DECREASE OF $C^{1,1}$ PROPERTY IN VECTOR OPTIMIZATION

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Abstract. In the paper we generalize sufficient and necessary optimality conditions obtained by Ginchev, Guerraggio, Rocca, and by authors with the help of the notion of ℓ-stability for vector functions.

1. Introduction, notations and preliminaries

Generalized second-order optimality conditions has been extensively studied since 80's of last century (see *e.g.* [2,3,7–11,15,16,21,25–27] and references therein).

Recently there appeared a number of interesting papers dealing with constrained and unconstrained vector optimization problems. Let us mention for example [12–14,18–20,22–24]. We were mainly inspired by the first one.

Moreover, we wish to continue our research done in our previous article [4], where we have proved a generalized sufficient condition for scalar case with the use of the generalized second-order directional derivative of the Peano type.

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More precisely, we would like to replace the assumption that the objective function f must be of class $C^{1,1}$ (recall that f is of class $C^{1,1}$ if it has locally Lipschitz derivative) by weakened assumptions, *i.e.* f is assumed to be ℓ-stable at x. Here the ℓ-stability substitutes the lipschitzness of first derivative on a neighbourhood of the point x.

Let us get some needed notations and definitions together before we will proceed. Throughout all of this text we will work with functions $f : \mathbb{R}^m \to \mathbb{R}^n$ where $m, n \in \mathbb{N}$. If X is an Euclidean space endowed with the Euclidean norm, then $S_X = \{x \in X : \|x\| = 1\}$, $B_X = \{x \in X : \|x\| < 1\}$ denote a unit sphere and an open unit ball of X respectively. $\langle \cdot, \cdot \rangle$ denotes a standard scalar product on X. Further, if $Y \subset X$, then int Y, $\overline{Y}$ denote a topological interior and topological closure of Y respectively. Symbols conv Y, $\overline{\text{conv}}\, Y$ stand for the convex hull and closed convex hull of a subset Y of X respectively. $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ will denote the space of all continuous linear operators $L : \mathbb{R}^m \to \mathbb{R}^n$. $f'(x)$ denotes the Fréchet derivative of the function $f : \mathbb{R}^m \to \mathbb{R}^n$ at $x \in \mathbb{R}^m$, *i.e.* $f'(x) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$.

By a cone $C \subset \mathbb{R}^n$ we will always mean a nonempty, closed, convex and pointed cone with int $C \neq \emptyset$. For definitions see *e.g.* [17,28,29]. The (positive) polar cone we denote by C' and define as follows: $C' := \{\xi \in \mathbb{R}^n : \langle \xi, y \rangle \geq 0, y \in C\}$. The bipolar cone is defined $C'' := (C')' = \{y \in \mathbb{R}^n : \langle \xi, y \rangle \geq 0, \xi \in C'\}$. From now we always put $\Gamma := C' \cap S_{\mathbb{R}^n}$. Under assumptions set out above, it is well known that:

   (a) C' is also nonempty, closed, convex and pointed cone with int $C' \neq \emptyset$;
   (b) for each $y \in C \setminus \{0\}$, $\xi \in$ int C', it holds $\langle \xi, y \rangle > 0$;
   (c) $C = C''$.

Definition 1.1. Let $C \subset \mathbb{R}^n$ be a cone, $f : \mathbb{R}^m \to \mathbb{R}^n$ be a function.

   (a) A point $x^0 \in \mathbb{R}^m$ is said to be a *weakly efficient* point (or simply *w-minimizer*) for f, if there is a neighbourhood U of x^0 such that

$$x \in U \implies f(x) - f(x^0) \notin -\text{int } C.$$

   (b) A point $x^0 \in \mathbb{R}^m$ is said to be an *efficient point* (or *e-minimizer*) for f if there is a neighbourhood U of x^0 such that

$$x \in U \implies f(x) - f(x^0) \notin -(C \setminus \{0\}).$$

   (c) A point $x^0 \in \mathbb{R}^m$ is said to be an *isolated minimizer of second-order for f* if there is a neighbourhood U of x^0 and a constant $A > 0$ such that

$$x \in U \implies \sup_{\xi \in \Gamma}(\langle \xi, f(x) - f(x^0) \rangle) \geq A \|x - x^0\|^2.$$

For more informations about isolated minimizer see for instance [13], where Ginchev *et al.* stated the following optimality conditions. They used a notion of the generalized second-order directional derivative $f''_D(x; u)$ defined for a function

$f : \mathbb{R}^m \to \mathbb{R}^n$ at $x \in \mathbb{R}^m$ in the direction $u \in \mathbb{R}^m$ as follows:

$$
\begin{aligned}
f_D''(x; u) &:= \text{Limsup}_{t \downarrow 0}\{(2/t^2)(f(x + tu) - f(x) - tf'(x)u)\} \\
&:= \{y \in \mathbb{R}^n : \exists \{t_k\}_{k=1}^\infty \text{ such that } t_k \downarrow 0 \text{ and} \\
&\qquad (2/t_k^2)(f(x + t_k u) - f(x) - t_k f'(x)u) \to y\}.
\end{aligned}
$$

Theorem 1.1. [[13], Thm. 5 (Necessary conditions)] Assume that $f : \mathbb{R}^m \to \mathbb{R}^n$ is a $C^{1,1}$ function and let $\triangle(x) := \{\xi \in \mathbb{R}^n : \xi f'(x) = 0, \|\xi\| = 1\}$. Let $x \in \mathbb{R}^m$ be a w–minimizer for f. Then for every $u \in S_{\mathbb{R}^m}$ the following two conditions are satisfied:

(i) $\triangle(x) \cap C' \neq \emptyset$;

(ii) if $f'(x)u \in -(C \setminus \text{int } C)$ then
$$\min_{y \in f_D''(x;u)} \max\{\langle \xi, y \rangle : \xi \in \triangle(x) \cap C'\} \geq 0.$$

Theorem 1.2. [[13] Thm. 5 (Sufficient conditions)] Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a function of class $C^{1,1}$, $x \in \mathbb{R}^m$, $\triangle(x) = \{\xi \in \mathbb{R}^n : \xi f'(x) = 0, \|\xi\| = 1\}$. Suppose that $\triangle(x) \cap C' \neq \emptyset$ and that for every $u \in S_{\mathbb{R}^m}$ one of the following two conditions is satisfied:

(a) $f'(x)u \notin -C$;

(b) $f'(x)u \in -(C \setminus \text{int } C)$ and
$$\min_{y \in f_D''(x;u)} \max\{\langle \xi, y \rangle : \xi \in \triangle(x) \cap C'\} > 0.$$

Then x is an isolated minimizer of second-order for f.

Both previously mentioned theorems has been stated also for the constrained optimization problems in [12] again with the $C^{1,1}$ assumption.

Later, Khanh and Tuan showed [20] that for Theorems 1.1 and 1.2 the local Lipschitz assumption of derivatives is not needed.

Recall that the Fréchet derivative of $f : \mathbb{R}^m \to \mathbb{R}^n$ is said to be *calm* at $x_0 \in \mathbb{R}^m$ if there are $L > 0$ and a neighbourhood U of x_0 such that

$$\|f'(x) - f'(x_0)\| \leq L\|x - x_0\|, \quad \forall x \in U.$$

Further, we recall the definition of *second-order Hadamard directional derivative of $f : \mathbb{R}^m \to \mathbb{R}$ at $x \in \mathbb{R}^m$ in the direction $u \in \mathbb{R}^m$*:

$$\mathrm{d}^2 f(x; u) = \lim_{t \downarrow 0, v \to u} \frac{f(x + tv) - f(x) - t\mathrm{d}f(x; u)}{t^2/2},$$

where

$$\mathrm{d}f(x; u) = \lim_{t \downarrow 0, v \to u} \frac{f(x + tv) - f(x)}{t}.$$

For a scalar function $f : \mathbb{R}^m \to \mathbb{R}$ we define *lower second-order Hadamard directional derivative at $x \in \mathbb{R}^m$ in the direction $u \in \mathbb{R}^m$* by the following way:

$$\mathrm{d}^2 f(x; u) = \liminf_{t \downarrow 0, v \to u} \frac{f(x + tv) - f(x) - t\mathrm{d}f(x; u)}{t^2/2}.$$

The second-order Hadamard directional derivative is closely connected with another generalized second-order derivative which is defined for a function $f : \mathbb{R}^m \to \mathbb{R}^n$ at $x \in \mathbb{R}^m$ in the direction $u \in \mathbb{R}^m$ as follows (we suppose that f is Fréchet differentiable at x):

$$
\begin{aligned}
f_H''(x; u) \quad &:= \quad \mathrm{Limsup}_{t \downarrow 0, v \to u}\{(2/t^2)(f(x + tv) - f(x) - tf'(x)u)\} \\
&:= \quad \{y \in \mathbb{R}^n : \exists\{t_k\}_{k=1}^\infty, \{v_k\}_{k=1}^\infty \text{ such that } t_k \downarrow 0, v_k \to u \text{ and} \\
&\qquad (2/t_k^2)(f(x + t_k v_k) - f(x) - t_k f'(x)u) \to y\}\,.
\end{aligned}
$$

It follows immediately from the definitions that $f_D''(x; u) \subset f_H''(x; u)$. The following two theorems are unconstrained versions of Theorems 4.1 and 4.2 from [20]. Thus we state them without proofs.

Theorem 1.3. Assume that $f : \mathbb{R}^m \to \mathbb{R}^n$ is a continuously differentiable function near $x \in \mathbb{R}^m$ and let $\triangle(x) := \{\xi \in \mathbb{R}^n : \xi f'(x) = 0, \|\xi\| = 1\}$. Let x be a w-minimizer for f. Then for every $u \in S_{\mathbb{R}^m}$ the following two conditions are satisfied:

(i) $\triangle(x) \cap C' \neq \emptyset$;
(ii) if $f'(x)u \in -(C \setminus \mathrm{int}\, C)$ then
$\qquad \min_{y \in f_H''(x; u)} \max\{\langle \xi, y \rangle : \xi \in \triangle(x) \cap C'\} \geq 0.$

Theorem 1.4. Let the Fréchet derivative of a function $f : \mathbb{R}^m \to \mathbb{R}^n$ be calm at $x \in \mathbb{R}^m$, $\triangle(x) = \{\xi \in \mathbb{R}^n : \xi f'(x) = 0, \|\xi\| = 1\}$. Suppose that $\triangle(x) \cap C' \neq \emptyset$ and that for every $u \in S_{\mathbb{R}^m}$ one of the following two conditions is satisfied:

(a) $f'(x)u \notin -C$;
(b) $f'(x)u \in -(C \setminus \mathrm{int}\, C)$ and
$\qquad \min_{y \in f_D''(x; u)} \max\{\langle \xi, y \rangle : \xi \in \triangle(x) \cap C'\} > 0.$

Then x is an isolated minimizer of second-order for f.

Now, Theorems 1.3 and 1.4 generalize Theorems 1.1 and 1.2 respectively. In the sequel we will show another generalization of Theorem 1.1 (see Thm. 3.2) and we will generalize Theorem 1.4 (see Thm. 3.4).

We say that $f : \mathbb{R}^m \to \mathbb{R}$ is ℓ–*stable* at $x \in \mathbb{R}^m$ if there is a neighbourhood U of x and a constant $K > 0$ such that

$$
|f^\ell(y; h) - f^\ell(x; h)| \leq K\|y - x\|, \quad \forall y \in U, \ \forall h \in S_{\mathbb{R}^m},
$$

where $f^\ell(y; h) = \liminf_{t \downarrow 0}\{(f(y + th) - f(y))/t\}$.

The properties of the scalar functions that are ℓ-stable at some point were studied in [4–6].

Theorem 1.5. [[6] Thm. 2)] Let $f : \mathbb{R}^m \to \mathbb{R}$ be an ℓ-stable function at $x \in \mathbb{R}^m$. Then f is continuous near x.

The following theorem was originally proved in [4], Theorem 6 for the functions that are ℓ-stable at some point and continuous near this point.

Theorem 1.6. [[6] Thm. 3] Let a function $f : \mathbb{R}^m \to \mathbb{R}$ be ℓ-stable at $x \in \mathbb{R}^m$. If $f^\ell(x; h) = 0$ and

$$\liminf_{t \downarrow 0} \frac{f(x + th) - f(x) - t f^\ell(x; h)}{t^2/2} > 0,$$

for every $h \in S_{\mathbb{R}^m}$, then x is an isolated minimizer of second-order for f. Conversely, each isolated minimizer of second-order satisfies these sufficient conditions.

2. ℓ-STABILITY FOR VECTOR FUNCTIONS

Let us generalize the notion of ℓ-stability for vector functions.

Definition 2.1. Let $f : \mathbb{R}^m \to \mathbb{R}^n$, $x, u \in \mathbb{R}^m$, $\xi \in \mathbb{R}^n$. We define a lower Dini derivative at x in the direction u with respect to ξ by

$$f^\ell(x; u)(\xi) := \liminf_{t \downarrow 0} (\langle \xi, f(x + tu) - f(x) \rangle)/t.$$

The function f is said to be ℓ-stable at x if there are a neighbourhood U of x and a constant $K > 0$ such that

$$y \in U, u \in S_{\mathbb{R}^m}, \xi \in \Gamma \implies |f^\ell(y; u)(\xi) - f^\ell(x; u)(\xi)| \leq K \|y - x\|. \qquad (2.1)$$

It follows immediately from Definition 2.1 that $f^\ell(y; u)(\xi)$ is finite for every y sufficiently near x, for every $u \in S_{\mathbb{R}^m}$, and for every $\xi \in \Gamma$. In fact, we can say more, see formula (2.4).

It is not difficult to observe that each function f which is of class $C^{1,1}$ on a neighbourhood of a point x is also ℓ-stable at x. The reverse implication is not true (see Ex. 3.1). The rest of Section 2 is devoted to some properties of functions which are ℓ-stable at some point. We will show that each such function is strictly differentiable at this point (Prop. 2.2) and that a certain inequality holds (see the inequality (2.10)).

At first, we will generalize Theorem 1.5 for vector functions.

Theorem 2.1. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a function that is ℓ-stable at $x \in \mathbb{R}^m$. Then f is continuous near x.

Proof. For an arbitrary $\xi \in \Gamma$ we can consider a function $g_\xi : \mathbb{R}^m \to \mathbb{R}$ given by

$$g_\xi(y) = \langle \xi, f(y) \rangle, \quad \forall y \in \mathbb{R}^m.$$

We can calculate that for every $y \in \mathbb{R}^m$, and for every $h \in \mathbb{R}^m$ we have

$$f^\ell(y; h)(\xi) = g_\xi^\ell(y; h).$$

Since f is ℓ-stable at x, the function g_ξ is also ℓ-stable at x. It follows from Theorem 1.5 that the function g_ξ is continuous on some neighbourhood of x. The

well known facts from convex analysis imply that *span* $\Gamma = \mathbb{R}^n$ (consult *e.g.* [17], p. 11). Then, for every $k \in \{1, 2, \ldots, n\}$, there exist vectors $\xi^1, \xi^2, \ldots \xi^r \in \Gamma$, and $\alpha_1, \alpha_2, \ldots, \alpha_r, r \leq n$ such that

$$(0, \ldots, \underbrace{1}_{k-th}, \ldots, 0\}) = \sum_{l=1}^r \alpha_l \xi^l.$$

We suppose that $f^1, f^2, \ldots, f^n$ are the components of f, *i.e.* $f = (f^1, f^2, \ldots, f^n)$. Then

$$
\begin{aligned}
f^k(y) &= \langle (0, \ldots, \underbrace{1}_{k-th}, \ldots, 0), f(y) \rangle = \langle \sum_{l=1}^r \alpha_l \xi^\ell, f(y) \rangle \\
&= \sum_{l=1}^r \alpha_l \langle \xi^l, f(y) \rangle = \sum_{l=1}^r \alpha_l g_{\xi_l}(y).
\end{aligned}
$$

The function f^k is continuous as a linear combination of continuous functions for every $k \in \{1, 2, \ldots, n\}$. Therefore the function $f = (f^1, f^2, \ldots, f^n)$ is continuous on some neighbourhood of x. $\qquad\square$

From the result [4], Lemma 4, we can easily derive the following lemma.

Lemma 2.1. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function on an open subset $U \subset \mathbb{R}^m$ containing a segment $[a, b] := \{x \in \mathbb{R}^m : x = \lambda a + (1 - \lambda)b, \lambda \in [0, 1]\}$ and let $\xi \in \mathbb{R}^n$. Then there are points $\gamma_1, \gamma_2 \in (a, b) := \{x \in \mathbb{R}^m : x = \lambda a + (1 - \lambda)b, \lambda \in (0, 1)\}$ such that

$$f^\ell(\gamma_1; b - a)(\xi) \leq \langle \xi, f(b) - f(a) \rangle \leq f^\ell(\gamma_2; b - a)(\xi). \qquad (2.2)$$

Lemma 2.2. Let
$$L := \min_{c \in S_{\mathbb{R}^n}} \max_{\xi \in \Gamma} |\langle \xi, c \rangle|.$$
Then $L > 0$ and for each $a, b \in \mathbb{R}^n$ there is $\xi \in \Gamma$ such that

$$\|a - b\| \leq (1/L)|\langle \xi, a - b \rangle|. \qquad (2.3)$$

Proof. To prove the result, let us introduce a sublinear functional defined by

$$p(c) := \max_{\xi \in \Gamma} |\langle \xi, c \rangle|, \quad c \in \mathbb{R}^n.$$

Now it is clear that p is continuous and $p(c) > 0$ for every $c \in S_{\mathbb{R}^n}$ because the interior of C' is nonempty. Since $S_{\mathbb{R}^n}$ is compact, one deduces $L > 0$. $\qquad\square$

Proposition 2.1. Let a function $f : \mathbb{R}^m \to \mathbb{R}^n$ be ℓ-stable at $x \in \mathbb{R}^m$. Then f is Lipschitz on a neighbourhood of x.

Proof.

Step 1. In the first step we will show that

$$\alpha := \sup\{|f^\ell(x;u)(\xi)| : u \in S_{\mathbb{R}^m}, \xi \in \Gamma\} < +\infty. \tag{2.4}$$

Suppose on the contrary that there are sequences $\{\xi_k\}_{k=1}^\infty \subset \Gamma$ and $\{u_k\}_{k=1}^\infty \subset S_{\mathbb{R}^m}$ with the property:

$$\lim_{k\to\infty} |f^\ell(x;u_k)(\xi_k)| = +\infty. \tag{2.5}$$

When passing to subsequences we may assume that $u_k \to u_0$, $\xi_k \to \xi_0$ as $k \to +\infty$, where $u_0 \in S_{\mathbb{R}^m}$, $\xi_0 \in \Gamma$. In accordance with (2.5) we may assume without loss of generality that

$$\lim_{k\to\infty} f^\ell(x;u_k)(\xi_k) = -\infty.$$

The second case (*i.e.*, $\lim_{k\to+\infty} f^\ell(x;u_k)(\xi_k) = +\infty$) can be treated similarly. Using Theorem 2.1, we can find an open neighbourhood U of x, $\delta > 0$, and $K > 0$ such that f is continuous on U, $x + \delta B_{\mathbb{R}^m} \subset U$ and

$$|f^\ell(y;u)(\xi) - f^\ell(x;u)(\xi)| \le K\|y - x\| \tag{2.6}$$

for any $y \in U$, $u \in S_{\mathbb{R}^m}$ and $\xi \in \Gamma$.

By Lemma 2.1 and (2.6) for every $k \in \mathbb{N}$ there is $\gamma_k \in (x, x + \delta u_k)$ such that

$$\begin{aligned}
\langle \xi_k, f(x + \delta u_k) - f(x) \rangle \le \delta f^\ell(\gamma_k; u_k)(\xi_k) &\le \delta(f^\ell(x;u_k)(\xi_k) + K\|x - \gamma_k\|) \\
&\le \delta(f^\ell(x;u_k)(\xi_k) + K\delta). \tag{2.7}
\end{aligned}$$

Letting $k \to \infty$ in (2.7), we get

$$\langle \xi_0, f(x + \delta u_0) - f(x) \rangle \le \delta \lim_{k\to\infty} (f^\ell(x;u_k)(\xi_k) + K\delta) = -\infty,$$

a contradiction, because $\langle \xi_0, f(x+\delta u_0)-f(x) \rangle \in \mathbb{R}$. This proves the property (2.4).

Step 2. Now we will attempt to show that f is Lipschitz on an open ball $x+\delta B_{\mathbb{R}^m}$. Let $a, b \in x + \delta B_{\mathbb{R}^m}$ be arbitrary, then using Lemmas 2.1 and 2.2 we get $\gamma \in (a, b)$, $\xi \in \Gamma$ such that

$$\begin{aligned}
\|f(b)-f(a)\| \le (1/L)|f^\ell(\gamma; b-a)(\xi)| &\le (1/L)(K\|\gamma-x\|\|b-a\|+|f^\ell(x;b-a)(\xi)|) \\
&\le \{(K\delta+\alpha)/L\}\|b-a\|.
\end{aligned}$$

This completes the proof. $\square$

Recall that if Fréchet differentiable at x function $f : \mathbb{R}^m \to \mathbb{R}^n$ satisfies

$$f'(x)u = \lim_{y\to x, t\downarrow 0}\{(f(y + tu) - f(y))/t\}, \quad \forall u \in S_{\mathbb{R}^m},$$

and this convergence is uniform for $u \in S_{\mathbb{R}^m}$, then f is said to be *strictly differentiable at x*.

Proposition 2.2. Let a function $f : \mathbb{R}^m \to \mathbb{R}^n$ be ℓ-stable at $x \in \mathbb{R}^m$. Then f is strictly differentiable at x.

Proof. It follows that f is Lipschitz on a neighbourhood of x and thus by the famous Rademacher theorem there is a sequence $\{x_i\}_{i=1}^{+\infty}$ in $\mathbb{R}^m$ such that $x_i \to x$ as $i \to +\infty$ and for every $i \in \mathbb{N}$ there exists the Fréchet derivative $f'(x_i)$.

We will show that for arbitrary $u \in \mathbb{R}^m$, $\{f'(x_i)u\}_{i=1}^\infty$ is Cauchy sequence. Using Lemma 2.2 and the ℓ-stability of f at x we can find for every $j, k \in \mathbb{N}$ a $\xi_{j,k} \in \Gamma$ such that

$$
\begin{aligned}
0 \;\leq\; & \|f'(x_j)u - f'(x_k)u\| \leq (1/L)|\langle \xi_{j,k}, f'(x_j)u - f'(x_k)u \rangle| \\
\leq\; & (1/L)|\langle \xi_{j,k}, f'(x_j)u \rangle - f^\ell(x; u)(\xi_{j,k})| \\
& + (1/L)|f^\ell(x; u)(\xi_{j,k}) - \langle \xi_{j,k}, f'(x_k)u \rangle| \\
\leq\; & (K/L)\|u\|(\|x_j - x\| + \|x_k - x\|).
\end{aligned}
$$

The last expression converges to zero as $j, k \to +\infty$, whence $\{f'(x_i)u\}_{i=1}^\infty$ is Cauchy sequence for each fixed $u \in \mathbb{R}^m$.

Then we can put $Tu = \lim_{i \to \infty} f'(x_i)u$ for each $u \in \mathbb{R}^m$. Note that the last formula defines T as an element of $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ and the definition of Tu does not depend on the choice of the sequence $\{x_i\}$.

Now, for any $\xi \in \Gamma$, $u \in \mathbb{R}^m$, and for any $i \in \mathbb{N}$, we have:

$$
\begin{aligned}
|\langle \xi, Tu \rangle - f^\ell(x; u)(\xi)| \leq & |\langle \xi, Tu - f'(x_i)u \rangle| + |\langle \xi, f'(x_i)u \rangle - f^\ell(x; u)(\xi)| \\
\leq & \|Tu - f'(x_i)u\| + K\|x_i - x\|\|u\|.
\end{aligned}
$$

Letting $i \to \infty$, we arrive at $\langle \xi, Tu \rangle = f^\ell(x; u)(\xi)$ for any $\xi \in \Gamma$ and $u \in \mathbb{R}^m$.

In what follows we show the strict differentiability of f at x. If y lies sufficiently close to the point x, $t > 0$ is sufficiently close to 0 and $u \in S_{\mathbb{R}^m}$, then due to Lemma 2.1 for every $\xi \in \Gamma$ there exist two points $\gamma_{1,\xi}, \gamma_{2,\xi} \in (y, y + tu)$ such that

$$
(1/t)f^\ell(\gamma_{1,\xi}; tu)(\xi) \leq \langle \xi, (f(y + tu) - f(y))/t \rangle \leq (1/t)f^\ell(\gamma_{2,\xi}; tu)(\xi).
$$

Then

$$
\begin{aligned}
f^\ell(\gamma_{1,\xi}; u)(\xi) - f^\ell(x; u)(\xi) \leq & \langle \xi, (f(y + tu) - f(y))/t \rangle - f^\ell(x; u)(\xi) \\
\leq & f^\ell(\gamma_{2,\xi}; u)(\xi) - f^\ell(x; u)(\xi). \qquad (2.8)
\end{aligned}
$$

Furthermore, due to the ℓ-stability at the point x we infer

$$
|f^\ell(\gamma_{i,\xi}; u)(\xi) - f^\ell(x; u)(\xi)| \leq K\|\gamma_{i,\xi} - x\|, \qquad (2.9)
$$

for $i = 1, 2$ and for each $u \in S_{\mathbb{R}^m}$.

Using inequalities (2.8), (2.9), and the fact that $\langle \xi, Tu \rangle = f^\ell(x; u)(\xi)$ for any $\xi \in \Gamma$ and $u \in \mathbb{R}^m$, we obtain

$$-K\|\gamma_{1,\xi} - x\| \leq \langle \xi, [(f(y + tu) - f(y))/t] - Tu \rangle \leq K\|\gamma_{2,\xi} - x\|,$$

for any $\xi \in \Gamma$, y sufficiently close to x, $u \in S_{\mathbb{R}^m}$, and for any $t > 0$ sufficiently close to 0, where $\gamma_{1,\xi}, \gamma_{2,\xi} \in (y, y + tu)$. Therefore if $t \to 0+$, $y \to x$, then due to Lemma 2.2, we have that

$$\|[(f(y + tu) - f(y))/t] - Tu\| \leq (1/L)\sup_{\xi \in \Gamma}\langle \xi, [(f(y + tu) - f(y))/t] - Tu \rangle \to 0,$$

as $t \to 0+$ uniformly for $u \in S_{\mathbb{R}^m}$. The previous formula yields

$$\lim_{y \to x, t \downarrow 0}\{(f(y + tu) - f(y))/t\} = Tu = f'(x)u,$$

and this limit is uniform for $u \in S_{\mathbb{R}^m}$. $\qquad\qquad\qquad\qquad\qquad\square$

The following lemma plays a very important role in optimality conditions presented in Section 3. It has been already shown in [13] that inequality (2.10) holds for $C^{1,1}$ functions, but here we generalize it also for ℓ-stable functions.

Lemma 2.3. Let a function $f : \mathbb{R}^m \to \mathbb{R}^n$ be ℓ-stable at $x \in \mathbb{R}^m$. Then there is $\alpha > 0$ such that

$$\forall u, w \in \mathbb{R}^m \; \exists \delta > 0 \; \forall t \in (0, \delta) : \|(2/t^2)(f(x + tu) - f(x) - tf'(x)u)$$
$$- (2/t^2)(f(x + tw) - f(x) - tf'(x)w)\|$$
$$\leq \alpha(\|u\| + \|w\|)\|u - w\|. \qquad (2.10)$$

Proof. Note that by Proposition 2.2 f is strictly differentiable at x. Having in mind Theorem 2.1, we suppose that U denotes a neighbourhood of x on which f is continuous and (2.1) is true. Let us consider an auxiliary function $g : \mathbb{R}^m \to \mathbb{R}^n$ defined by $g(z) := f(z) - f'(x)z, z \in \mathbb{R}^m$. If we fix $u, w \in \mathbb{R}^m$, then there are $\delta > 0$, $\tau > 0$ such that $x + \tau B_{\mathbb{R}^m} \subset U$ and $\delta u \in \tau B_{\mathbb{R}^m}$, $\delta w \in \tau B_{\mathbb{R}^m}$. So that for every $t \in (0, \delta)$ we have $x + tu \in x + \tau B_{\mathbb{R}^m}$, $x + tw \in x + \tau B_{\mathbb{R}^m}$. Due to Lemmas 2.1, 2.2 and ℓ-stability there exist $\gamma_t \in (x + tu, x + tw)$ and $\xi_t \in \Gamma$ such that

$$\|(2/t^2)(f(x + tu) - f(x) - tf'(x)u) - (2/t^2)(f(x + tw) - f(x) - tf'(x)w)\|$$
$$= (2/t^2)\|g(x + tu) - g(x + tw)\|$$
$$\leq (2/Lt^2)|\langle \xi_t, g(x + tu) - g(x + tw)\rangle|$$
$$\leq (2/Lt)|g^\ell(\gamma_t; u - w)(\xi_t)|$$
$$= (2/Lt)|f^\ell(\gamma_t; u - w)(\xi_t) - \langle \xi_t, f'(x)(u - w)\rangle|$$
$$\leq (2K/Lt)\|\gamma_t - x\|\|u - w\|.$$

Since for some $\mu \in (0, 1)$ we have $\gamma_t = \mu(x + tu) + (1 - \mu)(x + tw)$, then we can derive:

$$
\begin{aligned}
\|\gamma_t - x\| &= \|\mu(x + tu) + (1 - \mu)(x + tw) - x\| \\
&= t\|\mu u + (1 - \mu)w\| \\
&\leq t(\mu\|u\| + (1 - \mu)\|w\|) \\
&\leq t(\|u\| + \|w\|).
\end{aligned}
$$

Now, letting $\alpha := (2K/L) > 0$ we come to our inequality (2.10).   □

3. Optimality conditions

In the final paragraph, we will state some results from the article [13] in a more general form with the use of the ℓ-stability notion. We have omitted proofs of these results because they can be proved by similar methods as in the above mentioned papers. In [13], Theorem 4, the authors proved a result giving second-order necessary conditions for weak efficiency where the objective function f was assumed to be of class $C^{1,1}$. The proof of this result mainly relies on a certain inequality like the inequality (2.10) in Lemma 2.3. Thus, it is not surprising that the following version of that theorem also holds.

Theorem 3.1. Let a function $f : \mathbb{R}^m \to \mathbb{R}^n$ be ℓ-stable at $x \in \mathbb{R}^m$. Let x be a w–minimizer for f. Then the following two conditions are satisfied for each $u \in S_{\mathbb{R}^m}$:

    (i) $f'(x)u \notin -\mathrm{int}\ C$;
    (ii)   if $f'(x)u \in -(C \setminus \mathrm{int}\ C)$ then for all $y \in f''_D(x; u)$ it holds
          $\mathrm{conv}\{y, \mathrm{Im}f'(x)\} \cap (-\mathrm{int}\ C) = \emptyset.$

The proof of Theorem 1.1 is based on Theorem 3.1 which was originally proved for $C^{1,1}$ functions in [13]. In view of current assumptions of Theorem 3.1, we can reformulate Theorem 1.1 in a more general fashion.

Theorem 3.2. Let a function $f : \mathbb{R}^m \to \mathbb{R}^n$ be ℓ-stable at $x \in \mathbb{R}^m$. Let x be a w-minimizer of f. Then if we put

$$\triangle(x) := \{\xi \in \mathbb{R}^n : \xi f'(x) = 0, \|\xi\| = 1\},$$

for every $u \in \mathbb{R}^m$, the following two conditions are satisfied:

    (i) $\triangle(x) \cap C' \neq \emptyset$;
    (ii)   if $f'(x)u \in -(C \setminus \mathrm{int}\ C)$ then
          $\min_{y \in f''_D(x;u)} \max\{\langle \xi, y \rangle : \xi \in \triangle(x) \cap C'\} \geq 0.$

In the next theorem we provide a similar generalization of another necessary condition formulated in [13].

Theorem 3.3. Let a function $f : \mathbb{R}^m \to \mathbb{R}^n$ be ℓ-stable at $x \in \mathbb{R}^m$. Let x be an isolated minimizer of second-order for f. Then $\triangle(x) \cap C' \neq \emptyset$, and for every $u \in S_{\mathbb{R}^m}$ one of the following two conditions is satisfied:

  (i) $f'(x)u \notin -C$;
  (ii) $f'(x)u \in -(C \setminus \mathrm{int}\, C)$ and
           $\min_{y \in f''_D(x;u)} \max\{\langle \xi, y \rangle : \xi \in \triangle(x) \cap C'\} > 0.$

The second-order sufficient condition given in Theorems 1.2 and 1.4 can also be proven under an ℓ-stability assumption. With respect to Proposition 2.2, Theorem 1.6 is a special (scalar) case of Theorems 3.4 and 3.3.

Theorem 3.4. Let a function $f : \mathbb{R}^m \to \mathbb{R}^n$ be ℓ-stable at $x \in \mathbb{R}^m$. Let $\triangle(x) := \{\xi \in \mathbb{R}^n : \xi f'(x) = 0, \|\xi\| = 1\} \cap C' \neq \emptyset$ and suppose that for every $u \in S_{\mathbb{R}^m}$ one of the following two conditions is satisfied:

  (i) $f'(x)u \notin -C$;
  (ii) $f'(x)u \in -(C \setminus \mathrm{int}\, C)$ and
           $\min_{y \in f''_D(x;u)} \max\{\langle \xi, y \rangle : \xi \in \triangle(x) \cap C'\} > 0.$

Then x is an isolated minimizer of second-order for f.

The following example shows a function which satisfies the assumptions of Theorem 3.4 but it is not of class C^1 and thus we cannot use neither Theorem 1.2 nor Theorem 1.4.

Example 3.1. Consider a function $f : \mathbb{R} \to \mathbb{R}^2$ minimized with respect to a cone $C := \mathbb{R}^2_+$ and defined by

$$f(x) := (g(x), 0), \quad \forall x \in \mathbb{R},$$

where $g(x) := \int_0^{|x|} \varphi(u)\mathrm{d}u$, $x \in \mathbb{R}$, and

$$\varphi(u) := \begin{cases} 1, & \text{if } u > 1, \\ 1/(k+1), & \text{if } 1/(k+1) < u \leq 1/k, k = 1, 2, \ldots, \\ 0, & \text{if } u = 0. \end{cases}$$

The function $\varphi(u)$ is nondecreasing on $[0, \infty)$, hence we can consider that integral in a Riemann sense. Note that g is even and convex. Also observe that $f'(0) = (g'(0), 0) = (0, 0)$, and for the right and left derivative of g it holds respectively

$$g'_+(1/k) = 1/k, \quad g'_-(1/k) = 1/(k+1), \quad k = 1, 2, \ldots$$

Moreover, for each $x \in \mathbb{R}$ the following holds:

$$\int_0^{|x|} \varphi(u)\mathrm{d}u \geq \int_0^{|x|} (u/(1+u))\mathrm{d}u = |x| - \ln(1 + |x|).$$

Further, we have:

$$\langle \xi, f'(0)u \rangle = \langle (\xi_1, \xi_2), (0, 0) \rangle = 0, \quad \forall \xi \in \mathbb{R}^2, \ \forall u \in \mathbb{R}.$$

This implies that

$$\triangle(0) \cap C' = \{\xi \in \mathbb{R}^2 : \xi_1 \geq 0, \xi_2 \geq 0, \xi_1^2 + \xi_2^2 = 1\} \neq \emptyset.$$

If $u \in S_{\mathbb{R}^1} = \{-1, 1\}$, $\xi \in \Gamma = \{\xi \in \mathbb{R}^2 : \xi_1 \geq 0, \xi_2 \geq 0, \xi_1^2 + \xi_2^2 = 1\}$, $x > 0$, then

$$|f^\ell(x; u)(\xi) - f^\ell(0; u)(\xi)| = |f^\ell(x; u)(\xi)| \quad = |\xi_1 g'_\pm(x)| \leq |g'_\pm(x)| = \lim_{u \to x^\pm} \varphi(u) \leq x.$$

Similarly, for $x < 0$ we have:

$$|f^\ell(x; u)(\xi) - f^\ell(0; u)(\xi)| = |f^\ell(x; u)(\xi)| \leq -x.$$

Thus f is continuous on $\mathbb{R}$ and ℓ-stable at $x^0 = 0$. Let $u = +1 \in S_{\mathbb{R}^1}$, $\overline{\xi} = (1, 0) \in \triangle(0) \cap C'$, and let

$$y = \lim_{k \to \infty} (2/t_k^2) \left(f(0 + t_k \cdot 1) - f(0) - t_k f'(0) \cdot 1 \right) = \lim_{k \to \infty} (2/t_k^2) f(t_k),$$

where $t_k \downarrow 0$. Then $y \in f''_D(0; 1)$ and for every $k \in \mathbb{N}$ we have:

$$\langle \overline{\xi}, (2/t_k^2) f(t_k) \rangle = (2/t_k^2) g(t_k) \geq (2/t_k^2)(t_k - \ln(1 + t_k)).$$

Letting $k \to +\infty$, we get

$$1 \leq \langle \overline{\xi}, y \rangle \leq \max\{\langle \xi, y \rangle : \xi \in \triangle(0) \cap C'\},$$

where $y \in f''_D(0; 1)$ was arbitrary and hence

$$0 < 1 \leq \min_{y \in f''_D(0;1)} \max\{\langle \xi, y \rangle : \xi \in \triangle(0) \cap C'\}.$$

In a similar fashion we can do this for $u = -1$ and hence the condition (ii) from Theorem 3.4 is satisfied. Therefore $x^0 = 0$ is an isolated minimizer of second-order for f.

Finishing our paper, we would like to remark that in some papers published recently (*e.g.* [11,18,19]) the authors stated optimality conditions under weaker regularity of considered functions than ℓ-stability at some point but in terms of generalized second-order derivative of Hadamard type as for example $\mathrm{d}^2 f(x; u)$, $\mathrm{d}_\ell^2 f(x; u)$ or $f''_H(x; u)$.

We note that second-order Hadamard derivative $\mathrm{d}^2 f(x; u)$ do not coincide with the classical ones even in the case of C^2 functions, for more details see considerations in [13].

On the other hand, Theorem 3.4 can not be used to state that for example the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} -x, & \text{for } x < 0, \\ x^2, & \text{for } x \geq 0, \end{cases}$$

attains its strict local minimum at 0. In contrast to the following theorem which is a special case of a result proved in [11].

Theorem 3.5. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a function. If for each $u \in S_{\mathbb{R}^m}$ one of the following condition holds

   (i) $\mathrm{d}f(x; u) > 0$;

   (ii) $\mathrm{d}f(x; u) = 0$ and $\mathrm{d}_l^2 f(x; u) > 0$, then x is an isolated minimizer of second-order.

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