ESAIM: PS 18 (2014) 854–880
DOI: 10.1051/ps/2014008

ESAIM: Probability and Statistics
www.esaim-ps.org

COMPACT CONVEX SETS OF THE PLANE AND PROBABILITY THEORY *

Jean-François Marckert[1] and David Renault[1]

Abstract. The Gauss−Minkowski correspondence in $\mathbb{R}^2$ states the existence of a homeomorphism between the probability measures μ on $[0, 2\pi]$ such that $\int_0^{2\pi} e^{ix} d\mu(x) = 0$ and the compact convex sets (CCS) of the plane with perimeter 1. In this article, we bring out explicit formulas relating the border of a CCS to its probability measure. As a consequence, we show that some natural operations on CCS – for example, the Minkowski sum – have natural translations in terms of probability measure operations, and reciprocally, the convolution of measures translates into a new notion of convolution of CCS. Additionally, we give a proof that a polygonal curve associated with a sample of n random variables (satisfying $\int_0^{2\pi} e^{ix} d\mu(x) = 0$) converges to a CCS associated with μ at speed $\sqrt{n}$, a result much similar to the convergence of the empirical process in statistics. Finally, we employ this correspondence to present models of smooth random CCS and simulations.

Mathematics Subject Classification. 52A10, 60B05, 60D05, 60F17, 60G99.

Received December 11, 2012. Revised July 15, 2013.

1. Introduction

Convex sets are central in mathematics: they appear everywhere! Nice overviews of the topic have been provided by Busemann [8], Pólya [21] and Pogorelov [20]. In probability theory, *compact convex sets* (CCS) appear in 1865 with Sylvester's question [25]: for $n = 4$ points chosen independently and at random in the unit square K, what is the probability that these n points are in convex position? The question can be generalised to various shapes K, different values of n, and other dimensions. It has been recently solved by Valtr [27, 28] when K is a triangle or a parallelogram and by Marckert [17] when K is a circle (see also Bárány [1], Buchta [7] and Bárány [2]). Random CCS also show up as the cells of the Voronoï diagram of a Poisson point process (see Calka [9]), and in the problem of determining the distribution of convex polygonal lines subject to some constraints. For example, when the vertices are constrained to belong to a lattice, the problem has been widely investigated (Sinai [24], Bárány and Vershik [3], Vershik and Zeitouni [29], Bogachev and Zarbaliev [6]). Another combinatorial model related to this question is based on the *digitally convex polyominos* (DCPs). The DCP associated to a convex planar set C is the maximal convex polyomino with vertices in $\mathbb{Z}^2$ included in C. Let D_n be the set of DCPs with perimeter $2n$. In a recent paper, Bodini, Duchon and Jacquot [5] investigate the limit

Keywords and phrases. Random convex sets, symmetrisation, weak convergence, Minkowski sum.

* *Part of this work is supported by the ANR PRESAGE (ANR-11-BS02-003).*

[1] CNRS, LaBRI, Université de Bordeaux, 351 cours de la Libération, 33405 Talence cedex, France.
name@labri.fr; renault@labri.fr

shape of uniform DCPs taken in D_n under the uniform distribution $\mathbb{U}_n$. Even if not convex, these polyominos can be seen as discretisation of CCS.

All these models possess the same drawbacks: they are discrete models (polygonal, except for DCP) and their limit when the size parameter goes to $+\infty$ are deterministic shapes. To our knowledge, no model of random non-polygonal CCS have been investigated yet. One of the goals of this article is to develop tools that allow one to provide examples of such models, and this goal is attained in the following manner:

• First, we state a connection between the CCS of the plane and probability measures. Theorem 2.2 asserts that the set of CCS of the plane having perimeter 1, considered up to translation, is in one-to-one correspondence with the set $\mathcal{M}_{\mathcal{T}}^0$ of probability distributions μ on the circle $\mathbb{R}/(2\pi\mathbb{Z})$ satisfying $\int_0^{2\pi}\exp(ix)\mathrm{d}\mu(x) = 0$. This famous theorem, revisited in Section 2.2, is sometimes called in the literature the Gauss−Minkowski Theorem (*cf.* Vershik [29] and Busemann [8], Sect. 8), and the measure μ is called the surface area measure of the CCS [18]. Moreover, the bijection is an homeomorphism when both sets are equipped with natural topologies. In this article, we provide an explicit parametrisation of a CCS in terms of the distribution function of μ. This perspective brings out a new and important relation between the CCS with perimeter 1 and probability measures, differing in this from the more generic "arbitrary total mass" measures.

• This connection with probability theory appears therefore as a natural tool to define new operations on CCS and revisit numerous known results that were proved using geometrical arguments. For instance, the set $\mathcal{M}_{\mathcal{T}}^0$ is stable by convolution and mixture. This induces natural operations on CCS that one may also qualify of *convolution* and *mixture*. As a matter of fact, the mixture of CCS defined in this way coincides with the Minkowski addition (Sect. 3.1), and Minkowski symmetrisation simply maps a CCS associated to a measure μ onto the CCS associated with $\frac{1}{2}(\mu+\mu(2\pi- \,.\,)$ (Prop. 3.4). The notions of *convolution of CCS* and *symmetrisation by convolution* (Sects. 3.2 and 3.3) appear to be new and provide a new proof of the isoperimetric inequality (Thm. 3.6). Roughly, the CCS obtained by convolution of two CCS has a radius of curvature function equal to the convolution of the curvature functions of these two CCS.

• The probabilistic approach also allows one to prove stochastic convergence theorems for models that differ radically from the ones mentioned earlier. Consider for instance $\mu \in \mathcal{M}_{\mathcal{T}}^0$, and take n random variables $\{X_j, j = 0,\dots,n-1\}$ i.i.d. according to μ. Let $\{\widehat{X}_j, j = 0,\dots,n-1\}$ be the X_k's reordered in $[0,2\pi)$. Let B_n be the curve formed by the concatenation of the vectors $\mathrm{e}^{i\widehat{X}_j}$. We show that the curve B_n rescaled by n converges when $n\to\infty$ to the boundary $\mathcal{B}_\mu$ of a CCS associated with μ (Thm. 2.8 and Cor. 2.9). This convergence holds at speed $\sqrt{n}$ and has Gaussian fluctuations (Thm. 2.8). As a generalisation, every distribution on $\mathbb{C}$ with mean 0 can be sent on a CCS by a second correspondence (which is not bijective) (Sect. 4.2). Again, the appropriate point of view consists in considering the boundary of the CCS as the limit of the curve associated with a sample of n random variables (r.v.) sorted according to their argument.

• The last part of this paper (Sect. 5) is devoted to the investigation of models of random CCS that stem from the aforesaid connection. Our first model is a model of random polygons defined as follows: take $\{z_j, j = 0,\dots,n-1\}$ i.i.d. according to a distribution ν in $\mathbb{C}$. Let $\{y_i = z_{i+1 \bmod n} - z_i, i = 0,\dots,n-1\}$ and $\{\widehat{y}_j, j = 0,\dots,n-1\}$ the y_i's sorted according to their argument. The $\widehat{y}_i's$ are the consecutive vector sides of the polygonal CCS with vertices $\{\sum_{j=0}^{d}\widehat{y}_j, d = 0,\dots,n-1\}$. When $n\to\infty$, a rescaled version of this CCS converges in distribution to a deterministic CCS (Thms. 4.2 and 5.1). We discuss the finite case in Section 5.1.

• Another model results from the role that Fourier series play in the representation of the boundaries of CCS. For a r.v. X with values in $[0,2\pi]$ and distribution μ, the Fourier coefficients of μ, namely $a_n(\mu) = \mathbb{E}(\cos(nX))$ and $b_n(\mu) = \mathbb{E}(\sin(nX))$, are well defined for any $n \geq 0$. Our bijection between CCS and measures in hand, the question of designing a model of random CCS is equivalent to that of designing a model of random measure μ satisfying a.s.$\int_0^{2\pi}\exp(ix)\mathrm{d}\mu(x) = 0$ (equivalently $a_1(\mu) = b_1(\mu) = 0$ a.s.). Nevertheless to design a model of random measures μ satisfying these constraints is not equivalent to design random Fourier coefficients $(a_n, b_n, n \geq 0)$ since these latter may not correspond to those of a probability measure. In Section 5, we explain how this can be handled, and provide several models of random CCS that are not random polygons.

Notations. "CCS" will always be used for "compact convex set of the plane $\mathbb{R}^2$". We assume that all the mentioned r.v. are defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and denote by $\mathbb{E}$ the expectation. For any probability distribution μ, X_μ designates a r.v. with distribution μ. We write $X \sim \mu$ to say that X has distribution μ. The notations $\xrightarrow[n]{(d)}$, $\xrightarrow[n]{(proba.)}$, $\xrightarrow[n]{(weak)}$ stand for the convergence in distribution, in probability, and the weak convergence.

2. Correspondence between CCS and distributions

We start this section by recalling some simple facts concerning CCS and measures on the circle $\mathbb{R}/(2\pi\mathbb{Z})$. Thereafter we state the Gauss−Minkowski theorem (Thm. 2.2) which establishes a correspondence between measures and CCS, and we provide a new proof based on probabilistic arguments. In Section 2.4 we express the area of a CCS thanks to the Fourier coefficients of the associated measure. Finally in Section 2.5 we state one of the main results of the paper (Thm. 2.8): under some mild hypotheses, it ensures the convergence of the trajectory made of n i.i.d. increments sorted according to their arguments and rescaled by n to a limit CCS boundary at speed $\sqrt{n}$.

2.1. CCS of the plane

A subset S of $\mathbb{R}^2$ is a *convex* set if for any $z_1, z_2 \in S$, the segment $[z_1, z_2] \subset S$. In this paper, we are interested only in CCS of the Euclidean plane $\mathbb{R}^2$. Let Seg be the set of bounded closed segments with different extremities, and Nei be the set of CCS with non empty interiors. The set of CCS of $\mathbb{R}^2$ contains exactly Seg, Nei, the empty set, and the CCS reduced to a single point. In the sequel we focus on Seg ∪ Nei only.

For $S \in$ Nei, S° will designate the interior of S, and $\partial S = S \setminus S^\circ$ the boundary of S. We call *parametrisation* of ∂S, a map $\gamma : [a, b] \to \partial S$ for some interval $[a, b] \subset \mathbb{R}$, such that $\gamma(a) = \gamma(b)$ and such that γ is injective from $[a, b)$ to ∂S. The length of ∂S is well defined, finite and positive, and is called the *perimeter* of S and denoted $\mathsf{Peri}(S)$. It may be used to provide a *natural parametrisation* of ∂S, that is to say a function $\gamma : [0, |\partial S|] \to \partial S$, continuous and injective on $[0, |\partial S|]$, such that $\gamma(0) = \gamma(|\partial S|)$ and such that the length of $\{\gamma(t), t \in [0, s]\}$ is equal to s for any $s \in [0, |\partial S|]$. For $S \in$ Seg, the notion of natural parametrisation also exists, but it is different. For technical reasons, we choose the following one: the natural parametrisation of a segment $[a, b]$ is defined to be $\gamma(t) = a(1 - \frac{t}{|b-a|}) + b\frac{t}{|b-a|}$ on $[0, |b - a|]$ and $\gamma(t) = a(\frac{t}{|b-a|} - 1) + b(2 - \frac{t}{|b-a|})$ on $[|b - a|, 2|b - a|]$, as if the segments were thick and two-sided. In this case, we define $\mathsf{Peri}(S) = 2|b - a|$.

Definition 2.1. The *boundary* B of $C \in$ Nei is defined as $B = C \setminus C^\circ$. The boundary of $C = [a, b] \in$ Seg is defined as C itself.

By definition, the boundary of a CCS is equal to the path induced by its natural parametrisation, and its perimeter is the length of this path.

2.2. Measures on the circle

Let $\mathcal{T}$ be the circle $\mathbb{R}/(2\pi\mathbb{Z})$ equipped with the quotient topology, and $\mathcal{M}_\mathcal{T}$ be the set of probability distributions on $\mathcal{T}$. The weak convergence on $\mathcal{M}_\mathcal{T}$ is defined as usual: $(\mu_n, n \geq 0) \xrightarrow[n]{(weak)} \mu$ in $\mathcal{M}_\mathcal{T}$ if for any bounded continuous function $f : \mathcal{T} \to \mathbb{R}$, $\int_\mathcal{T} f \mathrm{d}\mu_n \to \int_\mathcal{T} f \mathrm{d}\mu$. Let $\mu \in \mathcal{M}_\mathcal{T}$, and consider

$$\begin{aligned} F_\mu : \mathcal{T} &\longrightarrow [0, 1] \\ x &\longmapsto \mu([0, x]) \end{aligned}$$

be the *cumulative distribution function* (CDF) of μ. Let $\mathcal{I}_\mu$ be the set of points of continuity of F_μ, where by convention, $0 \in \mathcal{I}_\mu$ if $F_\mu(0) = \mu(\{0\}) = 0$. If $\mu_n \xrightarrow[n]{(weak)} \mu$ in $\mathcal{M}_\mathcal{T}$, then it can not be deduced that $F_{\mu_n} \to F_\mu$ pointwise on $\mathcal{I}_\mu$ since $\delta_{2\pi} = \delta_0$ in $\mathcal{M}_\mathcal{T}$. What is still true, is that

$$F_{\mu_n}(y) - F_{\mu_n}(x) \to F_\mu(y) - F_\mu(x), \text{ for any } (x, y) \in \mathcal{I}_\mu.$$

A function $F : [0, 2\pi) \to \mathbb{R}$ is a CDF of some distribution $\mu \in \mathcal{M}_{\mathcal{T}}$ if it is right continuous, non decreasing on $[0, 2\pi]$, satisfies $0 \le F(0) \le 1$, $F(2\pi-) = 1$ (see Wilms [30], p. 4–5 for additional information and references).

Consider the continuous function

$$\begin{aligned} Z_\mu : [0,1] &\longrightarrow \mathbb{C} \\ t &\longmapsto Z_\mu(t) = \int_0^t \exp(iF_\mu^{-1}(u))\mathrm{d}u, \end{aligned} \tag{2.1}$$

where F_μ^{-1} is the *standard generalised inverse* of F_μ:

$$\begin{aligned} F_\mu^{-1} : [0,1] &\longrightarrow [0, 2\pi) \\ y &\longmapsto F_\mu^{-1}(y) := \inf\{x \ge 0 \ : \ F_\mu(x) \ge y\}. \end{aligned}$$

The range $\mathcal{B}_\mu$ of Z_μ is the central object here:

$$\mathcal{B}_\mu := \{Z_\mu(t), \ t \in [0,1]\}.$$

Since F_μ^{-1} is non decreasing, it admits at most a countable set of discontinuity points. Therefore Z_μ is differentiable on the complement of a countable subset of $[0, 1]$ and when it is the case, $Z'_\mu(t) = \mathrm{e}^{i\theta}$ represents the direction of the unique tangent to the convex at point $Z_\mu(t)$. Moreover, since the modulus of Z'_μ is 1, Z_μ is the natural parametrisation of $\mathcal{B}_\mu$ and $\mathcal{B}_\mu$ has length 1.

Let Conv be the set of CCS of the plane containing the origin, lying above the x-axis, and whose intersection with the x-axis is included in $\mathbb{R}^+$. Denote by $\mathsf{Conv}(1)$ the subset of Conv of CCS having perimeter 1, and by BConv the set of their corresponding boundaries. Set

$$\mathcal{M}_{\mathcal{T}}^0 = \left\{\mu \in \mathcal{M}[0, 2\pi] \ , \int_0^{2\pi-} \exp(i\theta)\mathrm{d}F_\mu(\theta) = 0\right\}$$

the subset of $\mathcal{M}_{\mathcal{T}}$ of measures having Fourier transform equal to 0 at time 1.

2.3. Probability measures and CCS

Probability distributions on $\mathbb{R}$ are characterised by their Fourier transform, and convergence of Fourier transforms characterises weak convergence by the famous Lévy's continuity Theorem. The following Theorem gives a similar characterisation of measures in $\mathcal{M}_{\mathcal{T}}^0$ by their representation as CCS of the plane.

Theorem 2.2.

1) *The map*

$$\begin{aligned} \mathcal{B} : \mathcal{M}_{\mathcal{T}}^0 &\longrightarrow \mathsf{BConv}(1) \\ \mu &\longmapsto \mathcal{B}_\mu \end{aligned}$$

is a bijection.

2) *$\mathcal{B}$ is an homeomorphism from $\mathcal{M}_{\mathcal{T}}^0$ (equipped with the weak convergence topology) to $\mathsf{BConv}(1)$ (equipped with the Hausdorff topology on compact sets).*

3) *The function Γ from $\mathsf{Conv}(1)$ to $\mathsf{BConv}(1)$ which sends a CCS to its boundary is an homeomorphism for the Hausdorff topology, and then*

$$\begin{aligned} \mathcal{C} : \mathcal{M}_{\mathcal{T}}^0 &\longrightarrow \mathsf{Conv}(1) \\ \mu &\longmapsto \mathcal{C}_\mu := \Gamma^{-1}(\mathcal{B}_\mu) \end{aligned}$$

is an homeomorphism.

This theorem sometimes called "Gauss−Minkowski" in the literature can be found in a slightly different form in Busemann ([8], Sect. 8). The integral formula (2.1) giving the parametrisation of the CCS in terms of F_μ^{-1}, which is central here, seems to be new. We provide a proof of Theorem 2.2 in probabilistic terms at the end of this section.

In Busemann, this theorem is stated more generally in $\mathbb{R}^n$, where the measures range over the unit sphere of $\mathbb{R}^n$ and verify a set of properties, which in $\mathbb{R}^2$ sum up to $\int_0^{2\pi} e^{ix} d\mu(x) = 0$. The measure μ is called the surface area measure [18] of the CCS $\mathcal{C}_\mu$, and is defined for more general convex sets in any dimension.

Remark 2.3. The map $\mathcal{B}$ that one may see as a "curve" transform, may be extended to $\mathcal{M}[0,2\pi]$, the set of measures on $[0,2\pi]$; in this case $\mathcal{B}(\mathcal{M}[0,2\pi])$ is the set of continuous almost everywhere differentiable curves of length 1, starting at the origin, having a positive argument in a neighbourhood of 0, and where along an injective parametrisation, the argument of the tangent is non decreasing[2].

There exists another formula for Z_μ in terms of expectations of r.v., that we will use as a guideline throughout the paper. Recall that if $U \sim \mathsf{uniform}[0,1]$ then $F_\mu^{-1}(U) \sim \mu$, and then

$$Z_\mu(t) = \mathbb{E}\left(\mathbf{1}_{U \le t} \exp(iF_\mu^{-1}(U))\right). \tag{2.2}$$

Since $x \le F_\mu(y)$ is equivalent to $F_\mu^{-1}(x) \le y$, we obtain that

$$\begin{aligned} Z_\mu(F_\mu(t)) &= \mathbb{E}\left(\mathbf{1}_{U \le F_\mu(t)} \exp(iF_\mu^{-1}(U))\right) \\ &= \mathbb{E}\left(\mathbf{1}_{F_\mu^{-1}(U) \le t} \exp(iF_\mu^{-1}(U))\right) \\ &= \mathbb{E}\left(\mathbf{1}_{X_\mu \le t} \exp(iX_\mu)\right). \end{aligned}$$

The function $t \mapsto Z_\mu(F_\mu(t))$ plays an important role since it encodes the extremal points of $\mathcal{B}_\mu$ (see below). The function Z_μ is somehow less pleasant since it can not be written directly in term of X_μ on $[0,1]$. To see this, let

$$I_\mu = \left\{t \in [0;2\pi) \text{ such that } \left\{u, u < t\right\} = \{F_\mu^{-1}(u) < F_\mu^{-1}(t)\}\right\}.$$

This corresponds to the set of t where $F_\mu^{-1}(t) > F_\mu^{-1}(t-h)$ for any $h > 0$ (or $t = 0$). It can be shown that $I_\mu = \{F(t), t \in [0,2\pi]\}$. Noticing that one can replace $\mathbf{1}_{U \le t}$ by $\mathbf{1}_{U<t}$ in (2.2), we have

$$Z_\mu(t) = \mathbb{E}\left(\mathbf{1}_{X_\mu < F_\mu^{-1}(t)} \exp(iX_\mu)\right) \text{ for } t \in I_\mu, \tag{2.3}$$

Now we can characterise $\mathsf{Ext}(C)$ the set of extremal points of C.

Lemma 2.4. *For any* $\mu \in \mathcal{M}_{\mathcal{T}}^0$, $\mathsf{Ext}(\mathcal{C}_\mu) = \{Z_\mu(F_\mu(t)), t \in [0,2\pi]\}$.

Proof. From (2.2), we see that Z_μ is linear on every interval inside the complement of I_μ in $[0,1]$: if (t_1,t_2) is such an interval, for any $t \in [t_1,t_2]$,

$$Z_\mu(t) = Z_\mu(t_1) + (t_2 - t)\frac{Z_\mu(t_2) - Z_\mu(t_1)}{t_2 - t_1}.$$

Therefore, the points in the complement of I_μ are not extremal, and reciprocally, every non-extremal point lies on a segment inside $\mathcal{B}_\mu$ and necessarily belongs to the complement of I_μ. Therefore $\mathsf{Ext}(\mathcal{C}_\mu)$ is equal to the closed set $\{Z_\mu(F_\mu(t), t \in [0,2\pi]\}$. □

[2]The Fourier transform $t \mapsto \Psi_\mu(t)$ also defines a curve $\{\Psi_\mu(t) : t \in A\}$ in the plane, for any interval A. This curve is different from $\mathcal{C}_\mu$, for any A.

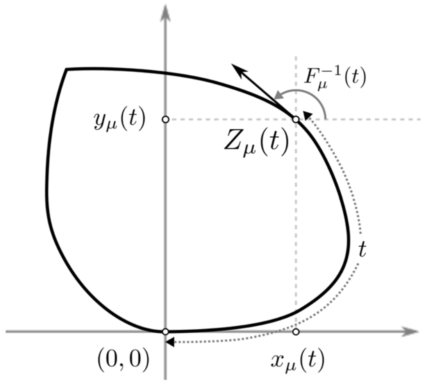


FIGURE 1. A CCS $\mathcal{C}_\mu$ for some measure μ, t gives the length of the curve $\mathcal{B}_\mu$ between 0 and $Z_\mu(t)$ (in the trigonometric order), $F_\mu^{-1}(t)$ is then the direction of the tangent at time t.

The curvature $k_\mu(t)$ of $\mathcal{C}_\mu$ at time t, is given by $\frac{1}{F'_\mu(F_\mu^{-1}(t))}$ when F_μ admits a derivative at $F_\mu^{-1}(t)$; in particular, this means that when μ admits a density f_μ, then $k_\mu(F_\mu(\theta)) = 1/f_\mu(F_\mu^{-1}(F_\mu(\theta))) = 1/f_\mu(\theta)$, which corresponds to the curvature at the point whose tangent has direction θ.

The real and imaginary parts $x_\mu(t) = \Re(Z_\mu(t))$ and $y_\mu(t) = \Im(Z_\mu(t))$ of $Z_\mu(t)$ satisfy

$$\begin{cases} x_\mu(t) = \int_0^t \cos\left(F_\mu^{-1}(u)\right) \mathrm{d}u = \int_0^{F_\mu^{-1}(t)} \cos(v) \mathrm{d}F(v) \\ y_\mu(t) = \int_0^t \sin\left(F_\mu^{-1}(u)\right) \mathrm{d}u = \int_0^{F_\mu^{-1}(t)} \sin(v) \mathrm{d}F_\mu(v). \end{cases} \tag{2.4}$$

the second equality in each line being valid only for $t \in I_\mu$.

Proof of Theorem 2.2.
1) The proof of 3) is immediate. We establish 1).

a) First, we prove that for any $\mu \in \mathcal{M}_{\mathcal{T}}^0$, $\mathcal{B}_\mu$ is the boundary of a CCS $\mathcal{C}_\mu \in \mathsf{Conv}(1)$. A *support half-plane* of $\mathcal{B}_\mu$ is a half-plane H intersecting $\mathcal{B}_\mu$ on its border and such that $\mathcal{B}_\mu \subset H$. The function Z_μ is continuous, and a simple analysis shows that y_μ is such that $y_\mu(0) = y_\mu(1) = 0$, and is increasing then decreasing over $[0,1]$. Therefore, $\mathcal{B}_\mu$ lies on the half plane above the x-axis, which is a support half-plane of $\mathcal{B}_\mu$. More generally, for any $\theta \in [0, 2\pi)$, $\mu_\theta(.) = \mu(. - \theta \mod 2\pi)$ is still in $\mathcal{M}_{\mathcal{T}}^0$, and $\mathcal{B}_{\mu_\theta}$ lies on the half plane above the x-axis. Therefore, for all $t \in [0,1)$, the line D_t passing through $Z_\mu(t)$ making an angle $F_\mu^{-1}(t)$ with the origin, is the border of a support half-plane of $\mathcal{B}_\mu$. Since F_μ^{-1} is right-continuous, $\mathcal{B}_\mu$ is even tangent to D_t.

We now show that $\mathcal{B}_\mu$ is a simple curve or a segment: let z be such that $z = Z_\mu(t_1) = Z_\mu(t_2)$, for $t_1 < t_2$. Then, by definition (2.1), $\int_{[t_1,t_2]} \exp(iF_\mu^{-1}(u))\mathrm{d}u = \int_{[0,t_1]\cup[t_2,1]} \exp(iF_\mu^{-1}(u)) = 0$. Each of these integrals is the weighted barycentre of a portion of the circle, both portions being disjoint except at their extremities t_1 and t_2. Since both barycentres are equal (to 0), the support of μ must be included in $\{t_1, t_2\}$. This implies that $F_\mu^{-1}(t_2) = \pi + F_\mu^{-1}(t_1)$ and $\mu(\{t_2\}) = \mu(\{t_1\}) = 1/2$. In other words, the CCS is a segment of length 1/2. Therefore, when $\mathcal{B}_\mu$ is not a segment, it is a bounded Jordan curve that encloses a bounded connected subset $\mathcal{C}_\mu$. In this last case, $\mathcal{B}_\mu$ is the border of $\mathcal{C}_\mu$ and every point of the border possesses a support half-plane, therefore $\mathcal{C}_\mu$ is convex (see for example 3.3.6 in [18]).

b) The injectivity of $\mathcal{B}$ is clear since if $F_\mu^{-1}(t) = F_\nu^{-1}(t)$ for all $t \in [0,1]$, then $\mu = \nu$. Now, let B be a CCS boundary in $\mathsf{BConv}(1)$ and consider the unique natural parametrisation Z of B in the counterclockwise direction such that $Z(0) = Z(1) = 0$. The map Z is 1-Lipschitz on $[0,1]$ and therefore absolutely continuous. Therefore Z is differentiable almost everywhere and satisfies $Z(t) = \int_0^t g(s)\mathrm{d}s$, where g coincides with the derivative of Z

on I, a subset of $[0,1)$ of measure 1 ([22], Thm. 7.18). Since Z is the natural parametrisation of B, $|g(t)|$ is equal to 1 almost everywhere. Since B is a CCS boundary, the argument of $g(t)$ is the direction of the unique supporting half-plane at $Z(t)$ and then $\arg g$ is non-decreasing over I.

Then $g(s)=\exp(iG(s))$ for some non decreasing function $G: I\to[0,2\pi)$. Let $G^*:[0,1)\to[0,2\pi)$ be defined by $G^*(x)=\inf\{G(y), y\geq x, y\in I\}$ (G^* is the largest non-decreasing function smaller than G over I). For all t, $Z(t)=\int_0^t \mathrm{e}^{iG^*(s)}\mathrm{d}s$. Since G^* is non-decreasing, it possesses a right-continuous modification $\tilde{G}$ which also satisfies $Z(t)=\int_0^t \mathrm{e}^{i\tilde{G}(s)}\mathrm{d}s$. The function $\tilde{G}$ is the inverse of a CDF F_ν for some ν in $\mathcal{M}^0_{\mathcal{T}}$. □

2) Consider first the continuity of $\mathcal{B}$. For any $t\in[0,2\pi)$ and any pair of distributions (μ,ν), since $x\to\exp(ix)$ is 1-Lipschitz,

$$\begin{aligned}|Z_\mu(t)-Z_\nu(t)| &= \left|\int_0^t \exp(iF_\mu^{-1}(u))-\exp\left(iF_\nu^{-1}(u)\right)\mathrm{d}u\right|\\ &\leq \int_0^t \mathrm{d}_{\mathcal{T}}\left(F_\mu^{-1}(u),F_\nu^{-1}(u)\right)\mathrm{d}u,\end{aligned}$$

where $\mathrm{d}_{\mathcal{T}}$ is the distance in $\mathcal{T}$, defined for $0\leq x\leq y<2\pi$ by $\mathrm{d}_{\mathcal{T}}(x,y)=\min\{y-x,2\pi-y+x\}$. This last quantity is then bounded above, uniformly in $t\in[0,1]$ by $\mathbb{E}(\mathrm{d}_{\mathcal{T}}(X_\mu,X_\nu))$, for

$$X_\mu := F_\mu^{-1}(U),\quad X_\nu := F_\nu^{-1}(U),$$

where $U\sim\mathsf{uniform}[0,2\pi]$. Now, $\mathbb{E}(\mathrm{d}_{\mathcal{T}}(X_\mu,X_\nu))$ is a Wasserstein like distance $W_1(\mu,\nu)$ between the distributions μ and ν in $\mathcal{T}$ (the standard Wasserstein distance is rather defined between measures on an interval, not on the circle). Now, it is classical that the convergence in distribution implies the convergence of the Wasserstein distance to 0 (see Dudley [10], Sect. 11.8). This property can be easily extended to the present case, considering that $X_n \xrightarrow[n]{(d)} X$ in $\mathcal{M}_{\mathcal{T}}$ iff there exists $\theta\in[0,2\pi]$ (any point of continuity of X does the job) for which $X_n-\theta \mod 2\pi \xrightarrow[n]{(d)} X-\theta \mod 2\pi$ in the standard sense.

Reciprocally, let $(B_n, n\geq 0)$ be a sequence of CCS boundaries B_n converging to $\mathcal{B}_\mu$ for the Hausdorff distance d_H. By Theorem 2.2 1), there exists $\mu_n\in\mathcal{M}^0_{\mathcal{T}}$ such that $\mathcal{B}_{\mu_n}=B_n$. We now establish that $(\mu_n,n\geq 0)$ possesses exactly one accumulation point, equal to μ. Consider a subsequence $F_{\mu_{n_k}}$ such that $F_{\mu_{n_k}}\xrightarrow{D_1} G$, where G is the CDF of a measure ν. Such a subsequence exists since $\mathcal{M}^0_{\mathcal{T}}$ is compact (and then sequentially compact, since it is a metric space). Now, for D_1 denoting the Skorokhod distance (see *e.g.* Billingsley [4], Chap. 3), $F_{\mu_{n_k}}\xrightarrow{D_1} G \quad\Rightarrow\quad F^{-1}_{\mu_{n_k}}\xrightarrow{D_1} G^{-1}$. According to the first part of this proof, the limit CCS boundary $\mathcal{B}_\nu$ must be equal to $\mathcal{B}_\mu$. Since by Theorem 2.2 1), the CCS characterise the measure, $\nu\overset{(d)}{=}\mu$. □

2.4. Fourier decomposition of the CCS curve

Fourier coefficients provide powerful tools to analyse the geometrical properties of the CCS curves.
Let f be a function from $[0,2\pi]$ with values in $\mathbb{R}$. The quantity $\frac{1}{2}a_0+\sum_{k\geq 1}a_k\cos(ku)+b_k\sin(ku)$ is the standard Fourier series of f, where

$$a_k=\pi^{-1}\int_0^{2\pi}\cos(ku)f(u)\mathrm{d}u,\quad b_k=\pi^{-1}\int_0^{2\pi}\sin(ku)f(u)\mathrm{d}u.$$

For μ in $\mathcal{M}_{\mathcal{T}}$ (or in $\mathcal{M}[0,2\pi]$), the Fourier coefficients of μ are defined, for any $k\geq 0$ by

$$a_0(\mu)=\frac{1}{\pi},\quad a_k(\mu)=\frac{1}{\pi}\mathbb{E}(\cos(kX_\mu)),\quad b_k(\mu)=\frac{1}{\pi}\mathbb{E}(\sin(kX_\mu)). \tag{2.5}$$

In this setting, the condition $\int_0^{2\pi} e^{iu} \mathrm{d}F_\mu(u) = 0$ coincides with

$$a_1(\mu) = \mathbb{E}(\cos(X_\mu)) = 0, \quad b_1(\mu) = \mathbb{E}(\sin(X_\mu)) = 0. \tag{2.6}$$

The following proposition, whose proof can be found in Wilms ([30], Thms. 1.6 and 1.7), states that probability measures are characterised by their Fourier coefficients, and establishes a continuity theorem.

Proposition 2.5.

1) *The function*

$$\begin{aligned} \mathsf{Coeffs} : \mathcal{M}_\mathcal{T} &\longrightarrow \mathbb{R}^\mathbb{N} \times \mathbb{R}^\mathbb{N} \\ \mu &\longmapsto ((a_k(\mu), k \geq 0), (b_k(\mu), k \geq 1)) \end{aligned}$$

is injective.

2) *Let $\mu, \mu_1, \mu_2, \ldots$ be a sequence of measures in $\mathcal{M}_\mathcal{T}$. The two following statements are equivalent: $\mu_n \xrightarrow[n]{(weak)} \mu$ and $\mathsf{Coeffs}(\mu_n)$ converges pointwise to $\mathsf{Coeffs}(\mu)$ (meaning that for any k, $a_k(\mu_n) \to a_k(\mu)$ and $b_k(\mu_n) \to b_k(\mu)$).*

Example 2.6. – If $\mu \sim \mathsf{uniform}[0, 2\pi]$ then $a_k(\mu) = b_k(\mu) = 0$ for any $k \geq 1$.
– If $\mu = \sum_{k=0}^{m-1} \frac{1}{m} \delta_{2\pi k/m}$ is the uniform distribution on the vertices of a regular m-gon (with a vertex at position $(0,0)$), then all the b_k are null, $a_0(\mu) = 1/\pi$, and $a_k(\mu) = \pi^{-1} \mathbf{1}_{k \in m\mathbb{N}^\star}$.

Of course, deciding whether a given pair $((a_k, k \geq 0), (b_k, k \geq 1))$ corresponds to a pair $((a_k(\nu), k \geq 0), (b_k(\nu), k \geq 1))$ for some $\nu \in \mathcal{M}_\mathcal{T}$ is a difficult task: there does not exist in the literature any characterisation of Fourier series of non negative measures. The case of measures having a density with respect to the Lebesgue measure is discussed in Section 5.3.

The area of a CCS $\mathcal{C}_\mu$ has an expression in terms of $\mathsf{Coeffs}(\mu)$. In this section, we consider a CCS with a smooth C^1 boundary that is equal to its Fourier expansion. The following formula can be deduced from Hurwitz ([13], pp. 372–373), where it is given using a parametrisation of the boundary of the CCS. In our settings, writing $\mathcal{A}(\mu)$ for the area of $\mathcal{C}_\mu$, it translates into:

$$\mathcal{A}(\mu) = \frac{1}{4\pi} - \frac{\pi}{2} \sum_{k \geq 2} \frac{a_k^2(\mu) + b_k^2(\mu)}{k^2 - 1}. \tag{2.7}$$

As did Hurwitz, this equation can be proved from Green's theorem stating that:

$$\mathcal{A}(\mu) = \int_0^1 x_\mu(t) \frac{\mathrm{d}y_\mu(t)}{\mathrm{d}t} \mathrm{d}t = -\int_0^1 y_\mu(t) \frac{\mathrm{d}x_\mu(t)}{\mathrm{d}t} \mathrm{d}t. \tag{2.8}$$

As a matter of fact, this formula remains valid for every CCS in $\mathsf{Conv}(1)$ (*cf.* Cor. 3.7). Rewriting (2.8) and using (2.4) gives

$$\begin{aligned} \mathcal{A}(\mu) &= \int_0^1 \int_0^t \cos(F_\mu^{-1}(u)) \mathrm{d}u \sin(F_\mu^{-1}(t)) \mathrm{d}t \\ &= \mathbb{E}\left(\cos(X) \sin(X') 1_{X \leq X'}\right). \end{aligned} \tag{2.9}$$

where X and X' are two independent copies of X_μ.

Remark 2.7. One can show that (2.7) implies (2.9) by noticing that $\mathbb{E}(\cos(kX))^2 + \mathbb{E}(\sin(kX))^2 = \mathbb{E}(\cos(k(X - X'))$ and using the general equality $\sum_{k \geq 2} \frac{\cos(kx)}{k^2-1} = \frac{\cos(x)}{4} - \frac{(\pi - (x \bmod 2\pi))}{2} \sin(x) + \frac{1}{2}$. Notice that Hurwitz [12] deduced the isoperimetric inequality from (2.9) with a proof which only requires an equivalent of Wirtinger's inequality.

2.5. Convergence of discrete CCS and an application to statistics

Consider $X_1, \dots, X_n$ i.i.d. having distribution μ with support in $[0, 2\pi)$. The empirical CDF associated with this sample is defined by $F_n(x) = n^{-1}\#\{i : X_i \leq x\}$. The law of large number ensures that $F_n \to F_\mu$ pointwise in probability, and $(n^{1/2}|F_n(x) - F_\mu(x)|, x \in [0, 2\pi])$ converges in distribution in $D[0, 2\pi]$, the set of cÃ dlÃ g function equipped with the Skorokhod topology, to $(\mathsf{b}(F_\mu(x)), x \in [0, 2\pi])$ where b is a standard Brownian bridge (see Billingsley [4], Thm. 14.3).

Now assume that the X_i take their values in $\mathcal{T}$, and let $\hat{X}_1, \dots, \hat{X}_n$ be the sequence $X_1, \dots, X_n$ sorted in increasing order (with the natural order on $[0, 2\pi)$). Consider the function $Z_n : [0, 1] \to \mathbb{C}$ defined by $Z_n(0) = 0$,

$$Z_n(k/n) = \frac{1}{n}\sum_{j=1}^{k} \exp(i\hat{X}_j), \quad \text{for } k \in \{1, \dots, n\},$$

and extended by linear interpolation between the points $(k/n, k \in \{0, \dots, n\})$. Also define the empirical curve B_n associated with the distribution μ, as $B_n := \{Z_n(t), t \in [0, 1]\}$. The curve B_n belongs to $\mathsf{BConv}(1)$ if and only if $\sum_{j=1}^{n} \mathrm{e}^{iX_j} = 0$; otherwise, since the steps are sorted, B_n is either simple or may contain at most 1 self-intersection point, that is a pair $t_1 < t_2$ such that $Z_n(t_1) = Z_n(t_2)$. For $\theta \in [0, 2\pi)$, let $N_n(\theta) = \#\{i, X_i \leq \theta\}$ be the number of variables smaller than θ. The set of extremal points of B_n is

$$\mathsf{Ext}(B_n) = \{Z_n(N_n(\theta)/n), \theta \in [0, 2\pi]\}\,. \tag{2.10}$$

Set for any $\theta \in [0, 2\pi)$,

$$W_n(\theta) := \sqrt{n}\,[Z_n(N_n(\theta)/n) - Z_\mu(F_\mu(\theta))]\,.$$

This process measures the difference between Z_n and its limit.

Denote by $\pi_1(z) = \Re(z)$, $\pi_2(z) = \Im(z)$ and $\pi(z) = (\pi_1(z), \pi_2(z))$.

Theorem 2.8.

1) *The following convergence*

$$\pi\,(W_n(\theta), \theta \in [0, 2\pi]) \xrightarrow[n]{(d)} (G_\theta, \theta \in [0, 2\pi]) \tag{2.11}$$

holds in $(D[0, 2\pi], \mathbb{R}^2)$*, where* G *is a centred Gaussian process whose finite dimensional distributions are given in Section A.1, in formula (A.5).*

2) *For any* $n \geq 1$*,* $d_H(B_n, \mathcal{B}_\mu) = \max_\theta |Z_n(N_n(\theta)/n) - Z_\mu(F_\mu(\theta))|$*, and then* $\sqrt{n}d_H(B_n, \mathcal{B}_\mu)$ *converges in distribution to* $\max_\theta |G_\theta|$.

See illustration in Figure 2. The following Corollary – which gives the asymptotic shape for our random polygons – is a direct consequence of Theorem 2.8.

Corollary 2.9. *If* $\mu \in \mathcal{M}^0_{\mathcal{T}}$ *then:*

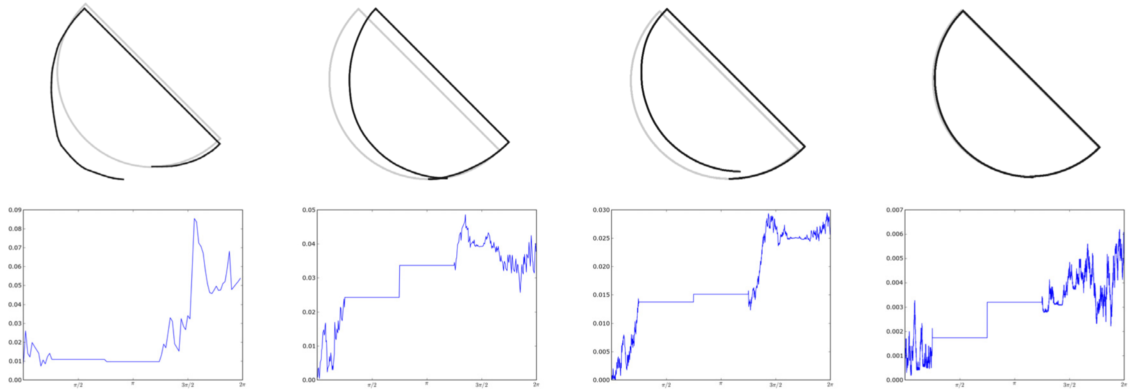
1) *The following convergence holds in distribution in* $D[0, 2\pi]$*:*

$$(Z_n(N_n(\theta)/n), \theta \in [0, 2\pi]) \xrightarrow[n]{(d)} (Z_\mu(F_\mu(\theta)), \theta \in [0, 2\pi]). \tag{2.12}$$

2) $d_H(B_n, \mathcal{B}_\mu) \to 0$ *in probability.*

Remark 2.10. A direct proof of Corollary 2.9 that ignores Theorem 2.8 is as follows: first, the convergence of the finite dimensional distributions (FDD) corresponding to 1) holds as a consequence of the law of large numbers. Then, for an $\varepsilon > 0$, choose k and the points $(\theta_1, \dots, \theta_k)$ such that the union of the segments $B_\varepsilon := \cup_{i=0..k-1}[Z_\mu(F_\mu(\theta_i)), Z_\mu(F_\mu(\theta_{i+1}))]$ has a length larger than $1 - \varepsilon$. From there, 2) follows since for n large enough, $|Z_n(N_n(\theta_i)/n) - Z_\mu(F_\mu(\theta_i))|$ goes to 0 in probability for any $i \leq k$. This implies that the union of the segments $B'_n = \cup_i[Z_n(N_n(\theta_i)/n), Z_n(N_n(\theta_{i+1})/n)]$ has total length larger than $1 - 2\varepsilon$ for n large enough, with probability going to 1. Since B_n has length 1, for those same n, $d_H(B_n, B'_n) \leq 2\varepsilon$.

The proof of Theorem 2.8 is postponed to the appendix.



$n = 100$ $n = 500$ $n = 2000$ $n = 10000$

FIGURE 2. Convergence towards the half-circle. The first row of figures describes the discrete CCS of size n (in black) compared to the limit CCS (in grey). The second row displays the distance between the discrete CCS and its limit ($\theta \to |W_n(\theta)|$).

3. Operations on measures and on CCS

Mixture and convolution are natural operations on $\mathcal{M}^0_{\mathcal{T}}$:

1) *Mixture*: if $\mu, \nu \in \mathcal{M}^0_{\mathcal{T}}$ then for any $\lambda \in [0,1]$, $\lambda\mu + (1-\lambda)\nu \in \mathcal{M}^0_{\mathcal{T}}$.
2) *Convolution*: if $\mu, \nu \in \mathcal{M}^0_{\mathcal{T}}$ then $\mu \underset{\mathcal{T}}{\star} \nu \in \mathcal{M}^0_{\mathcal{T}}$, where $(\underset{\mathcal{T}}{\star})$ denotes the convolution in $\mathcal{M}_{\mathcal{T}}$. This conclusion holds even if only μ is in $\mathcal{M}^0_{\mathcal{T}}$.

Then the maps $\mathcal{B}$ and $\mathcal{C}$ transport these operations on $\mathsf{Conv}(1)$:

Definition 3.1. Let $\mathcal{C}_\mu$ and $\mathcal{C}_\nu$ be two CCS in $\mathsf{Conv}(1)$ and $\lambda \in [0,1]$.

1) We call *mixture* of $\mathcal{C}_\mu$ and of $\mathcal{C}_\nu$ with weights $(\lambda, 1-\lambda)$, the CCS $\mathcal{C}_{\lambda\mu+(1-\lambda)\nu}$.
2) We call *convolution* of $\mathcal{C}_\mu$ and $\mathcal{C}_\nu$, the CCS $\mathcal{C}_\mu \star \mathcal{C}_\nu := \mathcal{C}_{\mu \underset{\mathcal{T}}{\star} \nu}$.

In this section we provide some facts which seem to be unknown: a mixture is sent by $\mathcal{C}$ on a Minkowski sum (Prop. 3.2) and the Minkowski symmetrisation can also be expressed in terms of mixtures (Thm. 3.5). The convolution of CCS acts somehow on the radius of curvature and seems to be a new operation, leading to a notion of symmetrisation by convolution that we introduce in Section 3.2.

3.1. Mixtures of CCS/Minkowski sum

Let A and B be two subsets of $\mathbb{R}^2$. The Minkowski sum of A and B is the set $A+B = \{a+b \ : a \in A, b \in B\}$. Further, for any λ, write $\lambda A = \{\lambda a \ : a \in A\}$. We have:

Proposition 3.2. *Let $\nu, \mu \in \mathcal{M}^0_{\mathcal{T}}$, $\lambda \in [0,1]$. Then $\mathcal{C}_{\lambda\mu+(1-\lambda)\nu}$ coincides with $\lambda\mathcal{C}_\mu + (1-\lambda)\mathcal{C}_\nu$. This means that the mixture of CCS and the Minkowski sum are the same, and that the CCS of a mixture corresponds to the mixture of the CCS (up to a translation).*

This property is already known, see *e.g.* Schneider [23], (4.3.1). This proposition (see Fig. 3) implies that the boundaries $\mathcal{B}_{\lambda\mu+(1-\lambda)\nu}$ and $\partial(\text{convex hull}(\lambda\mathcal{B}_\mu + (1-\lambda)\mathcal{B}_\nu))$ coincide.

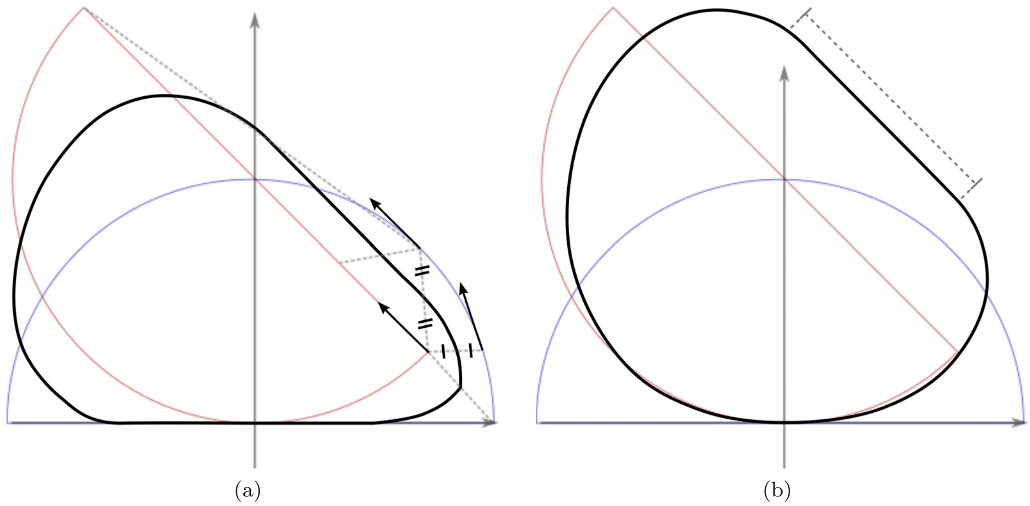


FIGURE 3. Construction of the (a) mixture and (b) convolution of two half-circles. Notice that every point of the mixture is the barycentre of two points of the original half-circles, and that the CCS obtained by convolution possesses a linear segment whose angle corresponds to the sum of the angles of the segments in the original half-circles.

Proof. We first give a proof when μ and ν have densities. Recall the characterisation given in Lemma 2.4. Write

$$\begin{aligned} Z_{\lambda\mu+(1-\lambda)\nu}(F_{\lambda\mu+(1-\lambda)\nu}(t)) &= \lambda \int_0^t \exp(it)\mathrm{d}\mu(t) + (1-\lambda)\int_0^t \exp(it)\mathrm{d}\nu(t) \\ &= \lambda Z_\mu(F_\mu(t)) + (1-\lambda)Z_\nu(F_\nu(t)). \end{aligned} \tag{3.1}$$

The extremal points of $\mathcal{C}_{\lambda\mu+(1-\lambda)\nu}$ are then obtained as particular barycentres of extremal points of $\mathcal{C}_\mu$ and $\mathcal{C}_\nu$. When both μ and ν have a density, this implies that the point in $\mathcal{B}_{\lambda\mu+(1-\lambda)\nu}$ where the tangent has direction θ is obtained as the barycentre of the corresponding points in $\mathcal{B}_\mu$ and $\mathcal{B}_\nu$. This implies that $\mathcal{C}_{\lambda\mu+(1-\lambda)\nu} \subset \lambda\mathcal{C}_\mu + (1-\lambda)\mathcal{C}_\nu$.

We establish the other inclusion by using the fact that CCS are characterised by their supporting half-planes: for every $t \in [0, 2\pi]$, let $D_\mu(t)$ be the line passing through $Z_\mu(F_\mu(t))$ making an angle t with the x-axis. The line $D_\mu(t)$ defines a supporting half-plane $H_\mu(t)$ for $\mathcal{C}_\mu$. Since $\mathcal{C}_\mu$ is a CCS, this half-plane is minimal for the inclusion with regard to the property of making an angle t with the x-axis. Considering that the points in (3.1) all belong to their associated half-plane, these half-planes verify:

$$H_{\lambda\mu+(1-\lambda)\nu}(t) = \lambda H_\mu(t) + (1-\lambda)H_\nu(t).$$

Now, the left-hand side represents a supporting half-plane for $\mathcal{C}_{\lambda\mu+(1-\lambda)\nu}$ and the right-hand side another supporting half-plane for $\lambda\mathcal{C}_\mu + (1-\lambda)\mathcal{C}_\nu$. We deduce that the CCS they enclose are equal.

When μ or ν have no densities, take a sequence (μ_n, ν_n) of measures having densities and which converges weakly to (μ, ν); we then obtain $\mathcal{C}_{\lambda\mu_n+(1-\lambda)\nu_n} = \lambda\mathcal{C}_{\mu_n} + (1-\lambda)\mathcal{C}_{\nu_n}$ and conclude by Theorem 2.2. □

Hence the CCS $\mathcal{C}_{\lambda\mu+(1-\lambda)\nu}$ has a perimeter equal to 1, as all CCS of $\mathsf{Conv}(1)$. This implies that the perimeter of the Minkowski sum $\lambda\mathcal{C}_\mu + (1-\lambda)\mathcal{C}_\nu$ is 1 (well known fact, obtained here without geometric arguments).

Remark 3.3. For μ and ν in $\mathcal{M}^0_{\mathcal{T}}$ and $\lambda \in [0,1]$, we have

$$\mathcal{A}(\lambda\mu + (1-\lambda)\nu)^{1/2} \geq \lambda\mathcal{A}(\mu)^{1/2} + (1-\lambda)\mathcal{A}(\nu)^{1/2}. \tag{3.2}$$

This is the so-called Brunn−Minkowski inequality; it implies that $\mathcal{A}(\lambda\mu + (1-\lambda)\nu) \geq \min\{\mathcal{A}(\mu), \mathcal{A}(\nu)\}$. It can be proved using Hurwitz formula (2.7) and the Cauchy–Schwarz inequality.

3.1.1. Minkowski symmetrisation and measure symmetrisation

Let K be a CCS of $\mathbb{R}^2$ and $u \in \mathbb{R}^2$, $|u| = 1$. We denote by $\pi_u \in O(2)$ the reflection with respect to the straight line passing through the origin and orthogonal to u, *i.e.* $\pi_u(x) = x - 2\langle x, u\rangle u$. The *Minkowski* (or Blaschke) *symmetrisation* of K is the CCS $S_u(K) = \frac{1}{2}(\pi_u K + K)$. The same operation can be defined over $\mathbb{C}$: for $u = \mathrm{e}^{i\theta}$, the Minkowski symmetrisation of K with respect to direction θ is the map $(K, \theta) \mapsto \frac{\mathrm{e}^{i\theta}}{2}(\overline{\mathrm{e}^{-i\theta}K} + \mathrm{e}^{-i\theta}K)$, where $\bar{z}$ is the complex conjugate of z.

Now, let $\theta \in [0, 2\pi]$, $\mu \in \mathcal{M}_{\mathcal{T}}^0$, and set $\mu(\theta)$ be the distribution of $X_\mu + \theta \mod 2\pi$. Since $\mathbb{E}(\exp(i(X_\mu + \theta))) = \mathrm{e}^{i\theta}\mathbb{E}(\exp(iX_\mu))$, $\mu(\theta)$ is in $\mathcal{M}_{\mathcal{T}}^0$. The CCS $\mathcal{C}_{\mu(\theta)}$ can be obtained from $\mathcal{C}_\mu$ by a rotation (of angle $-\theta$) followed by a translation.

For any $\nu \in \mathcal{M}_{\mathcal{T}}^0$, set $\overleftarrow{\nu} = \nu(2\pi - .)$. The symmetrisation of ν with respect to direction θ is the measure $S(\nu(\theta))$ defined by

$$S(\nu(\theta)) = \frac{1}{2}(\nu(\theta) + \overleftarrow{\nu(\theta)}). \tag{3.3}$$

Further the symmetrisation by mixture of $\mathcal{C}_\nu$ with respect to direction θ is defined to be $\mathcal{C}_{S(\nu(\theta))}$.

A direct consequence of Proposition 3.2 is the following:

Proposition 3.4. *The symmetrisation by mixture with respect to direction θ coincides with the Minkowski symmetrisation with respect to $u = \mathrm{e}^{i\theta}$, up to a translation.*

Again Theorem 2.8 provides a new point of view on this symmetrisation. Starting from a set of angles $\theta_1, \ldots, \theta_k$ and an initial measure $\nu \in \mathcal{M}_{\mathcal{T}}^0$, construct the sequence of measures ν_k defined by $\nu_0 = \nu$ and $\nu_{k+1} = S(\nu_k(\theta_k))$. This sequence consists in alternating rotations and symmetrisations of the initial measure ν.

Theorem 3.5. *For any $\theta \in [0, 2\pi]$, any $\nu \in \mathcal{M}_{\mathcal{T}}^0$, the following properties hold:*

1) *the CCS $\mathcal{C}_{S(\nu(\theta))}$ has the same perimeter as $\mathcal{C}_\nu$ (that is 1);*
2) *the area does not decrease: $\mathcal{A}(S(\nu(\theta))) \geq \mathcal{A}(\nu)$;*
3) *for any $k \geq 0$, there exists $\theta_1, \ldots, \theta_k \in [0, 2\pi]$ such that*

$$d_H(\mathcal{C}_{\nu_k}, \mathsf{Circle}(i/(2\pi), 1/(2\pi))) \leq 2^{-k}\pi,$$

where $\mathsf{Circle}(z, r)$ is the circle with centre z and radius r;

4) *among all CCS with perimeter 1, the circle has the largest area.*

Properties 1), 2), 4) are classical; we provide direct probabilistic proofs below. Statement 3) which gives a bound on the speed of convergence to the ball for well chosen directions of symmetrisation, is known in $\mathbb{R}^n$ (see Klartag [14], Thm. 1.3), but the proof we provide here in $\mathbb{R}^2$ is much simpler.

Proof. First, 4) is clearly a consequence of the three first points (to be honest, our proof uses (3.2), which implies directly the isoperimetric inequality). The first item follows from the fact that if $S(\nu(\theta)) \in \mathcal{M}_{\mathcal{T}}^0$, then $\mathcal{B}_{S(\nu(\theta))} \in \mathsf{BConv}(1)$. And (3.2) implies 2) since $\mathcal{A}(\nu) = \mathcal{A}(\nu(\theta)) = \mathcal{A}(\overleftarrow{\nu(\theta)})$.

Let us prove 3). If $L = [X_1, \ldots, X_l]$ for some $l \geq 1$, a list of r.v. with distribution $\nu_1, \ldots, \nu_l$, we say that ν is the *equi-mixture* of L if $\nu = \frac{1}{l}(\nu_1 + \ldots + \nu_l)$.

Take $X \sim \nu$. $\nu_1 := S(\nu(\theta_1))$ is the equi-mixture of $[X + \theta_1 \mod 2\pi, -X - \theta_1 \mod 2\pi]$. Therefore using that $(a \mod 2\pi) + b \mod 2\pi = (a + b) \mod 2\pi$, S_{ν_2} is the equi-mixture of $[X + \theta_1 \pm \theta_2 \mod 2\pi, -X - \theta_1 \pm \theta_2 \mod 2\pi]$. Iterating this, one observes that S_{ν_k} is the equi-mixture of $[X + \theta_1 \pm \theta_2 \pm \ldots \pm \theta_k \mod 2\pi, -X - \theta_1 \pm \theta_2 \pm \ldots \pm \theta_k \mod 2\pi]$. If $\theta_k = (2\pi)/2^{k-1}$ then S_{ν_k} is the equi-mixture of μ_1 and μ_2, where μ_1 and μ_2 are the respective equi-mixture of $[X + \theta_1 \pm \theta_2 \pm \ldots \pm \theta_k \mod 2\pi]$ and of $[-X - \theta_1 \pm \theta_2 \pm \ldots \pm \theta_k \mod 2\pi]$.

Now, both μ_1 and μ_2 converge to $\mathsf{uniform}[0,2\pi]$: to check this, consider the sequence of intervals $I_n = [2\pi n 2^{-k-1}, 2\pi(n+1)2^{-k-1})$, for $0 \le n \le 2^{k-1}-1$. For $j \in \{1,2\}$, $\mu_j(I_n) = 1/2^{k-1}$ for any n. Indeed, μ_1 (resp. μ_2) is the equi-mixture of all measures obtained from the distribution of X (resp. $-X$) by dyadic translation of depth k, then since all intervals I_n have depth k, they have the same weight. Hence $F_{\mu_1}(2\pi n 2^{-k+1}) = n2^{-k+1}$ for any n. Therefore, since F_{μ_1} is increasing, we have that $\|F_{\mu_j} - F\|_\infty \le 2^{-k+1}$, for $F_\upsilon(x) = x/(2\pi)$, the CDF of $\mathsf{uniform}[0,2\pi]$, which gives $\|F_{\nu_k} - F_\upsilon\|_\infty \le 2^{-k+1}$. Further, the right inverses $F_{\nu_k}^{-1}$ and F_υ^{-1} are close:

$$\|F_{\nu_k}^{-1} - F_\upsilon^{-1}\|_\infty \le 2^{-k+1}2\pi.$$

Thanks to (2.1),

$$\begin{aligned}|Z_{\nu_k}(t) - Z_\upsilon(t)| &\le \int_0^t \left|\exp(iF_{\nu_k}^{-1}(u)) - \exp(iF_\upsilon^{-1}(u))\right| \mathrm{d}u \\ &\le \int_0^t \left|F_{\nu_k}^{-1}(u) - F_\upsilon^{-1}(u))\right| \mathrm{d}u\end{aligned}$$

and therefore $\|Z_{\nu_k}(t) - Z_\upsilon(t)\|_\infty \le 2^{-k}\pi$. □

3.2. Convolution of measures/convolution of CCS

In fact, $\mathcal{B}_{\mu \underset{\mathcal{T}}{\star} \nu}$ is obtained as a kind of convolution of $\mathcal{B}_\mu$ and $\mathcal{B}_\nu$. As seen earlier if μ has a density f_μ then $f_\mu(\theta)$ represents the radius of curvature of $\mathcal{B}_\mu$ at time $F_\mu(\theta)$. Therefore the radius of curvature R_θ of $\mathcal{B}_{\mu \underset{\mathcal{T}}{\star} \nu}$ at time $F_{\mu \underset{\mathcal{T}}{\star} \nu}(\theta)$ is the convolution of the radii of curvature of $\mathcal{B}_\mu$ and $\mathcal{B}_\nu$ as follows:

$$R_\theta = \int_0^{2\pi} f_\mu(x) f_\nu((\theta - x) \mod 2\pi)\mathrm{d}x.$$

Theorem 3.6. *Let μ and ν in $\mathcal{M}_{\mathcal{T}}^0$. The convolution does not decrease the area*

$$\mathcal{A}\left(\mu \underset{\mathcal{T}}{\star} \nu\right) \ge \max\{\mathcal{A}(\mu), \mathcal{A}(\nu)\}.$$

Since $\mathsf{uniform}[0,2\pi]$ *is an absorbing point for* $\underset{\mathcal{T}}{\star}$, *and $\mathcal{C}_u$ is the circle of perimeter 1, this implies the isoperimetric inequality:* $\mathcal{A}(\mathsf{uniform}[0,2\pi]) \ge \mathcal{A}(\nu)$, $\forall \nu \in \mathcal{M}_{\mathcal{T}}^0$.

Proof. Consider X and Y two independent r.v. such that $X \sim \mu$, $Y \sim \nu$. Let $\eta = \mu \underset{\mathcal{T}}{\star} \nu$. By expansion of $\cos(n(X+Y))$ and $\sin(n(X+Y))$ we get

$$\begin{aligned}a_n(\eta) &= a_n(\mu)a_n(\nu) - b_n(\mu)b_n(\nu) \\ b_n(\eta) &= b_n(\mu)a_n(\nu) + a_n(\mu)b_n(\nu).\end{aligned}$$

Since $\cos(kX)$ and $\sin(kX)$ have non-negative variances,

$$a_n^2(\mu) + b_n^2(\mu) = \mathbb{E}(\cos(nX))^2 + \mathbb{E}(\sin(nX))^2 \le \mathbb{E}(\cos^2(nX) + \sin^2(nX)) = 1.$$

Hence,

$$\begin{aligned}a_n^2(\eta) + b_n^2(\eta) &= (a_n^2(\mu) + b_n^2(\mu))(a_n^2(\nu) + b_n^2(\nu)) \\ &\le \min\{a_n^2(\mu) + b_n^2(\mu), a_n^2(\nu) + b_n^2(\nu)\},\end{aligned}$$

The conclusion follows from (2.7). □

Corollary 3.7. *Let $\mu \in \mathcal{M}_{\mathcal{T}}^0$. Then the formula (2.7) for $\mathcal{A}(\mu)$ holds.*

Proof. Formula (2.7) is valid when μ admits a $\mathcal{C}^1$ density. Just assume that $\mathbb{E}(\mathrm{e}^{iX_\mu}) = 0$. Let N be a Gaussian centred r.v. with variance 1, and let $N_k = N/\sqrt{k} \mod 2\pi$ for $k \geq 1$, and $\mu_k = \mu * N_k$. Clearly $\mu_k \in \mathcal{M}_{\mathcal{T}}^0$, and $\mu_k \xrightarrow[n]{(weak)} \mu$ which implies $\mathcal{A}(\mu_k) \to \mathcal{A}(\mu)$. Now,

$$\forall n \in \mathbb{Z}, \qquad \mathbb{E}(\mathrm{e}^{inN_k}) = \mathbb{E}(\mathrm{e}^{in(N/\sqrt{k} \mod 2\pi)}) = \mathbb{E}(\mathrm{e}^{inN/\sqrt{k}}) = \mathrm{e}^{-\frac{n^2}{2k}}.$$

Then the Fourier coefficients of N_k verify $a_n = \mathrm{e}^{-\frac{n^2}{2k}}$ and $b_n = 0$. Since μ_k admits a $\mathcal{C}^\infty$ density function, and as a corollary of the proof of Theorem 3.6:

$$\mathcal{A}(\mu_k) = \frac{1}{4\pi} - \frac{\pi}{2}\sum_{n\geq 2}\frac{\left(a_n^2\left(\mu\right) + b_n(\mu)^2\right)\mathrm{e}^{-\frac{1}{2k}n^2}}{n^2-1}.$$

As a consequence of Lebesgue's dominated convergence theorem, $\mathcal{A}(\mu_k)$ converges to the right hand side of (2.7). □

Definition 3.8. A measure ν in $\mathcal{M}_{\mathcal{T}}$ is said to be *c-stable* (for some $c > 0$) if for X_ν and X_ν' two independent r.v. under ν,

$$X_\nu + X_\nu' \mod 2\pi \overset{(d)}{=} cX_\nu \mod 2\pi. \tag{3.4}$$

This qualification of "stable" comes from the standard notion of probability theory where the same question is studied without the mod 2π operation (see Feller [11], Sect. VI). The following Proposition due to Lévy ([16], p. 11) identifies the set of 1-stable distributions.

Proposition 3.9. *The only 1-stable measures are* $\mathsf{uniform}[0, 2\pi]$*, the Dirac measure at 0, and the family, indexed by $m \geq 1$, of uniform measures on $\{k2\pi/m, k = 0, \ldots, m-1\}$.*

We say that a distribution ν is in the 2π-domain of attraction of a distribution μ, and write $\nu \in \mathsf{DA}(\mu)$, if for a family $(X_i, i \geq 1)$ of i.i.d. r.v. under ν, there exists $\theta \in [0, 2\pi]$ such that

$$\sum_{i=1}^n (X_i - \theta) \mod 2\pi \xrightarrow[n]{(d)} X_\mu.$$

We let $\mathsf{DA} = \{\mu \ : \mathsf{DA}(\mu) \neq \emptyset\}$ be the set of measures μ whose domains of attraction are not empty.

Proposition 3.10.

1) *The set* DA *coincides with the set of 1-stable distributions.*
2) *For any $\nu \in \mathcal{M}_{\mathcal{T}}^0$, there exists $\theta \in [0, 2\pi]$ and a unique 1-stable measure μ s.t. $\nu \in \mathsf{DA}(\mu)$.*

Proof.
(1) If ν is a 1-stable distribution, and if $(X_i, i \geq 1)$ are i.i.d. and taken under ν, then it is easily seen that $X_1 + \ldots + X_n \mod 2\pi \overset{(d)}{=} X_1$. Therefore, every 1-stable distribution is in DA.
Conversely, assume that $(X_i, i \geq 1)$ are i.i.d., distributed according to ν, and that $\sum_{i=1}^n (X_i - \theta) \mod 2\pi \xrightarrow[n]{(d)} \mu$. Splitting the sum on the left-hand side into two parts, μ appears to be solution of $\mu = \mu \underset{\mathcal{T}}{\star} \mu$, and then μ is 1-stable.

(2) Take $(X_i, i \geq 1)$ i.i.d. r.v. under ν, $\theta \in [0, 2\pi]$, and compute the limit of the kth Fourier coefficient, for $k \geq 1$, of $\sum_{j=1}^n (X_j - \theta)$,

$$\mathbb{E}(\mathrm{e}^{ik\sum_{j=1}^n (X_j-\theta)}) = \mathbb{E}(\mathrm{e}^{ik(X_1-\theta)})^n.$$

This coefficient either converges to 0 or is of modulus 1 (which implies $X = \theta/k[2\pi/k]$ a.s.). In either case, the limit is a 1-stable distribution. More precisely, let k be the smallest Fourier coefficient of the limit of modulus 1. If $k = +\infty$, the limit is the uniform distribution on $[0, 2\pi]$, otherwise it is the uniform distribution on $\{\frac{2j\pi}{k}, j \in [0, k-1]\}$. (see also Wilms [30], Thms. 2.1 and 2.4). □

3.3. Symmetrisation of CCS by convolution

Let $\nu \in \mathcal{M}_{\mathcal{T}}^0$ and $\overleftarrow{\nu} = \nu(2\pi - .)$. The distribution

$$S_C(\nu) := \nu \underset{\mathcal{T}}{\star} \overleftarrow{\nu} \tag{3.5}$$

is clearly symmetric. We call it the *symmetrisation by convolution* of ν[3].

Denote by $\nu_1 = S_C(\nu)$, $\nu_2 = S_C(\nu_1)$, ... Let X_n be a r.v. under ν_n.

Proposition 3.11. *Let $\nu \in \mathcal{M}_{\mathcal{T}}^0$, and let μ be the unique measure such that $S_C(\nu)$ belongs to $\mathsf{DA}(\mu)$. For $\theta = \pi$ or $\theta = 0$ we have*

$$X_n - n\theta \mod 2\pi \xrightarrow[n]{(d)} \mu.$$

Proof. First, ν_n is the distribution of $\sum_{i=1}^n (X_i - X_i') \mod 2\pi$ for some i.i.d. copies $X_i's$ and X_i''s of X_ν. The Fourier coefficients of ν_n can then be computed, and they converge to those of a 1-stable distribution as in Proposition 3.10, for $\theta \in \{0, \pi\}$ since $X_i - X_i'$ is symmetric. □

4. Extensions

In this section are discussed two natural extensions of our model. In Section 4.1 we discuss CCS with an unconstrained perimeter. In Section 4.2 is investigated the convergence of a trajectory made by i.i.d. increments with values in $\mathbb{C}$ sorted according to their arguments. If ν is a centred distribution on $\mathbb{C}$, these trajectories converge to a CCS $\mathcal{C}_{\mathsf{K}(\nu)}$ for an operator K defined below.

4.1. CCS with an unconstrained perimeter

The perimeter of the CCS in the construction we gave is 1 because the total mass of all measures in $\mathcal{M}_{\mathcal{T}}^0$ is 1. Denote by $\overline{\mathcal{M}_{\mathcal{T}}}^0$ the set of positive measures ν with support $\mathcal{T}$ and such that $\nu(\mathcal{T}) < +\infty$. Formula (2.1), which defines the CCS associated with a probability measure extends to these measures, and the CCS perimeter $\mathsf{Peri}(\nu) = \nu(\mathcal{T})$. A lot of statements given before extend naturally to $\overline{\mathcal{M}_{\mathcal{T}}}^0$. Most notably

Proposition 4.1. *For any measures $\nu_1, \nu_2 \in \overline{\mathcal{M}_{\mathcal{T}}}^0$, any positive numbers λ_1, λ_2 we have:*

$$\mathsf{Peri}\left(\sum_{i=1}^n \lambda_i \nu_i\right) = \sum_{i=1}^n \lambda_i \, \mathsf{Peri}(\nu_i) \tag{4.1}$$

$$\mathsf{Peri}\,(\nu_1 \star \nu_2) = \mathsf{Peri}(\nu_1)\,\mathsf{Peri}(\nu_2). \tag{4.2}$$

The area of $\mathcal{C}_{\sum_{i=1}^n \lambda_i \nu_i}$ and of $\mathcal{C}_{\nu_1 \star \nu_2}$ are still given by the Fourier coefficients of the measures $\sum_{i=1}^n \lambda_i \nu_i$ and $\nu_1 \star \nu_2$, as can be easily checked.

As said before, (4.1) is a well known result.

[3]Notice that replacing 2π by some other θ in the definition of $\overleftarrow{\nu}$ only affects $S_C(\nu)$ by a simple rotation in $\mathcal{T}$.

4.2. Reordering of random vectors in $\mathbb{C}$

The Gauss−Minkowski correspondence can be seen thanks to Corollary 2.9 as a consequence of the convergence of polygonal lines corresponding to some reordered random segments. This reordering can be done even if the lengths are not all the same; nevertheless the condition $\mathbb{E}(e^{iX_\mu}) = 0$ is needed to get a closed convex curve at the limit. In this section we investigate a generalisation of this construction where the sides of the polygons are r.v. in $\mathbb{C}$.

Let μ be a distribution with support included in $\mathbb{C}$ with mean 0, but different from δ_0. Take a sequence $W := (W_1, \ldots, W_n)$ of i.i.d. r.v. with common distribution μ, and let $\hat{W} := (\hat{W}_1, \ldots, \hat{W}_n)$ the list W sorted according to the arguments of the W_i's (if several of them have the same argument but different modulus, then take a uniform random order among them). For $\theta \in [0, 2\pi)$, define $N_n(\theta) := \#\{i, W_i \le \theta\}$. Let $S := (S(k), k = 0, \ldots, n)$ be the sequence of partial sums

$$S(k) := \sum_{j=1}^{k} \hat{W}_j, \tag{4.3}$$

piecewise linearly interpolated between integer points, and let $\mathbf{B}_n = \{S(t), t \in [0, n]\}$ be the polygonal line corresponding to the graph of S extended to $[0, n]$.

The distribution μ induces a law $\mathbb{P}_{|W|,\arg(W)}$ for the pair $(|W|, \arg(W))$, and a law $\mathbb{P}_{\arg(W)}$ for $\arg(W)$; let $\mathbb{P}_{|W|,x}$ be a version of the distribution of $|W|$ conditioned on $\arg(W) = x$ (this is defined up to a null set under $\mathbb{P}_{\arg(W)}$; for the sake of completeness, take $\mathbb{P}_{|W|,x} = \delta_0$ on the complementary set). We denote by m_x the mean of $|W|$ under $\mathbb{P}_{|W|,x}$.

Let ν be the measure having density $m/\mathbb{E}(|W|)$ with respect to $\mathbb{P}_{\arg(W)}$, that is

$$\mathrm{d}\nu(x) = \frac{m_x}{\mathbb{E}(|W|)} \mathrm{d}\mathbb{P}_{\arg(W)}(x). \tag{4.4}$$

The map which sends μ onto ν will be denoted K:

$$\mathsf{K}(\mu) = \nu. \tag{4.5}$$

Denote by $F^{\arg}$ the CDF of $\arg(W)$, and by F_ν that of the measure ν. From now on, let W_θ denote a r.v. W under the condition $\{\arg(W) \le \theta\}$.

We here present a theorem stating the aforementioned convergence; we think that it provides an agreeable way to see the phenomenons into play.

Theorem 4.2. *Consider the model described in the present section. Assume that μ is centred ($\neq \delta_0$), and let $\nu = \mathsf{K}(\mu)$. We have:*

1) $d_H(\mathbf{B}_n/(n\mathbb{E}(|W|)), \mathcal{B}_\nu) \xrightarrow[n]{\text{(a.s.)}} 0.$

2) *For any θ,*

$$\frac{S(N_n(\theta))}{n\mathbb{E}(|W|)} \xrightarrow[n]{\text{(a.s.)}} \int_0^\theta e^{it} \mathrm{d}\nu(t) = Z_\nu(F_\nu(\theta)). \tag{4.6}$$

Remark 4.3.

(a) Prosaically, the previous Theorem says that if μ is a centred distribution on $\mathbb{C}$ the CCS associated with μ is $\mathcal{C}_{K(\mu)}$.

(b) According to (4.4) and Theorem 4.2, $\mathcal{B}_{\mathsf{K}(\nu)}$ is the circle (with radius $1/(2\pi)$) if and only if $\mathbb{P}_{\arg}$ admits a density $f_\nu(\cdot)$ with respect to the Lebesgue measure, and $\theta \mapsto f_\nu(\theta) m_\nu(\theta)$ is constant.

(c) The ellipse of equation $x^2/c^2 + y^2 = R^2$ with perimeter $2\pi Rc = 1$, is obtained in the case where

$$m_\nu(\theta) = \frac{1}{2\pi}\frac{c}{\cos(\theta)^2 + c^2\sin(\theta)^2}.$$

This can be shown using the following parametrisation: $x(t) = \sin(t)$, $y(t) = c(1-\cos(t))$.

Proof of Theorem 4.2.
2) The cardinality of $N_n(\theta)$ has the binomial $(n, F^{\arg}(\theta))$ distribution. It satisfies for any θ,

$$N_n(\theta)/n \xrightarrow[n]{\text{(a.s.)}} F_\nu(\theta). \tag{4.7}$$

Conditionally on $N_n(\theta) = m$ the (multi)set $\{\hat{W}_1, \ldots, \hat{W}_m\}$ is distributed as a set of m i.i.d. copies of W_θ. Therefore by the law of large number,

$$\frac{S(N_n(\theta))}{n\mathbb{E}(|W|)} \xrightarrow[n]{\text{(a.s.)}} \frac{F^{\arg}(\theta)\mathbb{E}(W_\theta)}{\mathbb{E}(|W|)} = \frac{\mathbb{E}(W1_{\arg(W)\le\theta})}{\mathbb{E}(|W|)} \tag{4.8}$$

$$= \frac{\mathbb{E}(|W|\mathrm{e}^{i\arg(W)}1_{\arg(W)\le\theta})}{\mathbb{E}(|W|)} \tag{4.9}$$

$$= \int_0^\theta \mathrm{e}^{it}\frac{m_t}{\mathbb{E}(|W|)}\mathrm{d}\mathbb{P}_{\arg(W)}(t) = Z_\nu(F_\nu(\theta)). \tag{4.10}$$

This ends the proof of 2) and shows the a.s. simple convergence of the extremal points of the random curve to those of the deterministic limit. □

1) Similarly, the length $L_n(\theta)$ of the curve composed by the segments between the points $(S(i), 0 \le i \le N_n(\theta))$ satisfies

$$L_n(\theta) \xrightarrow[n]{\text{(a.s.)}} L(\theta) := \frac{\mathbb{E}(|W|1_{\arg(W)\le\theta})}{\mathbb{E}(|W|)}, \tag{4.11}$$

where $L(\theta)$ is the length of the curve $t \mapsto Z_\mu(t)$ between times 0 and $F_\mu(\theta)$. Fix a small $\varepsilon > 0$. There exists $\theta_1 < \ldots < \theta_k$ such that the convex hull of the points $Z_\nu(F_\nu(\theta_i))$ is at distance at most ε of $\mathcal{B}_\nu$. Notice that such a property implies that the successive segments lengths $l_i = |Z_\nu(F_\nu(\theta_i)) - Z_\nu(F_\nu(\theta_{i-1}))|$ satisfies

$$L(\theta_i) - L(\theta_{i-1}) - 2\varepsilon \le l_i \le L(\theta_i) - L(\theta_{i-1})$$

since B_ν is convex and the graph of Z_ν must stay at distance at most ε of $[Z_\nu(F_\nu(\theta_i)), Z_\nu(F_\nu(\theta_{i-1}))]$ between times $F_\nu(\theta_i)$ and $F_\nu(\theta_{i-1})$. But for n large enough, up to an additional ε, the discrete curve has the same properties with high probability. By (4.8)

$$\sup_{1\le j\le n}\left|\frac{S(N_n(\theta_j))}{n\mathbb{E}(|W|)} - Z_\nu(F_\nu(\theta_j))\right| \xrightarrow[n]{\text{(a.s.)}} 0.$$

The length $L_n(\theta_i) - L_n(\theta_{i-1})$ of the curve between θ_{i-1} and θ_i converges a.s. to $L(\theta_i) - L(\theta_{i-1})$ by (4.11). This implies that the Hausdorff distance between $\mathbf{B}_n/(n\mathbb{E}(|W|))$ and the convex hull of the points $\frac{S(N_n(\theta_j))}{n\mathbb{E}(|W|)}$'s goes to zero a.s. □

We now consider convolution and mixture of CCS.

Proposition 4.4. *Let X and Y be independent r.v. in $\mathbb{C}$ with mean 0 (but not equal to 0 a.s.), and $\lambda \in [0,1]$. Let μ_X, μ_Y and $\mu_{X.Y}$ be the laws of X, Y and $X.Y$. We have*

$$\mathcal{C}_{\mathsf{K}(\mu_{X.Y})} = \mathcal{C}_{\mathsf{K}(\mu_X)} \star \mathcal{C}_{\mathsf{K}(\mu_Y)} \quad \textit{and} \quad \mathcal{C}_{\mathsf{K}(\lambda\mu_X + (1-\lambda)\mu_Y)} = \lambda\mathcal{C}_{\mathsf{K}(\mu_X)} + (1-\lambda)\mathcal{C}_{\mathsf{K}(\mu_Y)}.$$

Proof. The statement concerning the mixture is quite easy and follows Theorem 4.2 for example. For the other one, following (3.1), it suffices to see that $\mathsf{K}(\mu_{X.Y}) = \mathsf{K}(\mu_X) \underset{\mathcal{T}}{\star} \mathsf{K}(\mu_Y)$. Observe that for any measure μ on $\mathbb{C}$ (such that $0 < |X_\mu| < +\infty$),

$$\frac{\mathbb{E}(\mathrm{e}^{ix\arg(X_\mu)}|X_\mu|)}{\mathbb{E}(|X_\mu|)} = \int_0^{2\pi} \mathrm{e}^{ix\theta} \frac{m_{X_\mu}(\theta)}{\mathbb{E}(|X_\mu|)} \mathrm{d}\mathbb{P}_{\arg(X_\mu)}(\theta).$$

Indeed, according to (4.4), the Fourier transform of $\mathsf{K}(\mu)$ at position x is given by $\frac{\mathbb{E}(\mathrm{e}^{ix\arg(X_\mu)}|X_\mu|)}{\mathbb{E}(|X_\mu|)}$. Hence, the Fourier transform of $\mathsf{K}(\mu_{X.Y})$, for X and Y independent, is

$$\frac{\mathbb{E}(\mathrm{e}^{ix\arg(XY)}|XY|)}{\mathbb{E}(|XY|)} = \frac{\mathbb{E}(\mathrm{e}^{ix\arg(X)}|X|)}{\mathbb{E}(|X|)} \frac{\mathbb{E}(\mathrm{e}^{ix\arg(Y)}|Y|)}{\mathbb{E}(|Y|)},$$

which implies that the Fourier transform of $\mathsf{K}(\mu_{X.Y})$ and of $\mathsf{K}(\mu_X) \underset{\mathcal{T}}{\star} \mathsf{K}(\mu_Y)$ are the same. $\mathcal{C}_{\mathsf{K}(\mu_{X.Y})}$ and $\mathcal{C}_{\mathsf{K}(\mu_X)} \star \mathcal{C}_{\mathsf{K}(\mu_Y)}$ are equal by Definition 3.1. □

Remark 4.5. The CCS $\mathcal{C}_{\mathsf{K}(\mu)}$ characterises $\mathsf{K}(\mu)$ but not μ. For example the two following measures $\mu_1 = \frac{1}{3}\left(\delta(1) + \delta(\mathrm{e}^{2i\pi/3}) + \delta(\mathrm{e}^{4i\pi/3})\right)$ and $\mu_2 = \frac{1}{3}\left(\frac{1}{2}\delta(\frac{1}{2}) + \frac{1}{2}\delta(\frac{3}{2}) + \delta(\mathrm{e}^{2i\pi/3}) + \delta(\mathrm{e}^{4i\pi/3})\right)$ satisfy $\mathsf{K}(\mu_1) = \mathsf{K}(\mu_2)$ and $\mathcal{C}_{\mathsf{K}(\mu_i)}$ is an equilateral triangle. Every CCS $\mathcal{C}_\nu$ can therefore be seen as an equivalence class of measures over $\mathbb{C}$. However, $\mathsf{K}\left(\mu_1 \underset{\mathcal{T}}{\star} \mu_1\right)$ represents a polygon with 6 sides, whereas $\mathsf{K}\left(\mu_1 \underset{\mathcal{T}}{\star} \mu_2\right)$ a polygon with 7 sides, even though $\mathsf{K}(\mu_1) = \mathsf{K}(\mu_2)$. Hence $\mathsf{K}(\mu_1 \star \mu_2)$ is not a function of $\mathsf{K}(\mu_1)$ and $\mathsf{K}(\mu_2)$, and then the convolution of measures in $\mathbb{C}$ can not be turned into a nice operation on CCS.

5. Some models of random CCS

In this part, we consider the problem of finding natural distributions on the set of CCS. We first recall some classical considerations on simple models of random convex polygons. In a second part we take advantage of the representation of CCS by measures in $\mathcal{M}_{\mathcal{T}}^0$ to present models for the generation of smooth CCS based on random Fourier coefficients.

5.1. Reordering of closed polygons

Consider the problem of generating a convex polygon by specifying a finite set of vectors representing its edges. Let μ be a distribution on $\mathbb{C}$ whose support is not reduced to a point, and for some $n \geq 2$, let $(X_i, i = 1, \ldots, n)$ be n i.i.d. r.v. distributed according to μ, and set

$$W_i = X_{(i \bmod n)+1} - X_i, \quad 1 \leq i \leq n.$$

Naturally, $\sum_{i=1}^n W_i = 0$. Let $(\hat{W}_i, 1 \leq i \leq n)$ be the sequence $(W_i, 1 \leq i \leq n)$ sorted according to their arguments. Let now S be defined as in (4.3), and $\mathbf{B}_n$ defined as in Section 4.2. Further, let μ be the distribution of $W_1 = X_2 - X_1$, and $\nu = \mathsf{K}(\mu)$.
The following result analogous with Theorem 4.2 shows that $\mathbf{B}_n$ converges in distribution to $\mathcal{B}_\nu$:

Theorem 5.1. *Assume that μ is centred (different from δ_0). Then*

$$d_H\left(\mathbf{B}_n/(n\mathbb{E}(|W|)), \mathcal{B}_\nu\right) \xrightarrow[n]{\text{(a.s.)}} 0.$$

Moreover (4.6) *holds.*

Proof. We have $S(N_n(\theta)) = \sum_{i=1}^n (X_{(i \bmod n)+1} - X_i) 1_{\arg(X_{(i \bmod n)+1} - X_i) \leq \theta}$; the difference with the proof of Theorem 4.2 is the dependence between the r.v. in the sum. But these r.v. are only weakly dependent (each r.v. depends on the previous and following one); then strong law of large number applies to this case (since the sum can be split into two sums with i.i.d. r.v.), and the rest of the proof follows that of Theorem 4.2. □

5.2. Convex polygon by conditioning/convex polygon by chance

Another natural way to sample a convex polygon is to take some i.i.d. points $W_0, \dots, W_{n-1}$ in the plane according to a distribution μ with support not included in a line, and to condition $(W_0, \dots, W_{n-1})$ to be a convex polygon. Define the set of all possible convex polygons as

$$\mathbf{B}_n = \{\mathbf{w} := (w_0, \dots, w_{n-1}) : \arg(w_{i+1 \bmod n} - w_i) \text{ forms an increasing sequence in } [0, 2\pi)\}.$$

Hence, $\mathbf{w}$ represents the list of vertices of a convex polygons encountered when following its boundary in the counter-clockwise direction (with some conditions for w_0).

The value of $\mu^{\otimes n}(\mathbf{B}_n)$ is known only for μ equal to the uniform distribution in a triangle or in a parallelogram [27, 28] and in a circle [17]; when μ is the uniform distribution in a CCS, the limit behaviour for $\mathbf{w}$ under the condition $\mathbf{w} \in \mathbf{B_n}$ is described in Bárány [1]. We open here a parenthesis to explain the underlying difficulty. Consider $S_n := (w_0, \dots, w_n)$ a n-tuple of points in $\mathbb{R}^2$, not three of them being on the same line (this happens almost surely if μ admits a density on an open set in $\mathbb{R}^2$). When $w_i = (x_i, y_i)$ for any i, the algebraic area of the triangle (w_i, w_j, w_k) is

$$A_{i,j,k} = \frac{1}{2}(x_i y_j + x_j y_k + x_k y_i - y_i x_j - y_j x_k - y_k x_i). \tag{5.1}$$

The set $(s_{i,j,k} := \mathsf{sign}(A_{i,j,k}), 0 \le i < j < k \le n-1)$ is called the *chirotope* of S_n. An equivalence class for the chirotope, is called an *order type*. The sequence S_n forms a convex polygon iff all $s_{i,j,k}$ have the same sign. It is known that some order types are empty, and also that deciding if an order type is not empty, is a NP-complete problem (*cf.* Knuth [15], Sect. 6).

When $(W_j = (X_j, Y_j), j = 0, \dots, n-1)$ is a family of i.i.d. r.v., such that the X_i and Y_i are independent Gaussian centred r.v. with variance 1, it turns out that the Laplace transform of the joint law of the $A_{i,j,k}$'s (the areas of the triangles (W_i, W_j, W_k)) that is

$$\Phi(\lambda_{i,j,k}, 0 \le i < j < k \le n-1) := \mathbb{E}\left(\exp\left(\sum_{0 \le i < j < k \le n-1} \lambda_{i,j,k} A_{i,j,k}\right)\right)$$

is equal to $|\det(\Lambda)|^{-1/2}$, where $\Lambda = (\ell_{i,j})$ and $\ell_{i,j} = \sum_a \lambda_{i,j,a} + \lambda_{a,i,j} - \lambda_{i,a,j}$ (in a neighbourhood of the origin of $\mathbb{R}^{\binom{n}{3}}$). To get this result, the method is the same as the one for the computation of the Fourier transform of a Gaussian vector in $\mathbb{R}^d$.

Remark 5.2. As remarked by Andrea Sportiello in a private communication, $|\det(\Lambda)|$ is always a square of a polynomial in the coefficients $\bar{\lambda}_{i,j}$. Indeed, for $\Lambda' = \begin{bmatrix} -Id_n & 0 \\ 0 & Id_n \end{bmatrix} \Lambda$, Λ and Λ' have the same determinant (up to factor $(-1)^n$). But it can be shown that Λ' is a skew matrix, and then its determinant is the square of its Pfaffian, which is indeed a polynomial on its coefficients.

The Gaussian distribution is probably the simplest non trivial measure for which this computation is possible. The question of the emptiness of an order type $S = (s_{i,j,k}, i < j < k)$ can be translated in term of the support of the measure, but Knuth's result implies that it is a difficult task. If $n = 3$, only one triangle is present; the Laplace transform is $1/(1 - 3\lambda_{0,1,2}^2/4)$, the transform of a Gamma r.v. with a random sign; when $n = 4$, the Laplace transform is much more complex.

5.3. Generation of smooth random CCS

This part is mainly prospective. By Theorem 2.2, to conceive a model of random CCS in $\mathsf{Conv}(1)$ and to conceive a model of random measures with values in $\mathcal{M}_{\mathcal{T}}^0$ is the same problem. Since the condition "to be in $\mathcal{M}^0_{\mathcal{T}}$" has a simple expression in term of Fourier coefficients, and since the Fourier coefficients determine the measure (Prop. 2.5), a simple idea consists in describing random measures in $\mathcal{M}^0_{\mathcal{T}}$ using random Fourier coefficients.

This leads us to Szegö's Theorem [26]: if a trigonometric polynomial $P : \mathcal{T} \to \mathbb{R}^+$ admits only non-negative values, then there exists a polynomial D such that:

$$\forall t \in \mathcal{T}, \qquad P(t) = |D(\mathrm{e}^{it})|^2$$

Moreover D is unique up to multiplication by a complex of modulus 1. If we consider the Fourier expansion $D(\mathrm{e}^{it}) = \sum_{n\geq 0} \rho_n \mathrm{e}^{i\theta_n} \mathrm{e}^{\mathrm{int}}$, for some finite sequences of real numbers $(\rho_n), (\theta_n)$, the modulus of D is equal to:

$$|D(\mathrm{e}^{it})|^2 = A_0 + \sum_{n\geq 1} A_n \cos(nt) + B_n \sin(nt)$$

$$\text{with} \begin{cases} A_0 = \sum_{k\geq 0} \rho_k^2 \\ A_n = 2\sum_{k\geq 0} \rho_{k+n}\rho_k \cos(\theta_k - \theta_{k+n}) \text{ for } n \geq 1, \\ B_n = 2\sum_{k\geq 0} \rho_{k+n}\rho_k \sin(\theta_k - \theta_{k+n}) \text{ for } n \geq 1. \end{cases} \tag{5.2}$$

Hence, the trigonometric polynomial P is the density of a measure $\mu \in \mathcal{M}^0_{\mathcal{T}}$ iff the sequences (A_n) and (B_n) satisfy (i) the perimeter condition ($A_0 = \frac{1}{2\pi}$, ensuring that μ is a probability measure) and (ii) the closed path condition ($A_1 = B_1 = 0$, ensuring that $\int_0^{2\pi} \mathrm{e}^{ix}\mathrm{d}\mu(x) = 0$).

5.3.1. Generation of CCS via their Fourier coefficients

In order to generate a random pair $\mathcal{P} := ((\rho_k, k \geq 0), (\theta_k, k \geq 0))$ satisfying both conditions, two possibilities are open, depending on which condition should be satisfied first (but the question of finding natural distributions for CCS will remain open).

To satisfy $A_1 = B_1 = 0$ first, it suffices to generate ρ_j and θ_j for $j \geq 1$ at random then take ρ_0 and θ_0 such that:

$$\rho_0\rho_1\mathrm{e}^{i(\theta_0-\theta_1)} = -\sum_{k\geq 1} \rho_{k+1}\rho_k \mathrm{e}^{i(\theta_k-\theta_{k+1})}.$$

This is always possible if the sum converges and if ρ_1 is not 0. To satisfy $A_0 = 1/2\pi$ from here, a normalisation step can be applied: divide each ρ_n by $\sqrt{\sum_{k\geq 0} \rho_k^2}$.

Szegö's theorem ensures that the set of measures induced by this method has full support over $\mathcal{M}^0_{\mathcal{T}}$: indeed, each measures in $\mathcal{M}^0_{\mathcal{T}}$ can be weakly approached by a sequence of distributions with strictly positive density; these ones can be in turn approached by a sequence of positive trigonometric polynomials, and Szegö's theorem gives a representation of these polynomials. The results of such a generation can be seen on Figure 4.

Another solution consists in ensuring first $A_0 = 1/2\pi$, which comes down to producing $(\rho_k, k \geq 0)$ such that $\sum_{k\geq 0} \rho_k^2 = \frac{1}{2\pi}$. This can be done by choosing (generating) random reals r_j in $[0,1]$, and setting:

$$\rho_k^2 = \frac{1}{2\pi} r_k \prod_{j=0}^{k-1} (1 - r_j).$$

This is well defined if $\prod_k (1 - r_k)$ converges to 0 when k goes to infinity (for example, taking i.i.d. r_j's under $\mathsf{uniform}[0,1]$ does the job). From here, satisfying $A_1 = 0$ and $B_1 = 0$ by a right choice of θ's can become more difficult, and even impossible, for example if $\rho_0 = \rho_1 > 0$ and all other ρ_i's are 0. Nevertheless, it is possible to generate $\mathcal{P}$ satisfying all the constraints at once. Choose (at random or not) a subset F of $\mathbb{N}$ such that if $i \in F$, then $i + 1 \notin F$, and a sequence x_k such that $\sum_{k\geq 0} x_k^2 = \frac{1}{2\pi}$ as above. Now, let n_j be the $j + 1$th smallest element in F, with the convention that the smallest is n_0. Define the sequence (ρ_k) by:

$$\rho_{n_j} = r_j, \quad \rho_k = 0 \text{ otherwise}$$

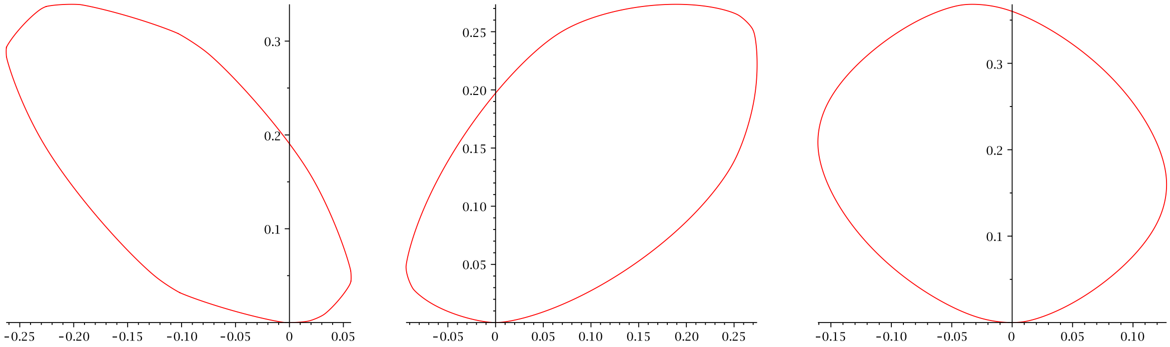


FIGURE 4. Examples of random CCS generated from trigonometric polynomials containing 25 non-zero coefficients (with $\rho_j \sim \mathsf{uniform}[0;1]$, and $\theta_j \sim \mathsf{uniform}[0;2\pi]$, all these r.v. being taken independently).

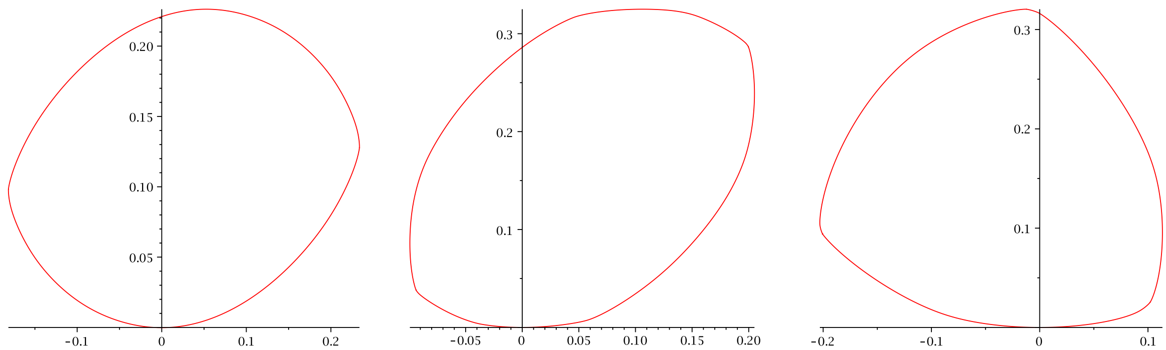


FIGURE 5. Examples of random CCS generated from polynomials containing 12 non-zero coefficients with sparse coefficients (the indices of the non-null Fourier coefficients of F are selected with probability 0 if the previous coefficient was selected, and with probability $\frac{1}{2}$ otherwise; $\rho_j \sim \mathsf{uniform}[0;1]$; $\theta_j \sim \mathsf{uniform}[0;2\pi]$, all these r.v. are taken independently).

Thanks to (5.2), $A_1 = B_1 = 0$ (since for all k, $\rho_k\rho_{k+1} = 0$), and this for any choice of (θ_k). Examples of CCS generated this way appear on Figure 5.

5.3.2. Generation of CCS with a given area

Consider the problem of generating a CCS in $\mathsf{Conv}(1)$ with a given area $\alpha = \frac{1}{4\pi} - \frac{\pi}{2}\beta \in [0, \frac{1}{2\pi^2}]$. Such a CCS corresponds to Fourier coefficients that satisfies:

$$\sum_{k\geq 2} \frac{a_k^2 + b_k^2}{k^2-1} = \beta.$$

As in the previous section, we consider a sequence of numbers (r_j) in $[0,1)$ for $j \geq 2$, such that $\prod_{j\geq 2}(1-r_j) = 0$, and define positive reals (c_k) such that:

$$\frac{c_k^2}{k^2-1} = \beta\ r_k \prod_{j=2}^{k-1}(1-r_j).$$

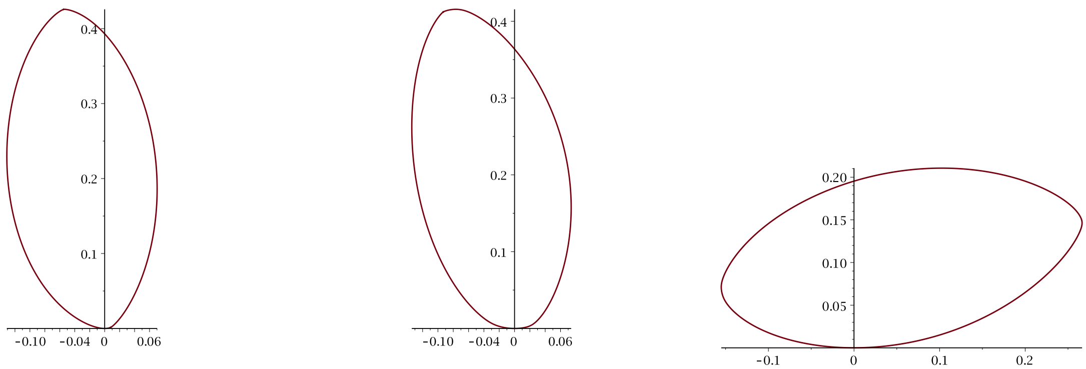


Figure 6. Examples of random CCS of perimeter 1 generated such that their area is equal to $\frac{1}{4\pi} - \frac{\pi}{2} \times 0.01$ (the polynomials possess 20 non-null coefficients, $\rho_j \sim \mathsf{uniform}[0;1]$, and $\theta_j \sim \mathsf{uniform}[0;2\pi]$, all these r.v. being taken independently).

Let $(\theta_k, k \geq 2)$ be a sequence of real numbers in $[0, 2\pi)$. Then the Fourier coefficients of the associated measure can be computed as follows:

$$a_k = \cos(\theta_k)c_k, \quad b_k = \sin(\theta_k)c_k.$$

It is still possible to take $a_1 = b_1 = 0$ and $a_0 = 1/(2\pi)$, but since we didn't use Szegö's theorem, the standard Fourier series associated to the a_i's and b_i's is unlikely to be a positive function. From here, it suffices to reject all series with a negative minimum. The results of such a generation appear on Figure 6. Experiments show that the rejection rate is very high, and that it is very difficult to generate CCS with $\beta > 0.01$ (the theoretical maximum being $\frac{1}{2\pi^2} \approx 0.05$).

Appendix A.

A.1. Proof of Theorem 2.8

Convergence of the FDD of W_n.

Let $\theta_0 := 0 \leq \theta_1 < \theta_2 < \ldots < \theta_\kappa = 2\pi$ for some $\kappa \geq 1$ be fixed. In the sequel, for any function (random or not) L indexed by θ, $\Delta L(\theta_j)$ will stand for $L(\theta_j) - L(\theta_{j-1})$. For any $\ell \leq \kappa$

$$W_n(\theta_\ell) = \sqrt{n} \sum_{j=0}^{\ell} \Delta \left[Z_n(N_n(\theta_j)) - Z_\mu(F_\mu(\theta_j)) \right] \tag{A.1}$$

where by convention $Z_n(N_n(\theta_{-1})) = Z_\mu(F_\mu(\theta_{-1})) = 0$. The convergence of the FDD of W_n follows from those of $(\sqrt{n}\Delta \left[Z_n(N_n(\theta_j)) - Z_\mu(F_\mu(\theta_j)) \right], 0 \leq i \leq \kappa)$. Notice that

$$\Delta Z_\mu(F_\mu(\theta_j)) = \mathbb{E}\left(\exp(iX) 1_{\theta_{j-1} < X \leq \theta_j} \right). \tag{A.2}$$

If for some j, θ_{j-1} and θ_j are chosen in such a way that $\Delta F_\mu(\theta_j) = 0$ then the jth increment in (A.1) is 0 almost surely (this is the case for the 0th increment if $\mu(\{0\}) = 0$). We now discuss the asymptotic behaviour of the other increments : let $J = \{j \in \{0, \ldots, \kappa\} : \Delta F_\mu(\theta_j) \neq 0\}$.

Let $(n_j, j \in J)$ be some fixed integers such that $n = \sum n_j$. Denote by $\mu_{\theta_{j-1},\theta_j}$ the law of X_μ conditioned on $\{\theta_{j-1} < X_\mu \leq \theta_j\}$, and by $X_{\theta_{j-1},\theta_j}$ a r.v. under this distribution. Conditionally on $(N_n(\theta_j) = n_j, j \in J)$, the r.v. $\Delta Z_n(N_n(\theta_j)), j \in J$ are independent. The law of $\Delta Z_n(N_n(\theta_j))$ is that of a sum of $n_j - n_{j-1}$ i.i.d. copies of r.v. under $\mu_{\theta_{j-1},\theta_j}$, denoted from now on $(X_{\theta_{j-1},\theta_j}(k), k \geq 1)$):

$$\mathbb{E}\left(\Delta Z_n(N_n(\theta_j))\middle|N_n(\theta_l) = n_l, l \in J\right) = n^{-1}\mathbb{E}\left(\sum_{m=1}^{n_j-n_{j-1}} \mathrm{e}^{iX_{\theta_{j-1},\theta_j}(m)}\right) = \frac{(n_j - n_{j-1})}{n}\frac{\Delta Z_\mu(F_\mu(\theta_j))}{\Delta F_\mu(\theta_j)}.$$

Since $(\Delta N_n(\theta_j), j \in J) \sim \mathsf{Multinomial}\,(n, (\Delta F_\mu(\theta_j), j \in J))$,

$$\left(\frac{\Delta N_n(\theta_j) - n\Delta F_\mu(\theta_j)}{\sqrt{n}}, j \in J\right) \xrightarrow[n]{(d)} (N_j, j \in J) \tag{A.3}$$

where $(N_j, j \in J)$ is a centred Gaussian vector with covariance function

$$\mathrm{cov}(N_k, N_l) = -\Delta F_\mu(\theta_k)\,.\,\Delta F_\mu(\theta_l),$$

formula valid for any $0 \leq k, l \leq \kappa$. Putting together the previous considerations, we have, conditioning first on the $N_n(\theta_j)$'s, and then integrating on the distribution of these r.v.,

$$\Delta W_n(\theta_j) = \sum_{l=1}^{\Delta N_n(\theta_j)} \frac{\mathrm{e}^{iX_{\theta_{j-1},\theta_j}(l)} - \mathbb{E}(\mathrm{e}^{iX_{\theta_{j-1},\theta_j}})}{\sqrt{n}} + \left(\frac{\Delta N_n(\theta_j) - n\Delta F_\mu(\theta_j)}{\sqrt{n}}\right)\mathbb{E}(\mathrm{e}^{iX_{\theta_{j-1},\theta_j}}). \tag{A.4}$$

Using (A.3) and the central limit theorem, we then get that

$$(\pi\Delta W_n(\theta_j), 0 \leq j \leq \kappa) \xrightarrow[n]{(d)} \sqrt{\Delta F_\mu(\theta_j)}\widetilde{N}_j + N_j\begin{bmatrix}\mathbb{E}(\cos(X_{\theta_{j-1},\theta_j}))\\ \mathbb{E}(\sin(X_{\theta_{j-1},\theta_j})\end{bmatrix}, \tag{A.5}$$

where the r.v. $N_j, \tilde{N}_j, j \leq \kappa$ are independent, and the r.v. $\tilde{N}_j$ are centred Gaussian r.v. with covariance matrix, the covariance matrix of $\begin{bmatrix}\cos(X_{\theta_{j-1},\theta_j})\\ \sin(X_{\theta_{j-1},\theta_j})\end{bmatrix}$.

Tightness of $\{W_n, n \geq 0\}$ *in* $D[0, 2\pi]$.

A criterion for tightness in $D[0, 2\pi]$ can be found in Billingsley ([4], Thm. 13.2) a sequence of processes $(W_n, n \geq 1)$ with values in $D[0, 2\pi]$ is tight if, for any $\varepsilon \in (0, 1)$, there exists $\delta > 0, N > 0$ such that

$$\lim_{\delta\to 0}\limsup_n \mathbb{P}(\omega'(W_n, \delta) \geq \varepsilon) = 0$$

where $\omega'(f, \delta) = \inf_{(t_i)} \max_i \sup_{s,t\in[t_{i-1},t_i)} |f(s) - f(t)|$, and the partitions (t_i) range over all partitions of the form $0 = t_0 < t_1 < \ldots < t_n \leq 2\pi$ with $\min\{t_i - t_{i-1}, 1 \leq i \leq n\} \geq \delta$.

Since only the tightness in $D[0, 2\pi]$ interests us, we will focus on $\Re(W)$ (since the imaginary part can be treated likewise, and since the tightnesses of both $\Re(W)$ and $\Im(W)$ implies that of W). For the sake of brevity, in the sequel, we will use W instead of $\Re(W)$.

The first step in our proof consists in comparing the distribution $\mathbb{P}_n$ of a set $\{X_1, \ldots, X_n\}$ of n i.i.d. copies of X_μ with a Poisson point process P_n on $[0, 2\pi]$ with intensity $n\mu$, denoted by $\mathbb{P}_{P_n}$. Conditionally on $\#P_n = k$, the k points $P_n := \{Y_1, \ldots, Y_k\}$ are i.i.d. and have distribution μ, and then $\mathbb{P}_{P_n}(\,\cdot\,|\#P = n) = \mathbb{P}_n$. The Poisson point process is naturally equipped with a filtration $\sigma := \{\sigma_t = \sigma(\{P \cap [0, t]\}), t \in [0, 2\pi]\}$.

We are here working under $\mathbb{P}_{P_n}$, and we let $N(\theta) = \#P_n \cap [0, \theta]$; notice that under $\mathbb{P}_n$, N and N_n coincide.

We will show the tightness of W under $\mathbb{P}_{P_n}$ first. Before doing this, let us see why it implies the same result under $\mathbb{P}_n$: Let m be a point in $[0, 2\pi]$ such that $F_\mu(x) > 1/4, 1 - F_\mu(x) > 1/4$ (it is a kind of median of μ). We need in the sequel $1-F_\mu(m)>0$; for measures in $\mathcal{M}^0_{\mathcal{T}}$ this is always the case, since if not, an atom with weight $>1/2$ would exist. We will see that the tightness under $\mathbb{P}_{P_n}$ implies that the sequence of processes W under $\mathbb{P}_n$ is tight in $D[0,m]$ (the same proof works on $D[m,2\pi]$ by a time reversing argument). We claim that for any event σ_m measurable,

$$\mathbb{P}_n(A)=\mathbb{P}_{P_n}(A\,|\,\#P=n)\le c\,\mathbb{P}_{P_n}(A) \tag{A.6}$$

for a constant c independent on n and of A (but which depends on μ). This in hand, the tightness under $\mathbb{P}_{P_n}$ of W on $D[0,m]$ implies that under $\mathbb{P}_n$. Let us prove (A.6). We have

$$\begin{aligned}\mathbb{P}_{P_n}(A\,|\,\#P=n)&=\sum_k\frac{\mathbb{P}_{P_n}(A,\#(P\cap[0,m])=k)\mathbb{P}(\#P\cap[m,2\pi]=n-k)}{\mathbb{P}(\#P=n)}\\&\le\sum_k\mathbb{P}_{P_n}(A,\#(P\cap[0,m])=k)\sup_{k'}\frac{\mathbb{P}(\#P\cap[m,2\pi]=n-k')}{\mathbb{P}(\#P=n)}\\&\le c\,\mathbb{P}_{P_n}(A)\end{aligned}$$

where $c=\sup_{n\ge1}\sup_{k'}\frac{\mathbb{P}(\#P\cap[m,2\pi]=n-k')}{\mathbb{P}(\#P=n)}$, which is indeed finite since:

- first $\#P\cap[m,2\pi]\sim\mathsf{Poisson}(n(1-F_\mu(m)))$, and then $\sup_{k'}\mathbb{P}(\#P\cap[m,2\pi]=n-k')$ is the mode of a Poisson distribution. When the parameter is λ, the mode is equivalent to $1/\sqrt{2\pi\lambda}$ when $\lambda\to+\infty$, so here it is equivalent to $1/\sqrt{2\pi n(1-F_\mu(m))}$,
- and by Stirling $\mathbb{P}(\#P=n)\sim(2\pi n)^{-1/2}$.

Working with a Poisson point process instead of working with n r.v. provides some independence between the number of r.v. X_i in disjoint intervals, and then on the fluctuations of W_n in disjoint intervals.

Before starting, recall that if $N\sim\mathsf{Poisson}(a)$, for any positive λ,

$$\mathbb{P}(N\ge x)=\mathbb{P}\left(\mathrm{e}^{\lambda N}\ge\mathrm{e}^{\lambda x}\right)\le\mathbb{E}(\mathrm{e}^{\lambda N-\lambda x})=\mathrm{e}^{-a+a\mathrm{e}^{\lambda}-\lambda x} \tag{A.7}$$

$$\mathbb{P}(N\le x)=\mathbb{P}\left(\mathrm{e}^{-\lambda N}\ge\mathrm{e}^{-\lambda x}\right)\le\mathbb{E}(\mathrm{e}^{-\lambda N+\lambda x})=\mathrm{e}^{-a+a\mathrm{e}^{-\lambda}+\lambda x}. \tag{A.8}$$

Let $A_\mu=\{x\in[0,2\pi],\mu(\{x\})>0\}$ be the set of positions of the atoms of μ. We now decompose $\mu=\mu|_{A_\mu}+\mu|_{\complement A_\mu}$; under $\mathbb{P}_n$ as well as under $\mathbb{P}_{P_n}$, the process W can be also decomposed under the form $W|_{A_\mu}+W|_{\complement A_\mu}$ using $N|_{A_\mu}(\theta)=\#P\cap[0,\theta]\cap A_\mu$, $Z|_{A_\mu}(N|_{A_\mu}(\theta))=\sum_{j=1}^N\mathrm{e}^{i\hat{X}_j}1_{\hat{X}_j\in A_\mu}$, *etc.* The fluctuations of $W=W|_{A_\mu}+W|_{\complement A_\mu}$ are then bounded by the sum of the fluctuations of both processes $W|_{A_\mu}$ and $W|_{\complement A_\mu}$. It is then sufficient to show the tightness for a purely atomic measure μ, and for a measure having no atom μ.

Case where μ is purely atomic.

Take some (small) $\eta\in(0,1)$, $\varepsilon>0$; we will show that one can find a finite partition $(t_i,i\in I)$ of $[0,2\pi]$ and a $\delta\in(0,1)$ such that

$$\limsup_n\mathbb{P}_n(\omega'(W_n,\delta)\ge\varepsilon)\le\eta, \tag{A.9}$$

which is sufficient for our purpose. In fact we will establish (A.9) under $\mathbb{P}_{P_n}$ instead, on $[0,m]$ and then on $[m,2\pi]$, since we saw that this was sufficient (replacing η by $c\eta$ in (A.9), suffices too).

Now, let $A_\mu^{\ge a}:=\{x\in A_\mu:\mu(\{x\})\ge a\}$. Clearly $\#A_\mu^{\ge a}\le1/a$ and $[0,2\pi]\setminus A_\mu^{\ge a}$ forms a finite union of open connected intervals $(O_x,x\in G)$, with extremities $(t'_i,i\in I)$. The intervals $(O_x,x\in G)$ can be further cut as follows:

- do nothing to those such that $\mu(O_x)<2a$;
- those such that $\mu(O_x)>2a$ are further split. Since they contain no atom with mass $>a$, they can be split into smaller intervals having all their weights in $[a,2a]$ except for at most one (in each interval O_x which may have a weight smaller than a).

Once all these splittings have been done, a list of at most $3/a$ intervals are obtained, all of them having a weight smaller than $2a$. Name $G_a = (O_x, x \in I_a)$ the collection of obtained open intervals, index by I_a, and by $(t_i^a, i \geq 0)$ the partitions obtained. Clearly

$$M_a := \max_{i \in I_a} \mathbb{E}(\cos(X_\mu)^2 1_{X_\mu \in O_i}) \leq M'_a := 2a.$$

Control of the fluctuations of W_n on an interval O_x.

In the sequel we take $a = \varepsilon^3$ and consider a unique interval $O_x = (\theta_{j-1}, \theta_j) \in G_a$, in which case we have $M_{\varepsilon^3} \leq 2\varepsilon^3$. We control first the last position of the random walk W_n. Under $\mathbb{P}_{P_n}$, $\mathcal{P}(n\mu\{\theta\}) := \#P_n \cap \{\theta\}$ has distribution $\mathsf{Poisson}(n\mu(\{\theta\}))$, the r.v. corresponding to different points being independent. Following (A.4), under $\mathbb{P}_{P_n}$, we get

$$\Delta W_n(\theta_j) = \sqrt{n} \sum_{\substack{\theta \in A_\mu \\ \theta_{j-1} \leq \theta < \theta_j}} \left(\frac{\mathcal{P}(n\mu\{\theta\})}{n} - \mu(\{\theta\}) \right) \cos(\theta). \tag{A.10}$$

These centred r.v. can be controlled as usual Poisson r.v. as recalled above. On the first hand,

$$\mathbb{P}(\Delta W_n(\theta_j) \geq \varepsilon) = \mathbb{P}\left(\sum_\theta \mathcal{P}(n\mu\{\theta\}) \cos(\theta) \geq y \right) \tag{A.11}$$

where

$$y = \varepsilon\sqrt{n} + n\mathbb{E}(\cos(X) 1_{X \in A_\mu, \theta_{j-1} < X \leq \theta_j}) \tag{A.12}$$

and where the set of summation is the same as before (from now on, it will be omitted).
Writing $\mathbb{P}\left(\sum_\theta \mathcal{P}(n\mu\{\theta\}) \cos(\theta) \geq y\right) \leq \inf_{\lambda > 0} \mathrm{e}^{-\lambda y} \prod_\theta \mathbb{E}(\mathrm{e}^{(\lambda \cos(\theta)) \mathcal{P}(n\mu\{\theta\})})$ one has

$$\mathbb{P}(\Delta W_n(\theta_j) \geq \varepsilon) \leq \inf_{\lambda > 0} \exp\left(-\sum_\theta n\mu\{\theta\} + \sum_\theta n\mu\{\theta\} \mathrm{e}^{\lambda \cos(\theta)} - \lambda y \right).$$

To get a bound we will take $\lambda = \varepsilon/(2\sqrt{n} M'_{\varepsilon^3})$. This allows one to bound $\mathrm{e}^{\lambda \cos(\theta)}$ by $1 + \lambda\cos(\theta) + \lambda^2 \cos(\theta)^2$ which is valid uniformly for any θ provided that n is large enough. Hence for n large enough replacing y by its value,

$$\begin{aligned}
\mathbb{P}(\Delta W_n(\theta_j) \geq \varepsilon) &\leq \inf_{\lambda > 0} \exp\left(\lambda^2 n \mathbb{E}(\cos^2(\theta) 1_{\theta \in I_x}) - \lambda\varepsilon\sqrt{n}\right) \\
&\leq \inf_{\lambda > 0} \exp\left(\lambda^2 n M'_{\varepsilon^3} - \lambda\varepsilon\sqrt{n}\right) \\
&\leq \exp(-1/(4\varepsilon))
\end{aligned}$$

this last equality being obtained for $\lambda = \varepsilon/(2M'_{\varepsilon^3}\sqrt{n})$.

The proof for the control of $\mathbb{P}(\Delta W_n(\theta_j) \leq -\varepsilon) \leq \inf_{\lambda > 0} \mathbb{E}\left(\mathrm{e}^{-\lambda \Delta W_n(\theta_j) - \lambda\delta}\right)$ for $\delta > 0$ gives rise to the same estimates, except that the bound $\mathrm{e}^{\lambda \cos(\theta)}$ by $1 - \lambda\cos(\theta) + \lambda^2 \cos(\theta)^2/4$ is taken to replace the other one, giving a bound $\exp(-1/(2\varepsilon))$ at the end.

Now we have to control the fluctuations, and not only the terminal value of the random walk. Theorem 12 page 50 in Petrov [19] allows one to control the first ones using the second ones.

Control of the fluctuations of W_n on all intervals.

The control of all intervals all together can be achieved using the union bound: since they are at most $3/\varepsilon^3$ such intervals by the union bound

$$\mathbb{P}_{P_n}(\sup_j \Delta W_n(\theta_j) \geq \varepsilon) \leq 3\varepsilon^{-3} \mathrm{e}^{-1/(4\varepsilon)}.$$

This indeed goes to 0 when $\varepsilon \to 0$.

Case where μ has no atom.

We now show the tightness of W under $\mathbb{P}_{P_n}$ when μ has no atom and use the same method as before: we work under $\mathbb{P}_{P_n}$, cut $[0,2\pi]$ under sub-intervals $[t_{j-1},t_j]'$s, control the differences between starting and ending values on these intervals, since we saw that it was sufficient.

First we cut $[0,2\pi]$ into n (tiny) equal parts ($[2\pi(j-1)/n, 2\pi j/n], j=1,\ldots,n$). From (A.4)

$$W(2\pi j/n) - W(2\pi j'/n) = \sum_{l=j'+1}^{j} \Gamma_l + \Theta_l \tag{A.13}$$

where, under $\mathbb{P}_{P_n}$, denoting further $\theta_j = 2\pi j/n$,

$$\Gamma_l = \sum_{m=1}^{\mathcal{P}(n\Delta(F_\mu(\theta_l)))} \frac{\cos(X_{\theta_{j-1},\theta_j}(m)) - \mathbb{E}(\cos(X_{\theta_{j-1},\theta_j}))}{\sqrt{n}}$$
$$\Theta_l = \frac{\mathcal{P}(n\Delta(F_\mu(\theta_l))) - n\Delta F_\mu(\theta_l)}{\sqrt{n}}\mathbb{E}(\cos(X_{\theta_{l-1},\theta_l}))$$

and $\mathcal{P}(\lambda) \sim \mathsf{Poisson}(\lambda)$ and the different Poisson r.v. appearing in the Γ_l and Θ_l are independent. Let $\varepsilon > 0$ be given and $N_{\varepsilon^3} = \lceil 1/\varepsilon^3 \rceil$. Since μ has no atom there exists some times $t_0 = 0 < t_1, \ldots < t_{N_\varepsilon} = 2\pi$ such that $\mu([t_{i-1},t_i]) \le \varepsilon^3$. We now control the fluctuations of W on these intervals.

Write $D_j := W(\frac{\lfloor 2\pi t_j n \rfloor}{n}) - W(\frac{\lfloor 2\pi t_{j-1} n \rfloor}{n})$ as a sum of r.v. Γ_l and Θ_l as in (A.13):

$$D_j = S_j + S'_j$$

where

$$S_j = \sum_{l=\lfloor 2\pi t_{j-1} n \rfloor + 1}^{\lfloor 2\pi t_j n \rfloor} \Gamma_l, \qquad S'_j = \sum_{l=\lfloor 2\pi t_{j-1} n \rfloor + 1}^{\lfloor 2\pi t_j n \rfloor} \Theta_l.$$

Each Γ_l is itself a sum which involves a Poisson number of terms: the total number of terms in S_j is $N_{t_j} - N_{t_{j-1}}$, a Poisson r.v. with parameter smaller than $\varepsilon^3 n$ under $\mathbb{P}_{P_n}$. From (A.7), $\mathbb{P}_{P_n}(N(t_j) - N(t_{j-1}) \ge 3\varepsilon^3 n) \le \mathrm{e}^{-c\varepsilon^3 n}$ for some positive c, this meaning that with high probability, S_j is a sum of less than $3\varepsilon^3 n$ centred and bounded r.v. of the form $\frac{\cos(X_{\theta_{j-1},\theta_j}(m)) - \mathbb{E}(\cos(X_{\theta_{j-1},\theta_j}))}{\sqrt{n}}$. By Hoeffding's inequality

$$\mathbb{P}(|S_j| \ge \varepsilon | N(t_j) - N(t_{j-1}) \le 3\varepsilon^3 n) \le c' \exp(-c/\varepsilon)$$

for some $c, c' > 0$.

The sum S'_j is controlled as above, in the atomic case (see (A.10) and below).

We now show 2); since $f \mapsto \max_\theta |f(\theta)|$ is continuous on $D[0,2\pi]$, we only need to prove $d_H(B_n, \mathcal{B}_\mu) = \max_\theta |Z_n(N_n(\theta)/n) - Z_\mu(F_\mu(\theta))|$.

Since B_n and $\mathcal{B}_\mu$ are compact, there exists (x_n, x) in $B_n \times \mathcal{B}_\mu$ realising this distance: $|x_n - x| = d(x_n, \mathcal{B}_\mu) = d(B_n, x) = d_H(B_n, \mathcal{B}_\mu)$. Consider now the set of directions Θ_n and Θ of the tangents at x_n on B_n and that at x on $\mathcal{B}_\mu$ (we call here a tangent at a on A a line l that passes by a and such that A is contained in one of the close half plane defined by l. The set of directions of these tangents is an interval). We claim that there exists in $\Theta_n \cap \Theta$ the direction $\theta^\star$ orthogonal to (x_n, x). If not, this means that at x_n (or at x) the line passing at x_n (or x) and orthogonal to (x_n, x) crosses B_n (or $\mathcal{B}_\mu$). This would imply that in a neighbourhood of x (or x_n) there exists a point x' (or x'_n) closer to x_n (resp. x) than x (resp. x_n), a contradiction.

To end the proof, we need to show that (x, x') corresponds to some $(S_n(N_n(\theta)/n), Z_\mu(F_\mu(\theta)))$. In other words, they are extremal points on their respective curves, and owns some parallel tangents. The second statement is clear. For the first one, we have to deal with the fact that B_n (and so do $\mathcal{B}_\mu$ for certain measures μ) have linear portions. But the distance between B_n and $\mathcal{B}_\mu$ is not reached inside the linear intervals since the Hausdorff distance between a segment $[a, b]$ and a CCS C is given by $\max\{d(a, C), d(b, C)\}$. □

Acknowledgements. We thank both referees for their numerous remarks that really helped to improve the paper.

References

[1] I. Bárány, Sylvester's question: The probability that n points are in convex position. *Ann. Probab.* **27** (1999) 2020–2034.
[2] I. Bárány, Random polytopes, convex bodies and approximation, in *Stochastic Geometry*, Vol. 1892 of *Lect. Notes Math.* Springer Berlin/Heidelberg (2007) 77–118.
[3] I. Bárány and A.M Vershik, On the number of convex lattice polytopes. *Geom. Func. Anal.* **2** (1992) 381–393.
[4] P. Billingsley, Convergence of probability measures. *Wiley Series in Probability and Statistics: Probability and Statistics*, 2nd edition. A Wiley-Interscience Publication. John Wiley & Sons Inc., New York (1999).
[5] O. Bodini, Ph. Duchon, A. Jacquot and L. Mutafchiev, Asymptotic analysis and random sampling of digitally convex polyominoes. In *Proc. of the 17th IAPR international conference on Discrete Geometry for Computer Imagery*, DGCI'13. Springer-Verlag, Berlin, Heidelberg (2013) 95–106.
[6] L.V. Bogachev and S.M. Zarbaliev, Universality of the limit shape of convex lattice polygonal lines. *Ann. Probab.* **39** (1992) 2271–2317.
[7] C. Buchta, On the boundray structure of the convex hull of random points. *Adv. Geom.* (2012). Available at: http://www.uni-salzburg.at/pls/portal/docs/1/1739190.PDF.
[8] H. Busemann, Convex Surfaces. *Interscience.* New York (1958).
[9] P. Calka, Precise formulae for the distributions of the principal geometric characteristics of the typical cells of a two-dimensional poisson-voronoi tessellation and a poisson line process. *Adv. Appl. Probab.* **35** (2003) 551–562. Available at http://www.univ-rouen.fr/LMRS/Persopage/Calka/publications.html.
[10] R.M. Dudley, Real Analysis and Probability. *Cambridge Studies in Advanced Mathematics.* Cambridge University Press (2002).
[11] W. Feller, An introduction to probability theory and its applications. Vol. II. 2nd edition. John Wiley & Sons Inc., New York (1971).
[12] M.A. Hurwitz, Sur le problème des isopérimètres. *C. R. Acad. Sci. Paris* **132** (1901) 401–403.
[13] M.A. Hurwitz, Sur quelques applications géométriques des séries de Fourier. *Annales Scientifiques de l'École Normale supérieure*, **19** (1902) 357–408. Available at http://archive.numdam.org/article/ASENS_1902_3_19__357_0.pdf.
[14] B. Klartag, On John-type ellipsoids, in Geometric aspects of functional analysis, vol. 1850 of *Lect. Notes Math.* Springer, Berlin (2004) 149–158.
[15] D.E. Knuth, Axioms and hulls. Vol. 606 of *Lect. Notes Comput. Sci.* Springer-Verlag, Berlin (1992). Available at: http://www-cs-faculty.stanford.edu/~uno/aah.html.
[16] P. Lévy, L'addition des variables aléatoires définies sur un circonférence. *Bull. Soc. Math. France* **67** (1939) 1–41. Available at http://archive.numdam.org/article/BSMF_1939__67__1_0.pdf.
[17] J.-F. Marckert, Probability that n random points in a disk are in convex position. Available at http://arxiv.org/abs/1402.3512 (2014).
[18] M. Moszyńska, Selected Topics in Convex Geometry. Birkhäuser (2006).
[19] V.V. Petrov, Sums of independent random variables. Translated from the Russian by A.A. Brown. Band 82, *Ergebnisse der Mathematik und ihrer Grenzgebiete.* Springer-Verlag, New York (1975).
[20] A.V. Pogorelov, Extrinsic geometry of convex surfaces. American Mathematical Society, Providence, R.I. (1973). Translated from the Russian by Israel Program for Scientific Translations, in vol. 35 Translations of *Mathematical Monographs.*
[21] G. Pólya, Isoperimetric Inequalities in Mathematical Physics. *Ann. Math. Stud.* Kraus (1965).
[22] W. Rudin, Real and Complex Analysis, 3rd edn. McGraw-Hill International Editions (1987).
[23] R. Schneider, Convex Bodies: The Brunn−Minkowski Theory. Cambridge University Press (1993).
[24] Ya.G. Sinai, Probabilistic approach to the analysis of statistics for convex polygonal lines. *Functional Anal. Appl.* **28** (1994) 1.
[25] J.J. Sylvester, On a special class of questions on the theory of probabilities. *Birmingham British Assoc. Rept.* (1865) 8–9.
[26] G. Szegö, Orthogonal polynomials. *Colloquium Publications*, 4th edition. American Mathematical Society (1939).
[27] P. Valtr, Probability that n random points are in convex position. *Discr. Comput. Geom.* **13** (1995) 637–643.
[28] P. Valtr, The probability that n random points in a triangle are in convex position. *Combinatorica* **16** (1996) 567–573.
[29] A. Vershik and O. Zeitouni, large deviations in the geometry of convex lattice polygons. *Israel J. Math.* **109** (1999) 13–27.
[30] R.J.G. Wilms, Fractional parts of random variables. Limit theorems and infinite divisibility, Dissertation. Technische Universiteit Eindhoven, Eindhoven (1994).