

MULTIDIMENSIONAL LIMIT THEOREMS FOR SMOOTHED EXTREME VALUE ESTIMATES OF POINT PROCESSES BOUNDARIES

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Abstract. In this paper, we give sufficient conditions to establish central limit theorems and moderate deviation principle for a class of support estimates of empirical and Poisson point processes. The considered estimates are obtained by smoothing some bias corrected extreme values of the point process. We show how the smoothing permits to obtain Gaussian asymptotic limits and therefore pointwise confidence intervals. Some unidimensional and multidimensional examples are provided.

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INTRODUCTION

The problem of estimating a set S given a sequence of finite sets S_n of random points drawn from its interior arises in classification [15], clustering problems [21], discriminant analysis [2], outliers detection [6], image analysis [25] and in econometrics [4]. In many cases, the unknown support can be written

$$S = \{(x, y) : x \in E; 0 \leq y \leq f(x)\}, \quad (1)$$

where f is an unknown function and E an arbitrary set (typically a subset of \mathbb{R}^d). Then, the problem reduces to estimating f .

In econometrics, the data consist of pairs (X_i, Y_i) where X_i represents the input, possibly multidimensional (labor, energy or capital), used to produce an output Y_i in a given firm. In such a framework, the value $f(x)$ can be interpreted as the maximum level of output which is attainable for the level of input x . Then, economical considerations suggest to suppose that f is increasing and concave and an adapted estimator, called the DEA (Data Envelopment Analysis) has been developed and studied. Its asymptotic distribution is established by [10].

In the case where we do not assume that f is monotone, Geffroy [9] proposed an estimator of f which is a kind of histogram based on the extreme values of the sample. Geffroy's estimator has been improved in several directions. On one hand, piecewise polynomials estimators were introduced and their optimality in an asymptotic minimax sense was proved under different regularity conditions on f (see [17,20,25–27,33]).

On the other hand, regularization and bias correction of Geffroy's estimate have been considered using different way of smoothing (see [8,11,12]). In [13], the multivariate central limit theorem has been obtained for a general class of estimates of this type including all the above. For technical reason (the maxima on disjoint

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cell of Poisson processes are independent) the results obtained in [8,11,12] and [13] were only given when S_n is generated by a Poisson point process. This paper encompasses the results obtained in [13] of several manners. Central limit theorems but also moderate deviation principles are investigated for a wider class of estimators than in [13]. Moreover, we show that our results hold in the context of Poisson process but also when S_n is generated by an empirical point process.

In the next section, the general class of estimators is given. Section 2 contains our main results and provides some examples of possible bias reduction. Some applications, including kernel and projection estimators are provided in Section 3. The proofs are given in Section 4.

1. THE BOUNDARY ESTIMATE

Let $f : (E, \mathcal{E}) \rightarrow (\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ be a measurable function on a probability space (E, \mathcal{E}, ν) , where $\mathcal{B}(\mathbb{R}^+)$ is the Borel σ -algebra on \mathbb{R}^+ . Consider the set

$$S = \{(x, y) \in E \times \mathbb{R}, 0 \leq y \leq f(x)\}.$$

Our aim is to estimate S from a sequence of S -valued random vectors

$$S_n = \{(X_{n,i}, Y_{n,i}), 1 \leq i \leq N_n(S)\},$$

with associated counting process

$$N_n = \{N_n(D) : D \in \mathcal{E} \otimes \mathcal{B}(\mathbb{R}^+)\}, \quad n \geq 1,$$

of mean measure

$$n \, c \, \mathbf{1}_S(x, y) \, \nu(dx) \, dy, \tag{2}$$

where $c > 0$. Two cases are considered below:

- (P) N_n is a Poisson point process,
- (E) N_n is an (n -sample) empirical point process.

Let $k_n \uparrow \infty$ and denote by $\{I_{n,r} : 1 \leq r \leq k_n\}$ a measurable partition of E . For all $1 \leq r \leq k_n$, set

$$\nu_{n,r} = \nu(I_{n,r}), \quad D_{n,r} = \{(x, y) : x \in I_{n,r}, 0 \leq y \leq f(x)\} \quad \text{and} \quad N_{n,r} = N_n(D_{n,r}).$$

We also introduce

$$Y_{n,r}^* = \max\{Y_{n,i} : (X_{n,i}, Y_{n,i}) \in D_{n,r}\},$$

if $D_{n,r} \neq \emptyset$ and $Y_{n,r}^* = 0$ otherwise (in the sequel, we use the convention $0 \times \infty = 0$) and

$$f_{n,r} = \nu_{n,r}^{-1} \int_{I_{n,r}} f \, d\nu,$$

denotes the mean value of f on $I_{n,r}$.

In that context, f can be estimated using Geffroy estimator

$$\bar{f}_n(x) = \sum_{r=1}^{k_n} \mathbf{1}_{I_{n,r}}(x) Y_{n,r}^*. \tag{3}$$

But several improvements of \bar{f}_n can be considered. First, it is well-known that $Y_{n,r}^*$ estimates $f_{n,r}$ with a negative bias which reveals to be of the order $(n c \nu_{n,r})^{-1}$. Hence, to reduce this bias we can introduce unbiased

versions of (3):

$$\widehat{f}_n^\circ(x; \widehat{c}_n) = \sum_{r=1}^{k_n} \mathbf{1}_{I_{n,r}}(x) \left(Y_{n,r}^* + (n\widehat{c}_n \nu_{n,r})^{-1} \right), \tag{4}$$

where \widehat{c}_n is a convenient estimator of c . Now, the non random conterpoint of \widehat{f}_n° is the step function

$$f_n^\circ(x) = \sum_{r=1}^{k_n} \mathbf{1}_{I_{n,r}}(x) f_{n,r} \tag{5}$$

and if f is regular, it is clear that smoothed version of (5) can lead to more efficient estimation of f .

Therefore, we will consider estimators of f of the form

$$\widehat{f}_n(x; \widehat{c}_n) = \sum_{r=1}^{k_n} \nu_{n,r} \kappa_{n,r}(x) \left(Y_{n,r}^* + (n\widehat{c}_n(x) \nu_{n,r})^{-1} \right) \quad (x \in E), \tag{6}$$

where $\kappa_{n,r} : E \rightarrow \mathbb{R}$ is a weighting function determining the nature of the smoothing introduced in the estimate. In the next section, some general conditions are imposed on $\kappa_{n,r}$ and \widehat{c}_n in order to obtain a central limit and a moderate deviation principle for \widehat{f}_n .

2. MAIN RESULTS

In the following, we consider the ν -essential infimum and supremum of f on E ,

$$m = \sup\{\alpha > 0 : \nu(\{f < \alpha\}) = 0\} \quad \text{and} \quad M = \inf\{\alpha > 0 : \nu(\{f > \alpha\}) = 0\}$$

and, for all $1 \leq r \leq k_n$, we denote by

$$m_{n,r} = \sup\{\alpha > 0 : \nu(\{f < \alpha\} \cap I_{n,r}) = 0\},$$

$$M_{n,r} = \inf\{\alpha > 0 : \nu(\{f > \alpha\} \cap I_{n,r}) = 0\},$$

respectively the ν -essential infimum, supremum of f on $I_{n,r}$ (in most of applications, E is a subset of \mathbb{R}^d , ν is absolutely continuous with respect to Lebesgue measure and f is continuous, hence all ess-inf and ess-sup considered below are just classical inf and sup). For all $x \in E$, we set

$$f_n(x) = \sum_{r=1}^{k_n} \nu_{n,r} \kappa_{n,r}(x) f_{n,r}, \quad \kappa_n(x) = \left(\sum_{r=1}^{k_n} \kappa_{n,r}^2(x) \right)^{1/2}$$

$$w_{n,r}(x) = \kappa_{n,r}(x) / \kappa_n(x) \quad \text{and} \quad \nu_n = \min\{\nu_{n,r}, 1 \leq r \leq k_n\}.$$

In the sequel, $(\varepsilon_n)_{n \geq 1}$ denotes a sequence of positive real numbers such that $(\varepsilon_n)_{n \geq 1} \equiv 1$ or $\varepsilon_n \downarrow \infty$.

The following assumptions will be needed below:

(H.1) $k_n \uparrow \infty$ and $(n\nu_n)^{-1} \max(\log(n), \varepsilon_n^{-1}) \rightarrow 0$ as $n \rightarrow \infty$.

(H.2) $0 < m \leq M < +\infty$ and

$$\delta_n := \max_{1 \leq r \leq k_n} \nu_{n,r} (M_{n,r} - m_{n,r}) = o(1/n) \text{ as } n \rightarrow \infty.$$

There exists $F \subset E$ such that:

(H.3) For each $\{x_1, \dots, x_p\} \subset F$, there exists a covariance matrix $\Sigma_{(x_1, \dots, x_p)} = [\sigma(x_i, x_j)]_{1 \leq i, j \leq p}$ in \mathbb{R}^p such that for all $1 \leq i, j \leq p$,

$$\sum_{r=1}^{k_n} w_{n,r}(x_i)w_{n,r}(x_j) \rightarrow \sigma(x_i, x_j) \text{ as } n \rightarrow \infty.$$

(H.4) For all $x \in F$,

$$\varepsilon_n^{-1/2} \max_{1 \leq r \leq k_n} |w_{n,r}(x)| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

(H.5) For all $x \in F$,

$$\varepsilon_n^{1/2} n \kappa_n(x)^{-1} |f_n(x) - f(x)| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

(H.6) For all $x \in F$,

$$\varepsilon_n^{1/2} \sum_{r=1}^{k_n} |w_{n,r}(x)| (n\delta_n)^2 = o(1) \text{ as } n \rightarrow \infty.$$

(C.1) For all $x \in F$, and any $\eta > 0$,

$$\limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P} \left(\varepsilon_n^{1/2} \left| \sum_{r=1}^{k_n} w_{n,r}(x) \right| \left| \widehat{c}_n(x)^{-1} - c^{-1} \right| \geq \eta \right) = -\infty.$$

(C.2) For all $x \in F$, and any $\eta > 0$,

$$\limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P} (|\widehat{c}_n(x) - c| \geq \eta) = -\infty.$$

Before proceeding, let us comment on the assumptions. (H.1)–(H.4) are devoted to the control of the conditionally centered estimator $\widehat{f}_n - \mathbb{E}(\widehat{f}_n | N_n)$. Assumption (H.1) imposes that the number of terms of the partition goes to infinity not to quickly so that the mean number of points in each $D_{n,r}$ goes to infinity. (H.2) requires the unknown function f to be bounded away from 0. It also imposes that the mean number of points in $D_{n,r}$ above $m_{n,r}$ converges to 0. Note that (H.1) and (H.2) force the ν -essential oscillation of f on $I_{n,r}$ to converge uniformly to 0: $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. (H.3) is devoted to the multivariate aspects of the limit theorems. (H.4) imposes to the weight functions $\kappa_{n,r}(x)$ in the linear combination (6) to be approximatively of the same order. This is a natural condition to obtain an asymptotic gaussian behavior. These assumptions are easy to verify in practice since they involve either f or $\kappa_{n,r}$ without mixing these two quantities. Assumptions (H.5) and (H.6) are devoted to the control of the residual conditional bias term $\mathbb{E}(\widehat{f}_n | N_n) - f$. (H.5) is natural since it implies that the non random part of the bias $f_n - f$ vanishes. (H.6) control of the residual conditional bias term $\mathbb{E}(\widehat{f}_n | N_n) - f_n$. These two assumptions involve both the unknown function f and the weight functions $\kappa_{n,r}$ and (H.6) can be looked at as a stronger version of (H.2). Condition (C.1) imposes the speed of convergence of \widehat{c}_n to the unknown parameter c to cancel the bias terms $(n\kappa_{n,r})^{-1}$. It appears as a minimal condition needed to estimate c in the debiasing procedure without affecting the limit properties of \widehat{f}_n in terms of central limit theorem and moderate deviation principle. Assumption (C.2) allows to replace c by its estimator in the asymptotic variance of \widehat{f}_n .

2.1. Central limit theorem

Our first result states the multivariate central limit theorem for \widehat{f}_n :

Theorem 2.1. *Assume that (H.1) – (H.6) hold for $(\varepsilon_n)_{n \geq 1} \equiv 1$, that (C.1) holds for $\widehat{c}_{1,n}$ and that (C.2) holds for $\widehat{c}_{2,n}$. Then, for all $\{x_1, \dots, x_p\} \subset F$,*

$$\left\{ \kappa_n(x_j)^{-1} n \widehat{c}_{2,n}(x_j) \left(\widehat{f}_n(x_j, \widehat{c}_{1,n}) - f(x_j) \right) : 1 \leq j \leq p \right\} \xrightarrow{\mathcal{D}} N(0, \Sigma_{(x_1, \dots, x_p)}),$$

where $\xrightarrow{\mathcal{D}}$ denotes the convergence in distribution and $N(0, \Sigma_{(x_1, \dots, x_p)})$ is the centered gaussian law in \mathbb{R}^p , with covariance matrix $\Sigma_{(x_1, \dots, x_p)}$.

This result can be used to obtain explicit asymptotic $\gamma\%$ confidence interval for $f(x)$ of the form:

$$\left[\widehat{f}_n(x, \widehat{c}_{1,n}) - z_\gamma \kappa_n(x) (n\widehat{c}_{2,n}(x))^{-1}, \widehat{f}_n(x, \widehat{c}_{1,n}) + z_\gamma \kappa_n(x) (n\widehat{c}_{2,n}(x))^{-1} \right],$$

where z_γ is the $(\gamma + 1)/2$ th quantile of the $N(0, 1)$ distribution.

2.2. Moderate deviation principle

We will now establish a family of large deviation principles for \widehat{f}_n which is sometimes referenced in the litterature as a moderate deviation principle. Recall that a sequence of random variable $(W_n)_{n \geq 1}$ is said to follow the large deviation principle in \mathbb{R}^p with speed $\varepsilon_n \downarrow 0$ and good rate function $I : \mathbb{R}^p \mapsto [0, \infty]$, whenever, for every set $A \in \mathcal{B}(\mathbb{R}^p)$,

$$\begin{aligned} -\inf \left\{ I(u) : u \in \overset{\circ}{A} \right\} &\leq \liminf_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P}(W_n \in A) \\ &\leq \limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P}(W_n \in A) \leq -\inf \left\{ I(u) : u \in \overline{A} \right\}, \end{aligned}$$

where $\overset{\circ}{A}$ (resp. \overline{A}) denotes the interior (resp. closure) of A in \mathbb{R}^p . This will be denoted by $(W_n) \in \text{LDP}(\varepsilon_n, I)$ in the sequel. We refer to [7] for general informations about large deviation theory.

Theorem 2.2. *Assume that (H.1)–(H.6) hold for some $\varepsilon_n \downarrow 0$, that (C.1) holds for $\widehat{c}_{1,n}$ and that (C.2) holds for $\widehat{c}_{2,n}$. Then, for all $\{x_1, \dots, x_p\} \subset F$ such that $\Sigma_{(x_1, \dots, x_p)}$ is regular,*

$$\left\{ \varepsilon_n^{1/2} \kappa_n(x_j)^{-1} n\widehat{c}_{2,n}(x_j) \left(\widehat{f}_n(x_j, \widehat{c}_{1,n}) - f(x_j) \right) : 1 \leq j \leq p \right\} \in \text{LDP}(\varepsilon_n, I_{(x_1, \dots, x_p)})$$

where

$$I_{(x_1, \dots, x_p)} : u \in \mathbb{R}^p \mapsto 2^{-1} {}^t u \Sigma_{(x_1, \dots, x_p)}^{-1} u.$$

Let us mention some applications of Theorem 2.2. First, it entails that for all $s > 0$ and all $x \in F$,

$$\mathbb{P} \left(\varepsilon_n^{1/2} n \left| \kappa_n(x)^{-1} \widehat{c}_{2,n}(x) \left(\widehat{f}_n(x, \widehat{c}_{1,n}) - f(x) \right) \right| \geq s \right) \simeq \exp(-2^{-1} s^2 \varepsilon_n^{-1})$$

(here $u_n \simeq v_n$ means that $\log u_n / \log v_n \rightarrow 1$ as $n \rightarrow \infty$). This fact and the Borel-Cantelli Lemma can be used to prove that under (H.1)–(H.6) with $\varepsilon_n = \log(n)^{-1}$, we have, for all $x \in F$,

$$\limsup_{n \rightarrow \infty} \left((2 \log(n))^{1/2} \kappa_n(x) \right)^{-1} n\widehat{c}_{2,n}(x) \left| \widehat{f}_n(x, \widehat{c}_{1,n}) - f(x) \right| \leq 1 \text{ a.s.}$$

Moreover, it is well known that moderate deviation principle is a key tool to prove laws of the iterated logarithm (see e.g. [5] Th. 1-4-1). In terms of confidence intervals, Theorem 2.2 can also be useful to compute the logarithmic asymptotic level of confidence intervals with asymptotic level 0. More precisely, for fixed $x \in E$ and $t > 0$, consider the interval

$$\mathbb{I}_n(x, t) = \left[\widehat{f}_n(x, \widehat{c}_{1,n}) - t\varepsilon_n^{-1/2} \kappa_n(x) (n\widehat{c}_{2,n}(x))^{-1}, \widehat{f}_n(x, \widehat{c}_{1,n}) + t\varepsilon_n^{-1/2} \kappa_n(x) (n\widehat{c}_{2,n}(x))^{-1} \right]$$

where $\varepsilon_n = o(1)$. Then, Theorem 2.1 entails that

$$\mathbb{P}(f(x) \notin \mathbb{I}_n(x, t)) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Now, if conditions of Theorem 2.2 hold for (ε_n) we readily obtain that

$$\limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P}(f(x) \notin \mathbb{I}_n(x, t)) = -2^{-1}t^2.$$

In other words, $(\mathbb{I}_n(x, t))_{n \geq 1}$ is a sequence of confidence intervals for $f(x)$ with logarithmic level $2^{-1}t^2$ and speed (ε_n) (see [29] Def. 1).

Finally, in estimation theory, Theorem 2.2 can be used to compute the Kallenberg efficiency of \widehat{f}_n (see [22,23]).

2.3. Estimation of c

At this stage, we present two estimates of f which belong to our general class (6) with some particular estimations of c .

The first example has been previously introduced and studied in term of central limit theorem in case (P) in [13] and is defined by

$$\widehat{f}_n^{\text{loc}}(x) = \sum_{r=1}^{k_n} \kappa_{n,r}(x) \nu_{n,r} Y_{n,r}^* (1 + N_{n,r}^{-1}).$$

It is readily seen that $\widehat{f}_n^{\text{loc}}$ can be written as in (6) with the localized estimator of c :

$$\widehat{c}_n(x) = \widehat{c}_n^{\text{loc}}(x) := k_n \bar{\kappa}_n(x) \left(n \sum_{r=1}^{k_n} \kappa_{n,r}(x) N_{n,r}^{-1} \nu_{n,r} Y_{n,r}^* \right)^{-1},$$

where

$$\bar{\kappa}_n(x) := k_n^{-1} \sum_{r=1}^{k_n} \kappa_{n,r}(x).$$

Now, as c is constant, it may appear interesting to use a more global estimation of it. To this aim, we consider the global estimator

$$\widehat{c}_n^{\text{glo}} = k_n \left(n \sum_{r=1}^{k_n} \nu_{n,r} Y_{n,r}^* N_{n,r}^{-1} \right)^{-1},$$

leading to

$$\widehat{f}_n^{\text{glo}}(x) := \widehat{f}_n(x; \widehat{c}_n^{\text{glo}}) = \sum_{r=1}^{k_n} (\kappa_{n,r}(x) + \bar{\kappa}_n(x) N_{n,r}^{-1}) \nu_{n,r} Y_{n,r}^*.$$

The following corollary shows that Theorems 2.1 and 2.2 can be applied to $\widehat{f}_n^{\text{loc}}$ and $\widehat{f}_n^{\text{glo}}$.

Corollary 2.1. *Theorems 2.1 and 2.2 hold when $\widehat{c}_{n,1} \in \{\widehat{c}_n^{\text{loc}}; \widehat{c}_n^{\text{glo}}\}$ and $\widehat{c}_{n,2} = \widehat{c}_n^{\text{glo}}$.*

Remark 2.1. For the limit theorems considered here, $\widehat{f}_n^{\text{loc}}$ and $\widehat{f}_n^{\text{glo}}$ are equivalent. Nethertheless, it can be shown that $\widehat{c}_n^{\text{glo}}$ is a better estimator of c than $\widehat{c}_n^{\text{loc}}$ and therefore $\widehat{f}_n^{\text{glo}}$ may be preferred to $\widehat{f}_n^{\text{loc}}$. On the other hand, if the intensity (2) of the point process is not totally uniform, it is clear that $\widehat{f}_n^{\text{loc}}$, which is based on a local estimation of c , is more robust than $\widehat{f}_n^{\text{glo}}$.

3. APPLICATIONS

We first introduce a general class of kernel estimators which will be shown to satisfy our main results given in Theorems 2.1 and 2.2. Then, we focus on the particular cases of Parzen-Rosenblatt and Dirichlet kernels.

3.1. General kernel estimates

Here, in order to smooth (4), a sequence $K_n : E \times E \rightarrow \mathbb{R}$, of general smoothing kernels is introduced. Conditions on this sequence will be imposed later. The general integrated kernel estimate is defined by

$$\begin{aligned} \widehat{f}_n(x; \widehat{c}_n) &= \int_E K_n(x, t) \widehat{f}_n^\circ(t; \widehat{c}_n) \nu(dt) \\ &= \sum_{r=1}^{k_n} \left(\int_{I_{n,r}} K_n(x, t) \nu(dt) \right) \left(Y_{n,r}^* + (n\widehat{c}_n \nu_{n,r})^{-1} \right). \end{aligned} \tag{7}$$

It appears that (7) is a particular case of (6) with $\kappa_{n,r}(x) = \nu_{n,r}^{-1} \int_{I_{n,r}} K_n(x, t) \nu(dt)$. In the case where the calculation of this mean value is computationally expansive, it can be approximated by $K_n(x, x_{n,r})$ for some $x_{n,r} \in I_{n,r}$, leading to the simplified estimate

$$\widetilde{f}_n(x; \widehat{c}_n) = \sum_{r=1}^{k_n} \nu_{n,r} K_n(x, x_{n,r}) \left(Y_{n,r}^* + (n\widehat{c}_n \nu_{n,r})^{-1} \right), \tag{8}$$

which is still a particular case of (6) with $\kappa_{n,r}(x) = K_n(x, x_{n,r})$.

In order to introduce the assumptions needed on K_n , we set, for all $x \in E$,

$$\Gamma_{n,r}(x) = \sup \{ K_n(x, t) - K_n(x, s) : (s, t) \in I_{n,r} \times I_{n,r} \},$$

$$\Xi_n(x) = k_n \left| \sum_{r=1}^{k_n} \int_{I_{n,r} \times I_{n,r}} K_n(x, t) (f(s) - f(t)) \nu(dt) \nu(ds) \right|,$$

and

$$\Psi_n(x) = \left| \int_E K_n(x, t) f(t) \nu(dt) - f(x) \right|.$$

For the sake of simplicity, assume that, for all $n \geq 1$, the partitions $\{I_{n,r} : 1 \leq r \leq k_n\}$ are such that $\nu_{n,r} = k_n^{-1}$ for all $r \leq k_n$. We also set

$$\Delta_n = \max_{r \leq k_n} (M_{n,r} - m_{n,r})$$

and, for all function $g : E \rightarrow \mathbb{R}$, we note

$$\|g\|_p = \left(\int_E |g(t)|^p \nu(dt) \right)^{1/p}, \text{ and } \|g\|_I = \sup_{t \in I} |g(t)| \text{ (} I \subset E \text{)}.$$

In this context, the general assumptions (H.1) – (H.6) can be expressed as:

(H.1) $k_n \uparrow \infty$ and $n^{-1} k_n \max(\log(n), \varepsilon_n^{-1}) \rightarrow 0$ as $n \rightarrow \infty$.

(H.2) $0 < m \leq M < +\infty$ and $n k_n^{-1} \Delta_n \rightarrow 0$ as $n \rightarrow \infty$.

(K.1) For all $n \geq 1$, $\int_{E \times E} |K_n(x, t)| \nu(dx) \nu(dt) < \infty$.

(K.2) For all $(x_1, x_2) \in F \times F$,

$$\sum_{r=1}^{k_n} \Gamma_{n,r}(x_1) \int_{I_{n,r}} |K_n(x_2, t)| \nu(dt) = o(\|K_n(x_1, \cdot)\|_2 \|K_n(x_2, \cdot)\|_2) \text{ as } n \rightarrow \infty.$$

(K.3) For all $(x_1, x_2) \in F \times F$,

$$\langle K_n(x_1, \cdot), K_n(x_2, \cdot) \rangle_2 (\|K_n(x_1, \cdot)\|_2 \|K_n(x_2, \cdot)\|_2)^{-1} \rightarrow \sigma(x_1, x_2) \text{ as } n \rightarrow \infty.$$

(K.4) For all $x \in F$,

$$(\varepsilon_n k_n)^{-1/2} \|K_n(x, \cdot)\|_2^{-1} \|K_n(x, \cdot)\|_E \rightarrow 0 \text{ as } n \rightarrow \infty.$$

(K.5) For all $x \in F$,

$$\varepsilon_n^{1/2} n k_n^{-1/2} \|K_n(x, \cdot)\|_2^{-1} \max(\Psi_n(x); \Xi_n(x)) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

(K.6) For all $x \in F$,

$$\varepsilon_n^{1/4} n k_n^{-3/4} \|K_n(x, \cdot)\|_2^{-1/2} \|K_n(x, \cdot)\|_1^{1/2} \Delta_n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

(K.7) For all $x \in F$,

$$\varepsilon_n^{1/2} n k_n^{-1/2} \|K_n(x, \cdot)\|_2^{-1} \left(\sum_{r=1}^{k_n} \int_{I_{n,r}} (K_n(x, t) - K_n(x, x_{n,r})) \nu(dt) \right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

In the following, we set

$$v_n(x) = n k_n^{-1/2} \|K_n(x, \cdot)\|_2^{-1},$$

and

$$\Sigma_{(x_1, \dots, x_p)} = [\sigma(x_i, x_j)]_{1 \leq i, j \leq p}.$$

The results established in Section 2 yield:

Theorem 3.1. *a) If (H.1), (H.2) and (K.1)–(K.6) hold with $(\varepsilon_n)_{n \geq 1} \equiv 1$, then, for all $\widehat{c}_{1,n}$ (resp. $\widehat{c}_{2,n}$) such that (C.1) (resp. (C.2)) hold and all $(x_1, \dots, x_p) \subset F$,*

$$\left\{ \widehat{c}_{2,n}(x_j) v_n(x_j) \left(\widehat{f}_n(x_j; \widehat{c}_{1,n}) - f(x_j) \right) : 1 \leq j \leq p \right\} \xrightarrow{\mathcal{D}} N(0, \Sigma_{(x_1, \dots, x_p)}). \quad (9)$$

b) If, moreover, (K.7) holds with $(\varepsilon_n)_{n \geq 1} \equiv 1$, then for all $\{x_1, \dots, x_p\} \subset F$,

$$\left\{ \widehat{c}_{2,n}(x_j) v_n(x_j) \left(\widetilde{f}_n(x_j; \widehat{c}_{1,n}) - f(x_j) \right) : 1 \leq j \leq p \right\} \xrightarrow{\mathcal{D}} N(0, \Sigma_{(x_1, \dots, x_p)}). \quad (10)$$

Theorem 3.2. *a) For all $\varepsilon_n \downarrow 0$ such that (H.1), (H.2) and (K.1)–(K.6) hold, for all $\widehat{c}_{1,n}$ (resp. $\widehat{c}_{2,n}$) such that (C.1) (resp. (C.2)) hold and all $(x_1, \dots, x_p) \subset F$ such that $\Sigma_{(x_1, \dots, x_p)}$ is regular,*

$$\left\{ \varepsilon_n^{1/2} \widehat{c}_{2,n}(x_j) v_n(x_j) \left(\widehat{f}_n(x_j; \widehat{c}_{1,n}) - f(x_j) \right) : 1 \leq j \leq p \right\} \in \text{LDP}(\varepsilon_n, \mathbf{I}_{(x_1, \dots, x_p)}) \quad (11)$$

with

$$\mathbf{I}_{(x_1, \dots, x_p)} : u \in \mathbb{R}^p \mapsto 2^{-1} {}^t u \Sigma_{(x_1, \dots, x_p)}^{-1} u.$$

b) If, moreover, (K.7) holds, then

$$\left\{ \varepsilon_n^{1/2} \widehat{c}_{2,n}(x_j) v_n(x_j) \left(\widetilde{f}_n(x_j; \widehat{c}_{1,n}) - f(x_j) \right) : 1 \leq j \leq p \right\} \in \text{LDP}(\varepsilon_n, \mathbf{I}_{(x_1, \dots, x_p)}). \quad (12)$$

Some illustrations of this result are now provided.

Remark 3.1. In all the applications below involving \widetilde{f}_n , $x_{n,r}$ will be defined as the center of $I_{n,r}$.

3.2. Parzen kernel estimates

Here, we take $E = [0, 1]^d$ ($d \in \mathbb{N}^*$), ν is the Lebesgue measure on E and $\{I_{n,r} : 1 \leq r \leq k_n\}$ is an equidistant partition of E such that $I_{n,r} = \prod_{j=1}^d J_{n,r,j}$ where the $J_{n,r,j}$ are intervals of $[0, 1]$ of length $k_n^{-1/d}$, leading to $\nu_{n,r} = k_n^{-1}$ for all $1 \leq r \leq k_n$.

The multivariate Parzen kernel estimate is defined by

$$K_n^{PR}(x, t) = h_n^{-d} K(h_n^{-1}(x - t)),$$

where $K : \mathbb{R}^d \rightarrow \mathbb{R}^+$ is a Parzen-Rosenblatt kernel such that $K \in L^2(\mathbb{R}^d)$, and (h_n) is a sequence of positive real numbers tending to zero (see e.g. [18]).

In a first time, we will assume that there exists $\alpha \in (0, 1]$ such that:

(PR.1a) f is α -Lipschitz,

(PR.1b) K is uniformly continuous except on a finite set $D \subset \mathbb{R}^d$ and $\int_{\mathbb{R}^d} \|u\|_{\mathbb{R}^d}^\alpha K(u) du < \infty$.

Denote by I_p the identity matrix of \mathbb{R}^p . We are now in position to prove a central limit theorem and a moderate deviation principle for \hat{f}_n .

Corollary 3.1. *Assume that (PR.1) holds and*

$$(i) \quad n^{-1}k_n \log(n) = o(1), \quad (ii) \quad h_n^d k_n \rightarrow \infty \quad \text{and} \quad (iii) \quad nk_n^{-1/2} h_n^{\alpha+d/2} = o(1).$$

Then, (9) holds for $F = (0, 1)^d$, $v_n = nh_n^{d/2} k_n^{-1/2}$ and $\Sigma_{(x_1, \dots, x_p)} = \|K\|_2^2 I_p$.

The best rate of convergence is obtained for $h_n = n^{-\frac{1}{\alpha+d}}$ and $k_n = n^{\frac{d}{\alpha+d}} u_n^2$, where $u_n \rightarrow \infty$ arbitrarily slowly. In this case, $v_n = n^{\frac{\alpha}{\alpha+d}} u_n^{-1}$.

Corollary 3.2. *Assume that (PR.1) holds and for some $\varepsilon_n \rightarrow 0$,*

$$(i) \quad n^{-1}k_n \max(\log(n), \varepsilon_n^{-1}) = o(1), \quad (ii) \quad h_n^d k_n \varepsilon_n \rightarrow \infty \quad \text{and} \quad (iii) \quad nk_n^{-1/2} h_n^{\alpha+d/2} \varepsilon_n^{1/2} = o(1). \quad (13)$$

Then, (11) holds for $F = (0, 1)^d$, $v_n = nh_n^{d/2} k_n^{-1/2}$ and $\Sigma_{(x_1, \dots, x_p)} = \|K\|_2^2 I_p$.

In particular, if (13) holds for $\varepsilon_n = \log(n)^{-1}$, then for all, $x \in F$,

$$\limsup_{n \rightarrow \infty} (2 \log(n))^{-1/2} nh_n^{d/2} k_n^{-1/2} \hat{c}_{2,n}(x) \left| \hat{f}_n(x, \hat{c}_{1,n}) - f(x) \right| \leq \|K\|_2^2 \text{ a.s.}$$

Remark 3.2. a) Corollary 3.1 shows that, when f is α -lipschitzian, the speed of convergence of \hat{f}_n can be chosen arbitrarily close to the minimax speed $n^{-\frac{\alpha}{\alpha+d}}$ (see [17]).

b) In the case of Poisson point process with $d = 1$, \hat{f}_n with Parzen Rosenblatt kernel has been studied in [12]. In particular [12] Theorem 6-2, gives a central limit theorem for \hat{f}_n with optimal rate $v_n = n^{\frac{\alpha}{\alpha+5/4}} u_n^{-1}$. Hence, from asymptotical point of view, \hat{f}_n is better than \tilde{f}_n .

Since our approach involves regularization of Geffroy's estimates it is natural to study the case where f is more regular than just α -lipschitzian. In the following, for simplicity, we take $E = [0, 1]$ and we only deal with the central limit theorem. Assume that:

(PR.2a) f is in $C^2(E)$

(PR.2b), There exists a finite set D such that

$$K \text{ is } C^1 \text{ on } D^c, \quad K' \in L^1(\mathbb{R}) \text{ and } \int_{\mathbb{R}} u^2 K(u) du < \infty. \quad (14)$$

Corollary 3.3. *a) If (PR.2) holds and*

$$(i) n^{-1}k_n \log(n) = o(1), (ii) h_n k_n \rightarrow \infty, (iii) nk_n^{-7/4}h_n^{1/4} = o(1) \text{ and } (iv) nk_n^{-1/2}h_n^{5/2} = o(1),$$

then, (9) is true for $F = (0, 1)$, $v_n = nh_n^{1/2}k_n^{-1/2}$ and $\Sigma_{(x_1, \dots, x_p)} = \|K\|_2^2 I_p$.

The choices of $h_n = n^{-5/17}$ and $k_n = n^{9/17}u_n^2$, where $u_n \rightarrow \infty$ arbitrarily slowly, lead to $v_n = n^{10/17}u_n^{-1}$.

b) If moreover, K is even, 1-Lipschitzian and C^2 on D^c with $K'' \in L^1(\mathbb{R})$, then, under (i)–(iv) and

$$(v) nk_n^{-5/2}h_n^{-3/2} = o(1), (10) \text{ is true for } F, v_n \text{ and } \Sigma_{(x_1, \dots, x_p)} \text{ defined as above.}$$

The choices of $h_n = n^{-2/7}$ and $k_n = n^{4/7}u_n^2$, where $u_n \rightarrow \infty$ arbitrarily slowly, lead to $v_n = n^{4/7}u_n^{-1}$.

3.3. Projection estimates: Dirichlet kernels

Let $f \in L^2(E, \nu)$ and $(e_j)_{j \in \mathbb{N}}$ be an orthonormal basis of $L^2(E, \nu)$. The expansion of f on this basis is

$$f(x) = \sum_{j=0}^{\infty} a_j e_j(x), \quad x \in E,$$

where each $a_j = \int_E e_j(t) f(t) \nu(dt)$ can be estimated by

$$\hat{a}_{j,k_n} = \sum_{r=1}^{k_n} \left(\int_{I_{n,r}} e_j(t) \nu(dt) \right) \left(Y_{n,r}^* + (n\hat{c}_n \nu_{n,r})^{-1} \right), \quad 1 \leq j \leq b_n.$$

This leads to an estimation of $f(x)$ via:

$$\hat{f}_n(x; \hat{c}_n) = \sum_{j=0}^{l_n} \hat{a}_{j,k_n} e_j(x) = \sum_{r=1}^{k_n} \left(\int_{I_{n,r}} K_n^D(x, t) \nu(dt) \right) \left(Y_{n,r}^* + (n\hat{c}_n \nu_{n,r})^{-1} \right), \tag{15}$$

where (l_n) is a sequence of integers tending to infinity and K_n^D the Dirichlet's kernel associated to the orthonormal basis $(e_j)_{j \in \mathbb{N}}$ defined by

$$K_n^D(x, t) = \sum_{j=0}^{l_n} e_j(x) e_j(t), \quad (x, t) \in E^2. \tag{16}$$

It appears that (15) is a particular case of (7t) with $K_n = K_n^D$. Of course, the, sometimes easier to handle, estimate

$$\tilde{f}_n(x; \hat{c}_n) = \sum_{r=1}^{k_n} \nu_{n,r} K_n^D(x, x_{n,r}) \left(Y_{n,r}^* + (n\hat{c}_n \nu_{n,r})^{-1} \right), \tag{17}$$

can also be defined. Below, we focus on the trigonometric basis on $E = [0, 1]$, ν is the Lebesgue measure on E , and for all $1 \leq r \leq k_n$, $I_{n,r}$ are disjoint intervals of $[0, 1]$ of length k_n^{-1} (and therefore $\nu_{n,r} = 1/k_n$).

This basis is defined for $x \in [0, 1]$ by

$$e_0(x) = 1, \quad e_{2j-1}(x) = 2^{1/2} \cos(2j\pi x), \quad e_{2j}(x) = 2^{1/2} \sin(2j\pi x), \quad j \geq 1.$$

It is easily seen in that case that the Dirichlet kernel is

$$\begin{aligned} K_n^D(x, t) &= \frac{\sin((1+l_n)\pi(x-t))}{\sin(\pi(x-t))} \quad \text{for } x \neq t \\ &= 1 + l_n \quad \text{if } x = t. \end{aligned} \tag{18}$$

Besides, we assume that: f is in $C^2([0, 1])$, $f(0) = f(1)$ and $f'(0) = f'(1)$. In particular,

$$\Delta_n = O(k_n^{-1}). \tag{19}$$

Corollary 3.4. a) Assume that

$$(i) n^{-1}k_n \log(n) = o(1), \quad (ii) k_n^{-1}l_n \log(l_n) = o(1), \quad (iii) nk_n^{-1/2}l_n^{-2} = O(1),$$

$$(iv) nk_n^{-5/2}l_n^{1/2} \log(l_n) = o(1) \quad \text{and} \quad (v) nl_n^{-1/4}k_n^{-7/4} \log(l_n)^{1/2} = o(1).$$

Then (9) holds for $F = [0, 1]$, $v_n = n(l_n k_n)^{-1/2}$ and $\Sigma_{(x_1, \dots, x_p)} = I_p$.

In particular, the choices of $l_n = n^{10/27}$ and $k_n = n^{14/27} \log(n)^{2/7} u_n^2$, where $u_n \rightarrow \infty$ arbitrarily slowly, lead to $v_n = n^{5/9} \log(n)^{-1/7} u_n^{-1}$.

b) If (i)–(iii) and (iv') $nk_n^{-5/2}l_n^{3/2} \log(l_n) = o(1)$ are true, then (10) holds for F , v_n and $\Sigma_{(x_1, \dots, x_p)}$ defined as above.

The choices of $l_n = n^{8/23}$ and $k_n = n^{14/23} \log(n)^{2/5} u_n^2$, where $u_n \rightarrow \infty$ arbitrarily slowly, lead to $v_n = n^{12/23} \log(n)^{-1/5} u_n^{-1}$.

Remark 3.3. a) From the asymptotical point of view, \widehat{f}_n is still better than \widetilde{f}_n . Nevertheless, since when f is C^2 , the minimax speed of convergence is $n^{-2/3}$ (see [17]), the above estimates are suboptimal.

b) In the case of Poisson point process, \widetilde{f}_n with the trigonometric Dirichlet kernel has been introduced by Girard and Jacob [11]. In that context, they get a pointwise central limit theorem (see [11] Cor. 3) but their result is not sharp and Corollary 3.4 (b) constitutes a significant improvement since, instead of $n^{12/23} \log(n)^{-1/5} u_n^{-1}$, they obtain an “optimal” speed of convergence of $n^{2/5} \log(n)^{-1/5} (\log \log(n))^\varepsilon$ ($\varepsilon > 0$).

3.4. Concluding remarks

In terms of applications, we have presented examples of integrated smoothed kernel estimates \widehat{f}_n and of discrete smoothed kernel estimates \widetilde{f}_n . In all the cases considered, \widehat{f}_n is strictly better than \widetilde{f}_n from asymptotical point of view (i.e. (K.7) is not implied by the other assumptions of Ths. 3.1 and 3.2). When $f \in C^2$, Parzen kernel leads to better speed of convergence than trigonometric Dirichlet kernel but it does not reach the minimax speed.

Of course, other kernels are allowed, in particular wavelet kernels can be treated in the framework of Theorems 3.1 and 3.2. This is part of our future work. More generally, the following problem should be of interest: given a class of function \mathcal{F} , and assuming that $f \in \mathcal{F}$, what is the best estimate of type (7) and in which case could we raise the minimax speed of convergence?

4. PROOFS

4.1. Proofs of the results of Section 2

Let us first introduce some notations useful for the sequel. For all $1 \leq r \leq k_n$, we define:

- $\lambda_{n,r} = nc\nu_{n,r} m_{n,r}, \quad \Lambda_{n,r} = nc\nu_{n,r} M_{n,r}, \quad \mu_{n,r} = nc\nu_{n,r} f_{n,r},$
- $b_{n,r} = b_{n,r}(\varepsilon_n) = \left(3\mu_{n,r}^{-1} \left(\varepsilon_n^{-1/2} (n\nu_{n,r})^{1/2} + \log(\nu_{n,r}^{-1}) \right) \right)^{1/2}, \quad b_n = b_n(\varepsilon_n) = \max_{r \leq k_n} b_{n,r},$
- $B_{n,r} = B_{n,r}(\varepsilon_n) = \{|N_{n,r} - \mu_{n,r}| \leq b_{n,r} \mu_{n,r}\}, \quad B_n = B_n(\varepsilon_n) = \bigcap_{r \leq k_n} B_{n,r},$
- $Z_{n,r}^* = nc\nu_{n,r} Y_{n,r}^*, \quad E_{n,r} = \mathbb{E}(Z_{n,r}^* | N_{n,r}), \quad Z_{n,r}^\circ = Z_{n,r}^* - E_{n,r},$
- $\beta_{n,r} = N_{n,r}^{-1} (N_{n,r} + 1) E_{n,r} - \mu_{n,r}, \quad \zeta_{n,r} = N_{n,r}^{-1} (N_{n,r} + 1) Z_{n,r}^\circ,$
- $\Theta_n(x) = \sum_{r=1}^{k_n} w_{n,r}(x) \zeta_{n,r}.$

Moreover $\mathcal{B}(n, p)$, $\mathcal{P}(\lambda)$ and $\mathcal{M}(n; p_1, \dots, p_k)$ stand respectively for the binomial, Poisson and multinomial distribution.

The proofs of the main results are built as follows. First, we use the fact that

$$\kappa_n^{-1} n c \left(\widehat{f}_n^{\text{loc}} - f \right) = \Theta_n + \rho_n, \tag{20}$$

where

$$\rho_n = \sum_{r=1}^{k_n} w_{n,r} \beta_{n,r} + \kappa_n^{-1} n c (f_n - f),$$

will be shown to be a remaining part which vanishes in terms of central limit theorem and of moderate deviation principle (see Lem. 4.5 below). Therefore, to get the intended result for $\widehat{f}_n^{\text{loc}}$, it will be sufficient to consider the dominant part Θ_n , which, as a sum of conditionally independent and centered random variables, will be studied thanks to general results of Section 5. In a second time, it will be proved that for all \widehat{c}_n such that (C.1) holds, $\widehat{f}_n(\cdot; \widehat{c}_n)$ and $\widehat{f}_n^{\text{loc}}$ are equivalent in terms of central limit theorem and of moderate deviation principle.

The next lemma contains some useful technical results:

Lemma 4.1. *Under assumptions (H.1) and (H.2):*

a) *For all $1 \leq r \leq k_n$, and all $t \in [0, M_{n,r}]$,*

$$\mathbb{P} (Y_{n,r}^* \leq t \mid N_{n,r}) = \left(\frac{\int_{I_{n,r}} \min(f, t) \, d\nu}{\int_{I_{n,r}} f \, d\nu} \right)^{N_{n,r}}.$$

In particular, for all $t \in [0, m_{n,r}]$,

$$\mathbb{P} (Y_{n,r}^* \leq t \mid N_{n,r}) = (t f_{n,r}^{-1})^{N_{n,r}}.$$

b) *For all large n ,*

$$\max_{1 \leq r \leq k_n} \left| \left(1 - (m_{n,r} f_{n,r}^{-1})^{N_{n,r}} \right) - \mu_{n,r}^{-1} N_{n,r} (\mu_{n,r} - \lambda_{n,r}) \right| \mathbf{1}_{B_n} \leq 2 (nc\delta_n)^2 \text{ a.s.}$$

c) *For all large n ,*

$$\max_{1 \leq r \leq k_n} \left(\int_{\lambda_{n,r}}^{\Lambda_{n,r}} \mathbb{P} (Z_{n,r}^* > s \mid N_{n,r}) \, ds \right) \mathbf{1}_{B_n} \leq 2 (nc\delta_n)^2 \text{ a.s.}$$

d) *For all large n ,*

$$\max_{1 \leq r \leq k_n} |\beta_{n,r}| \mathbf{1}_{B_n} \leq 4 (nc\delta_n)^2 \text{ a.s.}$$

e) *For all large n ,*

$$\max_{1 \leq r \leq k_n} |\mu_{n,r} - E_{n,r} - 1| \mathbf{1}_{B_n} \leq 4 (nc\delta_n)^2 + 2 \left(b_n + (mc\nu_n)^{-1} \right) \text{ a.s.}$$

f) *For all n large enough, all $1 \leq r \leq k_n$, and $s \geq 0$:*

$$\begin{aligned} \mathbb{P} (|Z_{n,r}^\circ| \geq s \mid N_{n,r}) \mathbf{1}_{B_{n,r}} &= (\mu_{n,r}^{-1} (E_{n,r} - s))^{N_{n,r}} \mathbf{1}_{[0, E_{n,r}]}(s) \mathbf{1}_{B_{n,r}} \\ &+ \left(1 - (\mu_{n,r}^{-1} (E_{n,r} + s))^{N_{n,r}} \right) \mathbf{1}_{[0, \lambda_{n,r} - E_{n,r}]}(s) \mathbf{1}_{B_{n,r}} \\ &+ \mathbb{P} (Z_{n,r}^* > s + E_{n,r} \mid N_{n,r}) \mathbf{1}_{[\lambda_{n,r} - E_{n,r}, \Lambda_{n,r} - E_{n,r}]}(s) \mathbf{1}_{B_{n,r}}. \end{aligned}$$

g) For all n large enough, all $1 \leq r \leq k_n$, and $l \geq 0$:

$$\begin{aligned} \mathbb{E} \left(|Z_{n,r}^\circ|^l \mid N_{n,r} \right) \mathbf{1}_{B_{n,r}} &= (\lambda_{n,r} - E_{n,r})^l \mathbf{1}_{B_{n,r}} \\ &+ l! \left(\mu_{n,r}^{-1} E_{n,r} \right)^{N_{n,r}} \frac{N_{n,r}! E_{n,r}^l}{(N_{n,r} + l)!} \mathbf{1}_{B_{n,r}} \\ &- \int_0^{\lambda_{n,r} - E_{n,r}} l s^{l-1} \left(\mu_{n,r}^{-1} (E_{n,r} + s) \right)^{N_{n,r}} ds \mathbf{1}_{B_{n,r}} \\ &+ \int_{\lambda_{n,r} - E_{n,r}}^{\Lambda_{n,r} - E_{n,r}} l s^{l-1} \mathbb{P} \left(Z_{n,r}^* > s + E_{n,r} \mid N_{n,r} \right) ds \mathbf{1}_{B_{n,r}}. \end{aligned}$$

Proof. a) Set

$$N_{n,r}^+(t) = N_n \left(D_{n,r} \cap (I_{n,r} \times (t, M_{n,r}]) \right).$$

Then

$$\mathbb{P} \left(Y_{n,r}^* \leq t \mid N_{n,r} \right) = \mathbb{P} \left(N_{n,r}^+(t) = 0 \mid N_{n,r} \right).$$

But, as easily seen, conditionally to $N_{n,r}$,

$$N_{n,r}^+(t) \rightsquigarrow \mathcal{B} \left(N_{n,r}, 1 - \frac{\int_{I_{n,r}} \min(f, t) d\nu}{\int_{I_{n,r}} f d\nu} \right).$$

Hence, we get the intended result.

b) For all $u \in [0, 1]$ and $\alpha \geq 2$, set $\varphi_\alpha(u) = (1 - u)^\alpha - (1 - \alpha u)$, then, the Taylor formula entails,

$$0 \leq \varphi_\alpha(u) \leq (\alpha u)^2.$$

Hence, for all large n and all $\omega \in B_n$,

$$\begin{aligned} &\max_{1 \leq r \leq k_n} \left| \left(1 - (m_{n,r} f_{n,r}^{-1})^{N_{n,r}} \right) - \mu_{n,r}^{-1} N_{n,r} (\mu_{n,r} - \lambda_{n,r}) \right| (\omega) \\ &= \max_{1 \leq r \leq k_n} \varphi_{N_{n,r}}(\omega) \left(\mu_{n,r}^{-1} (\mu_{n,r} - \lambda_{n,r}) \right) \\ &\leq \max_{1 \leq r \leq k_n} \left(N_{n,r}(\omega) \mu_{n,r}^{-1} (\mu_{n,r} - \lambda_{n,r}) \right)^2 \\ &\leq (1 + b_n)^2 (nc\delta_n)^2, \end{aligned}$$

where we have used the facts that

$$\max_{1 \leq r \leq k_n} (\mu_{n,r} - \lambda_{n,r}) \leq nc\delta_n = o(1) \tag{21}$$

and for all $\omega \in B_n$,

$$\max_{1 \leq r \leq k_n} N_{n,r}(\omega) \mu_{n,r}^{-1} \leq 1 + b_n.$$

Finally, we get the intended result since, by (H.1),

$$b_n \leq \left(3(cm)^{-1} \left((n\nu_n \varepsilon_n)^{-1/2} + (n\nu_n)^{-1} \log(\nu_n^{-1}) \right) \right)^{1/2} = o(1).$$

c) Using (b), we get that for all large n and all $\omega \in B_n$,

$$\begin{aligned}
& \max_{1 \leq r \leq k_n} n c \nu_{n,r} \left(\int_{m_{n,r}}^{M_{n,r}} \mathbb{P}(Y_{n,r}^* > t \mid N_{n,r}) dt \right) (\omega) \\
& \leq \max_{1 \leq r \leq k_n} n c \nu_{n,r} (M_{n,r} - m_{n,r}) \mathbb{P}(Y_{n,r}^* > m_{n,r} \mid N_{n,r}) (\omega) \\
& \leq n c \delta_n \max_{1 \leq r \leq k_n} \left(1 - (m_{n,r} f_{n,r}^{-1})^{N_{n,r}(\omega)} \right) \\
& \leq n c \delta_n \max_{1 \leq r \leq k_n} \mu_{n,r}^{-1} N_{n,r}(\omega) (\mu_{n,r} - \lambda_{n,r}) + (1 + b_n)^2 (n c \delta_n)^3 \\
& \leq (1 + b_n) (n c \delta_n)^2 + (1 + b_n)^2 (n c \delta_n)^3.
\end{aligned}$$

d) Using (a), we get

$$\begin{aligned}
& n c \nu_{n,r} \int_0^{m_{n,r}} \mathbb{P}(Y_{n,r}^* > t \mid N_{n,r}) dt \\
& = n c \nu_{n,r} \int_0^{m_{n,r}} \left(1 - (t f_{n,r}^{-1})^{N_{n,r}} \right) dt \\
& = \mu_{n,r} N_{n,r} (N_{n,r} + 1)^{-1} - (\mu_{n,r} - \lambda_{n,r}) N_{n,r} (N_{n,r} + 1)^{-1} \\
& \quad + \lambda_{n,r} (N_{n,r} + 1)^{-1} \left(1 - (m_{n,r} f_{n,r}^{-1})^{N_{n,r}} \right) \\
& = \mu_{n,r} N_{n,r} (N_{n,r} + 1)^{-1} - \mu_{n,r}^{-1} (\mu_{n,r} - \lambda_{n,r})^2 N_{n,r} (N_{n,r} + 1)^{-1} \\
& \quad + \lambda_{n,r} (N_{n,r} + 1)^{-1} \left[\left(1 - (m_{n,r} f_{n,r}^{-1})^{N_{n,r}} \right) - \mu_{n,r}^{-1} N_{n,r} (\mu_{n,r} - \lambda_{n,r}) \right].
\end{aligned}$$

Hence, by (b), for all large n and all $\omega \in B_n$,

$$\begin{aligned}
& \max_{1 \leq r \leq k_n} \left| n c \nu_{n,r} \int_0^{m_{n,r}} \mathbb{P}(Y_{n,r}^* > t \mid N_{n,r}) dt - \mu_{n,r} N_{n,r} (N_{n,r} + 1)^{-1} \right| (\omega) \\
& \leq \left((m c \nu_n)^{-1} + (1 - b_n) (1 + b_n)^2 \right) (n c \delta_n)^2 \\
& \leq 2 (n c \delta_n)^2,
\end{aligned}$$

which, combined with (c) and the fact that

$$E_{n,r} = n c \nu_{n,r} \int_0^{M_{n,r}} \mathbb{P}(Y_{n,r}^* > t \mid N_{n,r}) dt,$$

gives the result.

e)

$$\begin{aligned}
\max_{1 \leq r \leq k_n} |\mu_{n,r} - E_{n,r} - 1| \mathbf{1}_{B_n} &= \max_{1 \leq r \leq k_n} \left| \mu_{n,r} (N_{n,r} + 1)^{-1} - 1 - \beta_{n,r} (1 + N_{n,r}^{-1})^{-1} \right| \mathbf{1}_{B_n} \\
&\leq \max_{1 \leq r \leq k_n} \left| (\mu_{n,r} - N_{n,r}) (N_{n,r} + 1)^{-1} - (N_{n,r} + 1)^{-1} - \beta_{n,r} (1 + N_{n,r}^{-1})^{-1} \right| \mathbf{1}_{B_n} \\
&\leq (1 - b_n)^{-1} \left(b_n + (n c \nu_n m)^{-1} \right) + 4 (n c \delta_n)^2.
\end{aligned}$$

f) By (21) and (e),

$$\lim_{n \rightarrow \infty} \min_{1 \leq r \leq k_n} \min_{\omega \in B_n} (\lambda_{n,r} - E_{n,r}(\omega)) = 1.$$

Hence, eventually,

$$\min_{1 \leq r \leq k_n} \min_{\omega \in B_n} (\lambda_{n,r} - E_{n,r}(\omega)) > 0, \tag{22}$$

and, for all large n , all $1 \leq r \leq k_n$, and $s \geq 0$:

$$\begin{aligned} & \mathbb{P}(|Z_{n,r}^* - E_{n,r}| \geq s \mid N_{n,r}) \mathbf{1}_{B_n} \\ &= \mathbb{P}(Z_{n,r}^* \geq E_{n,r} + s \mid N_{n,r}) \mathbf{1}_{B_n} + \mathbb{P}(Z_{n,r}^* \leq E_{n,r} - s \mid N_{n,r}) \mathbf{1}_{B_n} \\ &= (\mu_{n,r}^{-1}(E_{n,r} - s))^{N_{n,r}} \mathbf{1}_{[0, E_{n,r}]}(s) \mathbf{1}_{B_n} \\ &+ \left(1 - (\mu_{n,r}^{-1}(E_{n,r} + s))^{N_{n,r}}\right) \mathbf{1}_{[0, \lambda_{n,r} - E_{n,r}]}(s) \mathbf{1}_{B_n} \\ &+ \mathbb{P}(Z_{n,r}^* > s + E_{n,r}) \mathbf{1}_{[\lambda_{n,r} - E_{n,r}, \Lambda_{n,r} - E_{n,r}]}(s) \mathbf{1}_{B_n}. \end{aligned}$$

g) Follows from the fact that, by (e), for all $l \geq 1$ and $r \leq k_n$,

$$\begin{aligned} \mathbb{E}(|Z_{n,r}^* - E_{n,r}|^l \mid N_{n,r}) &= \int_0^{+\infty} l s^{l-1} \mathbb{P}(|Z_{n,r}^* - E_{n,r}| > s \mid N_{n,r}) \, ds \\ &= \int_0^{E_{n,r}} l s^{l-1} (\mu_{n,r}^{-1}(E_{n,r} - s))^{N_{n,r}} \, ds \\ &+ \int_0^{\lambda_{n,r} - E_{n,r}} l s^{l-1} \left(1 - (\mu_{n,r}^{-1}(E_{n,r} + s))^{N_{n,r}}\right) \, ds \\ &+ \int_{\lambda_{n,r} - E_{n,r}}^{\Lambda_{n,r} - E_{n,r}} l s^{l-1} \mathbb{P}(Z_{n,r}^* > s + E_{n,r} \mid N_{n,r}) \, ds, \end{aligned}$$

and that

$$\begin{aligned} \int_0^{E_{n,r}} l s^{l-1} (\mu_{n,r}^{-1}(E_{n,r} - s))^{N_{n,r}} \, ds &= (\mu_{n,r}^{-1} E_{n,r})^{N_{n,r}} \int_0^1 l u^{l-1} (1 - u)^{N_{n,r}} \, du \\ &= (\mu_{n,r}^{-1} E_{n,r})^{N_{n,r}} ((N_{n,r} + l)!)^{-1} l! N_{n,r}! E_{n,r}^l. \end{aligned} \quad \square$$

We are now in position to prove all we need to study the dominant term Θ_n :

Lemma 4.2. *Under assumptions (H.1) and (H.2):*

a) *There exists a non negative real numbers $\eta_n \rightarrow 0$ such that, for all $n \geq 1$,*

$$\max_{1 \leq r \leq k_n} |\mathbb{E}(\zeta_{n,r}^2 \mid N_{n,r}) - 1| \mathbf{1}_{B_n} \leq \eta_n \text{ a.s.}$$

b)

$$\max_{l \geq 1} \max_{1 \leq r \leq k_n} \mathbb{E} \left((l!)^{-1} |\zeta_{n,r}|^l \mid N_{n,r} \right)^{1/l} \mathbf{1}_{B_n} \leq 4 \text{ a.s.}$$

c) *For all (ε_n) such that (H.1) holds,*

$$\limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P}(B_n^c) = -\infty.$$

Proof. a) Lemma 4.1 (g) for $l = 2$ entails

$$\begin{aligned}
 & \mathbb{E} (Z_{n,r}^{\circ 2} \mid N_{n,r}) \mathbf{1}_{B_{n,r}} \\
 &= (N_{n,r} + 1)^{-2} \mu_{n,r}^2 \mathbf{1}_{B_{n,r}} + \left((\lambda_{n,r} - E_{n,r})^2 - (N_{n,r} + 1)^{-2} \mu_{n,r}^2 \right) \mathbf{1}_{B_{n,r}} \\
 & \quad + 2 (\mu_{n,r}^{-1} \lambda_{n,r})^{N_{n,r}} (N_{n,r} + 2)^{-1} \lambda_{n,r} \left((N_{n,r} + 1)^{-1} (N_{n,r} + 2) E_{n,r} - \lambda_{n,r} \right) \mathbf{1}_{B_{n,r}} \\
 & \quad + 2 \int_{\lambda_{n,r} - E_{n,r}}^{\Lambda_{n,r} - E_{n,r}} s \mathbb{P} (Z_{n,r}^* > s + E_{n,r} \mid N_{n,r}) \, ds \mathbf{1}_{B_{n,r}}.
 \end{aligned} \tag{23}$$

Now, by Lemma 4.1 (c) and (e),

$$\begin{aligned}
 & \max_{1 \leq r \leq k_n} \int_{\lambda_{n,r} - E_{n,r}}^{\Lambda_{n,r} - E_{n,r}} 2s \mathbb{P} (Z_{n,r}^* > s + E_{n,r} \mid N_{n,r}) \, ds \mathbf{1}_{B_{n,r}} \\
 & \leq 2 \max_{1 \leq r \leq k_n} (\Lambda_{n,r} - E_{n,r}) \int_{\lambda_{n,r}}^{\Lambda_{n,r}} \mathbb{P} (Z_{n,r}^* > s \mid N_{n,r}) \, ds \mathbf{1}_{B_{n,r}} \\
 & \leq 8 (nc\delta_n)^2 \text{ eventually.}
 \end{aligned} \tag{24}$$

By Lemma 4.1 (d), for all large n ,

$$\begin{aligned}
 & \max_{1 \leq r \leq k_n} \left| (N_{n,r} + 1)^{-1} (N_{n,r} + 2) E_{n,r} - \lambda_{n,r} \right| \mathbf{1}_{B_{n,r}} \\
 &= \max_{1 \leq r \leq k_n} \left| (N_{n,r} + 1)^{-2} N_{n,r} (N_{n,r} + 2) \beta_{n,r} + (\mu_{n,r} - \lambda_{n,r}) - (N_{n,r} + 1)^{-2} \mu_{n,r} \right| \mathbf{1}_{B_{n,r}} \\
 & \leq 4 (nc\delta_n)^2 + nc\delta_n + (1 - b_n)^{-2} (nmc\nu_n)^{-1} = O (n\delta_n + (n\nu_n)^{-1}).
 \end{aligned} \tag{25}$$

Furthermore, since $a^2 - b^2 = (a - b)^2 + 2(a - b)b$, and

$$\begin{aligned}
 & \max_{1 \leq r \leq k_n} \left| \lambda_{n,r} - E_{n,r} - (N_{n,r} + 1)^{-1} \mu_{n,r} \right| \mathbf{1}_{B_{n,r}} \\
 &= \max_{1 \leq r \leq k_n} \left| (N_{n,r} + 1)^{-1} N_{n,r} \beta_{n,r} - (\mu_{n,r} - \lambda_{n,r}) \right| \mathbf{1}_{B_{n,r}} \\
 & \leq 4 (nc\delta_n)^2 + nc\delta_n \text{ eventually,}
 \end{aligned}$$

and we get that for all large n ,

$$\begin{aligned}
 & \max_{1 \leq r \leq k_n} \left| (\lambda_{n,r} - E_{n,r})^2 - \left((N_{n,r} + 1)^{-1} \mu_{n,r} \right)^2 \right| \mathbf{1}_{B_{n,r}} \\
 & \leq \left(4 (nc\delta_n)^2 + nc\delta_n \right)^2 + 2 \left(4 (nc\delta_n)^2 + nc\delta_n \right) \max_{1 \leq r \leq k_n} (N_{n,r} + 1)^{-1} \mu_{n,r} \\
 & \leq \left(4 (nc\delta_n)^2 + nc\delta_n \right) \left(4 (nc\delta_n)^2 + nc\delta_n + 2(1 - b_n)^{-1} \right)^2 \\
 & = O (n\delta_n).
 \end{aligned} \tag{26}$$

Now, (23)–(26) lead to

$$\begin{aligned} \max_{1 \leq r \leq k_n} \left| \mathbb{E} (\zeta_{n,r}^2 \mid N_{n,r}) - N_{n,r}^{-2} \mu_{n,r}^2 \right| \mathbf{1}_{B_n} &= \max_{1 \leq r \leq k_n} (1 + N_{n,r}^{-1})^2 \left| \mathbb{E} (Z_{n,r}^{\circ 2} \mid N_{n,r}) - (N_{n,r} + 1)^{-2} \mu_{n,r}^2 \right| \mathbf{1}_{B_n} \\ &\leq 4 \max_{1 \leq r \leq k_n} \left| \mathbb{E} (Z_{n,r}^{\circ 2} \mid N_{n,r}) - (N_{n,r} + 1)^{-2} \mu_{n,r}^2 \right| \mathbf{1}_{B_n} \\ &= O \left(n \delta_n + (n \nu_n)^{-1} \right) = o(1). \end{aligned} \tag{27}$$

And we get the result by (27) and the fact that,

$$\max_{1 \leq r \leq k_n} \left| N_{n,r}^{-2} \mu_{n,r}^2 - 1 \right| \mathbf{1}_{B_n} \leq (1 - b_n)^{-2} (2 + b_n) b_n = o(1).$$

b) Lemma 4.1 (g) and (22) entail that, for all large n , all $1 \leq r \leq k_n$, and all $l \geq 1$,

$$\begin{aligned} &\mathbb{E} \left((l!)^{-1} \left| Z_{n,r}^* - E_{n,r} \right|^l \mid N_{n,r} \right)^{1/l} \mathbf{1}_{B_{n,r}} \\ &\leq \left((N_{n,r} + 1)^{-1} E_{n,r} + (\lambda_{n,r} - E_{n,r}) + (\Lambda_{n,r} - E_{n,r}) \right) \mathbf{1}_{B_{n,r}} \leq 4. \end{aligned}$$

c) First we prove that, for all n large enough and all $1 \leq r \leq k_n$,

$$\mathbb{P} (B_{n,r}^c) \leq 2 \nu_{n,r} \exp \left(-\varepsilon_n^{-1/2} (n \nu_{n,r})^{1/2} \right). \tag{28}$$

In case (E), $N_{n,r} \rightsquigarrow \mathcal{B} (n, n^{-1} \mu_{n,r})$, Bernstein Inequality yields, for large n ,

$$\begin{aligned} \mathbb{P} (|N_{n,r} - \mu_{n,r}| > b_{n,r} \mu_{n,r}) &\leq 2 \exp \left(-\frac{b_{n,r}^2 \mu_{n,r}^2}{2 \mu_{n,r} + \frac{2}{3} b_{n,r} \mu_{n,r}} \right) \\ &\leq 2 \exp \left(-3^{-1} b_{n,r}^2 \mu_{n,r} \right). \end{aligned}$$

In case (P), $N_{n,r} \rightsquigarrow \mathcal{P} (\mu_{n,r})$, hence, using a classical inequality (see *e.g.* [31] p. 486), we get, for large n ,

$$\begin{aligned} \mathbb{P} (|N_{n,r} - \mu_{n,r}| > b_{n,r} \mu_{n,r}) &\leq \exp (b_{n,r} \mu_{n,r} - (1 + b_{n,r}) \mu_{n,r} \log (1 + b_{n,r})) \\ &\quad + \exp (-b_{n,r} \mu_{n,r} - (1 - b_{n,r}) \mu_{n,r} \log (1 - b_{n,r})) \\ &\leq \exp (-2^{-1} (1 - b_{n,r}) b_{n,r}^2 \mu_{n,r}) + \exp (-b_{n,r}^2 \mu_{n,r}) \\ &\leq 2 \exp (-3^{-1} b_{n,r}^2 \mu_{n,r}). \end{aligned}$$

Hence we get the result since, by definition of $b_{n,r}$,

$$\exp (-3^{-1} b_{n,r}^2 \mu_{n,r}) = \nu_{n,r} \exp \left(-\varepsilon_n^{-1/2} (n \nu_{n,r})^{1/2} \right).$$

Now, by (28),

$$\mathbb{P} (B_n^c) \leq \sum_{r \leq k_n} \mathbb{P} (B_{n,r}^c) \leq 2 \exp \left(-\varepsilon_n^{-1/2} (n \nu_n)^{1/2} \right).$$

And, since $n \nu_n \varepsilon_n \rightarrow \infty$,

$$\limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P} (B_n^c) \leq \limsup_{n \rightarrow \infty} \varepsilon_n \log (2) - (n \nu_n \varepsilon_n)^{1/2} = -\infty. \quad \square$$

Corollary 4.1. *a) Assume that (H.1)–(H.4) hold for $(\varepsilon_n)_{n \geq 1} \equiv 1$. Then, for all $\{x_1, \dots, x_p\} \subset F$,*

$$(\Theta_n(x_j))_{1 \leq j \leq p} \xrightarrow{\mathcal{D}} N(0, \Sigma_{(x_1, \dots, x_p)}).$$

b) Assume that (H.1)–(H.4) hold for some $\varepsilon_n \rightarrow 0$. Then, for all $\{x_1, \dots, x_p\} \subset F$ such that $\Sigma_{(x_1, \dots, x_p)}$ is regular,

$$\left(\varepsilon_n^{1/2} \Theta_n(x_j) \right)_{1 \leq j \leq p} \in \text{LDP}(\varepsilon_n, \mathbf{I}_{(x_1, \dots, x_p)}).$$

Proof. Take $\{x_1, \dots, x_p\} \subset F$ and set $W_{n,r} = (w_{n,r}(x_j))_{1 \leq j \leq p} \in \mathbb{R}^p$. Since

$$(\Theta_n(x_j))_{1 \leq j \leq p} = \sum_{r=1}^{k_n} W_{n,r} \zeta_{n,r},$$

(a) follows by Lemma 4.2 and Theorem 5.1 and (b) holds by Lemma 4.2 and Theorem 5.2. \square

The next lemma will be needed in the sequel:

Lemma 4.3. *Let $(M_r)_{r \leq k} \rightsquigarrow \mathcal{M}(\phi; p_1, \dots, p_k)$ and $(\Pi_r)_{r \leq k} \rightsquigarrow \bigotimes_{r \leq k} \mathcal{P}(\phi p_r)$.*

Then, for all $(a_1, \dots, a_k) \in (\mathbb{R}^+)^k$ and $\eta > 0$,

$$\mathbb{P} \left(\sum_{r \leq k} a_r M_r \geq \eta \right) \leq 2 \mathbb{P} \left(\sum_{r \leq k} a_r \Pi_r \geq \eta \right).$$

Proof. Without loss of generality, we can assume that there exist i.i.d. r.v. $(\zeta_i)_{i \geq 1}$ and disjoint sets $(A_r)_{1 \leq r \leq k_n}$ such that $p_r = \mathbb{P}(\zeta_1 \in A_r)$,

$$M_r = \# \{i \leq \phi : \zeta_i \in A_r\} \quad \text{and} \quad \Pi_r = \# \{i \leq \Pi : \zeta_i \in A_r\},$$

where $\Pi \rightsquigarrow \mathcal{P}(\phi)$ is independent of $(\zeta_i)_{i \geq 1}$. By independence of Π and $(M_r)_{r \leq k}$,

$$\mathbb{P} \left(\sum_{r \leq k} a_r M_r \geq \eta \right) \leq \mathbb{P}(\Pi < \phi) \mathbb{P} \left(\sum_{r \leq k} a_r M_r \geq \eta \right) + \mathbb{P} \left(\sum_{r \leq k} a_r \Pi_r \geq \eta \right).$$

Hence, since $\mathbb{P}(\Pi \geq \phi) \geq 1/2$ (see [1] Lemma 1), we get,

$$\mathbb{P} \left(\sum_{r \leq k} a_r M_r \geq \eta \right) \leq \mathbb{P}(\Pi \geq \phi)^{-1} \mathbb{P} \left(\sum_{r \leq k} a_r \Pi_r \geq \eta \right) \leq 2 \mathbb{P} \left(\sum_{r \leq k} a_r \Pi_r \geq \eta \right). \quad \square$$

The next lemma is useful to prove Corollary 2.1 (see also [14] and [28] for other applications).

Lemma 4.4. *a) Assume that (H.1) and (H.2) hold for some $(\varepsilon_n)_{n \geq 1}$. Set $(u_{n,r})_{r \leq k_n} \subset \mathbb{R}$ such that*

$$(i) \varepsilon_n^{1/2} (n\nu_n)^{-1} (n\delta_n)^2 u_n^{(1)} = o(1), \quad (ii) \varepsilon_n^{-1/2} u_n = o(1) \quad \text{and} \quad (iii) (n\nu_n)^{-1} u_n^{(2)} = o(1),$$

with

$$u_n := \max_{1 \leq r \leq k_n} |u_{n,r}|, \quad u_n^{(1)} := \sum_{r=1}^{k_n} |u_{n,r}| \quad \text{and} \quad u_n^{(2)} := \sum_{r=1}^{k_n} u_{n,r}^2.$$

Then, for all $\eta > 0$,

$$\limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P} \left(\varepsilon_n^{1/2} \left| \sum_{r=1}^{k_n} u_{n,r} (N_{n,r}^{-1} Z_{n,r} - 1) \right| \geq \eta \right) = -\infty.$$

b) (H.1), (H.2), (H.4) and (H.6), imply that (C.1) holds for $\widehat{c}_n^{\text{loc}}$ and $\widehat{c}_n^{\text{glo}}$.

c) (H.1) and (H.2) imply that (C.2) holds for $\widehat{c}_n^{\text{glo}}$.

Proof. a) By Lemma 4.2 (c) we just need to show that

$$\limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P} \left(\varepsilon_n^{1/2} \left| \sum_{r=1}^{k_n} u_{n,r} (N_{n,r}^{-1} Z_{n,r} - 1) \right| \mathbf{1}_{B_n} \geq \eta \right) = -\infty.$$

Now,

$$(N_{n,r}^{-1} Z_{n,r} - 1) \mathbf{1}_{B_{n,r}} = N_{n,r}^{-1} Z_{n,r}^\circ + N_{n,r}^{-1} \beta_{n,r} + \chi_{n,r},$$

with

$$\chi_{n,r} := \left(\mu_{n,r} (N_{n,r} + 1)^{-1} - 1 \right) \mathbf{1}_{B_{n,r}}.$$

Hence

$$\begin{aligned} & \varepsilon_n^{1/2} \left| \sum_{r=1}^{k_n} u_{n,r} (N_{n,r}^{-1} Z_{n,r} - 1) \right| \mathbf{1}_{B_n} \\ & \leq \varepsilon_n^{1/2} \left| \sum_{r=1}^{k_n} u_{n,r} N_{n,r}^{-1} Z_{n,r}^\circ \right| \mathbf{1}_{B_n} + \varepsilon_n^{1/2} \left| \sum_{r=1}^{k_n} u_{n,r} N_{n,r}^{-1} \beta_{n,r} \right| + \varepsilon_n^{1/2} \left| \sum_{r=1}^{k_n} u_{n,r} \chi_{n,r} \right|. \end{aligned}$$

But, Lemma 4.1 (d) and by (i),

$$\varepsilon_n^{1/2} \left| \sum_{r=1}^{k_n} u_{n,r} N_{n,r}^{-1} \beta_{n,r} \right| = O \left(\varepsilon_n^{1/2} (n\nu_n)^{-1} (n\delta_n)^2 \sum_{r=1}^{k_n} |u_{n,r}| \right) = o(1).$$

Moreover, by Lemma 4.2 (b), the fact that, eventually,

$$\max_{1 \leq r \leq k_n} N_{n,r}^{-1} |Z_{n,r}^\circ| \mathbf{1}_{B_n} \leq (1 - b_n)^{-1} \left(1 + m^{-1} \max_{1 \leq r \leq k_n} (M_{n,r} - m_{n,r}) \right) \leq 2,$$

and (conditional) Bernstein inequality, we get that there exists $A > 0$ such that, for all $\eta > 0$,

$$\begin{aligned} & \mathbb{P} \left(\varepsilon_n^{1/2} \left| \sum_{r=1}^{k_n} u_{n,r} N_{n,r}^{-1} Z_{n,r}^\circ \right| \mathbf{1}_{B_n} \geq \eta \mid \mathcal{B}_n \right) \\ & \leq 2 \exp \left(-A\eta^2 \varepsilon_n^{-1} \left(2 \sum_{r=1}^{k_n} u_{n,r}^2 \mathbb{E} (Z_{n,r}^{\circ 2} \mid N_{n,r}) N_{n,r}^{-2} \mathbf{1}_{B_n} + 2\eta \varepsilon_n^{-1/2} u_n \right)^{-1} \right) \\ & \leq 2 \exp \left(-A\eta^2 \varepsilon_n^{-1} \left((mcn\nu_n)^{-2} u_n^{(2)} + 2\eta \varepsilon_n^{-1/2} u_n \right)^{-1} \right). \end{aligned}$$

Therefore, by (ii) and (iii),

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P} \left(\varepsilon_n^{1/2} \left| \sum_{r=1}^{k_n} u_{n,r} N_{n,r}^{-1} Z_{n,r}^\circ \right| \mathbf{1}_{B_n} \geq \eta \right) \\ &= \limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{E} \left(\mathbb{P} \left(\varepsilon_n^{1/2} \left| \sum_{r=1}^{k_n} u_{n,r} N_{n,r}^{-1} Z_{n,r}^\circ \right| \mathbf{1}_{B_n} \geq \eta \mid \mathcal{B}_n \right) \right) \\ &\leq \log(2) - A\eta^2 \liminf_{n \rightarrow \infty} \left((mc\nu_n)^{-2} u_n^{(2)} + 2\eta \varepsilon_n^{-1/2} u_n \right)^{-1} = -\infty, \end{aligned}$$

and our last task is to show that

$$\limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P} \left(\varepsilon_n^{1/2} \left| \sum_{r=1}^{k_n} u_{n,r} \chi_{n,r} \right| \geq \eta \right) = -\infty. \quad (29)$$

a1) First, we consider case (P).

For all large n and all $r \leq k_n$,

$$\begin{aligned} |\chi_{n,r}| &\leq (1 - b_{n,r})^{-1} \mu_{n,r}^{-1} (|N_{n,r} - \mu_{n,r}| \mathbf{1}_{B_{n,r}} + 1) \\ &\leq (1 - b_{n,r})^{-1} b_{n,r} + ((1 - b_{n,r}) \mu_{n,r})^{-1} \leq 2b_{n,r}. \end{aligned}$$

Hence, $(\chi_{n,r} - \mathbb{E}(\chi_{n,r}))_{r \leq k_n}$ are independent, centered and eventually bounded by $4b_n$. Moreover,

$$\begin{aligned} \mathbb{V}(\chi_{n,r}) &\leq (1 - b_{n,r})^{-2} \mu_{n,r}^{-2} \mathbb{E} \left((|N_{n,r} - \mu_{n,r}| + 1)^2 \right) \\ &\leq (1 - b_{n,r})^{-2} \mu_{n,r}^{-2} \left(\mathbb{E} \left((N_{n,r} - \mu_{n,r})^2 \right)^{1/2} + 1 \right)^2 \\ &\leq (1 - b_{n,r})^{-2} \mu_{n,r}^{-2} \left(\mu_{n,r}^{1/2} + 1 \right)^2 \\ &\leq (1 - b_n)^{-2} \left(1 + (mc\nu_n)^{-1/2} \right)^2 (mc\nu_n)^{-1} \leq 2 (mc\nu_n)^{-1} \text{ eventually.} \end{aligned}$$

Therefore, by Bernstein inequality, there exists $A > 0$

$$\begin{aligned} \mathbb{P} \left(\varepsilon_n^{1/2} \left| \sum_{r=1}^{k_n} u_{n,r} (\chi_{n,r} - \mathbb{E}(\chi_{n,r})) \right| \geq \eta \right) &\leq 2 \exp \left(-A \eta^2 \varepsilon_n^{-1} \left(\sum_{r=1}^{k_n} u_{n,r}^2 \mathbb{V}(\chi_{n,r}) + \eta \varepsilon_n^{-1/2} 4b_n u_n \right)^{-1} \right) \\ &\leq 2 \exp \left(-A \eta^2 \varepsilon_n^{-1} \left(2 (mc\nu_n)^{-1} u_n^{(2)} + 4\eta b_n \varepsilon_n^{-1/2} u_n \right)^{-1} \right), \end{aligned}$$

and (ii) and (iii) entail

$$\limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P} \left(\varepsilon_n^{1/2} \left| \sum_{r=1}^{k_n} u_{n,r} (\chi_{n,r} - \mathbb{E}(\chi_{n,r})) \right| \geq \eta \right) = -\infty. \quad (30)$$

Moreover,

$$\left| \mathbb{E} \left(\left(\mu_{n,r} (N_{n,r} + 1)^{-1} - 1 \right) \right) \right| = e^{-\mu_{n,r}} \leq \exp(-mnc\nu_n) = o(n^{-1}),$$

and, using (28), we get

$$\left| \mathbb{E} \left(\left(\mu_{n,r} (N_{n,r} + 1)^{-1} - 1 \right) \mathbf{1}_{B_{n,r}^c} \right) \right| \leq \mu_{n,r} \mathbb{P} (B_{n,r}^c) = n^{-1} O \left((n\nu_n)^2 \exp \left(-\varepsilon_n^{-1/2} (n\nu_n)^{1/2} \right) \right) = o(n^{-1}),$$

thus,

$$\max_{1 \leq r \leq k_n} |\mathbb{E}(\chi_{n,r})| = o(n^{-1})$$

and

$$\varepsilon_n^{1/2} \left| \sum_{r=1}^{k_n} u_{n,r} \mathbb{E}(\chi_{n,r}) \right| = o \left(\varepsilon_n^{1/2} n^{-1} \sum_{r=1}^{k_n} |u_{n,r}| \right) = o \left(\varepsilon_n^{1/2} n^{-1} k_n u_n \right) = o(1),$$

which combined with (30) give the intended result.

a2) In case (E), we will use a coupling argument.

For all $n \geq 1$, define a sequence of i.i.d. random vectors $((X'_{n,i}, Y'_{n,i}))_{i \geq 1}$ with $(X'_{n,i}, Y'_{n,i})_{i \leq n} = (X_{n,i}, Y_{n,i})_{i \leq n}$.

Let $N'_n(S) \rightsquigarrow \mathcal{P}(n)$, be independent of $((X'_{n,i}, Y'_{n,i}))_{i \geq 1}$, and consider the counting process

$$N'_n : D \in \mathcal{E} \otimes \mathcal{B}(\mathbb{R}^+) \mapsto \# \{i \leq N'_n(S) : (X'_{n,i}, Y'_{n,i}) \in D\}.$$

It is not hard to see that N'_n is a Poisson point process with intensity measure defined in (2). Now, set

$$N'_{n,r} = N'_n(D_{n,r}), \quad B'_{n,r} = \{|N'_{n,r} - \mu_{n,r}| \leq b_{n,r} \mu_{n,r}\}, \\ \chi'_{n,r} = \left(1 - \mu_{n,r} (N'_{n,r} + 1)^{-1}\right) \mathbf{1}_{B'_{n,r}} \quad \text{and} \quad B'_n = \bigcap_{r \leq k_n} B'_{n,r}.$$

Since

$$|\chi_{n,r} - \chi'_{n,r}| \mathbf{1}_{B_n \cap B'_n} \leq (1 - b_n)^{-2} \mu_{n,r}^{-1} |N'_{n,r} - N_{n,r}|,$$

we have, for all large n ,

$$\left| \sum_{r=1}^{k_n} u_{n,r} \chi_{n,r} \right| \mathbf{1}_{B_n \cap B'_n} \leq \left| \sum_{r=1}^{k_n} u_{n,r} \chi'_{n,r} \right| + 2 \sum_{r=1}^{k_n} |u_{n,r}| \mu_{n,r}^{-1} |N'_{n,r} - N_{n,r}|. \tag{31}$$

By (a1),

$$\limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P} \left(\varepsilon_n^{1/2} \left| \sum_{r=1}^{k_n} u_{n,r} \chi'_{n,r} \right| \geq \eta \right) = -\infty. \tag{32}$$

Moreover, since by Cauchy-Schwarz inequality and (iii),

$$n^{-1/2} u_n^{(1)} \leq \left(n^{-1} \nu_n^{-1} u_n^{(2)} \right)^{1/2} = o(1),$$

we can define a sequence (α_n) such that

$$\alpha_n \uparrow \infty, \quad n^{-1/2} u_n^{(1)} \alpha_n = o(1) \quad \text{and} \quad \phi_n := n^{1/2} \varepsilon_n^{-1/2} \alpha_n \in \mathbb{N}.$$

Now, conditionally to $N'_n(S)$,

$$\left(|N'_{n,r} - N_{n,r}| \right)_{1 \leq r \leq k_n} \rightsquigarrow \mathcal{M}(|N'_n(S) - n|; n^{-1} \mu_{n,1}, \dots, n^{-1} \mu_{n,k_n}),$$

hence, Lemma 4.3 entails,

$$\mathbb{P} \left(\varepsilon_n^{1/2} \sum_{r=1}^{k_n} |u_{n,r}| \mu_{n,r}^{-1} |N'_{n,r} - N_{n,r}| \geq \eta \right) \leq 2 \mathbb{P} \left(\sum_{r=1}^{k_n} a_{n,r} \Pi_{n,r} \geq \eta \right) + \mathbb{P} \left(n^{-1/2} \varepsilon_n^{1/2} |N'_n(S) - n| \geq \alpha_n \right), \quad (33)$$

where

$$a_{n,r} = \varepsilon_n^{1/2} |u_{n,r}| \mu_{n,r}^{-1}, \quad \text{and} \quad (\Pi_{n,r})_{1 \leq r \leq k_n} \rightsquigarrow \bigotimes_{r \leq k_n} \mathcal{P}(\phi_n n^{-1} \mu_{n,r}).$$

Since $N'_n(S)$ is a sum of n i.i.d $\mathcal{P}(1)$ r.v. and $\alpha_n \uparrow \infty$, we have

$$\limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P} \left(n^{-1/2} \varepsilon_n^{1/2} |N'_n(S) - n| \geq \alpha_n \right) = -\infty \quad (34)$$

(for $(\varepsilon_n)_{n \geq 1} \equiv 1$, this follows by the central limit theorem, for $\varepsilon_n \downarrow 0$ such that $n\varepsilon_n \rightarrow \infty$, it holds by [7 Th. 3-7-1]). Moreover, for all large n ,

$$\begin{aligned} \mathbb{P} \left(\sum_{r=1}^{k_n} a_{n,r} \Pi_{n,r} \geq \eta \right) &\leq \exp \left(-\eta \varepsilon_n^{-3/2} u_n^{-1} \right) \prod_{r=1}^{k_n} \mathbb{E} \left(\exp \left(\varepsilon_n^{-3/2} u_n^{-1} a_{n,r} \Pi_{n,r} \right) \right) \\ &= \exp \left(-\eta \varepsilon_n^{-3/2} u_n^{-1} + \phi_n n^{-1} \sum_{r=1}^{k_n} \mu_{n,r} \left(\exp \left(\varepsilon_n^{-1} u_n^{-1} |u_{n,r}| \mu_{n,r}^{-1} \right) - 1 \right) \right) \\ &\leq \exp \left(-\eta \varepsilon_n^{-3/2} u_n^{-1} + 2\phi_n n^{-1} \varepsilon_n^{-1} u_n^{-1} \sum_{r=1}^{k_n} |u_{n,r}| \right) \\ &= \exp \left(-\varepsilon_n^{-3/2} u_n^{-1} \left(\eta - 2\alpha_n n^{-1/2} u_n^{(1)} \right) \right) \leq \exp \left(-2^{-1} \eta \varepsilon_n^{-3/2} u_n^{-1} \right), \end{aligned} \quad (35)$$

where (35) follows from the facts that

$$\max_{1 \leq r \leq k_n} \varepsilon_n^{-1} u_n^{-1} |u_{n,r}| \mu_{n,r}^{-1} = O \left((n\nu_n \varepsilon_n)^{-1} \right) = o(1) \quad \text{and} \quad e^t - 1 \leq 2t \quad \text{for small } t > 0.$$

Consequently,

$$\limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P} \left(\sum_{r=1}^{k_n} a_{n,r} \Pi_{n,r} \geq \eta \right) \leq -2^{-1} \eta \liminf_{n \rightarrow \infty} \varepsilon_n^{-1/2} u_n^{-1} = -\infty,$$

which, combined with (33) and (34) lead to

$$\limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P} \left(\varepsilon_n^{1/2} \sum_{r=1}^{k_n} |u_{n,r}| \mu_{n,r}^{-1} |N'_{n,r} - N_{n,r}| \geq \eta \right) = -\infty. \quad (36)$$

Finally, Lemma 4.2 (c) yields

$$\limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P} \left(B_n^c \cup B_n'^c \right) = -\infty, \quad (37)$$

and (29) follows by (31), (32), (36) and (37).

b) Note that, if $\widehat{c}_n \in \{\widehat{c}_n^{\text{loc}}, \widehat{c}_n^{\text{gl}}\}$, we can write

$$\varepsilon_n^{1/2} \left| \sum_{r=1}^{k_n} w_{n,r}(x) \right| \left| \widehat{c}_n(x)^{-1} - c^{-1} \right| = \varepsilon_n^{1/2} c \left| \sum_{r=1}^{k_n} u_{n,r}(x) (N_{n,r}^{-1} Z_{n,r}^* - 1) \right|,$$

where $u_{n,r}(x) = w_{n,r}(x)$ for $\widehat{c}_n = \widehat{c}_n^{\text{loc}}$ and $u_{n,r}(x) = k_n^{-1} \left| \sum_{s=1}^{k_n} w_{n,s}(x) \right|$ for $\widehat{c}_n = \widehat{c}_n^{\text{gl}}$.

In both cases, conditions (i) and (ii) of (a) hold respectively by (H.6) and (H.4) and condition (iii) is true by (H.1) since, here, $u_n^{(2)} \leq 1$. Hence, (a) shows that $\widehat{c}_n^{\text{loc}}$ and $\widehat{c}_n^{\text{glo}}$ satisfy (C.1).

c) Using the facts that for $a > 0$, $b > 0$, and $\eta > 0$,

$$|a - b| \geq \eta \Rightarrow |ab^{-1} - 1| \geq \eta(a + \eta)^{-1}$$

and that for all large n ,

$$k_n^{-1} \left| \sum_{r=1}^{k_n} (N_{n,r}^{-1} Z_{n,r}^* - 1) \right| \mathbf{1}_{B_n} \leq 2b_n < \eta(c + \eta)^{-1},$$

we get that, eventually,

$$\begin{aligned} \mathbb{P}(|c - \widehat{c}_n^{\text{glo}}| \geq \eta) &\leq \mathbb{P}\left(k_n^{-1} \left| \sum_{r=1}^{k_n} (N_{n,r}^{-1} Z_{n,r}^* - 1) \right| \geq \eta(c + \eta)^{-1}, B_n\right) + \mathbb{P}(B_n^c) \\ &\leq \mathbb{P}(B_n^c). \end{aligned}$$

Hence, by Lemma 4.2 (c),

$$\limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P}(|c - \widehat{c}_n^{\text{glo}}| \geq \eta) \leq \limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P}(B_n^c) = -\infty. \quad \square$$

Lemma 4.5. *Assume that (H.1) – (H.6) hold. Then, for all $x \in F$ and any $\eta > 0$,*

$$\limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P}\left(\varepsilon_n^{1/2} \left| \kappa_n(x)^{-1} nc \left(\widehat{f}_n(x, \widehat{c}_n) - f(x) \right) - \Theta_n(x) \right| \geq \eta\right) = -\infty.$$

Proof. Since

$$\begin{aligned} \varepsilon_n^{1/2} \left| \kappa_n^{-1} nc \left(\widehat{f}_n^{\text{loc}} - f \right) - \Theta_n \right| \mathbf{1}_{B_n} &\leq \varepsilon_n^{1/2} \sum_{r=1}^{k_n} |w_{n,r}| |\beta_{n,r}| \mathbf{1}_{B_n} + \kappa_n^{-1} nc |f_n - f| \\ &\leq 2\varepsilon_n^{1/2} (nc\delta_n)^2 \sum_{r=1}^{k_n} |w_{n,r}| + \varepsilon_n^{1/2} \kappa_n^{-1} nc |f_n - f| \\ &< 2^{-1}\eta \quad \text{eventually by (H.5) and (H.6),} \end{aligned}$$

we get that for large n ,

$$\limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P}\left(\varepsilon_n^{1/2} \left| \kappa_n^{-1} nc \left(\widehat{f}_n^{\text{loc}} - f \right) - \Theta_n \right| \geq 2^{-1}\eta\right) \leq \limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P}(B_n^c) = -\infty. \quad (38)$$

Moreover, since

$$\begin{aligned} \mathbb{P}\left(\varepsilon_n^{1/2} \kappa_n^{-1} nc \left| \widehat{f}_n^{\text{loc}} - \widehat{f}_n \right| \geq 2^{-1}\eta\right) &= \mathbb{P}\left(c\varepsilon_n^{1/2} \left| \sum_{r=1}^{k_n} w_{n,r} \right| \left| (\widehat{c}_n^{\text{loc}})^{-1} - \widehat{c}_n^{-1} \right| \geq 2^{-1}\eta\right) \\ &\leq \mathbb{P}\left(\varepsilon_n^{1/2} \left| \sum_{r=1}^{k_n} w_{n,r} \right| \left| (\widehat{c}_n^{\text{loc}})^{-1} - c^{-1} \right| \geq 4^{-1}\eta c^{-1}\right) \\ &\quad + \mathbb{P}\left(\varepsilon_n^{1/2} \left| \sum_{r=1}^{k_n} w_{n,r} \right| \left| \widehat{c}_n^{-1} - c^{-1} \right| \geq 4^{-1}\eta c^{-1}\right), \end{aligned}$$

Lemma 4.4 b), (C.1) and [7] Lemma 1-2-15 entail

$$\limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P} \left(\varepsilon_n^{1/2} \kappa_n^{-1} n c \left| \widehat{f}_n^{\text{loc}} - \widehat{f}_n \right| \geq 2^{-1} \eta \right) = -\infty. \quad (39)$$

Finally, since

$$\begin{aligned} & \mathbb{P} \left(\varepsilon_n^{1/2} \left| \kappa_n^{-1} n c \left(\widehat{f}_n - f \right) - \Theta_n \right| \geq \eta \right) \\ & \leq \mathbb{P} \left(\varepsilon_n^{1/2} \kappa_n^{-1} n c \left| \widehat{f}_n^{\text{loc}} - \widehat{f}_n \right| \geq 2^{-1} \eta \right) + \mathbb{P} \left(\varepsilon_n^{1/2} \left| \kappa_n^{-1} n c \left(\widehat{f}_n^{\text{loc}} - f \right) - \Theta_n \right| \geq 2^{-1} \eta \right), \end{aligned}$$

we get the intended result by (38), (39) and [7] Lemma 1-2-15. \square

Proof of Theorem 2.1. By Corollary 4.1 (a), Lemma 4.5 and [3] Theorem 4-1, we get that

$$\left\{ \kappa_n (x_j)^{-1} n c \left(\widehat{f}_n (x_j, \widehat{c}_{1,n}) - f (x_j) \right) : 1 \leq j \leq p \right\} \xrightarrow{\mathcal{D}} N \left(0, \Sigma_{(x_1, \dots, x_p)} \right),$$

which, combined with the fact that, $\widehat{c}_{2,n} \xrightarrow{\mathbb{P}} c$ by (C.2), gives the intended result. \square

Proof of Theorem 2.2. i) Corollary 4.1 (b), Lemma 4.5 and [7] Theorem 4-2-13, entail that the result hold for $\widehat{c}_{2,n} = c$.

ii) In the general case, for all $x \in F$, set

$$D_n = n \kappa_n^{-1} (\widehat{c}_{2,n} - c) \left(\widehat{f}_n - f \right).$$

By (i) and [7] Theorem 4-2-13, we just have to show that for all $x \in F$ and any $\eta > 0$,

$$\limsup_{n \rightarrow \infty} \varepsilon_n \log \left(\mathbb{P} \left(\varepsilon_n^{1/2} |D_n| \geq \eta \right) \right) = -\infty.$$

To this aim, note that, for all $\delta > 0$,

$$\mathbb{P} \left(\varepsilon_n^{1/2} |D_n| \geq \eta \right) \leq \mathbb{P} \left(|\widehat{c}_{2,n} - c| > \delta c \right) + \mathbb{P} \left(\varepsilon_n^{1/2} n \kappa_n^{-1} c \left| \widehat{f}_n (\cdot, \widehat{c}_{1,n}) - f \right| \geq \eta \delta^{-1} \right).$$

Therefore, by (C.2), [7] Lemma 1-2-15, and (i),

$$\limsup_{n \rightarrow \infty} \varepsilon_n \log \left(\mathbb{P} \left(\varepsilon_n^{1/2} |D_n| \geq \eta \right) \right) \leq -2^{-1} \eta^2 \delta^{-2},$$

and we conclude by letting $\delta \downarrow 0$. \square

4.2. Proofs of the results of Section 3

Proofs of Theorems 3.1 and 3.2. a) For all $x \in F$, we just verify (H.3)–(H.6) for $\kappa_{n,r}(x) := k_n \int_{I_{n,r}} K_n(x, t) \nu(dt)$. By (K.2),

$$\begin{aligned} \sum_{r=1}^{k_n} \kappa_{n,r}(x_1) \kappa_{n,r}(x_2) &= k_n^2 \sum_{r=1}^{k_n} \int_{I_{n,r}} \int_{I_{n,r}} K_n(x_1, s) K_n(x_2, t) \nu(ds) \nu(dt) \\ &= k_n \langle K_n(x_1, \cdot), K_n(x_2, \cdot) \rangle_2 \\ &\quad + k_n^2 \sum_{r=1}^{k_n} \int_{I_{n,r}} \int_{I_{n,r}} K_n(x_2, t) (K_n(x_1, s) - K_n(x_1, t)) \nu(dt) \nu(ds) \\ &= k_n \langle K_n(x_1, \cdot), K_n(x_2, \cdot) \rangle_2 + k_n O\left(\sum_{r=1}^{k_n} \Gamma_n(x_1) \int_{I_{n,r}} |K_n(x_2, t)| \nu(dt)\right) \\ &= k_n [\langle K_n(x_1, \cdot), K_n(x_2, \cdot) \rangle_2 + o(\|K_n(x_1, \cdot)\|_2 \|K_n(x_2, \cdot)\|_2)]. \end{aligned}$$

Hence,

$$\kappa_n(x) = k_n^{1/2} \|K_n(x, \cdot)\|_2 (1 + o(1)), \tag{40}$$

and (K.3) entails,

$$\sum_{r=1}^{k_n} w_{n,r}(x_1) w_{n,r}(x_2) = \sigma(x_1, x_2) + o(1),$$

which is (H.3). Now, (40) entails for all large n ,

$$\begin{aligned} \varepsilon_n^{-1/2} \max_{1 \leq r \leq k_n} |w_{n,r}(x)| &\leq 2(\varepsilon_n k_n)^{-1/2} \|K_n(x, \cdot)\|_2^{-1} k_n \max_{1 \leq r \leq k_n} \left| \int_{I_{n,r}} K_n(t, x) \nu(dt) \right| \\ &\leq 2(\varepsilon_n k_n)^{-1/2} \|K_n(x, \cdot)\|_2^{-1} \|K_n(x, \cdot)\|_E = o(1) \text{ by (K.4)}, \end{aligned}$$

i.e. (H.4) holds. In order to show (H.5), note that using (40) again, Fubini Theorem (which holds by (K.1)) and the triangle inequality yield

$$\begin{aligned} &\varepsilon_n^{1/2} n \kappa_n(x)^{-1} \left| \sum_{r=1}^{k_n} \nu_{n,r} \kappa_{n,r}(x) f_{n,r} - f(x) \right| \\ &= \varepsilon_n^{1/2} n \kappa_n(x)^{-1} \left| \sum_{r=1}^{k_n} k_n \int_{I_{n,r} \times I_{n,r}} K_n(x, t) f(s) \nu(dt) \nu(ds) - f(x) \right| \\ &\leq \varepsilon_n^{1/2} n \kappa_n(x)^{-1} (\Xi_n(x) + \Psi_n(x)) \\ &= O\left(\varepsilon_n^{1/2} n k_n^{-1/2} \|K_n(x, \cdot)\|_2^{-1} (\Xi_n(x) + \Psi_n(x))\right) = o(1) \text{ by (K.5)}. \end{aligned}$$

Finally, we show that (H.6) holds.

$$\begin{aligned} \varepsilon_n^{1/2} (n \delta_n)^2 \sum_{r=1}^{k_n} |w_{n,r}(x)| &= O\left(\varepsilon_n^{1/2} (n k_n^{-1} \Delta_n)^2 k_n^{-1/2} \|K_n(x, \cdot)\|_2^{-1} \sum_{r=1}^{k_n} k_n \left| \int_{I_{n,r}} K_n(t, x) \nu(dt) \right|\right) \\ &= O\left(\varepsilon_n^{1/2} n^2 k_n^{-3/2} \|K_n(x, \cdot)\|_2^{-1} \|K_n(x, \cdot)\|_1 \Delta_n^2\right) = o(1) \text{ by (K.6)}. \end{aligned}$$

b) For all $x \in F$, it is easy to see that

$$\begin{aligned} \frac{n}{\kappa_n(x)} \left| \widehat{f}_n(x) - \widetilde{f}_n(x) \right| &\leq 4Mn k_n^{-1/2} \|K_n(x, \cdot)\|_2^{-1} \left(\sum_{r=1}^{k_n} \int_{I_{n,r}} (K_n(x, t) - K_n(x, x_{n,r})) \nu(dt) \right) \\ &= o(1) \text{ by (K.7),} \end{aligned}$$

which, combined with (a), gives the intended result by standard arguments. □

It is easy to see that, for all $x \in F$,

$$\Xi_n(x) \leq \|K_n(x, \cdot)\|_1 \Delta_n. \tag{41}$$

Nevertheless, under regularity conditions, tighter bounds are possible for $\Xi_n(x)$:

Lemma 4.6. *Let $E = [0, 1]^d$ ($d \in \mathbb{N}^*$), ν be the Lebesgue measure on E and $\{I_{n,r} : 1 \leq r \leq k_n\}$ be a partition of E such that $I_{n,r} = \prod_{j=1}^d J_{n,r,j}$ where the $J_{n,r,j}$ are interval of $[0, 1]$ of length $k_n^{-1/d}$. Assume that f is in $C^2(E)$. Then,*

$$\Xi_n(x) = O \left(\left(\|K_n(x, \cdot)\|_1 + \sum_{r=1}^{k_n} \Gamma_{n,r}(x) \right) k_n^{-2/d} \right). \tag{42}$$

Proof. Denote by $x_{n,r}$ the center of the cell $I_{n,r}$. Since f is C^2 , the multivariate Taylor formula yields:

$$\begin{aligned} \max_{r \leq k_n} \max_{s \in I_{n,r}} |f(s) - f(x_{n,r}) - \langle f'(x_{n,r}), s - x_{n,r} \rangle_{\mathbb{R}^d}| &= O \left(\max_{r \leq k_n} \max_{s \in I_{n,r}} \|s - x_{n,r}\|_{\mathbb{R}^d}^2 \right) \\ &= O \left(k_n^{-2/d} \right). \end{aligned}$$

Moreover,

$$\int_{I_{n,r}} \langle f'(x_{n,r}), s - x_{n,r} \rangle_{\mathbb{R}^d} ds = 0.$$

Hence we get

$$\max_{r \leq k_n} \int_{I_{n,r}} (f(s) - f(x_{n,r})) ds = O \left(k_n^{-(d+2)/d} \right).$$

Now, the triangle inequality entails

$$\begin{aligned} \Xi_n(x) &\leq k_n \left| \sum_{r=1}^{k_n} \int_{I_{n,r}} K_n(x, t) \left(\int_{I_{n,r}} (f(s) - f(x_{n,r})) ds \right) dt \right| \\ &\quad + \left| \sum_{r=1}^{k_n} \int_{I_{n,r}} K_n(x, t) (f(t) - f(x_{n,r})) dt \right| \\ &= O \left(\|K_n(x, \cdot)\|_1 k_n^{-2/d} \right) + \left| \sum_{r=1}^{k_n} \int_{I_{n,r}} K_n(x, t) \langle f'(x_{n,r}), t - x_{n,r} \rangle_{\mathbb{R}^d} dt \right| \\ &= O \left(\|K_n(x, \cdot)\|_1 k_n^{-2/d} \right) + \left| \sum_{r=1}^{k_n} \int_{I_{n,r}} (K_n(x, t) - K_n(x, x_{n,r})) \langle f'(x_{n,r}), t - x_{n,r} \rangle_{\mathbb{R}^d} dt \right| \\ &= O \left(\|K_n(x, \cdot)\|_1 k_n^{-2/d} + \sum_{r=1}^{k_n} \Gamma_{n,r}(x) \int_{I_{n,r}} \|t - x_{n,r}\|_{\mathbb{R}^d} dt \right). \end{aligned} \tag{42} \quad \square$$

Proof of Corollaries 3.1 and 3.2. We just verify the assumptions of Theorems 3.1 and 3.2 with $K_n = K_n^{PR}$.

Set $x \in F$ and note that, by our assumptions on f and K ,

$$\Delta_n(x) = O\left(k_n^{-\alpha/d}\right), \quad \Xi_n(x) = O\left(k_n^{-\alpha/d}\right) \quad (\text{by } ()),$$

$$\|K_n(x, \cdot)\|_1 = \int_{h_n^{-1}(x-E)} K(u)du = 1 + o(1), \tag{43}$$

$$\text{and } \|K_n(x, \cdot)\|_2 = h_n^{-\frac{d}{2}} \left(\int_{h_n^{-1}(x-E)} K^2(u)du \right)^{1/2} = h_n^{-\frac{d}{2}} \|K\|_2 (1 + o(1)). \tag{44}$$

Moreover, for $\varepsilon > 0$ such that

$$(t - \varepsilon, t + \varepsilon) \subset E,$$

we have

$$\begin{aligned} \Psi_n(x) &= h_n^{-d} \left| \int_E K(h_n^{-1}(x-t)) (f(t) - f(x)) dt \right| \\ &= \left| \int_{h_n^{-1}(x-E)} K(u) (f(x - h_n u) - f(x)) du \right| \\ &\leq \left| \int_{\|u\|_{\mathbb{R}^d} \leq \varepsilon h_n^{-1}} K(u) (f(x - h_n u) - f(x)) du \right| \\ &\quad + \|f\|_E \left| \int_{\|u\|_{\mathbb{R}^d} > \varepsilon h_n^{-1}} (\varepsilon h_n^{-1})^{-\alpha} \|u\|_{\mathbb{R}^d}^\alpha K(u) du \right| \\ &= O(h_n^\alpha). \end{aligned} \tag{45}$$

To verify (H.2), note that

$$\begin{aligned} nk_n^{-1} \Delta_n &= O\left(nk_n^{-(1+\alpha/d)}\right) \\ &= O\left(\left(nk_n^{-1/2} h_n^{\alpha+d/2} \varepsilon_n^{1/2}\right) (h_n^d k_n \varepsilon_n)^{-\left(\frac{1}{2} + \frac{\alpha}{d}\right)} \varepsilon_n^{\alpha/d}\right) \\ &= o(1). \end{aligned}$$

(K.2) follows from the fact that

$$\begin{aligned} \sum_{r=1}^{k_n} \Gamma_{n,r}(x_1) \int_{I_{n,r}} |K_n(x_2, t)| dt &\leq \max_{r \leq k_n} \Gamma_{n,r}(x_1) \\ &= O\left(h_n^{-d} (h_n^d k_n)^{-\alpha/d}\right) = o(h_n^{-d}) = o(\|K_n(x_1, \cdot)\|_2 \|K_n(x_2, \cdot)\|_2). \end{aligned}$$

To check (K.3), observe that, for all fixed $u \in \mathbb{R}^d$, and all $x_1 \neq x_2$,

$$K(u) K(u + h_n^{-1}(x_1 - x_2)) = o(1).$$

Hence, by dominated convergence,

$$\begin{aligned} \langle K_n(x_1, \cdot), K_n(x_2, \cdot) \rangle_2 (\|K_n(x_1, \cdot)\|_2 \|K_n(x_2, \cdot)\|_2)^{-1} \\ = O \left(\int_{\mathbb{R}^d} K(u) K(u + h_n^{-1}(x_1 - x_2)) du \right) = o(1). \end{aligned}$$

For (K.4), note that,

$$\varepsilon_n^{-1/2} k_n^{-1/2} \|K_n(x, \cdot)\|_2^{-1} \|K_n(x, \cdot)\|_E = \varepsilon_n^{-1/2} k_n^{-1/2} h_n^{-d/2} \|K\|_2^{-1} \|K\|_E = o(1)$$

with (ii). (K.5) follows by (iii) since

$$nk_n^{-1/2} \varepsilon_n^{1/2} \|K_n(x, \cdot)\|_2^{-1} \max(\Psi_n(x); \Xi_n(x)) = O \left(nk_n^{-1/2} h_n^{\alpha+d/2} \varepsilon_n^{1/2} \right).$$

Finally, (K.6) follows by (ii) and (iii) since

$$nk_n^{-3/4} \varepsilon_n^{1/4} \|K_n(x, \cdot)\|_2^{-1/2} \|K_n(x, \cdot)\|_1^{1/2} \Delta_n = (k_n \varepsilon_n)^{-1/4} (h_n^d k_n)^{-\alpha/d} O \left(nk_n^{-1/2} h_n^{\alpha+d/2} \varepsilon_n^{1/2} \right).$$

□

Proof of Corollary 3.3. We just have to verify the assumptions of Theorem 3.1 with $K_n = K_n^{PR}$, $F = (0, 1)$ and $\Sigma_{(x_1, \dots, x_p)} = \|K\|_2^2 I_p$.

For $h_n^{-1}(x - t) \in D^c$, denote by $K'_n(x, t)$ and $K''_n(x, t)$ the first and second derivative of K_n with respect to t .

Set for all $x \in (0, 1)$,

$$R_n(x) = \{r \leq k_n : D \cap (h_n^{-1}(x - I_{n,r})) \neq \emptyset\}.$$

a) (K.1) holds trivially and assumption (i) gives (H.1). To show (H.2), note that since f is in $C^2(E)$, (ii) and (iii) yield

$$nk_n^{-1} \Delta_n = O(nk_n^{-2}) = O \left((nk_n^{-7/4} h_n^{1/4}) (k_n h_n)^{-1/4} \right) = o(1).$$

(K.2), (K.3) and (K.4) can be verified as in the proof of Corollary 3.1. We now consider (K.5). Observe that

$$\begin{aligned} \sum_{r=1}^{k_n} \Gamma_{n,r}(x) &= \sum_{r \in R_n(x)} \Gamma_{n,r}(x) + \sum_{r \notin R_n(x)} \Gamma_{n,r}(x) \\ &= \#R_n(x) O(h_n^{-1}) + \sum_{r \notin R_n(x)} \sup \left\{ \left| \int_s^t K'_n(x, u) du \right| : (s, t) \in I_{n,r} \times I_{n,r} \right\} \\ &\leq O(h_n^{-1}) + \int_0^1 |K'_n(x, u)| du = O(h_n^{-1}) + h_n^{-1} \int_{h_n^{-1}(x-1)}^{h_n^{-1}x} |K'(t)| dt \\ &= O(h_n^{-1}). \end{aligned} \tag{46}$$

Hence, by (46), Lemma 4.6 and (43),

$$\Xi_n(x) = O(h_n^{-1} k_n^{-2}). \tag{47}$$

Moreover, proceeding as in (45) and using Taylor formula with (14) we get,

$$\begin{aligned} \Psi_n(x) &\leq h_n^2 \|f''\|_E \int_{|u| \leq \varepsilon h_n^{-1}} u^2 K(u) du + h_n^2 \varepsilon^{-2} \|f\|_E \int_{|u| > \varepsilon h_n^{-1}} u^2 K(u) du \\ &= O(h_n^2). \end{aligned} \tag{48}$$

Now, using (48) and (47) we obtain, by (iv),

$$nk_n^{-1/2} \|K_n(x, \cdot)\|_2^{-1} \max(\Psi_n(x); \Xi_n(x)) = O\left(nk_n^{-1/2}h_n^{1/2} \max(h_n^2, h_n^{-1}k_n^{-2})\right) = o(1).$$

Finally, the fact that, by (iii),

$$nk_n^{-3/4} \|K_n(x, \cdot)\|_2^{-1/2} \|K_n(x, \cdot)\|_1^{1/2} \Delta_n \leq nk_n^{-7/4}h_n^{1/4} = o(1),$$

yields (K.6).

b) Since K is 1-Lipschitzian,

$$\sum_{r \in R_n(x)} \left| \int_{I_{n,r}} (K_n(x, t) - K_n(x, x_{n,r})) dt \right| = \#R_n(x) O\left((h_n k_n)^{-2}\right) = O\left((h_n k_n)^{-2}\right). \tag{49}$$

Now, take $r \notin R_n(x)$. Using Taylor formula, we get that

$$\begin{aligned} |K_n(x, t) - K_n(x, x_{n,r}) - (t - x_{n,r}) K'_n(x, x_{n,r})| &= 2^{-1} \left| \int_{\min(x_{n,r}; t)}^{\max(x_{n,r}; t)} (u - x_{n,r}) K''_n(x, u) du \right| \\ &\leq \int_{I_{n,r}} |u - x_{n,r}| |K''_n(x, u)| du \\ &\leq k_n^{-1} \int_{I_{n,r}} |K''_n(x, u)| du, \end{aligned}$$

which, combined with the fact that $x_{n,r}$ is the center of $I_{n,r}$, leads to

$$\begin{aligned} \left| \int_{I_{n,r}} (K_n(x, t) - K_n(x, x_{n,r})) dt \right| &\leq \int_{I_{n,r}} |K_n(x, t) - K_n(x, x_{n,r}) - (t - x_{n,r}) K'_n(x, x_{n,r})| dt \\ &\quad + \left| K'_n(x, x_{n,r}) \int_{I_{n,r}} (t - x_{n,r}) dt \right| \\ &\leq k_n^{-2} \int_{I_{n,r}} |K''_n(x, u)| du, \end{aligned}$$

and finally entails that,

$$\begin{aligned} \left| \sum_{r \notin R_n(x)} \int_{I_{n,r}} (K_n(x, t) - K_n(x, x_{n,r})) dt \right| &\leq k_n^{-2} \int_E |K''_n(x, u)| du \\ &= O\left((h_n k_n)^{-2}\right), \end{aligned} \tag{50}$$

where we have used the fact that

$$\begin{aligned} \int_E |K''_n(x, u)| du &= h_n^{-3} \int_E |K''(h_n^{-1}(x - u))| du \\ &= h_n^{-2} \int_{h_n^{-1}(x-E)} |K''(t)| dt = O(h_n^{-2}). \end{aligned}$$

Now, by (44), (49), (50), we obtain that

$$nk_n^{-1/2} \|K_n(x, \cdot)\|_2^{-1} \left| \sum_{r=1}^{k_n} \int_{I_{n,r}} (K_n(x, t) - K_n(x, x_{n,r})) dt \right| = O\left(nk_n^{-1/2} h_n^{1/2} (h_n k_n)^{-2}\right) = O\left(nk_n^{-5/2} h_n^{-3/2}\right),$$

and (K.7) follows by (vi). □

Proof of Corollary 3.4. We verify the assumptions of Theorem 3.1 with $K_n = K_n^D$. In what follows, the first and second derivative of $K_n(x, t)$ with respect to t are denoted by $K'_n(x, t)$ and $K''_n(x, t)$. a) (K.1) holds trivially. Assumptions (i) is (H.1). Since

$$nk_n^{-2} = \left(nl_n^{-1/4} k_n^{-7/4}\right) (k_n^{-1} l_n)^{1/4}$$

(H.2) follows by (ii) and (v). In order to verify (K.2)–(K.6) we will use the following well known facts (see e.g. [32])

$$\|K_n(x, \cdot)\|_E = 1 + l_n, \quad \|K_n(x, \cdot)\|_2 = (1 + l_n)^{1/2}, \quad \|K_n(x, \cdot)\|_1 = O(\log(l_n)), \tag{51}$$

$$\langle K_n(x_1, \cdot), K_n(x_2, \cdot) \rangle_2 = K_n(x_1, x_2) = o(l_n) \text{ for } x_1 \neq x_2, \tag{52}$$

$$\int_0^1 |K'_n(x, t)| dt = O(l_n \log(l_n)), \tag{53}$$

$$\int_0^1 |K''_n(x, t)| dt = O(l_n^2 \log(l_n)). \tag{54}$$

Since f is C^2 , and taking into account of $f(0) = f(1)$ and $f'(0) = f'(1)$, a double integration by parts yields that there exists a bounded sequence $(\alpha_j)_{j \geq 1}$, such that

$$\int_0^1 f(t) e_j(t) dt = \alpha_j j^{-2} \int_0^1 f''(t) e_j(t) dt.$$

Hence, using Cauchy Schwarz inequality, we get

$$\begin{aligned} \Psi_n(x) &= \left| \sum_{j>l_n} \alpha_j j^{-2} \int_0^1 f''(t) e_j(t) dt e_j(x) \right| \\ &= O\left(\sum_{j>l_n} j^{-4}\right)^{1/2} \left(\sum_{j>l_n} \left(\int_0^1 f''(t) e_j(t) dt\right)^2\right)^{1/2} \\ &= o\left(l_n^{-3/2}\right). \end{aligned} \tag{55}$$

Since $\max_{j \geq 1} j^{-1} \|e'_j\|_E = O(1)$, the Taylor formula gives

$$\begin{aligned} \max_{1 \leq r \leq k_n} \Gamma_{n,r}(x) &\leq \max_{1 \leq r \leq k_n} \sum_{j=0}^{l_n} |e_j(x)| \sup\{e_j(t) - e_j(s) : (s, t) \in I_{n,r} \times I_{n,r}\} \\ &= O\left(k_n^{-1} \sum_{j=0}^{l_n} j\right) = O(k_n^{-1} l_n^2). \end{aligned} \tag{56}$$

By (53), we get

$$\begin{aligned} \sum_{r=1}^{k_n} \Gamma_{n,r}(x) &= \sum_{r=1}^{k_n} \sup \left\{ \left| \int_s^t K'_n(x, u) du \right| : (s, t) \in I_{n,r} \times I_{n,r} \right\} \\ &\leq \int_0^1 |K'_n(x, u)| du = O(l_n \log(l_n)). \end{aligned}$$

Therefore, Lemma 4.6 entails

$$\Xi_n(x) = O(k_n^{-2} l_n \log(l_n)). \tag{57}$$

To verify (K.2), observe that by (51), (56) and (ii),

$$\begin{aligned} \sum_{r=1}^{k_n} \Gamma_{n,r}(x_1) \int_{I_{n,r}} |K_n(x_2, t)| dt &\leq \|K_n(x_2, \cdot)\|_1 \max_{1 \leq r \leq k_n} \Gamma_{n,r}(x_1) \\ &= O(k_n^{-1} l_n^2 \log(l_n)) \\ &= o(l_n) = o(\|K_n(x_1, \cdot)\|_2 \|K_n(x_2, \cdot)\|_2). \end{aligned}$$

(K.3) follows by (51) and (52). (K.4) holds by (51) and (ii).

In order to check (K.5), observe that (51), (55, 57), (iii) and (iv) yield

$$nk_n^{-1/2} \|K_n(x, \cdot)\|_2^{-1} \max(\Psi_n(x); \Xi_n(x)) = O\left(nk_n^{-1/2} l_n^{-1/2} \max(l_n^{-3/2}; k_n^{-2} l_n \log(l_n))\right) = o(1).$$

Finally, (51) and (v) give (K.6).

b) It is readily seen that (iv') implies (iv). Moreover, since

$$nl_n^{-1/4} k_n^{-7/4} \log(l_n)^{1/2} = k_n^{-1/4} \left(nk_n^{-1/2} l_n^{-2}\right)^{1/2} \left(nk_n^{-5/2} l_n^{3/2} \log(l_n)\right)^{1/2}$$

we get that (iv') and (iii) imply (v). Therefore (K.2)–(K.6) holds by a). To verify (K.7), note that, by Taylor formula, and since $x_{n,r}$ is the center of $I_{n,r}$, we get

$$\begin{aligned} \left| \int_{I_{n,r}} (K_n(x, t) - K_n(x, x_{n,r})) dt \right| &= 2^{-1} \left| \int_{I_{n,r}} \left(\int_{\min(x_{n,r}; t)}^{\max(x_{n,r}; t)} (u - x_{n,r}) K''_n(x, u) du \right) dt \right| \\ &\leq \int_{I_{n,r}} \left(\int_{I_{n,r}} |u - x_{n,r}| |K''_n(x, u)| du \right) dt \\ &\leq k_n^{-2} \int_{I_{n,r}} |K''_n(x, u)| du. \end{aligned}$$

This and (54) entails

$$\begin{aligned} \left| \sum_{r=1}^{k_n} \int_{I_{n,r}} (K_n(x, t) - K_n(x, x_{n,r})) dt \right| &\leq k_n^{-2} \int_0^1 |K''_n(x, t)| dt \\ &= O(k_n^{-2} l_n^2 \log(l_n)), \end{aligned}$$

which, combined with (iv') gives the result. □

5. APPENDIX: LIMIT THEOREMS FOR CONDITIONALLY INDEPENDENT RANDOM SUM

This part is devoted to provide general results about the central limit property and the moderate deviations principle of a sequence of random \mathbb{R}^p valued vectors which have been used in our proofs (see also [14] and [28]). Consider a triangular array $(\zeta_{n,r})_{r \leq k_n}$ of real valued random variables defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and set, for $n \geq 1$,

$$\theta_n = \sum_{r=1}^{k_n} W_{n,r} \zeta_{n,r},$$

where $(W_{n,r})_{r \leq k_n} \subset \mathbb{R}^p$. In the following, for all $u \in \mathbb{R}^p$, we set

$$w_{n,r}(u) = \langle W_{n,r}, u \rangle_{\mathbb{R}^d}^2 \text{ and } \theta_n(u) = \sum_{r=1}^{k_n} W_{n,r}(u) \zeta_{n,r}.$$

In order to state central limit theorem and moderate deviations principle for (θ_n) we consider a sequence $(\varepsilon_n)_{n \geq 1} \subset \mathbb{R}^+$ such that $(\varepsilon_n)_{n \geq 1} \equiv 1$ or $\varepsilon_n \downarrow 0$ and we assume:

$$(A.1) \quad k_n \xrightarrow{n \rightarrow \infty} 0 \text{ and } \varepsilon_n^{-1/2} w_n(u) \xrightarrow{n \rightarrow \infty} 0 \text{ where } w_n(u) := \max_{r \leq k_n} |w_{n,r}(u)|.$$

(A.2) There exists a covariance matrix Σ in \mathbb{R}^p such that for all $u \in \mathbb{R}^p$,

$$\sum_{r=1}^{k_n} w_{n,r}^2(u) \xrightarrow{n \rightarrow \infty} {}^t u \Sigma u := \sigma^2(u).$$

(A.3) For all $n \geq 1$, $(\zeta_{n,r})_{r \leq k_n}$ are independent conditionally to a sigma field $\mathcal{B}_n \subset \mathcal{A}$.

For all $n \geq 1$, there exists a set $B_n \in \mathcal{B}_n$ such that:

$$(A.4) \quad \limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P}(B_n^c) = -\infty.$$

(A.5) For all $r \leq k_n$, $\mathbb{E}(\zeta_{n,r} | \mathcal{B}_n) \mathbf{1}_{B_n} = 0$ a.s.

(A.6) For some sequence $\eta_n \downarrow 0$, $\max_{1 \leq r \leq k_n} |\mathbb{E}(\zeta_{n,r}^2 | \mathcal{B}_n) - 1| \mathbf{1}_{B_n} \leq \eta_n$ a.s.

(A.7) For some real $K_7 > 0$, $\limsup_{n \rightarrow \infty} \max_{1 \leq r \leq k_n} \mathbb{E}(|\zeta_{n,r}|^3 | \mathcal{B}_n) \mathbf{1}_{B_n} < K_7$ a.s.

(A.8) For some real $K_8 > 0$, $\limsup_{n \rightarrow \infty} \max_{l \geq 3} \max_{1 \leq r \leq k_n} \left(\frac{1}{l}\right)^{1/l} \mathbb{E}(|\zeta_{n,r}|^l | \mathcal{B}_n)^{1/l} \mathbf{1}_{B_n} < K_8$ a.s.

We are now in position to express the main results of this section:

Theorem 5.1. *If assumptions (A.1)–(A.7) hold with $\varepsilon_n = 1$, then $\theta_n \xrightarrow{\mathcal{D}} N(0, \Sigma)$.*

Proof. For all u in \mathbb{R}^p , set

$$\sigma_n^2(u) := \left(\sum_{r=1}^{k_n} w_{n,r}^2(u) \mathbb{E}(\zeta_{n,r}^2 | \mathcal{B}_n) \right) \text{ and } \tau_n(u) := \sum_{r=1}^{k_n} |w_{n,r}(u)|^3 \mathbb{E}(|\zeta_{n,r}|^3 | \mathcal{B}_n).$$

By the Berry Esseen Theorem (see [30] Th. 5.4) we get for an universal constant C_1 :

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}(\sigma_n^{-1}(u) \theta_n(u) \leq x | \mathcal{B}_n) - \Phi(x) \right| \mathbf{1}_{B_n} \leq C_1 \mathbb{E}(\sigma_n^{-3}(u) \tau_n(u) | \mathcal{B}_n) \mathbf{1}_{B_n}$$

where Φ denotes the standard normal distribution function. Now, by (A.2) and (A.6),

$$\left| \sigma_n^2(u) - \sigma(u)^2 \right| \mathbf{1}_{B_n} \leq \left| \sum_{r=1}^{k_n} w_{n,r}^2(u) - \sigma^2(u) \right| + \eta_n \sum_{r=1}^{k_n} w_{n,r}^2(u) = o(1).$$

In particular, for some real sequence $\gamma_n \downarrow 0$,

$$\sigma_n(u) \mathbf{1}_{B_n} \in [(1 - \gamma_n) \sigma(u) \mathbf{1}_{B_n}, (1 + \gamma_n) \sigma(u) \mathbf{1}_{B_n}]. \tag{58}$$

Moreover, by (A.7),

$$\tau_n(u) \mathbf{1}_{B_n} \leq w_n(u) K_7 \sum_{r=1}^{k_n} w_{n,r}^2(u).$$

Hence, for some C_2 all $n \geq 1$ and all $x \in \mathbb{R}$,

$$|\mathbb{P}(\sigma_n^{-1}(u) \theta_n(u) \leq x \mid \mathcal{B}_n) - \Phi(x)| \mathbf{1}_{B_n} \leq C_2 w_n(u)$$

and $\sigma_n^{-1}(u) \theta_n(u) \xrightarrow{D} N(0, 1)$, since

$$\begin{aligned} |\mathbb{P}(\sigma_n^{-1}(u) \theta_n(u) \leq x) - \Phi(x)| &\leq |\mathbb{P}(\sigma_n^{-1}(u) \theta_n(u) \leq x; B_n) - \Phi(x) \mathbb{P}(B_n)| + \mathbb{P}(B_n^c) \\ &\leq \mathbb{E}(|\mathbb{P}(\sigma_n^{-1}(u) \theta_n(u) \leq x \mid \mathcal{B}_n) - \Phi(x)| \mathbf{1}_{B_n}) + \mathbb{P}(B_n^c) \\ &\leq C_2 w_n(u) + \mathbb{P}(B_n^c) = o(1) \text{ by (A.2) and (A.4)}. \end{aligned}$$

Moreover, by (58), $\sigma_n(u) \xrightarrow{\mathbb{P}} \sigma(u)$ and we get the result by [3] Theorems 4-1 and 7-7. □

In order to deal with moderate deviation principle we assume that Σ is regular and we define

$$I : s \in \mathbb{R}^p \mapsto 2^{-1} t_s \Sigma^{-1} s.$$

Theorem 5.2. *Assume that Σ is regular and (A.1)–(A.8) hold for $\varepsilon_n \downarrow 0$. Then, $(\varepsilon_n^{1/2} \theta_n) \in \text{LDP}(\varepsilon_n, I)$.*

Proof. Using the conditional independance of $(\zeta_{n,r})_{r \leq k_n}$ and (A.5), we get that

$$\mathbb{E} \left(\exp \left(\varepsilon_n^{-1/2} \theta_n(u) \mathbf{1}_{B_n} \right) \mid \mathcal{B}_n \right) = \mathbf{1}_{B_n^c} + \prod_{r \leq k_n} (1 + 2^{-1} \varepsilon_n^{-1} w_{n,r}^2(u) (1 + \Gamma_{n,r}(u))) \mathbf{1}_{B_n},$$

where

$$\Gamma_{n,r}(u) := \left(\mathbb{E}(\zeta_{n,r}^2 \mid \mathcal{B}_n) - 1 + 2 \sum_{l \geq 3} (l!)^{-1} (\varepsilon_n^{-1/2} w_{n,r}(u))^{l-2} \mathbb{E}(\zeta_{n,r}^l \mid \mathcal{B}_n) \right) \mathbf{1}_{B_n}.$$

Now, for all large n , (A.6) and (A.8) lead to

$$\begin{aligned} \max_{r \leq k_n} |\Gamma_{n,r}(u)| &\leq \eta_n + 2K_8^2 \sum_{l \geq 3} \left| K_8 \varepsilon_n^{-1/2} w_{n,r}(u) \right|^{l-2} \\ &\leq \eta_n + 4K_8^3 \varepsilon_n^{-1/2} w_n(u) := \gamma_n(u) = o(1). \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{E} \left(\exp \left(\varepsilon_n^{-1/2} \theta_n(u) \mathbf{1}_{B_n} \right) \right) &\geq \mathbb{E} \left(\mathbf{1}_{B_n^c} + \left(\prod_{r \leq k_n} (1 + 2^{-1} \varepsilon_n^{-1} w_{n,r}^2(u) (1 - \gamma_n(u))) \right) \mathbf{1}_{B_n} \right) \\ &\geq \mathbb{P}(B_n) \prod_{r \leq k_n} (1 + 2^{-1} \varepsilon_n^{-1} w_{n,r}^2(u) (1 - \gamma_n(u))), \end{aligned}$$

$$\begin{aligned} \mathbb{E} \left(\exp \left(\varepsilon_n^{-1/2} \theta_n(u) \mathbf{1}_{B_n} \right) \right) &\leq \mathbb{E} \left(\mathbf{1}_{B_n^c} + \left(\prod_{r \leq k_n} (1 + 2^{-1} \varepsilon_n^{-1} w_{n,r}^2(u) (1 + \gamma_n(u))) \right) \mathbf{1}_{B_n} \right) \\ &\leq \mathbb{P}(B_n^c) + \prod_{r \leq k_n} (1 + 2^{-1} \varepsilon_n^{-1} w_{n,r}^2(u) (1 + \gamma_n(u))), \end{aligned}$$

and, therefore,

$$l_{n,1}(u) \leq \varepsilon_n \log \mathbb{E} \left(\exp \left(\varepsilon_n^{-1/2} \theta_n(u) \mathbf{1}_{B_n} \right) \right) \leq l_{n,2}(u) \tag{59}$$

with

$$l_{n,1}(u) = \varepsilon_n \log \mathbb{P}(B_n) + \sum_{r \leq k_n} \varepsilon_n \log (1 + 2^{-1} \varepsilon_n^{-1} w_{n,r}^2(u) (1 - \gamma_n(u)))$$

and

$$l_{n,2}(u) = \varepsilon_n \log \left(\mathbb{P}(B_n^c) + \prod_{r \leq k_n} (1 + 2^{-1} \varepsilon_n^{-1} w_{n,r}^2(u) (1 + \gamma_n(u))) \right).$$

Now, using (59) and Taylor expansion we get,

$$\lim_{n \rightarrow \infty} \varepsilon_n \log \mathbb{E} \left(\exp \left(\varepsilon_n^{-1/2} \theta_n(u) \mathbf{1}_{B_n} \right) \right) = 2^{-1} {}^t u \Sigma u,$$

hence, by the Gartner-Ellis Theorem (see [7] Th. 2-3-6), $(\varepsilon_n^{1/2} \theta_n \mathbf{1}_{B_n})_{n \geq 1} \in \text{LDP}(\varepsilon_n, \mathbf{I})$.

Finally, since, for all $\eta > 0$,

$$\limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P} \left(\varepsilon_n^{1/2} \|\theta_n - \theta_n \mathbf{1}_{B_n}\|_{\mathbb{R}^p} \geq \eta \right) \leq \limsup_{n \rightarrow \infty} \varepsilon_n \log \mathbb{P}(B_n^c) = -\infty,$$

we get the intended result by [7] Theorem 4-2-21. □

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