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APPROXIMATION OF THE FRACTIONAL BROWNIAN SHEET *VIA* ORNSTEIN-UHLENBECK SHEET

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Abstract. A stochastic "Fubini" lemma and an approximation theorem for integrals on the plane are used to produce a simulation algorithm for an anisotropic fractional Brownian sheet. The convergence rate is given. These results are valuable for any value of the Hurst parameters $(\alpha_1, \alpha_2) \in]0,1[^2, \alpha_i \neq \frac{1}{2}$. Finally, the approximation process is iterative on the quarter plane $\mathbb{R}^2_+$. A sample of such simulations can be used to test estimators of the parameters $\alpha_i, i = 1, 2$.

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1. Introduction

The aim of this paper is to produce and to study a simulation algorithm for an anisotropic fractional Brownian sheet. An important application of such a simulation is to supply a sample of fractional Brownian sheet almost sure approximations: thus estimators like these defined in [13] can be tested. Recall that the non necessarily Gaussian random fields were started by Samorodnitsky and Taqqu [17] and developed by Cohen [8]. For the $1-$dimensional fractional Brownian motion, Meyer *et al.* [15], Ayache and Taqqu [3] study an approximation of the fractional Brownian motion with Hurst parameter α, using a wavelet decomposition. Ayache *et al.* [2] also use a wavelet decomposition but for the anisotropic fractional Brownian sheet. Besides, Bardet *et al.* [4] did a careful comparison between the different algorithms. Here we recall and develop the results of [7], but there all the proofs are omitted. For the sake of completeness, and because these proofs enlighten the more general cases, we give them in Section 3. The scheme of [7] is as follows: Fubini's Lemma allows to get a representation of the fractional Brownian motion as a deterministic integral of Ornstein-Ulhenbeck processes; this integral is approximated by a finite sum over a geometric subdivision; besides, [7] introduce some operators on Hölder functions which – applied to the Brownian motion – give the integrands under the deterministic integral. This step allows them to obtain some fine results about these integrands regularity; they obtain a time iterative algorithm using Markov properties of Ornstein-Ulhenbeck processes. Gathering this algorithm and the deterministic integral approximation, they produce an approximation of the fractional Brownian motion. The rate convergence of this algorithm is studied in [7] (*cf.* p. 162) but without a temporal approximation nor an accuracy evaluation. Here we add the temporal approximation. Their rate convergence is about $O(\log N N^{1+\frac{\beta}{1+\beta}})$ where $\beta = \alpha \wedge (\frac{1}{2} - \alpha)$. More precisely concerning our algorithm, if we choose a simulation accuracy of about $N^{-\eta}, 0 \ < \eta < \frac{1}{2}$, to

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produce an I size image, we need to generate I independent random variables and a $2n$ Gaussian vector where $n = O(N^\eta \log N)$ and the algorithm complexity is $O(N^{1+\eta} \log N)$.

This study is to be done in higher dimensions. For the 2−dimensional case and when the value of one of the Hurst parameters (α_1, α_2) is more than $\frac{1}{2}$, for computation reasons and not because of the model, Chan et Wood's algorithms [19] failed. S. Léger used them in [13] to test some estimators of parameters α_i successfully when $\alpha_i < \frac{1}{2}$. But, when $\alpha_1 \vee \alpha_2 > \frac{1}{2}$, the extended 2–dimension circulant embedding of the covariance matrix should be theoretically non negative definite, but practically it is not; these matrices are not well-conditioned and so Choleski's method can't be used.

The generalization from 1−dimension to 2−dimension is not so easy: for instance, Taylor's formula in 2−dimension involves the cross derivatives and to control the constants we need to detail the computations. Moreover, as is pointed out in [10], in the one dimensional case, our method can be applied to any Gaussian sheet. It could also be used for any process written as a multiple integral of any function satisfying smooth assumptions. For instance look at $\int_{[0,+\infty[^2} g(x,y)\mu(\mathrm{d}x, \mathrm{d}y)$ where μ is a random measure and g is a Laplace transform. The aim here is to approximate this double integral.

Concerning our algorithm, if we choose a simulation accuracy of about $I^{-\eta}, 0 < \eta < \frac{1}{2}$, to produce a I^2 size image, we need to generate I^2 independent random variables, a $2n \times 2n$ Gaussian matrix, $I(1+I)$ $2n$-Gaussian vectors where n is about $[\log I] I^\eta$, then the algorithm complexity is $O([\log I]^2 I^{2(1+\eta)})$.

The paper is organized as follows: first the problem is set, the 2−dimensional Liouville Brownian sheet is defined, as is the 2−dimensional fractional Brownian sheet that we want to simulate. In Section 3, we first recall a set of deterministic tools built in [7] in order to obtain a discrete approximation of the 1−dimensional Brownian process. We follow the same scheme as the one in [7]: Section 4 extends all these results to our 2−dimensional Brownian sheet; first a theorem for deterministic integrals on the plane is proved, then operators on the set of Hölder 2−dimensional functions are defined and their properties are studied. All this is used to produce a discrete approximation of the 2−dimensional Brownian sheet and the errors are controlled. Finally, an iterative algorithm of the 2−dimensional Brownian sheet synthesis is given thanks to a kind of Markov property. This property relies on the fact that the fractional Brownian motion can be considered as an Ornstein-Uhlenbeck process superposition (*cf.* [5]). The rate of its convergence is given in Section 5 with a constant which is a random variable, the law of which is known (its extreme values have very low probability). This constant also belongs to any L^p so, as a byproduct, the approximating sheet uniformly converges to the fractional Brownian sheet in any L^p. The algorithm parameters are chosen with respect to a given accuracy of the approximation. The limit of this algorithm is stressed when the parameters α are very near $\frac{1}{2}, 0, 1$. The largest proofs are provided in Section 7.

2. Problem setting

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $\mathrm{d}B$ a white noise sheet on it. The "rectangular" fractional Brownian sheet W^{α_1,α_2} is defined in [13] or [14]:

$$W^{\alpha_1,\alpha_2}_{s,t} = \int_{\mathbb{R}^2} [(s-u)_+^{\alpha_1 - \frac{1}{2}} - (-u)_+^{\alpha_1 - \frac{1}{2}}][(t-v)_+^{\alpha_2 - \frac{1}{2}} - (-v)_+^{\alpha_2 - \frac{1}{2}}] \mathrm{d}B_{u,v}, \tag{1}$$

$(\alpha_1, \alpha_2) \in]0,1[^2$ and $(s,t) \in \mathbb{R}_+^2$. This random field is null on the axes. Similarly we introduce the Liouville Brownian sheet defined when $(s,t) \in \mathbb{R}_+^2$ by

$$V^{\alpha_1,\alpha_2}_{s,t} = \int_{[0,s]\times[0,t]} (s-u)^{\alpha_1 - \frac{1}{2}} (t-v)^{\alpha_2 - \frac{1}{2}} \mathrm{d}B_{u,v}, \tag{2}$$

but this one doesn't have stationary increments. Recall these two expressions in one-dimension:

$$W_t^\alpha = \int_{\mathbb{R}} [(t-u)_+^{\alpha-\frac{1}{2}} - (-u)_+^{\alpha-\frac{1}{2}}]\mathrm{d}B_u, \ \ V_t^\alpha = \int_0^t (t-u)^{\alpha-\frac{1}{2}}\mathrm{d}B_u, \ t \in \mathbb{R}. \tag{3}$$

We can't here approximate them directly by a sum since a recursive computation is not feasible (*cf.* Taqqu [17]). To produce an iterative process of the trajectories of these two random fields, we generalize Carmona *et al.*' method [7] to the 2−dimensional case. For the sake of clearness, we start with the Liouville Brownian sheet.

2.1. Liouville Brownian sheet as a superposition of Ornstein-Uhlenbeck processes

Recall the equality for $0 < \alpha < \frac{1}{2}$:

$$(s-u)^{\alpha-\frac{1}{2}} = \frac{1}{\Gamma(\frac{1}{2}-\alpha)} \int_0^\infty x^{-\alpha-\frac{1}{2}} \mathrm{e}^{-x(s-u)}\mathrm{d}x, \ s > u. \tag{4}$$

The key is the following so called stochastic Fubini lemma (*cf.* Carmona *et al.* [7] for instance). Let $(n,p) \in \mathbb{N}^2$, the mixed Lebesgue space and its norm [18] are:

$$\begin{aligned} \|f\|_{p_1,p_2} &= \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^p} |f(u,a)|^{p_1}\mathrm{d}u \right)^{\frac{p_2}{p_1}} \mathrm{d}a \right)^{\frac{1}{p_2}} \\ \mathcal{L}_{p_1,p_2}(\mathbb{R}^p \times \mathbb{R}^n) &= \{f \ : \ \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}, \text{ Borelian }, \|f\|_{p_1,p_2} < +\infty\}. \end{aligned} \tag{5}$$

Let us remark that if $f \in \mathcal{L}_{1,2}(\mathbb{R}^n \times \mathbb{R}^p)$ using the Cauchy-Schwartz' inequality:

$$\|f\|_{1,2}^2 = \left(\int_{\mathbb{R}^p} \left(\int_{\mathbb{R}^n} |f(u,a)|\mathrm{d}a \right)^2 \mathrm{d}u \right) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^p} |f(u,a)||f(u,b)|\mathrm{d}u\mathrm{d}a\mathrm{d}b \le \|f\|_{2,1}^2,$$

so yields the inclusion $\mathcal{L}_{2,1}(\mathbb{R}^p \times \mathbb{R}^n) \subset \mathcal{L}_{1,2}(\mathbb{R}^n \times \mathbb{R}^p)$.

Lemma 2.1. *Let $f \in \mathcal{L}_{2,1}(\mathbb{R}^p \times \mathbb{R}^n)$ and $\mathrm{d}B$ be a white noise sheet on $\mathbb{R}^p$, then almost surely :*

$$\int_{\mathbb{R}^n} (\int_{\mathbb{R}^p} f(u,a)\mathrm{d}B_u)\mathrm{d}a = \int_{\mathbb{R}^p} (\int_{\mathbb{R}^n} f(u,a)\mathrm{d}a)\mathrm{d}B_u. \tag{6}$$

Proof. The map $Y_1 : f \mapsto \int_{\mathbb{R}^n}\int_{\mathbb{R}^p} f(u,v)\mathrm{d}B_u\mathrm{d}v$ is a continuous linear map on the step functions in $\mathcal{L}_{2,1}$ taking its values in $L^2(\Omega)$. The set of these step functions is dense in $\mathcal{L}_{2,1}$ (*cf.* Lem. 6.2.11 p. 124 [18]) so this map admits a unique continuous linear extension on $\mathcal{L}_{2,1}$.

Let the map $Y_2(f) : f \mapsto \int_{\mathbb{R}^p}\int_{\mathbb{R}^n} f(u,v)\mathrm{d}v\mathrm{d}B_u$. It is a linear continuous map on $\mathcal{L}_{2,1}(\mathbb{R}^p \times \mathbb{R}^n) \subset \mathcal{L}_{1,2}(\mathbb{R}^n \times \mathbb{R}^p)$ with a norm 1 from $\mathcal{L}_{1,2}(\mathbb{R}^n \times \mathbb{R}^p)$ to $L^2(\Omega)$ so 'a fortiori' on $\mathcal{L}_{2,1}(\mathbb{R}^p \times \mathbb{R}^n)$:

$$\|Y_2(f)\|_2^2 = \int_{\mathbb{R}^p} (\int_{\mathbb{R}^n} f(u,a)da)^2\mathrm{d}u = \|f\|_{1,2}^2 \le \|f\|_{2,1}^2.$$

Finally, the maps $Y_i, i = 1,2$, are well defined and coincide on the step functions. □

A similar lemma is given in [11] (Lem. 4.1, p. 116) but the assumptions are quite different and are not satisfied in our cases.

This lemma and (4) allow us to prove:

Proposition 2.2. *Let $(\alpha_1,\alpha_2)\in]0,\frac{1}{2}[^2$; the process V^{α_1,α_2} admits the following representation $\forall(s,t)\in\mathbb{R}^2_+$, almost surely:*

$$V^{\alpha_1,\alpha_2}_{s,t}=\frac{1}{\Gamma(\frac{1}{2}-\alpha_1)}\frac{1}{\Gamma(\frac{1}{2}-\alpha_2)}\int_{\mathbb{R}^2_+}x^{-\alpha_1-\frac{1}{2}}y^{-\alpha_2-\frac{1}{2}}X(x,y,s,t)\mathrm{d}x\mathrm{d}y \tag{7}$$

where

$$X(x,y,s,t)=\int_{[0,s[\times[0,t[}\mathrm{e}^{-x(s-u)}\mathrm{e}^{-y(t-v)}\mathrm{d}B_{u,v}. \tag{8}$$

Proof. The stochastic Fubini Lemma 2.1 applied to $(s,t)\in\mathbb{R}^2_+$, $n=p=2$, $u=(u,v)$, $a=(x,y)$ and to $f(u,v,x,y)=x^{-\alpha_1-1/2}y^{-\alpha_2-1/2}\mathrm{e}^{-x(s-u)-y(t-v)}\mathbf{1}_{[0,t]}(v)\mathbf{1}_{[0,s]}(u)\mathbf{1}_{]0,+\infty[^2}(x,y)$, is correct since

$$\begin{aligned}\|f\|_{2,1}&=\int_{\mathbb{R}^2_+}x^{-\alpha_1-1/2}y^{-\alpha_2-1/2}\sqrt{\int_0^t\int_0^s\mathrm{e}^{-x2(s-u)-y2(t-v)}\mathrm{d}u\mathrm{d}v}\mathrm{d}x\mathrm{d}y\\&=\int_{\mathbb{R}_+}x^{-\alpha_1-1/2}\sqrt{\frac{1-\mathrm{e}^{-2xs}}{2x}}\mathrm{d}x\int_{\mathbb{R}_+}y^{-\alpha_2-1/2}\sqrt{\frac{1-\mathrm{e}^{-2yt}}{2y}}\mathrm{d}y<+\infty,\end{aligned}$$

and $-\alpha_i-1/2\in]-1,-\frac{1}{2}[,\ i=1,2,\ -\alpha_i-1<-1,\ i=1,2.$ □

Now for $\alpha\in]\frac{1}{2},1[$ and $s\geq u$, use the identity:

$$(s-u)^{\alpha-\frac{1}{2}}=\int_u^s\left(\alpha-\frac{1}{2}\right)(r-u)^{\alpha-\frac{3}{2}}\mathrm{d}r=-\frac{1}{\Gamma(\frac{1}{2}-\alpha)}\int_{\mathbb{R}^+}x^{\frac{1}{2}-\alpha}\int_u^s\mathrm{e}^{-x(r-u)}\mathrm{d}r\mathrm{d}x. \tag{9}$$

In the case $\alpha_1\vee\alpha_2\in]\frac{1}{2},1[$, let us introduce the notation

$$a_i=\alpha_i+\frac{1}{2}.\mathrm{sign}\left(\frac{1}{2}-\alpha_i\right),\ i=1,2. \tag{10}$$

Proposition 2.3. *Let $\alpha_1\vee\alpha_2\in]\frac{1}{2},1[$. The process V^{α_1,α_2} admits the following representation on $(\mathbb{R}^+)^2$, $\forall(s,t)$, almost surely:*

$$V^{\alpha_1,\alpha_2}_{s,t}=\frac{1}{\Gamma(\frac{1}{2}-\alpha_1)}\frac{1}{\Gamma(\frac{1}{2}-\alpha_2)}\int_{\mathbb{R}^2_+}x^{-a_1}y^{-a_2}U(x,y,s,t)\mathrm{d}x\mathrm{d}y \tag{11}$$

where $U=Y,T$ or Z and Y,T,Z are given by:

$$\begin{aligned}Y(x,y,s,t)&=\int_{[0,s[\times[0,t[}X(x,y,r_1,r_2)\mathrm{d}r_1\mathrm{d}r_2,\ \text{if } \alpha_i>\frac{1}{2},\ i=1,2,\\T(x,y,s,t)&=\int_0^tX(x,y,s,v)\mathrm{d}v,\ \text{if } \alpha_1<\frac{1}{2}<\alpha_2, &(12)\\Z(x,y,s,t)&=\int_0^sX(x,y,u,t)\mathrm{d}u\ \text{ if } \alpha_1>\frac{1}{2}>\alpha_2. &(13)\end{aligned}$$

Proof. We only detail the proof when $\alpha_i>\frac{1}{2},i=1,2$. Once again, the stochastic Fubini Lemma 2.1 is used ((s,t) being fixed) with $n=p=2$, $u=(u,v)\in[0,s]\times[0,t]$, $a=(x,y)\in\mathbb{R}^2_+$ and

$$f(u,v,x,y)=x^{\frac{1}{2}-\alpha_1}y^{\frac{1}{2}-\alpha_2}\int_u^s\int_v^t\mathrm{e}^{-x(r-u)-y(z-v)}\mathrm{d}r\mathrm{d}z$$

since $V_{s,t}^{\alpha_1,\alpha_2} = Y_2(f)$ and $\|f\|_{2,1} < \infty$. Thus $V_{s,t}^{\alpha_1,\alpha_2} = Y_1(f)$, meaning that

$$V_{s,t}^{\alpha_1,\alpha_2} = \int_{\mathbb{R}_+^2} (\int_0^s \int_0^t f(u,v,x,y)\mathrm{d}B_{u,v})\mathrm{d}x\mathrm{d}y.$$

But

$$\int_0^s \int_0^t f(u,v,x,y)\mathrm{d}B_{u,v} = x^{\frac{1}{2}-\alpha_1} y^{\frac{1}{2}-\alpha_2} \int_0^s \int_0^t \left(\int_u^s \int_v^t \mathrm{e}^{-x(r-u)} \mathrm{e}^{-y(z-v)} \mathrm{d}r\mathrm{d}z \right) \mathrm{d}B_{u,v}.$$

Using once again Lemma 2.1 (here the integrand is continuous with compact support):

$$\int_0^s \int_0^t \left(\int_u^s \int_v^t \mathrm{e}^{-x(r-u)} \mathrm{e}^{-y(z-v)} \mathrm{d}r\mathrm{d}z \right) \mathrm{d}B_{u,v} = \int_0^s \int_0^t X(x,y,r,z)\mathrm{d}r\mathrm{d}z,$$

and the proof is concluded. When $\alpha_1 \in]\frac{1}{2}, 1[$ or $\alpha_2 \in]\frac{1}{2}, 1[$, the proof is almost the same. □

Remark 2.4. The process X is continuous with respect the four parameters. Moreover, T, Z, Y are also continuous with respect the four parameters as integrals of X on the compact sets $[0,t]$ or $[0,s]$ or $[0,s] \times [0,t]$.

Proof. cf. Section 7. □

These processes, made discrete with respect to (s,t), look like ARMA processes. These fields can be seen as extended 2–dimensions Ornstein-Uhlenbeck processes in the sense of Proposition 2.5. Thus, the fractional Brownian sheet can be seen as a Ornstein-Uhlenbeck processes superposition.

Proposition 2.5. *Let $(x,y) \in \mathbb{R}_+^2$, $U = X, Y, Z$ or T then $U(x,y,.,.)$ is solution to the integral equation:*

$$\begin{aligned} U(x,y,s,t) &= U(0,0,s,t) + xy \int_0^s \int_0^t U(x,y,z,\tau)\mathrm{d}z\mathrm{d}\tau \\ &\quad -x \int_0^s U(x,0,z,t)\mathrm{d}z - y \int_0^t U(0,y,s,\tau)\mathrm{d}\tau,\ (s,t) \in \mathbb{R}_+^2. \end{aligned} \tag{14}$$

For instance, $U(x,0,s,t) = U(0,0,s,t) - x \int_0^s U(x,0,z,t)\mathrm{d}z$ and $U(0,y,s,t) = U(0,0,s,t) - \int_0^s U(0,y,s,\tau)d\tau$, where

$$U(0,0,s,t) = B_{s,t} 1_{\alpha_1 \vee \alpha_2 < \frac{1}{2}} + \int_0^t B_{s,u}\mathrm{d}u \mathbf{1}_{\alpha_1 < \frac{1}{2} < \alpha_2} + \int_0^s B_{u,t}\mathrm{d}u \mathbf{1}_{\alpha_2 < \frac{1}{2} < \alpha_1} + \int_0^s \int_0^t B_{u,v}\mathrm{d}u\mathrm{d}v \mathbf{1}_{\alpha_1 \wedge \alpha_2 > \frac{1}{2}}.$$

Proof. Above, we proved that any Gaussian field in this proposition admits a continuous modification. So it is enough to prove the result (s,t) being fixed.

Remark the identity when $0 \le u \le s, 0 \le v \le t$:

$$\mathrm{e}^{-x(s-u)} \mathrm{e}^{-y(t-v)} - 1 = xy \int_u^s \int_v^t \mathrm{e}^{-x(z-u)} \mathrm{e}^{-y(\tau-v)} \mathrm{d}z\mathrm{d}\tau - x \int_u^s \mathrm{e}^{-x(z-u)}\mathrm{d}z - y \int_v^t \mathrm{e}^{-y(\tau-v)}\mathrm{d}\tau.$$

This identity is integrated on $[0,s] \times [0,t]$ with respect to the white noise sheet $\mathrm{d}B$. Using the stochastic Fubini Lemma 2.1 we invert the two integrals (the lemma assumptions are satisfied: the integrands have compact support and are continuous).

So the first term in the decomposition of $X(x,y,s,t) - B_{s,t}$ is:

$$xy \int_0^s \int_0^t \left(\int_u^s \int_v^t \mathrm{e}^{-x(z-u)} \mathrm{e}^{-y(\tau-v)} \mathrm{d}z\mathrm{d}\tau \right) \mathrm{d}B_{u,v} = xy \int_0^s \int_0^t \left(\int_0^z \int_0^\tau \mathrm{e}^{-x(z-u)} \mathrm{e}^{-y(\tau-v)} \mathrm{d}B_{u,v} \right) \mathrm{d}z\mathrm{d}\tau$$

which is $xy\int_0^s\int_0^t X(x,y,z,\tau)\mathrm{d}z\mathrm{d}\tau$. The two following terms are identified with $-x\int_0^s X(x,0,z,t)\mathrm{d}z$ and $-y\int_0^t X(0,y,s,\tau)\mathrm{d}\tau$. This yields (14) for $U = X$. We now integrate this identity with respect to the two last arguments on $[0,s]\times[0,t]$, (respectively with respect to the last argument on $[0,t]$ or the third argument on $[0,s]$), yields (14) for $U = Y, Z, T$. □

2.2. Fractional Brownian sheet

Similar results can be obtained for the fractional Brownian sheet due to the equality, $s > u, 0 < \alpha < \frac{1}{2}$:

$$(s-u)_+^{\alpha-\frac{1}{2}} - (-u)_+^{\alpha-\frac{1}{2}} = \frac{1}{\Gamma(\frac{1}{2}-\alpha)}\int_0^\infty x^{-\alpha-\frac{1}{2}}[\mathrm{e}^{-x(s-u)}\mathbf{1}_{\{u<s\}} - \mathrm{e}^{xu}\mathbf{1}_{\{u<0\}}]\mathrm{d}x. \tag{15}$$

This equality allows us to show:

Proposition 2.6. *Let $(\alpha_1,\alpha_2)\in]0,\frac{1}{2}[^2$; the process W^{α_1,α_2} admits the following representation $\forall(s,t)\in\mathbb{R}_+^2$, almost surely:*

$$W_{s,t}^{\alpha_1,\alpha_2} = \frac{1}{\Gamma(\frac{1}{2}-\alpha_1)}\frac{1}{\Gamma(\frac{1}{2}-\alpha_2)}\int_{\mathbb{R}_+^2} x^{-\alpha_1-\frac{1}{2}}y^{-\alpha_2-\frac{1}{2}}\tilde{X}(x,y,s,t)\mathrm{d}x\mathrm{d}y, \tag{16}$$

where

$$\tilde{X}(x,y,s,t) = \int_{\mathbb{R}^2} f_s(x,u)f_t(y,v)\mathrm{d}B_{u,v} \tag{17}$$

and $f_s(x,u) = \mathbf{1}_{[0,s]}(u)\mathrm{e}^{-x(s-u)} + \mathbf{1}_{\mathbb{R}_-}(u)\mathrm{e}^{xu}(\mathrm{e}^{-xs}-1)$.

Remark 2.7. The product expansion $f_s(x,u)f_t(y,v)$ yields that $\tilde{X}(x,y,s,t)$ could also be defined as following:

$$X(x,y,s,t) + X_2(x,y,s,t) + X_3(x,y,s,t) + X_4(x,y,s,t)$$

where

$$\begin{aligned} X_2(x,y,s,t) &= (\mathrm{e}^{-xs}-1)\int_{]-\infty,0[\times]0,t[} \mathrm{e}^{xu}\mathrm{e}^{-y(t-v)}\mathrm{d}B_{u,v},\\ X_3(x,y,s,t) &= (\mathrm{e}^{-yt}-1)\int_{]0,s[\times]-\infty,0[} \mathrm{e}^{-x(s-u)}\mathrm{e}^{yv}\mathrm{d}B_{u,v},\\ X_4(x,y,s,t) &= (\mathrm{e}^{-xs}-1)(\mathrm{e}^{-yt}-1)\int_{]-\infty,0[\times]-\infty,0[} \mathrm{e}^{xu}\mathrm{e}^{yv}\mathrm{d}B_{u,v}. \end{aligned} \tag{18}$$

Proof. The function $f_{s,t} : (u,v,x,y) \mapsto x^{-\alpha_1-\frac{1}{2}}y^{-\alpha_2-\frac{1}{2}}f_s(x,u)f_t(y,v)$ belongs to $\mathcal{L}_{2,1}(\mathbb{R}^2,\mathbb{R}_+^2)$ (*cf.* (5)), (s,t) being fixed (indeed $\|f_{s,t}\|_{2,1} = \int_{\mathbb{R}_+} x^{-\alpha_1-\frac{1}{2}}\sqrt{\frac{1-\mathrm{e}^{-xs}}{x}}\,\mathrm{d}x.\int_{\mathbb{R}_+} y^{-\alpha_2-\frac{1}{2}}\sqrt{\frac{1-\mathrm{e}^{-yt}}{y}}\,\mathrm{d}y < \infty$) and we apply Lemma 2.1 to this function. □

In the case $\frac{1}{2} < \alpha < 1$, the relation (9) is applied to $s \geq u$ and $s = 0$, and solving the integrals with respect to r yields:

$$(*)\quad (s-u)_+^{\alpha-\frac{1}{2}} - (-u)_+^{\alpha-\frac{1}{2}} = -\int_{\mathbb{R}^+}\frac{x^{\frac{1}{2}-\alpha}}{\Gamma(\frac{1}{2}-\alpha)}\left[\mathbf{1}_{[0,s]}(u)\frac{1-\mathrm{e}^{-x(s-u)}}{x} + \mathbf{1}_{\mathbb{R}_-}(u)\mathrm{e}^{xu}\frac{1-\mathrm{e}^{-xs}}{x}\right]\mathrm{d}x.$$

This relation allows us to prove:

Proposition 2.8. *Let $\alpha_1 \vee \alpha_2 \in]\frac{1}{2}, 1[$; the process W^{α_1,α_2} admits the following representation on $\mathbb{R}_+^2$, denoting $a_i = \alpha_i + \frac{1}{2}sign(\frac{1}{2} - \alpha_i)$, almost surely:*

$$W_{s,t}^{\alpha_1,\alpha_2} = \frac{1}{\Gamma(\frac{1}{2}-\alpha_1)} \frac{1}{\Gamma(\frac{1}{2}-\alpha_2)} \int_{\mathbb{R}_+^2} x^{-a_1} y^{-a_2} U(x,y,s,t) \mathrm{d}x\mathrm{d}y, \tag{19}$$

where $U(x,y,s,t) = \int_{\mathbb{R}^2} h_s^1(x,u) h_t^2(y,v) \mathrm{d}B_{u,v}$, $h^i = f$ if $\alpha_i < \frac{1}{2}$, and $h^i = g$ if $\alpha_i > \frac{1}{2}$, $g_s(x,u) = \mathbf{1}_{[0,s]}(u)\frac{1-\mathrm{e}^{-x(s-u)}}{x} + \mathbf{1}_{\mathbb{R}_-}(u)\mathrm{e}^{xu}\frac{1-\mathrm{e}^{-xs}}{x}$, f is defined in Proposition 2.6.

Definition 2.9. Let us denote $\tilde{Y}(x,y,s,t) = B(g_s(x,.)g_t(y,.))$, $\tilde{Z}(x,y,s,t) = B(g_s(x,.)f_t(y,.))$, and $\tilde{T}(x,y,s,t) = B(f_s(x,.)g_t(y,.))$.

Using Remark 2.4, these fields $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{T}$ are continuous with respect to the four parameters, as sum of continuous fields.

2.3. Algorithm

In Section 5, we will provide a recursive algorithm to approximate $W_{s,t}^{\alpha_1,\alpha_2}$ by the following

Definition 2.10. Let $n \in \mathbb{N}^*$, $r, h, k > 0$:

$$\hat{W}_{n,r,h,k}^{\alpha_1,\alpha_2}(ih, jk) = \sum_{j_1,j_2=-n+1}^{n} c_{j_1}^1 c_{j_2}^2 U^{h,k}(r^{j_1-1}, r^{j_2-1}, ih, jk)$$

where $c_{j_l}^l = \frac{1}{\Gamma(\frac{1}{2}-\alpha_l)} \frac{(r^{1-a_l}-1)}{1-a_l} r^{(1-a_l)(j_l-1)}$, a_l is defined in (10) and $U^{h,k}$ will be defined below depending on the position of α_i with respect to $\frac{1}{2}$.

$$U^{h,k} = \tilde{X}^{h,k}\mathbf{1}_{]0,\frac{1}{2}[^2}(\alpha) + \tilde{Y}^{h,k}\mathbf{1}_{]\frac{1}{2},1[^2}(\alpha) + \tilde{T}^{h,k}\mathbf{1}_{]0,\frac{1}{2}[\times]\frac{1}{2},1[}(\alpha) + \tilde{Z}^{h,k}\mathbf{1}_{]\frac{1}{2},1[\times]0,\frac{1}{2}[}(\alpha). \tag{20}$$

The key of the recursive algorithm is Proposition 2.5. Let $(B_{ij}^{hk}, (i,j) \in \mathbb{N}^2)$ be a Gaussian white noise with variance hk, $(B_2^k(x), x \in \mathbb{R}^+)$ and $(B_3^h(y), y \in \mathbb{R}^+)$ be Gaussian vectors with covariance function equal to $\frac{k}{x+x'}$ respectively $\frac{h}{y+y'}$ Concerning the first term in U^{hk} (*cf.* (18)), we need a double induction as following, given $x = r^{j_1-1}$, $y = r^{j_2-1}$, $j_i \in \{-n+1, \cdots, n\}, i = 1, 2$:

$$\begin{aligned}
\forall j \in \mathbb{N},\ \mathcal{X}(x,0,jk) &= 0, \\
\mathcal{X}(x,ih,jk) &= \mathrm{e}^{-xh}\mathcal{X}(x,(i-1)h,jk) + \frac{1-\mathrm{e}^{-xh}}{xh} B_{ij}^{h,k}, \\
\hat{\mathcal{X}}(x,ih,jk) &= \frac{1}{x}[B_{ij}^{h,k} - \mathcal{X}(x,ih,jk)].
\end{aligned}$$

In a second step, we set $\forall i \in \mathbb{N},\ X(x,y,ih,0) = Z(x,y,ih,0) = 0$:

$$X(x,y,ih,(j+1)k) = \mathrm{e}^{-yk}X(x,y,ih,jk) + \frac{1-\mathrm{e}^{-yk}}{yk}\mathcal{X}(x,ih,jk), \ \left(\alpha_1 \vee \alpha_2 < \frac{1}{2}\right), \tag{21}$$

$$Z(x,y,ih,(j+1)k) = \mathrm{e}^{-yk}Z(x,y,ih,jk) + \frac{1-\mathrm{e}^{-yk}}{yk}\hat{\mathcal{X}}(x,ih,jk), \ \left(\alpha_2 < \frac{1}{2} < \alpha_1\right), \tag{22}$$

$$Y(x,y,ih,(j+1)k) = \frac{1}{y}\left[\sum_{l=1}^{j+1} \hat{\mathcal{X}}(x,ih,jk) - Z(x,y,ih,(j+1)k)\right], \ \left(\alpha_1 \wedge \alpha_2 > \frac{1}{2}\right). \tag{23}$$

We now deal with the three other terms in (18) but only in case $\alpha_1 \vee \alpha_2 < \frac{1}{2}$.

Concerning the fourth term in (18), an exact simulation is possible since $X_4(x,y,ih,jk) = (1-\mathrm{e}^{-xih})(1-\mathrm{e}^{-yjk})B_4(x,y)$ where B_4 is a centered Gaussian matrix with covariance function $\Gamma_4(x,x',y,y') = \frac{1}{x+x'}\frac{1}{y+y'}$.

The two other terms in (18), denoted as X_2 and X_3, are symmetrically obtained by induction:

$$\begin{aligned}
\forall i,\ X_2(x,y,ih,0) &= 0,\\
X_2(x,y,ih,(j+1)k) &= \mathrm{e}^{-yk}X_2(x,y,ih,jk) - \frac{1-\mathrm{e}^{-yk}}{yk}(1-\mathrm{e}^{-xih})B_2^k(x),
\end{aligned}$$

$$\begin{aligned}
\forall j,\ X_3(x,y,0,jk) &= 0,\\
X_3(x,y,(i+1)h,jk) &= \mathrm{e}^{-xh}X_3(x,y,ih,jk) - \frac{1-\mathrm{e}^{-xh}}{xh}(1-\mathrm{e}^{-yjk})B_3^h(x).
\end{aligned}$$

Finally, $\tilde{X}^{hk}$ is defined as the sum $(X+X_2+X_3+X_4)(x,h,ih,jk)$ (there is similar definitions of $\tilde{Y}^{hk}$, $\tilde{T}^{hk}$, $\tilde{Z}^{hk}$) and the following will be proved in Theorem 5.3.

Theorem. *For all $\varepsilon > 0$ there exist n,r,h,k so, $\forall T > 0$, there exists a random variable $C_{n,r,h,k}$ admitting exponential moments such that the error is uniformly bounded:*

$$\sup_{s,t\in[0,T]^2} |W^{\alpha_1,\alpha_2}(s,t) - \hat{W}^{\alpha_1,\alpha_2}_{r,n,h,k}(s,t)| \leq \varepsilon C_{n,r,h,k}. \tag{24}$$

3. Approximation of a fractional Brownian motion

This section develops results of [7] where all the proofs are omitted. Moreover here we bound the temporal approximation error. Using Fubini's Lemma, we get the representations of the Liouville Brownian motion. Almost surely, $\forall \alpha \in\ ]0,\frac{1}{2}[$, $\forall t \in \mathbb{R}_+$:

$$V_t^\alpha = \frac{1}{\Gamma(\frac{1}{2}-\alpha)}\int_{\mathbb{R}_+} x^{-\alpha-\frac{1}{2}}X(x,t)\mathrm{d}x \text{ where } X(x,t) = \int_{[0,t[} \mathrm{e}^{-x(t-u)}\mathrm{d}B_u; \tag{25}$$

$\forall \alpha \in\ ]\frac{1}{2},1[$, $\forall t \in \mathbb{R}_+$:

$$V_t^\alpha = \frac{1}{\Gamma(\frac{1}{2}-\alpha)}\int_{\mathbb{R}_+} x^{\frac{1}{2}-\alpha}Y(x,t)\mathrm{d}x \text{ where } Y(x,t) = \int_{[0,t[} X(x,z)\mathrm{d}z. \tag{26}$$

Then, we have similar results concerning the fractional Brownian motion, which has stationary increments. Almost surely, $\forall \alpha \in\ ]0,\frac{1}{2}[$, $\forall t \in \mathbb{R}_+$:

$$W_t^\alpha = \frac{1}{\Gamma(\frac{1}{2}-\alpha)}\int_{\mathbb{R}_+} x^{-\alpha-\frac{1}{2}}\tilde{X}(x,t)\mathrm{d}x \tag{27}$$

where

$$\tilde{X}(x,t) = \int_{[0,t[} \mathrm{e}^{-x(t-u)}\mathrm{d}B_u + (\mathrm{e}^{-xt}-1)\int_{[-\infty,0[} \mathrm{e}^{xu}\mathrm{d}B_u; \tag{28}$$

$\forall \alpha \in]\frac{1}{2},1[$ $\forall t \in \mathbb{R}_+$:

$$W_t^\alpha = \frac{1}{\Gamma(\frac{1}{2}-\alpha)}\int_{\mathbb{R}_+} x^{\frac{1}{2}-\alpha}\tilde{Y}(x,t)\mathrm{d}x \tag{29}$$

where

$$\tilde{Y}(x,t) = Y(x,t) + \frac{1-\mathrm{e}^{-xt}}{x}\int_{[-\infty,0[}\mathrm{e}^{xu}\mathrm{d}B_u. \tag{30}$$

The aim here is to approximate the integrals in the representations (27), (29) summarized as

$$W_t^\alpha = \frac{1}{\Gamma(\frac{1}{2}-\alpha)}\int_{\mathbb{R}_+} x^{-a}U(x,t)\mathrm{d}x, \tag{31}$$

where $U = \tilde{X}\mathbf{1}_{]0,\frac{1}{2}[}(\alpha) + \tilde{Y}\mathbf{1}_{]\frac{1}{2},1[}(\alpha)$ and a is $\alpha + \frac{1}{2}.sign(\frac{1}{2}-\alpha)$. We can deduce from the equation (25) that the process $X(x,.)$ is an Ornstein-Uhlenbeck process:

$$X(x,t) = B_t - x\int_0^t X(x,u)\mathrm{d}u, \quad t \geq 0. \tag{32}$$

3.1. Approximation of a deterministic integral

We recall Lemma 13 in [7] using the norm $\|.\|_{\infty,d,e}$, $(d,e) \in [0,1]^2,\ d<e$, on $C(]0,+\infty[,\mathbb{R})$ and the measure $\mu_{d,e}$ on $\mathbb{R}^+$ which defines the L^1-norm $\|f\|_{d,e}$:

$$\|g\|_{\infty,d,e} = \sup_{x\in]0,+\infty[}[x^d 1_{\{x<1\}} + x^e 1_{\{x\geq 1\}}]|g(x)|, \quad \mu_{d,e}(\mathrm{d}x) = \min(x^{-d},x^{-e}).\mathrm{d}x.$$

Proposition 3.1. *Let $(d,e)\in[0,1]^2, d<e$, a function $f\in L^1([0,+\infty[,\mathbb{R}^+,\mu_{d,e})$, and $g\in C^2(]0,+\infty),\mathbb{R})$ such that the norms $\|.\|_{\infty,d,e}$ of the maps g and $h_g : x\mapsto |x\nabla g(x)| + |D^2 g(x)|x^2$ are finite. Let $r\in]1,2[$, $n\in\mathbf{N}^*$ to define a geometric subdivision of $]0,+\infty[$, $\pi = (r^{-n},\cdots,r^n)$: $I_j = [r^{j-1},r^j]$ and:*

$$c_j = \int_{I_j} f(x)\mathrm{d}x,\ j = -n+1,\cdots,n.$$

Then:

$$\left|\int_{[0,+\infty[} g(x)f(x)\mathrm{d}x - \sum_{j=-n+1}^{n} c_j g(r^j)\right| \leq \frac{1}{4}r^e(r-1)\|h_g\|_{\infty,d,e}\|f\|_{\mu_{d,e}} + \|g\|_{\infty,d,e}\|f(\mathbf{1}_{]0,r^{-n}]} + \mathbf{1}_{]r^n,+\infty[})\|_{\mu_{d,e}}. \tag{33}$$

This proposition will be applied to $f : x \mapsto x^{-a},\ a = \alpha + \frac{1}{2}.sign(\frac{1}{2}-\alpha),\ g(x) = \tilde{X}(x,t)$ or $\tilde{Y}(x,t)$, to approximate $W^\alpha(t)$, respectively when $\alpha<\frac{1}{2},\ \alpha>\frac{1}{2}$.

The proof is omitted: it is quite similar to Theorem 4.1 proof (*cf.* Sect. 7).

The following corollary will be useful for the time discretization.

Corollary 3.2. *Let (d,e,f) and g,h satisfying the assumptions of Proposition 3.1, then*

$$\left|\sum_{i=-n+1}^{n} c_i[g(r^i) - h(r^i)]\right| \leq r^e\|f\|_{\mu_{d,e}}\|g-h\|_{\infty,d,e}, \tag{34}$$

where n et r are defined in Proposition 3.1.

Proof. Let $D_i = [g(r^i)-h(r^i)]|c_i|$. Using the norm $\|g-h\|_{\infty,d,e}$ definition, we bound D_i by

$$|c_i|(r^{-id}\mathbf{1}_{\{r^i<1\}} + r^{-ie}\mathbf{1}_{\{r^i\geq 1\}})\|g-h\|_{\infty,d,e} \leq |c_i|(r^{i-1})^{-d}\wedge(r^{i-1})^{-e}.\|g-h\|_{\infty,d,e}. \tag{35}$$

But $c_i = \int_{[r^{i-1},r^i]} f(x)\mathrm{d}x$ and $\mu_{d,e}(\mathrm{d}x) = x^{-d} \wedge x^{-e}\mathrm{d}x$ so

$$|D_i| \leq r^e \int_{[r^{i-1},r^i]} f(x)\mu_{d,e}(\mathrm{d}x)\|g-h\|_{\infty,d,e}.$$

To sum these bounds with respect to i get the conclusion. □

3.2. **1–dimensional operators on the Hölder functions**

In the aim to compute by approximation the fractional Brownian process W_t^α (31), t being fixed, we use Proposition 3.1 with $g(x) = U(x,t), f(x) = x^{-a}, (d,e)$ such that $a+d<1<a+e$. First, we have to study the smoothness of the Gaussian processes U and the associated functions h_U, and their norm $\|.\|_{\infty,d,e}$. That is to show the existence of (d,e) as above and moreover that:

(i) $U = \tilde{X}, \tilde{Y}$ belongs to $C^2(\mathbb{R}_+^*)$, t fixed;

(ii) $\sup_{x\in\mathbb{R}_+^*}(x^d\mathbf{1}_{x<1} + x^e\mathbf{1}_{x\geq 1})|x^i\partial^i_{x^i}U(x,t)| < \infty, i=0,1,2$ ($\partial^i_{x^i}U$ denotes $\frac{\partial^i}{\partial x^i}U$).

The point (i) will be a consequence of the fact that X is the image of the Brownian motion by an operator ψ defined below and some similar tricks to be shown for the other processes. The point (ii) (see Corollary 3.10 below) relies on deterministic properties of the operators ψ and θ defined below and on path-wise properties of the Brownian motion B. These operators are defined on the set of Hölder functions polynomially increasing at $-\infty$ (for instance as are the Brownian motion paths).

Define the set of α-Hölder functions on $I \subset \mathbb{R}$, taking value 0 at 0:

$$\mathcal{H}^0_\alpha(I) = \left\{ f \;:\; f(0)=0, \sup_{s,s'\in I^2, s\neq s'} \frac{|f(s)-f(s')|}{|s-s'|^\alpha} < \infty \right\}.$$

Finally, the Banach space $\mathcal{S}_\alpha$ is defined as the subset of the functions in $\mathcal{C}(]-\infty,T],\mathbb{R})$ such that:

$$\mathcal{S}_\alpha = \left\{ f \in \mathcal{C}(]-\infty,T],\mathbb{R}), \quad \sup_{x\leq -1} \frac{|f(x)|}{|x|^{1-\alpha}} < \infty \right\} \cap \mathcal{H}^0_\alpha([-1,T]),$$

and if $f \in \mathcal{S}_\alpha$ let us denote the norm:

$$\|f\|_\alpha = \sup_{s,s'\in[-1,T]^2, s\neq s'} \frac{|f(s)-f(s')|}{|s-s'|^\alpha} + \sup_{x\leq -1} \frac{|f(x)|}{|x|^{1-\alpha}}.$$

Remark that if $f \in \mathcal{S}_\alpha$, $\forall u \leq -1,\ |f(u)| \leq \|f\|_\alpha |u|^{1-\alpha}$ and $\forall u \geq -1,\ |f(u)| \leq \|f\|_\alpha |u|^\alpha$.

For instance, the Brownian motion process $B_t = \int_0^t \mathrm{d}B_u$ belongs to $\mathcal{S}_\alpha$, $\forall \alpha < \frac{1}{2}$. (*cf.* Revuz and Yor [16], Th. 2.2 in I.2 and Prop. 1.10(iv) in I.1.)

We introduce two linear operators on $\mathcal{S}_\alpha$, useful to manage the Liouville process since the Fubini Lemma shows that $X(x,s) = \psi(B)(x,s)$ and we will see below that $Y(x,s) = \hat{\psi}(B)(x,s)$ where:

$$\begin{aligned}
\psi(f) : (x,s) &\quad\mapsto\quad f(s)\mathrm{e}^{-xs} + x\int_0^s \mathrm{e}^{-xr}[f(s)-f(s-r)]\mathrm{d}r, \\
\hat{\psi}(f) : (x,s) &\quad\mapsto\quad \int_0^s \mathrm{e}^{-x(s-u)} f(u)\mathrm{d}u.
\end{aligned}$$

We now study the smoothness of these functions and we bound their partial derivatives with respect to x, uniformly on $\mathbb{R}_+^* \times [0,T]$.

Proposition 3.3. *Let $f \in \mathcal{C}((-\infty, T], \mathbb{R})$.*

(i) *For any $(x,s) \in [0,+\infty) \times [0,T]$, $\psi(f)(x,s) = f(s) - x\hat{\psi}(f)(x,s)$;*

(ii) *for any $(x,s) \in [0,+\infty[\times[0,T]$, $\psi(\int_0^\cdot f(u)\mathrm{d}u)(x,s) = \hat{\psi}(f)(x,s) = \int_0^s \psi(f)(x,u)\mathrm{d}u$.*

Proof. *cf.* Section 7. □

Remark 3.4. In the case $f = B$, the Brownian motion, notice that (i) and (ii) show that $X = \psi(B)$ and $Y = \hat{\psi}(B)$.

Note that (i) and (ii) imply the integral equation: function $\psi(f)(x,.)$ is solution to the integral equation $y(.) = f(.) - x\int_0^\cdot y(u)\mathrm{d}u$.

Finally, when $x \neq 0$, (i) implies that $\hat{\psi}(f)(x,t) = \frac{1}{x}(f(t) - \psi(f)(x,t))$.

Corollary 3.5. *For any function $f \in \mathcal{C}([0,T],\mathbb{R})$, the map $\psi(f)$ is indefinitely differentiable with respect to x, is continuous on $[0,+\infty[\times[0,T]$ and so are its partial derivatives, $\forall n \in \mathbf{N}$,*

$$\partial^n_{x^n}\psi(f)(x,s) = (-s)^n \mathrm{e}^{-xs} f(s) + \int_0^s [n(-r)^{n-1} + x(-r)^n]\mathrm{e}^{-xr}(f(s) - f(s-r))\mathrm{d}r.$$

Proof. We merely differentiate $x \mapsto \psi(f)(x,s)$ under the integral and we use the n-th derivative of $x \mapsto x\mathrm{e}^{-xr}$ which is $(n(-r)^{n-1} + x(-r)^n)\mathrm{e}^{-xr}$. □

Then we establish the continuity properties of these linear operators. So let $\mathcal{L}(\mathcal{S}_\alpha;\mathbb{R})$, the set of linear maps from $\mathcal{S}_\alpha$ to $\mathbb{R}$, endowed with the norm: when $A \in \mathcal{L}(\mathcal{S}_\alpha;\mathbb{R})$,

$$|||A||| = \sup_{f\in\mathcal{S}_\alpha, 0<\|f\|_\alpha\leq 1} \frac{|A(f)|}{\|f\|_\alpha}.$$

Theorem 3.6. *For any $\alpha \in]0,1]$,*

$$\psi : \begin{array}{ccc} \mathbb{R}^*_+ \times [0,T] & \to & \mathcal{L}(\mathcal{S}_\alpha;\mathbb{R}) \\ (x,s) & \mapsto & \psi(x,s)\ : (f \mapsto \psi(f)(x,s)) \end{array}$$

is a map on $\mathbb{R}^*_+ \times [0,T]$, indefinitely differentiable with respect to x. It and its derivatives are continuous. Moreover, let $C_0 = \Gamma(\alpha+1) \vee T^\alpha$, and $\forall n \geq 1$, $C_n = (2n+\alpha)(\frac{n+\alpha-1}{e})^{n+\alpha-1}2^{n+\alpha}(1 \vee \frac{T}{2})$, then for any function $f \in \mathcal{S}_\alpha$ with the Hölder norm $\|f\|_\alpha$,

$$\forall (x,t) \in [0,+\infty[\times[0,T], x^n|\partial^n_{x^n}\psi(f)(x,t)| \leq C_n\|f\|_\alpha[\mathbf{1}_{\{x<1\}} + x^{-\alpha}\mathbf{1}_{\{x\geq 1\}}].$$

Proof. Corollary 3.5 and the assumptions on f get

$$|\partial^n_{x^n}\psi(f)(x,s)| \leq \|f\|_\alpha[\mathrm{e}^{-xs}s^{n+\alpha} + x\int_0^s (r)^{n+\alpha}\mathrm{e}^{-xr}\mathrm{d}r + n\int_0^s (r)^{n+\alpha-1}\mathrm{e}^{-xr}\mathrm{d}r].$$

The integration by part formula shows that the sum of the first two terms is $(n+\alpha)\int_0^s (r)^{n+\alpha-1}\mathrm{e}^{-xr}\mathrm{d}r$. Then we obtain $x^n|\partial^n_{x^n}\psi(f)(x,s)| \leq \|f\|_\alpha(2n+\alpha)\int_0^s x^n r^n r^{\alpha-1}\mathrm{e}^{-xr}\mathrm{d}r$.

When $n = 0$, the bound becomes

$$|\psi(f)(x,s)| \leq \|f\|_\alpha \alpha x^{-\alpha}\int_0^s (xr)^{\alpha-1}\mathrm{e}^{-xr}x\mathrm{d}r$$

and $\alpha x^{-\alpha}\int_0^s (xr)^{\alpha-1}\mathrm{e}^{-xr}x\mathrm{d}r \leq x^{-\alpha}\Gamma(\alpha+1)\mathbf{1}_{\{x\geq1\}} + T^\alpha\mathbf{1}_{\{x<1\}}$.

When $n > 0$, using that the map $y \mapsto y^\gamma e^{-y}$, $\gamma > 0$, is bounded on $[0, +\infty[$ by a constant $c_\gamma = (\frac{\gamma}{e})^\gamma$, then, with $y = xr/2$,

$$\begin{aligned} x^n|\partial^n_{x^n}\psi(f)(x,s)| &\le \|f\|_\alpha(2n+\alpha)\left(\frac{n}{e}\right)^n 2^n \int_0^s r^{\alpha-1}e^{-xr/2}dr \\ &\le \|f\|_\alpha(2n+\alpha)\left(\frac{2n}{e}\right)^n \left[\frac{T^\alpha}{\alpha}\mathbf{1}_{\{x\le 1\}} + (\frac{x}{2})^{-\alpha}\Gamma(\alpha)\mathbf{1}_{\{x>1\}}\right] \end{aligned}$$

so $C_n = (2n+\alpha)(\frac{2n}{e})^n \max(\frac{T^\alpha}{\alpha}, 2^\alpha\Gamma(\alpha))$ with the convention $(\frac{2n}{e})^n = 1$ if $n = 0$. □

Note that $C_1 = \frac{2\alpha}{e}(2^\alpha\Gamma(\alpha+1) \vee \frac{T^\alpha}{\alpha})$, $C_2 = (4+\alpha)(\frac{4}{e})^2 2^{2+\alpha}(\frac{T^\alpha}{\alpha} \vee 2^\alpha\Gamma(\alpha))$.

Notice that $f \in \mathcal{S}_\alpha$ implies that $t \mapsto \int_0^t f(u)du$ is a Lipschitz-function with Lipschitz constant $T^\alpha\|f\|_\alpha$. This fact and the point (ii) of Proposition 3.3 ($\hat\psi(f) = \psi(\int f)$) and Theorem 3.6 applied to $\alpha = 1$ yield:

Corollary 3.7. *For any* $\alpha \in]0,1]$,

$$\hat\psi : \begin{array}{ccc} \mathbb{R}^*_+ \times [0,T] & \to & \mathcal{L}(\mathcal{S}_\alpha, \mathbb{R}) \\ (x,s) & \mapsto & \hat\psi(x,s)\ : (f \mapsto \hat\psi(f)(x,s)) \end{array}$$

is a map $\mathbb{R}^*_+ \times [0,T]$, *indefinitely differentiable with respect to* x. *It and its derivatives are continuous. Moreover,* $\forall n \in \mathbb{N}$, *for any function* $f \in \mathcal{S}_\alpha$, *with the Hölder norm* $\|f\|_\alpha$,

$$\forall (x,s) \in \mathbb{R}^*_+ \times [0,T],\ x^n|\partial^n_{x^n}\hat\psi(f)(x,s)| \le T^\alpha C_n\|f\|_\alpha[\mathbf{1}_{\{x<1\}} + x^{-1}\mathbf{1}_{\{x\ge 1\}}].$$

We now introduce two other operators on $\mathcal{S}_\alpha$ to get the fractional Brownian motion increments stationary, since the Fubini Lemma shows that $\tilde X(x,s) = (\psi+\theta)(B)(x,s)$:

$$\begin{aligned} \theta(f) : (x,s) &\mapsto -(e^{-xs}-1)x\int_{-\infty}^0 e^{xu}f(u)du, \\ \hat\theta(f) : (x,s) &\mapsto -\frac{1}{x}\theta(f)(x,s). \end{aligned} \tag{36}$$

Theorem 3.8. *For any* $\alpha \in]0, \frac{1}{2}[$,

$$\theta : \begin{array}{ccc} \mathbb{R}^*_+ \times [0,T] & \to & \mathcal{L}(\mathcal{S}_\alpha; \mathbb{R}) \\ (x,s) & \mapsto & \theta(x,s)\ : (f \mapsto \theta(f)(x,s)) \end{array}$$

is a map on $\mathbb{R}^*_+ \times [0,T]$, *indefinitely differentiable with respect to* x. *It and its derivatives are continuous. Moreover,* $\forall n \in \mathbb{N}$, *there exists a constant* C''_n *such that for any function* $f \in \mathcal{S}_\alpha$, *with the Hölder norm* $\|f\|_\alpha$,

$$\forall (x,s) \in \mathbb{R}^*_+ \times [0,T],\ x^n|\partial^n_{x^n}\theta(f)(x,s)| \le C''_n\|f\|_\alpha[x^\alpha\mathbf{1}_{\{x<1\}} + x^{-\alpha}\mathbf{1}_{\{x\ge 1\}}].$$

Proof. cf. Section 7. □

Note that $C''_0 = 1 \vee T$, $C''_1 = 10 \vee 6T$, $C''_2 = 50 \vee 26T$.

Using Leibnitz rule and the fact that $\alpha > 0$ implies $\frac{1}{x}[x^\alpha\mathbf{1}_{\{x<1\}} + x^{-\alpha}\mathbf{1}_{\{x\ge 1\}}] \le [x^{\alpha-1}\mathbf{1}_{\{x<1\}} + x^{-1}\mathbf{1}_{\{x\ge 1\}}]$, we get:

Corollary 3.9. *For any* $\alpha \in]0, \frac{1}{2}[$,

$$\hat\theta : \begin{array}{ccc} \mathbb{R}^*_+ \times [0,T] & \to & \mathcal{L}(\mathcal{S}_\alpha; \mathbb{R}) \\ (x,s) & \mapsto & \hat\theta(x,s)\ : (f \mapsto -\frac{1}{x}\theta(f)(x,s)) \end{array}$$

is a map on $\mathbb{R}_+^* \times [0,T]$, *indefinitely differentiable with respect to x. It and its derivatives are continuous. Moreover,* $\forall n \in \mathbb{N}$, *there exists a constant* C_n''' *such that for any function* $f \in \mathcal{S}_\alpha$, *with the Hölder norm* $\|f\|_\alpha$,

$$\forall (x,s) \in \mathbb{R}_+^* \times [0,T],\ x^n |\partial_{x^n}^n \hat{\theta}(f)(x,s)| \leq C_n''' \|f\|_\alpha [x^{\alpha-1}\mathbf{1}_{\{x<1\}} + x^{-1}\mathbf{1}_{\{x\geq 1\}}].$$

More precisely, $C_n''' = \sum_{k=0}^n \frac{n!}{k!} C_k''$.

We now deduce the following corollary to summarize Theorems 3.6, 3.8 and Corollaries 3.7 and 3.9:

Corollary 3.10. *Let* $\alpha \in]0, \frac{1}{2}[$. *For any* $n \in \mathbb{N}$, *there exists a constant* $D_n = C_n + C_n''$ *such that for any* $f \in \mathcal{S}_\alpha$, $x \in \mathbb{R}_*^+, s \in [0,T]$,

$$x^n |\partial_{x^n}^n (\psi + \theta)(f)(x,s)| \ \leq D_n \|f\|_\alpha [\mathbf{1}_{\{x<1\}} + x^{-\alpha}\mathbf{1}_{\{x\geq 1\}}].$$

For any $n \in \mathbb{N}$ *there exists a constant* $D_n' = C_n + C_n'''$ *such that* $f \in \mathcal{S}_\alpha$, $x \in \mathbb{R}_*^+, s \in [0,T]$,

$$x^n |\partial_{x^n}^n (\hat{\psi} + \hat{\theta})(f)(x,s)| \ \leq D_n' \|f\|_\alpha [x^{\alpha-1}\mathbf{1}_{\{x<1\}} + x^{-1}\mathbf{1}_{\{x\geq 1\}}].$$

This corollary shows the smoothness of the maps $X(.,s), \tilde{X}(.,s), Y(.,s), \tilde{Y}(.,s)$, and controls their growth and the one of their derivatives. More precisely, if we apply Corollary 3.10 to the Brownian motion $f = B$, yields $\forall x \in \mathbb{R}^+$, $\forall e \in]0, \frac{1}{2}[$ and $\forall d \in]\frac{1}{2}, 1[$:

$$\begin{aligned} \|\tilde{X}(x,s)\|_{\infty,0,e} \leq D_0 \|B\|_e,\ \|h_{\tilde{X}}(x,s)\|_{\infty,0,e} &\leq D_2 \|B\|_e, \\ \|\tilde{Y}(x,s)\|_{\infty,d,1} \leq D_0' \|B\|_{1-d},\ \|h_{\tilde{Y}}(x,s)\|_{\infty,d,1} &\leq D_2' \|B\|_{1-d}. \end{aligned} \tag{37}$$

3.3. 1-dimensional temporal approximation

Since $\psi(f)$ is solution to an integral equation (*cf.* Rem. 3.4), we can produce an iterative algorithm to define a function approximating $\psi(f)(x,t)$.

Concerning $\theta(f)(x,t)$, in case of $f = B$, an exact simulation is possible since $\theta(B)(x,t) = (\mathrm{e}^{-xt} - 1)X_0(x)$ where X_0 is a centered Gaussian process with covariance $\frac{1}{x+x'}$. Let $\pi = \{t_i = ih, 0 \leq i \leq N\}, h = T/N$, be a subdivision of the interval $[0,T]$. We define the linear interpolation of a function $f \in C([0,T]),\ f(0) = 0$:

$$\forall t \in [ih, (i+1)h],\ f^h(t) = f(ih) + \frac{t - ih}{h}[f((i+1)h) - f(ih)].$$

This function f^h is a piece-wise linear function and so belongs to the set $\mathcal{H}_1^0$. Using (ii) in Proposition 3.3 we obtain

$$\psi(f^h)(x,t) = \int_0^t \mathrm{e}^{-x(t-s)} (f^h)'(s) \mathrm{d}s$$

and so yields the induction for $i = 0, \cdots, N-1$:

$$\begin{aligned} \psi(f^h)(x,0) &= 0\ ; \\ \psi(f^h)(x,(i+1)h) &= \mathrm{e}^{-xh}\psi(f^h)(x,ih) + \frac{1-\mathrm{e}^{-xh}}{xh}[f((i+1)h) - f(ih)] \end{aligned} \tag{38}$$

since

$$\psi(f^h)(x,(i+1)h) = \int_0^{ih} \mathrm{e}^{-x(ih+h-s)} (f^h)'(s)\mathrm{d}s + \int_{ih}^{(i+1)h} \mathrm{e}^{-x(ih+h-s)} \frac{f((i+1)h) - f(ih)}{h} \mathrm{d}s.$$

We now study the smoothness of the function $f - f^h$ when f is β-Hölder.

Proposition 3.11. *Let* $f \in \mathcal{H}^0_\beta([0,T]), \beta \in]0,1[$ *and* $0 < \eta < \beta$. *Then* $f - f^h \in \mathcal{H}^0_{\beta-\eta}([0,T])$ *and* $\|f - f^h\|_{\mathcal{H}^0_{\beta-\eta}} \leq 4h^\eta \|f\|_{\mathcal{H}^0_\beta}$.

Proof. cf. Section 7. □

Remark 3.12. When $f \in \mathcal{H}^0_\beta([0,T])$ and $0 < \eta < \beta$, $f \in \mathcal{H}^0_{\beta-\eta}$, and we can bound $\|f^h\|_{\mathcal{H}^0_{\beta-\eta}}$ by $4h^\eta \|f\|_{\mathcal{H}^0_\beta} + \|f\|_{\mathcal{H}^0_{\beta-\eta}} \leq [4h^\eta + (1+T)^\eta]\|f\|_{\mathcal{H}^0_\beta}$.

From Theorem 3.6 and Corollary 3.7, we deduce:

Corollary 3.13. $\forall \beta \in]0, \frac{1}{2}[,\ 0 < \eta < \beta,\ f \in \mathcal{H}^0_\beta,\ h \leq 1,$ *then* $\forall (x,t) \in [0, \infty[\times [0,T]$,

$$\|\psi(f - f^h)\|_{\infty,0,\beta-\eta} = \max(1, x^{\beta-\eta})|(\psi)(f - f^h)(x,t)| \leq C_0(4h^\eta)\|f\|_{\mathcal{H}^0_\beta}.$$

Similarly: $\|\hat{\psi}(f - f^h)\|_{\infty,1,0} = \max(x,1)|(\hat{\psi})(f - f^h)(x,t)| \leq C_0'(4h^\eta)\|f\|_{\mathcal{H}^0_\beta}$.

3.4. Simulation of the fractional Brownian motion

As an application of the previous subsections to the function $f = B$, the Brownian motion, we propose the following algorithms, depending on two cases, if $\alpha < \frac{1}{2}$ or $\alpha > \frac{1}{2}$, using the formula (38) respectively Remark 3.4 and Definition (36).

Definition 3.14. Let $(B^h_i, i \in \mathbb{Z})$ be a Gaussian white noise, with variance h.

Let $a = \alpha + \frac{1}{2}\mathrm{sign}(\frac{1}{2} - \alpha),\ r \in]1,2[,\ n \in \mathbb{N}^*$ and $c_j = \frac{1}{\Gamma(\frac{1}{2}-\alpha)} r^{(j-1)(1-a)} \frac{r^{1-a}-1}{1-a}$.

When $\alpha < \frac{1}{2}$, we get $\forall t = ih$,

$$\hat{W}^\alpha_{n,r,h}(t) := \sum_{j=-n+1}^{n} c_j[\psi(B^h)(r^{j-1}, t) + (\mathrm{e}^{-r^{j-1}t} - 1)\int_{-\infty}^0 \mathrm{e}^{r^{j-1}u}\mathrm{d}B_u]$$

where $\psi(B^h)(r^{j-1}, 0) = 0$, $\int_{-\infty}^0 \mathrm{e}^{\cdot u}\mathrm{d}B_u$ is a centered Gaussian process with covariance function $\Gamma(x,x') = \frac{1}{x+x'}$. Note that $\psi(B^h)(x, .)$ satisfies the following induction:

$$\psi(B^h)(r^{j-1}, (i+1)h) = \mathrm{e}^{-r^{j-1}h}\psi(B^h)(r^{j-1}, ih) + \frac{1 - \mathrm{e}^{-r^{j-1}h}}{r^{j-1}h} B^h_i,\ i \geq 0. \tag{39}$$

When $\alpha > \frac{1}{2}$, we get $\forall t = ih$,

$$\hat{W}^\alpha_{n,r,h}(t) := \sum_{j=-n+1}^{n} c_j[\hat{\psi}(B^h)(r^{j-1}, t) + \frac{1 - \mathrm{e}^{-r^{j-1}t}}{r^{j-1}} \int_{-\infty}^0 \mathrm{e}^{r^{j-1}u}\mathrm{d}B_u],$$

where

$$\hat{\psi}(B^h)(r^{j-1}, t) = \frac{1}{r^{j-1}}\left[\sum_{l=1}^{i} B^h_l - \psi(B^h)(r^{j-1}, t)\right]. \tag{40}$$

Using the Orstein-Uhlenbeck stochastic differential equation (32) and the link $\tilde{X}(x,t) = X(x,t) + (\mathrm{e}^{xt} - 1)\int_{-\infty}^0 \mathrm{e}^{xu}\mathrm{d}B_u$ as the sum of two independent parts we get a simulation algorithm.

From the SDE we get for $X = \psi(B^h)$:

$$X(x, t+h) = \mathrm{e}^{-xh}X(x,t) + \frac{1 - \mathrm{e}^{-xh}}{xh}(B_{t+h} - B_t)$$

and for $\tilde{X}$

$$\tilde{X}(x,t+h) = X(x,t+h) + (\mathrm{e}^{-x(t+h)} - 1)\int_{-\infty}^{0} \mathrm{e}^{xu}\mathrm{d}B_u,$$

$$X(x,t) = \tilde{X}(x,t) - (\mathrm{e}^{-xt} - 1)\int_{-\infty}^{0} \mathrm{e}^{xu}\mathrm{d}B_u,$$

$$\tilde{X}(x,t+h) = \mathrm{e}^{-xh}\tilde{X}(x,t) + \frac{1-\mathrm{e}^{-xh}}{xh}(B_{t+h} - B_t) + (\mathrm{e}^{-xh} - 1)X_0(x),$$

the initial position is $\tilde{X}(x,0) = 0$ (*cf.* (28)).

Finally, using Proposition 3.1, Corollary 3.2, Corollary 3.10, (37), we get

Proposition 3.15. *The approximation $\hat{W}^{\alpha}_{n,r,h}(ih)$ converges almost surely to $W^{\alpha}(ih)$ uniformly when $i = 1,\cdots,I$, when n goes to infinity, $r-1$ and h go to zero. More precisely, if $\alpha < \frac{1}{2}$, $\eta < \alpha$, $\frac{1}{2} - \eta < e < \frac{1}{2}$, then the error is uniformly bounded $\forall t = ih$:*

$$|W^{\alpha}(t) - \hat{W}^{\alpha}_{n,r,h}(t)| \leq 4(2D_0 + D_1 + D_2)\left(\frac{1}{\frac{1}{2}-\alpha} + \frac{1}{e+\alpha-\frac{1}{2}}\right) r^e[r - 1 + r^{-n(\frac{1}{2}-\alpha)} + r^{-n(e+\alpha-\frac{1}{2})} + h^{\eta}]\|B\|_{e+\eta}.$$

The rate convergence of such algorithm is studied in [7] (*cf.* p. 162) but without a temporal approximation nor an accuracy evaluation. Here we add the temporal approximation. Their rate convergence is about $O(N^{1+\frac{\beta}{1+\beta}}\log N)$ where $\beta = \alpha \wedge (\frac{1}{2} - \alpha)$. Concerning our algorithm, if we choose a simulation accuracy of about $N^{-\eta}, 0 < \eta < \frac{1}{2}$, to produce an I size image, we need to generate I independent random variables and a $2n$ Gaussian vector where $n = O(N^{\eta}\log N)$ and the algorithm complexity is $O(\log N N^{1+\eta})$.

In [4], the authors have compared several methods for generating discretized simple path of long-range dependent processes such as fractional Brownian motion. They pointed out that the method summarized here is not exact but it is easy to implement and need not too much time.

4. Approximation of a sheet, 2 or more dimension

The generalization from 1−dimension to 2−dimension is not so easy: for instance, Taylor's formula in 2−dimension involves the cross derivatives and, to control the constants, we need to detail the computations. Moreover, note that this section can be applied to any Gaussian sheet. It also could be used for any process written as $W^{\alpha_1,\alpha_2}_{s,t}$ can be, as a multiple integral of any function satisfying the assumptions of Lemma 2.1 and belonging to a space $\mathcal{H}^{0,r}_{\alpha,\beta}$ (*cf.* 4.4). For instance look at $\int_{[0,+\infty[^2} g(x,y)f_1(x)f_2(y)\mathrm{d}x\mathrm{d}y$ with suitable assumptions on f_i and $g = B$, the Brownian sheet. Here the aim here is to approximate the double integral in the representations (7), (11), (16) or more generally (19) summarized as

$$W^{\alpha_1,\alpha_2}_{s,t} = \frac{1}{\Gamma(\frac{1}{2}-\alpha_1)\Gamma(\frac{1}{2}-\alpha_2)}\int\int_{\mathbb{R}^2_+} x^{-a_1}y^{-a_2}U(x,y,s,t)\mathrm{d}x\mathrm{d}y, \tag{41}$$

where U will be specified below depending on the cases and $a_i = \alpha_i + \frac{1}{2}.\mathrm{sign}(\frac{1}{2}-\alpha_i),\ i = 1,2$.

The tools are the composition of functional operators defined above in 3.2 and, as a by-product, we will obtain the error order and the convergence speed.

4.1. Approximation of a deterministic integral, dimension 2

In the following, we generalize the results in [1] recalled in Section 3.1. For any $(d,e) \in [0,1]^4, d_i < e_i, i = 1,2$, we define the norm $\|.\|_{\infty,d,e}$, on $C(]0,+\infty[^2,\mathbb{R})$ as follows:

$$\|g\|_{\infty,d,e} = \sup_{(x,y)\in]0,+\infty[^2} [x^{d_1}1_{\{x<1\}} + x^{e_1}1_{\{x\geq 1\}}][y^{d_2}1_{\{y<1\}} + y^{e_2}1_{\{y\geq 1\}}]|g(x,y)|,$$

and the measure on $(\mathbb{R}^+)^2$ $\mu_{d,e}(\mathrm{d}x)$ defined as $\Pi_{i=1,2}\min(x_i^{-d_i}, x_i^{-e_i})\mathrm{d}x$.

Theorem 4.1. *Let* $\alpha \in]0,1[^2$, $\alpha_i \neq \frac{1}{2}$, $a_i = \alpha_i + \frac{1}{2}sign(\frac{1}{2}-\alpha_i)$, $\beta_i \in]|\frac{1}{2}-\alpha_i|, \frac{1}{2}[$, $(d,e) \in [0,1]^4$, *with* $(d_i,e_i) = (0,\beta_i)$ *when* $\alpha_i < \frac{1}{2}$, *and* $(d_i,e_i) = (1-\beta_i, 1)$ *when* $\alpha_i > \frac{1}{2}$, $\gamma = \inf\{\beta_i - |\frac{1}{2}-\alpha_i|, |\frac{1}{2}-\alpha_i|,\ i=1,2\}$, $\varepsilon > 0$, $r = 1+\varepsilon$, $n = [\frac{-\log\varepsilon}{\gamma\log(1+\varepsilon)}]$, $c_j^i = r^{(1-a_i)(j-1)}\frac{r^{1-a_i}-1}{1-a_i}$, $(j = -n+1,\cdots,n)$; $(i=1,2)$.

Let g *and* f *be two functions in* $\mathcal{C}^2[(\mathbb{R}_*^+)^2,\mathbb{R}]$ *such that the norm* $\|.\|_{\infty,d,e}$ *of the maps* f, g, $\Delta g : (x,y) \mapsto x|\partial_x g(x,y)| + y|\partial_y g(x,y)|$ *and* $h_g : (x,y) \mapsto |\partial_{x^2}g(x,y)|x^2 + 2|\partial_{x,y}g(x,y)|xy + |\partial_{y^2}g(x,y)|y^2$ *are finite. Let* $C_\alpha = \frac{20}{\gamma^2}$, $D_\alpha = \frac{16}{\gamma^2}$, *then:*

$$(i) \quad \left|\int_{[0,+\infty[^2} g(x,y)x^{-a_1}y^{-a_2}\mathrm{d}x\mathrm{d}y - \sum_{j_1,j_2=-n+1}^{n} c_{j_1}^1 c_{j_2}^2 g(r^{j_1-1}, r^{j_2-1})\right|$$
$$\leq \varepsilon C_\alpha[\|g\|_{\infty,d,e} + \|\Delta g\|_{\infty,d,e} + \|h_g\|_{\infty,d,e}]$$

$$(ii) \quad \left|\sum_{j_1,j_2=-n+1}^{n} c_{j_1}^1 c_{j_2}^2 [g(r^{j_1-1}, r^{j_2-1}) - f(r^{j_1-1}, r^{j_2-1})]\right| \leq D_\alpha\|g-h\|_{\infty,d,e}.$$

Proof. cf. Section 7. □

4.2. Operators on Hölder functions depending on two variables

We now extend Theorems and Corollaries 3.6 to 3.9 to the two-dimensional case.

Define the operators $D_i, i=1,2,\ \Delta$:

$$D_1(f) : t \mapsto f(s,t) - f(s',t),\ D_2(f) : s \mapsto f(s,t) - f(s,t'),\ \Delta = D_1 \circ D_2. \tag{42}$$

Then let $\mathcal{H}^0_{\beta_1,\beta_2}$ be the set of real maps, continuous on $(-\infty,T]^2$, null on the axes, such that there exists a constant C satisfying: $\forall (s,t) \in [-1,T]^2, (u,v) \in [-1,0[^2$:

$$|\Delta f| \leq C\ \Pi_{s_i\neq t_i}|s_i - t_i|^{\beta_i}, \tag{43}$$

with the norm defined by:

$$\|f\|_{\mathcal{H}^0_{\beta_1,\beta_2}} = \inf\{C > 0,\ C \quad \text{satisfying} \quad (43)\}.$$

Now let $\mathcal{S}_{\beta_1,\beta_2} \subset \mathcal{H}^0_{\beta_1,\beta_2}$ be the Banach space with the norm:

$$\|f\|_{\beta_1,\beta_2} = \inf\{C > 0,\ C \quad \text{satisfying} \quad (43) \text{ and } (44)\},$$

where

$$\begin{array}{c} |D_2 f(u,s_2,t_2)| \leq C\ |u|^{1-\beta_1}|s_2-t_2|^{\beta_2}, \\ |D_1 f(s_1,t_1,v)| \leq C\ |v|^{1-\beta_2}|s_1-t_1|^{\beta_1}, \\ |f(u,v)| \leq C\ |u|^{1-\beta_1}|v|^{1-\beta_2}. \end{array} \tag{44}$$

Remark 4.2. Here and in the whole paper, the norms like $\|B\|_\beta$ or $\|B\|_{\mathcal{H}^0_{\beta_1,\beta_2}}$ admit an exponential moment so belong to any L^p: it is a consequence of Theorem 0.3.3 page 7 in Fernique [9], applied to the semi-norm sup of Gaussian processes.

We will apply all this section to the Brownian sheet B or to any other $f \in \mathcal{S}_{\beta_1,\beta_2}$.
On this vector space we define the eight operators:

$$\begin{array}{ll}\Phi_1(f) : (x,s,t) \mapsto \psi(f(.,t))(x,s); & \Phi_2(f) : (y,s,t) \mapsto \psi(f(s,.))(y,t);\\ \hat{\Phi}_1(f) : (x,s,t) \mapsto \hat{\psi}(f(.,t))(x,s); & \hat{\Phi}_2(f) : (y,s,t) \mapsto \hat{\psi}(f(s,.))(y,t);\\ \Theta_1(f) : (x,s,t) \mapsto \theta(f(.,t))(x,s); & \Theta_2(f) : (y,t,s) \mapsto \theta(f(s,.))(y,t);\\ \hat{\Theta}_1(f) : (x,s,t) \mapsto \hat{\theta}(f(.,t))(x,s); & \hat{\Theta}_2(f) : (y,t,s) \mapsto \hat{\theta}(f(s,.))(y,t).\end{array}$$

Let us denote $\mathcal{A}_i, i = 1,2$ the two operators sets above and the possible compositions:

$$\begin{aligned}\Phi &= (\Phi_1 + \Theta_1) \circ (\Phi_2 + \Theta_2),\\ \bar{\Phi} &= (\hat{\Phi}_1 + \hat{\Theta}_1) \circ (\Phi_2 + \Theta_2),\\ \underline{\Phi} &= (\Phi_1 + \Theta_1) \circ (\hat{\Phi}_2 + \hat{\Theta}_2),\\ \hat{\Phi} &= (\hat{\Phi}_1 + \hat{\Theta}_1) \circ (\hat{\Phi}_2 + \hat{\Theta}_2).\end{aligned}$$

In Lemma 4.6 and Corollary 4.7, we will show that these operators applied to the Brownian sheet respectively define the fields $\tilde{X}$, $\tilde{Z}$, $\tilde{T}$, $\tilde{Y}$.

The operators Φ, $\hat{\Phi}$, $\bar{\Phi}$, $\underline{\Phi}$ smoothness derives from the one-dimensional results and the following lemma.

Lemma 4.3. *Let $A_k \in \mathcal{A}_k$ and $\tilde{A}(x,y,s,t) = A_1(x,s) \circ A_2(y,t)$. Then the map $\tilde{A}$ is indefinitely differentiable with respect to (x,y), and $\forall (i,j) \in \mathbb{N}^2$,*

$$\partial^{i+j}_{x^i y^j} \tilde{A}(x,y,s,t) = \partial^i_{x^i} A_1(x,s) \circ \partial^j_{y^j} A_2(y,t) = \partial^j_{y^j} A_2(y,t) \circ \partial^i_{x^i} A_1(x,s). \tag{45}$$

Proof. cf. Section 7.

Theorem 4.4. *Let $\beta \in]0, \frac{1}{2}[^2$. The maps Φ, $\hat{\Phi}$, $\underline{\Phi}$, $\bar{\Phi}$ from $(\mathbb{R}^*_+)^2 \times]-\infty, T]^2$ to $\mathcal{L}(\mathcal{S}_{\beta_1,\beta_2}; \mathbb{R})$ are indefinitely differentiable with respect to x et y. Moreover for any $(i,j) \in \mathbb{N}^2$, there exists a constant $C_{i,j}$ such that $\forall (x,y,s,t) \in (\mathbb{R}^*_+)^2 \times [0,T]^2$:*

$$\begin{aligned} x^i y^j |||\partial^{i+j}_{x^i y^j} \Phi(x,y,s,t)||| &\le C_{i,j} [\mathbf{1}_{\{x<1\}} + x^{-\beta_1} \mathbf{1}_{\{x \ge 1\}}][\mathbf{1}_{\{y<1\}} + y^{-\beta_2} \mathbf{1}_{\{y \ge 1\}}],\\ x^i y^j |||\partial^{i+j}_{x^i y^j} \hat{\Phi}(x,y,s,t)||| &\le C_{i,j} [x^{\beta_1 - 1} \mathbf{1}_{\{x<1\}} + x^{-1} \mathbf{1}_{\{x \ge 1\}}][y^{\beta_2 - 1} \mathbf{1}_{\{y<1\}} + y^{-1} \mathbf{1}_{\{y \ge 1\}}],\\ x^i y^j |||\partial^{i+j}_{x^i y^j} \bar{\Phi}(x,y,s,t)||| &\le C_{i,j} [x^{\beta_1 - 1} \mathbf{1}_{\{x<1\}} + x^{-1} \mathbf{1}_{\{x \ge 1\}}][\mathbf{1}_{\{y<1\}} + y^{-\beta_2} \mathbf{1}_{\{y \ge 1\}}],\\ x^i y^j |||\partial^{i+j}_{x^i y^j} \underline{\Phi}(x,y,s,t)||| &\le C_{i,j} [\mathbf{1}_{\{x<1\}} + x^{-\beta_1} \mathbf{1}_{\{x \ge 1\}}][y^{\beta_2 - 1} \mathbf{1}_{\{y<1\}} + y^{-1} \mathbf{1}_{\{y \ge 1\}}].\end{aligned}$$

Proof. It is a consequence of Corollary 3.10, Lemma 4.3 and operators Φ, $\hat{\Phi}$, $\underline{\Phi}$, $\bar{\Phi}$ definitions.
More precisely, we note that $C_{i,j} = \sup\{D_i D_j,\ D'_i D'_j,\ D'_i D_j,\ D_i D'_j\}$. □

4.3. Approximation of the fractional Brownian sheet as a finite superposition of Ornstein-Uhlenbeck processes

The aim is to approximate the fractional Brownian sheet $W^{\alpha_1,\alpha_2}_{s,t}$ (1), so we use Theorem 4.1 with $g(x,y) = U(x,y,s,t)$. First, we have to study the smoothness of the Gaussian sheets U and the associated functions ΔU, h_U, and their norm $\|.\|_{\infty,d,e}$, (with (d,e) as in Th. 4.1) and prove that uniformly in (s,t):
(i) $U = \tilde{X}, \tilde{Y}, \tilde{T}, \tilde{Z}$ belongs to $C^2[(\mathbb{R}^*_+)^2]$;
(ii) $\sup_{x,y\in(\mathbb{R}^*_+)^2}(x^{d_1}\mathbf{1}_{x<1} + x^{e_1}\mathbf{1}_{x\geq 1})(y^{d_2}\mathbf{1}_{y<1} + y^{e_2}\mathbf{1}_{y\geq 1})|x^i y^j \partial^{i+j}_{x^i y^j} U(x,y,s,t)| < \infty, i+j = 0,1,2.$

The point (i) will be a consequence of the fact that $\tilde{X}$ is the image of the Brownian sheet by the operator Φ defined above and some similar tricks to be shown for the other processes. The point (ii) will be solved by Theorem 4.8 below.

So, (s,t) being fixed, we apply Theorem 4.1 to $g = U(.,.,s,t)$, U being the image, by one of the operators studied in the Section 4.2, of the continuous function B on $]-\infty, T]^2$:

$$B(s,t) = \int_0^s \int_0^t \mathrm{d}B_{u,v}.$$

Lemma 4.5. *For any $\beta \in]0, \frac{1}{2}[^2$, B admits a modification belonging to $\mathcal{S}_{\beta_1,\beta_2}$.*

Proof. We apply Kolmogorov Theorem to Brownian sheet $(B(s,t),\ \ (s,t) \in [-T,T]^2)$ and we note that the following processes follow the same law as B:
$(tB(s,\frac{1}{t}),\ (s,t) \in [0,+\infty[^2), (sB(\frac{1}{s},t),\ (s,t) \in [0,+\infty[^2)$ et $(stB(\frac{1}{s},\frac{1}{t}),\ (s,t) \in [0,+\infty[^2)$,
(*cf.* [6]). □

Lemma 4.6. *Almost surely for any $(x,y,s,t) \in [0,+\infty[^2 \times [0,T]^2$,*

$$\begin{aligned} X(x,y,s,t) &= \Phi_1 \circ \Phi_2(B)(x,y,s,t),\\ X_2(x,y,s,t) &= \Theta_1 \circ \Phi_2(B)(x,y,s,t),\\ X_3(x,y,s,t) &= \Phi_1 \circ \Theta_2(B)(x,y,s,t),\\ X_4(x,y,s,t) &= \Theta_1 \circ \Theta_2(B)(x,y,s,t). \end{aligned} \tag{46}$$

As a summary, the sum of the previous equalities yields $\tilde{X}(x,y,s,t) = \Phi(B)(x,y,s,t)$.

Proof. The two sides of the equalities (46) are continuous with respect to the four parameters so it is enough to set this identity almost surely with fixed (x,y,s,t).

The first equality is obtained using the remark that $\mathrm{e}^{-x(s-u)} = \mathrm{e}^{-xs} + x\int_{s-u}^s \mathrm{e}^{-xr}\mathrm{d}r$ and the stochastic Fubini Lemma 2.1.

The three other equalities are deduced from the identities

$$\mathrm{e}^{xu}\mathrm{e}^{-y(t-v)} = x\mathrm{e}^{-yt}\int_{-\infty}^u \mathrm{e}^{xr}\mathrm{d}r + xy\int_{-\infty}^u\int_{t-v}^t \mathrm{e}^{xr-yz}\mathrm{d}r\mathrm{d}z \text{ and } \mathrm{e}^{xu}\mathrm{e}^{yv} = xy\int_{-\infty}^u\int_{-\infty}^v \mathrm{e}^{xr+yz}\mathrm{d}r\mathrm{d}z.$$

We integrate them with respect to the Brownian sheet and once again we use the stochastic Fubini Lemma 2.1 since the maps $(u,r) \mapsto \mathrm{e}^{xr}\mathbf{1}_{\{r\leq u\leq 0\}}$, $u \mapsto \mathrm{e}^{xu}\mathbf{1}_{(-\infty,o]}(u)$, $(v,z) \mapsto \mathrm{e}^{-yz}\mathbf{1}_{[t-v,t]}(z) \in \mathcal{L}_{2,1}$. □

Successively using the definition of the fields $\tilde{Y}$, $\tilde{T}$, $\tilde{Z}$ (*cf.* Def. 2.9), the operators Φ, $\hat{\Phi}$, $\bar{\Phi}$, $\underline{\Phi}$ definitions, Lemmae 4.5, 4.6, the point (ii) in Proposition 3.3, yield

Corollary 4.7. *$\mathbb{P}$ almost surely, for any $(x,y,s,t) \in (\mathbb{R}_+^*)^2 \times [0,T]^2$,*

$$\begin{aligned} \tilde{X}(x,y,s,t) &= \Phi(B)(x,y,s,t),\ \tilde{Y}(x,y,s,t) = \hat{\Phi}(B)(x,y,s,t), \\ \tilde{Z}(x,y,s,t) &= \bar{\Phi}(B)(x,y,s,t),\ \tilde{T}(x,y,s,t) = \underline{\Phi}(B)(x,y,s,t). \end{aligned}$$

Globally denote the operator

$$\Psi = \Phi \mathbf{1}_{]0,\frac{1}{2}[^2}(\alpha) + \hat{\Phi}\mathbf{1}_{]\frac{1}{2},1[^2}(\alpha) + \overline{\Phi}\mathbf{1}_{]\frac{1}{2},1[\times]0,\frac{1}{2}[}(\alpha) + \underline{\Phi}\mathbf{1}_{]0,\frac{1}{2}[\times]\frac{1}{2},1[}(\alpha). \tag{47}$$

Lemma 4.6, Corollary 4.7 and Theorem 4.4 imply

Theorem 4.8. *Let $(d,e) \in [0,1]^4, d_i < e_i$, defined as $(0,0,e_1,e_2)\mathbf{1}_{]0,\frac{1}{2}[^2}(\alpha) + (d_1,d_2,1,1)\mathbf{1}_{]\frac{1}{2},1[^2}(\alpha)$ $+(d_1,0,1,e_2)\mathbf{1}_{]\frac{1}{2},1[\times]0,\frac{1}{2}[}(\alpha) + (0,d_2,e_1,1)\mathbf{1}_{]0,\frac{1}{2}[\times]\frac{1}{2},1[}(\alpha)$. Then $\forall p > 0$, the norms $\|.\|_{\infty,d,e}$ of $\Psi(B)$, $\Delta\Psi(B)$ and $h_{\Psi(B)}$ belong to L^p and these random variables admit an exponential moment.*

Proof. The field $B \in \mathcal{S}_\beta$, $\forall \beta \in]0,\frac{1}{2}[^2$ (*cf.* Lem. 4.5) and $\mathbb{P}-$almost surely, $\forall (x,y,s,t) \in (\mathbb{R}_+^*)^2 \times [0,T]^2$, $\tilde{X} = \Phi(B)(x,y,s,t)$ (*cf.* Lem. 4.6).

We then apply Theorem 4.4 first to $\tilde{X}$: $\forall (x,y,s,t) \in (\mathbb{R}_+^*)^2 \times [0,T]^2$,

$$x^i y^j |\partial^{i+j}_{x^i y^j} \tilde{X}(x,y,s,t)| \le C_{i,j}[\mathbf{1}_{\{x<1\}} + x^{-e_1}\mathbf{1}_{\{x\ge1\}}][\mathbf{1}_{\{y<1\}} + y^{-e_2}\mathbf{1}_{\{y\ge1\}}]\|B\|_{e_1,e_2}.$$

The case of the processes $\tilde{Y}$, $\tilde{Z}$, $\tilde{T}$ is solved substituting Lemma 4.6 by Corollary 4.7.

The theorem proof is concluded using Remark 4.2. □

This result shows the point (ii) asked at the top of this subsection as a corollary, choosing (d,e) as in Theorem 4.1:

Corollary 4.9. *Let $\beta \in]0,\frac{1}{2}[^2$ and $(d_i,e_i) = (0,\beta_i)$ when $\alpha_i < \frac{1}{2}$, $(d_i,e_i) = (1-\beta_i,1)$ when $\alpha_i > \frac{1}{2}$, the fields $\Psi(B)$ satisfy the following:*

$$\|\Psi(B)\|_{\infty,d,e} \le C_{0,0}\|B\|_\beta;\ \|\Delta\Psi(B)\|_{\infty,d,e} \le (C_{1,0}+C_{0,1})\|B\|_\beta;$$

$$\|\ h_{\Psi(B)}\|_{\infty,d,e} \le (C_{2,0}+C_{1,1}+C_{0,2})\|B\|_\beta.$$

We can apply Theorem 4.1 to the four parameters sets defined in Theorem 4.8 depending on the position of α_1 and α_2 with respect to $\frac{1}{2}$. Recall the notations:

$$a_i = \alpha_i + \frac{1}{2}.\text{sign}(\frac{1}{2} - \alpha_i), \alpha = (\alpha_1,\alpha_2). \tag{48}$$

We now get a corollary which defines an approximation converging to the fractional Brownian sheet almost surely uniformly and in any L^p.

Corollary 4.10. *Let $W^{\alpha_1,\alpha_2}_{s,t}$ as defined in Section 2 (1). For any $(\alpha_1,\alpha_2) \in]0,1[^2$, $\alpha_i \ne \frac{1}{2}, \beta_i \in]|\frac{1}{2}-\alpha_i|,\frac{1}{2}[, i = 1,2$, $(d,e) \in [0,1]^4$ such that $(d_i,e_i) = (0,\beta_i)$ when $\alpha_i < \frac{1}{2}$, $(d_i,e_i) = (1-\beta_i,1)$ when $\alpha_i > \frac{1}{2}$, and $\gamma = \inf(\beta_i - |\frac{1}{2}-\alpha_i|, |\frac{1}{2}-\alpha_i|,\ i=1,2)$.*

For any $\varepsilon > 0$, let $r = 1+\varepsilon$, $n = [\frac{-\log\varepsilon}{\gamma\log(1+\varepsilon)}]$, define the quantities ($c^i_j = \frac{1}{\Gamma(\frac{1}{2}-\alpha_i)} r^{(1-a_i)(j-1)}\frac{r^{1-a_i}-1}{1-a_i}$, $i=1,2$, $j = -n+1,\cdots,n$) (cf. Th. 4.1) and the process

$$\hat{W}^{\alpha_1,\alpha_2}_{n,r}(s,t) := \sum_{j_1,j_2=-n+1}^{n} c^1_{j_1} c^2_{j_2} \Psi(B)(r^{j_1-1}, r^{j_2-1}, s, t).$$

Then almost surely for all $(s,t)\in[0,T]^2$:

$$|W^{\alpha_1,\alpha_2}_{s,t}-\hat{W}^{\alpha_1,\alpha_2}_{n,r}(s,t)|\leq\varepsilon C_\alpha\sum_{i+j=0}^{2}C_{i,j}\|B\|_\beta$$

where C_α defined in Theorem 4.1.

Proof. The chosen pair (d,e) allows us to use Theorem 4.1. So we get:

$$|W^{\alpha_1,\alpha_2}_{s,t}-\hat{W}^{\alpha_1,\alpha_2}_{n,r}(s,t)|\leq\varepsilon C_\alpha[\|U\|_{\infty,d,e}+\|\Delta U\|_{\infty,d,e}+\|h_U\|_{\infty,d,e}],$$

where $U=\Psi(B)$. Moreover Corollary 4.9 controls the norms $\|\ .\ \|_{\infty,d,\varepsilon}$ of $U,\Delta U$ and h_U (which belong to any $L^p,\ \forall p$), so yields the result. □

4.4. 2-dimensional temporal approximation

The tools built in this section allow us to produce an iterative algorithm of the fractional Brownian sheet. So we obtained an approximation of $W^{\alpha_1,\alpha_2}_{s,t}$ on a grid $\{ih,jk\}$ by induction.

Definition 4.11. Let (h,k) be a double interpolation step on the plane, F^h_i the linear approximation with respect to the i^{th} component; we denote the "double" linear approximation: $f^{h,k}(s,t)=F^h_1\circ F^k_2(f)(s,t)$.

We mean that, if $I_i=[ih,(i+1)h]$, $J_j=[jk,(j+1)k]$, $\Delta_{i,j}(f)$ is the rectangular increment on $I_i\times J_j$, then $\forall(s,t)\in I_i\times J_j$,

$$\begin{aligned}f^{h,k}(s,t)\ &=\ f(ih,jk)+\frac{s-ih}{h}[f((i+1)h,jk)-f(ih,jk)]\\&\quad+\frac{t-jk}{k}[f(ih,(j+1)k)-f(ih,jk)]+\frac{s-ih}{h}\frac{t-jk}{k}\Delta_{i,j}(f).\end{aligned}$$

Proposition 4.12. *Let $(\beta_1,\beta_2)\in]0,\frac{1}{2}[^2$ and $(\varepsilon_1,\varepsilon_2),\ \varepsilon_i<\beta_i,\ i=1,2$. The map $f\mapsto f-f^{h,k}$ is a continuous linear map from $\mathcal{H}^0_{\beta_1,\beta_2}$ taking its values in $\mathcal{H}^0_{\beta_1-\varepsilon_1,\beta_2-\varepsilon_2}$. More precisely the norm $\|f-f^{h,k}\|_{\mathcal{H}^0_{\beta_1-\varepsilon_1,\beta_2-\varepsilon_2}}$ is bounded by:*

$$[4h^{\varepsilon_1}(1+T)^{\varepsilon_2}+4k^{\varepsilon_2}[(1+T)^{\varepsilon_1}+4h^{\varepsilon_1}]]\|f\|_{\mathcal{H}^0_{\beta_1,\beta_2}}.$$

Proof. Recall the operators $D_i, i=1,2,\ \Delta$:

$$D_1(f):t\mapsto f(s,t)-f(s',t),\ D_2(f):s\mapsto f(s,t)-f(s,t'),\ \Delta=D_1\circ D_2.$$

To bound the norm of a function g in $\mathcal{H}^0_{\beta_1,\beta_2}$, it is enough to bound quotients such as:

$$\frac{\Delta g(s,s',t,t')}{|s-s'|^{\beta_1}|t-t'|^{\beta_2}},\ s\neq s',t\neq t',$$

where $(s,s')\in[0,T]^2,\ (t,t')\in[-1,T]^2$.

So we get $f-f^{h,k}=(I-F^h_1\circ F^k_2)(f)$. The triangle inequality yields

$$\|(I-F^h_1\circ F^k_2)(f)\|_{\mathcal{H}^0_{\beta_1-\varepsilon_1,\beta_2-\varepsilon_2}}\leq\|(I-F^h_1)(f)\|_{\mathcal{H}^0_{\beta_1-\varepsilon_1,\beta_2-\varepsilon_2}}+\|F^h_1\circ(I-F^k_2)(f)\|_{\mathcal{H}^0_{\beta_1-\varepsilon_1,\beta_2-\varepsilon_2}}.$$

Proposition 3.11 and the fact that the operators D_2 and $D_1\circ(I-F^h_1)$ commute yield the bound:

$$\|(I-F^h_1\circ F^k_2)(f)\|_{\mathcal{H}^0_{\beta_1-\varepsilon_1,\beta_2-\varepsilon_2}}\leq 4h^{\varepsilon_1}\|f\|_{\mathcal{H}^0_{\beta_1,\beta_2-\varepsilon_2}}+4k^{\varepsilon_2}\|F^h_1(f)\|_{\mathcal{H}^0_{\beta_1-\varepsilon_1,\beta_2}}.$$

Remark 3.12 says that the $\mathcal{H}^0_{\beta-\varepsilon}$-norm of f^h is bounded by $\|f\|_{\mathcal{H}^0_\beta}[(T+1)^\varepsilon + 4h^\varepsilon]$, so $\|F^h_1(f)\|_{\mathcal{H}^0_{\beta_1-\varepsilon_1,\beta_2}} \leq [(T+1)^{\varepsilon_1} + 4h^{\varepsilon_1}]\|f\|_{\mathcal{H}^0_\beta}$, moreover $\|f\|_{\mathcal{H}^0_{\beta_1,\beta_2-\varepsilon_2}} \leq (T+1)^{\varepsilon_2}\|f\|_{\mathcal{H}^0_\beta}$. $\square$

Corollary 4.13. *Let* $\alpha \in]0,1[^2$, $\alpha_i \neq \frac{1}{2}$, $\beta_i \in]|\frac{1}{2} - \alpha_i|, \frac{1}{2}[$, $i = 1,2$, $\varepsilon_i < \beta_i$, *and* $(d_i, e_i) = (0, \beta_i - \varepsilon_i)$ *when* $\alpha_i < \frac{1}{2}$, $(d_i, e_i) = (1 - \beta_i + \varepsilon_i, 1)$ *when* $\alpha_i > \frac{1}{2}$. *Let*

$$\Psi_i = \Phi_i \mathbf{1}_{]0,\frac{1}{2}[}(\alpha_i) + \hat{\Phi}_i \mathbf{1}_{]\frac{1}{2},1[}(\alpha_i).$$

For any $f \in \mathcal{H}^0_{\beta_1,\beta_2}$, *the norms* $\|\Psi_1 \circ \Psi_2(f - f^{h,k})\|_{\infty,d,e}$, *are uniformly bounded: almost surely, for any* $(s,t) \in [0,T]^2$

$$\|\Psi_1 \circ \Psi_2(f - f^{h,k})\|_{\infty,d,e} \leq C_{0,0}[4h^{\varepsilon_1}(1+T)^{\varepsilon_2} + 4k^{\varepsilon_2}((1+T)^{\varepsilon_1} + 4h^{\varepsilon_1})]\|f\|_{\mathcal{H}^0_{\beta_1,\beta_2}}.$$

Proof. Since $f - f^{h,k} \in \mathcal{H}^0_{\beta_1-\varepsilon_1,\beta_2-\varepsilon_2}$ and $\Psi_1 \circ \Psi_2$ is a linear operator, then Theorem 4.4 ($i + j = 0$) and Proposition 4.12 show the result. $\square$

Now we can apply this corollary to $f = B$ thus $X^{h,k}(x,y,s,t) = \Phi_1 \circ \Phi_2(B^{h,k})$, $Y^{h,k}(x,y,s,t) = \hat{\Phi}_1 \circ \hat{\Phi}_2(B^{h,k})$, and so on, and we obtain the convergence with respect to the norm $\|.\|_{\infty,d,e}$ and its speed.

Corollary 4.14. *Let* $\varepsilon > 0$. *Using the same notations as in Corollary 4.13, let* $h = \varepsilon^{1/\varepsilon_1}$, $k = \varepsilon^{1/\varepsilon_2}$. *Let* B *be the Brownian sheet. Then almost surely for any* $(s,t) \in [0,T]^2$,

$$\|\Psi_1 \circ \Psi_2(B - B^{h,k})(.,.,s,t)\|_{\infty,d,e} \leq \varepsilon D \|B\|_{\mathcal{H}^0_{\beta_1,\beta_2}}$$

where $D = 4C_{0,0}((1+T)^{\varepsilon_2} + (1+T)^{\varepsilon_1} + 4\varepsilon)$. *Notice that* $\|B\|_{\mathcal{H}^0_{\beta_1,\beta_2}}$ *is in* $L^p, \forall p$.

5. Simulation algorithm

Finally, we gather all the results to propose a recursive algorithm to approximate the fractional Brownian sheet. Actually the trick here is that the fields X, Y, Z, T have a kind of Markov property as it will be seen using the induction formulae below.

5.1. The induction

Let $f \in \mathcal{H}^{0,r}_\beta$, $\beta \in]0, \frac{1}{2}[^2$. We obtain an approximation of $\Psi(f)$ by a recursive algorithm on a grid $((ih, jk)$, $i \geq 0$, $j \geq 0)$.

The linear interpolation $f^{h,k}$ is a piece-wise C^2-class function on the quadrant $[0,T]^2$. We use the operator Φ_1 then Φ_2 and we get as for the operator ψ:

$$\Phi_1 \circ \Phi_2(f^{h,k})(x,y,s,t) = \int_0^s \int_0^t \mathrm{e}^{-x(s-u)} \mathrm{e}^{-y(t-v)} \partial^2_{1,2} f^{h,k}(u,v) \mathrm{d}u \mathrm{d}v, \tag{49}$$

meaning that

$$\Phi_1 \circ \Phi_2(f^{h,k})(x,y,s,t) = \psi[\psi(f^{h,k}(.,t))(x,s)](y,t).$$

We use the notations D_i and Δ (42):

$$\mathcal{X}(x, ih, jk) = \psi(D_2(f^{h,k})(ih, jk))(x, ih),$$

$$\hat{\mathcal{X}}(x, ih, jk) = \hat{\psi}(D_2(f^{h,k})(ih, jk))(x, ih),$$

where $\mathcal{X}$ satisfies the following induction (*cf.* (38)) $\forall x, \forall j \in \mathbb{N},\ \mathcal{X}(x,0,jk)=0$:

$$\mathcal{X}(x,ih,jk)=\mathrm{e}^{-xh}\mathcal{X}(x,(i-1)h,jk)+\frac{1-\mathrm{e}^{-xh}}{xh}\Delta(f^{h,k})(ih,jk) \tag{50}$$

and $\hat{\mathcal{X}}(x,ih,jk)$ is defined as $\frac{1}{x}[D_2(f^{h,k})(ih,jk)-\mathcal{X}(x,ih,jk)]$.

We compose with ψ (respectively $\hat{\psi}$) on the second $f^{h,k}$ component, $\forall x, \forall y, \forall i \in \mathbb{N},\ X(x,y,ih,0)=Z(x,y,ih,0)=Y(x,y,ih,0)=0$:

$$X(x,y,ih,(j+1)k)=\mathrm{e}^{-yk}X(x,y,ih,jk)+\frac{1-\mathrm{e}^{-yk}}{yk}\mathcal{X}(x,ih,jk), \tag{51}$$

$$Z(x,y,ih,(j+1)k)=\mathrm{e}^{-yk}Z(x,y,ih,jk)+\frac{1-\mathrm{e}^{-yk}}{yk}\hat{\mathcal{X}}(x,ih,jk), \tag{52}$$

$$Y(x,y,ih,(j+1)k)=\frac{1}{y}[\sum_{l=1}^{j+1}\hat{\mathcal{X}}(x,ih,jk)-Z(x,y,ih,(j+1)k)]. \tag{53}$$

Now we apply this induction to Brownian sheet $f=B$ to approximate $W^{\alpha_1,\alpha_2}(ih,jk)$. Remark that $(\Delta(B^{h,k})(ih,jk))$ is exactly the white noise $(B^{hk}_{i,j})$ introduced after Definition 2.10.

Definition 5.1. At any point (ih,jk) of the grid, $i,j\geq 0$, W^{α_1,α_2} defined by

$$W^{\alpha_1,\alpha_2}_{ih,jk}=\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\frac{x^{-a_1}y^{-a_2}}{\Gamma(\frac{1}{2}-\alpha_1)\Gamma(\frac{1}{2}-\alpha_2)}U(x,y,ih,jk)\mathrm{d}x\mathrm{d}y$$

is approximated by

$$\hat{W}^{\alpha_1,\alpha_2}_{n,r,h,k}(ih,jk)=\sum_{j_1,j_2=-n+1}^{n}c^1_{j_1}c^2_{j_2}U^{h,k}(r^{j_1-1},r^{j_2-1},ih,jk) \tag{54}$$

where $n>0,\ r\in]1,2[,\ h,k>0,\ c^l_{j_l}=\frac{1}{\Gamma(\frac{1}{2}-\alpha_l)}\frac{(r^{1-a_l}-1)}{1-a_l}r^{(1-a_l)(j_l-1)}$ and $U^{h,k}$ is defined as

$$U^{h,k}=\Psi_1\circ\Psi_2(B^{h,k})+[T_1\circ\Psi_2+\Psi_1\circ T_2+T_1\circ T_2](B^{hk}), \tag{55}$$

Ψ_i being defined in Corollary 4.13 and similarly $T_i=\Theta_i\mathbf{1}_{]0,\frac{1}{2}[}(\alpha_i)+\hat{\Theta}_i\mathbf{1}_{]\frac{1}{2},1[}(\alpha_i)\ i=1,2$.

Concerning $T_1\circ T_2(B)$, an exact simulation is possible since

$$\Theta_1\circ\Theta_2(B)(x,y,ih,jk)=(1-\mathrm{e}^{-xih})(1-\mathrm{e}^{-yjk})B_4(x,y)$$

where B_4 is a centered Gaussian matrix with covariance function $\Gamma_4(x,x',y,y')=\frac{1}{x+x'}\frac{1}{y+y'}$. To obtain $T_1\circ T_2(B)$ more generally, using (36), we get $T_1\circ T_2(B)(x,y,ih,jk)=-1/x\Theta_1\circ\Theta_2(B)(x,y,ih,jk)$ or $-1/y\Theta_1\circ\Theta_2(B)(x,y,ih,jk)$ or $1/xy\Theta_1\circ\Theta_2(B)(x,y,ih,jk)$.

Besides, the terms $T_i\circ\Psi_j(B)$ are obtained recursively. For instance, let $\Theta_1\circ\Psi_2$: operator ψ is applied to a centered Gaussian process and we get

$$\Theta_1\circ\Psi_2(B)(x,y,ih,(j+1)k)=\mathrm{e}^{-yk}\Theta_1\circ\Psi_2(B)(x,y,ih,jk)-(1-\mathrm{e}^{-xih})\frac{1-\mathrm{e}^{-yk}}{yk}\int_{-\infty}^{0}x\mathrm{e}^{xu}\int_{jk}^{(j+1)k}\mathrm{d}B_{uv},$$

the last term is approximated by $(1-\mathrm{e}^{-xih})\frac{1-\mathrm{e}^{-yk}}{yk}$ times the centered Gaussian vector $B^k_2(x)$ introduced in Section 2.3.

Remark 5.2. Such an algorithm could be used for any stationary increments field belonging to $\mathcal{H}^0_{\beta_1,\beta_2}$ as B is.

5.2. **Precision and complexity of the algorithm**

We give the error in this approximation, gathering the results in Theorem 4.1, Corollaries 4.10 and 4.14 and using the same notations as in Corollary 4.13.

Theorem 5.3. *Let* $\varepsilon > 0,\ r = 1+\varepsilon,\ \beta_i \in]|\alpha_i - \frac{1}{2}|, \frac{1}{2}[,\ \eta < \beta_i, i = 1,2,\ \gamma = \inf\{|\frac{1}{2}-\alpha_i|, \beta_i - |\frac{1}{2}-\alpha_i|, i = 1,2\},\ n = [\frac{-\log\varepsilon}{\gamma\log(1+\varepsilon)}],\ h = k = \varepsilon^{1/\eta}$.

At any point $(s,t) \in [0,T]^2$, $|W^{\alpha_1,\alpha_2}(s,t) - \hat{W}^{\alpha_1,\alpha_2}_{r,n,h,k}(s,t)|$ *is uniformly bounded by*

$$\varepsilon\left[C_\alpha \sum_{i+j=0}^{2} C_{i,j}\|B\|_\beta + D_\alpha D\|B\|_{\mathcal{H}^0_{\beta_1,\beta_2}}\right] \tag{56}$$

where $C_\alpha = \frac{20}{\gamma^2},\ D_\alpha = \frac{16}{\gamma^2},\ D = 4C_{0,0}((1+T)^\eta + (1+T)^\eta + 4\varepsilon),\ I = T/h, J = T/k$.

Finally, for any p,

$$\|\sup_{(s,t)\in[0,T]^2} |W^{\alpha_1,\alpha_2}(s,t) - \hat{W}^{\alpha_1,\alpha_2}_{r,n,h,k}(s,t)|\|_p \le \varepsilon\left[C_\alpha \sum_{i+j=0}^{2} C_{i,j}\|\|B\|_\beta\|_p + D_\alpha D\|\|B\|_{\mathcal{H}^0_{\beta_1,\beta_2}}\|_p\right].$$

Proof. At any point (ih,jk) of a grid, the fractional Brownian sheet $W^{\alpha_1,\alpha_2}(ih,jk)$ is approximated by:

$$\sum_{j_1,j_2=-n+1}^{n} c^1_{j_1}c^2_{j_2}U^{h,k}(r^{j_1-1}, r^{j_2-1}, ih, jk).$$

Two types of errors occur: one of the integral approximation, one of the time interpolation.
(i) Corollary 4.10 gives the error bound uniformly when $(s,t) \in [0,T]^2$:

$$|W^{\alpha_1,\alpha_2}(s,t) - \hat{W}^{\alpha_1,\alpha_2}_{n,r}(s,t)| \le \varepsilon C_\alpha \sum_{i+j=2} C_{i,j}\|B\|_\beta$$

with $\hat{W}^{\alpha_1,\alpha_2}_{n,r}(s,t) = \sum_{j_1,j_2=-n+1}^{n} c^1_{j_1}c^2_{j_2}\Psi(B)(r^{j_1-1}, r^{j_2-1}, s, t)$.

(ii) The definition $\hat{W}^{\alpha_1,\alpha_2}_{n,r,h,k}(s,t) = \sum_{j_1,j_2=-n+1}^{n} c^1_{j_1}c^2_{j_2}U^{h,k}(r^{j_1-1}, r^{j_2-1}, s, t)$ implies an error specified in (ii) Theorem 4.1:

$$\left|\sum_{j_1,j_2=-n+1}^{n} c^1_{j_1}c^2_{j_2}\Psi_1\circ\Psi_2(B - B^{h,k})(r^{j_1-1}, r^{j_2-1}, s, t)\right| \le D_\alpha\|\Psi_1\circ\Psi_2(B-B^{h,k})\|_{\infty,d,e},$$

with (d,e) defined as in Corollary 4.13. The last norm is controlled in Corollary 4.14 using the choice of parameters in Corollary 4.13 and the fact that $B \in \mathcal{H}^0_\beta,\ \forall\beta \in]0,\frac{1}{2}[^2$:

$$\|\Psi_1\circ\Psi_2(B-B^{h,k})\|_{\infty,d,e} \le \varepsilon D\|B\|_{\mathcal{H}^0_{\beta_1,\beta_2}},\ D = 4C_{0,0}((1+T)^\eta + (1+T)^\eta + 4\varepsilon).$$

The choice of r,n,h,k yields the conclusion since $T = hI = kJ$. □

Remark 5.4. At last, it remains to know how many terms are to be computed with respect to the convergence speed: let $N > 0$, and choose an accuracy of about $\varepsilon = N^{-\eta}$. Then n, r, h, k are deduced:

$$n \sim \frac{\eta}{\gamma}N^\eta\log N,\ h = k = N^{-1}, I = J = N.$$

Moreover, to produce an I^2 size image, we need to generate I^2 independent random variables, a $2n \times 2n$ Gaussian matrix, $(1+I)I$ Gaussian vectors in $\mathbb{R}^{2n}$. Finally, looking at (50) to (54) the computations complexity is about $O(n^2 IJ)$, or (with respect to N):

$$O([\log N]^2 N^{2(1+\eta)}).$$

As an example, let $\varepsilon = 0.1$ and $\alpha_i = 3/4$, $\beta_i = 0.495$ and let us choose $\eta = \varepsilon_i = 0.49$. Then, n is about $\frac{1}{\gamma} 10 \log 10$ and even γ is very small, n could be not so huge.

6. Conclusion

Chan and Wood presented in [19] the failing cases when the circulant matrix deduced from the covariance Toeplitz matrix is not definite positive (see also Stéphanie Léger's thesis [13] where the author tried this algorithm) because of numerical computations (and not because of the model). A job is now to be done and is in progress: after managing this simulation, we verify this simulation robustness with numerical estimations on this fractional Brownian sheet synthesis. In a first approach, this seems to be better than these obtained by Stéphanie Léger in [13].

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7. Annex

Proof of Remark 2.4. To show the continuity of X, we first prove a so-called "Kolmogorov' Lemma". Perhaps it is well known, but for the moment we miss a precise reference. The standard result (*cf.* for instance [12] pp. 53–55) can be written in multi-indices case using rectangular increments as follows. Let σ_j the elements of $\mathcal{S} = \{-1, +1\}^d$, and a random field X on $[0,1]^d$, s and $t \in [0,1]^d$, let α_j the parity of the sequence σ_j, $\sigma_j(s,t) = (s_i^{\sigma_j(i)} t_i^{1-\sigma_j(i)}, i = 1, \cdots, d)$ let the operator Δ:

$$\Delta X_{s,t} = \sum_{\sigma_j \in \mathcal{S}} (-1)^{\alpha_j} X_{\sigma_j(s,t)} = D_d(D_{d-1}(\cdots(D_1 X)\cdots)), \tag{57}$$

where D_i is the finite difference operator on the ith coordinate. In the case $d = 2$, look at Theorem 5.1 p. 1266 in Bernam [6]; this theorem can be written in multi-indices case using ΔX (57):

Let X a stochastically continuous separable d-indices process and there exist positive constants r, C, ε, such that:

$$E[|\Delta X|^r] \leq C \Pi_{s_i \neq t_i} |t_i - s_i|^{1+\varepsilon}, \tag{58}$$

then X is almost surely continuous on $[0,1]^d$.

If these assumptions are satisfied by the field X (8), this one is continuous in L^2, so separable. Since X is a Gaussian field, it also satisfies the stochastic continuity.

By definition, $X(x,y,s,t) = B(f(x,s,.)f(y,t,.))$ with $f(x,s,u) = \mathbf{1}_{[0,s]}(u)\mathrm{e}^{-x(s-u)}$: between two points in $\mathbb{R}^4$, the $L^2(\Omega)$-norm of ΔX is the $L^2(\mathbb{R}^2)-$norm of $\Delta[f(x,s,.)f(y,t,.)]$ that we can summarize as the double variation: $\Delta f(x,s,.)f(y,t,.) = \Delta_s(\Delta_x(f)).\Delta_t(\Delta_y(f))$.

This increment is the product of two increments on different spaces so it is enough to verify the assumptions (58) for one of the factors.

Let us remark that $f(x,s,u) = f_1(x,s,u)f_2(s,u)$, $f_1(x,s,u) = \mathrm{e}^{-x(s-u)}$, $f_2(s,u) = \mathbf{1}_{[0,s]}(u)$ with $|f_i| \leq 1$.

Thus, $|\Delta_s(f)| \leq |\Delta_s(f_1)| + |\Delta_s(f_2)|$ and the L^2-norm is bounded: $\|\Delta_s(f)\|_2 \leq C\sqrt{|\Delta s|}$ since f_1 is C^∞-class and $\Delta_s(f_2) = \mathbf{1}_{[s',s]}(u)$. Moreover ΔX law is Gaussian so: $E[|\Delta_s X|^3] \leq C|\Delta s|^{3/2}$.

Moreover the fact that f_1 is C^∞-class yields $|\Delta_x(f)| = |f_2.\Delta_x(f_1)| \leq C|\Delta x|$ so: $E[|\Delta_s X|]^2 = \|\Delta_x(f)\|_2 \leq C|\Delta x|$.

Finally, $|\Delta_s(\Delta_x(f))| = |\Delta_s(\Delta_x(f_1)f_2)| \leq |\Delta_s\Delta_x(f_1)| + |\Delta_x(f_1).\Delta_s(f_2)|$, get the L^2-norm bound:

$$\|\Delta_x\Delta_s X\|_2 = \|\Delta_s\Delta_x(f)\|_2 \leq \|\Delta_s\Delta_x(f_1)\|_2 + \|\Delta_x(f_1).\Delta_s(f_2)\|_2 \leq C\Delta x(\Delta s + \sqrt{\Delta s}).$$

Bernam's theorem with $r = 3, \varepsilon = \frac{1}{2}$ shows that X admits a continuous modification with respect to the four parameters. □

Proof of Proposition 3.3.

(i) $\psi(f)(x,s)$ can be written $f(s)[\mathrm{e}^{-xs} + x\int_0^s \mathrm{e}^{-xr}\mathrm{d}r] - x\int_0^s \mathrm{e}^{-xr} f(s-r)\mathrm{d}r$ or $f(s) - x\int_0^s \mathrm{e}^{-x(s-u)} f(u)\mathrm{d}u$ after a change of variables.
We now use the following steps:

(a) the operators ψ and $\hat{\psi}$ commute: $\hat{\psi}\circ\psi = \psi\circ\hat{\psi}$,

(b) the function $\hat{\psi}(f)$ is $\mathcal{C}^1$–class on $[0,+\infty[\times[0,T]$ and $\partial_x\hat{\psi}(f) = -\hat{\psi}\circ\hat{\psi}(f)$; $\partial_s\hat{\psi}(f) = \psi(f)$, and when f is derivable satisfying $f(0) = 0$, $\psi(f) = \hat{\psi}(f')$,

(c) the function $\psi(f)$ is differentiable with respect to x and $\partial_x\psi(f) = -\hat{\psi}\circ\psi(f) = -\psi\circ\hat{\psi}(f)$,

(a) follows from the point (i) and Fubini theorem.

(b) The map $(x,s)\mapsto\hat{\psi}(f)(x,s)$ admits partial derivatives respectively with respect to s and x:

$$\partial_s\hat{\psi}(f)(x,s) = f(s) - x\int_0^s \mathrm{e}^{-x(s-u)} f(u)\mathrm{d}u = \psi(f)(x,s),$$

$$\partial_x\hat{\psi}(f)(x,s) = -\int_0^s (s-u)\mathrm{e}^{-x(s-u)} f(u)\mathrm{d}u = -\hat{\psi}\circ\hat{\psi}(f)(x,s).$$

(c) Moreover $(x,s)\mapsto\psi(f)(x,s)$ admits partial derivative with respect to x:

$$\partial_x\psi(f)(x,s) = -\hat{\psi}(f)(x,s) - x\partial_x\hat{\psi}(f)(x,s) = -\hat{\psi}(f)(x,s) - x\hat{\psi}\circ\hat{\psi}(f)(x,s)$$

which coincide with $-\psi\circ\hat{\psi}(f)(x,s)$ so (c) yields.

(ii) Let the function $F : s\mapsto\int_0^s f(u)\mathrm{d}u$, the point (b) applied to F proves (ii). □

Proof of Theorem 3.8. We will use the fact that $f\in\mathcal{S}_\alpha$ implies that $\forall u\leq -1$, $|f(u)|\leq\|f\|_\alpha|u|^{1-\alpha}$ and $\forall u\geq -1$, $|f(u)|\leq\|f\|_\alpha|u|^\alpha$.

When $n = 0$, we get:

$$|\theta(f)(x,s)| \leq \|f\|_\alpha x(1-\mathrm{e}^{-xs})[\int_0^1 |u|^\alpha\mathrm{e}^{-xu}\mathrm{d}u + \int_1^\infty |u|^{1-\alpha}\mathrm{e}^{-xu}\mathrm{d}u].$$

The second term integral in the right hand is less than $\Gamma(2-\alpha)x^{\alpha-2}$, and the first term integral is less than $\Gamma(1+\alpha)x^{-\alpha-1}$. Since $\alpha\in]0,\frac{1}{2}[$, $\Gamma(\alpha+1)$ and $\Gamma(2-\alpha)\leq\Gamma(1)\vee\Gamma(2) = 1$. So the bound is $\|f\|_\alpha(1-\mathrm{e}^{-xs})[x^{\alpha-1} + x^{-\alpha}]$.

Since $\alpha < \frac{1}{2}$, if $x\geq 1$, $(1-\mathrm{e}^{-xs})x^{\alpha-1}\leq x^{-\alpha}$, and if $x < 1$, $(1-\mathrm{e}^{-xs})x^{-\alpha}\leq x^{\alpha-1}xT = Tx^\alpha$, so the theorem is proved for $n = 0$, and $C'_0 = 2\vee(1+T)$.

Besides, when $n \geq 1$, let us note that $(1-\mathrm{e}^{-xs})\mathrm{e}^{xu} = x\int_{u-s}^{u} \mathrm{e}^{xr}\mathrm{d}r$. So we get

$$\theta(f)(x,s) = x(1-\mathrm{e}^{-xs})\int_{-\infty}^{0} f(u)\mathrm{e}^{xu}\mathrm{d}u = \int_{-\infty}^{0} f(u)(\int_{u-s}^{u} x^2\mathrm{e}^{xr}\mathrm{d}r)\mathrm{d}u.$$

We now use Leibnitz rule and Lebesgue derivation theorem to get:

$$\partial^n_{x^n}\theta(f)(x,s) = \int_{-\infty}^{0} f(u)(\int_{u-s}^{u} (x^2r^n + 2nxr^{n-1} + n(n-1)r^{n-2})\mathrm{e}^{xr}\mathrm{d}r)\mathrm{d}u.$$

Using that $f \in \mathcal{S}_\alpha$ and switching the variables u and $-u$, r and $-r$, we get:

$$x^n|\partial^n_{x^n}\theta(f)(x,s)| \leq \|f\|_\alpha x^n \int_0^\infty (u^\alpha \mathbf{1}_{u\leq 1} + u^{1-\alpha}\mathbf{1}_{u>1})\left(\int_u^{u+s}(x^2r^n + 2nxr^{n-1} + n(n-1)r^{n-2})\mathrm{e}^{-xr}\mathrm{d}r\right)\mathrm{d}u.$$

Note that the map on $\mathbb{R}^+$, $u \mapsto u^\gamma \mathrm{e}^{-u}$, when $\gamma > 0$, is bounded by $c_\gamma = (\frac{\gamma}{e})^\gamma$, for instance $c_1 = \mathrm{e}^{-1} < \frac{1}{2}$, $c_2 = 4\mathrm{e}^{-2} \leq 1$. This remark and careful bounds prove the theorem. We detail the proof only in cases $n = 1$ and $n = 2$.

When $n = 1$, the bound is

$$\|f\|_\alpha x\int_0^\infty (u^\alpha \mathbf{1}_{u\leq 1} + u^{1-\alpha}\mathbf{1}_{u>1})\int_u^{u+s}(x^2r + 2x)\mathrm{e}^{-xr}\mathrm{d}r)\mathrm{d}u.$$

First, as in the case $n = 0$, $x\int_0^\infty (u^\alpha \mathbf{1}_{u\leq 1} + u^{1-\alpha}\mathbf{1}_{u>1})(\int_u^{u+s} 2x\mathrm{e}^{-xr}\mathrm{d}r)\mathrm{d}u \leq 4x^{-\alpha}\mathbf{1}_{\{x\geq 1\}} + 2(1+T)x^\alpha \mathbf{1}_{\{x<1\}}$.

The first term in the integrand $x^2r\mathrm{e}^{-xr} = 2x\mathrm{e}^{-xr/2}(xr/2)\mathrm{e}^{-xr/2} \leq 2x\mathrm{e}^{-xr/2}\mathrm{e}^{-1}$ so we get the bound

$$x\int_0^\infty (u^\alpha \mathbf{1}_{u\leq 1} + u^{1-\alpha}\mathbf{1}_{u>1})(\int_u^{u+s} x^2r\mathrm{e}^{-xr}\mathrm{d}r)\mathrm{d}u \leq 2x^2\mathrm{e}^{-1}\int_0^\infty (u^\alpha \mathbf{1}_{u\leq 1} + u^{1-\alpha}\mathbf{1}_{u>1})(\int_u^{u+s} \mathrm{e}^{-xr/2}\mathrm{d}r)\mathrm{d}u =$$

$$4\mathrm{e}^{-1}x\int_0^\infty (u^\alpha \mathbf{1}_{u\leq 1} + u^{1-\alpha}\mathbf{1}_{u>1})(1-\mathrm{e}^{-xs/2})\mathrm{e}^{-xu/2}\mathrm{d}u \leq 4\mathrm{e}^{-1}x(1-\mathrm{e}^{-xs/2})[(2/x)^{\alpha+1} + (2/x)^{2-\alpha}] \leq$$

$$16x^{-\alpha}\mathbf{1}_{\{x\geq 1\}} + 8(1+T)x^\alpha \mathbf{1}_{\{x<1\}}.$$

Globally,

$$x|\partial_x\theta(f)(x,s)| \leq 10(2\vee(1+T))\|f\|_\alpha(x^{-\alpha}\mathbf{1}_{\{x\geq 1\}} + x^\alpha \mathbf{1}_{\{x<1\}}).$$

When $n = 2$, the bound is

$$\|f\|_\alpha x^2\int_0^\infty (u^\alpha \mathbf{1}_{u\leq 1} + u^{1-\alpha}\mathbf{1}_{u>1})(\int_u^{u+s}(x^2r^2 + 2xr + 2)\mathrm{e}^{-xr}\mathrm{d}r)\mathrm{d}u.$$

The last term is the same as in the case $n = 1$, so we get:

$$x^2\int_0^\infty \left(u^\alpha \mathbf{1}_{u\leq 1} + u^{1-\alpha}\mathbf{1}_{u>1}\right)\left(\int_u^{u+s} 2\mathrm{e}^{-xr}\mathrm{d}r\right)\mathrm{d}u \leq 2(1-\mathrm{e}^{-xs})x^{-\alpha} \leq 4x^{-\alpha}\mathbf{1}_{\{x\geq 1\}} + 2(1+T)x^\alpha \mathbf{1}_{\{x<1\}}.$$

The second term is twice the first of the case $n = 1$, so we get:

$$x^2\int_0^\infty \left(u^\alpha \mathbf{1}_{u\leq 1} + u^{1-\alpha}\mathbf{1}_{u>1}\right)\left(\int_u^{u+s} 2xr\mathrm{e}^{-xr}\mathrm{d}r\right)\mathrm{d}u \leq 32x^{-\alpha}\mathbf{1}_{\{x\geq 1\}} + 16(1+T)x^\alpha \mathbf{1}_{\{x<1\}}.$$

The bound of the first term is quite similar: $x^2r^2\mathrm{e}^{-xr} = 4\mathrm{e}^{-xr/2}(xr/2)^2\mathrm{e}^{-xr/2} \leq 4c_2\mathrm{e}^{-xr/2} \leq 4\mathrm{e}^{-xr/2}$, so

$$x^2\int_0^\infty (u^\alpha \mathbf{1}_{u\leq 1} + u^{1-\alpha}\mathbf{1}_{u>1})\left(\int_u^{u+s} x^2r^2\mathrm{e}^{-xr}\mathrm{d}r\right)\mathrm{d}u \leq x^2\int_0^\infty (u^\alpha \mathbf{1}_{u\leq 1} + u^{1-\alpha}\mathbf{1}_{u>1})\left(\int_u^{u+s} 4\mathrm{e}^{-xr/2}\mathrm{d}r\mathrm{d}u\right) =$$

$$8x\int_0^\infty (u^\alpha \mathbf{1}_{u\leq 1} + u^{1-\alpha}\mathbf{1}_{u>1})(1-\mathrm{e}^{-xs/2})\mathrm{e}^{-xu/2}\mathrm{d}u \leq$$

$$8x(1-\mathrm{e}^{-xs/2})[(2/x)^{\alpha+1} + (2/x)^{2-\alpha}] \leq 32(2x^{-\alpha}\mathbf{1}_{\{x\geq 1\}} + 1(+T)x^\alpha \mathbf{1}_{\{x<1\}}.$$

Globally, we get $x^2|\partial^2_{x^2}\theta(f)(x,s)| \leq 50(2\vee(1+T))\|f\|_\alpha(x^{-\alpha}\mathbf{1}_{\{x\geq 1\}} + x^\alpha\mathbf{1}_{\{x<1\}})$. □

Proof of Proposition 3.11. We have to bound the difference $|f(s)-f^h(s)-f(t)+f^h(t)|$ with respect to $|s-t|^{\beta-\eta}$. So let

$$Q = \frac{|f(s)-f^h(s)-f(t)+f^h(t)|}{|s-t|^{\beta-\eta}h^\eta\|f\|_{\mathcal{H}^0_\beta}}.$$

Suppose that $s<t, s\in I_i, t\in I_k$. Three cases occur.

(i) $i=k$: $t-s\leq h$ so the numerator of Q is:

$$|f(t)-f(s)+\frac{s-t}{h}(f((i+1)h)-f(ih))| \text{ bounded by } \|f\|_{\mathcal{H}^0_\beta}[(t-s)^\beta + (t-s)h^{\beta-1}].$$

Thus $Q\leq [(\frac{t-s}{h})^\eta + (\frac{t-s}{h})^{1-\beta+\eta}] \leq 2$.

(ii) $i<k$ and $t-s\geq h$: the numerator of Q is:

$$|f(ih)-f(s)+\frac{s-ih}{h}(f((i+1)h)-f(ih)) - [f(kh)-f(t)+\frac{t-kh}{h}(f((k+1)h)-f(kh))]|;$$

the first term is bounded by $\|f\|_{\mathcal{H}^0_\beta}[(s-ih)^\beta + (s-ih)h^{\beta-1}] \leq 2h^\beta\|f\|_{\mathcal{H}^0_\beta}$, the second one is bounded similarly and so $Q\leq 4(\frac{h}{t-s})^{\beta-\eta}\leq 4$.

(iii) $i<k$ and $t-s<h$: here we write the linear interpolation $f^h(s)$ with respect to the common point $(i+1)h$:

$$f^h(s) = f((i+1)h) + \frac{(i+1)h-s}{h}(f((i+1)h)-f(ih)).$$

Let us remark that $(i+1)h-s\leq t-s\leq h$ so yields the bound:

$$|f^h(s)-f(s)| \leq |f((i+1)h)-f(s)| + \frac{(i+1)h-s}{h}|f((i+1)h)-f(ih)| \leq \|f\|_{\mathcal{H}^0_\beta}[(t-s)^\beta + (t-s)h^{\beta-1}].$$

The bound of the second term is the same since $t-(i+1)h\leq t-s\leq h$.

So $Q\leq 2(\frac{t-s}{h}^\eta + \frac{t-s}{h}^{1+\eta-\beta}) \leq 4$. □

Proposition 7.1. *Let $(d,e)\in[0,1]^4$, two functions $f_k\in L^1([0,+\infty[,\mathbb{R}^+,\mu_{d_k,e_k}), k=1,2$, and $g\in C^2(]0,+\infty)^2,\mathbb{R})$ such that the norm $\|.\|_{\infty,d,e}$ of the maps g and*

$$h_g : (x,y)\mapsto |\partial_{x^2}g(x,y)|x^2 + 2|\partial_{x,y}g(x,y)|xy + |\partial_{y^2}g(x,y)|y^2$$

are finite. Let $r\in]1,2[$, $n\in\mathbf{N}^*$ to define a geometric subdivision of $]0,+\infty[$, $\pi=(t_{-n},\cdots,t_n), t_i=r^i$: $I_j=[t_{j-1},t_j]$; let:

$$c^i_j = \int_{I_j} f_i(x)\mathrm{d}x \ ; \ \eta^j_i = \frac{\int_{I_j} xf_i(x)\mathrm{d}x}{\int_{I_j} f_i(x)\mathrm{d}x}, j=-n+1,\cdots,n \ ; \ i=1,2.$$

Then

$$\left|\int_{[0,+\infty[^2} g(x,y)f_1(x)f_2(y)\mathrm{d}x\mathrm{d}y \quad - \sum_{i,j=-n+1}^{n} c_i^1 c_j^2 g(\eta_1^i,\eta_2^j)\right|$$
$$\leq \frac{3}{2}r^2(r-1)^2\|h_g\|_{\infty,d,e}\|f_1\|_{\mu_{d_1,e_1}}\|f_2\|_{\mu_{d_2,e_2}} \quad + \quad C\|g\|_{\infty,d,e}[\sum_{k=1,2}\|f_k(\mathbf{1}_{]0,t_{-n}]}+\mathbf{1}_{]t_n,+\infty[})\|_{\mu_{d_k,e_k}}]$$

where $C = \max[\|f_1\|_{\mu_{d_1,e_1}}, \|f_2\|_{\mu_{d_2,e_2}}]$.

Proof. First the integral of the function $(x,y) \mapsto g(x,y)f_1(x)f_2(y)$ on $]0,+\infty[\times[0,r^{-n}]$, $]0,+\infty[\times[r^n,+\infty[$, $[0,r^{-n}]\times[0,+\infty[$ et $[r^n,+\infty[\times]0,+\infty[$ is bounded as follows; for instance, the assumptions on f_k, $k=1,2$ and g yield

$$\int_0^{r^{-n}}\int_0^{\infty} g(x,y)f_1(x)f_2(y)\mathrm{d}x\mathrm{d}y =$$
$$\int_0^{r^{-n}}\int_0^{1} x^{d_1}y^{d_2}g(x,y)x^{-d_1}f_1(x)y^{-d_2}f_2(y)\mathrm{d}x\mathrm{d}y + \int_0^{r^{-n}}\int_1^{\infty} x^{d_1}y^{\varepsilon_2}g(x,y)x^{-d_1}f_1(x)y^{-\varepsilon_2}f_2(y)\mathrm{d}x\mathrm{d}y$$

which is bounded by:

$$\|g\|_{\infty,d,\varepsilon}\|f_2\|_{\mu_{d_2,\varepsilon_2}}\|f_1\mathbf{1}_{]0,r^{-n}]}\|_{\mu_{d_1,\varepsilon_1}}.$$

Similarly we get:

$$\int_{[r^n,\infty)\times]0,+\infty[} g(x,y)f_1(x)f_2(y)\mathrm{d}x\mathrm{d}y \leq \|g\|_{\infty,d,\varepsilon}\|f_2\|_{\mu_{d_2,\varepsilon_2}}\|f_1\mathbf{1}_{]r^n,\infty]}\|_{\mu_{d_1,\varepsilon_1}}.$$

The other bounds are obtained by inverting the indices 1 and 2.

Secondly we manage a bound of the difference on each $I_i\times I_j$ now called as $[a,b]\times[c,d]$, dropping the indices i et j. Let to bound :

$$D = |\int_{[a,b]\times[c,d]} g(x,y)f_1(x)f_2(y)\mathrm{d}x\mathrm{d}y - \int_{[a,b]} f_1(x)\mathrm{d}x\int_{[c,d]} f_2(x)\mathrm{d}x g(\eta_1,\eta_2)|.$$

To do that we introduce two independent random variables $X_k, k=1,2$, with a support respectively in $[a,b]$ and $[c,d]$ and a density with respect to Lebesgue measure $\frac{f_k}{\int f_k(x)\mathrm{d}x}$. We develop the function g with order 2 between points $X=(X_1,X_2)$ and $\eta=(\eta_1,\eta_2)$:

$$g(X)-g(\eta) = \sum_{k=1,2}(X_k-\eta_k)\partial_k g(\eta_1,\eta_2) + \frac{1}{2}\int_0^1 D^2 g(\theta X+(1-\theta)\eta)(X-\eta,X-\eta)\mathrm{d}\theta.$$

In this difference, the 1-order term is null because of the definition of η_k. The 2-order term, denoting $Y=(Y_i=\theta X_i+(1-\theta)\eta_i, i=1,2)$, is:

$$\partial_{x^2}g(Y)Y_1^2\frac{(X_1-\eta_1)^2}{Y_1^2} + \partial_{y^2}g(Y)Y_2^2\frac{(X_2-\eta_2)^2}{Y_2^2} + 2\partial_{x,y}g(Y)Y_1Y_2\frac{(X_1-\eta_1)(X_2-\eta_2)}{Y_1Y_2}.$$

The first term factor can be written

$$\partial_{x^2}g(Y)Y_1^2 = \Pi_{i=1}^2(Y_i^{d_i}\mathbf{1}_{Y_i<1}+Y_i^{\varepsilon_i}\mathbf{1}_{Y_i\geq 1})\partial_{x^2}g(Y)Y_1^2\Pi_{i=1}^2 Y_i^{-d_i}\wedge Y_i^{-\varepsilon_i}.$$

The sum of the absolute value of the two first terms is then bounded by

$$\|h_g\|_{\infty,d,\varepsilon}\left[\frac{(X_1-\eta_1)^2}{a^2}+\frac{(X_2-\eta_2)^2}{c^2}\right](a^{-d_1}\wedge a^{-\varepsilon_1})(c^{-d_2}\wedge c^{-\varepsilon_2}),$$

since Y takes its values in $[a,b]\times[c,d]$.

The absolute value of the last term is also bounded by

$$\|h_g\|_{\infty,d,\varepsilon}\left[\frac{(X_1-\eta_1)^2}{2a^2}+\frac{(X_2-\eta_2)^2}{2c^2}\right](a^{-d_1}\wedge a^{-\varepsilon_1})(c^{-d_2}\wedge c^{-\varepsilon_2}).$$

Globally, we obtain the following bound to the expectation of $g(X)-g(\eta)$:

$$D\le\frac{3}{2}\int_{[a,b]}f_1(x)(a^{-d_1}\wedge a^{-\varepsilon_1})\mathrm{d}x\int_{[c,d]}f_2(x)(c^{-d_2}\wedge c^{-\varepsilon_2})\mathrm{d}x\|h_g\|_{\infty,d,\varepsilon}\left(\frac{V(X_1)}{a^2}+\frac{V(X_2)}{c^2}\right).$$

Let us note that the variance maximum of random variables with a support in an interval is the squared length half of this interval:

$$D\le\frac{3}{4}\int_{[a,b]}f_1(x)(a^{-d_1}\wedge a^{-\varepsilon_1})\mathrm{d}x\int_{[c,d]}f_2(x)(c^{-d_2}\wedge c^{-\varepsilon_2})\mathrm{d}x\|h_g\|_{\infty,d,\varepsilon}\left[\frac{(b-a)^2}{a^2}+\frac{(d-c)^2}{c^2}\right].$$

Moreover, in one hand when $a=r^i$, $b=r^{i+1}$, $\frac{(b-a)^2}{a^2}=(r-1)^2$, and in the other hand $\int_{[a,b]}f_1(x)(a^{-d_1}\wedge a^{-\varepsilon_1})\mathrm{d}x\le r^{\varepsilon_1}\int_{[a,b]}f_1(x)\mathrm{d}\mu_{d_1,\varepsilon_1}(x)$, thus

$$D\le\frac{3}{2}r^{\varepsilon_1+\varepsilon_2}(r-1)^2\int_{[a,b]}f_1(x)\mathrm{d}\mu_{d_1,\varepsilon_1}(x)\int_{[c,d]}f_2(x)\mathrm{d}\mu_{d_2,\varepsilon_2}(x)\|h_g\|_{\infty,d,\varepsilon}.$$

We now sum all these bounds on all $I_i\times I_j$ plus the edge terms:

$$\begin{aligned}&\left|\int_{[0,+\infty[^2}g(x,y)f_1(x)f_2(y)\mathrm{d}x\mathrm{d}y\quad-\quad\sum_{i,j=1}^{n}c_{i,j}g(\eta_1^i\eta_2^j)\right|\\ \le\ &\frac{3}{2}r^{\varepsilon_1+\varepsilon_2}(r-1)^2\|h_g\|_{\infty,d,\varepsilon}\|f_1\|_{\mu_{d_1,\varepsilon_1}}\|f_2\|_{d_2,\varepsilon_2}\quad+\quad C\|g\|_{\infty,d,\varepsilon}[\sum_{k=1,2}\|f_k\mathbf{1}_{]0,t_{-n}]}\|_{\mu_{d_k,\varepsilon_k}}+\|f_k\mathbf{1}_{]t_{-n},+\infty[}\|_{\mu_{d_k,\varepsilon_k}}]\end{aligned}$$

since the sum of $\int_{I_i}f_1(x)\mathrm{d}\mu_{d_1,\varepsilon_1}(x)\int_{I_j}f_2(x)\mathrm{d}\mu_{d_2,\varepsilon_2}(x)$ is equal to $\|f_1\|_{\mu_{d_1,\varepsilon_1}}\|f_2\|_{\mu_{d_2,\varepsilon_2}}$. □

The following corollaries are deduced; they are useful for the time discretization.

Corollary 7.2. *Let (d,e,f_1,f_2) and g satisfying $\|\Delta g\|_{\infty,d,e}<\infty$ and the assumptions of Proposition 7.1, then*

$$|\sum_{i,j=-n+1}^{n}c_i^1c_j^2[g(\eta_1^i,\eta_2^j)-g(r^{i-1},r^{j-1})]|\le(r-1)r^2\|f_1\|_{\mu_{d_1,e_1}}\|f_2\|_{\mu_{d_2,e_2}}\|\Delta g\|_{\infty,d,e},$$

where c_i^k, η_k^i, $k=1,2$, $i=-n+1,\cdots,n$, n et r are defined in Proposition 7.1.

Proof. Let $D_{i,j}=c_i^1c_j^2[g(\eta_1^i,\eta_2^j)-g(r^{i-1},r^{j-1})]$ and use Taylor theorem:

$$|D_{i,j}|\le|c_i^1c_j^2|\sup_{x\in I_i,y\in I_j}\{|\partial_xg(x,y)||\eta_1^i-r^{i-1}|+|\partial_yg(x,y)||\eta_2^j-r^{j-1}|\}.$$

Note that $|\eta_1^i - r^{i-1}|$ is less than $(r-1)r^{i-1}$ and that $x \in I_i$ implies that $r^{i-1} \leq x$ so:

$$|D_{i,j}| \leq (r-1)|c_i^1 c_j^2| \sup_{x\in I_i, y\in I_j} \{x|\partial_x g(x,y)| + y|\partial_y g(x,y)|\}.$$

Using $\|\Delta g\|_{\infty,d,e}$ definition we get

$$|D_{i,j}| \leq (r-1)|c_i^1 c_j^2|(r^i)^{-d_1} \wedge (r^i)^{-e_1}.(r^j)^{-d_2} \wedge (r^j)^{-e_2}\|\Delta g\|_{\infty,d,e}.$$

But $c_k^i = \int_{[r^{k-1},r^k]} f_i(x)\mathrm{d}x$ and $\mu_{d_k,e_k}(\mathrm{d}x) = x^{-d_k} \wedge x^{-e_k}\mathrm{d}x$ so

$$|D_{i,j}| \leq (r-1)r^{e_1+e_2} \int_{[r^{i-1},r^i]} f_1(x)\mu_{d_1,e_1}(\mathrm{d}x) \int_{[r^{j-1},r^j]} f_2(x)\mu_{d_2,e_2}(\mathrm{d}x)\|\Delta g\|_{\infty,d,e}.$$

To sum these bounds with respect to i and j get the conclusion. □

Corollary 7.3. *Let (d,e,f_1,f_2,g,h) satisfying the assumptions of Proposition 7.1, then*

$$\left|\sum_{i,j=-n+1}^{n} c_i^1 c_j^2[g(r^{i-1},r^{j-1}) - h(r^{i-1},r^{j-1})]\right| \leq r^2\|f_1\|_{\mu_{d_1,e_1}}\|f_2\|_{\mu_{d_2,e_2}}\|g-h\|_{\infty,d,e},$$

where c_i^k, n et r are defined in Proposition 7.1.

Proof. Let $D_{i,j} = |[g(r^{i-1},r^{j-1}) - h(r^{i-1},r^{j-1})]c_i^1 c_j^2|$. Using the norm $\|g-h\|_{\infty,d,e}$ definition, we bound $D_{i,j}$ by

$$|c_i^1 c_j^2|\Pi_{k=1}^2 (r^{i-1})^{-d_1} \wedge (r^{i-1})^{-e_1}.(r^{j-1})^{-d_2} \wedge (r^{j-1})^{-e_2}\|g-h\|_{\infty,d,e}. \tag{59}$$

But $c_i^k = \int_{[r^{i-1},r^i]} f_k(x)\mathrm{d}x$ and $\mu_{d_k,e_k}(\mathrm{d}x) = x^{-d_k} \wedge x^{-e_k}\mathrm{d}x$ so

$$D_{i,j} \leq r^{e_1+e_2} \int_{[r^{i-1},r^i]} f_1(x)\mu_{d_1,e_1}(\mathrm{d}x) \int_{[r^{j-1},r^j]} f_2(x)\mu_{d_2,e_2}(\mathrm{d}x)\|g-h\|_{\infty,d,e}.$$

To sum these bounds with respect to i and j get the conclusion. □

Proof of Theorem 4.1. The chosen pair (d,e) allows us to use Proposition 7.1 and Corollary 7.2 with

$$f_i(x) = x^{-a_i},\ i = 1,2.$$

(i) First we get

$$\begin{aligned} D &= |\int_{[0,+\infty[^2} g(x,y)f_1(x)f_2(y)\mathrm{d}x\mathrm{d}y - \sum_{j_1,j_2=-n+1}^{n} c_{j_1}^1 c_{j_2}^2 g(r^{j_1-1},r^{j_2-1})| \\ &\leq \frac{3}{2}r^2(r-1)^2\|h_g\|_{\infty,d,e}\|f_1\|_{\mu_{d_1,e_1}}\|f_2\|_{\mu_{d_2,e_2}} + C\|g\|_{\infty,d,e}[\sum_{k=1,2}\|f_k(\mathbf{1}_{]0,r^{-n}]} + \mathbf{1}_{]r^n,+\infty[})\|_{\mu_{d_k,e_k}}] \\ &\quad +(r-1)r^2\|f_1\|_{\mu_{d_1,e_1}}\|f_2\|_{\mu_{d_2,e_2}}\|\Delta g\|_{\infty,d,e}. \end{aligned} \tag{60}$$

Besides, we compute

$$\|f_k(\mathbf{1}_{]0,r^{-n}]} + \mathbf{1}_{]r^n,+\infty[})\|_{\mu_{d_k,e_k}}.$$

An integration computation shows when $\alpha_k < \frac{1}{2}$:

$$\|f_k \mathbf{1}_{]0,r^{-n}]}\|_{\mu_{d_k,e_k}} = \frac{1}{\frac{1}{2}-\alpha_k} r^{-n(\frac{1}{2}-\alpha_k)};\ \|f_k \mathbf{1}_{[r^n,\infty)}\|_{\mu_{d_k,e_k}} = \frac{1}{\alpha_k+\beta_k-\frac{1}{2}} r^{-n(\alpha_k+\beta_k-\frac{1}{2})}$$

and when $\alpha_k > \frac{1}{2}$:

$$\|f_k \mathbf{1}_{]0,r^{-n}]}\|_{\mu_{d_k,e_k}} = \frac{1}{-\alpha_k+\beta_k+\frac{1}{2}} r^{-n(-\alpha_k+\beta_k+\frac{1}{2})};\ \|f_k \mathbf{1}_{[r^n,\infty)}\|_{\mu_{d_k,e_k}} = \frac{1}{\alpha_k-\frac{1}{2}} r^{-n(\alpha_k-\frac{1}{2})}.$$

So the sum on $k = 1, 2$ is bounded by $\frac{4}{\gamma} r^{-\gamma n}$ using the γ definition. Getting $r = 1$ we easily deduce that

$$\|f_i\|_{\mu_{d_i,e_i}} = \frac{2}{|1-2\alpha_i|} + \frac{1}{\beta_i - |\frac{1}{2}-\alpha_i|} \le \frac{2}{\gamma}.$$

Thus, recalling that $r = 1+\varepsilon \le 2$, the first term in the right bound of (60) is less than

$$6\varepsilon^2 \Pi_{i=1,2} \left(\frac{2}{|1-2\alpha_i|} + \frac{1}{\beta_i - |\frac{1}{2}-\alpha_i|} \right) \|h_g\|_{\infty,d,e} \le \varepsilon^2 \frac{24}{\gamma^2} \|h_g\|_{\infty,d,e}.$$

Then recalling that C defined in Proposition 7.1 is $\max_{i=1,2} \|f_i\|_{\mu_{d_i,e_i}}$ and that $r^{-n\gamma} \sim \varepsilon$, the second term in the right bound of (60) is less than

$$\varepsilon \frac{2}{\gamma} \max_{i=1,2} \left(\frac{2}{|1-2\alpha_i|} + \frac{1}{\beta_i - |\frac{1}{2}-\alpha_i|} \right) \|g\|_{\infty,d,e} \le \varepsilon \frac{4}{\gamma^2} \|g\|_{\infty,d,e}.$$

Finally, the third term in the right bound of (60) is less than

$$\varepsilon 4 \Pi_{i=1,2} \left(\frac{2}{|1-2\alpha_i|} + \frac{1}{\beta_i - |\frac{1}{2}-\alpha_i|} \right) \|\Delta g\|_{\infty,d,e} \le \varepsilon \frac{16}{\gamma^2} \|\Delta g\|_{\infty,d,e}.$$

So yields the constant $C_\alpha = \frac{20}{\gamma^2}$.
(ii) Let

$$E_{j_1,j_2} = |c^1_{j_1} c^2_{j_2} (g(r^{j_1-1}, r^{j_2-1}) - h(r^{j_1-1}, r^{j_2-1}))|.$$

We use Corollary 7.3 to bound E_{j_1,j_2} by

$$|c^1_{j_1} c^2_{j_2}| (r^{j_1-1})^{-d_1} \wedge (r^{j_1-1})^{-e_1} . (r^{j_2-1})^{-d_2} \wedge (r^{j_2-1})^{-e_2} \|g-h\|_{\infty,d,e}.$$

But $c^k_{j_k} = \int_{[r^{j_k-1}, r^{j_k}]} f_k(x) \mathrm{d}x$ and $\mu_{d_k,e_k}(\mathrm{d}x) = x^{-d_k} \wedge x^{-e_k} \mathrm{d}x$ so

$$E_{j_1,j_2} \le r^{e_1+e_2} \int_{[r^{j_1-1}, r^{j_1}]} f_1(x) \mu_{d_1,e_1}(\mathrm{d}x) \int_{[r^{j_2-1}, r^{j_2}]} f_2(x) \mu_{d_2,e_2}(\mathrm{d}x) \|g-h\|_{\infty,d,e}.$$

To sum these bounds with respect to j_1 and j_2 get $D_\alpha = r^2 \Pi_{k=1,2} \|f_k\|_{\mu_{d_k,e_k}} \le \frac{16}{\gamma^2}$, so yields the conclusion. □

Proof of Lemma 4.3. The key of the proof is the commutativity of any operator in $\mathcal{A}_1$ with any operator in $\mathcal{A}_2$.

For any $f \in \mathcal{S}_{\alpha_1,\alpha_2}$, $(x,y,s,t) \in (\mathbb{R}^*_+)^2 \times\ ]-\infty, T]^2$, $\partial^j_{y^j} A_2(f)(y,t)$ belongs to $\mathcal{S}_{\alpha_1}$ similarly $\partial^i_{x^i} A_1(f)(x,s) \in \mathcal{S}_{\alpha_2}$, so the operators compositions can be done. Then we do an induction on (i,j).

The result is true for $i = j = 0$ since any operator in $\mathcal{A}_1$ commute with any operator in $\mathcal{A}_2$ (it is a tedious but straightforward formal verification).

We now suppose that the result is proved for (i,j) and we prove it for $(i,j+1)$. The same will be true for $(i+1,j)$. Let $(x,y,s,t)\in(\mathbb{R}_+^*)^2\times]-\infty,T]^2$. On one hand by definition for any $f\in\mathcal{S}_{\alpha_1,\alpha_2}$

$$\partial^{i+j+1}_{x^i,y^{j+1}}\tilde{A}(x,y,s,t)(f)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[\partial^i_{x^i}A_1(\partial^j_{y^j}A_2(f)(y+\varepsilon,t))(x,s)-\partial^i_{x^i}A_1(\partial^j_{y^j}A_2(f)(y,t))(x,s)].$$

But the operator $\partial^i_{x^i}A_1(x,s)$ is continuous linear so it commutes with the limit, this limit moreover belongs to $\mathcal{S}_{\alpha_1,\alpha_2}$:

$$\begin{aligned}\partial^{i+j+1}_{x^i,y^{j+1}}\tilde{A}(x,y,s,t)(f) &= \partial^i_{x^i}A_1\left(\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[\partial^j_{y^j}A_2(f)(y+\varepsilon,t)-\partial^j_{y^j}A_2(f)(y,t)]\right)(x,s),\\ &= \partial^i_{x^i}A_1(\partial^{j+1}_{y^{j+1}}A_2(f)(y,t))(x,s),\end{aligned}$$

so we get $\partial^{i+j+1}_{x^iy^{j+1}}\tilde{A}(x,y,s,t)=\partial^i_{x^i}A_1(x,s)\circ\partial^{j+1}_{y^{j+1}}A_2(y,t)$.

On the other hand using the induction assumption:

$$\begin{aligned}\partial^{i+j+1}_{x^i,y^{j+1}}\tilde{A}(x,y,s,t)(f) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\partial^j_{y^j}A_2(\partial^i_{x^i}A_1(f)(x,s))(y+\varepsilon,t)-\partial^j_{y^j}A_2(\partial^i_{x^i}A_1(f)(x,s))(y,t),\\ &= \partial^{j+1}_{y^{j+1}}A_2(\partial^i_{x^i}A_1(f(.,t)(x,s))(y,t)\end{aligned}$$

and the lemma is proved. □

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