ESAIM: PS March 2010, Vol. 14, p. 65–92
DOI: 10.1051/ps:2008028

ESAIM: Probability and Statistics
www.esaim-ps.org

LOCAL LIMIT THEOREMS FOR BROWNIAN ADDITIVE FUNCTIONALS AND PENALISATION OF BROWNIAN PATHS, IX

Bernard Roynette[1] and Marc Yor[2, 3]

Abstract. We obtain a local limit theorem for the laws of a class of Brownian additive functionals and we apply this result to a penalisation problem. We study precisely the case of the additive functional: $\left(A_t^- := \int_0^t 1_{X_s<0} \mathrm{d}s, t \geq 0\right)$. On the other hand, we describe Feynman-Kac type penalisation results for long Brownian bridges thus completing some similar previous study for standard Brownian motion (see [B. Roynette, P. Vallois and M. Yor, *Studia Sci. Math. Hung.* **43** (2006) 171–246]).

Mathematics Subject Classification. 60F17, 60G44, 60J25, 60J35, 60J55, 60J57, 60J60,60J65.

Received March 17, 2008.

1. Notations and introduction

1.1. Notations

- $\left(\Omega, \left(X_t, \mathcal{F}_t\right)_{t\geq 0}, \mathcal{F}_\infty = \vee_{t\geq 0} \mathcal{F}_t, P_x(x \in \mathbb{R})\right)$ denotes the canonical realisation of the one-dimensional Wiener process. $\Omega = \mathcal{C}\big([0,\infty[\longrightarrow \mathbb{R}\big)$ is the space of continuous functions on $[0,\infty[$, $\left(X_t, t \geq 0\right)$ the coordinates process on this space, $\left(\mathcal{F}_t, t \geq 0\right)$ its natural filtration and $\left(P_x, x \in \mathbb{R}\right)$ the family of Wiener measures on $\left(\Omega, \mathcal{F}_\infty\right)$, with $P_x(X_0 = x) = 1$. When $x = 0$, we write simply P for P_0.
- We denote by $\left(L_t^x, t \geq 0, x \in \mathbb{R}\right)$ the jointly continuous family of the local times of $\left(X_t, t \geq 0\right)$. We denote $\left(L_t, t \geq 0\right)$ for $\left(L_t^0, t \geq 0\right)$, the (continuous) local time process at level 0 and by $\left(\tau_l, l \geq 0\right)$ its right-continuous inverse:

$$\tau_l := \inf\{s > 0; L_s > l\} \qquad (l \geq 0). \tag{1.1}$$

- To q a positive Radon measure on $\mathbb{R}$, $q \neq 0$, we associate the continuous additive functional:

$$A_t^q := \int_{\mathbb{R}} L_t^x q(\mathrm{d}x). \tag{1.2}$$

Keywords and phrases. Limit theorems for additive functionals, Feynman-Kac functionals, long Brownian bridges.

[1] Institut Elie Cartan, Université Henri Poincaré, BP 239, 54506 Vandoeuvre-lés-Nancy Cedex, France;
`Bernard.Roynette@iecn.u-nancy.fr`

[2] Laboratoire de Probabilités et Modèles Aléatoires, Université Paris VI et VII, 4 place Jussieu, Case 188, 75252 Paris Cedex 05, France

[3] Institut Universitaire de France.

Article published by EDP Sciences

When q admits a density with respect to Lebesgue measure, we keep the former notation by still writing q for the density; we have:

$$A_t^q = \int_0^t q(X_s)\mathrm{d}s \tag{1.3}$$

from the occupation density formula for Brownian motion.

Throughout the following, we shall assume that q satisfies one of the three following hypotheses:

H1. (The integrable case) $\displaystyle\int_{\mathbb{R}} \big(1+|x|\big)\, q(\mathrm{d}x) < \infty$.

H2. (The left unilateral case) $\displaystyle\int_{-\infty}^{0} \big(1+|x|\big)\, q(\mathrm{d}x) < \infty$ and there exists $\alpha < 1$ such that $\displaystyle\varliminf_{x\to\infty} x^{2\alpha} q^{(a)}(x) \geq b > 0$ where $q^{(a)}$ denotes the absolutely continuous part of q.

H3. (The right unilateral case) $\displaystyle\int_{0}^{\infty} \big(1+|x|\big)\, q(\mathrm{d}x) < \infty$ and there exists $\alpha < 1$ such that $\displaystyle\varliminf_{x\to-\infty} |x|^{2\alpha} q^{(a)}(x) \geq b > 0$.

Of course, if the pair $\big((X_t, A_t^q),\ t\geq 0\big)$ satisfies H2 (resp. H3), then the pair $\big((-X_t, A_t^q),\ t\geq\ 0\big)$ satisfies H3 (resp. H2).

1.2. **Introduction**

1.2.1.

In [14], we obtained the following results:

i) Under H1, H2 or H3, for any $\lambda > 0$:

$$\lim_{t\to\infty} \sqrt{t}\, E_x\Big[\mathrm{e}^{\left(-\frac{\lambda}{2}A_t^q\right)}\Big] := \varphi_{\lambda q}(x). \tag{1.4}$$

(Later, we shall give other presentations of $\varphi_{\lambda q}$.)

ii) For any $s \geq 0$ and $\Lambda_s \in \mathcal{F}_s$:

$$\lim_{t\to\infty} \frac{E\Big[1_{\Lambda_s}\mathrm{e}^{\left(-\frac{\lambda}{2}A_t^q\right)}\Big]}{E\Big[\mathrm{e}^{\left(-\frac{\lambda}{2}A_t^q\right)}\Big]} := Q^{(\lambda q)}\big(\Lambda_s\big). \tag{1.5}$$

where formula (1.5) induces a probability $Q^{(\lambda q)}$ on $\Big(\Omega, \mathcal{F}_\infty\Big)$ (see [9,11,13,14] for more details; see also [2] about this penalisation result).

The first part of this work consists in:

- Using the result i) to obtain a limit theorem relative to the law of the additive functional $\big(A_t^q, t\geq 0\big)$. This is the content of Theorem 2.1.
- Obtaining a penalisation result, which is more general than (1.5) *i.e.*, by replacing the exponential function $x \longrightarrow \mathrm{e}^{-\frac{\lambda x}{2}}$ by a more general function. This is the content of Theorem 2.4.

1.2.2.

In Section 3 of this work, we study in detail the situation where $q = 1_{]-\infty,0]}$ *i.e.*, $A_t^q := A_t^- := \int_0^t 1_{X_s<0}\mathrm{d}s$. In particular, we prove a penalisation theorem for long Brownian bridges: this is the content of Theorem 3.1.

1.2.3.

Section 4 of this work is devoted to the study of Feynman-Kac penalisation for long Brownian bridges, which generalizes what we have done in [14] for standard Brownian motion. This is the content of Theorem 4.1.

To summarize, this work extends, in the above directions, our preceding work [14].

2. A local limit theorem for the laws of some Brownian additive functionals and a penalisation result

2.1. A local limit theorem

Theorem 2.1. *Let q satisfy one of the hypotheses H1, H2 or H3, and let $(A_t^q, t \geq 0)$ be defined by (1.2) (or (1.3)). Then, for every $x \in \mathbb{R}$, there exists a positive, σ-finite measure ν_x, carried by $\mathbb{R}_+$, such that:*

$$\sqrt{t}\, P_x(A_t^q \in \mathrm{d}z) \underset{t\to\infty}{\longrightarrow} \nu_x(\mathrm{d}z). \tag{2.1}$$

The convergence in (2.1) is understood in the following sense: for any function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ Borel, and sub-exponential i.e.: there exist two positive constants C_1 and C_2 such that:

$$0 \leq f(x) \leq C_1\, \mathrm{e}^{-C_2 x}.$$

then

$$\sqrt{t}\, E_x\left[f(A_t^q)\right] \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}_+} f(z)\nu_x(\mathrm{d}z).$$

The measure ν_x is characterized by:

$$\int_0^\infty \mathrm{e}^{-\frac{\lambda}{2}y}\nu_x(\mathrm{d}y) = \varphi_{\lambda q}(x). \tag{2.2}$$

2.2. Proof of Theorem 2.1

We first begin with some precisions, taken from [14], (see also Kotani [6]) about $\varphi_{\lambda q}$, which was defined from (1.4) but admits at least another characterization, namely:
$\varphi_{\lambda q}$ is the unique solution of the Sturm-Liouville equation:

$$\varphi''(\mathrm{d}x) = \lambda\varphi(x)q(\mathrm{d}x). \tag{2.3}$$

This equation is taken in the sense of Schwartz distributions, and subject to the following boundary conditions:

$$\text{Under H1.}: \qquad \varphi'(+\infty) = -\varphi'(-\infty) = \sqrt{\frac{2}{\pi}} \tag{2.4}$$

$$\text{Under H2.}: \qquad \varphi'(-\infty) = -\sqrt{\frac{2}{\pi}} \text{ and } \varphi(+\infty) = 0 \tag{2.5}$$

$$\text{Under H3.}: \qquad \varphi'(+\infty) = \sqrt{\frac{2}{\pi}} \text{ and } \varphi(-\infty) = 0. \tag{2.6}$$

Theorem 2.1 is now an immediate consequence of the next lemma.

Lemma 2.2. *Under either of the hypotheses H1, H2, or H3, the function: $\lambda \longrightarrow \varphi_{\lambda q}(x)$ $(\lambda > 0)$ is, for any real x, completely monotone, i.e., it satisfies:*

$$(-1)^n \frac{\partial^n}{\partial\lambda^n}\varphi_{\lambda q}(x) \geq 0. \tag{2.7}$$

Consequently, there exists a positive, σ-finite measure ν_x, carried by $\mathbb{R}_+$, such that:

$$\varphi_{\lambda q}(x) = \int_0^\infty \mathrm{e}^{-\frac{\lambda}{2} z} \nu_x(\mathrm{d}z). \tag{2.8}$$

We shall give two proofs for Lemma 2.2.

2.3. A first proof of Lemma 2.2

We define, for every $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and every real $h \neq 0$:

$$D_h f(\lambda) := \frac{f(\lambda + h) - f(\lambda)}{h} \cdot \tag{2.9}$$

For $f(\lambda) := \exp - \dfrac{\lambda}{2} A_t^q$, we get:

$$(D_h)^n (f)(\lambda) = \mathrm{e}^{-\frac{\lambda A_t^q}{2}} \Big(\frac{\mathrm{e}^{-\frac{A_t^q h}{2}} - 1}{h} \Big)^n$$

and, hence for all $h \neq 0$:

$$(-1)^n (D_h)^n (f)(\lambda) \geq 0. \tag{2.10}$$

Consequently, taking the expectation of the LHS in (2.10), we obtain:

$$\sqrt{t} (-1)^n E_x \big[(D_h)^n \Big(\exp - \frac{\bullet}{2} A_t^q \Big) \big] \geq 0. \tag{2.11}$$

Hence, from (1.4):

$$\sqrt{t} (-1)^n E_x \Big[(D_h)^n \Big(\exp - \frac{\bullet}{2} A_t^q \Big) \Big] \underset{t \to \infty}{\longrightarrow} (-1)^n (D_h)^n \big(\varphi_{\bullet q}(x) \big).$$

Thus:

$$(-1)^n (D_h)^n \big(\varphi_{\bullet q}(x) \big)(\lambda) \geq 0. \tag{2.12}$$

Letting $h \to 0$ in (2.12), and using the fact that: $D_h f \underset{h \to 0}{\longrightarrow} f'$, we get:

$$(-1)^n \frac{\partial^n}{\partial \lambda^n} \big(\varphi_{\lambda q}(x) \big) \geq 0. \tag{2.13}$$

□

2.4. A second proof of Lemma 2.2

We shall only give this second proof under the hypothesis H1 and for $x = 0$. In [14], Proposition 4.13, formula (4.43), we have obtained the following explicit formula for $\varphi_{\lambda q}(0)$:

$$\begin{aligned} \varphi_{\lambda q}(0) = \frac{1}{\sqrt{2\pi}} \int_0^\infty & \Big[Q_l^{(0)}(\exp - \lambda \langle Y, q^- \rangle) Q_l^{(2)}(\exp - \lambda \langle Y, q^+ \rangle) \\ & + Q_l^{(2)}(\exp - \lambda \langle Y, q^- \rangle) Q_l^{(0)}(\exp - \lambda \langle Y, q^+ \rangle) \Big] dl \end{aligned} \tag{2.14}$$

where, in this formula (2.14), the process $(Y_x,\, x \geq 0)$ is, under $Q_l^{(0)}$, (resp. under $Q_l^{(2)}$), a squared Bessel process with dimension 0, (resp. 2), starting from l, and we denote:

$$\langle Y, q^+ \rangle = \int_0^\infty Y_x \ \ q(\mathrm{d}x) \ ; \quad \langle Y, q^- \rangle = \int_{-\infty}^0 Y_{-x} \ \ q(\mathrm{d}x). \tag{2.15}$$

It is then clear from (2.14) that: $\lambda \longrightarrow \varphi_{\lambda q}(0)$ is the Laplace transform of a positive measure, as an integral, with respect to the parameter l of the product of two Laplace transforms of positive measures (indexed by l). □

We shall now give some examples for which the measure ν_x may be computed explicitly. We recall that ν_x is characterized by: $\int_0^\infty \mathrm{e}^{-\frac{\lambda}{2}z}\nu_x(\mathrm{d}z) = \varphi_{\lambda q}(x)$ where $\varphi_{\lambda q}(x)$ is given by (2.3)...(2.6).

2.5. Computation of ν_x for $q(\mathrm{d}y) = \delta_0(\mathrm{d}y)$

In this case, the hypothesis H1 is verified and $A_t^q = L_t$, is the local time at level 0

$$\begin{aligned}\varphi_{\lambda q}(x) &= \sqrt{\frac{2}{\pi}}\Big(\frac{2}{\lambda} + |x|\Big) \quad \big(\textit{cf.}\ [14],\ \text{Ex. 4.8, pp. 199–200}\big)\\ &= \int_0^\infty \mathrm{e}^{-\frac{\lambda}{2}z}\Big(\sqrt{\frac{2}{\pi}}1_{z\geq 0}\mathrm{d}z + \sqrt{\frac{2}{\pi}}|x|\delta_0(\mathrm{d}z)\Big).\end{aligned} \tag{2.16}$$

Thus:

$$\nu_x(\mathrm{d}z) = \sqrt{\frac{2}{\pi}}1_{[0,\infty]}(z)\mathrm{d}z + \sqrt{\frac{2}{\pi}}|x|\delta_0(\mathrm{d}z). \tag{2.17}$$

2.6. Computation of ν_x for $q(\mathrm{d}y) = \delta_a(\mathrm{d}y) + \delta_b(\mathrm{d}y)$ with $(a < b)$

In this case, the hypothesis H1 is satisfied and $A_t^q = L_t^a + L_t^b$ where $(L_t^a,\ t \geq 0)$ resp. $(L_t^b,\ t \geq 0)$ denotes the local time at level a, resp. at level b. We know (see [14], Ex. 4.8, pp. 199–200) that

$$\begin{aligned}\varphi_{\lambda q}(x) &= \begin{cases} \sqrt{\frac{2}{\pi}}\left(\frac{1}{\lambda} + x - b\right) & \text{if } x > b\\ \sqrt{\frac{2}{\pi}}\,\frac{1}{\lambda} & \text{if } x \in [a,b]\\ \sqrt{\frac{2}{\pi}}\left(\frac{1}{\lambda} + a - x\right) & \text{if } x < a\end{cases}\\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \mathrm{e}^{-\frac{\lambda}{2}z}\Big\{\frac{1}{2}\,\mathrm{d}z + (x-b)1_{x>b}\delta_0(\mathrm{d}z) + (a-x)1_{x<a}\delta_0(\mathrm{d}z)\Big\}.\end{aligned} \tag{2.18}$$

Hence:

$$\nu_x(\mathrm{d}z) = \sqrt{\frac{2}{\pi}}\left\{\frac{1}{2}\,1_{[0,\infty[}(z)\mathrm{d}z + (x-b)^+\delta_0(\mathrm{d}z) + (a-x)^+\delta_0(\mathrm{d}z)\right\}. \tag{2.19}$$

2.7. Computation of ν_x, for $q(y) = \mathrm{e}^{2y}$

In this case, the hypothesis H2 is satisfied and $A_t^q = \int_0^t \mathrm{e}^{2X_s}\ \mathrm{d}s$.

To begin with, we show:

$$\varphi_{\lambda q}(x) = \sqrt{\frac{2}{\pi}}\,K_0\left(\sqrt{\lambda}\,\mathrm{e}^x\right) \tag{2.20}$$

where K_0 denotes the Bessel-Mc Donald function with index 0 (see [7], p. 108).

Let $\psi(x) := \sqrt{\frac{2}{\pi}}\,K_0\left(\sqrt{\lambda}\,\mathrm{e}^x\right)$. To check (2.20), it suffices to see that:

$$\psi''(x) = \lambda\mathrm{e}^{2x}\psi(x),\ \psi(x) \underset{x\to\infty}{\longrightarrow} 0,\ \psi'(x) \underset{x\to-\infty}{\longrightarrow} -\sqrt{\frac{2}{\pi}}\cdot \tag{2.21}$$

Now (2.21) follows from (see [7], p. 110):

$$K_0' = -K_1, \; -K_1'(z) = \frac{1}{z}\, K_1(z) + K_0(z)$$

and

$$\psi(x) \underset{x\to\infty}{\sim} \sqrt{\frac{2}{\pi}}\Big(\frac{\pi}{2\sqrt{\lambda}\,\mathrm{e}^{x}}\Big)\mathrm{e}^{-\sqrt{\lambda}\,\mathrm{e}^{x}} \underset{x\to\infty}{\longrightarrow} 0 \qquad \text{([7], p. 123)}$$
$$\psi'(x) = -\sqrt{\frac{2}{\pi}}\,\sqrt{\lambda}\,\mathrm{e}^{x} K_1(\sqrt{\lambda}\,\mathrm{e}^{x}) \underset{x\to-\infty}{\sim} -\sqrt{\frac{2}{\pi}}\,\sqrt{\lambda}\,\mathrm{e}^{x}\frac{1}{2}\,\frac{2}{\sqrt{\lambda}\,\mathrm{e}^{x}} \underset{x\to-\infty}{\longrightarrow} -\sqrt{\frac{2}{\pi}}\cdot \qquad \text{([7], p. 111)}$$

This proves (2.20). But, we also have:

$$K_0(\sqrt{\lambda}\,\mathrm{e}^{x}) = \frac{1}{2}\int_0^\infty \mathrm{e}^{-t-\frac{\lambda \mathrm{e}^{2x}}{4t}}\,\frac{\mathrm{d}t}{t} \qquad (cf.\ [7],\ \text{p. } 119)$$
$$= \frac{1}{2}\int_0^\infty \mathrm{e}^{-\frac{\lambda u}{2}-\frac{\mathrm{e}^{2x}}{2u}}\,\frac{\mathrm{d}u}{u}\cdot$$

Hence:

$$\nu_x(\mathrm{d}z) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{\mathrm{e}^{2x}}{2z}}\,1_{[0,\infty[}(z)\,\frac{\mathrm{d}z}{z}\cdot \tag{2.22}$$

2.8. Computation of ν_x for $q_0(\mathrm{d}x) = 1_{]-\infty,0]}(x)\mathrm{d}x$

Here, it is the hypothesis H3 which is satisfied, and

$$A_t^{q_0} = \int_0^t 1_{]-\infty,0]}(X_s)\mathrm{d}s.$$

By scaling, one has, under $P_0 : A_t^{q_0} \overset{\text{(law)}}{=} t\,A_1^{q_0}$, and it is well known that under $P_0, A_1^{q_0}$ follows the arc sine law, *i.e.*, the beta $\left(\frac{1}{2},\frac{1}{2}\right)$ law. We shall recall the law of $A_t^{q_0}$ under P_x for any $x \in \mathbb{R}$, (see Sect. 3.1 below), which will allow to obtain the following result:

$$\nu_x(\mathrm{d}z) = x_+\sqrt{\frac{2}{\pi}}\,\delta_0(\mathrm{d}z) + \frac{1}{\pi}\,\mathrm{e}^{-\frac{x^2}{2z}}1_{[0,\infty[}(z)\frac{\mathrm{d}z}{\sqrt{z}}\cdot \tag{2.23}$$

For the moment, we shall prove (2.23) without using the explicit law of $A_t^{q_0}$. For this purpose, we already observe that:

$$\varphi_{\lambda q_0}(x) = \sqrt{\frac{2}{\pi}}\left\{\mathrm{e}^{x\sqrt{\lambda}}\frac{1}{\sqrt{\lambda}}\,1_{x\le 0} + \left(x+\frac{1}{\sqrt{\lambda}}\right)1_{x>0}\right\}. \tag{2.24}$$

Indeed we have:

$$\varphi''_{\lambda q_0}(x) = \lambda\,1_{]-\infty,0]}(x)\varphi_{\lambda q_0}(x), \qquad \varphi'_{\lambda q_0}(+\infty) = \sqrt{\frac{2}{\pi}}, \qquad \varphi_{\lambda q_0}(-\infty) = 0.$$

Then, it remains to see that:

$$\int_0^\infty \mathrm{e}^{-\frac{\lambda}{2}z}\nu_x(\mathrm{d}z) = \varphi_{\lambda q_0}(x) \tag{2.25}$$

where ν_x is defined *via* (2.23) and $\varphi_{\lambda q_0}(x)$ by (2.24). Now, for $x>0$, one has:

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-\frac{\lambda}{2}z}\nu_x(\mathrm{d}z) &= x_+\sqrt{\frac{2}{\pi}}+\frac{1}{\pi}\int_0^\infty \mathrm{e}^{-\frac{\lambda z}{2}}\frac{\mathrm{d}z}{\sqrt{z}}\\ &= x_+\sqrt{\frac{2}{\pi}}+\frac{1}{\pi}\sqrt{\frac{2}{\lambda}}\,\Gamma(1/2)=x_+\sqrt{\frac{2}{\pi}}+\sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{\lambda}}=\varphi_{\lambda q_0}(x)\end{aligned}$$

whereas for $x<0$:

$$\int_0^\infty \mathrm{e}^{-\frac{\lambda}{2}z}\nu_x(\mathrm{d}z)=\frac{1}{\pi}\int_0^\infty \mathrm{e}^{-\frac{\lambda}{2}z-\frac{x^2}{2z}}\frac{\mathrm{d}z}{\sqrt{z}}=\frac{2}{\pi}K_{1/2}\left(|x|\sqrt{\lambda}\right)\left(\frac{x^2}{\lambda}\right)^{1/4}\qquad \text{(see [7], p. 119)}\,.$$

However, one has: $K_{1/2}\left(|x|\sqrt{\lambda}\right)=\left(\frac{\pi}{2|x|\sqrt{\lambda}}\right)^{1/2}\mathrm{e}^{-|x|\sqrt{\lambda}}$. Hence:

$$\int_0^\infty \mathrm{e}^{-\frac{\lambda}{2}z}\nu_x(\mathrm{d}z)=\frac{2}{\pi}\left(\frac{x^2}{\lambda}\right)^{1/4}\left(\frac{\pi}{2|x|\sqrt{\lambda}}\right)^{1/2}\mathrm{e}^{-|x|\sqrt{\lambda}}=\sqrt{\frac{2}{\pi}}\,\frac{1}{\sqrt{\lambda}}\,\mathrm{e}^{-|x|\sqrt{\lambda}}=\varphi_{\lambda q_0}(x).$$

2.9. Computation of ν_x when $q(y)=1_{[a,b]}(y)\quad (a<b)$

The hypothesis H1 is satisfied and $A_t^q=\int_0^t 1_{[a,b]}(X_s)\,\mathrm{d}s$. We shall prove that:

$$\nu_x^{(a,b)}(\mathrm{d}z)=\begin{cases}\sqrt{\frac{2}{\pi}}(x-b)\delta_0(\mathrm{d}z)+\frac{1}{\pi\sqrt{z}}\,1_{[0,\infty[}(z)\mathrm{d}z\left(1+2\sum_{n=1}^{\infty}\mathrm{e}^{-\frac{n^2(b-a)^2}{2z}}\right) & \text{if } x>b\\ \sqrt{\frac{2}{\pi}}\,(a-x)\delta_0(\mathrm{d}z)+\frac{1}{\pi\sqrt{z}}1_{[0,\infty[}(z)\mathrm{d}z\left(1+2\sum_{n=1}^{\infty}\mathrm{e}^{-\frac{n^2(b-a)^2}{2z}}\right) & \text{if } x<a\\ \frac{1}{\pi\sqrt{z}}\sum_{n=0}^{\infty}\left(\mathrm{e}^{-\frac{\left(n(b-a)+b-x\right)^2}{2z}}+\mathrm{e}^{-\frac{\left(n(b-a)+(x-a)\right)^2}{2z}}\right)1_{[0,\infty[}(z)\mathrm{d}z & \text{if } x\in[a,b].\end{cases}\tag{2.26}$$

Here, the explicit form of $\varphi_{\lambda q}^{(a,b)}(x)$ is (see [14], Ex. 4.7, p. 199):

$$\varphi_{\lambda q}^{(a,b)}(x)=\begin{cases}\sqrt{\frac{2}{\pi}}\left(\frac{1}{\sqrt{\lambda}\tanh\left(\sqrt{\lambda}\frac{b-a}{2}\right)}+x-b\right) & \text{if } x>b\\ \sqrt{\frac{2}{\pi}}\left(\frac{1}{\sqrt{\lambda}\tanh\left(\sqrt{\lambda}\frac{b-a}{2}\right)}+a-x\right) & \text{if } x<a\\ \sqrt{\frac{2}{\pi}}\left(\frac{\cosh\left(\sqrt{\lambda}\left(x-\frac{a+b}{2}\right)\right)}{\sqrt{\lambda}(\sinh\left(\sqrt{\lambda}\frac{b-a}{2}\right))}\right) & \text{if } x\in[a,b].\end{cases}\tag{2.27}$$

It now remains to prove that:

$$\int_0^\infty \mathrm{e}^{-\frac{\lambda}{2}z}\nu_x^{(a,b)}(\mathrm{d}z)=\varphi_{\lambda q}^{(a,b)}(x)\tag{2.28}$$

where $\nu_x^{(a,b)}$ is defined *via* (2.26) and $\varphi_{\lambda q}^{(a,b)}$ *via* (2.27). But, (2.28) follows, after some elementary computations from the identities, for every real u and $v > 0$:

$$\frac{\cosh(\sqrt{\lambda}u)}{\sqrt{\lambda}\sinh(\sqrt{\lambda}v)} = \sum_{n=0}^{\infty}\int_0^{\infty} \mathrm{d}h \left(\mathrm{e}^{-\sqrt{\lambda}\left(h+(2n+1)v-u\right)} + \mathrm{e}^{-\sqrt{\lambda}\left(h+(2n+1)v+u\right)}\right) \tag{2.29}$$

$$= \sum_{n=0}^{\infty}\int_0^{\infty} \mathrm{d}h \int_0^{\infty} \mathrm{d}s \left(H_{h+(2n+1)v-u}(s) + H_{h+(2n+1)v+u}(s)\right)\mathrm{e}^{-\lambda s} \tag{2.30}$$

with

$$H_a(u) := \frac{a}{2\sqrt{\pi u^3}}\,\mathrm{e}^{-a^2/4u} = \frac{-1}{\sqrt{\pi u}}\,\frac{\partial}{\partial a}\,(\mathrm{e}^{-\frac{a^2}{4u}}) \qquad (a > 0).$$

Passing from (2.29) to (2.30) is obtained by using the elementary formula:

$$\mathrm{e}^{-\sqrt{\lambda}a} = \int_0^{\infty} \mathrm{e}^{-\lambda u} H_a(u)\mathrm{d}u = \int_0^{\infty} \mathrm{e}^{-\lambda u} \frac{a}{2\sqrt{\pi u^3}}\,\mathrm{e}^{-\frac{a^2}{4u}}\mathrm{d}u. \tag{2.31}$$

(Note that (2.31) is nothing else but a translation of: $E\left(\mathrm{e}^{-\frac{\lambda^2}{2}T_a}\right) = \exp(-\lambda a)$, where T_a denotes the hitting time of level $a > 0$ by Brownian motion starting from 0, and H_a is the density of $T_{\frac{a}{\sqrt{2}}}$.)

We now show (2.29).

$$\begin{aligned}
\frac{\cosh(\sqrt{\lambda}u)}{\sqrt{\lambda}\sinh(\sqrt{\lambda}v)} &= \frac{1}{\sqrt{\lambda}}\,\mathrm{e}^{-\sqrt{\lambda}(v-u)}\,\frac{1+\mathrm{e}^{-2\sqrt{\lambda}u}}{1-\mathrm{e}^{-2\sqrt{\lambda}v}}\\
&= \frac{1}{\sqrt{\lambda}}\,\mathrm{e}^{-\sqrt{\lambda}(v-u)}(1+\mathrm{e}^{-2\sqrt{\lambda}u})\left(\sum_{n=0}^{\infty}\mathrm{e}^{-2n\sqrt{\lambda}v}\right)\\
&= \frac{1}{\sqrt{\lambda}}\left\{\sum_{n=0}^{\infty}\mathrm{e}^{-\sqrt{\lambda}\{v-u+2nv\}} + \sum_{n=0}^{\infty}\mathrm{e}^{-\sqrt{\lambda}\left\{2(u+nv)+(v-u)\right\}}\right\}\\
&= \frac{1}{\sqrt{\lambda}}\left\{\sum_{n=0}^{\infty}\left(\mathrm{e}^{-\sqrt{\lambda}\left((2n+1)v-u\right)} + \mathrm{e}^{-\sqrt{\lambda}\left(u+(2n+1)v\right)}\right)\right\}\\
&= \int_0^{\infty}\mathrm{e}^{-\sqrt{\lambda}h}\left\{\sum_{n=0}^{\infty}\mathrm{e}^{-\sqrt{\lambda}\left((2n+1)v-u\right)} + \mathrm{e}^{-\sqrt{\lambda}\left(u+(2n+1)v\right)}\right\}\mathrm{d}h\\
&= \sum_{n=0}^{\infty}\int_0^{\infty}\left(\mathrm{e}^{-\sqrt{\lambda}\left(h+(2n+1)v-u\right)} + \mathrm{e}^{-\sqrt{\lambda}\left(h+(2n+1)v+u\right)}\right)\mathrm{d}h.
\end{aligned}$$

Remark 2.3.

(i) If in formula (2.26), we take: $b = 0$, and we let a tend to $-\infty$, we obtain:

$$\lim_{a\to-\infty}\nu^{a,0}(\mathrm{d}z) = \begin{cases} \sqrt{\frac{2}{\pi}}x_+\delta_0(\mathrm{d}z) + \frac{1}{\pi\sqrt{z}}1_{[0,\infty[}(z)\mathrm{d}z & \text{if } x > 0\\ \frac{1}{\pi\sqrt{z}}\,\mathrm{e}^{-\frac{x^2}{2z}}1_{[0,\infty[}(z)\mathrm{d}z & \text{if } x \le 0.\end{cases} \tag{2.32}$$

We note that the RHS of (2.32) is nothing else but the measure ν_x associated with $q_0(y) = 1_{]-\infty,0]}$ (see (2.23)). This may be interpreted as "a continuity property" of $\varphi^{a,b}$, as $a \longrightarrow -\infty$.

(ii) In the same spirit, but taking up now the computation from Sect. 2.9, where we choose for q the function: $q^{(c)}(y) = \frac{1}{2c}\, 1_{[-c,+c]}(y)$, we have:

$$\int_0^\infty \mathrm{e}^{-\frac{\lambda z}{2}}\, \nu_x^{(c)}\,(\mathrm{d}z) \underset{c\to 0}{\longrightarrow} \int_0^\infty \mathrm{e}^{-\frac{\lambda z}{2}}\, \nu_x(\mathrm{d}z) = \sqrt{\frac{2}{\pi}}\left(\frac{2}{\lambda} + |x|\right) \tag{2.33}$$

where ν_x is the measure associated to $q(\mathrm{d}z) = \delta_0(\mathrm{d}z)$ (see (2.16)). In other terms, since: $\frac{1}{2c}\int_0^t 1_{[-c,c]}(X_s)\mathrm{d}s \underset{c\to 0}{\longrightarrow} L_t$ *a.s.*, we witness there also a "continuity property of $\nu_x^{(c)}$ as $c \longrightarrow 0$". Let us show (2.33) for $x = 0$; from (2.27):

$$\int_0^\infty \mathrm{e}^{-\frac{\lambda z}{2}}\, \nu_0^{(c)}(\mathrm{d}z) = \sqrt{\frac{2}{\pi}}\left\{\frac{\cosh\left(\sqrt{\frac{\lambda}{2c}}\, c\right)}{\sqrt{\frac{\lambda}{2c}}\,\sinh\left(\sqrt{\frac{\lambda}{2c}}\, c\right)}\right\} \underset{c\to 0}{\longrightarrow} \sqrt{\frac{2}{\pi}} \times \frac{2}{\lambda}$$

and for $x \neq 0$, and c small enough, we obtain from (2.27) that:

$$\int_0^\infty \mathrm{e}^{-\frac{\lambda z}{2}}\, \nu_x^{(c)}(\mathrm{d}z) = \sqrt{\frac{2}{\pi}}\left\{\frac{1}{\sqrt{\frac{\lambda}{2c}}\,\tanh\left(\sqrt{\frac{\lambda}{2c}}\, c\right)} + |x-c|\right\} \underset{c\to 0}{\longrightarrow} \sqrt{\frac{2}{\pi}}\left(\frac{2}{\lambda} + |x|\right).$$

2.10. Computation of ν_x when $q(y) = 1_{[0,\infty[}(y) y^\alpha$, $\alpha > 0$

The hypothesis H2 is satisfied, and we have: $A_t^q = \int_0^t \mathbf{1}_{(X_s>0)} X_s^\alpha \mathrm{d}s$.
We now show the existence of a constant $C_\alpha > 0$ such that:

$$\nu_0(\mathrm{d}z) = \frac{C_\alpha}{z^{\frac{1+\alpha}{2+\alpha}}}\, 1_{[0,\infty[}(z)\mathrm{d}z. \tag{2.34}$$

Indeed, thanks to the scaling property, we have:

$$\begin{aligned} E_0\left(\mathrm{e}^{-\frac{\lambda}{2}\int_0^t 1_{X_s>0} X_s^\alpha\,\mathrm{d}s}\right) &= E_0\left(\mathrm{e}^{\left(-\frac{\lambda}{2}\, t^{1+\alpha/2} A_1^q\right)}\right) \\ &= E_0\left(\exp\left(-\frac{1}{2}\, A^q_{\lambda^{\frac{2}{2+\alpha}} t}\right)\right). \end{aligned} \tag{2.35}$$

Thus, multiplying (2.35) by $\sqrt{t}$ and letting t tend to $+\infty$, we obtain:

$$\varphi_{\lambda q}(0) = \frac{1}{\lambda^{\frac{1}{2+\alpha}}}\, \varphi_{1\,q}(0) = \frac{1}{\lambda^{\frac{1}{2+\alpha}}}\, c'_\alpha = c_\alpha \int_0^\infty \mathrm{e}^{-\frac{\lambda}{2} z}\, \frac{\mathrm{d}z}{z^{\frac{1+\alpha}{2+\alpha}}}.$$

The same computations, performed this time with $x \neq 0$, lead to:

$$\varphi_{\lambda q}(x) = \frac{1}{\lambda^{\frac{1}{2+\alpha}}}\, \varphi_{1q}\left(x\lambda^{\frac{1}{2+\alpha}}\right),\ i.e.\ \nu_x = \frac{1}{\lambda^{\frac{1}{2+\alpha}}}\, \nu^{(\lambda)}_{x\lambda^{\frac{1}{2+\alpha}}},$$

where $\nu^{(\lambda)}$ is the image of ν by the application $z \longrightarrow \lambda z$.

Question 2.4. We know $\big($see [10], Chap. X$\big)$ that, if q is an integrable function, then:

$$\frac{1}{\sqrt{t}}\int_0^t q(x+X_s)\mathrm{d}s \xrightarrow[t\to\infty]{\text{law}} \left(\int q(x+y)\mathrm{d}y\right)|N| = \left(\int q(y)\mathrm{d}y\right)|N| \tag{2.36}$$

where N is a standard Gaussian variable, and on the LHS of (2.36), $(X_s,\ s\geq 0)$ is a Brownian motion starting from 0. Let g denote the density of the r.v. $\bar{q}|N|$ with $\bar{q} = \displaystyle\int q(y)\mathrm{d}y$ that is:

$$g(z) = \frac{1}{\bar{q}}\sqrt{\frac{2}{\pi}}\mathrm{e}^{-\frac{z^2}{2(\bar{q})^2}}1_{[0,\infty[}(z).$$

Let us now consider the *supplementary hypothesis* $\widetilde{H}$, which seems reasonable enough in view of (2.36), that the density $g_t(x,\cdot)$ of the r.v. $\displaystyle\frac{1}{\sqrt{t}}\int_0^t q(x+X_s)\,\mathrm{d}s$ converges, as $t\longrightarrow\infty$, uniformly on every compact, towards g.

However, this would imply that, for every function h, which is continuous with compact support, one would have:

$$\begin{aligned}
\sqrt{t}\,E_x\big[h(A_t^q)\big] &= \sqrt{t}\,E_x\Big[h\Big(\frac{A_t^q}{\sqrt{t}}\sqrt{t}\Big)\Big]\\
&= \sqrt{t}\int_0^\infty h(z\sqrt{t})\,g_t(x,z)\,\mathrm{d}z\\
&= \int_0^\infty h(y)g_t\Big(x,\frac{y}{\sqrt{t}}\Big)\mathrm{d}y \xrightarrow[t\to\infty]{} \int_0^\infty h(y)g(0)\mathrm{d}y.
\end{aligned}$$

But, from Theorem 2.1., we know that:

$$\sqrt{t}\,E_x\big[h(A_t^q)\big] \xrightarrow[t\to\infty]{} \int_{\mathbb{R}_+} h(z)\nu_x(\mathrm{d}z).$$

Thus, this would imply that the measure $\nu_x(\mathrm{d}z)$ would be equal to:

$$\frac{1}{\bar{q}}\sqrt{\frac{2}{\pi}}\,1_{[0,\infty[}\mathrm{d}z \tag{2.37}$$

so that, the measure ν_x would not depend on x, and would be proportional to Lebesgue measure on $\mathbb{R}_+$. But clearly, this is not the case for either of the examples in Sections 2.4 to 2.9. Consequently, the hypothesis $\widetilde{H}$ is not satisfied for the corresponding $q\,'s$. It would be of interest to know for which $q\,'s$, if any, it is satisfied.

2.11. Penalisation by $h(A_t^q)$

Let q satisfy one of the previous hypotheses H1, H2 or H3, and denote, as before:

$A_t^q = \displaystyle\int_{\mathbb{R}} L_t^x q(\mathrm{d}x)\left(=\int_0^t q(X_s)\mathrm{d}s \text{ if } q \text{ admits a density}\right)$. Let now $h:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ such that:

$$\sqrt{t}\,E_x\big[h(A_t^q)\big] \xrightarrow[t\to\infty]{} \int_0^\infty h(z)\nu_x(\mathrm{d}z). \tag{2.38}$$

Then, (2.38) is satisfied, from Theorem 2.1, as soon as h is sub-exponential (for example if h is continuous, with compact support). We shall now study the penalisation of Wiener measure by the functional $h(A_t^q)$, *i.e.*: we shall study the limit, as $t \longrightarrow \infty$, of:

$$\frac{E_x\big(1_{\Lambda_s}\, h(A_t^q)\big)}{E_x\big(h(A_t^q)\big)} \qquad (s \geq 0,\ \Lambda_s \in \mathcal{F}_s). \tag{2.39}$$

We have already made this study in two situations:

1) $q(\mathrm{d}y) = \delta_0(\mathrm{d}y)$ then $A_t^q = L_t$ (*cf.* [15]);

2) $A_t^q = \displaystyle\int_{\mathbb{R}} L_t^y\, q(\mathrm{d}y)$ and $h(u) = \exp\left(-\dfrac{\lambda}{2}\, u\right)$ (*cf.* [14]).

This time, Theorem 2.1 allows us to obtain:

Theorem 2.5. *Let q, A^q and h as above. Then:*

1) *For every $s \geq 0$, and every $\Lambda_s \in \mathcal{F}_s$:*

$$\lim_{t\to\infty} \frac{E_x\big(1_{\Lambda_s}\, h(A_t^q)\big)}{E_x\big(h(A_t^q)\big)} \quad \text{exists.} \tag{2.40}$$

2) *This limit equals $E_x(\mathbf{1}_{\Lambda_s} M_s^{h,q}) := Q^{h,q}(\Lambda_s)$, where*

$$M_s^{h,q} := \frac{\displaystyle\int_{\mathbb{R}_+} \nu_{X_s}(\mathrm{d}z)\, h(z + A_s^q)}{\displaystyle\int_{\mathbb{R}_+} \nu_x(\mathrm{d}z)\, h(z)}\cdot \tag{2.41}$$

Furthermore, $(M_s^{h,q}, s \geq 0)$ is a positive martingale. In the case when $h(u) := \mathrm{e}^{-\frac{\lambda}{2}u}$ $(u, \lambda \geq 0)$, we then obtain:

$$M_s^{h,q} = \frac{\varphi_{\lambda q}(X_s)}{\varphi_{\lambda q}(x)} \exp\Big(-\frac{\lambda}{2}\, A_s^q\Big). \tag{2.42}$$

Proof of Theorem 2.5. We have:

$$\frac{E_x\big(\mathbf{1}_{\Lambda_s}\, h(A_t^q)\big)}{E_x\big(h(A_t^q)\big)} = \frac{E_x\big(\mathbf{1}_{\Lambda_s} E_b\big(h(a + A_{t-s}^q)\big)}{E_x\big(h(A_t^q)\big)}$$

from the Markov property, where $b = X_s$ and $a = A_s^q$. Thus, from Theorem 2.1:

$$E_x\big(h(A_t^q)\big) \underset{t\to\infty}{\sim} \frac{1}{\sqrt{t}} \int_0^\infty \nu_x(\mathrm{d}z) h(z)$$

and

$$E_b\big(h(a + A_{t-s}^q)\big) \underset{t\to\infty}{\sim} \frac{1}{\sqrt{t-s}} \int_0^\infty \nu_b(\mathrm{d}z) h(a+z).$$

Hence:

$$\begin{aligned}\frac{E_x\big(\mathbf{1}_{\Lambda_s}\, h(A_t^q)\big)}{E_x\big(h(A_t^q)\big)} &\underset{t\to\infty}{\sim} \frac{\sqrt{t}}{\sqrt{t-s}}\, \frac{E_x\big(\mathbf{1}_{\Lambda_s} \int_{\mathbb{R}_+} \nu_{X_s}(\mathrm{d}z) h(z + A_s^q)\big)}{\int_{\mathbb{R}_+} h(z)\nu_x(\mathrm{d}z)} \\ &\underset{t\to\infty}{\longrightarrow} E_x(\mathbf{1}_{\Lambda_s} M_s^{h,q}).\end{aligned}$$

In the preceding lines, we have been a little careless concerning the exchange of limit and expectation. Likewise, although it is easy to see that $(M_s^{h,q},\ s \geq 0)$ is a local martingale, some care is needed in order to show that it is a true martingale. However, all this is correct as soon as h is sub-exponential. We leave details to the reader. □

3. A detailed study for $q_0 = 1_{]-\infty,0]}$, $A_t^- := \int_0^t 1_{(X_s<0)}\mathrm{d}s$

Throughout this section, we choose $q_0 = 1_{]-\infty,0]}$. Thus, the hypothesis H3 is now satisfied. We shall study this situation in detail, which we are able to do as we know (see [16]) the law of $A_t^{q_0} = \int_0^t q_0(X_s)\mathrm{d}s$ under P_x, for every real x (see (3.5) and (3.7) below). We shall, successively:

- compute explicitly the measure ν_x starting from the knowledge of the law of $A_t^{q_0}$ and we shall recover the result of Section 2.8 above;
- study the penalisation, not only of the process $(X_t,\, t \geq 0)$ by $h(A_t^{q_0})$, but also the penalisation of the "long bridges" by this functional;
- describe precisely the behavior of the canonical process under the probability Q^{h,q_0}, where Q^{h,q_0} is defined *via*:

$$Q^{h,q_0}(\Lambda_s) = E(1_{\Lambda_s} M_s^{h,q_0}) \qquad (s \geq 0,\ \Lambda_s \in \mathcal{F}_s). \tag{3.1}$$

3.1. The law of A_t^- and the computation of ν_x

To simplify notation, we denote:

$$A_t^- = \int_0^t 1_{(X_s<0)}\,\mathrm{d}s = \int_0^t q_0(X_s)\mathrm{d}s. \tag{3.2}$$

We recall the following result, which is found in [16]. For any $f : [0,1] \longrightarrow \mathbb{R}_+$, Borel, sub-exponential (see Thm. 2.1) and any $y > 0$:

$$E_0\left[f\left(\int_0^1 1_{(X_s<y)}\mathrm{d}s\right)\right] = \int_0^1 \frac{\mathrm{d}u}{\pi\sqrt{u(1-u)}}\,\mathrm{e}^{-\frac{y^2}{2u}} f(u) + f(1)\sqrt{\frac{2}{\pi}}\int_0^y \mathrm{e}^{-\frac{\alpha^2}{2}}\mathrm{d}\alpha \tag{3.3}$$

whereas, for any $y < 0$, we use:

$$\int_0^1 1_{(X_s<y)}\,\mathrm{d}s \overset{\text{law}}{=} \int_0^1 1_{(X_s>-y)}\mathrm{d}s \overset{\text{law}}{=} 1 - \int_0^1 1_{(X_s<-y)}\mathrm{d}s \tag{3.4}$$

and by the scaling property:

$$E_x\left[f\left(\int_0^t 1_{(X_s<0)}\mathrm{d}s\right)\right] = E_x\left(f(A_t^-)\right) = E_0\left(f\left(\int_0^t 1_{(X_s<-x)}\mathrm{d}s\right)\right) = E_0\left(f\left(t\int_0^1 1_{\left(X_s<-\frac{x}{\sqrt{t}}\right)}\mathrm{d}s\right)\right).$$

Hence, from (3.3) and (3.4), if $x \leq 0$:

$$E_x\left[f(A_t^-)\right] = \int_0^t \frac{\mathrm{d}v}{\pi\sqrt{v(t-v)}}\,\mathrm{e}^{-\frac{x^2}{2v}} f(v) + f(t)\sqrt{\frac{2}{\pi}}\int_0^{\frac{|x|}{\sqrt{t}}} \mathrm{e}^{-\frac{\alpha^2}{2}}\mathrm{d}\alpha \tag{3.5}$$

$$\underset{t\to\infty}{\sim} \frac{1}{\pi\sqrt{t}}\int_0^\infty \frac{\mathrm{d}v}{\sqrt{v}}\,\mathrm{e}^{-\frac{x^2}{2v}} f(v) \tag{3.6}$$

whereas, if $x > 0$:

$$E_x\left[f(A_t^-)\right] = f(0)\sqrt{\frac{2}{\pi}}\int_0^{\frac{x}{\sqrt{t}}} \mathrm{e}^{-\frac{\alpha^2}{2}}\mathrm{d}\alpha + \int_0^t \frac{\mathrm{d}v}{\pi\sqrt{v(t-v)}}\,\mathrm{e}^{-\frac{x^2}{2(t-v)}} f(v) \tag{3.7}$$

$$\underset{t\to\infty}{\sim} f(0)\sqrt{\frac{2}{\pi}}\int_0^{\frac{x}{\sqrt{t}}} \mathrm{e}^{-\frac{\alpha^2}{2}}\mathrm{d}\alpha + \frac{1}{\pi\sqrt{t}}\int_0^\infty \frac{\mathrm{d}v}{\sqrt{v}}\, f(v). \tag{3.8}$$

Thus, we obtain, for $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, Borel, sub-exponential (and then $\int_0^\infty \frac{\mathrm{d}v}{\sqrt{v}} f(v) < \infty$)

$$\sqrt{t}\, E_x\left[f(A_t^-)\right] \underset{t\to\infty}{\longrightarrow} \int_0^\infty f(z)\nu_x(\mathrm{d}z)$$

with

$$\nu_x(\mathrm{d}z) = x_+\sqrt{\frac{2}{\pi}}\delta_0(\mathrm{d}z) + \frac{1}{\pi}\,\mathrm{e}^{-\frac{x_-^2}{2z}} 1_{[0,\infty[}(z)\frac{\mathrm{d}z}{\sqrt{z}}$$

which is precisely (2.23).

3.2. Penalisation by $h(A_t^-)$. A study of "long bridges" and of the Q^h-process

We recall that, from (3.5), the density of A_t^- under P_0, which we denote by $p_{A_t^-}$, equals:

$$p_{A_t^-}(y) = \frac{1}{\pi}\,\frac{1}{\sqrt{y(t-y)}}\,\mathbf{1}_{[0,t]}(y) \qquad (\text{: the arc sine law}). \tag{3.9}$$

Throughout the following, h denotes a function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that:

$$\int_0^\infty \frac{\mathrm{d}y}{\sqrt{y}}\, h(y) < \infty.$$

And we assume, without loss of generality, that:

$$\int_0^\infty \frac{\mathrm{d}y}{\sqrt{y}}\, h(y) = 1. \tag{3.10}$$

Theorem 3.1. 1) *For every $s \geq 0$ and every $\Lambda_s \in \mathcal{F}_s$:*

$$\lim_{t\to\infty} E_0(1_{\Lambda_s}|A_t^- = a) = Q^{(a)}(\Lambda_s) \tag{3.11}$$

with

$$Q^{(a)}(\Lambda_s) := \sqrt{\frac{2}{\pi}}\,\frac{1_{a<s}}{\sqrt{s-a}}\; E_0(1_{\Lambda_s}X_s^+|A_s^- = a) + E\left[1_{\Lambda_s}\sqrt{\frac{a}{a-A_s^-}}\,1_{(A_s^-<a)}\,\mathrm{e}^{-\frac{(X_s^-)^2}{2(a-A_s^-)}}\right] \tag{3.12}$$

(recall that $X_s^+ = 0 \vee X_s$, $X_s^- = -(X_s \wedge 0)$ and $A_s^- = \int_0^s 1_{X_u<0}\,\mathrm{d}u$).

2) *For every function h which satisfies (3.10), for every $s \geq 0$ and any $\Lambda_s \in \mathcal{F}_s$:*

$$\lim_{t\to\infty} \frac{E_0\left(1_{\Lambda_s}h(A_t^-)\right)}{E_0\left(h(A_t^-)\right)} = E(1_{\Lambda_s}M_s^h) \tag{3.13}$$

where $(M_s^h,\, s \geq 0)$ is the positive martingale given by:

$$M_s^h := \sqrt{2\pi}\,X_s^+\,h(A_s^-) + \int_0^\infty \frac{\mathrm{d}y}{\sqrt{y}}\,\mathrm{e}^{-\frac{(X_s^-)^2}{2y}}\,h(A_s^- + y) \tag{3.14}$$

(note that $M_0^h = 1$).

3) *Formula (3.13) induces a probability Q^h on $(\Omega, \mathcal{F}_\infty)$, which admits the following disintegration:*

$$Q^h(\Lambda) = \int_0^\infty Q^{(a)}(\Lambda)\, \frac{h(a)}{\sqrt{a}}\, \mathrm{d}a \qquad (\Lambda \in \mathcal{F}_\infty) \tag{3.15}$$

where $Q^{(a)}$ is given by (3.12).

4) *Under Q^h, the canonical process $\big(X_t,\, t \geq 0\big)$ satisfies:*

$$i) \quad A_\infty^- \text{ is finite a.s., and admits as density } \frac{h(y)}{\sqrt{y}}\, 1_{y>0}; \tag{3.16}$$

$$ii) \quad \text{let } g = \inf\{t\,;\ A_t^- = A_\infty^-\} = \sup\{t\,;\ X_t \leq 0\}. \tag{3.17}$$

Then $Q^h(g < \infty) = 1$

iii) *the processes $(X_t,\, t \leq g)$ and $(X_{g+t},\, t \geq 0)$ are independent;*

iv) *the process $(X_{g+t},\ t \geq 0)$ is a 3-dimensional Bessel process starting from 0.*

Moreover, while proving Theorem 3.1, we shall give a precise description of the process $(X_t\,;\ t \leq g)$.

3.3. Proof of Theorem 3.1

3.3.1. *Proof of point 1) in Theorem 3.1*

For this purpose, we choose a function h, which is Borel, positive, and satisfies (3.10).

We first write:

$$E_0\big(1_{\Lambda_s} h(A_t^-)\big) = \int_0^t E_0(1_{\Lambda_s} | A_t^- = a) p_{A_t^-}(a) h(a) \mathrm{d}a. \tag{3.18}$$

Then, conditioning with respect to $\mathcal{F}_s$, we obtain:

$$E_0\left(1_{\Lambda_s} h(A_t^-)\right) = E_0\left(1_{\Lambda_s}\, E_0\left(h\left(a + \int_0^{t-s} 1_{(X_u < -x)}\, \mathrm{d}u\right)\right)\right) \tag{3.19}$$

with $a = A_s^-$ and $x = X_s$. Using now (3.5) and (3.6), we obtain:

$$\begin{aligned} E_0\left(1_{\Lambda_s} h(A_t^-)\right) &= E_0\left(1_{\Lambda_s} 1_{x<0}\left(\left(\int_0^{t-s} \frac{\mathrm{d}v}{\pi\sqrt{v(t-s-v)}}\, \mathrm{e}^{-\frac{x^2}{2v}} h(a+v)\right) + h(a+t-s)\psi\left(\frac{|x|}{\sqrt{t-s}}\right)\right)\right) \\ &\quad + E_0\left(1_{\Lambda_s} 1_{x>0}\left[h(a)\psi\left(\frac{x}{\sqrt{t-s}}\right) + \int_0^{t-s} \frac{\mathrm{d}v}{\pi\sqrt{v(t-s-v)}}\, \mathrm{e}^{-\frac{x^2}{2(t-s-v)}} h(a+v)\right]\right) \quad (3.20) \\ &:= (1)_t + (2)_t \quad (3.21) \end{aligned}$$

where $\psi\Big(\frac{x}{\sqrt{t}}\Big) := P\Big(|N| \leq \frac{x}{\sqrt{t}}\Big) = \sqrt{\frac{2}{\pi}} \int_0^{\frac{x}{\sqrt{t}}} \mathrm{e}^{-\frac{\alpha^2}{2}} \mathrm{d}\alpha \underset{t\to\infty}{\sim} \sqrt{\frac{2}{\pi}}\, \frac{x}{\sqrt{t}}\cdot$

We now study successively $(1)_t$ and $(2)_t$. We rewrite $(1)_t$ in the form:

$$\begin{aligned}(1)_t &= \int_0^s p_{A_s^-}(a)\mathrm{d}a\, E_0\left(1_{\Lambda_s}1_{X_s<0}\left(\int_0^{t-s}\frac{\mathrm{d}v}{\pi\sqrt{v(t-s-v)}}\,\mathrm{e}^{-\frac{X_s^2}{2v}}h(a+v)\right.\right.\\ &\qquad\left.\left.+\,h(a+t-s)\,\psi\left(\frac{|X_s|}{\sqrt{t-s}}\right)\right)\Big|\,A_s^-=a\right)\\ &= \int_0^{t-s}\frac{\mathrm{d}v}{\pi\sqrt{v(t-s-v)}}\int_v^{v+s}\mathrm{d}a\,p_{A_s^-}(a-v)\,E_0\big(1_{\Lambda_s}1_{X_s<0}\,\mathrm{e}^{-\frac{X_s^2}{2v}}\,|A_s^-=a-v\big)h(a)\\ &\quad+\int_{t-s}^t p_{A_s^-}(a+s-t)E_0\Big(1_{\Lambda_s}1_{X_s<0}\,\psi\Big(\frac{|X_s|}{\sqrt{t-s}}\Big)\Big|A_s^-=a+s-t\Big)h(a)\mathrm{d}a. \end{aligned} \tag{3.22}$$

Similarly:

$$\begin{aligned}(2)_t &= \int_0^s p_{A_s^-}(a)\,E_0\Big(1_{\Lambda_s}1_{X_s>0}\,\psi\Big(\frac{|X_s|}{\sqrt{t-s}}\Big)\Big|A_s^-=a\Big)h(a)\mathrm{d}a \qquad (3.23)\\ &\quad+\int_0^s\mathrm{d}a\,p_{A_s^-}(a)\int_0^{t-s}\frac{\mathrm{d}v}{\pi\sqrt{v(t-s-v)}}\,E_0\big(1_{\Lambda_s}1_{X_s>0}\,\mathrm{e}^{-\frac{X_s^2}{2(t-s-v)}}\,\big|A_s^-=a)h(a+v)\\ &= \int_0^s p_{A_s^-}(a)\,E_0\Big(1_{\Lambda_s}1_{X_s>0}\,\psi\Big(\frac{|X_s|}{\sqrt{t-s}}\Big)\Big|A_s^-=a\Big)h(a)\mathrm{d}a \qquad (3.24)\\ &\quad+\int_0^{t-s}\frac{\mathrm{d}v}{\pi\sqrt{v(t-s-v)}}\int_v^{v+s}p_{A_s^-}(a-v)\,E_0\Big(1_{\Lambda_s}1_{X_s>0}\,\mathrm{e}^{-\frac{X_s^2}{2(t-s-v)}}\,\big|A_s^-=a-v\Big)h(a)\mathrm{d}a.\end{aligned}$$

Then, comparing (3.18), (3.22), (3.24) and identifying the "coefficient of $h(a)$", it follows that:

$$E_0\big(1_{\Lambda_s}|A_t^-=a\big)=(\tilde{1})_t+(\tilde{2})_t$$

with:

$$\begin{aligned}(\tilde{1})_t &= \frac{1}{p_{A_t^-}(a)}\int_0^{t-s}\frac{\mathrm{d}v}{\pi\sqrt{v(t-s-v)}}\,1_{v<a<v+s}\,p_{A_s^-}(a-v)\,E_0\Big(1_{\Lambda_s}1_{X_s<0}\,\mathrm{e}^{-\frac{X_s^2}{2v}}\big|A_s^-=a-v\Big)\\ &\quad+\frac{1}{p_{A_t^-}(a)}\,1_{t-s<a<t}\,p_{A_s^-}(a+s-t)\,E_0\Big(1_{\Lambda_s}1_{X_s<0}\,\psi\Big(\frac{|X_s|}{\sqrt{t-s}}\Big)\big|A_s^-=a+s-t\Big)\\ &\underset{t\to\infty}{\longrightarrow}\int_0^s\frac{\sqrt{a}}{\sqrt{a-w}}\,p_{A_s^-}(w)\,E_0\big(1_{\Lambda_s}1_{X_s<0}\,\mathrm{e}^{-\frac{X_s^2}{2(a-w)}}\,\big|A_s^-=w\big)\mathrm{d}w\\ &= E_0\Big(1_{\Lambda_s}1_{X_s<0}\,\mathrm{e}^{-\frac{X_s^2}{2(a-A_s^-)}}\,\frac{\sqrt{a}}{\sqrt{a-A_s^-}}\,1_{A_s^-<a}\Big)\end{aligned} \tag{3.25}$$

since $p_{A_t^-}(a) = \dfrac{1}{\pi\sqrt{a(t-a)}}\, 1_{[0,t]}(a)$. Similarly, one has:

$$
\begin{aligned}
(\tilde{2})_t \quad &= \quad \frac{p_{A_s^-}(a)}{p_{A_t^-}(a)}\, 1_{a<s}\, E_0\left(1_{\Lambda_s} 1_{X_s>0}\psi\left(\frac{|X_s|}{\sqrt{t-s}}\right)\middle|\, A_s^- = a\right) \\
&\quad + \frac{1}{p_{A_t^-}(a)} \int_0^{t-s} 1_{v<a<v+s} \frac{\mathrm{d}v}{\pi\sqrt{v(t-s-v)}}\, p_{A_s^-}(a-v)\, E_0\left(1_{\Lambda_s} 1_{X_s>0} \mathrm{e}^{-\frac{X_s^2}{2(t-s-v)}} \big| A_s^- = a-v\right) \\
&\underset{t\to\infty}{\longrightarrow} \quad \frac{1_{a<s}}{\sqrt{s-a}} \sqrt{\frac{2}{\pi}}\, E_0\big(1_{\Lambda_s} X_s^+ \big| A_s^- = a\big) + E_0\left(1_{\Lambda_s} 1_{X_s>0} \sqrt{\frac{a}{a-A_s^-}}\, 1_{A_s^-<a}\right).
\end{aligned}
$$

Hence, point 1 of Theorem 3.1 follows. □

3.3.2. *Proof of points 2 and 3 in Theorem 3.1*

In fact, point 2 has already been shown while proving Theorem 2.5. With the help of the form (2.41) of M^h and the explicit computation of ν_x (see formula (2.23)), we obtain:

$$
\begin{aligned}
M_s^h \quad &= \quad \frac{\displaystyle\int_0^\infty \nu_{X_s}(\mathrm{d}y)\, h(A_s^- + y)}{\displaystyle\int_0^\infty \nu_0(\mathrm{d}y)\, h(y)} = \frac{\displaystyle\int_0^\infty h(A_s^- + y)\Big[X_s^+ \sqrt{\frac{2}{\pi}} \delta_0(\mathrm{d}y) + \frac{1}{\pi}\, \mathrm{e}^{-\frac{(X_s^-)^2}{2y}} \frac{\mathrm{d}y}{\sqrt{y}}\Big]}{\displaystyle\frac{1}{\pi}\int_0^\infty h(y)\, \frac{\mathrm{d}y}{\sqrt{y}}} \\
&= \quad \sqrt{2\pi}\, X_s^+\, h(A_s^-) + \int_0^\infty \frac{\mathrm{d}y}{\sqrt{y}}\, \mathrm{e}^{-\frac{(X_s^-)^2}{2y}}\, h(A_s^- + y). \qquad (3.26)
\end{aligned}
$$

Now, clearly, this point 1 of Theorem 3.1 which we just proved implies also point 2 of the same Theorem 3.1. Indeed, we have:

$$
\frac{E_0\big(1_{\Lambda_s} h(A_t^-)\big)}{E_0\big(h(A_t^-)\big)} = \frac{\displaystyle\int_0^t E_0\big(1_{\Lambda_s} | A_t^- = a\big)\, h(a)\, p_{A_t^-}(a)\mathrm{d}a}{\displaystyle\int_0^t h(a)\, p_{A_t^-}(a)\mathrm{d}a}.
$$

From the above point 1, and with the help of the explicit form of $p_{A_t^-}(a)$ as given by (3.9) the above quantity converges, as $t \to \infty$, towards:

$$
\frac{\displaystyle\int_0^\infty \frac{\mathrm{d}a}{\sqrt{a}}\, Q^{(a)}(\Lambda_s) h(a)}{\displaystyle\int_0^\infty \frac{h(a)\mathrm{d}a}{\sqrt{a}}} = \int_0^\infty \frac{h(a)}{\sqrt{a}}\, Q^{(a)}(\Lambda_s)\mathrm{d}a \qquad (3.27)
$$

since we assumed: $\displaystyle\int_0^\infty \frac{h(a)\mathrm{d}a}{\sqrt{a}} = 1.$

It now remains to compute, to prove point 3,

$$\begin{aligned}\int_0^\infty \frac{h(a)}{\sqrt{a}}\, Q^{(a)}(\Lambda_s)\mathrm{d}a &= \sqrt{\frac{2}{\pi}}\int_0^s \frac{1}{\sqrt{a(s-a)}}\, E_0\big(1_{\Lambda_s} X_s^+ | A_s^- = a\big) h(a)\mathrm{d}a \qquad (3.28)\\ &\quad + \int_0^\infty \frac{h(a)}{\sqrt{a}}\, E_0\left(1_{\Lambda_s}\sqrt{\frac{a}{a-A_s^-}}\, 1_{A_s^- < a}\, \mathrm{e}^{-\frac{(X_s^-)^2}{2(a-A_s^-)}}\right)\mathrm{d}a \quad \big(\text{from } (3.12)\big)\\ &= \sqrt{2\pi}\int_0^s p_{A_s^-}(a)\, E_0\big(1_{\Lambda_s} X_s^+ | A_s^- = a\big)\, h(a)\mathrm{d}a\\ &\quad + \int_0^\infty \frac{\mathrm{d}y}{\sqrt{y}}\, E_0\left(1_{\Lambda_s}\mathrm{e}^{-\frac{(X_s^-)^2}{2y}} h(A_s^- + y)\right)\\ &\quad \big(\text{after the change of variable } a - A_s^- = y\big)\\ &= \sqrt{2\pi}\, E_0\big(1_{\Lambda_s} X_s^+\, h(A_s^-)\big) + \int_0^\infty E_0\left(1_{\Lambda_s}\mathrm{e}^{-\frac{(X_s^-)^2}{2y}} h(A_s^- + y)\right)\frac{\mathrm{d}y}{\sqrt{y}}\\ &= E_0(1_{\Lambda_s} M_s^h) \qquad (\text{from } (3.14)). \qquad (3.29)\end{aligned}$$

We now remark that point 3 in Theorem 3.1 states precisely formula (3.29) we just established. □

3.3.3. *Proofs of points 4i) and 4ii) in Theorem 3.1*

a) From formula (3.15) and from Doob's optional sampling theorem, we deduce:

$$Q^h(A_\infty^- > a) = E[M_{\sigma_a}^h], \quad \text{with } \sigma_a := \inf\{t\,;\ A_t^- > a\}. \qquad (3.30)$$

But:

$$\begin{aligned}M_{\sigma_a}^h &= \sqrt{2\pi}\, h(a)\, X_{\sigma_a}^+ + \int_0^\infty \frac{\mathrm{d}y}{\sqrt{y}}\, \mathrm{e}^{-\frac{(X_{\sigma_a}^-)^2}{2y}} h(a+y)\\ &= \int_0^\infty \frac{\mathrm{d}y}{\sqrt{y}}\, \mathrm{e}^{-\frac{(X_{\sigma_a}^-)^2}{2y}} h(a+y) \qquad \text{since } X_{\sigma_a}^+ = 0.\end{aligned}$$

We recall that the *process* $(X_{\sigma_a}^-\,;\ a \geq 0)$ is distributed as the reflecting Brownian motion $\big(|X_a|,\ a \geq 0\big)$, where $(X_a,\ a > 0)$ is a standard Brownian motion starting from 0 (see, *e.g.* [5], Thm. 3.1, p. 419). Hence, we obtain:

$$\begin{aligned}E[M_{\sigma_a}^h] &= \int_0^\infty \frac{\mathrm{d}y}{\sqrt{y}}\, h(a+y)\, E(\mathrm{e}^{-\frac{X_a^2}{2y}})\\ &= \int_0^\infty \frac{\mathrm{d}y}{\sqrt{y}}\, h(a+y)\sqrt{\frac{y}{y+a}} = \int_0^\infty \frac{\mathrm{d}y}{\sqrt{y+a}}\, h(a+y) = \int_a^\infty \frac{\mathrm{d}y}{\sqrt{y}}\, h(y).\end{aligned}$$

b) We now remark that it is easy to recover the law of A_∞^- under Q^h from points 1 and 2 in Theorem 3.1. We may already prove that, under $Q^{(a)}$, one has $A_\infty^- = a$ a.s. Indeed, this follows from:

$$\begin{aligned}\text{if } b > a, \qquad Q^{(a)}(A_s^- > b) &= \sqrt{\frac{2}{\pi}}\, \frac{1_{a<s}}{\sqrt{s-a}}\, E_0\big(X_s^+ 1_{A_s^- > b} | A_s^- = a\big)\\ &\quad + E_0\left(\sqrt{\frac{a}{a-A_s^-}}\, 1_{b<A_s^-<a}\, \mathrm{e}^{-\frac{(X_s^-)^2}{2(a-A_s^-)}}\right) = 0.\end{aligned}$$

Hence, passing to the limit as $s \to \infty$, if $b > a$: $Q^a(A_\infty^- > b) = 0$.

On the other hand, it is clear that $E_0\big(1_{A_s^-\le a}|A_t^- = a\big) = 1 \quad (t > s)$, hence, passing to the limit as $t \to \infty$, and then, letting $s \to \infty$ we obtain:

$$Q^{(a)}(A_\infty^- \le a) = 1.$$

Finally, from (3.15), we get:

$$Q^h(A_\infty^- \le b) = \int_0^\infty \frac{h(a)}{\sqrt{a}}\, Q^{(a)}(A_\infty^- \le b)\mathrm{d}a = \int_0^b \frac{h(a)}{\sqrt{a}}\, \mathrm{d}a.$$

□

3.3.4. *Computation of Azéma's supermartingale* $Z_t := Q^h(g > t|\mathcal{F}_t)$

Our proof of points 4iii) and 4iv) in Theorem 3.1 is based on the theory of enlargements of filtration (*cf.* [3] or [4]). In order to apply this theory, we need to calculate Azéma's supermartingale $Q^h(g > t|\mathcal{F}_t)$. We start with this computation.

$$\text{Let} \quad g = \inf\{t \ge 0\,;\ A_t^- = A_\infty^-\} = \sup\{t \ge 0\,;\ X_t \le 0\}. \tag{3.31}$$

Lemma 3.2. *The following explicit formula holds:*

$$Z_t := Q^{(h)}\big(g > t|\mathcal{F}_t\big) = 1_{(X_t<0)} + 1_{(X_t>0)}\, \frac{\displaystyle\int_0^\infty \frac{\mathrm{d}v}{\sqrt{v}}\, h(A_t^- + v)}{M_t^h}. \tag{3.32}$$

Proof of Lemma 3.2. We note that, for $\Lambda_t \in \mathcal{F}_t$:

$$Q^h(1_{g>t}1_{\Lambda_t}) = Q^h(1_{\Lambda_t}1_{X_t<0}) + Q^h(1_{\Lambda_t}1_{X_t>0}1_{d_t<\infty})$$

(where d_t denotes the first return time to 0 after time t)

$$= Q^h(1_{\Lambda_t}1_{X_t<0}) + E(1_{\Lambda_t}1_{X_{t>0}}M_{d_t}^h).$$

We have:

$$M_{d_t}^h = \sqrt{2\pi}\, h(A_{d_t}^-)X_{d_t}^+ + \int_0^\infty \frac{\mathrm{d}v}{\sqrt{v}}\, h(A_{d_t}^- + v) = \int_0^\infty \frac{\mathrm{d}v}{\sqrt{v}}\, h(A_t^- + v) \quad (\text{from} \quad (3.14))$$

since $X_{d_t} = 0$ and $A_{d_t}^- = A_t^-$ on the set $\Big(X_t > 0\Big)$.

Hence:

$$Q^h(1_{g>t}1_{\Lambda_t}) = Q^h\left(1_{\Lambda_t}\left(1_{X_t<0} + 1_{X_t>0}\, \frac{\displaystyle\int_0^\infty \frac{\mathrm{d}w}{\sqrt{w}}\, h(A_t^- + w)}{M_t^h}\right)\right).$$

This proves (3.32), hence Lemma 3.2. □

3.3.5. *Proof of* $Q^h(g<\infty)=1$

We deduce from (3.32) that:

$$\begin{aligned} Q[g<t] &= 1-Q[g>t] \\ &= 1-E\left[1_{X_t<0}M_t^h + 1_{X_t>0}\int_0^\infty \frac{\mathrm{d}v}{\sqrt{v}}\,h(A_t^-+v)\right] \\ &= \sqrt{2\pi}\,E\big[X_t^+ h(A_t^-)\big] \end{aligned}$$

(from (3.26) and since $(M_t^h,\ t\geq 0)$ is a martingale s.t. $E(M_t^h)=1$)

$$\begin{aligned} &= \frac{\sqrt{2\pi}}{2}\,E\left(\int_0^t h(A_s^-)\,\mathrm{d}\,L_s\right) \quad \text{(from Itô-Tanaka formula)} \\ &\underset{t\to\infty}{\longrightarrow} \sqrt{\frac{\pi}{2}}E\left(\int_0^\infty h(A_s^-)\,\mathrm{d}\,L_s\right) = \sqrt{\frac{\pi}{2}}E\left(\int_0^\infty h(a)\,\mathrm{d}L_{\sigma_a}\right) \end{aligned}$$

(where $(\sigma_a,\ a\geq 0)$ denotes the right continuous inverse of $(A_t^-,t\geq 0)$)

$$= 2\sqrt{\frac{\pi}{2}}\,E\left(\int_0^\infty h(a)\,\mathrm{d}L_a\right)$$

(since $(X_{\sigma_a}^-,\ a\geq 0)$ is distributed as $\big(|X_a|,\ a\geq 0;$ *cf.* point $a\big)$ of Sect. 3.3.3)

$$= 2\sqrt{\frac{\pi}{2}}\int_0^\infty h(a)\,E(\mathrm{d}L_a) = \int_0^\infty \frac{h(a)}{\sqrt{a}}\,\mathrm{d}a = 1$$

since $E(L_a)=\sqrt{a}\sqrt{\frac{2}{\pi}}$. □

3.3.6. *Description of the canonical process* $(X_t,\ t\geq 0)$ *under* Q^h

For this purpose, we shall use the technique of enlargement of filtrations. Thus, let $(\mathcal{G}_t,\ t\geq 0)$ denote the smallest filtration which makes g a $(\mathcal{G}_t\ t\geq 0)$ stopping time, and which contains $(\mathcal{F}_t,\ t\geq 0)$.

The application of Girsanov's Theorem and (3.14) imply the existence of a $(\mathcal{F}_t,\ Q^h)$ Brownian motion $(\beta_t,\ t\geq 0)$ such that, under Q^h:

$$X_t = \beta_t + \int_0^t \frac{1}{M_s^h}\left\{\sqrt{2\pi}\,h(A_s^-)\,1_{X_s>0} - \left(\int_0^\infty \frac{\mathrm{d}w}{w^{3/2}}\,\mathrm{e}^{-\frac{(X_s^-)^2}{2w}}\,h(A_s^-+w)\right)X_s^-\right\}\mathrm{d}s. \tag{3.33}$$

We now apply the enlargement formulae $\big($*cf.* [3,4,8]$\big)$. We first observe that:

$$\mathrm{d}Z_t = -\frac{\displaystyle\int_0^\infty \frac{\mathrm{d}w}{\sqrt{w}}\,h(A_t^-+w)}{(M_t^h)^2}\,\big(\sqrt{2\pi}\,h(A_t^-)1_{X_t>0}\,\mathrm{d}X_t\big) + \mathrm{d}V_t \tag{3.34}$$

where $(V_t,t\geq 0)$ has bounded variations and therefore:

$$\mathrm{d}\langle Z,\,X\rangle_t = -\frac{\displaystyle\int_0^\infty \frac{\mathrm{d}w}{\sqrt{w}}\,h(A_t^-+w)}{(M_t^h)^2}\,\sqrt{2\pi}\,h(A_t^-)\,1_{(X_t>0)}\mathrm{d}t. \tag{3.35}$$

Thus, there exists a $\big((\mathcal{G}_t,\ t\geq 0),\ Q^h\big)$ Brownian motion $(\widetilde{\beta}_t,\ t\geq 0)$ such that:

$$\begin{aligned} \mathrm{d}X_t &= \mathrm{d}\widetilde{\beta}_t + \frac{1}{M_t^h}\left\{\sqrt{2\pi}\,h(A_t^-)\,1_{X_t>0} - \Big(\int_0^\infty \frac{\mathrm{d}w}{w^{3/2}}\,\mathrm{e}^{-\frac{(X_t^-)^2}{2w}}\,h(A_t^-+w)\Big)\,X_t^-\right\}\mathrm{d}t \\ &\quad +1_{t<g}\left[-\frac{\int_0^\infty \frac{\mathrm{d}w}{\sqrt{w}}\,h(A_t^-+w)}{(M_t^h)^2}\big(\sqrt{2\pi}\,h(A_t^-)\,1_{X_t>0}\big)\times\frac{M_t^h}{\int_0^\infty\frac{\mathrm{d}w}{\sqrt{w}}\,h(A_t^-+w)}\right]\mathrm{d}t \\ &\quad -1_{t>g}\left[-\frac{\int_0^\infty \frac{\mathrm{d}w}{\sqrt{w}}\,h(A_t^-+w)}{(M_t^h)^2}\big(\sqrt{2\pi}\,h(A_t^-)\,1_{X_t>0}\big)\times\frac{M_t^h}{\sqrt{2\pi}\,h(A_t^-)\,X_t^+}\right]\mathrm{d}t. \end{aligned}$$

This yields, after some simplifications:

$$X_t = \widetilde{\beta}_t - \int_0^{t\wedge g}\frac{1}{M_s^h}\left(\int_0^\infty\frac{\mathrm{d}w}{w^{3/2}}\,\mathrm{e}^{-\frac{(X_s^-)^2}{2w}}h(A_s^-+w)\right)\,X_s^-\,\mathrm{d}s + \int_{t\wedge g}^t\frac{\mathrm{d}s}{X_s} \tag{3.36}$$

since, after g, $X_t^-=0$, hence $X_t^+=X_t$.

Points 4 iii) and iv) of Theorem 3.1 now follow immediately from (3.36). □

Remark 3.3. When $h(x)=\mathrm{e}^{-\frac{\lambda x}{2}}\quad(\lambda>0,\ x\geq 0)$, the equation (3.36) simplifies as:

$$X_t = \widetilde{\beta}_t - \int_0^{t\wedge g}\frac{\sqrt{X_s^-}\,\lambda^{\frac12}}{\lambda^{\frac14}\sqrt{2\pi}\,X_s^+ - \sqrt{X_s^-}}\,\mathrm{d}s + \int_{t\wedge g}^t\frac{\mathrm{d}s}{X_s}\cdot \tag{3.37}$$

This formula (3.37) follows from:

$$\begin{aligned} \int_0^\infty\frac{\mathrm{d}w}{w^{3/2}}\,\mathrm{e}^{-\frac{(X_s^-)^2}{2w}-\frac{\lambda}{2}w}\mathrm{d}w &= \left(\frac{(X_s^-)^2}{\lambda}\right)^{-1/4} 2\,K_{-1/2}(\sqrt{\lambda}X_s^-) \\ \int_0^\infty\frac{\mathrm{d}w}{w^{1/2}}\mathrm{e}^{-\frac{(X_s^-)^2}{2w}-\frac{\lambda}{2}w}\mathrm{d}w &= \left(\frac{(X_s^-)^2}{\lambda}\right)^{+1/4} 2\,K_{1/2}(\sqrt{\lambda}X_s^-) \end{aligned}$$

and from: $K_{-\frac12}(z)=K_{\frac12}(z)=\left(\dfrac{\pi}{2z}\right)^{\frac12}\mathrm{e}^{-z}$ ([7], p. 112 and p. 119).

3.3.7. *Markovian limit process*

Theorem 3.1 shows that the process $(X_t,\ t\geq 0)$ is not Markovian under Q^h, whereas the 2-dimensional process $\big((X_t,A_t^-),\ t\geq 0\big)$ is Markovian.

Indeed, g is not a $(\mathcal{F}_t,\ t\geq 0)$ stopping time and the dynamics of (X_t) is not the same before and after g. On the other hand, we know $\big($see [14]$\big)$ that if $h(x):=\mathrm{e}^{-\frac{\lambda}{2}x}\quad(\lambda,x\geq 0)$, then the Q^h-process is Markovian. It is the diffusion with infinitesimal generator L^h:

$$L^hf(x)=\frac12\,f''(x)+\frac{\varphi'}{\varphi}\,(x)f'(x),\qquad f\in C_b^2$$

where φ denotes the unique solution of $\varphi''=\lambda\varphi,\varphi(-\infty)=0\ ;\ \varphi'(+\infty)=\sqrt{\dfrac{2}{\pi}}\cdot$ In this case, the solution of this equation (see (2.24)) takes the explicit form:

$$\varphi_\lambda(x)=\sqrt{\frac{2}{\pi}}\left\{\mathrm{e}^{x\sqrt{\lambda}}\,\frac{1}{\sqrt{\lambda}}\,1_{x\leq 0}+\left(x+\frac{1}{\sqrt{\lambda}}\right)1_{x>0}\right\}. \tag{3.38}$$

Under Q^h, we obtain:

$$X_t = B_t + \int_0^t \frac{\mathrm{d}u}{X_u^+ + \frac{1}{\sqrt{\lambda}}} \qquad \big(\text{compare with } (3.37)\big) \tag{3.39}$$

where $(B_t,\ t \geq 0)$ is a $\big((\mathcal{F}_t,\ t \geq 0),\ Q^h\big)$ Brownian motion. The martingale $(M_s^h,\ s \geq 0)$ is equal to:

$$M_s^h = \varphi_\lambda(X_s) \exp\left(-\frac{\lambda}{2}\int_0^s 1_{]-\infty,0]}(X_u)\,\mathrm{d}u\right). \tag{3.40}$$

This example motivated us to raise the question: which are the functions h such that the Q^h-process is Markovian? The answer is given by the following:

Proposition 3.4. *Let h be regular, bounded, satisfying equation (3.10) i.e., $\int_0^\infty \frac{\mathrm{d}y}{\sqrt{y}} h(y) = 1$ and such that the process $(X_t,\ t \geq 0)$ is Markov under Q^h. Then, there exists $\lambda \geq 0$ such that $h(x) = \sqrt{\frac{\lambda}{2\pi}}\mathrm{e}^{-\frac{\lambda}{2}x}$ $(x \geq\ 0)$.*

Proof of Proposition 3.4. To answer this question, we come back to equation (3.33). The problem is to find under which conditions the drift term:

$$\frac{\sqrt{2\pi}\, h(A_t^-)\, 1_{X_t>0} - \left(\int_0^\infty \frac{\mathrm{d}w}{w^{3/2}}\, \mathrm{e}^{-\frac{(X_t^-)^2}{2w}} h(A_t^- + w)\right) X_t^-}{\sqrt{2\pi}\, h(A_t^-)\, X_t^+ + \int_0^\infty \frac{\mathrm{d}w}{w^{1/2}}\, \mathrm{e}^{-\frac{(X_t^-)^2}{2w}} h(A_t^- + w)} \tag{3.41}$$

does not depend on A_t^-. Considering this expression when $X_t < 0$, the problem amounts to study the functions h for which:

$$\frac{x\int_0^\infty \frac{\mathrm{d}w}{w^{3/2}}\, \mathrm{e}^{-\frac{x^2}{2w}} h(a+w)}{\int_0^\infty \frac{\mathrm{d}w}{w^{1/2}}\, \mathrm{e}^{-\frac{x^2}{2w}} h(a+w)} := k(x) \tag{3.42}$$

does not depend on a. (3.42) may be written: $\frac{\partial}{\partial x}\log\big(\theta(x,a)\big) = -k(x)$ where we have denoted: $\theta(x,a) := \int_0^\infty \frac{\mathrm{d}w}{\sqrt{w}}\, \mathrm{e}^{-\frac{x^2}{2w}} h(a+w)$.

Hence, by integration we obtain the existence of two functions φ_1 and φ_2 such that:

$$\int_0^\infty \frac{x}{\sqrt{2\pi w^3}}\, \mathrm{e}^{-\frac{x^2}{2w}} h(a+w)\,\mathrm{d}w = \varphi_1(a)\varphi_2(x). \tag{3.43}$$

Letting $x \longrightarrow 0$ in (3.43), we obtain $h(a) = \varphi_1(a)\varphi_2(0)$. Note that the LHS in (3.43) writes $E\big(h(a+T_x)\big)$, where $(T_x,\ x \geq 0)$ is the $\frac{1}{2}$-stable subordinator of Brownian first hitting times. Hence we have:

$$E\big[h(a+T_x)\big] = P_x(h)(a) = E\left[\varphi_1(a+T_x)\varphi_2(0)\right] = \varphi_1(a)\varphi_2(0) \tag{3.44}$$

where $(P_x, x \geq 0)$ denotes the semi-group associated with the subordinator $(T_x,\ x \geq 0)$, whose infinitesimal generator is $\left(\frac{\partial^2}{\partial x^2}\right)^{\frac{1}{2}}$. In other terms, from (3.43), we get:

$$P_x\varphi_1(a) = \frac{\varphi_2(x)}{\varphi_2(0)}\,\varphi_1(a). \tag{3.45}$$

φ_1 is an eigenfunction of P_x, and consequently an eigenfunction of $\frac{\partial^2}{\partial x^2}$. φ_1 being positive and bounded: $\varphi_1(a) = c\,\mathrm{e}^{-\frac{\lambda}{2}\,a}$ $(a, \lambda \geq 0)$ and $h(a) = c\,\mathrm{e}^{-\frac{\lambda}{2}a}\varphi_2(0) = c'\,\mathrm{e}^{-\frac{\lambda}{2}a}$. □

4. A local limit theorem for a class of additive functionals of the "long Brownian bridges"

4.1. **Statement of Theorem 4.1**

In this section, our aim is to obtain results similar to those in Section 2, but, now, Brownian motion $(X_s,\ s \geq 0)$ is being replaced by the Brownian bridge with length t, with $t \to \infty$. q denotes a function from $\mathbb{R}$ to $\mathbb{R}_+$, which is Borel, and such that:

$$0 < \int_{-\infty}^{\infty} (1 + x^2)\, q(x)\, \mathrm{d}x < \infty. \tag{4.1}$$

We let:

$$A_t^q := \int_0^t q(X_s)\, \mathrm{d}s. \tag{4.2}$$

Theorem 4.1.
1) *For every x and $y \in \mathbb{R}$, and $\mu > 0$:*

$$E_x\Big(\exp\Big(-\frac{\mu}{2}\int_0^t q(X_s)\,\mathrm{d}s\Big)\Big|X_t = y\Big) \underset{t\to\infty}{\sim} \frac{\pi}{2}\,\frac{\varphi_{\mu q}(x)\varphi_{\mu q}(y)}{t} \tag{4.3}$$

where $\varphi_{\mu q}$ denotes the unique solution of:

$$\varphi'' = (\mu q)\,\varphi,\ \lim_{x\to+\infty}\varphi'(x) = -\lim_{x\to-\infty}\varphi'(x) = \sqrt{\frac{2}{\pi}} \tag{4.4}$$

2) $\lim_{t\to\infty} \; t\, P_x\big(A_t^q \in \mathrm{d}z | X_t = y\big) = \nu_x * \nu_y\,(\mathrm{d}z)$ (4.5)
where ν_x and ν_y have been defined in Theorem 2.1. The convergence in (4.5) has the same meaning as in Theorem 2.1.

4.2. **Proof of Theorem 4.1**

Without loss of generality, we shall assume that $\mu = 1$.

4.2.1. *An auxiliary lemma*

Lemma 4.2. *There exists a constant $C > 0$, depending only on q, such that:*

$$E_x\Big(\exp\Big(-\frac{1}{2}\int_0^t q(X_s)\,\mathrm{d}s\Big)\big|X_t = y\Big) \leq C\,\mathrm{e}^{\frac{(x-y)^2}{2t}}\,\frac{(1+|x|)(1+|y|)}{1+t}. \tag{4.6}$$

4.2.2. *Proof of Lemma 4.2*

1) *As an intermediary result, we already show that*:

$$E_x\Big(\exp\Big(-\frac{1}{2}\int_0^t q(X_s)\,\mathrm{d}s\Big)\big|X_t = y\Big) \leq C\,\mathrm{e}^{\frac{(x-y)^2}{2t}}\,\frac{1+|x|}{\sqrt{1+t}} \tag{4.7}$$

for a constant C which does not depend on x, y, t.

To prove (4.7), we condition with respect to $X_{t/2}$, and we get:

$$E_x\Big(\exp\Big(-\frac{1}{2}\int_0^t q(X_s)\,\mathrm{d}s\Big)\big|X_t=y\Big)\,\mathrm{e}^{-\frac{(x-y)^2}{2t}} = \frac{1}{\sqrt{\pi t}}\int_{-\infty}^{\infty} E_x\Big(\exp-\frac{1}{2}A^q_{t/2}\big|X_{t/2}=c\Big)$$
$$E_c\Big(\exp-\frac{1}{2}A^q_{t/2}\big|X_{\frac{t}{2}}=y\Big)\,\mathrm{e}^{-\frac{(x-c)^2}{t}-\frac{(y-c)^2}{t}}\mathrm{d}c. \tag{4.8}$$

In (4.8), we majorize $E_c\Big(\exp-\frac{1}{2}A^q_{t/2}|X_{t/2}=y\Big)$ by 1, and we get:

$$\begin{aligned} E_x\Big(\exp-\frac{1}{2}A^q_t|X_t=y\Big)\,\mathrm{e}^{-\frac{(x-y)^2}{2t}} &\leq \frac{1}{\sqrt{\pi t}}\int_{-\infty}^{\infty} E_x\Big(\exp-\frac{1}{2}A^q_{t/2}|X_{t/2}=c\Big)\,\mathrm{e}^{-\frac{(x-c)^2}{2t/2}}\mathrm{d}c \\ &\leq E_x\Big(\exp-\frac{1}{2}A^q_{t/2}\Big)\leq C\,\frac{1+|x|}{\sqrt{1+t}} \end{aligned}$$

from Lemma 4.3 in [14]. Thus, we have obtained (4.7).

2) *Then, plugging the estimate (4.7) in (4.8), we obtain:*

$$\begin{aligned} E_x\Big(\exp\Big(-\frac{1}{2}\,A^q_t\Big)\big|X_t=y\Big) &\leq \mathrm{e}^{\frac{(x-y)^2}{2t}}\,\frac{C\big(1+|y|\big)}{\sqrt{1+t}}\int_{-\infty}^{\infty}E_x\Big(\exp-\frac{1}{2}A^q_{t/2}|X_{t/2}=c\Big)\,\frac{\mathrm{e}^{-\frac{(x-c)^2}{2t/2}}}{\sqrt{2\pi t/2}}\,\mathrm{d}c \\ &\leq \frac{C\big(1+|x|\big)\,\big(1+|y|\big)}{1+t}\,\mathrm{e}^{\frac{(x-y)^2}{2t}} \end{aligned}$$

since:

$$E_c\Big(\exp-\frac{1}{2}A^q_{t/2}|X_{t/2}=y\Big)=E_y\Big(\exp-\frac{1}{2}A^q_{t/2}|X_{t/2}=c\Big)$$

and

$$\mathrm{e}^{\frac{(x-y)^2}{2t}}\,\frac{C\big(1+|y|\big)}{\sqrt{1+t}}\,E_x\Big(\exp-\frac{1}{2}A^q_{t/2}|X_{t/2}=c\Big)\leq \mathrm{e}^{\frac{(x-y)^2}{2t}}\,C\,\frac{\big(1+|y|\big)\big(1+|x|\big)}{1+t}$$

by applying once again Lemma 4.3 in [14]. □

4.2.3. *Another auxiliary lemma*

Lemma 4.3. *Let* $Z(t,x,y):=E_x\Big(\exp-\frac{1}{2}A^q_t|X_t=y\Big)$. *We also denote by* $U(t,x,y)$ *the solution of:*

$$\left\{\begin{array}{l} \dfrac{\partial U}{\partial t}\,(t,x,y)-\dfrac{1}{2}\,\dfrac{\partial^2 U}{\partial x^2}\,(t,x,y)+\dfrac{1}{2}\,U(t,x,y)\,q(x)=0 \\ U(0,\bullet,y)=\delta_y. \end{array}\right. \tag{4.9}$$

Then: $\quad Z(t,x,y)=\sqrt{2\pi t}\,\mathrm{e}^{\frac{(x-y)^2}{2t}}\,U(t,x,y)$.

In particular, it follows from Lemma 4.2, that:

$$U(t,x,y)\leq C\,\frac{\big(1+|x|\big)\big(1+|y|\big)}{(1+t)^{3/2}}\quad (t\geq 1). \tag{4.10}$$

4.2.4. *Proof of Lemma 4.3*

We know that, for every regular function f:

$$Z^f(t,x):=E_x\Big[\exp\Big(-\frac{1}{2}\,A^q_t\Big)\,f(X_t)\Big]$$

is solution of:

$$\frac{\partial Z^f}{\partial t} - \frac{1}{2}\,\frac{\partial^2 Z^f}{\partial x^2} + \frac{1}{2}\,Z^f q = 0, \quad Z^f(0,x) = f(x). \tag{4.11}$$

It suffices, in order to obtain Lemma 4.3, to write:

$$Z(t,x,y) = \lim_{\varepsilon\downarrow 0} \frac{E_x\Big[\Big(\exp\Big(-\frac{1}{2}\,A_t^q\Big)\Big)\; f_\varepsilon(X_t)\Big]}{E_x\big(f_\varepsilon(X_t)\big)}$$

where f_ε is a family of functions which converges weakly towards δ_y, and to use (4.11). □

4.2.5. *Use of the Laplace transform*

We define, for every $\lambda > 0$:

$$A(\lambda,x,y) = \int_0^\infty \mathrm{e}^{-\lambda t} U(t,x,y)\,\mathrm{d}t. \tag{4.12}$$

Since $Z(t,x,y)$ is a decreasing function of t, we deduce the following equivalences from the Tauberian theorem (see [1])

$$\begin{aligned}
&i) \quad Z(t,x,y) \underset{t\to\infty}{\sim} \frac{\pi}{2}\,\frac{\varphi_q(x)\varphi_q(y)}{t}\\
&ii) \quad U(t,x,y) \underset{t\to\infty}{\sim} \frac{\sqrt{\pi}}{2\sqrt{2}}\,\frac{\varphi_q(x)\varphi_q(y)}{t^{3/2}}\\
&iii) \quad \Big|\frac{\partial}{\partial\lambda}\,A(\lambda,x,y)\Big| = -\frac{\partial}{\partial\lambda}\,A(\lambda,x,y) \underset{\lambda\to 0}{\sim} \frac{1}{2\sqrt{2}}\,\frac{1}{\sqrt{\lambda}}\,\varphi_q(x)\varphi_q(y).
\end{aligned} \tag{4.13}$$

We shall now show (4.13). We already deduce from Lemmas 4.2 and 4.3 that:

$$\lim_{\lambda\to 0} A(\lambda,x,y) = \int_0^\infty U(t,x,y)\,\mathrm{d}t < \infty \tag{4.14}$$

$$A(\lambda,x,y) \le C\,\big(1+|x|\big)\big(1+|y|\big) \tag{4.15}$$

$$\Big|\frac{\partial}{\partial\lambda}A(\lambda,x,y)\Big| \le \frac{C}{\sqrt{\lambda}}\,\big(1+|x|\big)\big(1+|y|\big). \tag{4.16}$$

To prove (4.13) we shall show that: $\psi(x,y) := \lim_{\lambda\to 0}\sqrt{\lambda}\,\frac{\partial}{\partial\lambda}\,A(\lambda,x,y)$ satisfies the Sturm-Liouville equation (for any fixed y):

$$\frac{\partial^2}{\partial x^2}\,\psi = \psi\,q, \quad \text{with adequate limit conditions in } x = \pm\infty. \tag{4.17}$$

4.2.6. *Convergence of the Laplace transform: first step*

We get, from (4.9):

$$U(t,x,y) = \frac{1}{\sqrt{2\pi t}}\,\mathrm{e}^{-\frac{(x-y)^2}{2t}} - \frac{1}{2}\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\frac{(x-z)^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}}\,U(s,z,y)\,q(z)\mathrm{d}z. \tag{4.18}$$

Thus, after taking the Laplace transform in the variable t of the two sides of (4.18), we obtain:

$$A(\lambda,x,y) = g_\lambda(x,y) - \frac{1}{2}\int_{-\infty}^{\infty} g_\lambda(x,z)\,A(\lambda,z,y)\,q(z)\mathrm{d}z \tag{4.19}$$

where g_λ denotes the density of the resolvent kernel of Brownian motion:

$$g_\lambda(x,z) = \frac{1}{\sqrt{2\lambda}}\, \mathrm{e}^{-|x-z|\,\sqrt{2\lambda}}.$$

We write (4.19) in the form:

$$A(\lambda,x,y) = G_\lambda\Big[\delta_y - \frac{1}{2}\,\big(A(\lambda,\bullet,y)\,q(\bullet)\Big](x) \tag{4.20}$$

with for any Radon measure $\mu(\mathrm{d}z)$:

$$G_\lambda\mu\,(x) := \int_{-\infty}^{\infty} g_\lambda(x,z)\,\mu(\mathrm{d}z) \tag{4.21}$$

and we use the resolvent equation: $\dfrac{\partial^2}{\partial x^2}\,G_\lambda\,\mu = -2\mu + 2\lambda G_\lambda\,\mu$, to obtain:

$$\frac{\partial^2}{\partial x^2}\,A(\lambda,x,y) = 2\lambda\,A(\lambda,x,y) - \big[2\delta_y - A(\lambda,x,y)\,q(x)\big]. \tag{4.22}$$

As a consequence, differentiating with respect to λ, then multiplying by $\sqrt{\lambda}$, we obtain:

$$\frac{\partial^2}{\partial x^2}\Big(\sqrt{\lambda}\,\frac{\partial A}{\partial\lambda}\,(\lambda,x,y)\Big) - \sqrt{\lambda}\,\frac{\partial A}{\partial\lambda}(\lambda,x,y)\,q(x) = 2\sqrt{\lambda}\,A(\lambda,x,y) + 2\lambda^{3/2}\frac{\partial A}{\partial\lambda}\,(\lambda,x,y). \tag{4.23}$$

Hence, from (4.16) and (4.15), and denoting $\widetilde{A}(\lambda,x,y) := \sqrt{\lambda}\,\dfrac{\partial A}{\partial\lambda}\,(\lambda,x,y)$, it follows that:

$$\Big|\frac{\partial^2}{\partial x^2}\big(\widetilde{A}(\lambda,x,y)\big) - \widetilde{A}(\lambda,x,y)\,q(x)\Big| \le C\,\sqrt{\lambda}\big(1+|x|\big)\big(1+|y|\big) \qquad (\lambda\to 0) \tag{4.24}$$

(4.24) is the first step to prove that $\widetilde{A}(\lambda,x,y)$ converges, as $\lambda\to 0$, to a solution of the Sturm-Liouville equation (4.17).

4.2.7. *Convergence of the Laplace transform: limit conditions in* $x=\pm\infty$

We now examine the limit conditions in $x=\pm\infty$.
We come back to equation (4.19) which we differentiate with respect to λ, then we multiply by λ:

$$\begin{aligned}\sqrt{\lambda}\widetilde{A}(\lambda,x,y) \;&=\; -\frac{1}{2}\,A(\lambda,x,y) - \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{e}^{-|x-z|\sqrt{2\lambda}}|x-z|\,\big(\delta_y(\mathrm{d}z) - A(\lambda,z,y)\,q(z)\mathrm{d}z\big)\\ &\quad -\frac{1}{\sqrt{2}}\int_{-\infty}^{\infty}\mathrm{e}^{-|x-z|\,\sqrt{2\lambda}}\widetilde{A}(\lambda,x,z)\,q(z)\mathrm{d}z.\end{aligned} \tag{4.25}$$

From (4.16) and (4.14), respectively we deduce that:

$$\sqrt{\lambda}\,\widetilde{A}(\lambda,x,y) \underset{\lambda\to 0}{\longrightarrow} 0 \qquad \text{and} \quad A(\lambda,x,y) \quad \text{converges as} \quad \lambda\to 0.$$

Hence, from (4.25), since $\displaystyle\int_{-\infty}^{\infty}(1+x^2)\,q\,(\mathrm{d}z) < \infty$:

$$\lim_{\lambda\to 0}\int_{-\infty}^{\infty}\mathrm{e}^{-|x-z|\,\sqrt{2\lambda}}\widetilde{A}(\lambda,x,z)\,q(z)\mathrm{d}z \quad \text{exists.} \tag{4.26}$$

On the other hand, differentiating (4.25) with respect to x, we obtain:

$$\frac{\partial \widetilde{A}}{\partial x}(\lambda, x, y) = \frac{\partial B}{\partial x} - \frac{1}{2}\left\{\int_{-\infty}^{x} \mathrm{e}^{-|x-z|\sqrt{2\lambda}}\widetilde{A}(\lambda, z, y)\, q(z)\mathrm{d}z + \int_{x}^{\infty} \mathrm{e}^{-|x-z|\sqrt{2\lambda}}\widetilde{A}(\lambda, z, y)\, q(z)\mathrm{d}z\right\} \tag{4.27}$$

with

$$B := -\frac{1}{2\sqrt{\lambda}}\left\{A + \int_{-\infty}^{\infty} \mathrm{e}^{-|x-z|\sqrt{2\lambda}}\big(A(\lambda, z, y)\, q(z)\mathrm{d}z - \delta_y(\mathrm{d}z)\big)\right\}. \tag{4.28}$$

We deduce from (4.27), (4.26) and (4.28) that:

$$\lim_{\substack{\lambda\to 0\\ x\to\infty}} \frac{\partial}{\partial x}\widetilde{A}(\lambda, x, y) = -\lim_{\substack{\lambda\to 0\\ x\to-\infty}} \frac{\partial}{\partial x}\widetilde{A}(\lambda, x, y) = C(y) \tag{4.29}$$

(*cf.* [14], pp. 194–197 for similar computations).

4.2.8. *End of the proof of point 1 of Theorem 4.1*

Thus, from the equivalence between i), ii) and iii) which we recalled in (4.13), we get:

$$E_x\Big[\exp - \frac{1}{2}\int_0^t q(X_s)\mathrm{d}s \big| X_t = y\Big] \underset{t\to\infty}{\sim} \frac{\psi(x,y)}{t} \tag{4.30}$$

where ψ is solution to:

$$\frac{\partial^2 \psi}{\partial x^2}(x,y) = \psi(x,y)\, q(x), \ \lim_{x\to+\infty}\frac{\partial \psi}{\partial x}(x,y) = -\lim_{x\to-\infty}\frac{\partial \psi}{\partial x}(x,y) = C(y). \tag{4.31}$$

Thus, from the definition of φ_q $\big($see (4.4)$\big)$, we get:

$$\psi(x,y) = C(y)\sqrt{\frac{\pi}{2}}\,\varphi_q(x).$$

Now, since $Z(t,x,y)$ is symmetric in x and y:

$$\psi(x,y) = K\varphi_q(x)\varphi_q(y). \tag{4.32}$$

It remains to determine the value of K. For this purpose, we write:

$$\varphi_q(x) = E_x\left(\left(\exp\left(-\frac{1}{2}\int_0^t \mathrm{d}s\, q(X_s)\right)\right)\varphi_q(X_t)\right)$$

(since $\varphi_q(X_t)\exp\left(-\frac{1}{2}A_t^q\right)$, $t \geq 0$ is a martingale)

$$\begin{aligned}
&= \int_{-\infty}^{\infty} E_x\left(\exp\left(-\frac{1}{2}A_t^q\right)\big| X_t = y\right)\frac{\mathrm{e}^{-\frac{(x-y)^2}{2t}}}{\sqrt{2\pi t}}\,\varphi_q(y)\mathrm{d}y \\
&\underset{t\to\infty}{\sim} K\int_{-\infty}^{\infty}\frac{\varphi_q(x)\varphi_q(y)}{t}\,\varphi_q(y)\,\frac{\mathrm{e}^{-\frac{(x-y)^2}{2t}}}{\sqrt{2\pi t}}\,\mathrm{d}y \\
&= \frac{K\,\varphi_q(x)}{t}\,E_x\big(\varphi_q^2(X_t)\big) \underset{t\to\infty}{\sim} \frac{K\,\varphi_q(x)}{t}\,\frac{2}{\pi}\,t
\end{aligned}$$

since $\varphi_q(z) \underset{|z|\to\infty}{\sim} \sqrt{\frac{2}{\pi}}\,|z|$. Hence $K\frac{2}{\pi} = 1$, that is: $K = \frac{\pi}{2}\cdot$

Thus, we have obtained point 1 of Theorem 4.1.

4.2.9. *Point 2 of Theorem 4.1 may be proven*

with the help of (4.3), exactly as Theorem 2.1. □

Remark 4.4. Under our hypothesis H1 on q, there is the equivalence:

$$Z(t,x,y) \equiv E_x\Big(\exp - \frac{1}{2}\int_0^t q(X_s)\mathrm{d}s \big| X_t = y\Big) \underset{t\to\infty}{\sim} \frac{\pi}{2t}\,\varphi_q(x)\varphi_q(y). \tag{4.33}$$

Intuitively, we may think of the bridge of duration t going from x to y as "resembling", as $t \to \infty$, to the concatenation of two Brownian motions each being defined on a time interval $\Big[0, \frac{t}{2}\Big]$, with the first one starting from x and the second one, after time reversal, starting from y, these two parts being independent. If this were true, then:

$$\begin{aligned} Z(t,x,y) &= E_x\Big(\exp - \frac{1}{2}\,A_t^q | X_t = y\Big) = E_x\Big(\exp - \frac{1}{2}\,A_{t/2}^q\Big)\,E_y\Big(\exp - \frac{1}{2}\,A_{t/2}^q\Big) \\ &\underset{t\to\infty}{\sim} \frac{\varphi_q(x)}{\sqrt{t/2}}\,\frac{\varphi_q(y)}{\sqrt{t/2}} = \frac{4}{\pi}\,\Big(\frac{\pi}{2}\,\frac{\varphi_q(x)\varphi_q(y)}{t}\Big)\cdot \end{aligned}$$

Thus, comparing with (4.33) the factor $\frac{4}{\pi}$ which we just obtained measures, in some sense, the default of independence of these two Brownian components.

Remark 4.5. Theorem 4.1 allows to "penalize long Brownian Bridges". More precisely, for every $s \geq 0$ and $\Lambda_s \in \mathcal{F}_s$:

$$\frac{E_x\Big(1_{\Lambda_s}\exp\Big(-\frac{1}{2}\,A_t^q\Big)\big|X_t = y\Big)}{E_x\Big(\exp\Big(-\frac{1}{2}\,A_t^q\Big)\big|X_t = y\Big)} \underset{t\to\infty}{\longrightarrow} E_x(1_{\Lambda_s}M_s^{\varphi}) \tag{4.34}$$

with $M_s^{\varphi} := \frac{\varphi_q(X_s)}{\varphi_q(x)}\,\exp\Big(-\frac{1}{2}\,A_s^q\Big)$, and $(M_s^{\varphi},\ s \geq 0)$ is a positive martingale. In other terms, comparing with Theorem 5.1 in [14], the penalisation is the same for "long bridges" as for Brownian motion itself. Once more (see [12]), we obtain that a long bridge of duration t, as $t \to \infty$, behaves as a standard Brownian motion.

4.2.10. *Finally, we show (4.34)*

$$\begin{aligned} \frac{E_x\Big(1_{\Lambda_s}\exp\Big(-\frac{1}{2}\,A_t^q\Big)\big|X_t = y\Big)}{E_x\Big(\exp\Big(-\frac{1}{2}\,A_t^q\Big)\big|X_t = y\Big)} &= \frac{E_x\Big(1_{\Lambda_s}\Big(\exp\Big(-\frac{1}{2}A_s^q\Big)\Big)\,E_{X_s,s}\Big(\exp\Big(-\frac{1}{2}\int_s^t q(X_u)\mathrm{d}u\Big)\big|X_t = y\Big)\Big)}{E_x\Big(\exp\Big(-\frac{1}{2}\,A_t^q\Big)\big|X_t = y\Big)} \\ &\underset{t\to\infty}{\sim} \frac{E_x\Big(1_{\Lambda_s}\exp\Big(-\frac{1}{2}\,A_s^q\Big)\,\frac{\varphi_q(X_s)\varphi_q(y)}{t-s}\Big)}{\frac{\varphi_q(x)\varphi_q(y)}{t}} \underset{t\to\infty}{\longrightarrow} E_x(1_{\Lambda_s}M_s^{\varphi}). \end{aligned}$$

References

[1] W. Feller, *An introduction to probability theory and its applications*, volume II. John Wiley & Sons Inc., New York (1966).
[2] Y. Hariya and M. Yor, Limiting distributions associated with moments of exponential Brownian functionals. *Studia Sci. Math. Hung.* **41** (2004) 193–242.
[3] T. Jeulin, *Semimartingales et grossissement d'une filtration.* Lect. Notes Maths **833**. Springer (1980).
[4] T. Jeulin and M. Yor, Eds., *Grossissements de filtrations: exemples et applications.* Lect. Notes Maths **1118**. Springer (1985).
[5] I. Karatzas and S. Shreve, *Brownian motion and Stochastic Calculus.* Springer (1991).
[6] S. Kotani, Asymptotics for expectations of multiplicative functionals of 1-dimensional Brownian motion. Preprint (2006).
[7] N.N. Lebedev, *Special functions and their applications.* Dover (1972).
[8] R. Mansuy and M. Yor, *Random Times and Enlargement of Filtrations in a Brownian Setting.* Lect. Notes Maths **1873**. Springer (2006).
[9] J. Najnudel, B. Roynette and M. Yor, *A global view of Brownian penalisations.* MSJ Memoirs, volume 19. Mathematical Society of Japan, Tokyo (2009).
[10] D. Revuz and M. Yor, *Continuous Martingales and Brownian Motion.* Third edition. Springer (1999).
[11] B. Roynette and M. Yor, *Penalising Brownian paths.* Lect. Notes Maths **1969**. Springer (2009).
[12] B. Roynette, P. Vallois and M. Yor, Penalisation of a Brownian motion with drift by a function of its one-sided maximum and its position, III. *Periodica Math. Hung.* **50** (2005) 247–280.
[13] B. Roynette, P. Vallois and M. Yor, Some penalisations of the Wiener measure. *Japan J. Math.* **1** (2006) 263–299.
[14] B. Roynette, P. Vallois and M. Yor, Limiting laws associated with Brownian motion perturbed by normalized exponential weights. *Studia Sci. Math. Hung.* **43** (2006) 171–246.
[15] B. Roynette, P. Vallois and M. Yor, Limiting laws associated with Brownian motions perturbed by its maximum, minimum, and local time, II. *Studia Sci. Math. Hung.* **43** (2006) 295–360.
[16] M. Yor, The distribution of Brownian quantiles. *J. Appl. Prob.* **32** (1995) 405–416.