THE BEHAVIOR OF A MARKOV NETWORK WITH RESPECT TO AN ABSORBING CLASS: THE TARGET ALGORITHM

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Abstract. In this paper, we face a generalization of the problem of finding the distribution of how long it takes to reach a "target" set T of states in Markov chain. The graph problems of finding the number of paths that go from a state to a target set and of finding the n-length path connections are shown to belong to this generalization. This paper explores how the state space of the Markov chain can be reduced by collapsing together those states that behave in the same way for the purposes of calculating the distribution of the hitting time of T. We prove the existence and the uniqueness of a optimal projection for this aim which extends the results given in [G. Aletti and E. Merzbach, *J. Eur. Math. Soc. (JEMS)* **8** (2006) 49–75], together with the existence of a polynomial algorithm which reaches this optimum. Some applied examples are presented. Markov complexity is defined an tested on some classical problems to demonstrate the deeper understanding that is made possible by this approach.

Keywords. Markov time of the first passage, stopping rules, Markov complexity, graphs and networks.

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1. INTRODUCTION

Let $(X_n)_n$ be an homogeneous Markov chain on the state set E. E is assumed to be at most countable. Fixed a subset $T \subseteq E$, called *target*, we face in this

paper the problem of finding the distribution of the hitting time of T for any initial distribution, called *target problem* (TP). A TP is hence defined as a triple (E, P, T), where P is the transition probability matrix.

The problem of finding general closed-forms for different kinds of waiting problems is widely studied. As an example, Ebneshahrashoob and Sobel [6] derived distributional results for the random variables in the case of Bernoulli trials. Several extensions have appeared recently to Markov-dependent trials via combinatorial or Markov chain embedding (see, *e.g.* Aki and Hirano [3]; Antzoulakos and Philippou [7]; Koutras and Alexandrou [1]) and in general closed-forms by Stefanov [9].

Stefanov and Pakes [10] explicitly derive the joint distributions of various quantities associated with the time of reaching an arbitrary pattern of zeros and ones in a sequence of Bernoulli (or dependent) trials. Their methodology is based on first embedding the problem into a more general framework for an appropriate finite-state Markov chain with one absorbing state and then treating that chain using tools of exponential families. A new approach was given in [2], where it was proved that there exists an optimal projection for any given TP which leaves the probability of reaching the target set unchanged. In [2], Section 2, some examples had been solved underlying the new method, giving solutions to some cumbersome combinatorial problems. It was stated that numerically efficient algorithms for the optimal reduction have meaning of real "chaos reduction" algorithms. In fact, their proof was essentially based on the class of equivalence relationship on a set, and the Markov chain may be in practice so big that numerical computations can be not practicable, since a nontrivial reducing map is not a local search. In the framework of [2], a projection map is an equivalence relationship S on the indexing set E s.t.

- $\forall e_i \in T,\ e_i\, S\, e_j \iff e_j \in T$;
- for any $\{e_i, e_j, e_k\} \subseteq E$: $e_i\, S\, e_j$ we have

$$\sum_{e_l\, S\, e_k} P(e_i, e_l) = \sum_{e_l\, S\, e_k} P(e_j, e_l)$$

  where T (the absorbing target class) and $P : E \times E \to \mathbb{R}_+$ are given (P is the Markov matrix of the network).

Note that we may have $P(e_1, e_3) \neq P(e_2, e_3)$ but $P(e_1, e_3) + P(e_1, e_4) = P(e_2, e_3) + P(e_2, e_4)$, which means that $e_1\, S\, e_2$ may be found if *we know* that $e_3\, S\, e_4$. Moreover, it is not difficult to build examples where the only nontrivial compressing map corresponds to the optimal nontrivial projection. Therefore, searching for a compressing map appears as a non–polynomial search, in the sense that we have to look at the whole set of equivalent relations on E. The Hidden Markov Model framework may help to understand what a projection map is. Given a TP (E, P, T), an equivalency $S : E \to E/S$ is a projection map if one class is formed by T and the process $Y_n = X_n/S$ is still a Markov chain, for any initial distribution.
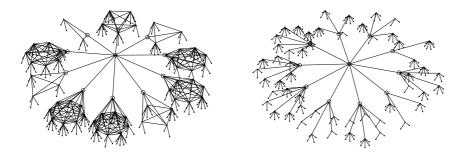
FIGURE 1. Networks that may be compressed (see [2]).

On the converse, the framework in [2] regards a huge class of problems which occur in many real situations. We recall here how this class of problem may appear.

(1) In finance the filter rule for trading is a special case of the Markov chain stopping rule suggested in [2] (see, *e.g.*, [8]).

(2) "When enough is enough"! For example, an insured has an accident only occasionally. How many accidents in a specified number of years should be used as a stopping time for the insured (in other words, when the insurance contract should be discontinued).

(3) *State dependent markov chains.* Namely, the transition probabilities are given in terms of the history. In many situations, the matrix of the embedded problem may be reduced. This is the applied part of the paper: When a big matrix can be reduced then this paper provides a polynomial algorithm for reaching the optimal compressed problem.

(4) *Small-world Networks.* Given one of the networks as in Figure 1 (either as Markov network or as a graph), is it possible to reduce it in polynomial time and to preserve the law of reaching a given absorbing state?

There are of course many other such examples (*e.g.*, records: Arnold *et al.* [4] and optimization: Cairoli and Dalang [5]). This fact motivates the research of both exact solutions and ε-approximations in polynomial time for the "target problem".

In this paper, we first extend the TP framework given in [2] to a more general framework, where a semiring R replaces $\mathbb{R}_+$ and the indexing set of the network may be countable. The structure of semiring allows us to choose both rings (see $\widetilde{\mathbb{R}}$ and $\widetilde{\mathbb{N}}$ in Exs. 2.3 and 2.4 below) and (boolean) lattices, as in Example 2.5. The target problem will therefore include stopping laws' problems on Markov finite/countable network (see Ex. 2.3), counting problems on how many paths of length n reach a given target (see Ex. 2.4) and general graph connection (see Ex. 2.4).

The existence and the uniqueness of the optimal projection for any TP is proven in Section 3. This is in fact an extension of the results in [2] to the new framework and, moreover, the new proof underlines links between topological properties of the TP and the projection maps.

We note that, in huge chains, P may be computable while P^n cannot, due to sparsity problems. The main contribution of this article is stated in Section 4, where a polynomial algorithm for reaching the optimal projection of any given target problem (E, P, T) is found. In the last section, we first extend this method to multi–target problems (where $\mathbf{T} = \{T_1, \ldots, T_k\}$) and then we test it on some classical stopping Markov problems. The behavior of the given chain with respect to the target T may be better understood by looking at the optimal projection (see Ex. 5.2).

2. R-NETWORKS

Let E be an at most countable indexing set. We denote by $\mathcal{E}$ the discrete topology on E, *i.e.* we take $\mathcal{E} = \mathfrak{P}(E)$ where $\mathfrak{P}(set)$ is the power set of *set*. Recall that to each topological space (set, top), it is possible to associate a unique closure operator $cl_{\mathrm{top}} : \mathfrak{P}(set) \to \mathfrak{P}(set)$ where the closed sets are the fixed points of the closure operator. Throughout the whole paper, we denote by $|A|$ the cardinality of a given set A.

Let $(R, +, 0, *, 1)$ be a semiring with identity and zero[1]. Moreover, we assume that (a) R is closed, commutative and distributive under $|E|$-sums; (b) R is equipped with a $T0$-topology $\mathcal{R}$ of closed sets. Finally, we denote by $\mathrm{Hom}(E \times E, R)$ the set of the all functions from $E \times E$ to R.

Remark 2.1. The structure of topological space will play a central rôle in the next section. In fact, a topological proof of the existence and uniqueness theorem is presented. We made this choice since this can be a way for projecting some uncountable indexed problems.

Definition 2.2. A monoid $(\mathcal{M}_E(R), \cdot, \mathbf{1})$ is called *R-monoid over E* if
- $\mathcal{M}_E(R) \subseteq \mathrm{Hom}(E \times E, R)$;
- the product $\cdots$ is defined as the matrix multiplication, *i.e.*

$$(P_1 \cdot P_2)(e_i, e_j) = \sum_{e \in E}(P_1(e_i, e) * P_1(e, e_j)); \qquad (2.1)$$

We call *R-network over* $\mathcal{M}_E(R)$ any couple (E, P), where $P \in \mathcal{M}_E(R)$.

As a consequence of the previous definition, for any $P \in \mathcal{M}_E(R)$,

(1) there exists the nth power

$$P^n = \underbrace{P \cdot P \cdot \cdots \cdot P \cdot P}_{n \text{ times}} \in \mathcal{M}_E(R), \qquad n \in \mathbb{N},$$

and therefore (E, P^n) is again a R-network over $\mathcal{M}_E(R)$.

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[1]$(R, +, 0, *, 1)$ is a semiring with identity and zero if $(R, *, 1)$ is a monoid, $(R, +, 0)$ is a commutative monoid and $x * (y + z) = (x * y) + (x * z)$, $(y + z) * x = (y * x) + (z * x)$, and $0 * x = x * 0 = 0$.

(2) we can set $P^0 = \mathbf{1}$, and the diagonal of units of the ring is unity $\mathbf{1}$ of the monoid:

$$\text{if} \quad \mathbf{1}((e_i, e_j)) = \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{if } i \neq j; \end{cases} \quad \text{then} \quad \mathbf{1} \cdot P = P \cdot \mathbf{1} = P$$

(3) there exists the function $\mathbb{P}_P : (E \times \mathcal{E}) \to R$ so defined

$$\mathbb{P}_P(e, A) = \sum_{a \in A} P(e, a), \tag{2.2}$$

s.t.

$$\mathbb{P}_{P_1 \cdot P_2}(e, A) = \sum_{e_i \in E} P_1(e, e_i) * \mathbb{P}_{P_2}(e_i, A), \qquad \forall e \in E, A \subseteq E. \tag{2.3}$$

Equation (2.3) is a (Chapman–Kolmogorov type) convolution equation. It is a consequence of the fact that R is closed, commutative and distributive under $|E|$-sums.

Example 2.3 (Markov matrices). Let $\widetilde{\mathbb{R}} = \mathbb{R}_+ \cup \{0\} \cup \{\infty\}$, we can put $*$ as the real product, $+$ the sum, and we define a) $\infty\, x = x\, \infty = \infty\ \forall x \neq 0$ (while, by definition, $\infty\, 0 = 0\, \infty = 0$); b) $\infty + x = x + \infty = \infty\ \forall x \in \mathbb{R}$. A nontrivial monoid $\mathcal{M}_E(\widetilde{\mathbb{R}})$ strictly contained in the ring of nonnegative matrices is the space of stochastic matrices, *i.e.* $P \in \mathcal{M}_E(\widetilde{\mathbb{R}})$ if $\sum_{e_i \in E} P(e, e_i) = 1\ \forall e \in E$. Note that $\mathcal{M}_E(\widetilde{\mathbb{R}}) \subsetneq \mathrm{Hom}(E \times E, \widetilde{\mathbb{R}})$. When P is a stochastic matrix, P^n is clearly the n-time transition matrix; while $\mathbb{P}_P(e, A)$ represent the probability of reaching the set A starting from e.

Now, let E be the set of the vertexes of graphs. A (symmetric) $E \times E$–matrix with values in $\{0, 1\}$ may be seen as the adjacency matrix of a simple (un)directed graph. If the matrix has values in $\mathbb{N}$, it may be seen as the adjacency matrix of a pseudograph (*i.e.*, a $\mathbb{N}$ edge-labelled directed graph).

Different semirings may be used for different aims.

Example 2.4 (graphs on the ring $\widetilde{\mathbb{N}}$). Take $\widetilde{\mathbb{N}} = \mathbb{N}_+ \cup \{0\} \cup \{\infty\}$, $*$ and $+$ as in Example 2.3 and take $\mathcal{M}_E(R) = Hom(E \times E, R)$. In this case, if P is an adjacency matrix, $P^n(e_i, e_j)$ counts how many paths of length n start in e_i and reach e_j; while $\mathbb{P}_P(e, A)$ represent number of paths that reach the set A starting from e.

Another interesting example on graphs is the following, where the semiring is a lattice (it is not a ring).

Example 2.5 (graphs on the boolean lattice). Take $R = \{0, 1\}$, $*$ and $+$ as the logical AND and OR, respectively. In this case, if P is the adjacency matrix of a simple (un)directed graph, then $P^n(e_i, e_j)$ counts if there exists a n-length path starting in e_i and reaching e_j; while $\mathbb{P}_P(e, A)$ represent if e is connected with the set A.

3. Target problem

Form now on, let (E, P) be a fixed $(R, +, 0, *, 1)$-network over a given network monoid $(\mathcal{M}_E(R), \cdot, \mathbf{1})$.

Definition 3.1. A closed subset $T \in \mathcal{E}$ is a *target set on* (E, P) if it is an absorbing class: $\mathbb{P}_P(t, T) = 1$ and $\mathbb{P}_P(t, E \setminus T) = 0$, for any $t \in T$. We call *target problem* any (E, P, T), where T is a target set on (E, P).

Note that one may always change the matrix P in order to satisfy Definition 3.1, by defining

$$P'(e_1, e_2) = \begin{cases} P(e_1, e_2), & \text{if } e_1 \notin T; \\ 1, & \text{if } e_1 \in T \text{ and } e_1 = e_2; \\ 0, & \text{if } e_1 \in T \text{ and } e_1 \neq e_2. \end{cases}$$

We define now the problem of reducing the number of the state space E, in terms of equivalence relationships on E.

Let $\widetilde{E}$ be the set of all equivalence relations on E and let $V, S \in \widetilde{E}$. We say that $V \preceq S$ if $a_1 V a_2$ implies $a_1 S a_2$. The relation $\preceq$ is just set-theoretic inclusion between equivalence relations, since any relation is a subset of $E \times E$.

Definition 3.2. Let (E, P, T) be a target problem. An equivalence relationship $S \in \widetilde{E}$ is called *compatible projection with respect to the target problem* (E, P, T) if

(1) $\forall e \in T, e S e_j \iff e_j \in T$;
(2) there exists $P^* \in \mathcal{M}_{E/S}(R)$ s.t. the following diagram commutes

$$
\begin{array}{ccc}
E \times \mathcal{E} & & \\
\Big\uparrow{\scriptstyle (Id_E, \pi^{-1})} & \searrow^{\mathbb{P}_P} & \\
E \times E/S & \xrightarrow[\mathbb{P}_P \circ (Id_E, \pi^{-1})]{\quad\quad} & R \\
\Big\downarrow{\scriptstyle (\pi, Id_{E/S})} & \nearrow_{P^*} & \\
E/S \times E/S & &
\end{array}
\qquad (3.1)
$$

where $\pi : E \to E/S$ is the canonical projection.

We call $\mathcal{S} = \mathcal{S}(E, P, T)$ the set of all compatible projections.

Definition 3.2 states when we can project our target problem (E, P, T) in the smaller one $(E/F, P^*, t = \pi(T))$. Note that the set $\mathcal{S}$ is trivially nonempty: the following relationship

$$e_1 S e_2 \iff \{e_1, e_2\} \subseteq T \text{ or } e_1 = e_2$$

belongs to $\mathcal{S}$. The problem is: is there a maximum element of the $\preceq$–partially ordered set $\mathcal{S}$?

Before stating the main theorem of this section, we underline the strong link between the set of the topologies on E and $\widetilde{E}$. In fact, note that each topology $\mathcal{P}$ on E induces a natural equivalence relationship on E, given by the relationship $e_i \, V \, e_j \iff cl_{\mathcal{P}}(e_i) = cl_{\mathcal{P}}(e_j)^2$. Moreover, fixed $V \in \widetilde{E}$, we denote by $(E/V, \widetilde{\mathcal{E}}_V)$ the topological quotient space: $\widetilde{\mathcal{E}}_V$ is the finest topology for which the canonical projection $\pi_V : (E, \mathcal{E}) \to (E/V, \widetilde{\mathcal{E}}_V)$ is continuous. Define $\mathcal{P}_V$ as the coarsest topology for which the canonical projection $\pi_V : (E, \mathcal{P}_V) \to (E/V, \widetilde{\mathcal{E}}_V)$ is continuous. The topology $\mathcal{P}_V$ on E will be called *canonical topology associated to V*.

We have the following lemma.

Lemma 3.3. *For any $V \in \widetilde{E}$ and $\mathcal{P}_V$ as above,*

$$e_i \, V \, e_j \iff cl_{\mathcal{P}_V}(e_i) = cl_{\mathcal{P}_V}(e_j) \,, \forall e_i, e_i \in E. \tag{3.2}$$

Proof. Since $\mathcal{E} = \mathfrak{P}(E)$, then $\widetilde{\mathcal{E}}_V = \mathfrak{P}(E/F)$. Therefore, $\mathcal{P}_V$ is generated by the union of the disjoint classes of equivalence, which is the thesis. $\square$

Now we state and prove the theorem of existence and uniqueness of the optimal projection.

Theorem 3.4. *For any target problem (E, P, T), there exists the optimal projection, i.e. $\exists S \in \mathcal{S}$ s.t. $V \preceq S, \forall V \in \mathcal{S}$, where $\mathcal{S} = \mathcal{S}(E, P, T)$.*

Proof. For any $V \in \mathcal{S}$, let $\mathcal{P}_V$ the canonical topology associated to V. We define the topology $\mathcal{P}$:

$$\mathcal{P} = \bigcap \{ \mathcal{P}_V, V \in \mathcal{S} \}, \tag{3.3}$$

and the associated equivalence relationship $\overline{S}$ on E:

$$e_i \, \overline{S} \, e_j \iff cl_{\mathcal{P}}(e_i) = cl_{\mathcal{P}}(e_j), \qquad \forall e_i, e_j \in E. \tag{3.4}$$

The closed sets of $(E, \mathcal{P})$ are the common fixed points of all $\{cl_V, V \in \mathcal{S}\}$, identified by the fixed points of the closure operator $cl_{\mathcal{P}} : \mathfrak{P}(E) \to \mathfrak{P}(E)$. Therefore, by (3.2) and (3.3), a subset $F \subseteq E$ belongs to $\mathcal{P}$ if and only if it is a union of equivalence classes of $E/V, \forall V \in \mathcal{S}$. Moreover, since $\mathcal{P} \subseteq \mathcal{P}_V$, then $V \preceq \overline{S}, \forall V \in \mathcal{S}$. We are going to show now the main step of the proof, namely that $\overline{S} \in \mathcal{S}$.

By Definition 3.2.1 $cl_{\mathcal{P}_V}(e) = T, \forall V \in \mathcal{S} \, \forall e \in T$. Therefore, $cl_{\mathcal{P}}(e) = T \, \forall e \in T$, *i.e.* Definition 3.2.1 holds for $\overline{S}$.

Now, define $\overline{P}(e_1, e) = \mathbb{P}_P(e_1, cl_{\mathcal{P}}(e))$ (clearly, $\overline{P} : E \times E \to R$). Since $cl_{\mathcal{P}}(e)$ is an equivalency class, $\overline{P} = \mathbb{P}_P \circ (Id_E, \pi^{-1} \circ \pi)$, where $\pi : E \to E/\overline{S}$ is the canonical projection. What remains to prove is $\overline{P}(e_1, e) = \overline{P}(e_2, e)$, if $cl_{\mathcal{P}}(e_1) = cl_{\mathcal{P}}(e_2)$.

Take $e, e_1, e_2 \in E$ s.t.

$$r_1 = \overline{P}(e_1, e) \neq \overline{P}(e_2, e) = r_2. \tag{3.5}$$

---

[2]Here and in the sequel, we use the notation $cl.(e_i)$ instead of $cl.(\{e_i\})$ for reading simplicity.

Now, by Definition 3.2.2, for any $V \in \mathcal{S}$, $\exists P_V^* : E/V \times E/V \to R$ s.t. (3.1) holds. Since $\mathcal{P} \subseteq \mathcal{P}_V$, then $C = \pi_V^{-1}(\pi_V(C))$, for any C closed set in $\mathcal{P}$.

Again, by Definition 3.2.2,

$$\widetilde{P}_e(\cdot) := \overline{P}(\cdot, e) = \mathbb{P}_P(\cdot, cl_{\mathcal{P}}(E)) = \mathbb{P}_P(\cdot, \pi_V^{-1}(\pi_V(cl_{\mathcal{P}}(e)))) \qquad (3.6)$$

$$= \sum_{f \in \pi_V(cl(e))} P_V^*(\pi_V(\cdot), f) =: Q_{e,V}(\pi_V(\cdot)), \qquad (3.7)$$

where again $\pi_V : E \to E/V$ is the canonical projection.

We recall that if $\widetilde{\mathcal{E}}_V = \mathfrak{P}(E/F)$ denotes the discrete topology on E/V, $\pi_V : (E, \mathcal{P}_V) \to (E/V, \widetilde{\mathcal{E}}_V)$ is continuous by definition of $\mathcal{P}_V$. Therefore, the function $Q_{e,V} \circ \pi_V : (E, \mathcal{P}_V) \to (R, \mathcal{R})$ is continuous. Since $(R, \mathcal{R})$ is $T0$, $\exists C \in \mathcal{R}$ s.t. either $r_1 \in C$ and $r_2 \notin C$, or $r_2 \in C$ and $r_1 \notin C$; r_1 and r_2 are given in (3.5)). $\pi_V^{-1}(Q_{e,V}^{-1}(C))$ is hence a closed set in $(E, \mathcal{P}_V), \forall V \in \mathcal{S}$. Note that, since $\widetilde{P}_e^{-1}(C) = \pi_V^{-1}(Q_{e,V}^{-1}(C))$, the set $\pi_V^{-1}(Q_{e,V}^{-1}(C))$ does not depend on V. As a consequence, since

$$\widetilde{P}_e^{-1}(C) \in \mathcal{P}_V, \forall V \in \mathcal{S}$$

then $\widetilde{P}_e^{-1}(C) \in \mathcal{P}$, i.e. $cl(e_1) \neq cl(e_2)$. $\qquad \square$

4. Target algorithm

The proof of the optimal solution's existence was based on the fact that the set of compatible equivalence $\widetilde{E}$ has its $\preceq$–join in $\widetilde{E}$.

We act again on the set of equivalence relations on a set, but we will focus our attention on E/S instead of on E, where S is the optimal projection, whose existence and uniqueness is given in Theorem 3.4.

We state the following trivial lemma without proof.

Lemma 4.1. *Let A be a set. $|\cdot|$ is monotone with respect to $\preceq$ in $\widetilde{A}$, i.e.*

$$\forall S_1, S_2 \in \widetilde{A}, \qquad S_1 \preceq S_2 \Longrightarrow |A/S_1| \geq |A/S_2|. \qquad (4.1a)$$

Moreover, if $|A| < \infty$, $|\cdot|$ is strictly monotone:

$$|A/S_1| = |A/S_2|, S_1 \preceq S_2 \Longrightarrow S_1 = S_2. \qquad (4.1b)$$

Let (E, T, P) be a a target problem. We denote by $F_\pi \in \widetilde{E}$ the optimal projection, by $\pi : E \to E/F_\pi$ the canonical projection, by F the quotient set E/F_π and by

P^* the function defined in (3.1):

$$
\begin{array}{ccc}
E \times \mathcal{E} & & \\
\big\uparrow {\scriptstyle (Id_E,\pi^{-1})} & \searrow {\scriptstyle \mathbb{P}_P} & \\
E \times E/F_\pi & \xrightarrow{\ \ \mathbb{P}_P \circ (Id_E,\pi^{-1})\ \ } & R \\
\big\downarrow {\scriptstyle (\pi, Id_{E/F_\pi})} & \nearrow {\scriptstyle P^*} & \\
E/F_\pi \times E/F_\pi & &
\end{array}
\tag{4.2}
$$

Let $\widetilde{F}_t$ be the set of all equivalence relations on F such that the target state $t \in F$ is left "alone": *i.e.* $R \in \widetilde{F}_t$ if $t\,R\,f \iff f = t$.

Note that $\widetilde{F}_t \hookrightarrow \widetilde{E}$; more precisely, since $E \xrightarrow{\pi} F$, we have:

$$
\widetilde{F}_t \xhookrightarrow{\ j\ } \mathfrak{P}(F \times F) \xrightarrow{\ (\pi,\pi)^{-1}\ } \mathfrak{P}(E \times E).
$$

It is obvious that $(\pi,\pi)^{-1} \circ j : \widetilde{F}_t \to \mathfrak{P}(E \times E)$ defines an equivalence relationship on E. With this inclusion in mind, we can state that $\widetilde{F}_t \subseteq \widetilde{E}$:

$$
\widetilde{F}_t \longleftrightarrow \{R \in \widetilde{E} \colon F_\pi \preceq R\},
\tag{4.3}
$$

and hence we refer to $\widetilde{F}_t$ both as a class of equivalence relations on F and on E. We call I_F the identity relationship on F:

$$
f_1\, I_F\, f_2 \iff f_1 = f_2
$$

i.e. I_F is just F_π on $\widetilde{F}_t$, and let M_E be maximal relationship on $\widetilde{F}_t$. As a relationship in $\widetilde{E}$ it becomes

$$
e_1\, M_E\, e_2 \iff \{e_1, e_2\} \subseteq T \text{ or } \{e_1, e_2\} \subseteq (E \setminus T).
$$

Clearly, $M_E \in \widetilde{F}_t$ and $I_F \preceq S \preceq M_E$, $\forall S \in \widetilde{F}_t$ (*i.e.*, I_F and M_E are the minimal and maximal relationships on $\widetilde{F}_t$). Note that we can compute M_E without knowing F.

We build now a monotone operator $\mathcal{F}$ on $\widetilde{E}$ (the algorithm's idea will be to reach F_π – unknown – starting from M_E – known –).

Let $\mathcal{F} : \widetilde{E} \to \widetilde{E}$ so defined: for any $S \in \widetilde{E}$, let $s_1, s_2, \ldots$ be the classes of equivalence of E induced by S. Define

$$
e_1 \mathcal{F}_{s_i} e_2 \iff \mathbb{P}_P(e_1, s_i) = \mathbb{P}_P(e_2, s_i)
$$

$$
\mathcal{F}(S) = \bigcap_{i=1,2,\ldots} \mathcal{F}_{s_i} \cap S.
$$

Now, we focus our attention on the action of $\mathcal{F}$ on $\widetilde{F}_t$. First, we prove that $\mathcal{F}: \widetilde{F}_t \to \widetilde{F}_t$ and then we will show that the unique fixed point of $\mathcal{F}: \widetilde{F}_t \to \widetilde{F}_t$ is I_F.

Lemma 4.2. $\mathcal{F}_{|_{\widetilde{F}_t}}: \widetilde{F}_t \to \widetilde{F}_t$ is a $\preceq$-monotone operator on $\widetilde{F}_t$.
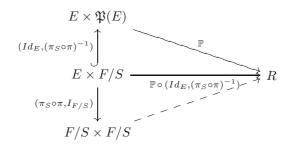
Proof. First, note that $\widetilde{F}_t$ is trivially closed for intersection. Let $S \in \widetilde{F}_t \hookrightarrow \widetilde{E}$. Since every $s_i \in E/S$ is a subset of F, the existence of P^* in (4.2) ensures that $\mathcal{F}_{s_i} \in \widetilde{F}_t$. Therefore, $\mathcal{F}(S) \in \widetilde{F}_t$. Since $\mathcal{F}(S) \subseteq S$, it is a monotone operator. $\qquad\square$

Theorem 4.3. I_F is the unique fixed point of $\mathcal{F}: \widetilde{F}_t \to \widetilde{F}_t$.

Proof. F_π is trivially a fixed point for $\mathcal{F}: \widetilde{E} \to \widetilde{E}$ by (4.2) and therefore I_F is a fixed point for $\mathcal{F}: \widetilde{F}_t \to \widetilde{F}_t$.

Now, let $S \in \widetilde{F}_t$ s.t. $S = \mathcal{F}(S)$. Define the canonical map $\pi_S: F \to F/S$. We have

- $(\pi_S \circ \pi)(E) = F/S$;
- $(\pi_S \circ \pi)(T) = \pi_S(\pi(T)) = \pi_S(t) = t$,
  $(\pi_S \circ \pi)^{-1}(t) = \pi^{-1}(\pi_S^{-1}(t)) = \pi^{-1}(t) = T$;
- $S \subseteq \mathcal{F}_{s_i}$ $\forall i$, and hence the following diagram commutes:

$$
\begin{array}{ccc}
E \times \mathfrak{P}(E) & & \\
\Big\uparrow{\scriptstyle (Id_E,(\pi_S\circ\pi)^{-1})} & \searrow^{\mathbb{P}} & \\
E \times F/S & \xrightarrow{\mathbb{P}\circ(Id_E,(\pi_S\circ\pi)^{-1})} & R \\
\Big\downarrow{\scriptstyle (\pi_S\circ\pi,I_{F/S})} & \nearrow & \\
F/S \times F/S & &
\end{array}
$$

Since π is the optimal projection such that (3.1) holds, then $F/S \equiv F$, *i.e.* $S = I_F$.
$\qquad\square$

4.1. Polynomial-time algorithm

In this section, we assume that $|E| = M < \infty$[3]. As a consequence, $|E/S| \leq M$ for any equivalence relationship $S \in \widetilde{E}$.

Theorem 4.4. *Let* (E, T, P) *be given and let* N *be the cardinality of* F, *i.e.* $N = |E/F_\pi|$. *Then* $\mathcal{F}^{N-2}(M_E) = F_\pi$, *where* $\mathcal{F}^n := (\mathcal{F} \circ \mathcal{F}^{n-1})$ *and* $\mathcal{F}^0$ *is the identity operator (i.e.* $\mathcal{F}^0(R) = R$, $\forall R$).

Proof. First, note that $\mathcal{F}^n(M_E) \in \widetilde{F}_t$ $\forall n$ (by Lem. 4.2). Therefore, we may consider $\mathcal{F}^n: \widetilde{F}_t \to \widetilde{F}_t$. We have $I_F \preceq \mathcal{F}^{n+1}(M_E) \preceq \mathcal{F}^n(M_E) \preceq M_E$ $\forall n$.

---

[3]Note that in this case, assumption (a) (page 234) on the semiring is superfluous.

Let $C_n = |E/(\mathcal{F}^n(M_E))|$. We now prove by induction on n that

$$\mathcal{F}^n(M_E) \neq I_F \quad \Longrightarrow \quad C_n > n+1. \tag{4.4}$$

For $n = 0$, $C_0 = 2$ (otherwise $E = T$ and the problem is trivial). For the induction step, if $\mathcal{F}^n(M_E) \neq I_F$, then $C_n > n+1$. $\mathcal{F}$ is a monotone operator, then $\mathcal{F}^{n+1}(M_E) \preceq \mathcal{F}^n(M_E)$ and hence $C_{n+1} \geq C_n$ by (4.1a). Now, if $C_{n+1} = C_n$, then $\mathcal{F}^{n+1}(M_E) = \mathcal{F}^{n+1}(M_E)$ by (4.1b) which means that $\mathcal{F}^n(M_E) = I_F$ by Theorem 4.3. Therefore, (4.4) holds.

If $\mathcal{F}^{N-3}(M_E) = I_F$, then $\mathcal{F}^{N-2}(M_E) = I_F$ by Theorem 4.3. As a consequence of (4.4), if $\mathcal{F}^{N-3}(M_E) \neq I_F$, then $C_{N-2} \geq N = |F|$. Therefore, since $\mathcal{F}^{N-1}(M_E) \in \widetilde{F}_t$, we have $\mathcal{F}^{N-2}(M_E) = I_F$ by (4.1b), $i.e.$ $\mathcal{F}^{N-2}(M_E) = F_\pi$. $\square$

Remark 4.5. Note that the operator $\mathcal{F}$ may be computed in a $|E|$-polynomial time. Corollary 4.4 ensures that

$$\underbrace{\mathcal{F} \circ \mathcal{F} \circ \cdots \circ \mathcal{F}}_{\text{at most } |E/F_\pi| - 2 \text{ times } (\leq |E|)}$$

will reach F, given any triple (E, T, P).

5. Extension to multiple targets and examples of Markov networks

The previous results and those in [2] may be extended to multiple target problems. More precisely, let $\mathbf{T} = \{T_1, T_2, \ldots\}$ be target disjoint sets on the same R-network (E, P) over $\mathcal{M}_E(R)$. We are interested in the optimal $\{T_1, T_2, \ldots\}$-compatible relationship S such that (3.1) holds.

The answer is trivial, since each target class T_i defines its equivalence relationship S_i. It is not difficult to show that the required set S is just $S = \cap S_i$. In the sequel, we call *Markov complexity* of the problem $(E, \mathbf{T}, P)$ the cardinality of the optimal set E/S.

Example 5.1 (negative binomial distribution)**.** Repeat independently a game with probability p of winning until you win n games.

Let $S_n = \sum_{i=1}^n Y_i$, where $\{Y_i, i \in \mathbb{N}\}$ is a sequence of i.i.d. bernoulli random variable with $Prob(\{Y_i = 1\}) = 1 - Prob(\{Y_i = 0\}) = p$. Our interest is engaged by the computation of the probability of reaching n starting from 0. Let $E =$

$\{0, 1, \ldots, n\}$ be the set of levels we have reached. We have

	0	1	2	$\ldots$	$n-1$	$n = T$
0	$(1-p)$	p	0	$\ldots$	0	0
1	0	$(1-p)$	p	$\ddots$	0	0
2	0	0	$(1-p)$	$\ddots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$
$n-1$	0	0	0	$\ldots$	$(1-p)$	p
$n = T$	0	0	0	$\ldots$	0	1

$=: P.$

Since the length of the minimum path for reaching the target state n from different states is different, the problem is irreducible by [2], Proposition 31. Its Markov complexity is $n + 1$. We show how the target algorithm reaches this solution. It starts from M_E which divides E into the two classes $C_0^0 = T = \{n\}$ and $C_1^0 = E \setminus T = \{0, \ldots, n-1\}$. For any $e \in E$, the algorithm computes $p_e = (\mathbb{P}(e, C_0^0), \mathbb{P}(e, C_1^0))$, finding

$$p_e = \begin{cases} (1, 0) & \text{if } e = n; \\ (p, 1-p) & \text{if } e = n-1; \\ (0, 1) & \text{if } e \in \{0, \ldots, n-2\}. \end{cases}$$

Therefore, $\mathcal{F}(M_E)$ divides E into three classes $C_0^1 = T = \{n\}$ and $C_1^1 = \{n-1\}$ and $C_2^1 = \{0, \ldots, n-1\}$. Since the number of classes is increased, the algorithm does not stop. It computes again $p_e = (\mathbb{P}(e, C_0^1), \mathbb{P}(e, C_1^1), \mathbb{P}(e, C_2^1))$ and it finds four classes. It stops (in this case) only when it divides all the states (in $n + 1$ steps, the maximum allowed).

Example 5.2 (random walk on a cube). A particle performs a symmetric random walk on the vertices of a unit cube, *i.e.*, the eight possible positions of the particle are $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 1, 0)$,…,$(1, 1, 1)$, and from its current position, the particle has a probability of $1/3$ of moving to each of the 3 neighboring vertices. This process ends when the particle reaches $(0, 0, 0)$ or $(1, 1, 1)$.

Let $T_1 = (0, 0, 0)$, $T_2 = (1, 1, 1)$. The following transition matrix

	$(0,0,0)$	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$	$(1,1,0)$	$(1,0,1)$	$(0,1,1)$	$(1,1,1)$
$(0,0,0)$	1	0	0	0	0	0	0	0
$(1,0,0)$	1/3	0	0	0	1/3	1/3	0	0
$(0,1,0)$	1/3	0	0	0	1/3	0	1/3	0
$(0,0,1)$	1/3	0	0	0	0	1/3	1/3	0
$(1,1,0)$	0	1/3	1/3	0	0	0	0	1/3
$(1,0,1)$	0	1/3	0	1/3	0	0	0	1/3
$(0,1,1)$	0	0	1/3	1/3	0	0	0	1/3
$(1,1,1)$	0	0	0	0	0	0	0	1

can be easily reduced on

	t_1	f_1	f_2	t_2
t_1	1	0	0	0
f_1	1/3	0	2/3	0
f_2	0	2/3	0	1/3
t_2	0	0	0	1

where $t_i = T_i$ and $f_i = \{e = (e_1, e_2, e_3) : \sum e_j = i\}$, *i.e.* its Markov complexity is 4. We show how the target algorithm reaches this results on the more general matrix

	$(0,0,0)$	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$	$(1,1,0)$	$(1,0,1)$	$(0,1,1)$	$(1,1,1)$
$(0,0,0)$	1	0	0	0	0	0	0	0
$(1,0,0)$	1/3	0	0	0	α_1	$2/3 - \alpha_1$	0	0
$(0,1,0)$	1/3	0	0	0	α_2	0	$2/3 - \alpha_2$	0
$(0,0,1)$	1/3	0	0	0	0	α_3	$2/3 - \alpha_3$	0
$(1,1,0)$	0	α_4	$2/3 - \alpha_4$	0	0	0	0	1/3
$(1,0,1)$	0	α_5	0	$2/3 - \alpha_5$	0	0	0	1/3
$(0,1,1)$	0	0	α_6	$2/3 - \alpha_6$	0	0	0	1/3
$(1,1,1)$	0	0	0	0	0	0	0	1

where $0 \leq \alpha_i \leq 2/3$, for any $i = 1, \ldots, 6$. The algorithm starts from M_E which divides E into the three classes $C_0^0 = T_1 = \{(0,0,0)\}$, $C_1^0 = T_2 = \{(1,1,1)\}$ and $C_2^0 = E \backslash (T_1 \cup T_2)$. For any $e \in E$, the algorithm computes $p_e = (\mathbb{P}(e, C_0^0), \mathbb{P}(e, C_1^0), \mathbb{P}(e, C_2^0))$, finding

$$
p_e = \begin{cases}
(1,0,0) & \text{if } e \in f_0 = C_0^0; \\
(1/3, 0, 2/3) & \text{if } e \in f_1 = \{(e_1, e_2, e_3) : \sum e_j = 1\}; \\
(0, 1/3, 2/3) & \text{if } e \in f_2 = \{(e_1, e_2, e_3) : \sum e_j = 2\}; \\
(0, 1, 0) & \text{if } e \in f_3 = C_1^0.
\end{cases}
$$

Therefore, $\mathcal{F}(M_E)$ divides E into the four classes $\{f_i, i = 0, \ldots, 3\}$. Since the number of classes is increased, the algorithm does not stop. It computes again $p_e = (\mathbb{P}(e, f_0), \mathbb{P}(e, f_1), \mathbb{P}(e, f_2), \mathbb{P}(e, f_3))$ and it finds:

$$
p_e = \begin{cases}
(1, 0, 0, 0) & \text{if } e \in f_0; \\
(1/3, 0, 2/3, 0) & \text{if } e \in f_1; \\
(0, 2/3, 0, 1/3) & \text{if } e \in f_2; \\
(0, 0, 0, 1) & \text{if } e \in f_3.
\end{cases}
$$

The algorithm stops, since the number of classes is not increased, and in p_e we may read the optimal projected matrix P^*.

If we are only interested in the time of stopping (*i.e.* $\mathbf{T} = T_1 \cup T_2$), the previous problem may be reduced to a geometrical one (Markov complexity equal to 2). Clearly, this results hold also for random walk on a d-dimensional cube or symmetric groups.

Example 5.3 (coupon collector's problem). Let n objects $\{e_1, \ldots, e_n\}$ be picked repeatedly with probability p_i that object e_i is picked on a given try, with $\sum_i p_i = 1$. Find the earliest time at which all n objects have been picked at least once.

Let Λ be the set of permutations of the n objects. For a fixed permutation $\lambda = (e_{\lambda_1}, e_{\lambda_2}, \ldots, e_{\lambda_n}) \in \Lambda$ we denote by $E^{\lambda^i} = \{e_{\lambda_1}, e_{\lambda_2}, \ldots, e_{\lambda_i}\}$ the set of the first i-objects in λ (without order!).

Now, let $\mathcal{A}_\lambda$ be the set of all the paths that have picked all the n objects with the order given by λ. In Pattern–Matching Algorithms framework (see [2], Sect. 3 and Rem. 18), the *stopping λ-rule* we consider here is denoted by

$$T_\lambda = e_{\lambda_1}\{E^{\lambda^1}\}^* e_{\lambda_2}\{E^{\lambda^2}\}^* \cdots e_{\lambda_{n-1}}\{E^{\lambda^{n-1}}\}^* e_{\lambda_n},$$

and it becomes a target state of an embedded Markov problem on a graph (see [2], Sect. 3). The stopping class for the Coupon Collector's Problem is accordingly $\mathbf{T} = \cup_{\lambda \in \Lambda} T_\lambda$.

It is not difficult to show that the general Coupon Collector's Problem may be embedded into a Markow network of $2^n - 1$-nodes (its general Markov hard complexity), where $E = \{\mathbf{T}, \{E^{\lambda^i} : \lambda \in \Lambda, 1 < i < n\}$, the transition matrix is given by

$$P(E^{\lambda^i}, E^{\zeta^j}) = \begin{cases} \sum_{k \in \lambda^i} p_k, & \text{if } E^{\lambda^i} = E^{\zeta^j}; \\ p_k, & \text{if } j = i+1 \text{ and } E^{\zeta^j} = \{E^{\lambda^i}, e_k\}; \\ 0, & \text{otherwise}; \end{cases}$$

$\lambda, \zeta \in \Lambda$ and $\text{Prob}(\{X^1 = e_k\}) = p_k$. Note that this matrix is not in general reducible.

If some p_i are equal, *i.e.*, when some states act with the same law with respect to the problem, the set E can be projected into a minor one. The easiest case (namely, $p_i = 1/n \; \forall i$) is projected into a n-state problem:

	f_1	f_2	f_3	$\ldots$	f_{n-1}	T
f_1	$1/n$	$1 - 1/n$	0	$\ldots$	0	0
f_2	0	$2/n$	$1 - 2/n$	$\ddots$	0	0
f_3	0	0	$3/n$	$\ddots$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$
f_{n-1}	0	0	0	$\ldots$	$(n-1)/n$	$1/n$
T	0	0	0	$\ldots$	0	1

$\Big\} =: P$

with $\text{Prob}(\{X^1 = f_1\}) = 1$. Here, $f_i = \{E^{\lambda^i}, \lambda \in \Lambda\}$. The problem is again irreducible by [2], Proposition 31 and its Markov complexity is n. In general, when we have m different values of $\{p_i, i = 1, \ldots, n\}$ (namely, $q_1, \ldots, q_m$), if $n_m = |k : p_k = q_m|$, then the Markov complexity can be easily proven to be $\prod_{k=1}^m (n_k + 1) - 1$.

6. Conclusions

The behavior of a Markov chain with respect to a target T may be better understood by looking at the optimal projection. In this work we consider a general framework and we describe a polynomial-time algorithm in order to reach the minimal equivalent target problem. Theorem 3.4 can be viewed as an extension of the analogue result stated in [2]. Different examples are provided with their minimal set of states. A MATLAB Version 6.5 of such an algorithm for multitarget **T** may be downloaded at http://www.mat.unimi.it/~aletti/.

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