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LEXICOGRAPHIC α-ROBUSTNESS: AN APPLICATION TO THE 1-MEDIAN PROBLEM

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Abstract. In the last decade, several robustness approaches have been developed to deal with uncertainty. In decision problems, and particularly in location problems, the most used robustness approach rely either on maximal cost or on maximal regret criteria. However, it is well known that these criteria are too conservative. In this paper, we present a new robustness approach, called *lexicographic α-robustness*, which compensates for the drawbacks of criteria based on the worst case. We apply this approach to the 1-median location problem under uncertainty on node weights and we give a specific algorithm to determine robust solutions in the case of a tree. We also show that this algorithm can be extended to the case of a general network.

Keywords. Robustness, 1-median location problem, minmax cost/ regret.

Mathematics Subject Classification. 90B50.

1. Introduction

Robustness analysis looks for solutions in a context where the imprecise, uncertain and generally ill-defined parameters of a problem make inappropriate the search for optimal solutions [28,34]. In such a case, uncertainty or imprecision on parameters is modelled by scenarios represented either by a discrete set or by a

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cartesian product of intervals. The most used robustness criteria rely either on the maximal cost or on the maximal regret [19]: a robust solution is one that minimizes the maximal cost or regret among all scenarios. Nevertheless, grasping the notion of robustness through only one measure (the maximal cost or regret) is questionable, since this leads to favor only the worst case scenario which is quite conservative. Furthermore, no tolerance is considered in this measure.

These major drawbacks of the criteria founded on the worst case suggest considering alternative robustness criteria. In the case of deterministic public location problems where a maximum cost function has to be minimized, Ogryczak departs from considering only the worst case by introducing the notion of *lexicographic minimax* [22]. In this paper, we use and extend this idea in order to define a new robustness approach when the set of scenarios is discrete. Such a representation of uncertainty is commonly used in strategic problems like location problems to represent future alternatives [14,25,31].

Our paper is organized as follows. In Section 2, we define the *1-median problem* and review the main works on robustness for this problem. In Section 3, we introduce a relation called α*-leximax*, and use it to define a set of robust solutions. In Section 4, we apply our robustness approach to the *1-median problem* on a tree for which we present a specific algorithm that finds the robust points of the tree. In Section 5, we extend our results to general networks. In the final section, we summarize the important points of this work and suggest some perspectives.

2. Literature review

Network location problems are aimed at locating new facilities in order to meet the demand of a certain number of customers [8]. Demand and travel between demand sites and facilities are assumed to occur only on a graph $G = (V, E)$ composed of a set $V = \{v_i, i = 1, \ldots, n\}$ of n nodes (or vertices) and a set E of m edges. The length of each edge (v_i, v_j), *i.e.* the distance between site v_i and site v_j, is denoted c_{ij}. We assume that demands occur only at the nodes of the network and that they can be characterized by a weight vector $W = (w_1, w_2, \ldots, w_n)$ where w_i is the weight associated with node v_i for $i = 1, \ldots, n$.

The *absolute 1-median problem* is to locate the *absolute median* of a graph G, that is the point of G which minimizes the total weighted distance to all nodes of the graph. A *point* of the graph corresponds either to a node or to any point on an edge. Let us denote $d(a, b)$ the minimum distance between two points a and b of G. The 1-median problem is formulated as follows:

$$\min_{x \in G} C(x) = \sum_{i=1}^{n} w_i d(x, v_i). \tag{2.1}$$

As shown in [13], an absolute median of a graph necessarily lies on a vertex of the graph. The absolute 1-median problem is then equivalent to the *vertex 1-median* *problem* which can be written as:

$$\min_{v\in V} C(v) = \sum_{i=1}^{n} w_i d(v, v_i). \tag{2.2}$$

Consequently, given all distances $d(v_i, v_j)$, the problem can be solved in $O(n^2)$ time by enumerating and evaluating the n possible solutions.

Deterministic approaches assume that the problem parameters (node weights and edge lengths) are fixed and well known. In practice, however, it often appears difficult to determine in a reliable and irrevocable way all the data of a given problem. The decision-maker is often confronted by uncertainty that makes the deterministic reasoning inappropriate.

Let us assume that the node weights and the edge lengths can take many different values and that there is a set S of possible scenarios (possible values of the parameters). For a given scenario s and a point x of G, the cost function under scenario s is defined as follows:

$$C^s(x) = \sum_{i=1}^{n} w_i^s d^s(x, v_i) \tag{2.3}$$

where w_i^s and $d^s(a, b)$ denote respectively the weight of node v_i and the minimum distance between points a and b under scenario s. The regret of solution x (also called *opportunity loss* or *absolute deviation* [19]) is the difference between the cost of x under scenario s and the cost of the best solution under the same scenario:

$$R^s(x) = C^s(x) - C^s(x^{*s}) \tag{2.4}$$

where x^{*s} is the optimal solution of the 1-median problem under scenario s.

To determine the robust solutions for the 1-median problem, authors often attempted to optimize the worst case performance of the system by minimizing the maximal cost or the maximal regret, see for example [1–3,7,19,33]. The *minmax 1-median problem* is defined as follows:

$$\min_{x\in G}\ \max_{s\in S} C^s(x) \tag{2.5}$$

and the *minmax regret 1-median problem* has the following expression:

$$\min_{x\in G}\ \max_{s\in S} R^s(x) = \min_{x\in G}\ \max_{s\in S} \{C^s(x) - C^s(x^{*s})\}. \tag{2.6}$$

In the literature on minmax (regret) 1-median problem, the authors distinguished many models according to the graph structure (tree, network), the location sites (on nodes or on edges) as well as the nature of the scenario set. In the case of a discrete set of scenarios, Kouvelis and Yu [19] proposed an $O(nq)$ algorithm for the problem on a tree, where n is the number of nodes and q the number of scenarios. In the case of uncertainty modelled by intervals, many researchers studied the minmax-regret 1-median problem on a tree with uncertain node weights: Kouvelis *et al.* [18] first proposed an $O(n^4)$ algorithm, Chen and Lin [7] gave an algorithm in $O(n^3)$ time, and Averbakh and Berman [2,3] proposed two algorithms in $O(n^2)$ and $O(n \log^2 n)$ time. Recently, Brodal *et al.* [5] presented an $O(n \log n)$ version of the Averbakh and Berman's algorithm for this specific problem. When weights can be negative, Burkard and Dollani gave an algorithm in $O(n^2)$ for the problem on a tree [6]. As for the minmax regret 1-median problem on a general network, Averbakh and Berman presented in [2] two approaches in $O(nm^4)$ time and $O(mn^2 \log n)$ time for the absolute problem (location anywhere on the graph), the vertex problem having an order of complexity of $O(n^3)$. When edge lengths are uncertain, Chen and Lin [7] showed that, in the case of a tree, the problem can be reduced to the deterministic problem under the scenario with maximal lengths. On the other hand, on a general network, the problem with uncertain lengths becomes NP-hard [1].

It is generally admitted that minmax cost and minmax regret criteria are too conservative since they are based only on the worst case. Besides, the worst case performance is often reached for a scenario with a small likelihood of occurrence, especially when uncertainty is represented by intervals. Many authors tried to remedy the conservatism of the minmax model by proposing alternative approaches [4,21,23,24,35]. In the context of location and specifically for the P-median problem, Daskin *et al.* [9] introduced a new variant of this problem in which the decision-maker associates a probability with each scenario. The model then identifies the solution that minimizes the maximum regret with respect to a subset of scenarios whose total probability of occurrence is less than a fixed *reliability level*. The main drawback of this approach is that it uses subjective probabilities. In a recent work, Puerto *et al.* [26] focused on the uncertainty about the objective function to be optimized, and applied a minmax regret approach to a family of ordered median functions which includes notably the median and center objectives functions.

Following a different perspective, Snyder and Daskin [32] used a measure called *p-robustness* which was first introduced by Kouvelis *et al.* in [17]. This measure imposes a constraint dictating that the cost under each scenario must be within $(100 + p)\%$ of the optimal cost for that scenario, where $p \geq 0$ is an external parameter[1]. Moreover, the authors assign a probability to each scenario. Thus, they build a new robustness measure consisting in determining the p-robust solutions which minimize the expected-cost.

According to this review, we can distinguish two families of approaches to find robust solutions for a given problem. The first family looks for solutions which optimize a certain objective function (*e.g.* minmax approaches) whereas the second one imposes conditions that solutions must satisfy in order to be considered as robust (*e.g.* p-robustness). In the following, we define a new robustness approach which belongs to the second family of approaches.

[1] p is completely independent of the number of facilities P.

3. Definition of a new robustness approach

Let us suppose that, for a given problem, one (or several) of the parameters cannot be determined in a certain and definite way and that there is a finite set S of scenarios. Let X denote the set of feasible solutions and q the number of scenarios. Since the reasoning and the results are valid both for costs and for regrets, we use in what follows the term "cost" and the notation C indifferently for cost or regret. A robust solution according to the maximal cost criterion is a solution that verifies:

$$\min_{x\in X} \max_{s\in S} C^s(x). \tag{3.1}$$

As mentioned above, this criterion gives too much weight to the worst scenario. It is reasonable to look for an approach which takes into account the other costs, while keeping the maximal cost as the most important one. Moreover, this approach should offer some tolerance in relation to cost values. This tolerance can be interpreted as an *indifference threshold* [29].

In the next subsections, we introduce a new preference relation that we call α-*leximax* and use it to define a set of robust solutions.

3.1. The α-leximax relation

Let x be a solution of X. We associate to x a cost vector denoted by $C(x) = (C^{s^1}(x), \ldots, C^{s^q}(x))$ where $C^{s^j}(x)$ is the cost of solution x under scenario s^j, $1 \le j \le q$. By ordering the coordinates of $C(x)$ in a non-increasing order, we get a vector $\hat{C}(x)$ called *disutility vector* [20]. We have $\hat{C}^1(x) \ge \hat{C}^2(x) \ge \ldots \ge \hat{C}^q(x)$. Thus, $\hat{C}^j(x)$ is the j^{th} largest cost of x.

Definition 3.1. Let x and y be two solutions of X, $\hat{C}(x)$ and $\hat{C}(y)$ the associated disutility vectors. The *leximax* relation, denoted by $\succsim_{lex}$, is defined as follows [11,12]:

$$x \succ_{lex} y \quad \Leftrightarrow \quad \begin{cases} \exists k \in \{1,\ldots,q\} : \hat{C}^k(x) < \hat{C}^k(y), \text{ and} \\ \forall j \le k-1,\ \hat{C}^j(x) = \hat{C}^j(y) \end{cases}$$

x is said to be (strictly) preferred to y in the sense of the *leximax* relation.

$$x \sim_{lex} y \quad \Leftrightarrow \quad \forall k \in \{1,\ldots,q\},\ \hat{C}^k(x) = \hat{C}^k(y)$$

x and y are said to be equivalent in the sense of the *leximax* relation.

In other words, comparing two cost vectors in the sense of the *leximax* relation is equivalent to comparing the first distinct coordinates of the disutility vectors. Remark that reordering cost vector implies that we implicitly assume that the vector obtained by the permutation of the cost vector coordinates is equivalent to the original cost vector (the *leximax* relation is said to be *anonymous* [20]). This is justified by the fact that, in a situation of true uncertainty, none of the scenarios can be distinguished.

The previous definition of the *leximax* relation requires a perfect equality between the disutility vector coordinates of two solutions in order to consider them equivalent. Taking an indifference threshold α into account leads to the following definition:

Definition 3.2. Let x and y be two solutions of X, $\hat{C}(x)$ and $\hat{C}(y)$ the associated disutility vectors, and α a positive real value. The α-*leximax* relation, denoted by $\succsim_{lex}^{\alpha}$, is defined as follows:

$$x \succ_{lex}^{\alpha} y \quad \Leftrightarrow \quad \begin{cases} \exists k \in \{1,\ldots,q\} : \hat{C}^k(x) < \hat{C}^k(y) - \alpha, \text{ and} \\ \forall j \leq k-1,\ |\hat{C}^j(y) - \hat{C}^j(x)| \leq \alpha \end{cases}$$

x is said to be (strictly) preferred to y in the sense of the α-*leximax* relation.

$$x \sim_{lex}^{\alpha} y \quad \Leftrightarrow \quad \forall k \in \{1,\ldots,q\},\ |\hat{C}^k(y) - \hat{C}^k(x)| \leq \alpha$$

x and y are said to be indifferent in the sense of the α-*leximax* relation.

In the following subsection, we define a new robustness approach based on the α-*leximax* relation.

3.2. Lexicographic α-robust solutions

Let x^* be an ideal solution, most of the time fictitious, such that:

$$\hat{C}(x^*) = (\hat{C}^1(x_1^*), \hat{C}^2(x_2^*), \ldots, \hat{C}^q(x_q^*)) \tag{3.2}$$

where $x_k^* = \arg\min_{x \in X} \hat{C}^k(x)$ for all $k \in \{1,\ldots,q\}$. Let us consider the following set:

$$\begin{aligned} A(\alpha) &= \{x \in X : not(x^* \succ_{lex}^{\alpha} x)\} \\ &= \{x \in X : x \sim_{lex}^{\alpha} x^*\} \end{aligned} \tag{3.3}$$

where the second equality results from the fact that $\succsim_{lex}^{\alpha}$ is complete and that we cannot have $x \succ_{lex}^{\alpha} x^*$ by definition of x^*. Using the definition of α-*leximax* relation, the set $A(\alpha)$ can also be written as follows:

$$A(\alpha) = \{x \in X : \forall k \leq q,\ \hat{C}^k(x) - \hat{C}^k(x_k^*) \leq \alpha\}. \tag{3.4}$$

Any solution of $A(\alpha)$ performs well with regard to the disutility vector since $A(\alpha)$ is the set of solutions whose kth largest cost is close to the minimum for all $k \leq q$. If we consider this last condition as a robustness property, then we can consider $A(\alpha)$ as a set of robust solutions that we will call set of *lexicographic α-robust solutions*. Notice that this set is monotonic with regard to parameter α:

Property 3.1. $\alpha \leq \alpha' \Rightarrow A(\alpha) \subseteq A(\alpha')$.

Proof. Let $x \in A(\alpha)$. Then, we have for all $k \leq q$, $\hat{C}^k(x) - \hat{C}^k(x_k^*) \leq \alpha \leq \alpha'$ which gives the result. □

Moreover, it is obvious that for small values of α, this set can be empty. We consider that it is legitimate to require such a property of a robustness approach since, in some situations, the different instances of a problem can be such that no solution can be considered as robust.

Property 3.2. *The minimum value of α that guarantees the existence of lexicographic α-robust solutions is:*

$$\alpha_{min} = \min_{x \in X} \max_{1 \leq k \leq q} \{\hat{C}^k(x) - \hat{C}^k(x_k^*)\}. \tag{3.5}$$

Proof. Any solution x in $A(\alpha)$ should satisfy $\max_{1 \leq k \leq q}\{\hat{C}^k(x) - \hat{C}^k(x_k^*)\} \leq \alpha$. Then, $\alpha_{min} = \min_{x \in X} \max_{1 \leq k \leq q}\{\hat{C}^k(x) - \hat{C}^k(x_k^*)\}$. □

From property 3.1, we deduce that all solutions in $A(\alpha_{min})$ belong to any set of lexicographic α-robust solutions when it is not empty.

In general, set $A(\alpha)$ can be found using an iterative procedure which determines, at each iteration $k \in \{1, \ldots, q\}$, the subset:

$$A^k(\alpha) = \{x \in A^{k-1}(\alpha) : \hat{C}^k(x) - \hat{C}^k(x_k^*) \leq \alpha\}. \tag{3.6}$$

In the case of a finite set of solutions, the procedure requires $O(|X|q)$ elementary operations where $|X|$ is the number of elements of X and q the number of scenarios. For instance, if we consider the vertex 1-median problem defined in Section 2, we have $X = V$ and $|X| = n$ where V is the set of all nodes of the graph and n the number of nodes. For this problem, $A(\alpha)$ can be determined in $O(nq)$ time if we assume all solutions costs already computed.

In Section 4, we present an algorithm to solve the lexicographic α-robust 1-median problem on a tree, and, in Section 5, we extend it to general graphs. In both cases, X is infinite.

4. Lexicographic α-robust 1-median problem on a tree

4.1. Preliminaries

We consider the 1-median problem on a tree T in the case of uncertainty on node weights. The removal of any edge (v_i, v_j) of T partitions the tree into two connected components made up of node subsets V_i and V_j. For each point x on edge (v_i, v_j) of length c_{ij}, we denote by y the distance between node v_i and x $(0 \leq y \leq c_{ij})$. The cost under scenario s of point x is indifferently denoted by $C_{ij}^s(x)$ or $C_{ij}^s(y)$. $C^s(x^{*s})$ is the minimum cost under scenario s. For the 1-median problem on a tree, cost $C_{ij}^s(y)$ on edge (v_i, v_j) can be written as [19]:

$$C_{ij}^s(y) = \lambda_{ij}^s + \mu_{ij}^s y \tag{4.1}$$

where:

$$\mu_{ij}^s = \sum_{v_k \in V_i} w_k^s - \sum_{v_k \in V_j} w_k^s \tag{4.2}$$

$$\lambda_{ij}^s = \sum_{v_k \in V_i} w_k^s d(v_i, v_k) + \sum_{v_k \in V_j} w_k^s \Big(d(v_j, v_k) + c_{ij} \Big). \tag{4.3}$$

Equation (4.3) is suitable when C represents the cost. In the case of regret, λ_{ij}^s has the following expression:

$$\lambda_{ij}^s = \sum_{v_k \in V_i} w_k^s d(v_i, v_k) + \sum_{v_k \in V_j} w_k^s \Big(d(v_j, v_k) + c_{ij} \Big) - C^s(x^{*s}). \tag{4.4}$$

Kouvelis and Yu [19] consider the minmax cost and the minmax regret versions of this problem. In their approach, they determine, for a given edge (v_i, v_j), the solution y_{ij}^* which minimizes the maximal cost on the edge. They describe a procedure that computes y_{ij}^* by solving:

$$C_{ij}(y_{ij}^*) = \min_{0 \le y \le c_{ij}} \max_{s \in S} C_{ij}^s(y). \tag{4.5}$$

This procedure, which takes advantage of the convexity of the function $\max_{s \in S} C_{ij}^s(.)$ on $[0, c_{ij}]$, is applied to all edges of the tree, and a solution of minimal maximal cost (regret) is selected among all solutions y_{ij}^* found.

Instead of finding a unique robust 1-median on the tree, we want to determine the lexicographic α-robust set $A(\alpha)$, if it is not empty, in order to define *robust segments* of the tree. We present, hereafter, a specific algorithm for the *lexicographic α-robust 1-median* problem on a tree. We remark again that the notation C and the word "cost" refer indifferently to cost or to regret.

4.2. Determination of the robust segments of the tree

4.2.1. *Principle and notations*

We want to find the robust segments of a tree T, that is the set of lexicographic α-robust solutions when $X = T$. For a given edge (v_i, v_j) of length c_{ij} and a point $x \in (v_i, v_j)$, $\hat{C}_{ij}^k(x)$ represents the kth largest cost of x, $1 \le k \le q$. Unlike cost functions $C_{ij}^s(.)$, costs $\hat{C}_{ij}^k(.)$ are not linear functions on $[0, c_{ij}]$.

We define the following subsets for $k \in \{1, \ldots, q\}$ and $(v_i, v_j) \in E$:

$$I_{ij}^k(\alpha) = \{y \in [0, c_{ij}] : \hat{C}_{ij}^k(y) - \hat{C}^k(x_k^*) \le \alpha\} \tag{4.6}$$

where $x_k^* = \arg\min_{x \in X} \hat{C}^k(x)$. Subsets $I_{ij}^k(\alpha)$ are called *acceptable intervals of order k*.

Let $A_{ij}^k(\alpha)$ be the acceptable subsets defined as follows:

$$A_{ij}^k(\alpha) = \bigcap_{t=1}^{k} I_{ij}^t(\alpha). \tag{4.7}$$

Then, the acceptable subset $A^k(\alpha), k \geq 1$, defined in equation (3.6) can be written as:

$$A^k(\alpha) = \bigcup_{(v_i,v_j)\in E} A_{ij}^k(\alpha). \tag{4.8}$$

Therefore, in order to determine the set of lexicographic α-robust solutions, we use the following algorithm.

Algorithm α-LEXROB(1MT)

```
begin
    $A^0(\alpha) \leftarrow X$;
    $A_{ij}^0(\alpha) \leftarrow [0, c_{ij}]$ for all $(v_i, v_j) \in E$;
    for $k \leftarrow 1$ to $q$ do compute $x_k^*$;
    $k \leftarrow 1$;
    while ($k \leq q$ and $A^{k-1}(\alpha) \neq \varnothing$) do
        for all $(v_i, v_j) \in E$ do
            Determine $I_{ij}^k(\alpha)$;
            Determine $A_{ij}^k(\alpha) \leftarrow A_{ij}^{k-1}(\alpha) \cap I_{ij}^k(\alpha)$;
        Determine $A^k(\alpha) \leftarrow \bigcup_{(v_i,v_j)\in E} A_{ij}^k(\alpha)$;
        $k \leftarrow k + 1$;
end
```

If for a given $k \leq q$, $A^k(\alpha) = \varnothing$, then it is obvious that $A(\alpha) = \varnothing$.

In the following, we explain the algorithm in more detail.

4.2.2. *Determination of all points x_k^**

For each $k \leq q$, let $y_{ij}^{*(k)}$ be the solution which minimizes the cost $\hat{C}_{ij}^k$ on edge (v_i, v_j). Hence, the point x_k^* corresponds to the solution $y_{ij}^{*(k)}$ with minimum cost $\hat{C}_{ij}^k$ among all edges (v_i, v_j). Functions $\hat{C}_{ij}^k(.)$ are piecewise linear; they are convex only for $k = 1$. As a result, it is not possible to use Kouvelis and Yu's procedure which is based on the convexity of $\hat{C}_{ij}^1(.)$. Let us notice that, if $y_{ij}^{*(k)}$ is different from 0 and c_{ij}, it is bound to be one of the points where the function slope changes (see Fig. 1). We call these points $z_{ij}^{h(k)}$, $1 \leq h \leq h_{ij}^k$, where h_{ij}^k is the number of breakpoints of function $\hat{C}_{ij}^k(.)$ on $[0, c_{ij}]$. Points $z_{ij}^{1(k)}$ and $z_{ij}^{h_{ij}^k(k)}$ correspond respectively to 0 and c_{ij}.

In computational geometry [30], the kth largest of the numbers $C_{ij}^{s^1}(x), \ldots,$ $C_{ij}^{s^q}(x)$ is called the *k-level* of point x. Therefore, function $\hat{C}_{ij}^k(.)$ corresponds to

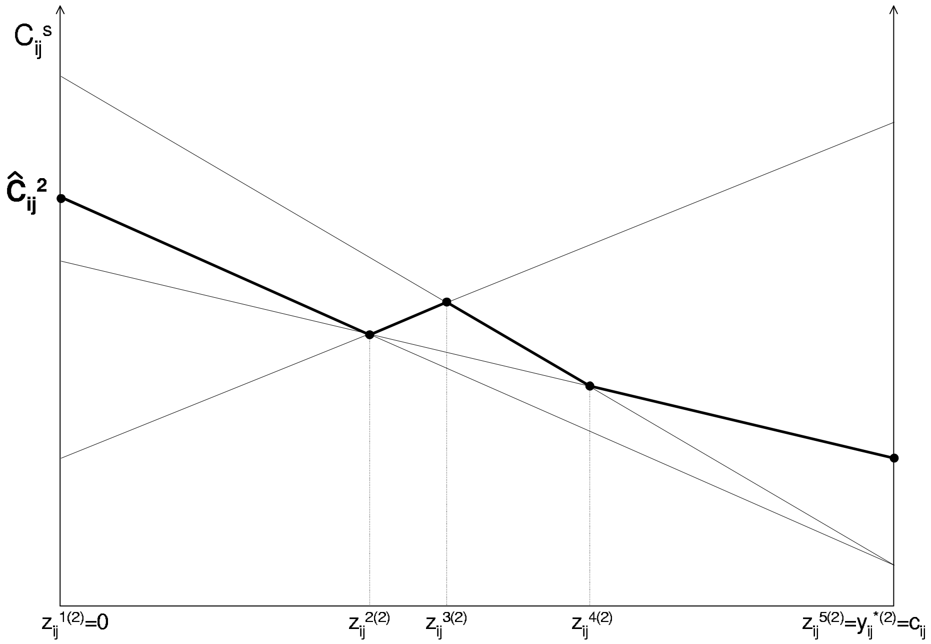

FIGURE 1. Illustration of cost $\hat{C}_{ij}^{k}(.)$ when $k = 2$.

the *k-level* of cost functions $C_{ij}^{s^1}(.), \ldots, C_{ij}^{s^q}(.)$. In [10], Edelsbrunner and Guibas presented an approach based on a topological plane sweep, to compute all intersection points of an arrangement of q lines, as well as their levels, in $O(q^2)$ time[2]. Using this approach, we can, hence, determine all points $z_{ij}^{h(k)}$, $h \in \{1, \ldots, h_{ij}^{k}\}$ and $k \in \{1, \ldots, q\}$, in $O(q^2)$ time. For a given edge (v_i, v_j) and for each k, $y_{ij}^{*(k)}$ is determined among points $z_{ij}^{h(k)}$, $h \in \{1, \ldots, h_{ij}^{k}\}$. Then, determining all points $y_{ij}^{*(k)}$ on edge (v_i, v_j) requires $O(q^2)$ elementary operations since there are $O(q^2)$ intersection points. As a tree has $n - 1$ edges and

$$x_k^* = \underset{x \in X}{\arg\min} \hat{C}^k(x) = \underset{(v_i,v_j) \in E}{\arg\min} \hat{C}_{ij}^{k}(y_{ij}^{*(k)}). \tag{4.9}$$

We have the following result:

Lemma 4.1. *Finding all points x_k^* requires $O(nq^2)$ elementary operations.*

[2] In [10], the $O(q^2)$ time complexity was presented in the case of *simple* arrangements, that is if any two lines intersect at a point, but no three do so. Nevertheless, this result was extended to the degenerate cases such that parallel or multiple concurrent lines in [27].

4.2.3. *Determination of acceptable intervals* $I_{ij}^k(\alpha)$

Unlike acceptable intervals of order 1, subsets $I_{ij}^k(\alpha), 2 \leq k \leq q$, are not necessarily connected because of the non convexity of functions $\hat{C}_{ij}^k(.)$. Nevertheless, for the sake of convenience, we will continue to call them *acceptable intervals*.

If it is not empty, the acceptable interval $I_{ij}^k(\alpha)$ can be represented by the union of p_{ij}^k elementary intervals as follows:

$$I_{ij}^k(\alpha) = \left[y_{ij}^{1(k)}, y_{ij}^{2(k)}\right] \cup \ldots \cup \left[y_{ij}^{2p_{ij}^k-1(k)}, y_{ij}^{2p_{ij}^k(k)}\right] \tag{4.10}$$

with $0 \leq y_{ij}^{1(k)} \leq \ldots \leq y_{ij}^{2p_{ij}^k(k)} \leq c_{ij}$ and $p_{ij}^k \in \mathbb{N}^*$.

Remind that $z_{ij}^{h(k)}, 1 \leq h \leq h_{ij}^k$, are the breakpoints of $\hat{C}_{ij}^k(.)$ on $[0, c_{ij}]$. It is obvious that for a given $h \in \{1, \ldots, h_{ij}^k-1\}$, if we have $\hat{C}_{ij}^k(z_{ij}^{h(k)}) > \hat{C}_{ij}^k(x_k^*)+\alpha$ and $\hat{C}_{ij}^k(z_{ij}^{h+1(k)}) > \hat{C}_{ij}^k(x_k^*)+\alpha$, that is $z_{ij}^{h(k)}$ and $z_{ij}^{h+1(k)}$ do not belong to $I_{ij}^k(\alpha)$, then $[z_{ij}^{h(k)}, z_{ij}^{h+1(k)}] \cap I_{ij}^k(\alpha) = \varnothing$. Similarly, if $z_{ij}^{h(k)}$ and $z_{ij}^{h+1(k)}$ belong to $I_{ij}^k(\alpha)$, then $[z_{ij}^{h(k)}, z_{ij}^{h+1(k)}] \subset I_{ij}^k(\alpha)$ (see Fig. 2). On the other hand, if one of them belongs to $I_{ij}^k(\alpha)$ and not the other, then only a subinterval of $[z_{ij}^{h(k)}, z_{ij}^{h+1(k)}]$ is included in $I_{ij}^k(\alpha)$. This subinterval is delimited by $z_{ij}^{h(k)}$ (or $z_{ij}^{h+1(k)}$) and the crosspoint between function $\hat{C}_{ij}^k(.)$ and the horizontal line of value $\hat{C}_{ij}^k(x_k^*) + \alpha$. Therefore, we just have to scan points $z_{ij}^{h(k)}$ in a non-decreasing order for h varying from 1 to $h_{ij}^k - 1$ in order to determine the parts of intervals $[z_{ij}^{h(k)}, z_{ij}^{h+1(k)}]$ which belong to $I_{ij}^k(\alpha)$ and afterwards deduce the set $I_{ij}^k(\alpha)$.

Since for each k, we scan all breakpoints $z_{ij}^{h(k)}, 1 \leq h \leq h_{ij}^k$, the determination of all acceptable intervals on a given edge (v_i, v_j) requires $O(q^2)$ elementary operations.

Lemma 4.2. *Given all points x_k^*, finding all acceptable intervals of the tree for k varying in $\{1, \ldots, q\}$ requires $O(nq^2)$ elementary operations.*

4.2.4. *Determination of the acceptable subsets* $A_{ij}^k(\alpha)$

We have $A_{ij}^1(\alpha) = I_{ij}^1(\alpha)$. For $k \geq 2$,

$$A_{ij}^k(\alpha) = A_{ij}^{k-1}(\alpha) \cap I_{ij}^k(\alpha). \tag{4.11}$$

If it is not empty, $A_{ij}^k(\alpha)$ is the union of r_{ij}^k subintervals of $[0, c_{ij}]$:

$$A_{ij}^k(\alpha) = [a_{ij}^{1(k)}, a_{ij}^{2(k)}] \cup \ldots \cup [a_{ij}^{2r_{ij}^k-1(k)}, a_{ij}^{2r_{ij}^k(k)}] \tag{4.12}$$

with $0 \leq a_{ij}^{1(k)} \leq \ldots \leq a_{ij}^{2r_{ij}^k(k)} \leq c_{ij}$ and $r_{ij}^k \in \mathbb{N}^*$.

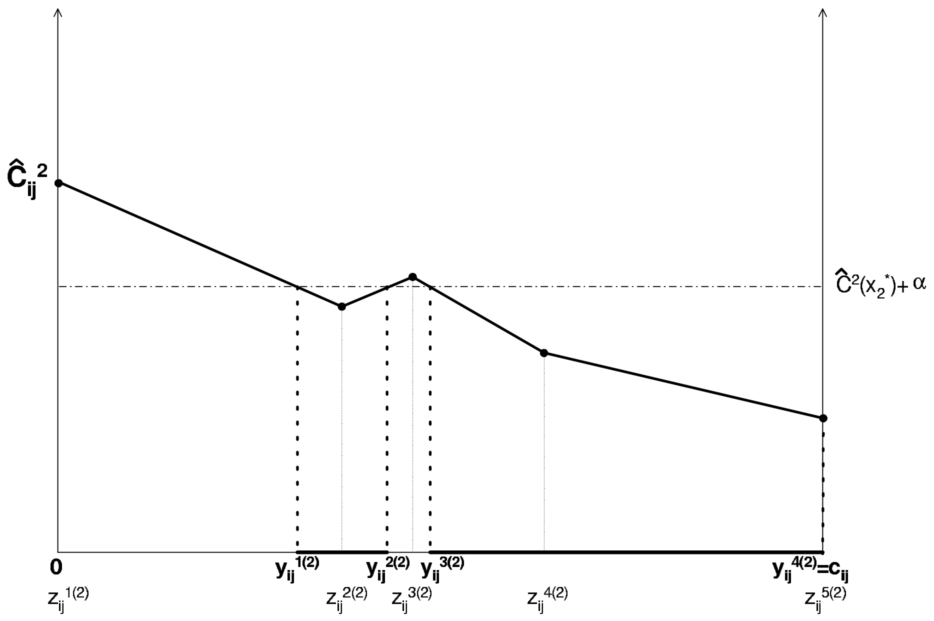

FIGURE 2. Illustration of acceptable interval $I_{ij}^k(\alpha)$ when $k = 2$.

To find the bounds of $A_{ij}^k(\alpha)$, we have to look for them among those of $A_{ij}^{k-1}(\alpha)$ and $I_{ij}^k(\alpha)$ (see Fig. 3). A detailed procedure, *Find(A_{ij}^k)*, is presented in Appendix I. The complexity of this procedure is $O(q^2)$ as shown in Appendix II. Using equations (3.6) and (4.8), we conclude that:

Lemma 4.3. *Given x_k^* for all $k \in \{1,\ldots,q\}$, finding the set $A(\alpha)$ requires $O(nq^3)$ elementary operations.*

4.2.5. *Complexity of lexicographic α-robust 1-median on a tree*

Theorem 4.1. *Lexicographic α-robust 1-median on a tree can be solved in $O(nq^3)$ time.*

Proof: Theorem 4.1 follows immediately from Lemmas 4.1, 4.2 and 4.3. □

4.3. Example

Let us consider the tree T of Figure 4 where values on edges represent lengths. Uncertainty on node weights is modelled by four scenarios as shown in Table 1 (v_s^* is the median under scenario s, $s \in \{s_1, s_2, s_3, s_4\}$).

The minmax median x^* of the tree is the point of (v_1, v_4) at a distance 2.5 from node v_1 and the minmax cost is 197. We present in Figure 5 the cost functions on edge (v_1, v_4) (C_{14}^s denotes the cost function on interval $[0, 7]$ under scenario s).

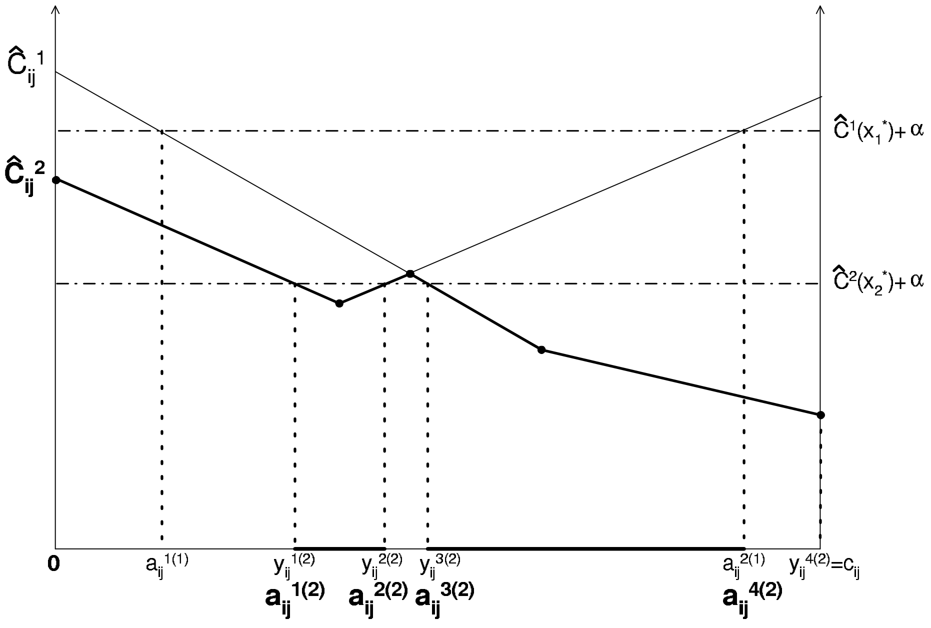

FIGURE 3. Illustration of acceptable subset $A_{ij}^{k}(\alpha)$ when $k = 2$.

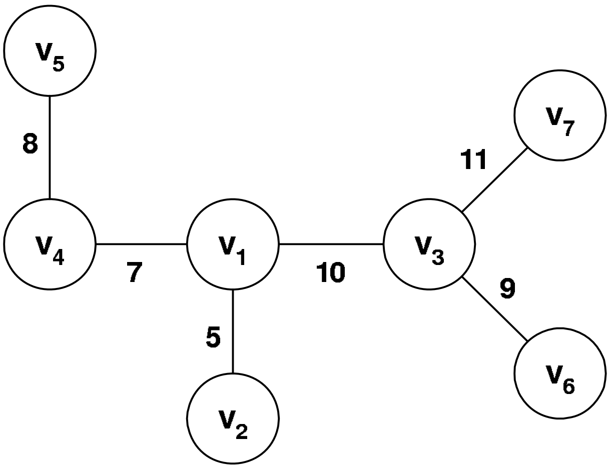

FIGURE 4. Example.

The points which minimize costs $\hat{C}^k, k = 1, \ldots, 4$, are $x_1^* = x^*$, x_2^* the point of edge (v_1, v_3) at a distance 2.5 from node v_1, $x_3^* = v_3$ and $x_4^* = v_1$. If we choose a

TABLE 1. *Weight scenarios.*

Weights	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_s^*
s_1	1	1	1	1	1	1	1	v_1
s_2	1	10	1	1	1	1	1	v_2
s_3	1	1	10	1	1	1	1	v_3
s_4	1	1	1	1	10	1	1	v_5

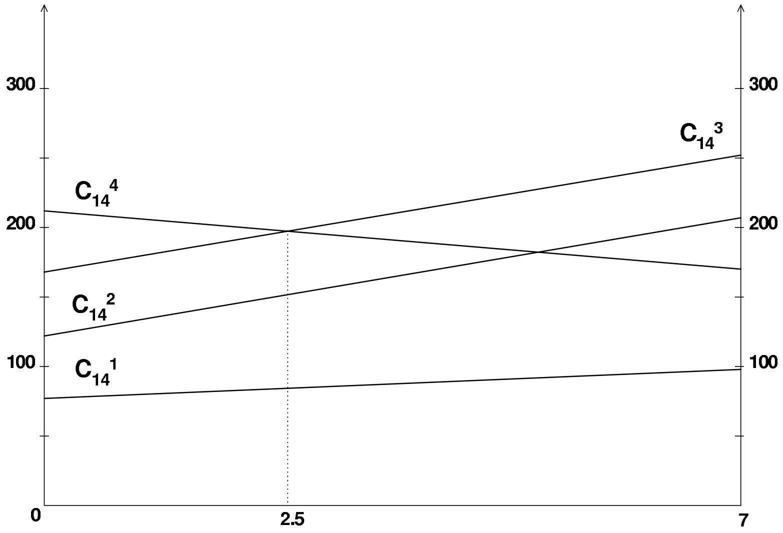

FIGURE 5. Cost functions on edge (v_1, v_4).

threshold $\alpha = 45$, the first iteration of algorithm α-*LEXROB(1MT)* gives the set:

$$A^1(45) = \{x \in T : \hat{C}^1(x) - 197 \leq 45\} \tag{4.13}$$

represented by bold segments in Figure 6.

After four iterations, we get the set $A(45)$ of lexicographic α-robust solutions of the tree. $A(45)$ is represented by the union of three segments (v_1, v_2'), (v_1, v_3') and (v_1, v_4') where v_2' is the point of edge (v_1, v_2) at a distance 1.78 from node v_1, v_3' the point of edge (v_1, v_3) at a distance 1 from node v_1 and v_4' the point of edge (v_1, v_4) at a distance 0.83 from node v_1 (see Fig. 7). Remark that x^*, the minmax robust solution, does not belong to the lexicographic α-robust set. Indeed, it performs well for the maximal cost function, but not well enough for $\hat{C}^2, \hat{C}^3$ and $\hat{C}^4$, compared with node v_1 for example as shown in Table 2.

The minimum value of parameter α which guarantees the existence of lexicographic α-robust solutions is $\alpha_{\min} = 35$, that is $\alpha < 35 \Rightarrow A(\alpha) = \varnothing$ and $\alpha \geq 35 \Rightarrow A(\alpha) \neq \varnothing$. For $\alpha = \alpha_{\min}$, the set of robust solutions is reduced to node v_1.

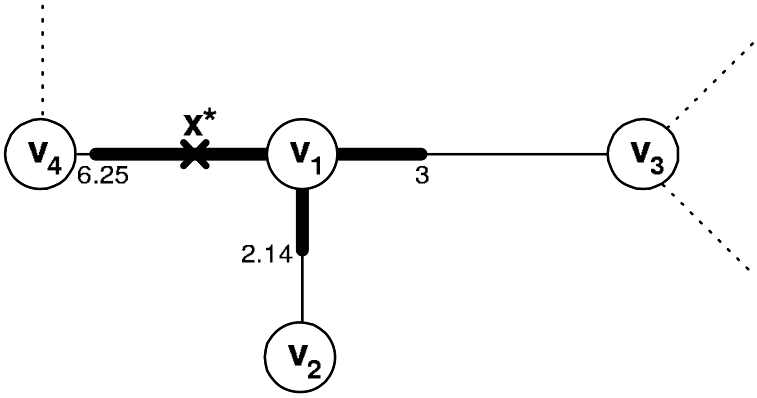

FIGURE 6. Set $A^1(45)$.

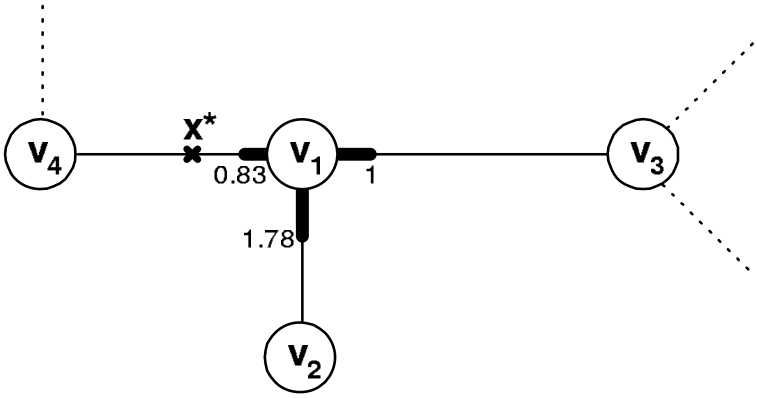

FIGURE 7. Minmax robust solution (x^*) and lexicographic α-robust solutions of the tree for $\alpha = 45$.

TABLE 2. *Comparison of x^* and v_1.*

Disutility costs	$k=1$	$k=2$	$k=3$	$k=4$
$\hat{C}^k(x^*)$	197	197	152	84.5
$\hat{C}^k(v_1)$	212	167	122	77
$\hat{C}^k(x^*) - \hat{C}^k(v_1)$	-15	30	30	7.5

5. Lexicographic α-robust 1-median problem on general graphs

We consider now the problem on a general network $G = (V, E)$. We want to determine the lexicographic α-robust segments of G. We recall that for any two

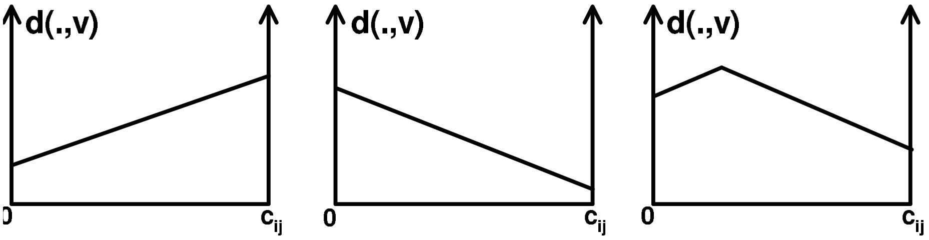

FIGURE 8. Plots of $d(.,v)$ on a given edge (v_i,v_j).

points a and b, $d(a,b)$ denotes the minimum distance between a and b. Observe that, on a general network, the distance $d(x,v)$ between a point x varying on an edge (v_i,v_j) and a given node $v \in V$ has three possible plots as shown in Figure 8. The third case occurs when none of the shortest paths from v to v_i and from v to v_j contains edge (v_i,v_j). In the two other cases, the point of (v_i,v_j) farthest from v is either node v_i or node v_j. In [2], Averbakh and Berman call all these farthest points *pseudonodes*. They also define a *basic interval* as a subinterval $[a,b]$ of an edge such that a and b are pseudonodes and there are no pseudonodes inside $[a,b]$.

Since the distance $d(x,.)$ as a function of x is linear on a given basic interval $[a,b]$, cost (or regret) functions $C^s(.)$ are linear on such a subinterval, for all $s \in S$. Thus, algorithm *α-LEXROB(1MT)* can be easily extended to a general graph using these basic intervals instead of the edges.

Theorem 5.1. *Lexicographic α-robust 1-median on a general network can be solved in $O(mnq^3)$ time.*

Proof. Given an edge e of G, the number of pseudonodes of e cannot exceed the number of nodes by definition. Therefore, the number of basic intervals of the graph is at most mn, where n is the number of nodes and m the number of edges. The remainder of the proof follows from Lemmas 4.1, 4.2 and 4.3. □

6. Conclusions and perspectives

In this paper, we introduced a new robustness approach, called *lexicographic α-robustness*, suitable when uncertainty is represented by a discrete set of scenarios. We applied this approach to the 1-median location problem and presented a specific algorithm to solve the problem when the underlying graph is a tree. We showed that the algorithm is easily extended to the case of a general network. These algorithms are polynomial if the number of scenarios is polynomially bounded, and especially when it is constant. We would like to point out that the algorithm presented here can be adapted to the 1-center location problem using a decomposition similar to the one presented in Section 5 [15].

Compared with minmax criteria, the new approach has several advantages. First, it considers several measures, that is to say costs or regrets, from the worst one to the best one respecting the aversion of the decision maker to risk. Second, it offers some tolerance since it includes an indifference threshold α, taking into account the subjective dimension of robustness. Finally, it can give an empty set of robust solutions depending on the threshold chosen. Indeed, it seems desirable to highlight that some instances do not contain any solution that can be considered as robust.

The approach presented here could be extended in many ways. The threshold α could be variable and differentiated for each measure. Moreover, robustness could be studied in relation to the k ($k \leq q$) largest costs instead of all ordered costs. Since the search for robust solutions is not always possible, such studies are in line with the determination of what Roy calls *robust conclusions* [28].

It is obvious that lexicographic α-robustness is more arduous to implement than minmax criteria. Nevertheless, we showed in this paper that this approach remains attractive in the case of 1-median location problem which is a polynomial problem. It is interesting to notice that this is also the case for some combinatorial problems. Indeed, Kalaï and Vanderpooten [16] and Kalaï [15] proved that lexicographic α-robustness approach does not change the complexity of the knapsack and the shortest path problems compared with their minmax (regret) versions. Therefore, we consider the application of this new robustness approach to some other polynomial and combinatorial problems to be an avenue for future research.

Appendix I

```
Procedure Find($A_{ij}^k$)
Input: $p_{ij}^k$, $E_{ij}^k = \{y_{ij}^{t(k)}, t \in \{1, \ldots, 2p_{ij}^k\}\}$, $r_{ij}^{k-1}$, $B_{ij}^{k-1} = \{a_{ij}^{l(k-1)}, l \in \{1, \ldots, 2r_{ij}^{k-1}\}\}$.
Output: $r_{ij}^k$, $B_{ij}^k = \{a_{ij}^{l(k)}, l \in \{1, \ldots, 2r_{ij}^k\}\}$.
begin
  if $E_{ij}^k = \varnothing$ then $B_{ij}^k \leftarrow \varnothing$
  else
=       $B_{ij}^k \leftarrow \varnothing$;
    $r \leftarrow 0$;
    $a_{ij}^{0(k-1)} \leftarrow 0$;
    for $t \leftarrow 1$ to $p_{ij}^k$ do
      Find the largest $a_{ij}^{l(k-1)}$ $(0 \leq l \leq 2r_{ij}^{k-1})$ such that $a_{ij}^{l(k-1)} < y_{ij}^{2t-1(k)}$;
      Find the largest $a_{ij}^{h(k-1)}$ $(0 \leq h \leq 2r_{ij}^{k-1})$ such that $a_{ij}^{h(k-1)} < y_{ij}^{2t(k)}$;
      if $l$ is odd then
        Put($y_{ij}^{2t-1(k)}$);
        if $y_{ij}^{2t(k)} \leq a_{ij}^{l+1(k-1)}$ then Put($y_{ij}^{2t(k)}$)
        else
          for $s \leftarrow l+1$ to $h$ do Put($a_{ij}^{s(k-1)}$);
          if $h$ is odd then Put($y_{ij}^{2t(k)}$);
      if $l$ is even then
```

```
        if $y_{ij}^{2t(k)} \geq a_{ij}^{l+1(k-1)}$ then Put($a_{ij}^{l+1(k-1)}$);
            if $y_{ij}^{2t(k)} \leq a_{ij}^{l+2(k-1)}$ then Put($y_{ij}^{2t(k)}$)
            else
                for $s \leftarrow l+2$ to $h$ do Put($a_{ij}^{s(k-1)}$);
                if $h$ is odd then Put($y_{ij}^{2t(k)}$);
    $r_{ij}^k \leftarrow r/2$;
end
```

Procedure Put(x) is "Put x in B_{ij}^k and increment r".

Appendix II

Proof of lemma 4.3. Let us consider an edge (v_i, v_j) and a fixed order k. p_{ij}^k is the number of subintervals of $[0, c_{ij}]$ given by the intersection of a straight line $(D = \hat{C}^k(x_k^*) + \alpha)$ with at most q lines and possibly the lines $y_{ij} = 0$ and $y_{ij} = c_{ij}$. Therefore, $p_{ij}^k \leq \frac{q+2}{2}$.

Let $B_{ij}^k = \{a_{ij}^{l(k)}, l \in \{1, \ldots, 2r_{ij}^k\}\}$ and $E_{ij}^k = \{y_{ij}^{t(k)}, t \in \{1, \ldots, 2p_{ij}^k\}\}$. Since we have $B_{ij}^k \subset (E_{ij}^k \cup B_{ij}^{k-1})$, then:

$$r_{ij}^k \leq p_{ij}^k + r_{ij}^{k-1} \leq \frac{q+2}{2} + r_{ij}^{k-1} \leq \ldots \leq (k-1)\left(\frac{q+2}{2}\right) + r_{ij}^1.$$

As $r_{ij}^1 = 1$ and $k \leq q$, we get $r_{ij}^k \leq q(\frac{q+2}{2}) + 1$, for all $k \in \{1, \ldots, q\}$.

As a result, we have $O(p_{ij}^k) = O(q)$ and $O(r_{ij}^k) = O(q^2)$ for all $k \leq q$.

To determine the largest $a_{ij}^{l(k-1)}$ such that $a_{ij}^{l(k-1)} < y_{ij}^{2t-1(k)}$ (t being fixed), one can use a binary search on the set $\{a_{ij}^{0(k-1)}, \ldots, a_{ij}^{2r_{ij}^{k-1}(k-1)}\}$. Such an approach has a complexity of $O(\log(r_{ij}^{k-1})) = O(\log(q^2)) = O(\log q)$. Idem for $a_{ij}^{h(k-1)}$.

In addition, the elements of B_{ij}^{k-1} are compared to those of E_{ij}^k only one time during the whole procedure.

Consequently, the procedure *Find(A_{ij}^k)* is in $O(p_{ij}^k \log(r_{ij}^{k-1}) + r_{ij}^{k-1}) = O(q^2)$ time. □

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