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ON THE TAILS OF THE DISTRIBUTION OF THE MAXIMUM OF A SMOOTH STATIONARY GAUSSIAN PROCESS *

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Abstract. We study the tails of the distribution of the maximum of a stationary Gaussian process on a bounded interval of the real line. Under regularity conditions including the existence of the spectral moment of order 8, we give an additional term for this asymptotics. This widens the application of an expansion given originally by Piterbarg [11] for a sufficiently small interval.

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1. Introduction and main result

Let $X = \{X(t), t \in [0,T]\}$, $T > 0$ be a real-valued centered Gaussian process and denote

$$M := \max_{t\in[0,T]} X(t).$$

The precise knowledge of the distribution of the random variable M is essential in many of statistical problems; for example, in Methodological Statistics (see Davies [8]), in Biostatistics (see Azaïs and Cierco–Ayrolles [4]). But a closed formula based upon natural parameters of the process is only known for a very restricted number of stochastic processes X: for instance, the Brownian motion, the Brownian bridge or the Ornstein–Uhlenbeck process (a list is given in Azaïs and Wschebor [6]). An interesting review of the problem could be found in Adler [2].

We are interested here in a precise expansion of the tail of the distribution of M for a smooth Gaussian stationary process. First, let us specify some notations

- $r(t) := \mathbb{E}\big(X(s)X(s+t)\big)$ denotes the covariance function of X. With no loss of generality we will also assume $\lambda_0 = r(0) = 1$;
- μ its spectral measure and λ_k $(k = 0, 1, 2, \dots)$ its spectral moments whenever they exist;
- $\phi(x) = \dfrac{1}{\sqrt{2\pi}} \exp\left(-\dfrac{x^2}{2}\right)$ and $\Phi(x) = \displaystyle\int_{-\infty}^{x} \phi(t)\mathrm{d}t$.

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Throughout this paper we will assume that $\lambda_8 < \infty$ and for every pair of parameter values s and t , $0 \le s \ne t \le T$, the six-dimensional random vector $(X(s), X'(s), X''(s), X(t), X'(t), X''(t))$ has a non-degenerate distribution.

Piterbarg [11] (Th. 2.2) proved (under the weaker condition $\lambda_4 < \infty$ instead of $\lambda_8 < \infty$) that for each $T > 0$ and any $u \in R$:

$$\left|1 - \Phi(u) + \sqrt{\frac{\lambda_2}{2\pi}} T\phi(u) - P(M > u)\right| \le B \ \exp\left(-\frac{u^2(1+\rho)}{2}\right), \tag{1}$$

for some positive constants B and ρ. It is easy to see (see for example Miroshin [10]) that the expression inside the modulus is non-negative, so that in fact:

$$0 \le 1 - \Phi(u) + \sqrt{\frac{\lambda_2}{2\pi}} T\phi(u) - P(M > u) \le B \ \exp\left(-\frac{u^2(1+\rho)}{2}\right) \cdot \tag{2}$$

The problem of improving relation (2) does not seem to have been solved in a satisfactory manner until now. A crucial step has been done by Piterbarg in the same paper (Th. 3.1) in which he proved that if T is small enough, then as $u \to +\infty$:

$$P(M > u) = 1 - \Phi(u) + \sqrt{\frac{\lambda_2}{2\pi}} T\phi(u) - \left(\frac{3\sqrt{3}(\lambda_4 - \lambda_2^2)^{9/2}}{2\pi\lambda_2^{9/2}(\lambda_2\lambda_6 - \lambda_4^2)}\right) \frac{T}{u^5} \phi\left(u\sqrt{\frac{\lambda_4}{\lambda_4 - \lambda_2^2}}\right) [1 + o(1)] \,. \tag{3}$$

The same result has been obtained by other methods (Azaïs and Bardet [3]; see also Azaïs *et al.* [5]).

However Piterbarg equivalent (3) is of limited interest for applications since it contains no information on the meaning of the expression "T small enough".

The aim of this paper is to show that formula (3) is in fact valid **for any** length T under appropriate conditions that will be described below.

Consider the function $F(t)$ defined by

$$F(t) := \frac{\lambda_2\big(1 - r(t)\big)^2}{\lambda_2\big(1 - r^2(t)\big) - r'^2(t)} \cdot$$

Lemma 1. *The even function F is well defined, has a continuous extension at zero and*

1. $F(0) = \dfrac{\lambda_2^2}{\lambda_4 - \lambda_2^2}$;
2. $F'(0) = 0$;
3. $0 < F''(0) = \dfrac{\lambda_2(\lambda_2\lambda_6 - \lambda_4^2)}{9(\lambda_4 - \lambda_2^2)} < \infty.$

Proof.

1. The denominator of $F(t)$ is equal to $\big(1 - r^2(t)\big)\mathrm{Var}\big(X'(0)|X(0), X(t)\big)$ thus non zero due to the non degeneracy hypothesis. A direct Taylor expansion gives the value of $F(0)$.
2. The expression of $F'(t)$ below shows that $F'(0) = 0$:

$$F'(t) = \frac{2\lambda_2\big(1 - r(t)\big)r'(t)\Big(r'^2(t) - \big(\lambda_2 - r''(t)\big)\big(1 - r(t)\big)\Big)}{\Big(\lambda_2\big(1 - r^2(t)\big) - r'^2(t)\Big)^2} \cdot \tag{4}$$

3. A Taylor expansion of (4) provides the value of $F''(0)$. Note that $\lambda_4 - \lambda_2^2$ can vanish only if there exists some real ω such that $\mu(\{-\omega\}) = \mu(\{\omega\}) = 1/2$. Similarly, $\lambda_2\lambda_6 - \lambda_4^2$ can vanish only if there exists some

real ω and $p \geq 0$ such that $\mu(\{-\omega\}) = \mu(\{\omega\}) = p$, $\mu(\{0\}) = 1 - 2p$. These cases are excluded by the non degeneracy hypothesis. □

We will say that the function F satisfies hypothesis (H) if it has a unique minimum at $t = 0$. The next proposition contains some sufficient conditions for this to take place.

Proposition 1. *(a) If $r'(t) < 0$ for $0 < t \leq T$ then (H) is satisfied.*
(b) Suppose that X is defined on the whole line and that

1. $\lambda_4 > 2\lambda_2^2$;
2. $r(t), r'(t) \to 0$ *as* $t \to \infty$;
3. *there exists no local maximum of $r(t)$ (other than at $t = 0$) with value greater or equal to* $\dfrac{\lambda_4 - 2\lambda_2^2}{\lambda_4}$.

Then (H) is satisfied for every $T > 0$.

An example of a process satisfying condition (b) but not condition (a) is given by the covariance

$$r(t) := \frac{1 + \cos(\omega t)}{2} \mathrm{e}^{-t^2/2}$$

if we choose ω sufficiently small. In fact, a direct computation gives $\lambda_2 = 1 + \omega^2/2$; $\lambda_4 = 3 + 3\omega^2 + \omega^4/2$ so that

$$\frac{\lambda_4 - 2\lambda_2^2}{\lambda_4} = \frac{1 + \omega^2}{3 + 3\omega^2 + \omega^4/2} \cdot$$

On $[0, \infty)$, the covariance attains its second largest local maximum in the interval $\left[\frac{\pi}{\omega}, \frac{2\pi}{\omega}\right]$, so that its value is smaller than $\exp\left(-\frac{\pi^2}{2\omega^2}\right)$. Hence, choosing ω is sufficiently small the last condition in (b) is satisfied.

The main result of this article is the following:

Theorem 1. *If the process X satisfies hypothesis (H), then (3) holds true.*

2. Proofs

Notations.

- $p_\xi(x)$ is the density (when it exists) of the random variable ξ at the point $x \in \mathbb{R}^n$.
- $\mathbb{1}_C$ denotes the indicator function of the event C.
- $U_u([a, b])$, $u \in \mathbb{R}$ is the number of upscrossings on the interval $[a, b]$ of the level u by the process X defined as follows:

$$U_u([a, b]) = \#\{t \in [a, b], X(t) = u, X'(t) > 0\} \cdot$$

- For k a positive integer, $\nu_k(u, [a, b])$ is the kth order factorial moment of $U_u([a, b])$

$$\nu_k(u, [a, b]) = \mathbb{E}\left(\big(U_u([a, b])\big)\big(U_u([a, b]) - 1\big) \ldots \big(U_u([a, b]) - k + 1\big)\right).$$

We define also

$$\bar{\nu}_k(u, [a, b]) = \mathbb{E}\left(\big(U_u([a, b])\big)\big(U_u([a, b]) - 1\big) \ldots \big(U_u([a, b]) - k + 1\big)\mathbb{1}_{\{X(0) \geq u\}}\right).$$

- $a^+ = a \vee 0$ denotes the positive part of the real number a.
- (const) denotes a positive constant whose value may vary from one occurrence to another.

We will repeatedly use the following lemma, that can be obtained using a direct generalization of Laplace's Method (see Dieudonné [9], p. 122).

Lemma 2. *Let f (respectively g) be a real-valued function of class C^2 (respectively C^k for some integer $k \geq 1$) defined on the interval $[0, T]$ of the real line verifying the conditions:*

1. *f has a unique minimum on $[0, T]$ at the point $t = t^*$, and $f'(t^*) = 0$, $f''(t^*) > 0$.*
2. *Let $k = \inf \{j : g^{(j)}(t^*) \neq 0\}$.*

Define

$$h(u) = \int_0^T g(t) \ \exp\left[-\frac{1}{2}u^2 f(t)\right] \ \mathrm{d}t.$$

Then, as $u \to \infty$:

$$h(u) \simeq \left(\frac{g^{(k)}(t^*)}{k!} \int_J x^k \ \exp\left[-\frac{1}{4}f''(t^*)x^2\right] \ \mathrm{d}x\right) \frac{1}{u^{k+1}} \exp\left[-\frac{1}{2}u^2 f(t^*)\right], \tag{5}$$

where $J = [0, +\infty)$, $J = (-\infty, 0]$ or $J = (-\infty, +\infty)$ according as $t^* = 0$, $t^* = T$ or $0 < t^* < T$ respectively.

We will use the following well-known expansion as (Abramovitz and Stegun [1], p. 932). For each $a_0 > 0$ as $u \to +\infty$

$$\int_u^\infty \exp\left[-\frac{1}{2}ay^2\right] \mathrm{d}y = \left(\frac{1}{au} - \frac{1}{a^2u^3} + \frac{3}{a^3u^5} + \mathcal{O}\left(\frac{1}{u^7}\right)\right) \exp\left[-\frac{1}{2}au^2\right], \tag{6}$$

for all $a \geq a_0$ where $\mathcal{O}\left(\frac{1}{u^7}\right)$ should be interpreted as bounded by $\frac{K}{u^7}$, K a constant depending only on a_0.

Proof of of Theorem 1.

Step 1: The proof is based on an extension of Piterbarg's result to intervals of any length. Let $\tau > 0$, the following relation is clear

$$\begin{aligned} P(M_{[0,\tau]} > u) &= P(X(0) > u) + P(U_u([0,\tau]).1\!\!1_{\{X(0) \leq u\}} \geq 1) \\ &= 1 - \Phi(u) + P(U_u([0,\tau]) \geq 1) - P(U_u([0,\tau]).1\!\!1_{\{X(0) > u\}} \geq 1). \end{aligned}$$

In the sequel a term will be called negligible if it is $\mathcal{O}\left(u^{-6} \exp\left(-\frac{1}{2}\frac{\lambda_4 u^2}{\lambda_4 - \lambda_2^2}\right)\right)$ as $u \to +\infty$. We use the following relations to be proved later:

(i) $P(U_u([0,T]).1\!\!1_{\{X(0)>u\}} \geq 1)$ is negligible;
(ii) Let $2\tau \leq T$. Then $P\left(\{U_u([0,\tau])U_u([\tau,2\tau]) \geq 1\}\right)$ is negligible.

With these relations, for $2\tau \leq T$, we have

$$\begin{aligned} P(M_{[0,2\tau]} > u) - \big(1 - \Phi(u)\big) &= P(U_u([0,2\tau]) \geq 1) + N_1 \\ &= P(U_u([0,\tau]) \geq 1) + P(U_u([\tau,2\tau]) \geq 1) + N_2 = 2P(U_u([0,\tau]) \geq 1) + N_3 \\ &= 2P(M_{[0,\tau]} > u) - 2\big(1 - \Phi(u)\big) + N_4, \end{aligned} \tag{7}$$

$N_1 \cdots N_4$ being negligible. Applying (7) repeatedly and on account of Piterbarg's theorem that states that (3) is valid if T is small enough, one gets the result.

Step 2: Proof of (i). Using Markov's inequality:

$$P(U_u([0,T]).1\!\!1_{\{X(0)>u\}} \geq 1) \leq \bar{\nu}_1(u, [0,T]),$$

where $\bar{\nu}_1$ is evaluated using the Rice formula (Cramér and Leadbetter [7])

$$\bar{\nu}_1(u,[0,T]) = \int_u^{+\infty} \mathrm{d}x \int_0^T \mathbb{E}\big(X'^{+}(t)|X(0)=x, X(t)=u\big) p_{X(0),X(t)}(x,u)\mathrm{d}t. \tag{8}$$

Also if Z is a real-valued random variable with a Normal-(m,σ^2) distribution,

$$\mathbb{E}(Z^+) = \sigma\phi\left(\frac{m}{\sigma}\right) + m\Phi\left(\frac{m}{\sigma}\right),$$

and plugging into (8) one obtains (see details in Azaïs *et al.* [5]):

$$\begin{aligned}\bar{\nu}_1(u,[0,T]) = \frac{\phi(u)}{2\pi}\int_0^T \mathrm{d}t \Bigg(\sqrt{\lambda_2 F}\int_u^\infty \mathrm{e}^{-\frac{1}{2}Fy^2}\mathrm{d}y \\ - \frac{r'^2\sqrt{F}}{\sqrt{\lambda_2}(1-r^2)}\exp\left[-\frac{(1-r)u^2}{2(1+r)}\right]\int_u^\infty \exp\left[-\frac{r'^2Fy^2}{2\lambda_2(1-r^2)}\right]\Bigg) = \int_0^T B(t,u)\mathrm{d}t,\end{aligned}$$

where r, r' and F stand for $r(t)$, $r'(t)$ and $F(t)$ respectively. Clearly, since $r''(0) = -\lambda_2 < 0$, there exists T_0 such that $r' < 0$ on $(0,T_0]$. Divide the integral into two parts: $[0,T_0]$ and $[T_0,T]$. Using formula (6) on $[0,T_0]$ we get

$$B(t,u) = \frac{\phi(u)}{2\pi}\sqrt{\lambda_2}F^{-5/2}\frac{\lambda_2(1-r)^2}{r'^2}u^{-3} + \mathcal{O}\big(u^{-5}\phi(u)\big),$$

and since, as $t\to 0$, $(1-r)^2r'^{-2} = \mathcal{O}(t^2)$, Lemma 2 shows that

$$\int_0^{T_0} B(t,u)\mathrm{d}t = \mathcal{O}\left(u^{-6}\exp\left(-\frac{\lambda_4u^2}{2(\lambda_4-\lambda_2^2)}\right)\right).$$

On the other hand, since $\inf_{t\in[T_0,T]} F(t)$ is strictly larger than $F(0)$, it follows easily from

$$\int_u^\infty \exp\left(-\frac{ay^2}{2}\right)\mathrm{d}y \le (\text{const})\frac{1}{\sqrt{a}}\exp\left(-\frac{au^2}{2}\right) \quad a>0,\ u\ge 0,$$

that

$$\int_{T_0}^T B(t,u)\mathrm{d}t$$

is negligible.

Step 3: Proof of (ii). Use once more Markov's inequality:

$$P\left(U_u([0,\tau])U_u([\tau,2\tau]) \ge 1\ \right) \le \mathbb{E}\left(U_u([0,\tau])U_u([\tau,2\tau])\right).$$

Because of Rice formula (Cramér and Leadbetter [7]):

$$\mathbb{E}\left(U_u([0,\tau])U_u([\tau,2\tau])\right) = \int_0^\tau\int_\tau^{2\tau} A_{t_2-t_1}(u)\mathrm{d}t_2\mathrm{d}t_1 = \int_0^{2\tau}(t\wedge(2\tau-t))A_t(u)\mathrm{d}t, \tag{9}$$

with

$$A_t(u) = E\Big(X'^{+}(0)X'^{+}(t)|X(0)=X(t)=u\Big)p_{X(0),X(t)}(u,u).$$

It is proved in Azaïs *et al.* [5], that

$$A_t(u) = \frac{1}{\sqrt{1-r^2}}\phi^2\left(\frac{u}{\sqrt{1+r}}\right)\left[T_1(t,u)+T_2(t,u)+T_3(t,u)\right], \tag{10}$$

with

- $T_1(t,u) = \sigma\sqrt{1+\rho}\ \phi(b)\phi(kb)$;
- $T_2(t,u) = 2(\sigma^2\rho - \mu^2)\int_b^{+\infty}\Psi(kx)\phi(x)\mathrm{d}x$;
- $T_3(t,u) = 2\mu\sigma\Psi(kb)\phi(b)$;
- $\mu = \mu(t,u) = \mathbb{E}\big(X'(0)|X(0)=X(t)=u\big) = -\frac{r'}{1+r}u$;
- $\sigma^2 = \sigma^2(t) = \mathrm{Var}\big(X'(0)|X(0),X(t)\big) = \lambda_2 - \frac{r'^2}{1-r^2}$;
- $\rho = \rho(t) = \mathrm{Cor}\big(X'(0),X'(t)|X(0),X(t)\big) = \frac{-r''(1-r^2)-rr'^2}{\lambda_2(1-r^2)-r'^2}$;
- $k = k(t) = \sqrt{\frac{1+\rho}{1-\rho}}$; $b = b(t,u) = \mu/\sigma$;
- $\Psi(z) = \int_0^z\phi(v)\mathrm{d}v$;
- r, r', r'' again stand for $r(t), r'(t), r''(t)$.

As in Step 2, we divide the integral (9) into two parts: $[0,T_0]$ and $[T_0,2\tau]$. For $t<T_0$, $b(t,u)$ and $k(t)$ are positive, thus using expansion (6), we get the formula p. 119 of Azaïs *et al.* [5]:

$$\begin{aligned} T_1(u)+T_2(u)+T_3(u) = {} & \frac{2\sigma^2}{b}\Psi(kb)(1+\rho)\phi(b) - \frac{4\sigma^2}{b^3}\Psi(kb)\phi(b) - (1+\rho)\sigma^2k\phi(kb)\phi(b) \\ & + 2\sigma^2k^3\phi(kb)\phi(b) + \frac{4\sigma^2}{b^2}k\phi(kb)\phi(b) + \mathcal{O}\left(\sigma^2\phi(kb)\phi(b)\left(\frac{1}{b^7}+\frac{k}{b^6}+\frac{k^3}{b^4}\right)\right)\cdot \end{aligned}$$

Since $T_1(u)+T_2(u)+T_3(u)$ is non negative, majorizing $\phi(kb)$ and $\Psi(kb)$ by 1 we get

$$A_t(u) \le (\mathrm{const})\frac{\sigma^2}{\sqrt{1-r^2(t)}}\left(\frac{(1+\rho)}{b}+k^3+\frac{k}{b^2}+\frac{1}{b^7}+\frac{k}{b^6}+\frac{k^3}{b^4}\right)\exp\left[-\frac{1}{2}\big(1+F(t)\big)u^2\right]\cdot$$

Now it is easy to see that, as $t\to 0$

$$\sigma^2 \simeq (\mathrm{const})t^2,\ (1+\rho)\simeq(\mathrm{const})t^2,\ \sqrt{1-r^2(t)}\simeq(\mathrm{const})t,\ b\simeq(\mathrm{const})u,$$

so that

$$\begin{aligned} \frac{\sigma^2(1+\rho)}{b\sqrt{1-r^2(t)}} &\simeq (\mathrm{const})\frac{t^3}{u}; \\ \frac{\sigma^2k^3}{\sqrt{1-r^2(t)}} &\simeq (\mathrm{const})t^4; \\ \frac{\sigma^2k}{b^2\sqrt{1-r^2(t)}} &\simeq (\mathrm{const})t^2u^{-2}; \end{aligned}$$

and also that the other terms are negligible. Then, applying Lemma 2:

$$\int_0^{T_0} (t \wedge (2\tau - t)) A_t(u) \mathrm{d}t \simeq (\mathrm{const}) u^{-6} \exp\left(-\frac{\lambda_4 u^2}{2(\lambda_4 - \lambda_2^2)}\right),$$

thus negligible.

For $t \geq T_0$ remark that $T_1(u) + T_2(u) + T_3(u)$ does not change when μ (and consequently b) changes of sign. Thus μ and b can supposed to be non-negative. Forgetting negative terms in formula (10) and majorizing Ψ by 1; $1 - \Phi(b)$ by (const)$\phi(b)$ and μ by (const)u, we get:

$$A_t(u) \leq (\mathrm{const}) \phi^2 \left(\frac{u}{\sqrt{1+r}}\right) \big(\phi(b)(1+u)\big) = (\mathrm{const})(1+u) \exp\left[-\frac{1}{2}\big(1 + F(t)\big) u^2\right] \cdot$$

We conclude as in Step 2. □

Proof of of Proposition 1. Let us prove statement (a). The expression (4) of F' shows that it is positive for $0 < t \leq T$, since $r'(t) < 0$ and

$$\big(r'^2(t) - (\lambda_2 - r''(t))(1 - r(t))\big) = -\frac{1}{4}\Big[\mathrm{Var}\big(X(t) - X(0)\big)\mathrm{Var}\big(X'(t) + X'(0)\big) - \mathrm{Cov}^2\Big(X(t) - X(0), X'(t) + X'(0)\Big)\Big] < 0. \tag{11}$$

Thus the minimum is attained at zero.

(b) Note that $F(0) = \dfrac{\lambda_2^2}{\lambda_4 - \lambda_2^2} < 1 = F(+\infty)$. If F has a local minimum at $t = t^*$, equation (4) shows that r has a local maximum at $t = t^*$ so that

$$F(t^*) = \frac{1 - r(t^*)}{1 + r(t^*)} > \frac{\lambda_2^2}{\lambda_4 - \lambda_2^2}$$

due to the last condition in (b). This proves (b). □

Remark. The proofs above show that even if hypothesis (H) is not satisfied, it is still possible to improve inequality (2). In fact it remains true for every ρ such that

$$\rho < \min_{t \in [0,T]} F(t).$$

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