta_1\bar{p}+\zeta_1 p\left(\mu_{0,1}+\sum\limits_{r=1}^{M}\alpha_r\mu_{r,1}\right)+A(\bar{p})(1-\zeta_1\bar{p})-1}{A(\bar{p})}.$$

3. *The mean number of customers in the orbit is given by,*

$$\begin{aligned}E[N]&=\Psi'(1)\\&=\frac{\begin{array}{l}\zeta_2\left(\zeta_1 p\left(\mu_{0,1}+\sum\limits_{r=1}^{M}\alpha_r\mu_{r,1}\right)+\zeta_1\bar{p}(1-A(\bar{p}))\right)\\+2\zeta_1^3\bar{p}p(1-A(\bar{p}))\left[\left(\mu_{0,1}+\sum\limits_{r=1}^{M}\alpha_r\mu_{r,1}\right)-1\right]\\+\zeta_1^3p^2\left(\mu_{0,2}+\sum\limits_{r=1}^{M}\alpha_r\mu_{r,2}\right)\end{array}}{2\zeta_1\left(1-\zeta_1 p\left(\mu_{0,1}+\sum\limits_{r=1}^{M}\alpha_r\mu_{r,1}\right)-\zeta_1\bar{p}(1-A(\bar{p}))\right)}.\end{aligned}$$

4. *The mean number of customers in the system is given by,*

$$\begin{aligned}E[L]&=\Phi'(1)\\&=\frac{\begin{array}{l}\zeta_2\left(\zeta_1 p\left(\mu_{0,1}+\sum\limits_{r=1}^{M}\alpha_r\mu_{r,1}\right)+\zeta_1\bar{p}(1-A(\bar{p}))\right)\\-2\zeta_1^3\bar{p}p(1-A(\bar{p}))+\zeta_1^3p^2\left(\mu_{0,2}+\sum\limits_{r=1}^{M}\alpha_r\mu_{r,2}\right)\\+2\zeta_1^2p\left(\mu_{0,1}+\sum\limits_{r=1}^{M}\alpha_r\mu_{r,1}\right)\left[1-\zeta_1 p\left(\mu_{0,1}+\sum\limits_{r=1}^{M}\alpha_r\mu_{r,1}\right)\right]\end{array}}{2\zeta_1\left(1-\zeta_1 p\left(\mu_{0,1}+\sum\limits_{r=1}^{M}\alpha_r\mu_{r,1}\right)-\zeta_1\bar{p}(1-A(\bar{p}))\right)}.\end{aligned}$$

5. *The mean time a customer spends in the system is given by,*

$$E[W]=\frac{E[L]}{p}.$$

Remark 3.5 (three special cases)**.** First case is a retrial queue with single arrival, the second case is system with one additional option for service and the third case is system with no additional option for service.
(a) Setting $C(z)=z$, the present model reduces to Geo/G/1 retrial queue with

general times and M-additional options for service. The generating functions presented Theorem 3.3 are reduced to

$$\phi_0(x,z) = \frac{A(x)-A(\bar{p})}{x-\bar{p}} \frac{px\left(z\gamma - zS_0(\gamma)\left[\alpha_0 + \sum_{r=1}^{M}\alpha_r S_r(\gamma)\right]\right)\pi_{0,0}}{\left\{[z+\bar{p}A(\bar{p})(1-z)]\left[\alpha_0 + \sum_{r=1}^{M}\alpha_r S_r(\gamma)\right]S_0(\gamma) - z\gamma\right\}}$$

$$\Omega_0(x,z) = \frac{S_0(x)-S_0(\gamma)}{x-\gamma} \frac{px\gamma A(\bar{p})(1-z)\pi_{0,0}}{\left\{[z+\bar{p}A(\bar{p})(1-z)]\left[\alpha_0 + \sum_{r=1}^{M}\alpha_r S_r(\gamma)\right]S_0(\gamma) - z\gamma\right\}}$$

$$\Omega_r(x,z) = \frac{S_r(x)-S_r(\gamma)}{x-\gamma} \frac{\alpha_r S_0(\gamma) px\gamma A(\bar{p})(1-z)\pi_{0,0}}{\left\{[z+\bar{p}A(\bar{p})(1-z)]\left[\alpha_0 + \sum_{r=1}^{M}\alpha_r S_r(\gamma)\right]S_0(\gamma) - z\gamma\right\}} \qquad r = 1,2,3,\ldots M$$

where

$$\pi_{0,0} = \frac{1 - p\left(\mu_{0,1} + \sum_{r=1}^{M}\alpha_r\mu_{r,1}\right) - \bar{p}(1-A(\bar{p}))}{A(\bar{p})}.$$

(b) If only one additional optional service is considered (*i.e.* $r = 1$), the present model reduces to $\mathrm{Geo}^{[X]}/\mathrm{G}/1$ retrial queue with second optional service, the probability generating function of number of customers in the system presented in Corollary 3.3 is reduced as follows:

$$\begin{aligned}\Phi(z) &= \pi_{0,0} + \phi_0(1,z) + z\Omega_1(1,z)\\ &= \frac{\gamma A(\bar{p})(1-z)(\alpha_0+\alpha_1 S_1(\gamma))S_0(\gamma)\pi_{0,0}}{([C(z)+\bar{p}A(\bar{p})(1-C(z))]\left[\alpha_0+\alpha_1 S_1(\gamma)\right]S_0(\gamma) - z\gamma)}\end{aligned}$$

where

$$\pi_{0,0} = \frac{1-\zeta_1 p(\mu_{0,1}+\alpha_1\mu_{1,1}) - \zeta_1\bar{p}(1-A(\bar{p}))}{A(\bar{p})}.$$

(c) If there is no additional optional service, then the probability generating function presented in Corollary 3.3 is reduced to

$$\Phi(z) = \pi_{0,0} + \phi_0(1,z) + z\Omega_1(1,z) \; = \frac{\gamma A(\bar{p})(1-z)S_0(\gamma)\pi_{0,0}}{([C(z)+\bar{p}A(\bar{p})(1-C(z))]\,S_0(\gamma) - z\gamma)}$$

where

$$\pi_{0,0} = \frac{1 - \zeta_1 p(\mu_{0,1}) - \zeta_1 \bar{p}(1 - A(\bar{p}))}{A(\bar{p})}.$$

This result coincides the probability generating function of number of customers in the system, derived by Hassan *et al.* [3].

4. Stochastic decomposition

In this section, the stochastic decomposition property of the system size distribution for the proposed model is derived. Fuhrmann and Cooper [16] presented a stochastic decomposition law for the *M/G/1* queueing model with generalized vacations, which affirms that the number of customers in any system with vacations in the stationary regime is distributed as the sum of two independent random variables: one is the number of customers in the corresponding standard model *M/G/1* and the other is the number of customer in the system with vacations given that the server is on vacations. The stochastic decomposition law for retrial queues has been studied by Yang and Templeton [33]. In the proposed model, the probability generating function of the system size can be decomposed as follows:

$$\Phi(z) = Q(z)\Theta(z)$$

where

$$Q(z) = \frac{\left[1 - \zeta_1 p\left(\mu_{0,1} + \sum_{r=1}^{M} \alpha_r \mu_{r,1}\right)\right](1-z)\left[\alpha_0 + \sum_{r=1}^{M} \alpha_r S_r(\gamma)\right] S_0(\gamma)}{\left[\alpha_0 + \sum_{r=1}^{M} \alpha_r S_r(\gamma)\right] S_0(\gamma) - z}$$

and

$$\Theta(z) = \frac{\gamma A(\bar{p})\left(\left[\alpha_0 + \sum_{r=1}^{M} \alpha_r S_r(\gamma)\right] S_0(\gamma) - z\right)\pi_{0,0}}{\left\{[C(z) + \bar{p}A(\bar{p})(1 - C(z))]\left[\alpha_0 + \sum_{r=1}^{M} \alpha_r S_r(\gamma)\right] S_0(\gamma) - z\gamma\right\}\left[1 - \zeta_1 p\left(\mu_{0,1} + \sum_{r=1}^{M} \alpha_r \mu_{r,1}\right)\right]}$$
$$= \frac{\pi_{0,0} + \phi_0(1,z)}{\pi_{0,0} + \phi_0(1,1)}.$$

It can be shown that $Q(z)$ is the probability generating function of the number of customers in the standard Geo$^{[X]}$/G/1 queue with multi-optional service and $\Theta(z)$ is the probability generating function of the number of customers in the present model given that the server is idle. This can be explained by the following decomposition law.

Theorem 4.1. *In the stationary regime, the total number L of customers in the system under study is distributed as the sum of two independent random variables: one is the total number L' of customers in the corresponding standard model $Geo^{[X]}/G/1$ with multi-optional service and the other is the total number M' of customers in the system given that the server is idle. That is, $L = L' + M'$*

The above theorem is used to compute a measure of proximity between the distributions of the system size in the standard $\mathrm{Geo}^{[X]}/\mathrm{G}/1$ queue and present queueing system. The following theorem is provided using this result.

Theorem 4.2. *The following inequalities hold:*

$$
\begin{aligned}
&2\frac{(1-A(\bar{p}))\left(\zeta_1 p\left(\mu_{0,1}+\sum\limits_{r=1}^{M}\alpha_r\mu_{r,1}\right)+\zeta_1\bar{p}-1\right)}{A(\bar{p})} \\
&\qquad\qquad \le \sum_{j=0}^{\infty}\left|P[L=j]-P[L^1=j]\right| \\
&\qquad\qquad \le 2\frac{(1-A(\bar{p}))\left(\zeta_1 p\left(\mu_{0,1}+\sum\limits_{r=1}^{M}\alpha_r\mu_{r,1}\right)+\zeta_1\bar{p}-1\right)}{A(\bar{p})\left(1-\zeta_1 p\left(\mu_{0,1}+\sum\limits_{r=1}^{M}\alpha_r\mu_{r,1}\right)\right)}.
\end{aligned}
$$

Proof. Using the decomposition law,

$$
\begin{aligned}
P[L=j] &= \sum_{k=0}^{j} P[L^1=k]P[M^1=j-k] \\
&= \sum_{k=0}^{j-1} P[L^1=k]P[M^1=j-k] + P[L^1=j]P[M^1=0] \\
&= P[L=j-1] + P[L^1=j]P[M^1=0].
\end{aligned}
$$

Based on the previous results, the following can be obtained:

$$
\begin{aligned}
\left|P[L=j]\right. &\left.-P[L^1=j]\right| \le P[L=j-1] + P[L^1=j](1-P[M^1=0]) \\
&= P[L=j] - P[L^1=j]P[M^1=0] + P[L^1=j](1-P[M^1=0]) \\
&= P[L=j] + P[L^1=j](1-2P[M^1=0]).
\end{aligned}
$$

Summing over all states, we get the upper bound:

$$\begin{aligned}\sum_{j=0}^{\infty}\left|P[L=j]-P[L^1=j]\right| &\leq \sum_{j=0}^{\infty} P[L=j-1]+P[L^1=j](1-P[M^1=0])\\ &= 2(1-2P[M^1=0])\\ &= 2\frac{(1-A(\bar{p}))\left(\zeta_1 p\left(\mu_{0,1}+\sum\limits_{r=1}^{M}\alpha_r\mu_{r,1}\right)+\zeta_1\bar{p}-1\right)}{A(\bar{p})\left(1-\zeta_1 p\left(\mu_{0,1}+\sum\limits_{r=1}^{M}\alpha_r\mu_{r,1}\right)\right)}.\end{aligned}$$

Using the inequality $\|a-b\|\geq a-b$

$$\begin{aligned}\sum_{j=0}^{\infty}|P[L=j] \; -P[L^1=j]| &\geq \left|P[L=0]-P[L^1=0]\right|+\sum_{j=1}^{\infty}\left(P[L=j]-P[L^1=j]\right)\\ &= P[L^1=0](1-P[M^1=0])+1-P[L=0]-(1-P[L^1=0])\\ &= 2P[L^1=0](1-P[M^1=0])\\ &= 2\frac{(1-A(\bar{p}))\left(\zeta_1 p\left(\mu_{0,1}+\sum\limits_{r=1}^{M}\alpha_r\mu_{r,1}\right)+\zeta_1\bar{p}-1\right)}{A(\bar{p})}.\end{aligned}$$

This completes the proof of the inequality. □

5. Numerical results

In this section, the numerical examples are presented to study the impact of the parameters on the mean orbit size $E[N]$ and mean waiting time $E[W]$. Assume that there are two additional optional services (*i.e.* $M=2$).

5.1. *Mean number of customers in orbit for various scenarios:*

Let us consider the following various scenarios:

Scenario-1: $\alpha_0=0.1, \alpha_1=0.8$, and $\alpha_2=0.1$
Scenario-2: $\alpha_0=0.2, \alpha_1=0.6$, and $\alpha_2=0.2$
Scenario-3: $\alpha_0=0.3, \alpha_1=0.4$, and $\alpha_2=0.3$
Scenario-4: $\alpha_0=0.4, \alpha_1=0.8$, and $\alpha_2=0.4$.

Let retrial times follow the geometric distribution with generating function $A(x)=\frac{1-\beta}{1-\beta x}$. Figure 1 depicts the effect of the various scenarios on mean orbit size $E[N]$ *versus* β. Moreover, by considering the retrial time as binomial distribution with

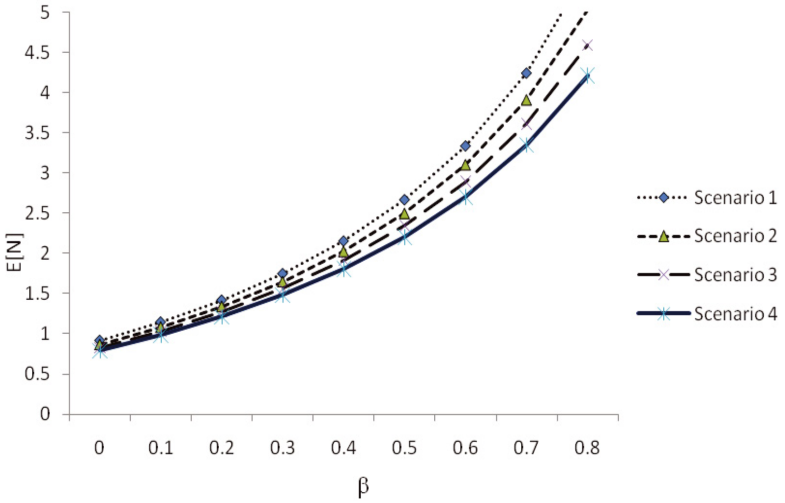

FIGURE 1. $E[N]$ *vs.* β for different scenarios ($p = 0.1$; $\zeta_1 = 1.5$; $\mu_{0,1} = 1$; $\mu_{1,1} = 2$; $\mu_{3,1} = 3$).

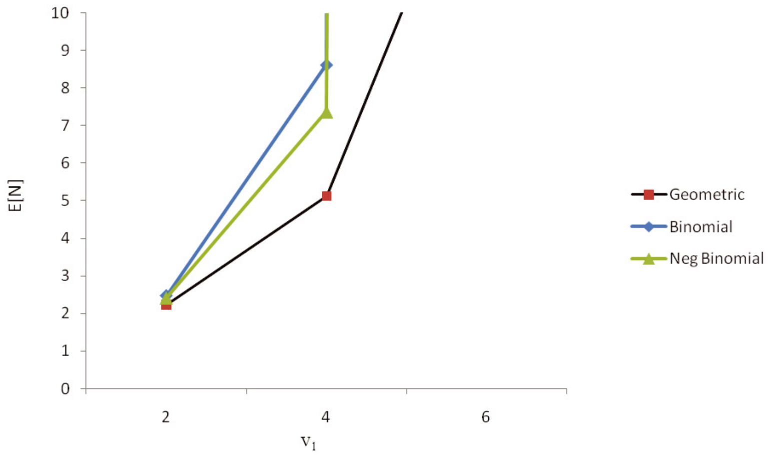

FIGURE 2. $E[N]$ *vs.* v_1 for different retrial distributions ($p = 0.1$; $\zeta_1 = 1.5$; $\mu_{0,1} = 1$; $\mu_{1,1} = 2$).

generating function $A(x) = ((1-\beta)+\beta x)^n$,and negative binomial distribution $A(x) = \left(\frac{1-\beta}{1-\beta x}\right)^n$,the effect of $\alpha_0, \alpha_1,$ and α_2 on $E[N]$ is analyzed. Figure 2 depicts the effect of Scenario-2 on $E[N]$ *versus* the mean retrial time v_1 for various types of retrial time distributions. In the case of binomial retrial time distribution and binomial service time distribution, $E[N]$ is increasing if n increases for varying

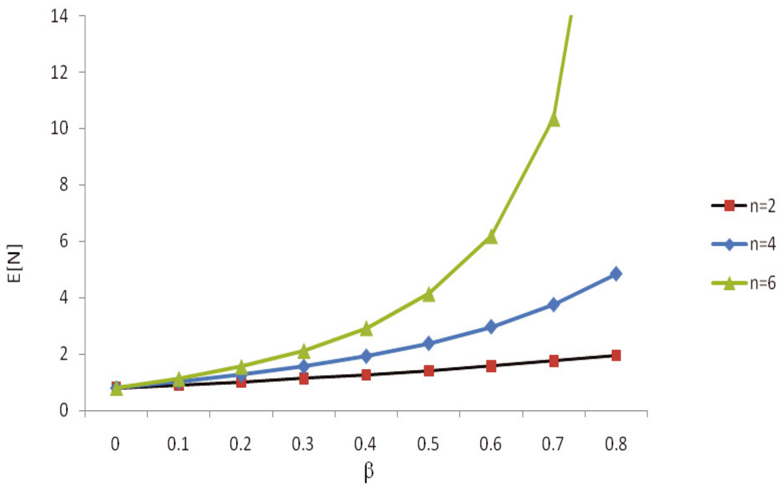

FIGURE 3. $E[N]$ *vs.* β for binomial retrial distribution with various values of n ($p = 0.1$; $\zeta_1 = 2$; $\mu_{0,1} = 1$; $\mu_{1,1} = 2$; $\mu_{3,1} = 5$).

values of β. In Figure 3, we can observe the impact of n on $E[N]$. From Figures 1 and 3, it is observed that

- mean number of customers in the orbit $E[N]$ is increased when β increases;
- mean number of customers in the orbit $E[N]$ is decreased when the probability of second additional optional service α_2 increases.

5.2. *Effect of mean batch size ζ_1 on mean waiting time $E[W]$ and on mean number of customers $E[N]$*

Figure 4 depicts the effect of mean batch size ζ_1 on mean waiting time $E[W]$ for the Scenario-2 by considering *geometric service time* for FES and two multi-optional services. Figure 5 depicts the effect of ζ_1 on mean number of customers in the retrial group $E[N]$ for Scenario-2. In both the cases, it is observed that $E[W]$ and $E[N]$ are increased as ζ_1 increases for the small value of $p = 0.1$. *Increasing ζ_1 rapidly decreases the upper bound of stability region.* Increasing the value of p yields a stability region which is almost empty for any $\zeta_1 \geq 1$. This implies that there is some significance impact of batch arrivals on system performance measures.

5.3. *Effect of various retrial time distributions on busy probability $(1 - \pi_{0,0})$ and $E[W]$*

In Scenario-2, Figure 6 illustrates that the busy probability $(1-\pi_{0,0})$ is increasing if mean time of retrial υ_1 *increases for various retrial time distributions* and

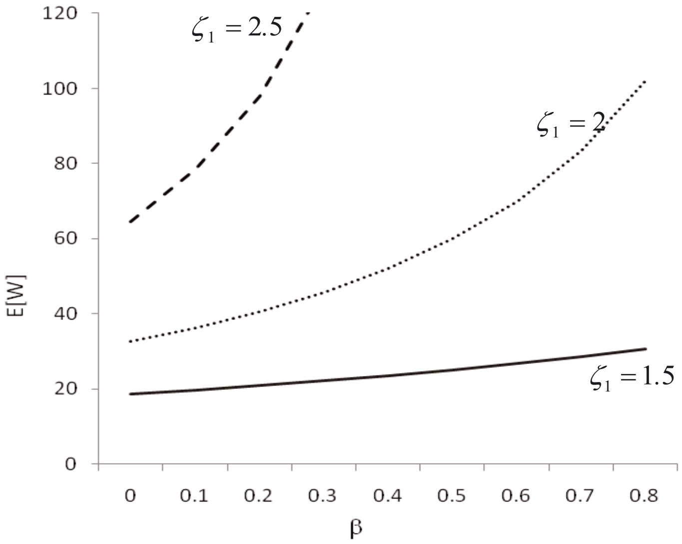

Figure 4. $E[W]$ *vs.* β for different ζ_1 with binomial service times for FES and two multi-optional services ($p = 0.1$; $\mu_{0,1}$; $\mu_{1,1} = 2$; $\mu_{3,1} = 3$).

Figure 7 illustrates the evolution of $E[W]$ as a function of υ_1 *for three different types of retrial time distributions.*

5.4. *Effect of service rate* $\mu_{0,1}$ *on* $E[W]$

Figure 8 depicts that the effect of FES rate $\mu_{0,1}$ on mean waiting time $E[W]$. For different values of $\mu_{0,1}$ (=1, 2 and 3) $E[W]$ is increasing as $\mu_{0,1}$ increases.

6. Conclusion

This paper concerns about steady state analysis of discrete time single server batch arrival retrial queue with general retrial times, and second M-multi-optional services. For such systems, numerical illustrations are clearly carried out to illustrate the influence of various system parameters on important performance measures. Some interesting particular cases are also discussed.

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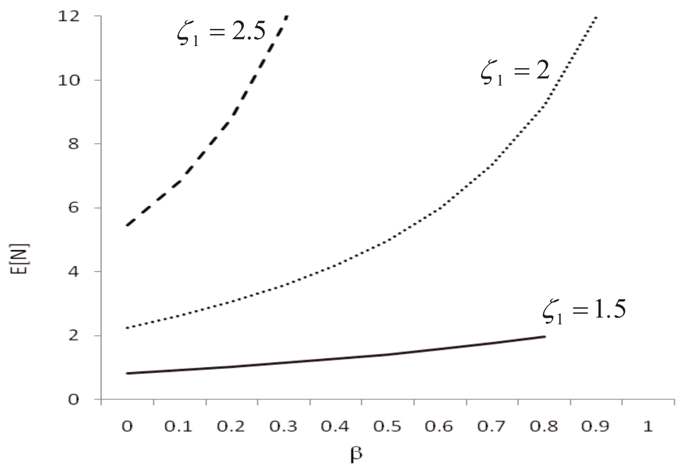

FIGURE 5. $E[N]$ *vs.* β for different ζ_1 with binomial service times for FES and two multi-optional services ($p = 0.1$; $\mu_{0,1} = 1$; $\mu_{1,1} = 2$; $\mu_{3,1} = 3$).

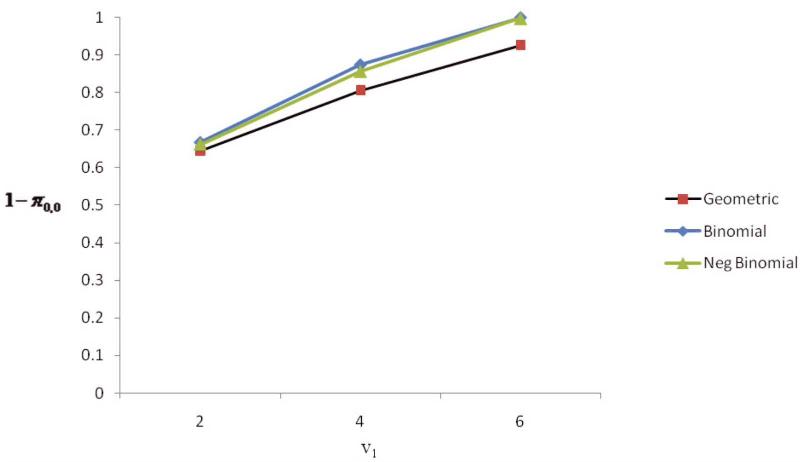

FIGURE 6. Busy probability $(1 - \pi_{0,0})$ *vs.* v_1 for different retrial time distributions ($p = 0.1$; $\mu_{0,1} = 1$; $\mu_{1,1} = 2$; $\mu_{3,1} = 3$).

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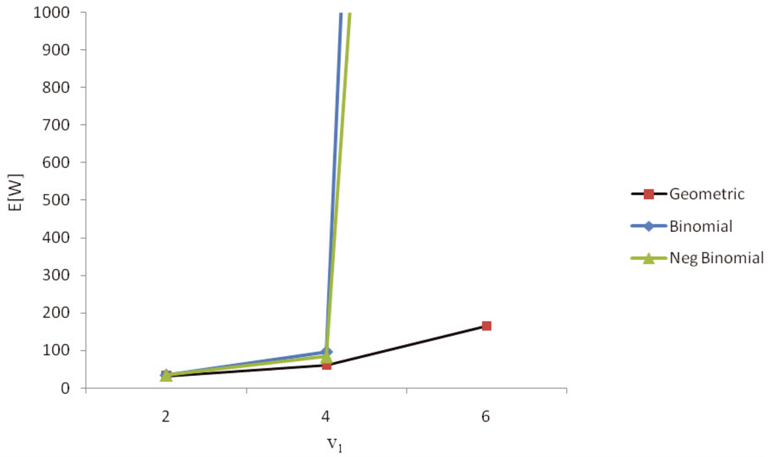

FIGURE 7. Mean waiting time $E[W]$ *vs.* v_1 for various retrial time distributions.

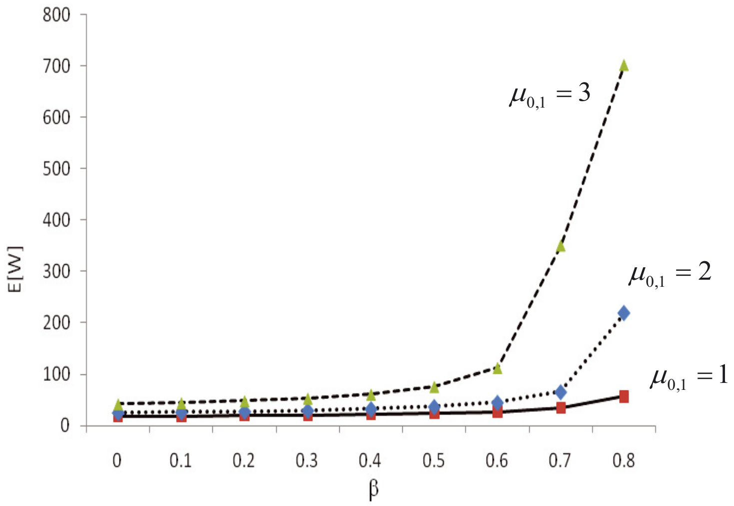

FIGURE 8. Mean waiting time $E[W]$ *vs.* β for various $\mu_{0,1}$.

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