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EXTREMES OF γ-REFLECTED GAUSSIAN PROCESSES WITH STATIONARY INCREMENTS

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Abstract. For a given centered Gaussian process with stationary increments $X(t), t \geq 0$ and $c > 0$, let $W_\gamma(t) = X(t) - ct - \gamma \inf_{0\leq s\leq t} (X(s) - cs)$, $t \geq 0$ denote the γ-reflected process, where $\gamma \in (0,1)$. This process is important for both queueing and risk theory. In this contribution we are concerned with the asymptotics, as $u \to \infty$, of $\mathbb{P}\left(\sup_{0\leq t\leq T} W_\gamma(t) > u\right)$, $T \in (0,\infty]$. Moreover, we investigate the approximations of first and last passage times for given large threshold u. We apply our findings to the cases with X being the multiplex fractional Brownian motion and the Gaussian integrated process. As a by-product we derive an extension of Piterbarg inequality for threshold-dependent random fields.

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1. Introduction

The seminal contribution [1] derived the exact asymptotics, as the initial capital u tends to infinity, of the ruin probability

$$\psi_{0,\infty}(u) = \mathbb{P}\left(\sup_{t\geq 0} W_0(t) > u\right), \quad W_0(t) := X(t) - ct, c > 0$$

for some general centered Gaussian processes $X(t), t \geq 0$. A key merit of the aforementioned paper is that it paved the way for the study of the tail asymptotics of supremum of Gaussian processes with trend over unbounded intervals. With a strong impetus from [1] a wide range of asymptotic results for supremum of such threshold dependent families of Gaussian processes were obtained in [2–9].

This paper is devoted to the analysis of extremes of γ-reflected processes W_γ, defined as

$$W_\gamma(t) = X(t) - ct - \gamma \inf_{0\leq s\leq t} (X(s) - cs), \quad \gamma \in [0,1),$$

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where X is a centered Gaussian process with stationary increments and $c > 0$. The analysis of γ-reflected processes is of interest for both queueing and risk theory. In risk theory γ is related to a fixed tax-payment rate, with

$$\psi_{\gamma,\infty}(u) = \mathbb{P}\left(\inf_{0\le t<\infty}\Big(u - W_\gamma(t)\Big) < 0\right) = \mathbb{P}\left(\sup_{0\le t<\infty} W_\gamma(t) > u\right) \tag{1.1}$$

representing the infinite-time ruin probability with initial capital u, see *e.g.*, [10]. For $\gamma = 1$, W_1 has also interpretation as a transient queue length process in a fluid queueing system fed by X and emptied with constant rate $c > 0$, see *e.g.*, [11–14].

More importantly, investigation of extremes of such processes is related to investigation of extremes of Gaussian random fields with interesting structures as already shown in [15]. Therein the asymptotics of (1.1) for $X = B_H$ a fractional Brownian motion with Hurst index $H \in (0,1)$ has been investigated. Using the self-similarity of B_H, for any $u > 0$ and $X = B_H$ we have

$$\begin{aligned}\psi_{\gamma,\infty}(u) &= \mathbb{P}\left(\sup_{t\ge 0}\left(X(t) - ct - \gamma \inf_{s\in[0,t]}(X(s) - cs)\right) > u\right)\\ &= \mathbb{P}\left(\sup_{0\le s\le t<\infty} \frac{X(tu) - \gamma X(su)}{1 + c(t-\gamma s)} > u\right)\\ &= \mathbb{P}\left(\sup_{0\le s\le t<\infty} Y(s,t) > u^{1-H}\right),\end{aligned} \tag{1.2}$$

where $Y(s,t) := \frac{X(t)-\gamma X(s)}{1+ct-c\gamma s}$. In view of (1.2) it is clear that for X being an fBm, the approximation of $\psi_{\gamma,\infty}(u)$ as $u \to \infty$ is closely related to the study of supremum of the Gaussian random field Y. The fact that Y does not depend on the threshold u is crucial and leads to substantial simplifications of the problem at hand. However, for a general centered Gaussian process X with stationary increments, due to the lack of self-similarity, one has to analyse the tail behaviour of threshold-dependent random field

$$Y_u(s,t) = \frac{X(tu) - \gamma X(su)}{1 + ct - c\gamma s}, \quad s,t \in [0,\infty), \tag{1.3}$$

which significantly increases the complexity of the problem due to the explicit dependence on the threshold u. We overcome this difficulty by deriving extensions of two classical results in extreme value theory of Gaussian processes. In particular, Lemma 5.3 provides a uniform (with respect to local behavior of variance-covariance structure of family of processes X_u) version of the celebrated Pickands-Piterbarg lemma, as given in, *e.g.*, Theorem D.2 in [16]. Lemma 5.1 extends Piterbarg inequality to threshold-dependent Gaussian random fields. The generality of these findings makes them also applicable to other problems related with extremes of threshold-dependent families of Gaussian random fields.

Under some conditions on the variance function σ^2, assuming in particular that it is regularly varying with index $2\alpha_0$ and $2\alpha_\infty$ at 0 and ∞, respectively, our main result presented in Theorem 2.1 below gives an asymptotic expansion of $\psi_{\gamma,\infty}(u)$ as $u \to \infty$. It turns out that three different types of asymptotics of $\psi_{\gamma,\infty}(u)$ take place, mainly determined by the following limit (which we assume to exist)

$$\varphi := \lim_{u\to\infty} \frac{\sigma^2(u)}{u} \in [0,\infty], \tag{1.4}$$

where $\sigma^2(t) = Var(X(t))$. Interestingly, this trichotomy is tightly related with the dependence structure of X. For example, if $X = B_H$, we can distinguish the case of $\varphi \in (0,\infty)$, *i.e.*, X is a standard Brownian motion, $\varphi = 0$ if $H \in (0,1/2)$ which is the well-known case of *short range dependent* fBm and $\varphi = \infty$ corresponding to $H \in (1/2,1]$, *i.e.*, the case of *long range dependent* fBm.

Comparing our findings with those obtained for $\gamma = 0$ in [4], using $\sim$ to denote the asymptotic equivalence, we obtain the following *asymptotic tax equivalence* (derived for $X = B_H$ in [15])

$$\psi_{\gamma,\infty}(u) \sim \mathcal{P}_{V_\varphi}^{\overline{\gamma}} \psi_{0,\infty}(u), \quad \overline{\gamma} := (1-\gamma)/\gamma, \quad \gamma \in (0,1) \tag{1.5}$$

as $u \to \infty$, with

$$V_\varphi = \frac{\sqrt{2}c\gamma}{\varphi} X, \quad \text{if } \varphi \in (0,\infty), \quad V_\varphi = B_{\alpha_\varphi}, \text{ if } \varphi \in \{0,\infty\}. \tag{1.6}$$

In our notation

$$\mathcal{P}_Z^a = \mathbb{E}\left\{ \sup_{t\in[0,\infty)} \mathrm{e}^{\sqrt{2}Z(t)-(1+a)\mathrm{Var}(Z(t))} \right\}, \quad a > 0$$

denotes the generalised Piterbarg constant, where Z is a centered Gaussian process with stationary increments and continuous sample paths. Note in passing that by Theorem 1.1 in [17] a.s. continuity of Z at each $t \in [0,S]$ is equivalent to the sample-continuous assumption above. Further, the constants $\mathcal{P}_{B_H}^a$, with B_H a standard fBm, are known only for

$$\mathcal{P}_{B_{1/2}}^a = 1 + \frac{1}{a} \quad \text{and} \quad \mathcal{P}_{B_1}^a = \frac{1}{2}\left(1 + \sqrt{1+\frac{1}{a}}\right), \tag{1.7}$$

see *e.g.*, [16, 18, 19]. For general $H \in (0,1)$, bounds for $\mathcal{P}_{B_H}^a$ are derived in [19, 20].

The asymptotics in (1.5) shows that the generalised Piterbarg constant governs the relation between the two ruin probabilities corresponding to the model with tax and without tax, *i.e.*, it defines what we call *the asymptotic tax equivalence*. However, in view of [21, 22] we know that for the case $X = B_H$, the tax rate γ does not influence the limiting distribution of the first and the last passage times. We investigate these problems in more general models for X. Define therefore the first and last passage times of W_γ given that the ruin occurs by

$$(\tau_1^*(u), \tau_2^*(u)) \stackrel{d}{=} (\tau_1(u), \tau_2(u)) \Big| (\tau_1(u) < \infty), \tag{1.8}$$

where

$$\tau_1(u) = \inf\{t \geq 0, W_\gamma(t) > u\} \quad \text{and} \quad \tau_2(u) = \sup\{t \geq 0, W_\gamma(t) > u\},$$

with the convention that $\inf\{\emptyset\} = \infty$ and $\sup\{\emptyset\} = 0$. Here $\stackrel{d}{=}$ stands for equality of the distribution functions.

Complementary, in this contribution we address also finite-time horizon counterparts of the introduced above problems. Namely

$$\psi_{\gamma,T}(u) := \mathbb{P}\left(\sup_{0\leq t\leq T} W_\gamma(t) > u \right) \tag{1.9}$$

for any finite $T > 0$ is analysed, extending partial results on $\psi_{0,T}$ given in [23]. Moreover, we shall deal also with the approximation of the conditional first passage time

$$\tau_1(u) \big| (\tau_1(u) < T)$$

as $u \to \infty$ (see Thm. 2.5), which shows that the approximating random variable is exponentially distributed.

The family of Gaussian processes X with stationary increments, considered in this contribution, covers general classes such as

(A) Multiplex fBm model, *i.e.*,

$$X(t) = \sum_{i=1}^{n} B_{H_i}(t), \quad t \geq 0,$$

with B_{H_i}'s being independent fBm's;

(B) Gaussian integrated process model, that is the case where $X(t) = \int_0^t Y(s)ds, t \geq 0$ with Y being a centered stationary Gaussian process with a.s. continuous sample paths.

Organization of the paper: In Section 2 we present some preliminaries, followed by the main results for the approximation of $\psi_{\gamma,T}(u), T \in (0, \infty]$, the approximating joint distribution for conditional scaled first and last passage times for $T \in (0, \infty]$. Section 3 is dedicated to applications related to model (A) and (B) mentioned above. For reader's convenience, we postpone all the proofs to Section 4; whereas some very technical claims are presented in Appendix.

2. Main Results

In the rest of this paper $X(t), t \geq 0$ is a centered Gaussian process with stationary increments, a.s. continuous sample paths and variance function $\sigma^2(t)$. An canonical example is $X = B_H, H \in (0, 1]$ for which we have $\sigma^2(t) = t^{2H}$. For a given centered Gaussian process Z with a.s. continuous sample paths set

$$\mathcal{H}_Z[0, S] = \mathbb{E}\left\{\sup_{t\in[0,S]} \mathrm{e}^{\sqrt{2}Z(t)-\mathrm{Var}(Z(t))}\right\}$$

and define (whenever the limit exits) the generalised Pickands constant $\mathcal{H}_Z$ by

$$\mathcal{H}_Z = \lim_{S\to\infty} S^{-1}\mathcal{H}_Z[0, S].$$

See [2, 4, 16, 24–38] for various definitions, existence and basic properties of Pickands constant.

2.1. Infinite-time horizon

First we focus on the infinite-time horizon case. Due to the stationarity of increments, the covariance of X is directly defined by σ^2, therefore our assumptions on X shall be reduced to assumptions on the variance function, namely:

AI : $\sigma^2(0) = 0$ and $\sigma^2(t)$ is regularly varying at ∞ with index $2\alpha_\infty \in (0, 2)$. Further, $\sigma^2(t)$ is twice continuously differentiable on $(0, \infty)$ with its first derivative $\dot{\sigma}^2(t) := \frac{\mathrm{d}\sigma^2}{\mathrm{d}t}(t)$ and second derivative $\ddot{\sigma}^2(t) := \frac{\mathrm{d}^2\sigma^2}{\mathrm{d}t^2}(t)$ being ultimately monotone at ∞.

AII : $\sigma^2(t)$ is regularly varying at 0 with index $2\alpha_0 \in (0, 2]$ and its first derivative $\dot{\sigma}^2(t)$ is ultimately monotone as $t \to 0$.

AIII : $\sigma^2(t)$ is increasing and $\frac{\sigma^2(t)}{t^2}$ is decreasing over $(0, \infty)$.

Define φ by (1.4) assuming that the limit exists. For notational simplicity we set

$$t_* = \frac{\alpha_\infty}{c(1 - \alpha_\infty)} > 0$$

and

$$\Delta_\gamma(u) = \begin{cases} \overleftarrow{\sigma}\left(\dfrac{\sqrt{2}\sigma^2(ut_*)}{\gamma u(1 + ct_*)}\right), & \text{if} \quad \varphi = \infty \ \textit{or} \ 0, \\ 1, & \text{if} \quad \varphi \in (0, \infty), \end{cases} \tag{2.1}$$

where $\overleftarrow{\sigma}$ is the asymptotic inverse of σ (see *e.g.*, [39, 40] for details).

Let t_u be a maximizer of $\frac{\sigma(ut)}{1+ct}$ over $t \geq 0$. In view of Lemma 4.1 for u large enough t_u is unique and

$$\lim_{u\to\infty} t_u = t_* \in (0,\infty).$$

Hereafter Ψ stands for the survival function of an $N(0,1)$ random variable.
Before stating our main result, let us observe that

$$\psi_{\gamma,\infty}(u) = \mathbb{P}\left(\sup_{0\leq s\leq t<\infty} \frac{X(tu)-\gamma X(su)}{u(1+ct-c\gamma s)} > 1\right)$$

is valid for any $u>0$. Typically the most likely point to reach high value u for a centered Gaussian random field corresponds to the point that maximizes its variance function, *i.e.*, in our case

$$(s_u,t_u) := \mathrm{argsup}_{(s,t):\ 0\leq s\leq t<\infty} \mathrm{Var}\left(\frac{X(tu)-\gamma X(su)}{u(1+ct-c\gamma s)}\right).$$

It will be shown in Lemma 4.1 that $s_u=0$ for u large and thus $t_u = \mathrm{argsup}_{t:\ t\geq 0}\frac{\sigma(ut)}{u(1+ct)}$. This explains the exponential term in the derived asymptotics. The following theorem extends results derived in [15], where the special case $X=B_H$ is considered.

Theorem 2.1. *If* **AI-AIII** *are satisfied, then for any* $\gamma\in(0,1)$ *and* $\varphi\in[0,\infty]$ *we have*

$$\psi_{\gamma,\infty}(u) \sim \frac{1}{c}\sqrt{\frac{2\alpha_\infty\pi}{1-\alpha_\infty}}\mathcal{H}_{V_\varphi}\mathcal{P}_{V_\varphi}^{\overline{\gamma}}\frac{\sigma(ut_*)}{\Delta_1(u)}\Psi\left(\frac{u(1+ct_u)}{\sigma(ut_u)}\right),$$

with $V_\varphi = \frac{\sqrt{2}c}{\varphi}X$ *if* $\varphi\in(0,\infty)$, $V_\varphi = B_{\alpha_\varphi}$ *if* $\varphi\in\{0,\infty\}$ *and* $\overline{\gamma} := (1-\gamma)/\gamma$.

An immediate application of the above theorem, together with the known results in [4] for the case $\gamma=0$, yields that, as $u\to\infty$

$$\psi_{\gamma,\infty}(u) \sim \mathcal{P}_{V_\varphi}^{\overline{\gamma}}\psi_{0,\infty}(u).$$

The above asymptotic tax equivalence shows that $\psi_{\gamma,\infty}(u)$ is proportional to $\psi_{0,\infty}(u)$ as $u\to\infty$, where the proportionality constant is determined by the generalised Piterbarg constant $\mathcal{P}_{V_\varphi}^{\overline{\gamma}}$.

Theorem 2.2. *If* **AI-AIII** *are satisfied, then for any* $\gamma\in(0,1)$ *and* $\varphi\in[0,\infty]$ *we have the convergence in distribution*

$$\left(\frac{\tau_1^*-ut_u}{A(u)},\frac{\tau_2^*-ut_u}{A(u)}\right) \xrightarrow{d} (\mathcal{N},\mathcal{N}),\quad u\to\infty,$$

where $A(u)=\frac{\sigma(ut_*)}{c}\sqrt{\frac{\alpha_\infty}{1-\alpha_\infty}}$ *and* $\mathcal{N}\sim N(0,1)$.

The above result implies that the standardized conditional first pasage time $\frac{\tau_1^*-ut_u}{A(u)}$ and last passage time $\frac{\tau_2^*-ut_u}{A(u)}$ both weakly converge to standard normal random variables and $\frac{\tau_2^*(u)-\tau_1^*(u)}{A(u)}\to 0$ in probability as $u\to\infty$.

2.2. Finite-time horizon

Next, we consider the finite-time horizon ruin probability, investigating $\psi_{\gamma,T}$ for T a finite positive constant. Since we consider the finite-time horizon, we shall impose weaker assumptions on the variance function σ^2, namely:

BI : $\sigma^2(0) = 0$ and $\sigma^2(t)$ is twice differentiable over interval $(0, T]$.
BII : $\sigma^2(t)$ is regularly varying at 0 with index $2\alpha_0 \in (0, 2]$.
BIII : For $t \in (0, T]$, the first derivative $\dot{\sigma^2}(t) > 0$ and $\frac{\sigma^2(t)}{t^2}$ is decreasing.

For notational simplicity we set below

$$q(u) = \overleftarrow{\sigma}\left(\frac{\sqrt{2}\sigma^2(T)}{u + cT}\right).$$

Theorem 2.3. *Suppose that* **BI–BIII** *hold and* $\gamma \in (0, 1)$.

(i) *If* $s = o(\sigma^2(s))$ *as* $s \to 0$*, then*

$$\psi_{\gamma,T}(u) \sim \mathcal{H}_{B_{\alpha_0}} \mathcal{P}_{B_{\alpha_0}}^{\overline{\gamma}} \frac{2\sigma^4(T)}{\dot{\sigma^2}(T) q(u) u^2} \Psi\left(\frac{u + cT}{\sigma(T)}\right).$$

(ii) *If* $\lim_{s\to 0} \frac{\sigma^2(s)}{s} = b \in (0, \infty)$*, then*

$$\psi_{\gamma,T}(u) \sim \mathcal{P}_{B_{1/2}}^{\frac{\dot{\sigma^2}(T)}{b}} \mathcal{P}_{B_{1/2}}^{\beta(b,\gamma)} \Psi\left(\frac{u + cT}{\sigma(T)}\right), \quad \beta(b, \gamma) := \frac{b(\gamma - \gamma^2) + \gamma\dot{\sigma^2}(T)}{b\gamma^2}.$$

(iii) *If* $\sigma^2(s) = o(s)$ *as* $s \to 0$*, then*

$$\psi_{\gamma,T}(u) \sim \Psi\left(\frac{u + cT}{\sigma(T)}\right).$$

Remarks 2.4.

(i) From the proof of Theorem 2.3, we can similarly get the asymptotics of $\psi_{0,T}(u)$ (see also [23]), which compared with $\psi_{\gamma,T}(u), \gamma \in (0, 1)$, gives

$$\psi_{\gamma,T}(u) \sim \mathcal{K}\psi_{0,T}(u), \quad u \to \infty,$$

with

$$\mathcal{K} = \begin{cases} \mathcal{P}_{B_{\alpha_0}}^{\overline{\gamma}}, & if\, s = o(\sigma^2(s)), \\ \mathcal{P}_{B_{1/2}}^{\beta(b,\gamma)}, & \text{if} \quad \lim_{s\to 0} \frac{\sigma^2(s)}{s} = b \in (0, \infty), \\ 1, & if\, \sigma^2(s) = o(s). \end{cases}$$

(ii) The approach used in the proofs of Theorem 2.1 and Theorem 2.3 enables us to find exact asymptotics of $\psi_{\gamma,T_u}(u)$ as $u \to \infty$, for some scenarios where the time-horizon T_u is a deterministic function of u. For example, if $ut_u = o(T_u)$ as $u \to \infty$, then by the proof of Theorem 2.1 we have $\psi_{\gamma,T_u}(u) \sim \psi_{\gamma,\infty}(u)$, $u \to \infty$. Additionally, if $T_u \to T$ as $u \to \infty$, then the asymptotics of $\psi_{\gamma,T_u}(u)$ can be obtained by replacing T_u with T in the corresponding formulas of Theorem 2.3. On the other side, the case $T_u \sim t^*u$ as $u \to \infty$, is out of the approach given in this paper. We suspect that it leads to the asymptotics of qualitatively other type than derived in Theorems 2.1, 2.3.

Next we consider a finite-time counterpart of Theorem 2.2. While for the infinite-time horizon the limit distribution in Theorem 2.2 is Gaussian, as shown below, this is not the case for finite-time horizon, where the limit distribution is exponential. The intuitive explanation for this is that the local behaviour of variance function of the considered Gaussian field in neighbourhood of the variance maximizer plays the key role for the type of the limit distribution. In particular, if the first derivative of the variance function is positive at this point, then the limiting distribution is exponential, while the first derivative equal to 0 at that point leads to limit with Normal distribution; compare Lemma 4.1 with Lemma 4.3.

Theorem 2.5. *If* **BI–BIII** *are satisfied and* $\lim_{s\to 0}\frac{\sigma^2(s)}{s}\in[0,\infty]$*, then the convergence in distribution*

$$\frac{\dot{\sigma^2}(T)}{2\sigma^4(T)}u^2(T-\tau_1)\big|(\tau_1\le T)\overset{d}{\to}\mathcal{E}$$

holds, as $u\to\infty$*, with* $\mathcal{E}$ *a unit exponential random variable.*

3. Applications

In this section, we shall focus on two important classes of processes with stationary increments. We discuss first the sum of independent fBm's with different Hurst parameters and then investigate Gaussian integrated processes.

3.1. Multiplex fBm

Let next $B_{H_i}, 1\le i\le n$ be independent standard fBm's with index $0<H_1<H_2\le\cdots\le H_{n-1}<H_n<1$ and define for $t\ge 0$

$$X(t)=M_{\boldsymbol{H}}(t):=\sum_{i=1}^n B_{H_i}(t),\quad \boldsymbol{H}=(H_1,\cdots,H_n). \tag{3.1}$$

A motivation to consider such a process stems from the insurance models with tax, where B_{H_i} represents the aggregated claims of the sub-portfolios of the insurance company. We have that

$$\sigma^2(t)=\sigma^2_{M_{\boldsymbol{H}}}(t)=\sum_{i=1}^n t^{2H_i}$$

satisfies **AI-AIII** with $\alpha_0=H_1$, $\alpha_\infty=H_n$. Further,

$$\varphi=\begin{cases}\infty,\ 1/2<H_n<1,\\ 1,\quad\ H_n=1/2,\\ 0,\ \ 0<H_n<1/2\end{cases}$$

implying the following result:

Corollary 3.1. *Suppose that* X *is defined by* (3.1).

(i) *If* $0<H_n<1/2$*, then*

$$\psi_{\gamma,\infty}(u)\sim\mathcal{H}_{B_{H_1}}\mathcal{P}^{\overline{\gamma}}_{B_{H_1}}2^{\frac{H_1-1}{2H_1}}\sqrt{\pi}c^{\frac{2H_n-H_1H_n-H_1}{H_1}}H_n^{\frac{H_1-4H_n+2H_1H_n}{2H_1}}(1-H_n)^{\frac{4H_n-H_1-2H_1H_n-2}{2H_1}}$$
$$\times u^{\frac{H_1H_n-2H_n+1}{H_1}}\Psi\left(\inf_{t>0}\frac{u(1+ct)}{\sigma_{M_{\boldsymbol{H}}}(ut)}\right).$$

(ii) *If $H_n = 1/2$, then*

$$\psi_{\gamma,\infty}(u) \sim \mathcal{H}_{\sqrt{2}cM_{\boldsymbol{H}}} \mathcal{P}^{\overline{\gamma}}_{\sqrt{2}c\gamma M_{\boldsymbol{H}}} \sqrt{\frac{2\pi u}{c^3}} \Psi\left(\inf_{t>0} \frac{u(1+ct)}{\sigma_{M_{\boldsymbol{H}}}(ut)}\right).$$

(iii) *If $1/2 < H_n < 1$, then*

$$\psi_{\gamma,\infty}(u) \sim \mathcal{H}_{B_{H_n}} \mathcal{P}^{\overline{\gamma}}_{B_{H_n}} 2^{\frac{H_n-1}{2H_n}} \sqrt{\pi} c^{1-H_n} H_n^{\frac{2H_n-3}{2}} (1-H_n)^{\frac{3H_n-2-2H_n^2}{2H_n}} u^{\frac{(1-H_n)^2}{H_n}} \Psi\left(\inf_{t>0} \frac{u(1+ct)}{\sigma_{M_{\boldsymbol{H}}}(ut)}\right).$$

Moreover, since **BI-BIII** are satisfied for $M_{\boldsymbol{H}}(t)$, we obtain for any $T > 0$.

Corollary 3.2. *Suppose that X is defined by (3.1).*

(i) *If $0 < H_1 < 1/2$, then*

$$\psi_{\gamma,T}(u) \sim \mathcal{H}_{B_{H_1}} \mathcal{P}^{\overline{\gamma}}_{B_{H_1}} 2^{-\frac{1}{2H_1}} \frac{\left(\sum_{i=1}^n T^{2H_i}\right)^{\frac{2H_1-1}{H_1}}}{\sum_{i=1}^n H_i T^{2H_i-1}} u^{\frac{1-2H_1}{H_1}} \Psi\left(\frac{u+cT}{\sqrt{\sum_{i=1}^n T^{2H_i}}}\right).$$

(ii) *If $H_1 = 1/2$, then*

$$\psi_{\gamma,T}(u) \sim \mathcal{P}_{B_{1/2}}^{2\sum_{i=1}^n H_i T^{2H_i-1}} \mathcal{P}_{B_{1/2}}^{\frac{\gamma-\gamma^2+2\gamma\sum_{i=1}^n H_i T^{2H_i-1}}{\gamma^2}} \Psi\left(\frac{u+cT}{\sqrt{\sum_{i=1}^n T^{2H_i}}}\right).$$

(iii) *If $1/2 < H_1 < 1$, then*

$$\psi_{\gamma,T}(u) \sim \Psi\left(\frac{u+cT}{\sqrt{\sum_{i=1}^n T^{2H_i}}}\right).$$

Remark 3.3. In the above corollaries, the main contribution to the asymptotics depends on all H_i's while the polynomial terms depend on the properties of variance function at time 0 and ∞ which is determined by Hurst parameters H_1 and H_n. It follows from the fact that the formula under $\Phi(\cdot)$ comes from global optimum of the variance function of the appropriate Gaussian field, while the polynomial part of the asymptotics follows from the asymptotic relation between local behavior of variance and correlation in the neighbourhood of the variance optimizer.

3.2. Gaussian integrated processes

Suppose that

$$X(t) = \int_0^t Y(s)ds, t \geq 0, \tag{3.2}$$

where Y is a stationary centered Gaussian process with unit variance and a.s. continuous sample paths.

Let $R(t) = Cov\left(Y(s), Y(s+t)\right), s, t \geq 0$. In this subsection, we shall consider two scenarios:

SRD (short-range dependent), *i.e.*, we shall assume that

(i) $R(t)$ is decreasing over $[0, \infty)$,
(ii) $\int_0^\infty R(t)dt = G \in (0, \infty)$.

LRD (long-range dependent), *i.e.*, we shall suppose that

(i) $R(t)$ is decreasing over $[0, \infty)$,

(ii) $R(t)$ is regularly varying at infinity with index $2H-2$, $H \in (1/2, 1)$. It follows that **AI-AIII** are satisfied if X is **SRD** or **LRD**, implying our next results.

Corollary 3.4. *Suppose that X is defined by* (3.2).

(i) *If X is* **SRD**, *then*

$$\psi_{\gamma,\infty}(u) \sim \mathcal{H}_{\frac{\sqrt{2}c}{G}X} \mathcal{P}^{\overline{\gamma}}_{\frac{\sqrt{2}\gamma c}{G}X} \sqrt{\frac{2\pi G u}{c^3}} \Psi\left(\inf_{t>0} \frac{u(1+ct)}{\sigma(ut)}\right).$$

(ii) *If X is* **LRD**, *then*

$$\psi_{\gamma,\infty}(u) \sim \mathcal{H}_{B_H} \mathcal{P}^{\overline{\gamma}}_{B_H} 2^{\frac{H-1}{2H}} \sqrt{\pi} c^{1-H} H^{\frac{1-4H+2H^2}{2H}} (1-H)^{\frac{3H-2-2H^2}{2H}} (2H-1)^{\frac{1-H}{2H}} \frac{u\sqrt{R(u)}}{\mathcal{R}^*(u)} \Psi\left(\inf_{t>0} \frac{u(1+ct)}{\sigma(ut)}\right),$$

with $\mathcal{R}^*$ the asymptotic inverse function of $u\sqrt{R(u)}$.

Since, **BI-BIII** are satisfied (note that $\sigma^2(t) \sim t^2 = o(t)$ as $t \to 0$) for $R(t)$ positive and decreasing on $[0, T]$, applying Theorem 2.3 we arrive at the following corollary.

Corollary 3.5. *If X is defined by* (3.2) *with $R(t)$ positive and decreasing on $[0, T]$, then*

$$\psi_{\gamma,T}(u) \sim \Psi\left(\frac{u+cT}{\sigma(T)}\right), \quad u \to \infty.$$

4. Proofs

We begin with introduction of some useful notation. Namely we write

$$D := \{(s,t) : 0 \le s \le t < \infty\}, \quad \sigma_\gamma^2(s,t) := \mathrm{Var}(X(t) - \gamma X(s)),$$

$$\sigma_{\gamma,u}(s,t) := \frac{\sigma_\gamma(us, ut)}{1 + c(t - \gamma s)}$$

and set further for $(s,t), (s_1, t_1) \in D$

$$r_u(s,t,s_1,t_1) := Cor(X(ut) - \gamma X(us), X(ut_1) - \gamma X(us_1)).$$

Hereafter, Q, $Q_i, i = 1, 2, \ldots$ are positive constants that may change from line to line. For any non-zero random variable X we shall define

$$\overline{X} := \frac{X}{\sqrt{\mathrm{Var}(X)}}.$$

In our proofs multiple limits appear; the order when passing to limit is important. We shall write for instance

$$a_u(S, S_1) \to 0, \quad u \to \infty, S \to \infty, S_1 \to \infty$$

to mean that

$$\lim_{S_1 \to \infty} \lim_{S \to \infty} \lim_{u \to \infty} a_u(S, S_1) = 0.$$

This convention applies for other instances of double or triple limits.

We briefly comment on some useful properties of σ. For $\lambda \in \mathbb{R}$, by **AI** and **AII**, the function

$$g_\lambda(t) := \frac{\sigma^2(t)}{t^\lambda} \tag{4.1}$$

is regularly varying at 0 with index $2\alpha_0 - \lambda$ and at infinity with index $2\alpha_\infty - \lambda$.

Further, by uniform convergence theorem (UCT) in [40–42], we have that for any $T > 0$ and $0 < \lambda < \min(2\alpha_0, 2\alpha_\infty)$

$$\lim_{u\to 0} \sup_{t\in(0,T]} \left| \frac{g_\lambda(ut)}{g_\lambda(u)} - |t|^{2\alpha_0-\lambda} \right| = 0$$

implying that for any $T > 0$, when u is sufficiently small

$$\frac{\sigma^2(ut)}{\sigma^2(u)} = \frac{g_\lambda(ut)}{g_\lambda(u)} |t|^\lambda \le 2|T|^{2\alpha_0-\lambda}|t|^\lambda, \quad t \in [0,T]. \tag{4.2}$$

Moreover, Potter's bounds (see *e.g.*, [40–42]) show that for any $0 < \epsilon < 2\alpha_0$, there exists $T > 0$ and $Q_1, Q_2 > 0$ such that for all $0 < s,t < T$

$$Q_1 \min\left(\left(\frac{t}{s}\right)^{2\alpha_0-\epsilon}, \left(\frac{t}{s}\right)^{2\alpha_0+\epsilon}\right) \le \frac{\sigma^2(t)}{\sigma^2(s)} \le Q_2 \max\left(\left(\frac{t}{s}\right)^{2\alpha_0-\epsilon}, \left(\frac{t}{s}\right)^{2\alpha_0+\epsilon}\right). \tag{4.3}$$

4.1. Proof of Theorem 2.1

First, we re-write for any $u > 0$ the ruin probability $\psi_{\gamma,\infty}(u)$ as

$$\begin{aligned}
\psi_{\gamma,\infty}(u) &= \mathbb{P}\left(\sup_{t\ge 0}\left(X(t) - ct - \gamma \inf_{s\in[0,t]}(X(s) - cs)\right) > u\right)\\
&= \mathbb{P}\left(\sup_{0\le s\le t<\infty} \frac{X(tu) - \gamma X(su)}{1 + c(t-\gamma s)} > u\right)\\
&= \mathbb{P}\left(\sup_{(s,t)\in D} Z_u(s,t) > m(u)\right),
\end{aligned}$$

with

$$Z_u(s,t) = \left(\frac{X(ut) - \gamma X(us)}{1 + c(t-\gamma s)}\right)\left(\frac{1 + ct_u}{\sigma(ut_u)}\right), \quad (s,t) \in D, u > 0, \tag{4.4}$$

and

$$m(u) = \inf_{t\ge 0} \frac{u(1+ct)}{\sigma(ut)} = \frac{u(1+ct_u)}{\sigma(ut_u)}, \quad u > 0. \tag{4.5}$$

Hereafter we shall denote

$$E(u) := E_1(u) \times E_2(u),\ E_1(u) = \left[0, \frac{\overleftarrow{\sigma}\left(u^{-1}\sigma^2(u)\ln u\right)}{u}\right),\ E_2(u) = \left(t_u - \frac{\sigma(u)\ln u}{u}, t_u + \frac{\sigma(u)\ln u}{u}\right)\cdot \tag{4.6}$$

As it will be shown below, the set $E(u)$ covers sufficiently large neighbourhood of the maximizer of variance of Z_u in order to determine the asymptotics of $\psi_{\gamma,\infty}(u)$ by supremum of $Z_u(s,t)$ over $E(u)$. More formally, for any $u > 0$ we write

$$\Theta(u) \le \psi_{\gamma,\infty}(u) \le \Theta(u) + \Theta_0(u), \tag{4.7}$$

with

$$\Theta(u) = \mathbb{P}\left(\sup_{(s,t)\in E(u)} Z_u(s,t) > m(u)\right), \quad \Theta_0(u) = \mathbb{P}\left(\sup_{(s,t)\in D\setminus E(u)} Z_u(s,t) > m(u)\right).$$

The strategy of the proof is to derive first the exact asymptotics of $\Theta(u)$ as $u \to \infty$ and then to show that (recall (4.7)) that $\lim_{u\to\infty} \Theta_0(u)/\Theta(u) = 0$.

Before proceeding to details of these steps of the proof, we summarize some dependence properties of the analyzed Gaussian field which will be needed in our proofs.

4.1.1. Dependence structure of Z_u

Proofs of the following lemmas are deferred to Appendix.

Lemma 4.1. *If the variance function σ^2 of X satisfies* **AI-AII**, *then for u large enough, the unique maximizer of $\sigma_{\gamma,u}(s,t)$ over D is $(0,t_u)$ and $\lim_{u\to\infty} t_u = t_* \in (0,\infty)$. Moreover, for any $0<\epsilon<\min(a_1,a_2)$, when u is large enough and δ is small enough*

$$(a_1-\epsilon)(t-t_u)^2+(a_2-\epsilon)\frac{\sigma^2(us)}{\sigma^2(u)} \le 1-\frac{\sigma_{\gamma,u}(s,t)}{\sigma_{\gamma,u}(0,t_u)} \le (a_1+\epsilon)(t-t_u)^2+(a_2+\epsilon)\frac{\sigma^2(us)}{\sigma^2(u)}, \quad |t-t_u|<\delta, 0\le s<\delta,$$

with

$$a_1 =: \frac{c^2(1-\alpha_\infty)^3}{2\alpha_\infty}, \quad a_2 =: \frac{\gamma(1-\gamma)}{2}\left[\frac{c(1-\alpha_\infty)}{\alpha_\infty}\right]^{2\alpha_\infty}.$$

Lemma 4.2. *If* **AI-AIII** *are satisfied and $\delta_u>0, u>0$ are such that $\lim_{u\to\infty}\delta_u=0$, then we have*

$$\lim_{u\to\infty} \sup_{(s,t)\neq(s_1,t_1)\in[0,\delta_u)\times(t_u-\delta_u,t_u+\delta_u)} \left| \frac{1-r_u(s,t,s_1,t_1)}{\frac{\sigma^2(u|t-t_1|)+\gamma^2\sigma^2(u|s-s_1|)}{2\sigma^2(ut_*)}} - 1 \right| = 0.$$

4.1.2. Asymptotic upper bound for $\Theta_0(u)$

For notational simplicity we define next (recall that $D=\{(s,t): 0\le s\le t<\infty\}$)

$$D_T=\{(s,t): 0\le s\le t\le T\}, \quad D_T^c = D\setminus D_T, \quad D_{\delta,u}=D_T\setminus([0,\delta]\times[t_u-\delta,t_u+\delta])$$

and

$$D_{\delta,u}^* = ([0,\delta]\times[t_u-\delta,t_u+\delta])\setminus E(u).$$

For any $u>0$

$$\begin{aligned}
&\mathbb{P}\left(\sup_{(s,t)\in D\setminus E(u)} Z_u(s,t)>m(u)\right)\\
&\le \mathbb{P}\left(\sup_{(s,t)\in D_T^c} Z_u(s,t)>m(u)\right)+\mathbb{P}\left(\sup_{(s,t)\in D_{\delta,u}} Z_u(s,t)>m(u)\right)+\mathbb{P}\left(\sup_{(s,t)\in D_{\delta,u}^*} Z_u(s,t)>m(u)\right)\\
&:= p_1(u)+p_2(u)+p_3(u).
\end{aligned}$$

Lemma 5.1 leads to

$$p_i(u) = o\left(\frac{u}{m(u)\Delta_1(u)}\Psi(m(u))\right), \quad i=1,2,3 \tag{4.8}$$

implying that

$$\Theta_0(u) = o\left(\frac{u}{m(u)\Delta_1(u)}\Psi(m(u))\right), \quad u\to\infty. \tag{4.9}$$

Since the proof of (4.8) is quite technical, we shall present it in Appendix.

4.1.3. Asymptotics of $\Theta(u)$

We shall distinguish three scenarios: $\varphi = 0$, $\varphi \in (0,\infty)$ and $\varphi = \infty$. The reason for this is that after rescaling the time of the correlation function in Lemma 4.2, we get

$$m^2(u)\left(1 - r_u\left(\frac{\Delta_\gamma(u)s}{u}, \frac{\Delta_1(u)t}{u}, \frac{\Delta_\gamma(u)s_1}{u}, \frac{\Delta_1(u)t_1}{u}\right)\right) \sim \frac{\sigma^2(\Delta_1(u)|t-t_1|)}{\sigma^2(\Delta_1(u))} + \frac{\sigma^2(\Delta_\gamma(u)|s-s_1|)}{\sigma^2(\Delta_\gamma(u))}\cdot \quad (4.10)$$

If $\varphi = 0$, then $\lim_{u\to\infty} \Delta_\gamma(u) = 0$ for $\gamma \in (0,1]$, implying that only the local behaviour of σ^2 at 0 contributes to the limit in (4.10). If $\varphi \in (0,\infty)$, then $\lim_{u\to\infty} \Delta_\gamma(u) \in (0,\infty)$, indicating that the whole function σ^2 determines the limit in (4.10). If $\varphi = \infty$, then $\lim_{u\to\infty} \Delta_\gamma(u) = \infty$, which means that the value of $\sigma^2(t)$ as $t \to \infty$ is sufficient for the limit in (4.10).

Case $\varphi = 0$. We shall apply the *uniform double sum* technique which is based on appropriate division of the set $E(u)$ on "tiny" intervals for which one can uniformly derive exact asymptotics utilising our novel result in Lemma 5.3 in Appendix. For this purpose we define

$$F_{k,S}(u) = \left[t_u + k\frac{\Delta_1(u)}{u}S, t_u + (k+1)\frac{\Delta_1(u)}{u}S\right], \quad k \in \mathbb{Z}, S > 0$$

$$L_{l,S}(u) = \left[l\frac{\Delta_\gamma(u)}{u}S, (l+1)\frac{\Delta_\gamma(u)}{u}S\right], \quad l \in \mathbb{N}\cup\{0\}, S > 0$$

and set

$$I_{k,l,S,S_1}(u) = L_{l,S_1}(u) \times F_{k,S}(u), \quad I_k(u) := I_{k,0,S,S_1}, \quad (4.11)$$

with $k \in \mathbb{N}, l \in \mathbb{N}\cup\{0\}, S, S_1 > 0$. Recall that, due to (2.1), $\Delta_\gamma(u) = \overleftarrow{\sigma}\left(\frac{\sqrt{2}\sigma^2(ut_*)}{\gamma u(1+ct_*)}\right)$. Further, let

$$N_{S,u} = \left[\frac{\sigma(u)\ln u}{\Delta_1(u)S}\right] + 1, \quad N^{(1)}_{S_1,u} = \left[\frac{\overleftarrow{\sigma}\left(u^{-1}\sigma^2(u)\ln u\right)}{\Delta_\gamma(u)S_1}\right] + 1 \quad (4.12)$$

and put

$$\mathbb{V}_1 = \{(k,k_1), -N_{S,u} \le k < k_1 \le N_{S,u}, |k-k_1| > 1\},$$
$$\mathbb{V}_2 = \{(k,k_1), -N_{S,u} \le k < k_1 \le N_{S,u}, k+1 = k_1\}.$$

We begin with the derivation of an upper estimate for $\Theta(u)$, as $u \to \infty$.

Upper bound of $\Theta(u)$. Bonferroni inequality yields

$$\begin{aligned}\Theta(u) \le & \sum_{k=-N_{S,u}}^{N_{S,u}} \mathbb{P}\left(\sup_{(s,t)\in I_k(u)} Z_u(s,t) > m(u)\right) \\ & + \sum_{k=-N_{S,u}}^{N_{S,u}} \sum_{l=1}^{N^{(1)}_{S_1,u}} \mathbb{P}\left(\sup_{(s,t)\in I_{k,l,S,S_1}(u)} Z_u(s,t) > m(u)\right) \\ := & \Theta_1(u) + \Theta_2(u). \end{aligned} \quad (4.13)$$

In light of Lemma 4.1 for u large enough

$$\Theta_1(u) \le \sum_{k=-N_{S,u}}^{N_{S,u}} \mathbb{P}\left(\sup_{(s,t)\in I_k(u)} \frac{\overline{Z}_u(s,t)}{1+(a_2-\epsilon)\frac{\sigma^2(us)}{\sigma^2(u)}} > m^{-\epsilon}_{k,0}(u)\right), \quad (4.14)$$

with $\varepsilon \in (0, \min(a_1, a_2))$ and

$$m_{k,0}^{\pm\epsilon}(u) = m(u)\left(1 + (a_1 - \epsilon)\left(k^* \frac{\Delta_1(u)}{u} S\right)^2\right), \quad k^* = \min(|k|, |k+1|).$$

In order to derive an upper bound for $\Theta_1(u)$, we apply Lemma 5.3 in Appendix, which gives uniform asymptotics for all terms in (4.14). For this purpose, let

$$g_{u,k} = m_{k,0}^{-\epsilon}(u), \quad \xi_{u,k} = \frac{Z_{u,k}(s,t)}{1 + f_{u,k}(s,t)}, \quad (s,t) \in E = [0, S_1] \times [0, S], \tag{4.15}$$

with $\quad k \in K_u = \{k : -N_{S,u} \le k \le N_{S,u}\}$, where

$$Z_{u,k}(s,t) = \overline{Z}_u\left(\frac{\Delta_\gamma(u)}{u} s, t_{u,k} + \frac{\Delta_1(u)}{u} t\right), \quad f_{u,k}(s) = (a_2 - \epsilon)\frac{\sigma^2(\Delta_\gamma(u)s)}{\sigma^2(u)}, \quad s \in [0, S_1]$$

and for $u > 0$

$$t_{u,k} = t_u + k\frac{\Delta_1(u)}{u} S.$$

We check that the conditions of Lemma 5.3 hold with the above introduced notation. We start off with proving that **P1**–**P3** (see Appendix) hold with

$$V(s,t) = B_{\alpha_0}(s) + B_{\alpha_0}^*(t), \quad (s,t) \in [0, S_1] \times [0, S],$$

where B_{α_0} and $B_{\alpha_0}^*$ are independent fBm's with index α_0. It is straightforward that condition **P1** holds. For **P2**, by Lemma 4.2 and the fact that

$$g_{u,k} \sim m(u), \quad u \to \infty$$

uniformly with respect to $k \in K_u$, we have that for all $k \in K_u$ and $(s,t), (s_1, t_1) \in E$, as $u \to \infty$

$$\begin{aligned}
(g_{u,k})^2 \mathrm{Var}\left(Z_{u,k}(s,t) - Z_{u,k}(s_1,t_1)\right) &= 2(g_{u,k})^2\left(1 - r_u\left(\frac{\Delta_\gamma(u)}{u}s, \frac{\Delta_1(u)}{u}t, \frac{\Delta_\gamma(u)}{u}s_1, \frac{\Delta_1(u)}{u}t_1\right)\right)\\
&\sim (g_{u,k})^2 \frac{\sigma^2(\Delta_1(u)|t - t_1|) + \gamma^2\sigma^2(\Delta_\gamma(u)|s - s_1|)}{\sigma^2(ut^*)}\\
&\sim 2\left(\frac{\sigma^2(\Delta_1(u)|t - t_1|)}{\sigma^2(\Delta_1(u))} + \frac{\sigma^2(\Delta_\gamma(u)|s - s_1|)}{\sigma^2(\Delta_\gamma(u))}\right),
\end{aligned}$$

implying that we can set

$$\theta_{u,k}(s,t,s_1,t_1) = \frac{\sigma^2(\Delta_1(u)|t - t_1|)}{\sigma^2(\Delta_1(u))} + \frac{\sigma^2(\Delta_\gamma(u)|s - s_1|)}{\sigma^2(\Delta_\gamma(u))}, \quad (s,t), (s_1,t_1) \in E, k \in K_u. \tag{4.16}$$

Moreover, since

$$\lim_{u\to\infty} \Delta_\gamma(u) = 0, \quad \gamma \in (0,1]$$

by UCT

$$\begin{aligned}
&\lim_{u\to\infty} \sup_{k\in K_u} \sup_{(s,t),(s_1,t_1)\in E} \left|\theta_{u,k}(s,t,s_1,t_1) - |s - s_1|^{2\alpha_0} - |t - t_1|^{2\alpha_0}\right|\\
&= \lim_{u\to\infty} \sup_{(s,t),(s_1,t_1)\in E} \left|\frac{\sigma^2(\Delta_\gamma(u)|s - s_1|)}{\sigma^2(\Delta_\gamma(u))} + \frac{\sigma^2(\Delta_1(u)|t - t_1|)}{\sigma^2(\Delta_1(u))} - |s - s_1|^{2\alpha_\infty} - |t - t_1|^{2\alpha_0}\right| = 0.
\end{aligned}$$

This means that **P2** holds. For **P3**, by (4.2) we have that for u sufficiently large

$$\theta_{u,k}(s,t,s_1,t_1)) = \frac{\sigma^2(\varDelta_1(u)|t-t_1|)}{\sigma^2(\varDelta_1(u))} + \frac{\sigma^2(\varDelta_\gamma(u)|s-s_1|)}{\sigma^2(\varDelta_\gamma(u))} \le 2\left(S^{2\alpha_0-\lambda} + S_1^{2\alpha_0-\lambda}\right)\left(|s-s_1|^\lambda + |t-t_1|^\lambda\right)$$

for $(s,t),(s_1,t_1)\in E$ and all $k\in K_u$ with $0<\lambda<\min(2\alpha_0, 2\alpha_\infty)$. By UCT, we have for all $(s,t),(s_1,t_1)\in E$

$$\begin{aligned}
&\sup_{|(s,t)-(s_1,t_1)|<\epsilon} |\theta_{u,k}(s,t,0,0)-\theta_{u,k}(s_1,t_1,0,0)|\\
&\le \sup_{|(s,t)-(s_1,t_1)|<\epsilon} \left|\frac{\sigma^2(\varDelta_1(u)t)-\sigma^2(\varDelta_1(u)t_1)}{\sigma^2(\varDelta_1(u))} + \frac{\sigma^2(\varDelta_\gamma(u)s)-\sigma^2(\varDelta_\gamma(u)s_1)}{\sigma^2(\varDelta_\gamma(u))} - (t^{2\alpha_0}-t_1^{2\alpha_0}+s^{2\alpha_0}-s_1^{2\alpha_0})\right|\\
&\quad + \sup_{|(s,t)-(s_1,t_1)|<\epsilon} |t^{2\alpha_0}-t_1^{2\alpha_0}+s^{2\alpha_0}-s_1^{2\alpha_0}|\\
&\le 2\epsilon + \sup_{|(s,t)-(s_1,t_1)|<\epsilon} |t^{2\alpha_0}-t_1^{2\alpha_0}+s^{2\alpha_0}-s_1^{2\alpha_0}| \le \mathbb{C}\epsilon^{\alpha_0}, \quad u\to\infty, \qquad (4.17)
\end{aligned}$$

with $\mathbb{C}$ depending only on α_0 (but not on $k\in K_u$). Moreover, using UCT, we have for $(s,t),(s_1,t_1)\in E$, $|(s,t)-(s_1,t_1)|<\epsilon$ and $k\in K_u$

$$\begin{aligned}
\left|(g_{u,k})^2\left(1-r_u\left(s_{u,l}+\frac{\varDelta_\gamma(u)}{u}s, t_{u,k}+\frac{\varDelta_1(u)}{u}t, s_{u,l}, t_{u,k}\right)\right) - \theta_{u,k}(s,t,0,0)\right| &\le \epsilon|\theta_{u,k}(s,t,0,0)|\\
&\le 2(S^{2\alpha_0}+S_1^{2\alpha_0})\epsilon
\end{aligned}$$

for all u large. Consequently, as $u\to\infty$

$$\begin{aligned}
&(g_{u,k})^2\mathbb{E}\{[Z_{u,k}(s,t)-Z_{u,k}(s_1,t_1)]\,Z_{u,k}(0,0)\}\\
&\le \left|(g_{u,k})^2\left(1-r_u(s_{u,l}+\frac{\varDelta_\gamma(u)}{u}s, t_{u,k}+\frac{\varDelta_1(u)}{u}t, s_{u,l}, t_{u,k})\right) - \theta_{u,k}(s,t,0,0)\right|\\
&+\left|(g_{u,k})^2\left(1-r_u(s_{u,l}+\frac{\varDelta_\gamma(u)}{u}s_1, t_{u,k}+\frac{\varDelta_1(u)}{u}t_1, s_{u,l}, t_{u,k})\right) - \theta_{u,k}(s,t,0,0)\right|\\
&+|\theta_{u,k}(s,t,0,0)-\theta_{u,k}(s,t,0,0)|\\
&\le \mathbb{C}\epsilon^{\alpha_0} + 4(S^{2\alpha_0}+S_1^{2\alpha_0})\epsilon
\end{aligned}$$

uniformly for $(s,t),(s_1,t_1)\in E$, $|(s,t)-(s_1,t_1)|<\epsilon$ and $k\in K_u$. Letting $\epsilon\to 0$, we confirm that **P3** holds. Hence we can conclude that **P1**–**P3** hold with $V(s,t)=B_{\alpha_0}(s)+B^*_{\alpha_0}(t), (s,t)\in E$, where B_{α_0} and $B^*_{\alpha_0}$ are independent fBm's with index α_0. Therefore, by the fact that for all $\varepsilon>0$ sufficiently small (hereafter $\Rightarrow$ means uniform convergence)

$$g^2_{u,k}f_{u,k}(s) \Rightarrow \gamma_\epsilon s^{2\alpha_0}, s\in[0,S_1], \quad \text{with} \quad \gamma_\epsilon = \frac{a_2-\epsilon}{a_2}\overline{\gamma},$$

and Lemma 5.3 we have

$$\frac{\mathbb{P}\left(\sup_{(s,t)\in E}\xi_{u,k}(s,t)>g_{u,k}\right)}{\varPsi(g_{u,k})} \to \mathcal{R}_V^{\gamma_\epsilon s^{2\alpha_0}}(E) = \mathcal{H}_{B_{\alpha_0}}[0,S]\mathcal{P}^{\gamma_\epsilon}_{B_{\alpha_0}}[0,S_1], \quad u\to\infty \qquad (4.18)$$

uniformly with respect to $-N_{S,u} \le k \le N_{S,u}$. From (4.14) and (4.18) it follows that

$$\begin{aligned}\Theta_1(u) &\le \sum_{k=-N_{S,u}}^{N_{S,u}} \mathcal{H}_{B_{\alpha_0}}[0,S]\mathcal{P}_{B_{\alpha_0}}^{\gamma_\epsilon}[0,S_1]\Psi(m_{k,0}^{-\epsilon}(u))(1+o(1))\\ &\le \sum_{k=-N_{S,u}}^{N_{S,u}} \mathcal{H}_{B_{\alpha_0}}[0,S]\mathcal{P}_{B_{\alpha_0}}^{\gamma_\epsilon}[0,S_1]\Psi(m(u))\mathrm{e}^{-(a_1-\epsilon)\left(k^* m(u)\frac{\Delta_1(u)}{u}S\right)^2}(1+o(1))\\ &\le \frac{\mathcal{H}_{B_{\alpha_0}}[0,S]}{S}\mathcal{P}_{B_{\alpha_0}}^{\gamma_\epsilon}[0,S_1](a_1-\epsilon)^{-1/2}\Psi(m(u))\frac{u}{m(u)\Delta_1(u)}\int_{-\infty}^{\infty}\mathrm{e}^{-x^2}\mathrm{d}x(1+o(1))\\ &\sim (a_1-\epsilon)^{-1/2}\sqrt{\pi}\mathcal{H}_{B_{\alpha_0}}\mathcal{P}_{B_{\alpha_0}}^{\gamma_\epsilon}\Psi(m(u))\frac{u}{m(u)\Delta_1(u)},\end{aligned} \tag{4.19}$$

as $u\to\infty, S, S_1\to\infty$ (in this order).

Next, we deal with $\Theta_2(u)$. By UCT, for any $\epsilon>0$ sufficiently small

$$(a_2-\epsilon)\sup_{s\in E_1(u)}\frac{\sigma^2(us)}{\sigma^2(u)}\to 0,\quad u\to\infty.$$

Moreover, by (4.3) for u large enough

$$\begin{aligned}\inf_{s\in L_{l,S}(u)}\left(m_{k,0}^{-\epsilon}(u)\right)^2\frac{\sigma^2(us)}{\sigma^2(u)} &\ge \frac{1}{2}\inf_{s\in[lS_1,(l+1)S_1]}\frac{\sigma^2(\Delta_\gamma(u)s)}{\sigma^2(\Delta_\gamma(u))}\frac{\sigma^2(\Delta_\gamma(u))}{\sigma^2(u)}m^2(u)\\ &\ge Q\inf_{s\in[lS_1,(l+1)S_1]}\frac{\sigma^2(\Delta_\gamma(u)s)}{\sigma^2(\Delta_\gamma(u))}\\ &\ge Q(lS_1)^\lambda,\quad 1\le l\le N_{S_1,u}^{(1)}, 0<\lambda<\min(2\lambda_0,2\alpha_\infty).\end{aligned}$$

Consequently, by Lemma 4.1 and 5.3 (note that we can similarly show the validity of **P1**–**P3** for $\overline{Z}_u(s,t)$) we have for any $\epsilon>0$

$$\begin{aligned}\Theta_2(u) &\le \sum_{k=-N_{S,u}}^{N_{S,u}}\sum_{l=1}^{N_{S_1,u}^{(1)}}\mathbb{P}\left(\sup_{(s,t)\in I_{k,l,S,S_1}(u)}\overline{Z}_u(s,t)>m_{k,0}^{-\epsilon}(u)\left(1+(a_2-\epsilon)\inf_{s\in L_{l,S}(u)}\frac{\sigma^2(us)}{\sigma^2(u)}\right)\right)\\ &\le \sum_{k=-N_{S,u}}^{N_{S,u}}\sum_{l=1}^{N_{S_1,u}^{(1)}}\mathcal{H}_{B_{\alpha_0}}[0,S]\mathcal{H}_{B_{\alpha_0}}[0,S_1]\Psi(m_{k,0}^{-\epsilon}(u))\mathrm{e}^{-Q_1(lS_1)^\lambda}(1+o(1))\\ &\le \sum_{k=-N_{S,u}}^{N_{S,u}}\mathcal{H}_{B_{\alpha_0}}[0,S]\mathcal{H}_{B_{\alpha_0}}[0,S_1]\Psi(m_{k,0}^{-\epsilon}(u))\mathrm{e}^{-Q_2S_1^\lambda}(1+o(1))\\ &= o\left(\Psi(m(u))\frac{u}{m(u)\Delta_1(u)}\right),\end{aligned} \tag{4.20}$$

as $u\to\infty, S, S_1\to\infty$. Combining (4.19) and (4.20), and letting $\epsilon\to 0$, we derive the upper bound of $\Theta(u)$.

Lower bound of $\Theta(u)$. By Bonferroni inequality we obtain

$$\Theta(u)\ge\sum_{k=-N_{S,u}+1}^{N_{S,u}-1}\mathbb{P}\left(\sup_{(s,t)\in I_k(u)}Z_u(s,t)>m(u)\right)-\Sigma_1(u)-\Sigma_2(u):=J(u)-\Sigma_1(u)-\Sigma_2(u), \tag{4.21}$$

with

$$\varSigma_i(u) = \sum_{(k,k_1)\in\mathbb{V}_i} \mathbb{P}\left(\sup_{(s,t)\in I_k(u)} Z_u(s,t) > m(u), \sup_{(s,t)\in I_{k_1}(u)} Z_u(s,t) > m(u)\right), \quad i=1,2. \tag{4.22}$$

With similar arguments as in the derivation of (4.19) we have

$$J(u) \ge (a_1+\epsilon)^{-1/2}\sqrt{\pi}\mathcal{H}_{B_{\alpha_0}}\mathcal{P}^{\gamma_{-\epsilon}}_{B_{\alpha_0}}\varPsi(m(u))\frac{u}{m(u)\varDelta_1(u)}(1+o(1)), \quad u\to\infty, S, S_1\to\infty. \tag{4.23}$$

In light of Lemma 4.2 and (4.3) we have for $(s,t,s_1,t_1)\in I_k(u)\times I_{k_1}(u)$ with $(k,k_1)\in\mathbb{V}_1$

$$\begin{aligned} 2 \le \mathrm{Var}(\overline{Z}_u(s,t)+\overline{Z}_u(s_1,t_1)) &= 4-2(1-r_u(s,t,s_1,t_1))\\ &\le 4-\frac{\gamma^2\sigma^2(u|s-s_1|)+\sigma^2(u|t-t_1|)}{2\sigma^2(ut_*)}\\ &\le 4-\frac{1}{2m^2(u)}\frac{\sigma^2(\varDelta_1(u)|u(t-t_1)/\varDelta_1(u)|)}{\sigma^2(\varDelta_1(u))}\\ &\le 4-Q_3\frac{|k_1-k|^\lambda S^\lambda}{m^2(u)}, \end{aligned}$$

where $0<\lambda<\min(2\alpha_0,2\alpha_\infty)$, implying thus

$$\begin{aligned} \varSigma_1(u) &\le \sum_{(k,k_1)\in\mathbb{V}_1}\mathbb{P}\left(\sup_{(s,t)\in I_k(u)}\overline{Z}_u(s,t) > m^{-\epsilon}_{k,0}(u), \sup_{(s,t)\in I_{k_1}(u)}\overline{Z}_u(s,t) > m^{-\epsilon}_{k_1,0}(u)\right)\\ &\le \sum_{(k,k_1)\in\mathbb{V}_1}\mathbb{P}\left(\sup_{(s,t,s_1,t_1)\in I_k(u)\times I_{k_1}(u)}\left(\overline{Z}_u(s,t)+\overline{Z}_u(s_1,t_1)\right) > 2\widetilde{m}^{-\epsilon}_{k,k_1,0}(u)\right)\\ &\le \sum_{(k,k_1)\in\mathbb{V}_1}\mathbb{P}\left(\sup_{(s,t,s_1,t_1)\in I_k(u)\times I_{k_1}(u)}\left(\overline{\overline{Z}_u(s,t)+\overline{Z}_u(s_1,t_1)}\right) > \frac{2\widetilde{m}^{-\epsilon}_{k,k_1,0}(u)}{\sqrt{4-Q_3\frac{|k_1-k|^\lambda S^\lambda}{m^2(u)}}}\right), \end{aligned}$$

with $\widetilde{m}^{-\epsilon}_{k,k_1,0}(u) = \min(m^{-\epsilon}_{k,0}(u), m^{-\epsilon}_{k_1,0}(u))$.
Let next

$$r_u(t,s,t_1,s_1,t',s',t_1',s_1') = Cor(\overline{Z_u}(s,t)+\overline{Z_u}(s_1,t_1), \overline{Z_u}(s',t')+\overline{Z_u}(s_1',t_1')).$$

By Lemma 4.2 and (4.3), for u sufficiently large

$$\begin{aligned} 1-r_u(s,t,s_1,t_1,s',t',s_1',t_1') &\le \frac{\mathrm{Var}(\overline{Z_u}(s,t)+\overline{Z_u}(s_1,t_1)-\overline{Z_u}(s',t')-\overline{Z_u}(s_1',t_1'))}{2\sqrt{\mathrm{Var}(\overline{Z_u}(s,t)+\overline{Z_u}(s_1,t_1))}\sqrt{\mathrm{Var}(\overline{Z_u}(s',t')+\overline{Z_u}(s_1',t_1')}}\\ &\le 1-r_u(s,t,s',t')+1-r_u(s_1,t_1,s_1',t_1')\\ &\le \frac{2}{m^2(u)}\frac{\sigma^2(\varDelta_\gamma(u)|u(s-s_1)/\varDelta_\gamma(u)|)+\sigma^2(\varDelta_\gamma(u)|u(s'-s_1')/\varDelta_\gamma(u)|)}{\sigma^2(\varDelta_\gamma(u))}\\ &\quad+\frac{2}{m^2(u)}\frac{\sigma^2(\varDelta_1(u)|u(t-t_1)/\varDelta_1(u)|)+\sigma^2(\varDelta_1(u)|u(t'-t_1')/\varDelta_1(u)|)}{\sigma^2(\varDelta_1(u))}\\ &\le \frac{Q_4(S^*)^2}{m^2(u)}\left[\left(\frac{u}{\varDelta_\gamma(u)}\right)^\kappa\left(|s-s'|^\kappa+|s_1-s_1'|^\kappa\right)+\left(\frac{u}{\varDelta_1(u)}\right)^\kappa\left(|t-t'|^\kappa+|t_1-t_1'|^\kappa\right)\right] \end{aligned}$$

holds for (t,s,t_1,s_1), $(t',s',t_1^{'},s_1^{'}) \in I_k(u)\times I_{k_1}(u)$ with $0<\kappa<\min(2\alpha_\infty,2\alpha_0)$ and $S^*=\max(S,S_1)\ge 1$. Define the following homogeneous Gaussian field

$$X_u^*(s,t,s_1,t_1)=2^{-1}(X_u^1(s)+X_u^2(t)+X_u^3(s_1)+X_u^4(t_1)),\quad (s,t,s_1,t_1)\in\mathbb{R}^4,$$

with $X_u^i(s), 1\le i\le 4$, being independent with the correlation functions

$$r_u^{(i)}(s,s')=\mathrm{e}^{-8Q_4(S^*)^2\left(\frac{u}{\Delta_1(u)}\right)^\kappa\frac{|s-s'|^\kappa}{m^2(u)}},\quad i=1,3,$$

$$r_u^{(i)}(s,s')=\mathrm{e}^{-8Q_4(S^*)^2\left(\frac{u}{\Delta_\gamma(u)}\right)^\kappa\frac{|s-s'|^\kappa}{m^2(u)}},\quad i=2,4.$$

We denote the correlation function of X_u^* by r_u^*. Clearly, for (t,s,t_1,s_1), $(t',s',t_1^{'},s_1^{'})\in I_k(u)\times I_{k_1}(u)$ and u large enough

$$1-r_u(s,t,s_1,t_1,s',t',s_1^{'},t_1^{'})\le 1-r_u^*(s,t,s_1,t_1,s',t',s_1^{'},t_1^{'}).$$

In light of Slepian's inequality (see *e.g.*, Thm. 2.2.1 in [43]; note in passing that there is a remarkable extension of this inequality for stable processes, see [44]) and Lemma 5.3 we have

$$\begin{aligned}
\varSigma_1(u)&\le\sum_{(k,k_1)\in\mathbb{V}_1}\mathbb{P}\left(\sup_{(s,t,s_1,t_1)\in I_k(u)\times I_{k_1}(u)}X_u^*(s,t,s_1,t_1)>\frac{2\widetilde{m}_{k,k_1,0}^{-\epsilon}(u)}{\sqrt{4-Q_3\frac{|k_1-k|^\lambda S^\lambda}{m^2(u)}}}\right)\\
&\le\sum_{(k,k_1)\in\mathbb{V}_1}\left(\mathcal{H}_{B_{\kappa/2}}[0,S_2]\right)^2\left(\mathcal{H}_{B_{\kappa/2}}[0,S_3]\right)^2\varPsi\left(\frac{2\widetilde{m}_{k,k_1,0}^{-\epsilon}(u)}{\sqrt{4-Q_3\frac{|k_1-k|^\lambda S^\lambda}{m^2(u)}}}\right)(1+o(1))\\
&\le\sum_{k=-N_{S,u}}^{N_{S,u}}\left(\frac{\mathcal{H}_{B_{\kappa/2}}[0,S_2]}{S_2}\right)^2\left(\frac{\mathcal{H}_{B_{\kappa/2}}[0,S_3]}{S_3}\right)^2\varPsi(m_{k,0}^{-\epsilon}(u))S_2^{-2}S_3^{-2}\sum_{j\ge1}\mathrm{e}^{-Q_5(jS)^\lambda}(1+o(1))\\
&\le Q_6\varPsi(m(u))\frac{u}{m(u)\varDelta_1(u)}S_1^{-2}\mathrm{e}^{-Q_7S^\lambda}(1+o(1))\\
&=o\left(\varPsi(m(u))\frac{u}{m(u)\varDelta_1(u)}\right),
\end{aligned}\tag{4.24}$$

with $S_2=(2Q_4(S^*)^2)^{1/\kappa}S$ and $S_3=(2Q_4(S^*)^2)^{1/\kappa}S_1$, as $u\to\infty, S\to\infty$ (in this order). Further, we obtain

$$\begin{aligned}
\varSigma_2(u)&=\sum_{k=-N_{S,u}}^{N_{S,u}}\mathbb{P}\left(\sup_{(s,t)\in I_k(u)}Z_u(s,t)>m(u),\sup_{(s,t)\in I_{k+1}(u)}Z_u(s,t)>m(u)\right)\\
&\le\sum_{k=-N_{S,u}}^{N_{S,u}}\left[\mathbb{P}\left(\sup_{(s,t)\in I_k(u)}Z_u(s,t)>m(u)\right)+\mathbb{P}\left(\sup_{(s,t)\in I_{k+1}(u)}Z_u(s,t)>m(u)\right)\right.\\
&\qquad\left.-\mathbb{P}\left(\sup_{(s,t)\in I_k(u)\cup I_{k+1}(u)}Z_u(s,t)>m(u)\right)\right]\\
&\le\sum_{k=-N_{S,u}}^{N_{S,u}}\left(2\mathcal{H}_{B_{\alpha_0}}[0,S]-\mathcal{H}_{B_{\alpha_0}}[0,2S]\right)\mathcal{P}_{B_{\alpha_0}}^{\gamma_\epsilon}[0,S_1]\varPsi(\widetilde{m}_{k,k+1,0}^{-\epsilon}(u))(1+o(1))\\
&=o\left(\varPsi(m(u))\frac{u}{m(u)\varDelta_1(u)}\right)
\end{aligned}\tag{4.25}$$

as $u \to \infty, S_1 \to \infty, S \to \infty$. Combining (4.23)–(4.25) and letting $\epsilon \to 0$, the lower bound of $\Theta(u)$ is derived. Since the upper and lower bound coincide, then we have

$$\Theta(u) \sim \sqrt{\frac{\pi}{a_1}} \mathcal{H}_{B_{\alpha_0}} \mathcal{P}_{B_{\alpha_0}}^{\overline{\gamma}} \Psi(m(u)) \frac{u}{m(u)\Delta_1(u)}$$

and therefore the claim follows by (4.7) and (4.9)–

Case $\varphi \in (0, \infty)$. The main difference to the above proof is that $\Delta_\gamma(u) = 1$ and $\gamma \in (0, 1]$, which influences (4.16) and (4.18) and hence the resulting Pickands or Piterbarg constants that show up in the result. Therefore, in order to avoid repetitions, we present only the counterpart of the derivations of (4.16) and (4.18). Next, we check **P2**–**P3** (conditions **P1** is easy to verify) by using the same notation as in (4.15) and (4.16). In order to prove **P2**, in light of Lemma 4.2 and the fact that $g_{u,k} \sim m(u)$ as $u \to \infty$ uniformly with respect to $k \in K_u$, we have that for all $k \in K_u$ and $(s, t), (s_1, t_1) \in E$, as $u \to \infty$

$$\begin{aligned}
(g_{u,k})^2 \mathrm{Var}\left(Z_{u,k}(s,t) - Z_{u,k}(s_1,t_1)\right) &= 2(g_{u,k})^2 \left(1 - r_u\left(\frac{\Delta_\gamma(u)}{u}s, \frac{\Delta_1(u)}{u}t, \frac{\Delta_\gamma(u)}{u}s_1, \frac{\Delta_1(u)}{u}t_1\right)\right) \\
&\sim (m(u))^2 \frac{\sigma^2(\Delta_1(u)|t-t_1|) + \gamma^2\sigma^2(\Delta_\gamma(u)|s-s_1|)}{\sigma^2(ut^*)} \\
&\sim 2\left(\frac{2c^2\gamma^2}{\varphi^2}\sigma^2(|s-s_1) + \frac{2c^2}{\varphi^2}\sigma^2(|t-t_1|)\right) \cdot
\end{aligned}$$

Hence, we can set that

$$\theta_{u,k}(s,t,s_1,t_1) = \frac{2c^2\gamma^2}{\varphi^2}\sigma^2(|s-s_1) + \frac{2c^2}{\varphi^2}\sigma^2(|t-t_1|), \quad (s,t),(s_1,t_1) \in E, k \in K_u,$$

which ensures that **P2** holds. Next, for **P3**, in light of (4.3) we derive that for u sufficiently large and $\lambda \in (0, \min(2\alpha_0, 2\alpha_\infty))$,

$$\theta_{u,k}(s,t,s_1,t_1) = \frac{2c^2}{\varphi^2}|\sigma^2(|s-s_1|) + \sigma^2(|t-t_1|)| \le Q\left(|s-s_1|^\lambda + t - t_1|^\lambda\right),$$

with $k \in K_u, (s,t),(s_1,t_1) \in E$. In addition, for $(s,t),(s_1,t_1) \in E$, $|(s,t)-(s_1,t_1)| < \epsilon$, $k \in K_u$ and u sufficiently large we have

$$\begin{aligned}
&(g_{u,k})^2 \mathbb{E}\{[Z_{u,k,l}(s,t) - Z_{u,k,l}(s_1,t_1)]\, Z_{u,k,l}(0,0)\} \\
&\le \left|(g_{u,k})^2\left(1 - r_u(s_{u,l} + \frac{1}{u}s, t_{u,k} + \frac{1}{u}t, s_{u,l}, t_{u,k})\right) - \theta_{u,k}(s,t,0,0)\right| \\
&\quad + \left|(g_{u,k})^2\left(1 - r_u(s_{u,l} + \frac{1}{u}s_1, t_{u,k} + \frac{1}{u}t_1, s_{u,l}, t_{u,k})\right) - \theta_{u,k}(s_1,t_1,0,0)\right| \\
&\quad + |\theta_{u,k}(s,t,0,0) - \theta_{u,k}(s_1,t_1,0,0)| \\
&\le \epsilon|\theta_{u,k}(s,t,0,0) + \theta_{u,k}(s_1,t_1,0,0)| + |\theta_{u,k}(s,t,0,0) - \theta_{u,k}(s_1,t_1,0,0)| \\
&\le \mathbb{C}_2\left(\epsilon + \left|\sigma^2(t) - \sigma^2(t_1)\right| + \left|\sigma^2(s) - \sigma^2(s_1)\right|\right) \to 0, \quad \epsilon \to 0.
\end{aligned}$$

Thus **P3** is satisfied. Next let

$$V(s,t) = \frac{1+ct_*}{\sqrt{2}\varphi {t_*}^{2\alpha_\infty}}\left[\gamma X(s) + X^*(t)\right] = \frac{\sqrt{2}c}{\varphi}\left[\gamma X(s) + X^*(t)\right], \quad (s,t) \in E,$$

with X^* an independent copy of X. Hence by Lemma 5.3 and the fact that (recall that $\gamma_\epsilon = \frac{a_2-\epsilon}{a_2}\overline{\gamma}$)

$$(g_{u,k})^2 f_{u,k}(s,t) \Rightarrow \frac{\gamma_\epsilon \gamma^2(1+ct_*)^2}{2{t_*}^{4\alpha_\infty}\varphi^2}\sigma^2(s) = \frac{2\gamma_\epsilon c^2\gamma^2}{\varphi^2}\sigma^2(s), \quad (s,t)\in E,\ u\to\infty,$$

we have

$$\frac{\mathbb{P}\left(\sup_{(s,t)\in E}\xi_{u,k}(s,t) > g_{u,k}\right)}{\Psi(g_{u,k})} \to \mathcal{R}_V^{\frac{2\gamma_\epsilon c^2\gamma^2}{\varphi^2}\sigma^2(s)}(E) = \mathcal{H}_{\frac{\sqrt{2}c}{\varphi}X}[0,S]\mathcal{P}^{\gamma_\epsilon}_{\frac{\sqrt{2}c\gamma}{\varphi}X}[0,S_1], \quad u\to\infty$$

uniformly with respect to $k\in K_u$. Repeating the derivations of (4.19)–(4.25), we conclude that the claim follows with the generalised Pickands and Piterbarg constants above instead of those for case $\varphi = 0$. Note that the existence of $\mathcal{H}_{X^*}$ has been proved, see *e.g.* [2, 4, 16]; the proof of the finiteness of the generalised Piterbarg constants $\lim_{S_1\to\infty}\mathcal{P}^{\gamma_\epsilon}_{\frac{\sqrt{2}c\gamma}{\varphi}X}[0,S_1]$ is postponed to Lemma 5.4 in the Appendix.

Case $\varphi = \infty$. Since $\Delta_\gamma(u)$ is the same as in the case $\varphi = 0$, the proof is very similar to that case. The main difference is that the limiting Gaussian process V that appears in (4.18) is here different, namely **P1**–**P3** hold with

$$V(s,t) = B_{\alpha_\infty}(s) + B^*_{\alpha_\infty}(t), \quad (s,t)\in[0,S_1]\times[0,S],$$

where B_{α_∞} and $B^*_{\alpha_\infty}$ are independent fBm's with index α_∞. We omit details. □

4.2. Proof of Theorem 2.2

We begin with transformation of the distribution of the conditional passage time to the ratio of two tail probabilities of supremum of γ-reflected Gaussian process over appropriately chosen intervals. Using the same notation as introduced in the proof of Theorem 2.1, first we focus on $\tau_1^*(u)$. Let $D_{x,u} = \{(s,t) : 0\le s\le t\le xu^{-1}A(u)+t_u\}$. For all u large we have

$$\begin{aligned}\mathbb{P}\left(\frac{\tau_1^*(u)-ut_u}{A(u)}\le x\right) &= \frac{\mathbb{P}\left(\tau_1(u)\le xA(u)+ut_u\right)}{\mathbb{P}\left(\tau_1(u)<\infty\right)} = \frac{\mathbb{P}\left(\sup_{t\in[0,xA(u)+ut_u]}W_\gamma(t)>u\right)}{\psi_{\gamma,\infty}(u)}\\ &= \frac{\mathbb{P}\left(\sup_{(s,t)\in D_{x,u}}Z_u(s,t)>m(u)\right)}{\psi_{\gamma,\infty}(u)},\end{aligned} \tag{4.26}$$

with $Z_u(s,t)$ defined in (4.4) and $m(u)$ defined in (4.5). By Theorem 2.1, it suffices to find the asymptotics of $\mathbb{P}\left(\sup_{(s,t)\in D_{x,u}}Z_u(s,t)>m(u)\right)$, for which we write

$$\pi^x(u) \le \mathbb{P}\left(\sup_{(s,t)\in D_{x,u}}Z_u(s,t)>m(u)\right) \le \pi^x(u) + \mathbb{P}\left(\sup_{(s,t)\in D\setminus E(u)}Z_u(s,t)>m(u)\right), \tag{4.27}$$

where

$$\pi^x(u) = \mathbb{P}\left(\sup_{(s,t)\in E_1(u)\times E_2^x(u)}Z_u(s,t)>m(u)\right), \quad E_2^x(u) = \left(t_u - \frac{\sigma(u)\ln u}{u}, t_u + xu^{-1}A(u)\right)$$

with D defined in (4.4) and $E_1(u), E(u)$ defined in (4.6). Moreover,

$$J^x(u) - \Sigma_1(u) - \Sigma_2(u) \le \pi^x(u) \le \pi_1^x(u) + \Theta_2(u), \tag{4.28}$$

where

$$\pi_1^x(u) = \sum_{k=-N_{S,u}}^{N_{S,u}^x} \mathbb{P}\left(\sup_{(s,t)\in I_k(u)} Z_u(s,t) > m(u) \right), \quad J^x(u) = \sum_{k=-N_{S,u}+1}^{N_{S,u}^x-1} \mathbb{P}\left(\sup_{(s,t)\in I_k(u)} Z_u(s,t) > m(u) \right)$$

with $N_{S,u}^x = \left[\frac{xA(u)}{\Delta_1(u)S}\right] + 1$, $I_k(u)$ defined in (4.11), $N_{S,u}$ in (4.12), $\Theta_2(u)$ in (4.13) and $\Sigma_i(u), i = 1,2$ in (4.22).

Case $\varphi = 0$. Similarly as in (4.19), with $\epsilon \in (0, a_1)$ and $k^* = \min(|k|, |k+1|)$, we have that

$$\begin{aligned}\pi_1^x(u) &\le \sum_{k=-N_{S,u}}^{N_{S,u}^x} \mathcal{H}_{B_{\alpha_0}}[0,S]\mathcal{P}_{B_{\alpha_0}}^{\gamma_\epsilon}[0,S_1]\Psi(m(u))\mathrm{e}^{-(a_1-\epsilon)\left(k^* m(u)\frac{\Delta_1(u)}{u}S\right)^2}(1+o(1)) \\ &= \frac{\mathcal{H}_{B_{\alpha_0}}[0,S]}{S}\mathcal{P}_{B_{\alpha_0}}^{\gamma_\epsilon}[0,S_1](a_1-\epsilon)^{-1/2}\Psi(m(u))\frac{u}{m(u)\Delta_1(u)}\int_{-\infty}^{\sqrt{\frac{a_1-\epsilon}{2a_1}}x} \mathrm{e}^{-y^2}\mathrm{d}y(1+o(1)) \\ &\sim \sqrt{\pi/a_1}\Phi(x)\mathcal{H}_{B_{\alpha_0}}\mathcal{P}_{B_{\alpha_0}}^{\overline{\gamma}}\Psi(m(u))\frac{u}{m(u)\Delta_1(u)}, \end{aligned} \tag{4.29}$$

as $u \to \infty, S, S_1 \to \infty, \epsilon \to 0$, and

$$J^x(u) \ge \sqrt{\pi/a_1}\Phi(x)\mathcal{H}_{B_{\alpha_0}}\mathcal{P}_{B_{\alpha_0}}^{\overline{\gamma}}\Psi(m(u))\frac{u}{m(u)\Delta_1(u)}(1+o(1)). \tag{4.30}$$

By (4.9)

$$\mathbb{P}\left(\sup_{(s,t)\in D\setminus E(u)} Z_u(s,t) > m(u) \right) = o(\pi_1^x(u)) = o(J^x(u)).$$

Furthermore, it follows from (4.20), (4.24) and (4.25) that $\Theta_2(u)$, $\Sigma_1(u)$ and $\Sigma_2(u)$ are all negligible in comparison with $\pi_1^x(u)$ and $J^x(u)$ for $x \in (-\infty, \infty]$. Therefore, as $u \to \infty$,

$$\mathbb{P}\left(\sup_{(s,t)\in D_{x,u}} Z_u(s,t) > m(u) \right) \sim \sqrt{\pi/a_1}\Phi(x)\mathcal{H}_{B_{\alpha_0}}\mathcal{P}_{B_{\alpha_0}}^{\gamma_\epsilon}\Psi(m(u))\frac{u}{m(u)\Delta_1(u)} \sim \Phi(x)\psi_{\gamma,\infty}(u), \tag{4.31}$$

which together with (4.26) implies

$$\lim_{u\to\infty} \mathbb{P}\left(\frac{\tau_1^*(u) - ut_u}{A(u)} \le x \right) = \Phi(x), \quad x \in (-\infty, \infty].$$

Next, we investigate the last passage time. Similarly as above, for $x \in (-\infty, \infty]$ we have

$$\begin{aligned}\mathbb{P}\left(\frac{\tau_2^*(u) - ut_u}{A(u)} \le x \right) &= 1 - \mathbb{P}\left(\frac{\tau_2(u) - ut_u}{A(u)} > x \Big| \tau_1(u) < \infty \right) \\ &= 1 - \frac{\mathbb{P}\left(\sup_{t\in[xA(u)+ut_u,\infty)} W_\gamma(t) > u \right)}{\mathbb{P}(\tau_1(u) < \infty)} \\ &= 1 - \frac{\mathbb{P}\left(\sup_{t\in[xu^{-1}A(u)+t_u,\infty)} Z_u(s,t) > m(u) \right)}{\mathbb{P}(\tau_1(u) < \infty)} \\ &\to 1 - \Psi(x) = \Phi(x) \end{aligned} \tag{4.32}$$

as $u \to \infty$. Hence application of Lemma 2.1 in [21] (recall that $\tau_1(u) \le \tau_2(u)$) yields that for any $x, y \in \mathbb{R}$

$$\mathbb{P}\left(\frac{\tau_1^* - ut_u}{A(u)} \le x, \frac{\tau_2^* - ut_u}{A(u)} \le y \right) \to \mathbb{P}\left(\mathcal{N} \le \min(x,y)\right), \quad u \to \infty.$$

Case $\varphi \in (0, \infty)$. Note that (4.29) and (4.30) are also valid by replacing $\mathcal{H}_{B_{\alpha_0}}[0, S]$ with $\mathcal{H}_{\frac{\sqrt{2}c}{\varphi}X}[0, S]$ and $\mathcal{P}^{\gamma_\epsilon}_{B_{\alpha_0}}[0, S_1]$ with $\mathcal{P}^{\gamma_\epsilon}_{\frac{\sqrt{2}c\gamma}{\varphi}X}[0, S_1]$. As shown in the proof of i) in Theorem 2.1, $\Theta_2(u)$, $\Sigma_1(u)$, $\Sigma_1(u)$ and $\mathbb{P}\left(\sup_{(s,t)\in D\setminus E(u)} Z_u(s,t) > m(u)\right)$ are all negligible in comparison with $J^x(u), x \in (-\infty, \infty]$ and $\pi_1^x(u)$. Hence

$$\mathbb{P}\left(\sup_{(s,t)\in D_{x,u}} Z_u(s,t) > m(u)\right) \sim \sqrt{\pi/a_1}\Phi(x)\mathcal{H}_{\frac{\sqrt{2}c\gamma}{\varphi}X}\mathcal{P}^{\gamma_\epsilon}_{\frac{\sqrt{2}c\gamma}{\varphi}X}\Psi(m(u))\frac{u}{m(u)\Delta_1(u)}$$
$$\sim \Phi(x)\psi_{\gamma,\infty}(u), \quad u \to \infty.$$

In light of (4.26), we have

$$\lim_{u\to\infty} \mathbb{P}\left(\frac{\tau_1^*(u) - ut_u}{A(u)} \le x\right) = \Phi(x), \quad x \in (-\infty, \infty].$$

Further, (4.32) can be proven using the same arguments. The joint weak convergence of the passage times follows now by a direct application of Lemma 2.1 in [21].

Case $\varphi = \infty$. The proof of this case follows line by line the same as the proof of case $\varphi = 0$ with the exception that we have to substitute B_{α_0} with B_{α_∞} throughout the proof of case $\varphi = 0$. This completes the proof.□

4.3. Proof of Theorem 2.3

For any u positive

$$\psi_{\gamma,T}(u) = \mathbb{P}\left(\sup_{0\le t\le T}\left(X(t) - ct - \gamma \inf_{s\in[0,t]}(X(s) - cs)\right) > u\right)$$
$$= \mathbb{P}\left(\sup_{0\le t\le T} Z_{1,u}(s,t) > m_1(u)\right),$$

where $m_1(u) = \frac{u+cT}{\sigma(T)}$ and

$$Z_{1,u}(s,t) = \left(\frac{X(t) - \gamma X(s)}{u + c(t - \gamma s)}\right) m_1(u).$$

Below, for notational simplicity we set

$$\sigma^2_{1,u}(s,t) := Var\left(Z_{1,u}(s,t)\right),$$
$$r_1(s,t,s_1,t_1) := \mathrm{Cor}(Z_{1,u}(s,t), Z_{1,u}(s_1,t_1)) = \mathrm{Cor}(X(t) - \gamma X(s), X(t_1) - \gamma X(s_1)).$$

Let $D_T = \{(s,t), 0 \le s \le t \le T\}$ and $A_\delta = [0,\delta] \times [T-\delta, T]$ with $0 < \delta < T/2$. For all large u

$$\pi^*(u) \le \psi_{\gamma,T}(u) \le \pi^*(u) + \pi^{**}(u) \tag{4.33}$$

holds with

$$\pi^*(u) := \mathbb{P}\left(\sup_{(s,t)\in A_\delta} Z_{1,u}(s,t) > m_1(u)\right), \quad \pi^{**}(u) := \mathbb{P}\left(\sup_{(s,t)\in D_T\setminus A_\delta} Z_{1,u}(s,t) > m_1(u)\right).$$

The idea of the proof is to apply to $\pi^*(u)$ Theorem 3.1 in [45] which gives the tail asymptotics of supremum of Gaussian random fields with unique maximum variance point and to show that $\pi^{**}(u)$ is asymptotically negligible compared to $\pi^*(u)$. For this we need to know the dependence structure of the random field $Z_{1,u}$, which is analyzed in the next step of the proof.

4.3.1. Dependence structure of $Z_{1,u}$

Proofs of the following lemmas are postponed to Appendix.

Lemma 4.3. *If σ^2 satisfies* **BI** *and* **BIII***, then the unique maximizer of $\sigma_{1,u}(s,t)$ over $\{(s,t) : 0 \le s \le t \le T\}$ is $(0,T)$. Moreover, for u large enough and as $(s,t) \to (0,T)$*

$$1 - \sigma_{1,u}(s,t) = \left(\frac{\dot{\sigma^2}(T)}{2\sigma^2(T)} - a_3(u) \right) (T-t)(1+o(1)) + \begin{cases} \left(\frac{\gamma \dot{\sigma^2}(T)}{2\sigma^2(T)} - \gamma a_3(u) \right) s(1+o(1)), & \text{if } \quad \sigma^2(s) = o(s) \\ \left(\frac{b(\gamma-\gamma^2) + \gamma \dot{\sigma^2}(T)}{2\sigma^2(T)} - \gamma a_3(u) \right) s(1+o(1)), & \text{if } \ vf(s) \sim bs \\ \frac{\gamma - \gamma^2}{2\sigma^2(T)} \sigma^2(s)(1+o(1)), & \text{if } \quad s = o(\sigma^2(s)), \end{cases} \tag{4.34}$$

where $a_3(u) = \frac{c}{u+cT} \to 0$, as $u \to \infty$.

Lemma 4.4. *If σ^2 satisfies* **BI–BII** *and $t^2 = o(\sigma^2(t))$ as $t \to 0$, then*

$$1 - r_1(s,t,s_1,t_1) \sim \frac{\sigma^2(|t-t_1|) + \gamma^2 \sigma^2(|s-s_1|)}{2\sigma^2(T)} \tag{4.35}$$

holds for $(s,t),(s_1,t_1) \to (0,T)$.

4.3.2. Upper estimate of $\pi^{**}(u)$

By Lemma 4.3, there exists a positive constant $0 < \eta < 1$ such that

$$\sup_{(s,t)\in D_T \setminus A_\delta} Var\left(Z_{1,u}(s,t)\right) \le 1 - \eta.$$

In addition, it follows from **BII** that

$$\mathrm{Var}(Z_{1,u}(s,t) - Z_{1,t}(s',t')) \le Q_1 \left(|t-t'|^{\alpha_0} + |s-s'|^{\alpha_0} \right), \quad (s,t) \in D_T.$$

Using Lemma 5.1 for u large enough we obtain

$$\mathbb{P}\left(\sup_{(s,t)\in D_T \setminus A_\delta} Z_{1,u}(s,t) > m_1(u) \right) \le Q_2 T^2 (m_1(u))^{\frac{4}{\alpha_0}} \Psi\left(\frac{m_1(u)}{\sqrt{1-\eta}} \right). \tag{4.36}$$

4.3.3. Asymptotics of $\pi^*(u)$

Case $s = o(\sigma^2(s))$ as $s \to 0$. In light of Lemma 4.3, for any positive δ and ϵ sufficiently small we have

$$\mathbb{P}\left(\sup_{(s,t)\in A_\delta} Z_{2,\epsilon}(s,t) > m_1(u) \right) \le \pi^*(u) \le \mathbb{P}\left(\sup_{(s,t)\in A_\delta} Z_{2,-\epsilon}(s,t) > m_1(u) \right),$$

where

$$Z_{2,\pm\epsilon}(s,t) = \frac{\overline{X(t) - \gamma X(s)}}{\left(1 + \frac{\dot{\sigma^2}(T) \pm \epsilon}{2\sigma^2(T)}(T-t)\right)\left(1 + \frac{\gamma - \gamma^2 \pm \epsilon}{2\sigma^2(T)} \sigma^2(s)\right)}, \quad (s,t) \in A_\delta,$$

where $\overline{Z}$ means standardisation of Z, *i.e.*, $\overline{Z(t)} = Z(t)/\sqrt{Var(Z(t)}$. In view of Lemma 4.4 and using Theorem 3.1 in [45], we derive

$$\mathbb{P}\left(\sup_{(s,t)\in A_\delta} Z_{2,\pm\epsilon}(s,t) > m_1(u)\right) \sim \mathcal{H}_{B_{\alpha_0}}\mathcal{P}_{B_{\alpha_0}}^{\frac{1-\gamma\pm\epsilon/\gamma}{\gamma}} \frac{2\sigma^2(T)}{\dot{\sigma}^2(T)\pm\epsilon}\frac{\Psi(m_1(u))}{q(u)m_1^2(u)}, \quad u\to\infty. \tag{4.37}$$

Letting $\delta \to 0$, $\epsilon \to 0$ leads to

$$\pi^*(u) \sim \mathcal{H}_{B_{\alpha_0}}\mathcal{P}_{B_{\alpha_0}}^{\overline{\gamma}} \frac{2\sigma^2(T)}{\dot{\sigma}^2(T)}\frac{\Psi(m_1(u))}{q(u)m_1^2(u)}, \quad u\to\infty,$$

which together with (4.33) and (4.36) establishes the claim.

Case $\boldsymbol{\sigma^2(s) \sim bs}$ as $\boldsymbol{s \to 0}$: In light of Theorem 3.1 in [45], in this case (4.37) is changed to

$$\mathbb{P}\left(\sup_{(s,t)\in A_\delta} Z_{2,\pm\epsilon}(s,t) > m_1(u)\right) \sim \mathcal{P}_{B_{1/2}}^{\frac{\dot{\sigma}^2(T)\pm\epsilon}{b}}\mathcal{P}_{B_{1/2}}^{\beta(b,\gamma)\pm\frac{\epsilon}{b\gamma^2}}\Psi(m_1(u)), \quad u\to\infty,$$

with

$$Z_{2,\pm\epsilon}(s,t) = \frac{\overline{X(t)-\gamma X(s)}}{\left(1+\frac{\dot{\sigma}^2(T)\pm\epsilon}{2\sigma^2(T)}(T-t)\right)\left(1+\frac{b(\gamma-\gamma^2)+\gamma\dot{\sigma}^2(T)\pm\epsilon}{2\sigma^2(T)}s\right)}, \quad (s,t)\in A_\delta.$$

Thus letting $\delta \to 0$, $\epsilon \to 0$ and using (4.33) and (4.36) establishes the claim.

Case $\boldsymbol{\sigma^2(s) = o(s)}$ as $\boldsymbol{s \to 0}$: For any $\epsilon > 0$, if δ is sufficiently small, then

$$1 - r_1(s,t,s_1,t_1) \le \frac{2\left(\sigma^2(|t-t_1|)+\sigma^2(|s-s_1|)\right)}{\sigma^2(T)} \le \epsilon\left(|t-t_1|+|s-s_1|\right), \quad (s,t),(s_1,t_1)\in A_\delta.$$

Let $Z_\epsilon^*(s,t)$ be a stationary Gaussian field over $[0,T]^2$ with variance 1 and correlation function

$$\frac{\mathrm{e}^{-4\epsilon s}+\mathrm{e}^{-4\epsilon t}}{2}, \quad s,t\in[0,T].$$

It follows that

$$1 - r_1(s,t,s_1,t_1) < 1 - \frac{\mathrm{e}^{-4\epsilon|s-s_1|}+\mathrm{e}^{-4\epsilon|t-t_1|}}{2}, \quad (s,t),(s_1,t_1)\in A_\delta.$$

In light of Lemma 4.3, by Slepian's inequality (see *e.g.*, Thm. 2.2.1 in [43]) and Theorem 3.1 in [45], we have, for positive δ sufficiently small

$$\begin{aligned}\pi^*(u) &\le \mathbb{P}\left(\sup_{(s,t)\in A_\delta}\frac{Z_\epsilon^*(s,t)}{\left(1+\frac{\dot{\sigma}^2(T)}{4\sigma^2(T)}(T-t)\right)\left(1+\frac{\gamma\dot{\sigma}^2(T)}{4\sigma^2(T)}s\right)} > m_1(u)\right)\\ &\sim \mathcal{P}_{B_{1/2}}^{\frac{\dot{\sigma}^2(T)}{8\epsilon\sigma^2(T)}}\mathcal{P}_{B_{1/2}}^{\frac{\gamma\dot{\sigma}^2(T)}{8\epsilon\sigma^2(T)}}\Psi(m_1(u)), \quad u\to\infty.\end{aligned} \tag{4.38}$$

Moreover,

$$\pi^*(u) \ge \mathbb{P}\left(Z_{1,u}(0,T) > m_1(u)\right) \sim \Psi(m_1(u)), \quad u\to\infty.$$

Thus letting $\epsilon \to 0$ in (4.38) leads to

$$\pi^*(u) \sim \Psi(m_1(u)), \quad u\to\infty,$$

which together with (4.33) and (4.36) completes the proof. □

Proof of Theorem 2.5 For $x>0$, let $T_{x,u}=T-\frac{2\sigma^4(T)x}{\dot{\sigma^2}(T)u^2}$. For all the three cases, using Theorem 2.3 and Remark 2.4 ii) we have

$$\mathbb{P}\left(\frac{\dot{\sigma^2}(T)u^2(T-\tau_1)}{2\sigma^4(T)}>x\Big|\tau_1\le T\right)=\frac{\psi_{T_{x,u}}(u)}{\psi_T(u)}\sim\frac{\Psi\left(\frac{u+cT_{x,u}}{\sigma(T_{x,u})}\right)}{\Psi\left(\frac{u+cT}{\sigma(T)}\right)}\sim \mathrm{e}^{\frac{(u+cT)^2}{2\sigma^2(T)}-\frac{(u+cT_{x,u})^2}{2\sigma^2(T_{x,u})}},\quad u\to\infty,$$

where for any $x>0$

$$\begin{aligned}\frac{(u+cT)^2}{2\sigma^2(T)}-\frac{(u+cT_{x,u})^2}{2\sigma^2(T_{x,u})}&=\frac{(u+cT)^2}{2\sigma^2(T)}\left(1-\frac{(1-\frac{c(T-T_{x,u})}{u+cT})^2}{\frac{\sigma^2(T_{x,u})}{\sigma^2(T)}}\right)\\&\sim\frac{(u+cT)^2}{2\sigma^2(T)}\left(1-\frac{(1-\frac{c(T-T_{x,u})}{u+cT})^2}{1-\frac{\dot{\sigma^2}(T)(T-T_{x,u})}{\sigma^2(T)}}\right)\\&\to -x,\quad u\to\infty.\end{aligned}$$

Thus the claim is established. □

5. Appendix

In this section we present an extension of Theorem 8.1 in [16] to threshold-dependent Gaussian fields, followed by a uniform version of Pickands–Piterbarg lemma motivated by Lemma 2 in [4]. Then we give lemma that deals with existence and positivity of generalized Piterbarg constants, which is followed by the proof of (4.8). Finally, we display the proofs of Lemmas 4.1-4.5. Before proceeding to the proofs in Appendix, we first present some regularly varying properties on σ^2. By **AI** and Theorem 1.7.2 in [41], we have that

$$\lim_{u\to\infty}\frac{u\dot{\sigma^2}(u)}{\sigma^2(u)}=2\alpha_\infty, \tag{5.1}$$

$$\lim_{u\to\infty}\frac{u\ddot{\sigma^2}(u)}{\sigma^2(u)}=2\alpha_\infty(2\alpha_\infty-1). \tag{5.2}$$

Lemma 5.2 in [46] shows that **AI** implies that in a neighborhood of 0

$$\sigma^2(t)\ge Ct^2, \tag{5.3}$$

then the function

$$\frac{1}{g_2(t)}=\frac{t^2}{\sigma^2(t)},\quad t\in(0,\infty) \tag{5.4}$$

is regularly varying at infinity with index $2(1-\alpha_\infty)>0$ and is bounded in a neighborhood of zero. By (5.4) and uniform convergence theorem in [41] we have that for any $T>0$

$$\lim_{u\to\infty}\sup_{t\in(0,T]}\left|\frac{g_2(u)}{g_2(ut)}-|t|^{2-2\alpha_\infty}\right|=0. \tag{5.5}$$

Moreover, Potter's bound in [41] shows that for any $0<\epsilon<2\alpha_\infty$, there exists $T>0$ and $Q_1,Q_2>0$ such that for all $s,t>T>0$

$$Q_1\min\left(\left(\frac{t}{s}\right)^{2\alpha_\infty-\epsilon},\left(\frac{t}{s}\right)^{2\alpha_\infty+\epsilon}\right)\le\frac{\sigma^2(t)}{\sigma^2(s)}\le Q_2\max\left(\left(\frac{t}{s}\right)^{2\alpha_\infty-\epsilon},\left(\frac{t}{s}\right)^{2\alpha_\infty+\epsilon}\right) \tag{5.6}$$

By UCT, similarly as in (4.2) we have that for $0 < \lambda < \min(2\alpha_0, 2\alpha_\infty)$ as u sufficiently large,

$$\frac{\sigma^2(ut)}{\sigma^2(u)} = \frac{g_\lambda(ut)}{g_\lambda(u)}|t|^\lambda \le 2|T|^{2\alpha_\infty-\lambda}|t|^\lambda, \quad t \in [0,T]. \tag{5.7}$$

By **AII** and Theorem 1.7.2 in [41] that

$$t\dot{\sigma^2}(t) \sim 2\alpha_0\sigma^2(t), \quad t \to 0,$$

which combined with (5.3) gives that $t/\dot{\sigma^2}(t)$ is bounded in a neighbourhood of zero. Therefore if **AI**–**AII** hold, we have from (5.1) that

$$K(t) := t(\dot{\sigma^2}(t))^{-1}, \quad t \in (0,\infty) \tag{5.8}$$

is a regularly varying function at infinity with index $2(1-\alpha_\infty) > 0$ and bounded in a neighbourhood of zero.

5.1. Extensions of Piterbarg inequality and Pickands-Piterbarg lemma

Piterbarg inequality, *e.g.* [16], (Thm. 8.1), provides a precise upper bound for tail distribution of supremum for a wide class of Gaussian processes. Our next result extends Piterbarg inequality to threshold-dependent Gaussian random fields with general covariance structure, allowing for supremum to be taken on sets that depend on u.

Lemma 5.1. *Let $X_{u,\tau}(t), t \in \mathbb{R}^d, \tau \in K_u, u > 0$ be a centered Gaussian field with variance $\sigma^2_{u,\tau}(t), t \in E_{u,\tau}$ and a.s. continuous sample paths where K_u are some index sets. Let further $E_{u,\tau} \subset \prod_{i=1}^d[-M_{u,i}, M_{u,i}], u > 0, \tau \in K_u$ be compact sets, and put $\sigma_{u,\tau} := \sup_{t\in E_{u,\tau}} \sigma_{u,\tau}(t)$. Suppose that $0 < a < \sigma_{u,\tau} < b < \infty$ holds for $\tau \in K_u$ and all large u. If for all u large and for any $s, t \in E_{u,\tau}$*

$$Var\Big(X_{u,\tau}(t) - X_{u,\tau}(s)\Big) \le C\sum_{i=1}^d |t_i - s_i|^{\gamma_i}, \quad s = (s_1,\dots,s_d), \quad t = (t_1,\dots,t_d), \tau \in K_u, \tag{5.9}$$

with $\gamma_i \in (0,2], 1 \le i \le d$, then for some $C_1 > 0$ and $u_0 > 0$ not depending on u and $\tau \in K_u$

$$\mathbb{P}\left(\sup_{t\in E_{u,\tau}} |X_{u,\tau}(t)| > u\right) \le C_1\prod_{i=1}^d \Big(M_{u,i}u^{\frac{2}{\gamma_i}} + 1\Big)\Psi\Big(u/\sigma_{u,\tau}\Big), \quad u > u_0. \tag{5.10}$$

Proof of Lemma 5.1. Let $E^{(1)}_{u,\tau} = \{t \in E_{u,\tau} : \sigma_{u,\tau}(t) > \sigma_{u,\tau}/2\}$ and $E^c_{u,\tau} := E_{u,\tau} \setminus E^{(1)}_{u,\tau}$. Then for $s, t \in E^{(1)}_{u,\tau}$,

$$1 - \mathrm{Cor}(X_{u,\tau}(t)X_{u,\tau}(s)) \le \frac{\mathrm{Var}(X_{u,\tau}(t) - X_{u,\tau}(s))}{2\sigma_{u,\tau}(t)\sigma_{u,\tau}(s)} \le \frac{2C}{a^2}\sum_{i=1}^d |t_i - s_i|^{\gamma_i}.$$

Let $Y(t), t \in \mathbb{R}^d$ be a homogeneous Gaussian process with variance 1 and correlation function

$$r_Y(t) = Cov\left(Y(s), Y(s+t)\right) = \mathrm{e}^{-\frac{4C}{a^2}\sum_{i=1}^d |t_i|^{\gamma_i}}, \quad s, t \in \mathbb{R}^d,$$

and let

$$L_{\boldsymbol{k}}(u) = \prod_{i=1}^d \left[k_i u^{-\frac{2}{\gamma_i}}, (k_i+1)u^{-\frac{2}{\gamma_i}}\right],$$

with $\boldsymbol{k} = (k_1, \ldots, k_d)$ and $k_i \in \mathbb{Z}, i = 1, \ldots, n$. By Slepian's inequality (see *e.g.*, Thm 2.2.1 in [43]) for u large enough we have

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in E^{(1)}_{u,\tau}} |X_{u,\tau}(t)| > u\right) &\le 2\mathbb{P}\left(\sup_{t\in E^{(1)}_{u,\tau}} \overline{X}_{u,\tau}(t) > \frac{u}{\sigma_{u,\tau}}\right)\\ &\le \sum_{\boldsymbol{k}: L_{\boldsymbol{k}}(u)\cap E^{(1)}_{u,\tau}\neq\emptyset} 2\mathbb{P}\left(\sup_{t\in L_{\boldsymbol{k}}(u)\cap E^{(1)}_{u,\tau}} \overline{X}_{u,\tau}(t) > \frac{u}{\sigma_{u,\tau}}\right)\\ &\le \sum_{\boldsymbol{k}: L_{\boldsymbol{k}}(u)\cap E^{(1)}_{u,\tau}\neq\emptyset} 2\mathbb{P}\left(\sup_{t\in L_{\boldsymbol{k}}(u)\cap E^{(1)}_{u,\tau}} Y(t) > \frac{u}{\sigma_{u,\tau}}\right)\\ &\le \sum_{\boldsymbol{k}: L_{\boldsymbol{k}}(u)\cap E^{(1)}_{u,\tau}\neq\emptyset} 2\mathbb{P}\left(\sup_{t\in L_{\boldsymbol{0}}(u)} Y(t) > \frac{u}{\sigma_{u,\tau}}\right).\end{aligned}$$

Further, by Lemma 6.1 in [16] and the fact that

$$\inf_{\tau\in K_u} \frac{u}{\sigma_{u,\tau}} \to \infty, \quad u\to\infty,$$

we have

$$\lim_{u\to\infty}\sup_{\tau\in K_u}\left|\frac{\mathbb{P}\left(\sup_{t\in L_{\boldsymbol{0}}(u)} Y(t) > \frac{u}{\sigma_{u,\tau}}\right)}{\Psi(\frac{u}{\sigma_{u,\tau}})} - \prod_{i=1}^d \mathcal{H}_{B_{\gamma_i/2}}\left[0, \left(4ca^{-2}\right)^{1/\gamma_i}\right]\right| = 0.$$

Consequently, for u sufficiently large

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in E^{(1)}_{u,\tau}} |X_{u,\tau}(t)| > u\right) &\le 2\prod_{i=1}^d \left[\mathcal{H}_{B_{\gamma_i/2}}[0, \left(4ca^{-2}\right)^{1/\gamma_i}] \left([2M_{u,i}u^{\frac{2}{\gamma_i}}]+1\right)\right]\Psi(u/\sigma_{u,\tau})\\ &\le C_1\prod_{i=1}^d \left([2M_{u,i}u^{\frac{2}{\gamma_i}}]+1\right)\Psi(u/\sigma_{u,\tau}) \qquad (5.11)\end{aligned}$$

uniformly with respect to $\tau \in K_u$. By (5.9) for any $0 < h \le 1$

$$\sup_{|t_i-s_i|\le h, 1\le i\le d} \sqrt{Var(X_{u,\tau}(t) - X_{u,\tau}(s))} \le \left(C\sum_{i=1}^d |t_i - s_i|^{\gamma_i}\right)^{1/2} \le (Cd)^{1/2} h^{\gamma_0/2},$$

with $\gamma_0 = \min_{1\le i\le d}\gamma_i$. Thus by (2.2) in [47] and (5.9), for any $E^c_{u,\tau}\cap L_{\boldsymbol{k}}(1) \neq \emptyset$, we have

$$\mathbb{P}\left(\sup_{t\in E^c_{u,\tau}\cap L_{\boldsymbol{k}}(1)} |X_{u,\tau}(t)| > \left[b + (2+\sqrt{2})(Cd)^{1/2}\int_1^\infty 2^{-\frac{\gamma_0 y^2}{2}} dy\right] x\right) \le \frac{5}{2} 2^{2d}\sqrt{2\pi}\Psi(x) \qquad (5.12)$$

for all $x \ge (1 + 4d\ln 2)^{1/2}$, which implies that we can find a constant y such that

$$\mathbb{P}\left(\sup_{t\in E^c_{u,\tau}\cap L_{\boldsymbol{k}}(1)} X_{u,\tau}(t) > y\right) < 1/2$$

holds for all $\boldsymbol{k}$ with $E^c_{u,\tau} \cap L_{\boldsymbol{k}}(1) \neq \emptyset$. Further, using Borell-TIS inequality, see *e.g.*, [43, 48, 49]

$$\begin{aligned}\mathbb{P}\left(\sup_{t\in E^c_{u,\tau}} X_{u,\tau}(t) > u\right) &\le \sum_{\boldsymbol{k}: E^c_{u,\tau}\cap L_{\boldsymbol{k}}(1)\neq\emptyset} \mathbb{P}\left(\sup_{t\in E^c_{u,\tau}\cap L_{\boldsymbol{k}}(1)} |X_{u,\tau}(t)| > u\right)\\ &\le 2^d \prod_{i=1}^d (M_{u,i}+1)\Psi\Big(2(u-y)/\sigma_{u,\tau}\Big)\\ &= o\left(\prod_{i=1}^d \left(M_{u,i}u^{\frac{2}{\gamma_i}}+1\right)\Psi\big(u/\sigma_{u,\tau}\big)\right),\end{aligned}$$

hence the claim is established by considering also (5.11). □

Remarks 5.2.

(i) In case $X_{u,\tau} = X$ and $E_{u,\tau} = E$ for all u and $\gamma_i = \gamma, i \le d$, the claim of Lemma 5.1 coincides with that of Theorem 8.1 in [16]. Note in passing that Piterbarg inequality gives sharper bounds than Borell-TIS inequality. The later however holds under weaker assumptions.

(ii) In the formulation of Lemma 5.1 we write $(M_{u,i}u^{2/\gamma_i}+1)$ and not simply $M_{u,i}u^{2/\gamma_i}$ since we want to cover also the case that $\lim_{u\to\infty} M_{u,i}u^{2/\gamma_i} = 0$.

The classical Pickands lemma gives the exact asymptotics of Gaussian processes on short intervals. We present below an extension of that lemma; our result is uniform with respect to some parameter $\tau_u \in K_u$. Let therefore $E \subset \mathbb{R}^d$ be a compact set with positive Lebesgue measure containing the origin and let K_u some index sets. We denote $C_0(E)$ the space of all continuous functions f on E, such that $f(0) = 0$, equipped with the sup-norm. For $f_{u,\tau} \in C_0(E)$ define

$$\xi_{u,\tau}(t) = \frac{Z_{u,\tau}(t)}{1+f_{u,\tau}(t)}, \quad t \in E, \quad \tau := \tau_u \in K_u,$$

with $Z_{u,\tau}$ a centered Gaussian field with unit variance and a.s. continuous sample paths. In the following lemma we derive the uniform asymptotics of

$$p_{u,\tau}(E) := \mathbb{P}\left(\sup_{t\in E}\xi_{u,\tau}(t) > g_{u,\tau}\right), \quad u\to\infty,$$

with respect to $\tau \in K_u$. We need the following assumptions, which are similar to those imposed in [46], (Lem. 5.1) and [4], (Lem. 2).

P1: $\inf_{\tau\in K_u} g_{u,\tau} \to \infty$ as $u \to \infty$.

P2: Let $\theta_{u,\tau}(s,t)$ be a function such that

$$\lim_{u\to\infty}\sup_{\tau\in K_u}\sup_{s\neq t\in E}\left| g^2_{u,\tau}\frac{Var\left(Z_{u,\tau}(t)-Z_{u,\tau}(s)\right)}{2\theta_{u,\tau}(s,t)} - 1\right| = 0.$$

There exists a centered Gaussian random field $V(t), t \in \mathbb{R}^d$ with $V(0) = 0$, covariance function $(\sigma^2_V(t)+\sigma^2_V(s)-\sigma^2_V(t-s))/2, s,t \in \mathbb{R}^d$ and a.s. continuous sample paths such that

$$\lim_{u\to\infty}\sup_{\tau\in K_u}|\theta_{u,\tau}(s,t)-\sigma^2_V(t-s)| = 0, \quad \forall s,t \in E.$$

P3: There exists some $a > 0$ such that

$$\limsup_{u\to\infty}\sup_{\tau\in K_u}\sup_{s\neq t, s,t\in E}\frac{\theta_{u,\tau}(s,t)}{\sum_{i=1}^d |s_i-t_i|^a} < \infty$$

and further

$$\lim_{\epsilon\downarrow 0}\limsup_{u\to\infty}\sup_{\tau\in K_u}\sup_{\|t-s\|<\epsilon,s,t\in E} g_{u,\tau}^2\mathbb{E}\left\{[Z_{u,\tau}(t)-Z_{u,\tau}(s)]\,Z_{u,\tau}(0)\right\}=0.$$

Lemma 5.3. *Let $g_{u,\tau}, V, \theta_{u,\tau}$ satisfy* **P1–P3**. *Assume that $f_{u,\tau}\in C_0(E), u>0, \tau\in K_u$*

$$\lim_{u\to\infty}\sup_{\tau\in K_u,t\in E}|g_{u,\tau}^2 f_{u,\tau}(t)-f(t)|=0. \tag{5.13}$$

Then we have

$$\lim_{u\to\infty}\sup_{\tau\in K_u}\left|\frac{p_{u,\tau}(E)}{\Psi(g_{u,\tau})}-\mathcal{R}_V^f(E)\right|=0, \tag{5.14}$$

with $\mathcal{R}_V^f(E):=\mathbb{E}\left\{\sup_{t\in E}\mathrm{e}^{\sqrt{2}V(t)-\sigma_V^2(t)-f(t)}\right\}\in(0,\infty)$.

Proof of Lemma 5.3. By conditioning on $\xi_{u,\tau}(0)=g_{u,\tau}-\frac{w}{g_{u,\tau}}, w\in\mathbb{R}$ for all $u>0$ large we obtain

$$\sqrt{2\pi}g_{u,\tau}\mathrm{e}^{\frac{g_{u,\tau}^2}{2}}\mathbb{P}\left(\sup_{t\in E}\xi_{u,\tau}(t)>g_{u,\tau}\right)=\int_{\mathbb{R}}\mathrm{e}^{w-\frac{w^2}{2g_{u,\tau}^2}}\mathbb{P}\left(\sup_{t\in E}\chi_{u,\tau}(t)>w\right)\mathrm{d}w=:I_{u,\tau},$$

where

$$\chi_{u,\tau}(t)=\zeta_{u,\tau}(t)|\zeta_{u,\tau}(0)=0,\quad \zeta_{u,\tau}(t)=g_{u,\tau}(\xi_{u,\tau}(t)-g_{u,\tau})+w.$$

In order to establish the proof we need to show that

$$\lim_{u\to\infty}\sup_{\tau\in K_u}\left|I_{u,\tau}-\mathcal{R}_V^f(E)\right|=0. \tag{5.15}$$

It follows that

$$\begin{aligned}\sup_{\tau\in K_u}|I_{u,\tau}-\mathcal{R}_V^f(E)|\le{}&\sup_{\tau\in K_u}\left|\int_{-M}^{M}\left[\mathrm{e}^{w-\frac{w^2}{2g_{u,\tau}^2}}\mathbb{P}\left(\sup_{t\in E}\chi_{u,\tau}(t)>w\right)-\mathrm{e}^{w}\mathbb{P}\left(\sup_{t\in E}V(t)>w\right)\right]\mathrm{d}w\right|\\&+\sup_{\tau\in K_u}\int_{|w|>M}\mathrm{e}^{w-\frac{w^2}{2g_{u,\tau}^2}}\mathbb{P}\left(\sup_{t\in E}\chi_{u,\tau}(t)>w\right)\mathrm{d}w\\&+\int_{|w|>M}\mathrm{e}^{w}\mathbb{P}\left(\sup_{t\in E}V(t)>w\right)\mathrm{d}w.\end{aligned}$$

Next, we give an upper bound of each term in the right hand side of the above inequality. Clearly, $\chi_{u,\tau}(0)=0$ almost surely, and the finite-dimensional distributions of $\chi_{u,\tau}(t), t\in E$ coincide with that of

$$\frac{1}{1+f_{u,\tau}(t)}\left(g_{u,\tau}\Big(Z_{u,\tau}(t)-r_{Z_{u,\tau}}(t,0)Z_{u,\tau}(0)\Big)+\mu_{u,\tau,w}(t)\right),\quad t\in E,$$

where

$$\mu_{u,\tau,w}(t)=-g_{u,\tau}^2\left(1-r_{Z_{u,\tau}}(t,0)+f_{u,\tau}(t)\right)+w(1-r_{Z_{u,\tau}}(t,0)+f_{u,\tau}(t)),\quad r_{Z_{u,\tau}}(t,s):=\mathrm{Cor}(Z_{u,\tau}(t),Z_{u,\tau}(s)).$$

Consequently, by **P1–P3** and (5.13) we have that uniformly with respect to $t\in E,\tau\in K_u, w\in[-M,M]$

$$\mu_{u,\tau,w}(t)\to-(\sigma_V^2(t)+f(t)),\ u\to\infty \tag{5.16}$$

and also for any $(s,t)\in E$ uniformly with respect to $\tau\in K_u, w\in[-M,M]$

$$\begin{aligned} v_u(s,t) &:= \mathrm{Var}\Big((1+f_{u,\tau}(t))\chi_{u,\tau}(t)-(1+f_{u,\tau}(s))\chi_{u,\tau}(s)\Big)\\ &= g_{u,\tau}^2\left[\mathbb{E}\left\{\Big(Z_{u,\tau}(t)-Z_{u,\tau}(s)\Big)^2\right\}-\left(r_{Z_{u,\tau}}(t,0)-r_{Z_{u,\tau}}(s,0)\right)^2\right]\\ &\to 2\mathrm{Var}(V(t)-V(s)),\ u\to\infty. \end{aligned} \tag{5.17}$$

Note that $v_u(s,t)$ does not depend on w and $f\in C_0(E)$. Consequently, following the proof of Lemma 4.1 in [50], the finite-dimensional distributions of $(1+f_{u,\tau}(t))\chi_{u,\tau}(t)$ converge uniformly for $\tau\in K_u, w\in[-M,M]$ where $M>0$ is fixed. By **P3**, the uniform convergence in (5.16), (5.17) and

$$\lim_{u\to\infty}\sup_{\tau\in K_u,t\in E}|f_{u,\tau}(t)|=0 \tag{5.18}$$

imply along the lines of the proof of second part of Lemma 4.1 in [50] that for arbitrary $M>0,\varepsilon>0$

$$\lim_{u\to\infty}\sup_{\tau\in K_u,w\in[-M,M],w\notin[-\varepsilon,\varepsilon]}\left|\mathbb{P}\left(\sup_{t\in E}\chi_{u,\tau}(t)>w\right)-\mathbb{P}\left(\sup_{t\in E}V(t)>w\right)\right|=0,$$

where we use the fact that $\sup_{t\in E}V(t)$ has a continuous distribution $H(t),t\geq 0$ for all $t>0$, see *e.g.*, [51] (Thm. 7.1) (recall that since $0\in E$ and $V(0)=0$, then $\sup_{t\in E}V(t)\geq 0$). Further, by **P1**

$$\lim_{u\to\infty}\sup_{\tau\in K_u,w\in[-M,M]}\mathrm{e}^w[1-\mathrm{e}^{-\frac{w^2}{2g_{u,\tau}^2}}]\leq\frac{\mathrm{e}^M M^2}{2\liminf_{u\to\infty}\inf_{\tau\in K_u}g_{u,\tau}^2}\to 0,\quad u\to\infty$$

we obtain by the fact that $\varepsilon>0$ can be chosen arbitrary small

$$\lim_{u\to\infty}\sup_{\tau\in K_u}\left|\int_{-M}^{M}\left[\mathrm{e}^{w-\frac{w^2}{2g_{u,\tau}^2}}\mathbb{P}\left(\sup_{t\in E}\chi_{u,\tau}(t)>w\right)-\mathrm{e}^w\mathbb{P}\left(\sup_{t\in E}V(t)>w\right)\right]\mathrm{d}w\right|=0.$$

Using (5.16) for $\delta\in(0,1/2)$, $|w|>M$ with M sufficiently large and all u large we have

$$\sup_{\tau\in K_u,t\in E}\mathbb{E}\left\{(1+f_{u,\tau}(t))\chi_{u,\tau}(t)\right\}\leq\delta|w|.$$

It follows from **P3** that for u large enough,

$$\mathrm{Var}\left((1+f_{u,\tau}(t))\chi_{u,\tau}(t)-(1+f_{u,\tau}(s))\chi_{u,\tau}(s)\right)\leq Q\sum_{i=1}^d|s_i-t_i|^a,\quad (s,t)\in E.$$

Thus by (5.18) and the result of Lemma 5.1, we obtain for some $\varepsilon,\delta\in(0,1/2)$ and all u and M large

$$\begin{aligned} &\int_{|w|>M}\mathrm{e}^{w-\frac{w^2}{2g_{u,\tau}^2}}\mathbb{P}\left(\sup_{t\in E}\chi_{u,\tau}(t)>w\right)\mathrm{d}w\\ &\leq\int_{|w|>M}\mathrm{e}^w\mathbb{P}\left(\sup_{t\in E}\left((1+f_{u,\tau}(t))\chi_{u,\tau}(t)-\mathbb{E}\left\{(1+f_{u,\tau}(t))\chi_{u,\tau}(t)\right\}\right)\right.\\ &>\left.w/2-\sup_{t\in E,\tau\in K_u}\mathbb{E}\left\{(1+f_{u,\tau}(t))\chi_{u,\tau}(t)\right\}\right)\mathrm{d}w\\ &\leq\int_{-\infty}^{-M}\mathrm{e}^w\mathrm{d}w+\int_M^\infty\mathrm{e}^w\mathbb{P}\left(\sup_{t\in E}(\chi_{u,\tau}(t)-\mathbb{E}\left\{\chi_{u,\tau}(t)\right\})>w/2-\delta w\right)\mathrm{d}w\\ &\leq\mathrm{e}^{-M}+\int_M^\infty\mathrm{e}^w\Psi\big((1-\varepsilon)(1/2-\delta)w\big)\,\mathrm{d}w\\ &=:A_1(M)\to 0,\quad M\to\infty. \end{aligned}$$

Moreover, using Borell-TIS inequality (see *e.g.*, [43, 49])

$$A_2(M) := \int_{|w|>M} e^w \mathbb{P}\left(\sup_{t\in E} V(t) > w\right) dw \to 0, \quad M \to \infty.$$

Hence (5.15) follows since

$$\begin{aligned}\sup_{\tau\in K_u} |I_{u,\tau} - \mathcal{R}_V^f(E)| &\le \sup_{\tau\in K_u}\left|\int_{-M}^{M}\left[e^{w-\frac{w^2}{2g_{u,\tau}^2}}\mathbb{P}\left(\sup_{t\in E}\chi_{u,\tau}(t) > w\right) - e^w\mathbb{P}\left(\sup_{t\in E} V(t) > w\right)\right] dw\right| + A_1(M) + A_2(M) \\ &\to A_1(M) + A_2(M), \quad u \to \infty, \\ &\to 0, \quad M \to \infty.\end{aligned}$$

Since further

$$\lim_{u\to\infty} \sup_{\tau\in K_u} \left|\sqrt{2\pi} g_{u,\tau} e^{\frac{g_{u,\tau}^2}{2}} \Psi(g_{u,\tau}) - 1\right| = 0,$$

the proof is completed. □

5.2. Piterbarg-type constant

In this subsection we prove the existence and positivity of the generalized Piterbarg constant that appears in Theorem 2.1. Let X be a centered Gaussian process with stationary increments, a.s. continuous sample paths and variance function satisfying the following two assumptions:

C0: $\sigma^2(t)$ is regularly varying at infinity with index $2\alpha_\infty \in (0, 2)$ and its first derivative is continuously differentiable over $(0,\infty)$ with $\dot{\sigma}^2(t)$ being ultimately monotone at infinity.

C1: $\sigma^2(t)$ is regularly varying at zero with index $2\alpha_0 \in (0, 2]$.

Then we have

$$1 - \mathrm{Cor}\,(X(ut), X(us)) = \frac{\sigma^2(u|t-s|) - (\sigma(ut) - \sigma(us))^2}{2\sigma(ut)\sigma(us)} = \frac{\sigma^2(u|t-s|) - (u\dot{\sigma}(u\theta)(t-s))^2}{2\sigma(ut)\sigma(us)},$$

with $\theta \in [s, t]$. Note that (5.1) implies

$$\lim_{u\to\infty} \frac{u\dot{\sigma}(u)}{\sigma(u)} = \alpha_\infty,$$

which together with (5.5) implies that

$$\begin{aligned}1 - \mathrm{Cor}\,(X(ut), X(us)) &\sim \frac{\sigma^2(u|t-s|)}{2\sigma(ut)\sigma(us)}\left(1 - \frac{\alpha_\infty^2}{\theta^2}\frac{\sigma^2(u\theta)(t-s)^2}{\sigma^2(u|t-s|)}\right) \\ &= \frac{\sigma^2(u|t-s|)}{2\sigma(ut)\sigma(us)}\left(1 - \alpha_\infty^2 \frac{g_2(u\theta)}{g_2(u|t-s|)}\right) \qquad (5.19) \\ &\sim \frac{\sigma^2(u|t-s|)}{2\sigma^2(u)} \qquad (5.20)\end{aligned}$$

as $u \to \infty$ for $s, t \in [1, 1 + u^{-1} \ln u]$.

Lemma 5.4. *If X is a centered Gaussian process with stationary increments and a.s. continuous sample paths such that its variance function satisfies* **C0**,**C1**, *then*

$$\mathcal{P}_X^a = \lim_{S\to\infty} \mathcal{P}_X^a[0,S] < \infty$$

holds for any $a \in (0,\infty)$.

Proof of Lemma 5.4. We first introduce some notation. For $S > 0, u > 1$ define

$$Y_u(t) = \frac{\overline{X}(u(t+1))}{1+\frac{a\sigma^2(ut)}{2\sigma^2(u)}}, \quad t \in [0, u^{-1}\ln u],$$

$$I_k(u) = [ku^{-1}S, u^{-1}(k+1)S], \quad 0 \le k \le N(u), \text{ with } N(u) := [S^{-1}\ln u] + 1.$$

It follows that for S sufficiently large

$$p_0(u) \le \mathbb{P}\left(\sup_{t\in[0,u^{-1}\ln u]} Y_u(t) > \sqrt{2}\sigma(u)\right) \le p_0(u) + \sum_{k=1}^{N(u)} p_k(u), \tag{5.21}$$

where

$$p_0(u) = \mathbb{P}\left(\sup_{t\in I_0(u)} Y_u(t) > \sqrt{2}\sigma(u)\right),$$

$$p_k(u) = \mathbb{P}\left(\sup_{t\in I_k(u)} \overline{X}(u(t+1)) > \sqrt{2}\sigma(u)\left(1+\frac{a\sigma^2(kS)}{4\sigma^2(u)}\right)\right), \quad k \ge 1.$$

In order to apply Lemma 5.3, by (5.19) we set

$$K_u = \{k : 0 \le k \le N(u)\}, \quad E = [0,S], \quad g_{u,k} = \sqrt{2}\sigma(u)\left(1+\frac{a\sigma^2(kS)}{4\sigma^2(u)}\right), k \in K_u, \tag{5.22}$$

$$Z_{u,k}(t) = \overline{X}(u(u^{-1}kS + u^{-1}t + 1)), \quad k \in K_u,$$

$$\theta_{u,k}(s,t) = g_{u,k}^2 \frac{\sigma^2(|t-s|)}{2\sigma^2(u)}, \quad s,t \in E, k \in K_u,$$

$$f_{u,0}(t) = \frac{a\sigma^2(t)}{2\sigma^2(u)}, \quad t \in E, \quad f_{u,k} = 0, \quad k \in K_u \setminus \{0\}, \quad V = X.$$

Since **P1**–**P2** are obviously fulfilled, we shall verify next **P3**. By **C1** we have, for u sufficiently large

$$\theta_{u,k}(s,t) = g_{u,k}^2 \frac{\sigma^2(|t-s|)}{2\sigma^2(u)} \le 2\sigma^2(|t-s|) \le Q|t-s|^{\alpha_0}, \quad s,t \in E, k \in K_u.$$

Moreover, by (5.19)

$$\begin{aligned}
&\sup_{k\in K_u} \sup_{\|t-s\|<\epsilon, s,t\in E} g_{u,k}^2 \mathbb{E}\left\{[Z_{u,k}(t) - Z_{u,\tau}(s)]\, Z_{u,k}(0)\right\} \\
&\le \sup_{k\in K_u} \sup_{\|t-s\|<\epsilon, s,t\in E} g_{u,k}^2 \left(\frac{\sigma^2(t)}{2\sigma^2(u)}(1+o(1)) - \frac{\sigma^2(s)}{2\sigma^2(u)}(1+o(1))\right) \\
&\le \sup_{k\in K_u} \sup_{\|t-s\|<\epsilon, s,t\in E} \frac{g_{u,k}^2}{2\sigma^2(u)} \left(|\sigma^2(t) - \sigma^2(s)| + o(1)\right) \to 0, \quad u \to \infty, \epsilon \downarrow 0.
\end{aligned}$$

Thus **P3** is satisfied. Hence

$$g_{u,0}^2 f_{u,0}(t) \to a\sigma^2(t), \quad u \to \infty$$

uniformly with respect to $t \in E$ and

$$g_{u,k}^2 f_{u,k}(t) = 0, \quad t \in E, k \in K_u \setminus \{0\}, \quad u > 0,$$

implying that

$$\lim_{u\to\infty} \frac{p_0(u)}{\Psi(\sqrt{2}\sigma(u))} = \mathcal{P}_X^a[0,S]$$

and

$$\lim_{u\to\infty} \sup_{k\in K_u/\{0\}} \left| \frac{p_k(u)}{\Psi\left(\sqrt{2}\sigma(u)\left(1+\frac{a\sigma^2(kS)}{4\sigma^2(u)}\right)\right)} - \mathcal{H}_X[0,S] \right| = 0. \tag{5.23}$$

Dividing (5.21) by $\Psi(\sqrt{2}\sigma(u))$, letting $u \to \infty$ and applying (5.6) for sufficiently large S_1 we have

$$\begin{aligned}
\mathcal{P}_X^a[0,S] &\leq \mathcal{P}_X^a[0,S_1] + \mathcal{H}_X[0,S_1]\sum_{k=1}^{\infty} \mathrm{e}^{-\frac{a\sigma^2(kS_1)}{2}} \\
&\leq \mathcal{P}_X^a[0,S_1] + \mathcal{H}_X[0,S_1]\sum_{k=1}^{\infty} \mathrm{e}^{-Q_1(kS_1)^{\alpha_\infty}} \\
&\leq \mathcal{P}_X^a[0,S_1] + \mathcal{H}_X[0,S_1]\mathrm{e}^{-Q_2 S_1^{\alpha_\infty}}.
\end{aligned}$$

Letting $S \to \infty$ leads to

$$\lim_{S\to\infty} \mathcal{P}_X^a[0,S] \leq \mathcal{P}_X^a[0,S_1] + \mathcal{H}_X[0,S_1]\mathrm{e}^{-Q_2 S_1^{\alpha_\infty}} < \infty$$

establishing the proof. □

5.3. Proofs of (4.8)

We begin with p_1, assuming that $T \in \mathbb{N}$ is sufficiently large. For $(s,t) \in [k,k+1] \times [l,l+1]$ with $l \geq T$ and $0 \leq k \leq l$, by (5.6), we have

$$\begin{aligned}
\mathrm{Var}(Z_u(s,t)) &= \frac{[(1-\gamma)\sigma^2(ut) + (\gamma^2-\gamma)\sigma^2(us) + \gamma\sigma^2(u|t-s|)](1+ct_u)^2}{(1+c(t-\gamma s))^2\,\sigma^2(ut_u)} \\
&\leq Q\frac{t^{2\alpha_\infty+\epsilon}}{(1+c(1-\gamma)t)^2} \\
&\leq Qt^{-(2-2\alpha_\infty-\epsilon)} \\
&\leq Ql^{-(2-2\alpha_\infty-\epsilon)}
\end{aligned}$$

for u sufficiently large, with $0 < \epsilon < \min(2\alpha_\infty, 2-2\alpha_\infty)$. Moreover, in view of (5.7), for $(s,t),(s_1,t_1) \in [k,k+1]\times[l,l+1]$ with $l \ge T$ and $0 \le k \le l$ and u large enough

$$\begin{aligned}\mathrm{Var}(\overline{Z}_u(s,t)-\overline{Z}_u(s_1,t_1)) &= 2-2Cov\left(\frac{X(ut)-\gamma X(us)}{\sigma_\gamma(us,ut)}, \frac{X(ut_1),\gamma X(us_1)}{\sigma_\gamma(us_1,ut_1)}\right)\\ &= \frac{\mathrm{Var}\left(X(ut)-X(ut_1)+\gamma X(us_1)-\gamma X(us)\right)-(\sigma_\gamma(us,ut)-\sigma_\gamma(us_1,ut_1))^2}{\sigma_\gamma(us,ut)\sigma_\gamma(us_1,ut_1)}\\ &\le 2\frac{\sigma^2(u|t-t_1|)+\sigma^2(u|s-s_1|)}{\sigma_\gamma(us,ut)\sigma_\gamma(us_1,ut_1)}\\ &\le \frac{4}{(1-\gamma)^2}\frac{\sigma^2(u|t-t_1|)+\sigma^2(u|s-s_1|)}{\sigma^2(ul)}\\ &\le Q_T\left(|s-s_1|^\lambda+|t-t_1|^\lambda\right),\end{aligned}$$

where Q_T is a positive constant depending on T and $0<\lambda<\min(2\alpha_0,2\alpha_\infty)$. Thus from the above results and using further Lemma 5.1, for T large enough we have

$$\begin{aligned}p_1(u) &\le \sum_{l=T}^{\infty}\sum_{k=0}^{l}\mathbb{P}\left(\sup_{(s,t)\in[k,k+1]\times[l,l+1]} Z_u(s,t)>m(u)\right)\\ &\le \sum_{l=T}^{\infty}\sum_{k=0}^{l}\mathbb{P}\left(\sup_{(s,t)\in[0,1]^2}\overline{Z}_u(s+k,t+l)>\frac{m(u)}{\sqrt{Ql^{-(2-2\alpha_\infty-\epsilon)}}}\right)\\ &\le \sum_{l=T}^{\infty} Q_2 l\left(m^2(u)l^{2-2\alpha_\infty-\epsilon}\right)^{2/\lambda}\mathrm{e}^{-Q_1 m^2(u)l^{2-2\alpha_\infty-\epsilon}}\\ &\le \mathrm{e}^{-Q_3m^2(u)T^{2-2\alpha_\infty-\epsilon}}\\ &= o\left(\frac{u}{m(u)\varDelta_1(u)}\varPsi(m(u))\right).\end{aligned}$$

Next, we show that $p_2(u)$ is also negligible. By UCT, we have

$$\mathrm{Var}(Z_u(s,t)) \to \frac{[(1-\gamma)t^{2\alpha_\infty}+(\gamma^2-\gamma)s^{2\alpha_\infty}+\gamma|t-s|^{2\alpha_\infty}](1+ct_*)^2}{\left(1+c(t-\gamma s)\right)^2 {t_*}^{2\alpha_\infty}} = \frac{f(s,t)}{f(0,t_*)}, \quad u\to\infty$$

uniformly over $D_{\delta,u}$, where $f(s,t)$ is defined in (5.25) with $(0,t_*)$ the unique maximum point over D. Consequently, there exists a constant $0<b_\delta<1$ such that for u large enough

$$\sup_{(s,t)\in D_{\delta,u}}\mathrm{Var}(Z_u(s,t))<b_\delta.$$

Furthermore, by (5.7) for u large enough we have

$$\begin{aligned}\mathrm{Var}(Z_u(s,t)-Z_u(s_1,t_1)) &= \frac{(1+ct_u)^2}{\sigma^2(ut_u)}\mathbb{E}\Big\{\Big(\frac{X(ut)-\gamma X(us)}{1+c(t-\gamma s)}-\frac{X(ut_1)-\gamma X(us_1)}{1+c(t_1-\gamma s_1)}\Big)^2\Big\}\\ &\le Q_4\left(\frac{\sigma^2(u|t-t_1|)}{\sigma^2(ut_u)}+\frac{\sigma^2(u|s-s_1|)}{\sigma^2(ut_u)}+(t-t_1)^2+(s-s_1)^2\right)\\ &\le Q_5(|t-t_1|^\lambda+|s-s_1|^\lambda), \quad (s,t),(s_1,t_1)\in D_T,\end{aligned} \tag{5.24}$$

with $\lambda\in(0,\min(2\alpha_0,2\alpha_\infty))$. Consequently, by Lemma 5.1

$$p_2(u)\le Q_6T^2(m(u))^{4/\lambda}\varPsi\left(\frac{m(u)}{b_\delta}\right)=o\left(\frac{u}{m(u)\varDelta_1(u)}\varPsi(m(u))\right).$$

Finally, we focus on $p_3(u)$. In light of Lemma 4.1, we know that for δ sufficiently small and u sufficiently large

$$\begin{aligned}\sup_{(s,t)\in D^*_{\delta,u}} \mathrm{Var}(Z_u(s,t)) &\le \sup_{(s,t)\in D^*_{\delta,u}} \left(1-\frac{a_1}{2}(t-t_u)^2-\frac{a_2}{2}\frac{\sigma^2(us)}{\sigma^2(u)}\right)\\ &\le \sup_{(s,t)\in D^*_{\delta,u}} \left(1-\frac{a_1}{2}(t-t_u)^2\right)\\ &\le 1-Q_7\left(\frac{\ln m(u)}{m(u)}\right)^2,\end{aligned}$$

which together with (5.24) and the application of Lemma 5.1 leads to

$$p_3(u)\le Q_8(m(u))^{\frac{4}{\lambda}}\Psi\left(\frac{m(u)}{\sqrt{1-Q_7\left(\frac{\ln m(u)}{m(u)}\right)^2}}\right)=o\left(\frac{u}{m(u)\Delta_1(u)}\Psi(m(u))\right),\quad u\to\infty$$

establishing (4.8). □

5.4. Proofs of Lemmas 4.1–4.4

In this section we present details of the proof of Lemmas 4.1–4.4.

Proof of Lemma 4.1. For any $u>0$ we have

$$\sigma^2_{\gamma,u}(s,t)=\frac{(1-\gamma)\sigma^2(ut)+(\gamma^2-\gamma)\sigma^2(us)+\gamma\sigma^2(u(t-s))}{(1+c(t-\gamma s))^2}.$$

By UCT we have as $u\to\infty$

$$\frac{\sigma^2_{\gamma,u}(s,t)}{\sigma^2(u)}\to\frac{(1-\gamma)t^{2\alpha_\infty}+(\gamma^2-\gamma)s^{2\alpha_\infty}+\gamma(t-s)^{2\alpha_\infty}}{(1+c(t-\gamma s))^2}=:f(s,t) \tag{5.25}$$

uniformly for $0\le s\le t\le T$ with T any positive constant. Using (5.6) for any $0<\epsilon<\min(2\alpha_\infty,2-2\alpha_\infty)$ there exists a constant $u_\epsilon>0$ such that for all $0\le s\le t<\infty, t>T>1$ and $u>u_\epsilon$, we have

$$\frac{\sigma^2_{\gamma,u}(s,t)}{\sigma^2(u)}\le Q\frac{((1-\gamma)t^{2\alpha_\infty+\epsilon}+\gamma t^{2\alpha_\infty+\epsilon})}{(1+c(t-\gamma s))^2}\le\frac{Q}{t^{2-2\alpha_\infty-\epsilon}}\Rightarrow 0,\quad t\to\infty. \tag{5.26}$$

It follows from [15] that $f(s,t)$ has one unique maximum point $(0,t_*)$ over D, which combined with (5.25) and (5.26) yields that for u large enough, the maximum point of $\sigma^2_{\gamma,u}(s,t)$ denoted by (s_u,t_u) must be attained over $0\le s\le t\le T$ with $T>t_*$ large enough. Further, $(s_u,t_u)\to(0,t_*)$. By contradiction, suppose that $(s_u,t_u)\to(s_1^*,t_1^*)\ne(0,t_*)$. Hence, by (5.25), we have that

$$f(s_1^*,t_1^*)=\lim_{u\to\infty}\frac{\sigma^2_{\gamma,u}(s_u,t_u)}{\sigma^2(u)}\ge\lim_{u\to\infty}\frac{\sigma^2_{\gamma,u}(0,t_*)}{\sigma^2(u)}=f(0,t_*).$$

This contradicts the fact that $(0,t_*)$ is the unique maximum point of $f(s,t)$ over D. Next, we prove that the maximum point is unique. It follows that for $0<s<t<\infty$

$$\begin{aligned}\frac{\partial\sigma^2_{\gamma,u}(s,t)}{\partial s}&=\gamma A^{-4}(s,t)\left\{\left((\gamma-1)\dot{\sigma^2}(us)u-\dot{\sigma^2}(u(t-s))u\right)A^2(s,t)+2c\sigma^2_\gamma(us,ut)A(s,t)\right\},\\ \frac{\partial\sigma^2_{\gamma,u}(s,t)}{\partial t}&=A^{-4}(s,t)\left\{\left((1-\gamma)\dot{\sigma^2}(ut)u+\gamma\dot{\sigma^2}(u(t-s))u\right)A^2(s,t)-2c\sigma^2_\gamma(us,ut)A(s,t)\right\},\end{aligned} \tag{5.27}$$

with $A(s,t)=1+c(t-\gamma s)$. Suppose that $s_u>0$, then by the continuous differentiability of $\sigma^2_{\gamma,u}(s,t)$, we have

$$\frac{\partial \sigma^2_{\gamma,u}(s,t)}{\partial s}\Big|_{(s,t)=(s_u,t_u)}=\frac{\partial \sigma^2_{\gamma,u}(s,t)}{\partial t}\Big|_{(s,t)=(s_u,t_u)}=0,$$

which implies that

$$\dot{\sigma^2}(us_u)=\dot{\sigma^2}(ut_u)-\dot{\sigma^2}(u(t_u-s_u))=\ddot{\sigma^2}(u\theta_u)us_u,$$

with $\theta_u\in(t_u-s_u,t_u)$. For $K(t)=t/\dot{\sigma^2}(t)$ defined in (5.8), the last equation can be re-written as

$$\frac{u\theta_u\ddot{\sigma^2}(u\theta_u)}{\dot{\sigma^2}(u\theta_u)}\frac{K(us_u)}{K(u\theta_u)}=1. \tag{5.28}$$

Since **AI** holds, then by (5.1-5.2) and using UCT, we have

$$\lim_{u\to\infty}\frac{u\theta_u\ddot{\sigma^2}(u\theta_u)}{\dot{\sigma^2}(u\theta_u)}=2\alpha_\infty-1,\quad \lim_{u\to\infty}\frac{K(us_u)}{K(u\theta_u)}=0.$$

Hence, for u large enough

$$\frac{u\theta_u\ddot{\sigma^2}(u\theta_u)}{\dot{\sigma^2}(u\theta_u)}\frac{K(us_u)}{K(u\theta_u)}<1,$$

which contradicts (5.28). Consequently, for u large enough then $s_u=0$ and t_u is the maximum point of $\sigma^2_{\gamma,u}(0,t)=\frac{\sigma^2(ut)}{(1+ct)^2}$. It follows that (the following derivatives are all with respect to t)

$$\frac{\dot{\sigma^2_{\gamma,u}}(0,t)}{\sigma^2(u)}\to\dot{f}(0,t),\quad\text{and}\quad \frac{\ddot{\sigma^2_{\gamma,u}}(0,t)}{\sigma^2(u)}\to\ddot{f}(0,t)<0,\quad u\to\infty$$

hold uniformly over $[t_*-\delta,t_*+\delta]$ for $\delta>0$ small enough. This implies that $\frac{\dot{\sigma^2_{\gamma,u}}(0,t)}{\sigma^2(u)}$ is decreasing over $[t_*-\delta,t_*+\delta]$ for $\delta>0$. Thus t_u is unique and then $(0,t_u)$ is unique. We also have that the first derivative of $\sigma^2_{\gamma,u}(0,t)$ with respect to t at point t_u equals zero (see (5.27)), *i.e.*,

$$\frac{\partial \sigma^2_{\gamma,u}(0,t)}{\partial t}\Big|_{t=t_u}=A^{-4}(0,t_u)\left\{\left((1-\gamma)\dot{\sigma^2}(ut_u)u+\gamma\dot{\sigma^2}(ut_u)u\right)A^2(0,t_u)-2c\sigma^2_\gamma(0,ut_u)A(0,t_u)\right\}=0,$$

which is equivalent to

$$u\dot{\sigma^2}(ut_u)(1+ct_u)^2=2c\sigma^2(ut_u)(1+ct_u). \tag{5.29}$$

For any $u>0$

$$(1+ct_u+c(t-t_u-\gamma s))^2\sigma^2(ut_u)=(1+ct_u)^2\sigma^2(ut_u)+2c(1+ct_u)(t-t_u-\gamma s)\sigma^2(ut_u)+c^2(t-t_u-\gamma s)^2\sigma^2(ut_u)$$

and by Taylor expansion

$$\sigma^2(ut)=\sigma^2(ut_u)+\dot{\sigma^2}(ut_u)u(t-t_u)+\frac{1}{2}\ddot{\sigma^2}(u\theta_{1,u})u^2(t-t_u)^2,$$

$$\sigma^2(u(t-s))=\sigma^2(ut_u)+\dot{\sigma^2}(ut_u)u(t-t_u-s)+\frac{1}{2}\ddot{\sigma^2}(u\theta_{2,u})u^2(t-t_u-s)^2,$$

with $\theta_{1,u} \in (t, t_u)$ and $\theta_{2,u} \in (t-s, t_u)$. Inserting the above expansions to the following equation and using (5.29), we have

$$\begin{aligned}
1 - \frac{\sigma^2_{\gamma,u}(s,t)}{\sigma^2_{\gamma,u}(0,t_u)} &= \frac{(1+c(t-\gamma s))^2\sigma^2(ut_u) - \left[(1-\gamma)\sigma^2(ut) + (\gamma^2-\gamma)\sigma^2(us) + \gamma\sigma^2(u(t-s))\right](1+ct_u)^2}{(1+c(t-\gamma s))^2\sigma^2(ut_u)} \\
&= \frac{(\gamma-\gamma^2)(1+ct_u)^2\sigma^2(us) - \frac{1-\gamma}{2}u^2\ddot{\sigma^2}(u\theta_{1,u})(1+ct_u)^2(t-t_u)^2}{(1+c(t-\gamma s))^2\sigma^2(ut_u)} \\
&\quad + \frac{\sigma^2(ut_u)c^2(t-t_u-\gamma s)^2 - \frac{\gamma}{2}\ddot{\sigma^2}(u\theta_{2,u})u^2(1+ct_u)^2(t-t_u-s)^2}{(1+c(t-\gamma s))^2\sigma^2(ut_u)} \\
&= \frac{(\gamma-\gamma^2)(1+ct_u)^2\sigma^2(us) + \left(\sigma^2(ut_u)c^2 - \frac{1-\gamma}{2}u^2\ddot{\sigma^2}(u\theta_{1,u})(1+ct_u)^2 - \frac{\gamma}{2}\ddot{\sigma^2}(u\theta_{2,u})u^2(1+ct_u)^2\right)(t-t_u)^2}{(1+c(t-\gamma s))^2\sigma^2(ut_u)} \\
&\quad + \frac{\sigma^2(ut_u)c^2(\gamma^2 s^2 - 2\gamma(t-t_u)s) - \frac{\gamma}{2}\ddot{\sigma^2}(u\theta_{2,u})u^2(1+ct_u)^2(s^2-2(t-t_u)s)}{(1+c(t-\gamma s))^2\sigma^2(ut_u)}
\end{aligned} \tag{5.30}$$

It follows from (5.5) that for any $\delta > 0$ and u large enough

$$\frac{s^2}{\frac{\sigma^2(us)}{\sigma^2(ut_u)}} = t_u^2 \frac{g_2(ut_u)}{g_2(us)} \le 2t_*^{2\alpha_\infty}\delta^{2-2\alpha_\infty}, \quad s \in (0,\delta]. \tag{5.31}$$

Following (5.2), we have that

$$\frac{\ddot{\sigma^2}(u\theta_{i,u})u^2}{\sigma^2(ut_u)} \sim \frac{2\alpha_\infty(2\alpha_\infty - 1)}{(t_*)^2}, \quad u \to \infty, i = 1,2. \tag{5.32}$$

Moreover, for $\delta > 0$ sufficiently small and u sufficiently large

$$\begin{aligned}
|t-t_u|s &\le \delta^{(1-\alpha_\infty)/2}|t-t_u|\delta^{-(1-\alpha_\infty)/2}s \\
&\le \delta^{1-\alpha_\infty}(t-t_u)^2 + \delta^{\alpha_\infty-1}s^2 \\
&\le Q\delta^{1-\alpha_\infty}\left(\frac{\sigma^2(us)}{\sigma^2(u)} + (t-t_u)^2\right), \quad s \in (0,\delta].
\end{aligned} \tag{5.33}$$

Hence inserting (5.31)–(5.33) into (5.30), we have that for u sufficiently large

$$\begin{aligned}
(2a_1 - Q\delta^{1-\alpha_\infty})(t-t_u)^2 + (2a_2 - Q\delta^{1-\alpha_\infty})\frac{\sigma^2(us)}{\sigma^2(u)} &\le 1 - \frac{\sigma^2_{\gamma,u}(s,t)}{\sigma^2_{\gamma,u}(0,t_u)} \\
&\le (2a_1 + Q\delta^{1-\alpha_\infty})(t-t_u)^2 + (2a_2 + Q\delta^{1-\alpha_\infty})\frac{\sigma^2(us)}{\sigma^2(u)}
\end{aligned}$$

for $0 < s < \delta$ and $|t - t_u| < \delta$ with $\delta > 0$ sufficiently small and Q a fixed constant, which establishes the claim. □

Proof of Lemma 4.2. It follows from the direct calculation that

$$1 - r_u(s,t,s_1,t_1) = \frac{D_{1,u}(s,t,s_1,t_1) - D_{2,u}(s,t,s_1,t_1) + \gamma D_{3,u}(s,t,s_1,t_1)}{2\sigma_\gamma(us,ut)\sigma_\gamma(us_1,ut_1)},$$

with

$$\begin{aligned}
&D_{1,u}(s,t,s_1,t_1) = \sigma^2(u|t-t_1|) + \gamma^2\sigma^2(u|s-s_1|), \quad D_{2,u}(s,t,s_1,t_1) = (\sigma_\gamma(us,ut) - \sigma_\gamma(us_1,ut_1))^2, \\
&D_{3,u}(s,t,s_1,t_1) = \sigma^2(u|t-s|) + \sigma^2(u|t_1-s_1|) - \sigma^2(u|t_1-s|) - \sigma^2(u|t-s_1|).
\end{aligned}$$

Using Taylor expansion, we have

$$\begin{aligned} D_{3,u}(s,t,s_1,t_1) &= u\dot{\sigma^2}(u(t_1-s))(t-t_1) + \frac{1}{2}u^2\ddot{\sigma^2}(u\theta_1)(t-t_1)^2 \\ &\quad + u\dot{\sigma^2}(u(t-s_1))(t_1-t) + \frac{1}{2}u^2\ddot{\sigma^2}(u\theta_2)(t-t_1)^2 \\ &= \frac{1}{2}u^2\ddot{\sigma^2}(u\theta_1)(t-t_1)^2 + \frac{1}{2}u^2\ddot{\sigma^2}(u\theta_2)(t-t_1)^2 \\ &\quad + u^2\ddot{\sigma^2}(u\theta_3)(t-t_1)(t_1-t+s_1-s) \\ &\le u^2\left(\frac{1}{2}\ddot{\sigma^2}(u\theta_1) + \frac{1}{2}\ddot{\sigma^2}(u\theta_2) + 2\ddot{\sigma^2}(u\theta_3)\right)(t-t_1)^2 + 2u^2\ddot{\sigma^2}(u\theta_3)(s-s_1)^2, \end{aligned}$$

where θ_1, θ_2 and θ_3 are some positive constants (depending on u) satisfying $\frac{t_*}{2} < \theta_i < \frac{3}{2}t_*, i = 1,2,3$ for u sufficiently large. From (5.2) and (5.4), we have that for $\delta > 0$

$$\sup_{t\in(0,\delta)}\left|\frac{u^2\ddot{\sigma^2}(u)t^2}{\sigma^2(ut)}\right| \le Q\sup_{t\in(0,\delta)}\frac{\sigma^2(u)t^2}{\sigma^2(ut)} = Q\sup_{t\in(0,\delta)}\frac{g_2(u)}{g_2(ut)},$$

which together with (5.5) implies that if $\delta_u \to 0$ as $u \to \infty$

$$\sup_{t\in(0,\delta_u)}\left|\frac{u^2\ddot{\sigma^2}(u)t^2}{\sigma^2(ut)}\right| \le Q\sup_{t\in(0,\delta_u)}\frac{g_2(u)}{g_2(ut)} \to 0, \quad u\to\infty.$$

Therefore we get that uniformly for $(s,t) \neq (s_1,t_1) \in [0,\delta_u)\times(t_u-\delta_u, t_u+\delta_u)$

$$\frac{D_{3,u}(s,t,s_1,t_1)}{D_{1,u}(s,t,s_1,t_1)} \to 0, \quad u\to\infty.$$

By (5.4) and **AIII** we have for any $x\in(0,\infty)$ and any $y\in[0,1]$

$$1 \ge \frac{\sigma^2(xy)}{\sigma^2(x)} = \frac{g_2(xy)}{g_2(x)}y^2 \ge y^2.$$

Hence by UCT for $0 \le s_1 < s < \delta_u$ with $\delta_u \to 0$

$$\frac{\left(\sigma^2(us)-\sigma^2(us_1)\right)^2}{\sigma^2(u|s-s_1|)\sigma^2(u)} = \frac{\sigma^2(us)}{\sigma^2(u)}\frac{\left(1-\frac{\sigma^2(us_1)}{\sigma^2(us)}\right)^2}{\frac{\sigma^2(us(1-s_1/s))}{\sigma^2(us)}} \le \frac{\sigma^2(us)}{\sigma^2(u)}(1+s_1/s)^2 \le 4\frac{\sigma^2(us)}{\sigma^2(u)} \to 0, \quad u\to\infty. \tag{5.34}$$

By (5.1) and (5.4) we have

$$\begin{aligned} &\frac{D_{2,u}(s,t,s_1,t_1)}{D_{1,u}(s,t,s_1,t_1)} \\ &\le 4\frac{(1-\gamma)^2(\sigma^2(ut)-\sigma^2(ut_1))^2 + \gamma^2(\sigma^2(u(t-s))-\sigma^2(u(t_1-s_1)))^2 + (\gamma-\gamma^2)^2(\sigma^2(us)-\sigma^2(us_1))^2}{D_{1,u}(s,t,s_1,t_1)\left(\sigma_\gamma(us,ut)+\sigma_\gamma(us_1,ut_1)\right)^2} \\ &\le Q\left(\frac{(u\dot{\sigma^2}(u))^2(t-t_1)^2}{\sigma^2(u)\sigma^2(u|t-t_1|)} + \frac{(u\dot{\sigma^2}(u))^2(s-s_1)^2}{\sigma^2(u)\sigma^2(u|s-s_1|)} + \frac{\left(\sigma^2(us)-\sigma^2(us_1)\right)^2}{\sigma^2(u|s-s_1|)\sigma^2(u)}\right) \\ &\le Q_1\left(\frac{g_2(u)}{g_2(u|t-t_1|)} + \frac{g_2(u)}{g_2(u|s-s_1|)} + \frac{\left(\sigma^2(us)-\sigma^2(us_1)\right)^2}{\sigma^2(u|s-s_1|)\sigma^2(u)}\right). \end{aligned}$$

Further, it follows from (5.5) and (5.34) that

$$\frac{D_{2,u}(s,t,s_1,t_1)}{D_{1,u}(s,t,s_1,t_1)} \to 0,$$

as $u \to \infty$ uniformly for $(s,t) \neq (s_1,t_1) \in [0,\delta_u) \times (t_u - \delta_u, t_u + \delta_u)$ with $\delta_u \to 0$. This completes the proof. □

Proof of Lemma 4.3. We have

$$\begin{aligned}\sigma^2_{1,u}(s,t) &= \frac{(1-\gamma)\sigma^2(t) + (\gamma^2-\gamma)\sigma^2(s) + \gamma\sigma^2(t-s)}{\sigma^2(T)} \frac{(u+cT)^2}{(u+c(t-\gamma s))^2}\\ &=: f_1(s,t) f_{2,u}(s,t), \quad (s,t) \in D_T = \{(s,t), 0 \le s \le t \le T\}.\end{aligned}$$

In light of **BIII**, $f_1(s,t)$ is strictly increasing with respect to t and strictly decreasing with respect to s for $(s,t) \in D_T$. Moreover,

$$\lim_{u\to\infty} \sup_{(s,t)\in D_T} |f_{2,u}(s,t) - 1| = 0.$$

Thus we conclude that the maximum value of $\sigma^2_{1,u}(s,t)$ over D_T must be attained in a sufficiently small neighbourhood of $(0,T)$ for u large enough. Further, as $(s,t) \to (0,T)$

$$1 - f_1(s,t) = \frac{\dot{\sigma^2}(T)}{\sigma^2(T)}(T-t)(1+o(1)) + \begin{cases} \dfrac{\gamma\dot{\sigma^2}(T)}{\sigma^2(T)} s(1+o(1)), & \text{if } \sigma^2(s) = o(s),\\ \dfrac{b(\gamma-\gamma^2) + \gamma\dot{\sigma^2}(T)}{\sigma^2(T)} s(1+o(1)), & \text{if } \sigma^2(s) \sim bs,\\ \dfrac{\gamma-\gamma^2}{\sigma^2(T)}\sigma^2(s)(1+o(1)), & \text{if } s = o(\sigma^2(s)),\end{cases}$$

and for $u > 1$

$$1 - f_{2,u}(s,t) = \frac{-2c}{u+cT}\left(T - t + \gamma s\right)(1+o(1)),$$

which imply that (4.34) holds and further the maximum point of $\sigma_{1,u}(s,t)$ in a neighbourhood of $(0,T)$ is $(0,T)$. Thus the claim is established. □

Proof of Lemma 4.4. The proof is similar to that of Lemma 4.2. We have

$$1 - r_1(s,t,s_1,t_1) = \frac{D_1(s,t,s_1,t_1) - D_2(s,t,s_1,t_1) + \gamma D_3(s,t,s_1,t_1)}{2\sigma_\gamma(s,t)\sigma_\gamma(s_1,t_1)},$$

with

$$\begin{aligned}&D_1(s,t,s_1,t_1) = \sigma^2(|t-t_1|) + \gamma^2\sigma^2(|s-s_1|), \quad D_2(s,t,s_1,t_1) = (\sigma_\gamma(s,t) - \sigma_\gamma(s_1,t_1))^2,\\ &D_3(s,t,s_1,t_1) = \sigma^2(|t-s|) + \sigma^2(|t_1-s_1|) - \sigma^2(|t_1-s|) - \sigma^2(|t-s_1|).\end{aligned}$$

Using Taylor expansion and the fact that $t^2 = o(\sigma^2(t))$ as $t \downarrow 0$, we have

$$\begin{aligned}D_3(s,t,s_1,t_1) &= \dot{\sigma^2}(t_1-s)(t-t_1) + \frac{1}{2}\ddot{\sigma^2}(\theta_4)(t-t_1)^2 + \dot{\sigma^2}(t-s_1)(t_1-t) + \frac{1}{2}\ddot{\sigma^2}(\theta_5)(t-t_1)^2\\ &= \frac{1}{2}\ddot{\sigma^2}(\theta_4)(t-t_1)^2 + \frac{1}{2}\ddot{\sigma^2}(\theta_5)(t-t_1)^2 + \ddot{\sigma^2}(\theta_6)(t-t_1)(t_1-t+s_1-s)\\ &\le \left(\frac{1}{2}\ddot{\sigma^2}(\theta_4) + \frac{1}{2}\ddot{\sigma^2}(\theta_5) + 2\ddot{\sigma^2}(\theta_6)\right)(t-t_1)^2 + 2\ddot{\sigma^2}(\theta_6)(s-s_1)^2\\ &= o\left(D_1(s,t,s_1,t_1)\right), \quad s,s_1 \to 0, t, t_1 \to T,\end{aligned}$$

where θ_4, θ_5 and θ_6 are some positive constants satisfying $\frac{T}{2} < \theta_i < \frac{3}{2}T, i = 4,5,6$. By (5.4) and **BIII** we have for any $x \in (0,\infty)$ and any $y \in [0,1]$

$$1 \geq \frac{\sigma^2(xy)}{\sigma^2(x)} = \frac{g_2(xy)}{g_2(x)} y^2 \geq y^2,$$

hence for $0 \leq s_1 < s < T/2$

$$\begin{aligned}\frac{\left(\sigma^2(s) - \sigma^2(s_1)\right)^2}{\sigma^2(|s - s_1|)} &= \sigma^2(s) \frac{\left(1 - \frac{\sigma^2(s_1)}{\sigma^2(s)}\right)^2}{\frac{\sigma^2(s(1-s_1/s))}{\sigma^2(s)}} \\ &\leq \sigma^2(s)(1 + s_1/s)^2 \\ &\leq 4\sigma^2(s) \to 0, \quad s \to 0. \end{aligned} \tag{5.35}$$

By (5.4), (5.35) and the fact that $t^2 = o(\sigma^2(t))$ as $t \downarrow 0$, we have

$$\begin{aligned} D_2(s,t,s_1,t_1) &= \frac{(\sigma_\gamma^2(s,t) - \sigma_\gamma^2(s_1,t_1))^2}{(\sigma_\gamma(s,t) + \sigma_\gamma(s_1,t_1))^2} \\ &= \frac{\left((1-\gamma)(\sigma^2(t) - \sigma^2(t_1)) + (\gamma^2 - \gamma)(\sigma^2(s) - \sigma^2(s_1)) + \gamma(\sigma^2(t-s) - \sigma^2(t_1 - s_1))\right)^2}{(\sigma_\gamma(s,t) + \sigma_\gamma(s_1,t_1))^2} \\ &\leq \frac{8}{\sigma^2(T)} \left((\dot{\sigma^2}(T))^2(t-t_1)^2 + (\dot{\sigma^2}(T))^2(t - t_1 - s + s_1)^2 + (\sigma^2(s) - \sigma^2(s_1))^2\right) \\ &= o\left(D_1(s,t,s_1,t_1)\right), \quad s, s_1 \to 0, t, t_1 \to T. \end{aligned}$$

Therefore, we have

$$1 - r_1(s,t,s_1,t_1) \sim \frac{\sigma^2(|t-t_1|) + \gamma^2\sigma^2(|s-s_1|)}{2\sigma^2(T)}, \quad s, s_1 \to 0, t, t_1 \to T,$$

which completes the proof. □

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