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A NEW PROOF OF KELLERER'S THEOREM

Francis Hirsch[1] and Bernard Roynette[2]

Abstract. In this paper, we present a new proof of the celebrated theorem of Kellerer, stating that every integrable process, which increases in the convex order, has the same one-dimensional marginals as a martingale. Our proof proceeds by approximations, and calls upon martingales constructed as solutions of stochastic differential equations. It relies on a uniqueness result, due to Pierre, for a Fokker-Planck equation.

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1. Introduction

1.1. First we fix the terminology.

We say that two $\mathbb{R}$-valued processes are *associated*, if they have the same one-dimensional marginals. A process which is associated with a martingale is called a 1-*martingale*.

An $\mathbb{R}$-valued process $(X_t,\ t \geq 0)$ is called a *peacock* (see [2] for the origin of this term and many examples) if:

(i) it is *integrable*, that is:

$$\forall t \geq 0, \quad \mathbb{E}[|X_t|] < \infty;$$

(ii) it *increases in the convex order*, meaning that, for every convex function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$, the map:

$$t \geq 0 \longrightarrow \mathbb{E}[\psi(X_t)] \in (-\infty, +\infty]$$

is increasing.

Actually, it may be noted that, in the definition of a peacock, only the family $(\mu_t,\ t \geq 0)$ of its one-dimensional marginals is involved. In the following, we shall also call a *peacock*, a family $(\mu_t,\ t \geq 0)$ of probability measures

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[1] Laboratoire d'Analyse et Probabilités, Université d'Évry, Val d'Essonne, Boulevard F. Mitterrand, 91025 Évry Cedex, France.
francis.hirsch@univ-evry.fr

[2] Institut Elie Cartan, Université Henri Poincaré, B.P. 239, 54506 Vandœuvre-lès-Nancy Cedex, France.
bernard.roynette@iecn.u-nancy.fr

on $\mathbb{R}$ such that:

(i) $\forall t \geq 0, \quad \int |x| \, \mu_t(\mathrm{d}x) < \infty;$

(ii) for every convex function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$, the map:

$$t \geq 0 \longrightarrow \int \psi(x) \, \mu_t(\mathrm{d}x) \in (-\infty, +\infty]$$

is increasing.

Likewise, a family $(\mu_t, \ t \geq 0)$ of probability measures on $\mathbb{R}$ and an $\mathbb{R}$-valued process $(Y_t, \ t \geq 0)$ will be said to be *associated* if, for every $t \geq 0$, the law of Y_t is μ_t, *i.e.* if $(\mu_t, \ t \geq 0)$ is the family of the one-dimensional marginals of $(Y_t, \ t \geq 0)$.

1.2. It is an easy consequence of Jensen's inequality that an $\mathbb{R}$-valued process $(X_t, \ t \geq 0)$ which is a 1-martingale, is a peacock. A remarkable result due to Kellerer [3] states that, conversely, any $\mathbb{R}$-valued process $(X_t, \ t \geq 0)$ which is a peacock, is a 1-martingale. More precisely, Kellerer's result states that any peacock admits an associated martingale which is *Markovian*.

Recently, Lowther [4] stated that if $(\mu_t, \ t \geq 0)$ is a peacock such that the map: $t \longrightarrow \mu_t$ is weakly continuous (*i.e.* for any $\mathbb{R}$-valued, bounded and continuous function f on $\mathbb{R}$, the map: $t \longrightarrow \int f(x) \, \mu_t(\mathrm{d}x)$ is continuous), then $(\mu_t, \ t \geq 0)$ is associated with a strongly Markovian martingale which moreover is "almost-continuous" (see [4] for the definition).

1.3. In this paper, our aim is to present a new proof of the above mentioned theorem of Kellerer, which eventually identifies peacocks and 1-martingales. Our method is inspired from the "Fokker-Planck equation method" ([2], Sect. 6.2) and appears then as a new application of Pierre's uniqueness theorem for a Fokker-Planck equation ([2], Thm. 6.1]).

1.4. The remainder of this paper is organised as follows:

- in Section 2, we define as usual the *call function* C_μ of the law μ of an integrable random variable X, by:

$$\forall x \in \mathbb{R}, \quad C_\mu(x) = \int (y - x)^+ \, \mu(\mathrm{d}y) = \mathbb{E}[(X - x)^+]$$

  and we present some properties of the correspondence: $\mu \longrightarrow C_\mu$, which are useful in the study of peacocks;
- in Section 3, we prove that a family $(\mu_t, \ t \geq 0)$ of probability measures on $\mathbb{R}$, is associated to a *right-continuous* martingale, if and only if, $(\mu_t, \ t \geq 0)$ is a peacock such that the map: $t \longrightarrow \mu_t$ is *weakly right-continuous* on $\mathbb{R}_+$;
- in Section 4, by approximation from the previous result, we deduce Kellerer's theorem in the general case.

2. Call functions and peacocks

In this section, we fix the notation and the terminology, and we gather some preliminary results.

2.1. Call functions

In the sequel, we denote by $\mathcal{M}$ the set of probability measures on $\mathbb{R}$, equipped with the topology of weak convergence (with respect to the space of $\mathbb{R}$-valued, bounded, continuous functions on $\mathbb{R}$).

We denote by $\mathcal{M}_f$ the subset of $\mathcal{M}$ consisting of measures $\mu \in \mathcal{M}$ such that $\int |x| \ \mu(\mathrm{d}x) < \infty$. $\mathcal{M}_f$ is also equipped with the topology of weak convergence.

We define, for $\mu \in \mathcal{M}_f$, the *call function* C_μ by:

$$\forall x \in \mathbb{R}, \quad C_\mu(x) = \int (y - x)^+ \ \mu(\mathrm{d}y).$$

Proposition 2.1. *If $\mu \in \mathcal{M}_f$, then C_μ satisfies the following properties:*

(a) *C_μ is a convex, nonnegative function on $\mathbb{R}$;*
(b) *$\lim_{x\to+\infty} C_\mu(x) = 0$;*
(c) *there exists $a \in \mathbb{R}$ such that $\lim_{x\to-\infty}(C_\mu(x) + x) = a$.*

Conversely, if a function C satisfies the above three properties, then there exists a unique $\mu \in \mathcal{M}_f$ such that $C = C_\mu$. This measure μ is the second derivative, in the sense of distributions, of the function C.

Proof. Clearly, if $\mu \in \mathcal{M}_f$, then C_μ satisfies properties (a), (b) and (c). For example, (c) follows directly from:

$$\forall x \in \mathbb{R}, \quad C_\mu(x) + x = \int \sup(y, x) \ \mu(\mathrm{d}y)$$

which tends to $a = \int y \ \mu(\mathrm{d}y)$ as $x \to -\infty$. Moreover, it is easy to see that the measure μ is the second derivative, in the sense of distributions, of the function C_μ.

Conversely, let C be a function satisfying properties (a), (b) and (c). We define μ as the second derivative, in the sense of distributions, of the function C. Then μ is a positive measure. Denote by $C'(x)$ the right derivative, at x, of the convex function C. By properties (a) and (b),

$$\forall x \in \mathbb{R}, \ \ C'(x) \le 0 \quad \text{and} \quad \lim_{x\to+\infty} C'(x) = 0.$$

Therefore, for $x \in \mathbb{R}$,

$$C'(x) = -\int 1_{(x,+\infty)}(y) \ \mu(\mathrm{d}y).$$

By property (c), $\lim_{x\to-\infty} C'(x) = -1$ and then $\mu \in \mathcal{M}$.

Besides,

$$C(x) = -\int_x^{+\infty} C'(y) \ \mathrm{d}y = \int (y - x)^+ \ \mu(\mathrm{d}y)$$

and

$$C(x) + x = \int \sup(y, x) \ \mu(\mathrm{d}y).$$

Using again property (c), we see that $\mu \in \mathcal{M}_f$ and $C = C_\mu$. □

Proposition 2.2. *Let $\mu \in \mathcal{M}_f$ and set $\mathbb{E}[\mu] = \int x \ \mu(\mathrm{d}x)$. Then C_μ satisfies the following additional properties:*

(i) $\forall x \le y, \quad 0 \le C_\mu(x) - C_\mu(y) \le y - x;$
(ii) $\forall x, \quad C_\mu(x) + x - \mathbb{E}[\mu] = \int (x - y)^+ \ \mu(\mathrm{d}y);$
(iii) $\lim_{x\to-\infty}(C_\mu(x) + x) = \mathbb{E}[\mu].$

Proof. The proposition follows from the following equalities, already seen in the previous proof:

$$C'_\mu(x) = -\int 1_{(x,+\infty)}(y)\ \mu(\mathrm{d}y),$$

$$C_\mu(x) + x = \int \sup(y, x)\ \mu(\mathrm{d}y).$$ □

To state the next proposition, we now recall that a subset $\mathcal{H}$ of $\mathcal{M}$ is said to be *uniformly integrable* if

$$\lim_{c\to+\infty} \sup_{\mu\in\mathcal{H}} \int_{\{|x|\geq c\}} |x|\ \mu(\mathrm{d}x) = 0.$$

We remark that, if $\mathcal{H}$ is uniformly integrable, then

$$\mathcal{H} \subset \mathcal{M}_f \quad \text{and} \quad \sup\left\{\int |x|\ \mu(\mathrm{d}x);\ \mu\in\mathcal{H}\right\} < \infty.$$

Proposition 2.3. *Let I be a set and let $\mathcal{E}$ be a filter on I. Consider a uniformly integrable family $(\mu_i,\ i\in I)$ in $\mathcal{M}$, and $\mu\in\mathcal{M}$. The following properties are equivalent:*

(1) $\lim_{\mathcal{E}} \mu_i = \mu$ *with respect to the topology on $\mathcal{M}$;*

(2) $\mu\in\mathcal{M}_f$ *and*

$$\forall x\in\mathbb{R},\quad \lim_{\mathcal{E}} C_{\mu_i}(x) = C_\mu(x);$$

(3) $\mu\in\mathcal{M}_f$ *and, for every $\mathbb{R}$-valued continuous function f on $\mathbb{R}$ such that*

$$\exists a>0, b>0,\ \ \forall x\in\mathbb{R},\qquad |f(x)|\leq a + b\,|x|,$$

one has:

$$\lim_{\mathcal{E}} \int f(x)\ \mu_i(\mathrm{d}x) = \int f(x)\ \mu(\mathrm{d}x).$$

Proof. We first assume that property (1) holds. Then

$$\int |x|\ \mu(\mathrm{d}x) \leq \sup\left\{\int |x|\ \mu_i(\mathrm{d}x);\ i\in I\right\} < \infty,$$

and $\mu\in\mathcal{M}_f$. Let f be an $\mathbb{R}$-valued continuous function on $\mathbb{R}$ such that

$$\exists a>0, b>0,\ \ \forall x\in\mathbb{R},\qquad |f(x)|\leq a + b\,|x|.$$

We set, for $n\in\mathbb{N}$ and $x\in\mathbb{R}$, $f_n(x) = \max[\min(f(x), n), -n]$. Since f_n is continuous and bounded,

$$\lim_{\mathcal{E}} \int f_n(x)\ \mu_i(\mathrm{d}x) = \int f_n(x)\ \mu(\mathrm{d}x).$$

On the other hand, for $n\geq a$,

$$|f(x) - f_n(x)| = (|f(x)| - n)^+ \leq (b\,|x| + a - n)^+ \leq b\,|x|\,1_{\{|x|\geq \frac{n-a}{b}\}},$$

and hence

$$\sup_{i\in I}\left|\int f(x)\ \mu_i(\mathrm{d}x) - \int f_n(x)\ \mu_i(\mathrm{d}x)\right| \leq b\sup_{i\in I}\int_{\{|x|\geq\frac{n-a}{b}\}} |x|\ \mu_i(\mathrm{d}x).$$

By uniform integrability, we then obtain:

$$\lim_{n\to\infty}\sup_{i\in I}\left|\int f(x)\,\mu_i(\mathrm{d}x)-\int f_n(x)\,\mu_i(\mathrm{d}x)\right|=0.$$

Finally,

$$\begin{aligned}\int f(x)\,\mu(\mathrm{d}x)&=\lim_{n\to\infty}\lim_{\mathcal{E}}\int f_n(x)\,\mu_i(\mathrm{d}x)\\&=\lim_{\mathcal{E}}\lim_{n\to\infty}\int f_n(x)\,\mu_i(\mathrm{d}x)=\lim_{\mathcal{E}}\int f(x)\,\mu_i(\mathrm{d}x),\end{aligned}$$

and property (3) is satisfied.

Obviously, property (3) entails property (2).

Suppose then that property (2) holds. By equicontinuity (property (i) in Prop. 2.2),

$$\lim_{\mathcal{E}} C_{\mu_i}(x)=C_\mu(x)$$

uniformly on compact sets of $\mathbb{R}$, and hence in the sense of distributions. Consequently, since μ_i (resp. μ) is the second derivative, in the sense of distributions, of the function C_{μ_i} (resp. C_μ),

$$\lim_{\mathcal{E}}\mu_i=\mu$$

in the sense of distributions. As μ_i and μ are probability measures, this entails property (1). □

2.2. Peacocks

In this subsection, we fix a family $(\mu_t,\ t\geq 0)$ in $\mathcal{M}_f$ and we define a function $C(t,x)$ on $\mathbb{R}_+\times\mathbb{R}$ by:

$$C(t,x)=C_{\mu_t}(x).$$

We recall (see Sect. 1.1) that the family $(\mu_t,\ t\geq 0)$ is called a *peacock*, if

(i) $\forall t\geq 0,\quad \int |x|\,\mu_t(\mathrm{d}x)<\infty$;

(ii) for every convex function $\psi:\mathbb{R}\longrightarrow\mathbb{R}$, the map:

$$t\geq 0\longrightarrow\int\psi(x)\,\mu_t(\mathrm{d}x)\in(-\infty,+\infty]$$

is increasing.

The following characterization is easy to prove and is stated in [2], Exercise 1.7.

Proposition 2.4. *The family $(\mu_t,\ t\geq 0)$ is a peacock if and only if:*

(1) *the expectation $\mathbb{E}[\mu_t]$ does not depend on t;*
(2) *for every $x\in\mathbb{R}$, the function $t\geq 0\longrightarrow C(t,x)$ is increasing.*

The following proposition plays an important role in the sequel.

Proposition 2.5. *Assume that $(\mu_t,\ t\geq 0)$ is a peacock, and let $T>0$. Then,*

(1) *the set $\{\mu_t;\ 0\leq t\leq T\}$ is uniformly integrable;*
(2) $\lim_{|x|\to\infty}\sup\{C(t,x)-C(s,x);\ 0\leq s\leq t\leq T\}=0.$

Proof. Property (1) is stated in [2], Exercise 1.1. Actually, it suffices to remark that, if $c \geq 0$,

$$|x| \, 1_{\{|x| \geq c\}} \leq (2\,|x| - c)^+.$$

As the function $x \longrightarrow (2\,|x| - c)^+$ is convex,

$$\sup_{t \in [0,T]} \int_{\{|x| \geq c\}} |x| \; \mu_t(\mathrm{d}x) \leq \int (2\,|x| - c)^+ \; \mu_T(\mathrm{d}x).$$

Now, by dominated convergence,

$$\lim_{c \to +\infty} \int (2\,|x| - c)^+ \; \mu_T(\mathrm{d}x) = 0.$$

We have:

$$\sup\{C(t,x) - C(s,x);\ 0 \leq s \leq t \leq T\} \leq C(T,x).$$

Hence, by property (b) in Proposition 2.1,

$$\lim_{x \to +\infty} \sup\{C(t,x) - C(s,x);\ 0 \leq s \leq t \leq T\} = 0.$$

On the other hand, since $\mathbb{E}[\mu_t]$ does not depend on t,

$$C(t,x) - C(s,x) = [C(t,x) + x - \mathbb{E}[\mu_t]] - [C(s,x) + x - \mathbb{E}[\mu_s]].$$

Now, by property (ii) in Proposition 2.2,

$$C(t,x) + x - \mathbb{E}[\mu_t] = \int (x-y)^+ \; \mu_t(\mathrm{d}y),$$

is therefore nonnegative and increases with respect to t. Hence

$$\sup\{C(t,x) - C(s,x);\ 0 \leq s \leq t \leq T\} \leq C(T,x) + x - \mathbb{E}[\mu_T]$$

and, by property (iii) in Proposition 2.2,

$$\lim_{x \to -\infty} \sup\{C(t,x) - C(s,x);\ 0 \leq s \leq t \leq T\} = 0.$$

□

3. Right-continuous peacoks

In this section, we shall prove Kellerer's theorem for right-continuous peacoks. We proceed by regularization, using, for regularized peacocks, the Fokker-Planck equation method as in [2], Chapter 6. This method relies heavily on Pierre's uniqueness theorem for a Fokker-Planck equation ([2], Thm. 6.1).

We first recall the main result in the Fokker-Planck equation method, namely Theorem 6.2 in [2]. The next statement is a slightly extended version of this theorem.

Theorem 3.1 (see Thm. 6.2 in [2])**.** *Let $U = (0, +\infty) \times \mathbb{R}$ and $\overline{U}$ the closure of U ($\overline{U} = \mathbb{R}_+ \times \mathbb{R}$). Let σ be a continuous function on $\overline{U}$ such that $\sigma(t,x) > 0$ for every $(t,x) \in U$. Let $\mu \in \mathcal{M}_f$.*

(1) *The stochastic differential equation*

$$Z_t = Z_0 + \int_0^t \sigma(s, Z_s) \, \mathrm{d}B_s$$

(where Z_0 is a random variable with law μ, independent of the Brownian motion $(B_s, s \geq 0)$) admits a weak non-exploding solution $(Y_t,\ t \geq 0)$, which is unique in law;

(2) *let $p(t, \mathrm{d}x)$ be the law of Y_t. Then, $(p(t,\mathrm{d}x),\ t \geq 0)$ is the* unique *family in $\mathcal{M}$ such that:*

$$t \longrightarrow p(t, \mathrm{d}x) \in \mathcal{M} \quad \textit{is continuous and} \quad p(0, \mathrm{d}x) = \mu(\mathrm{d}x),$$

$$\frac{\partial p}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2 p) = 0 \quad \textit{in the sense of distributions on } U.$$

We now present our proof of Kellerer's theorem for right-continuous peacoks.

Theorem 3.2. *Let $(\mu_t,\ t \geq 0)$ be a family in $\mathcal{M}$. Then the following properties are equivalent:*

(1) *there exists a right-continuous martingale associated to $(\mu_t,\ t \geq 0)$;*
(2) *$(\mu_t,\ t \geq 0)$ is a peacock and the map:*
$$t \geq 0 \longrightarrow \mu_t \in \mathcal{M}$$
is right-continuous.

Proof. We first assume that property (1) is satisfied. Then, the fact that $(\mu_t,\ t \geq 0)$ is a peacock follows classically from Jensen's inequality. Let $(M_t,\ t \geq 0)$ be a right-continuous martingale associated to $(\mu_t,\ t \geq 0)$. Then, if f is a bounded continuous function, we obtain by dominated convergence that, for any $t \geq 0$,
$$\lim_{s \to t, s>t} \int f(x)\ \mu_s(\mathrm{d}x) = \lim_{s \to t, s>t} \mathbb{E}[f(M_s)] = \mathbb{E}[f(M_t)] = \int f(x)\ \mu_t(\mathrm{d}x).$$
Therefore, the map:
$$t \geq 0 \longrightarrow \mu_t \in \mathcal{M}$$
is right-continuous, and property (2) is satisfied.

Conversely, we now assume that property (2) is satisfied. We set, as in Section 2.2, $C(t,x) = C_{\mu_t}(x)$. We shall regularize, in space and time, $p(t, \mathrm{d}x) := \mu_t(\mathrm{d}x)$ considered as a distribution on U. Thus, let α be a density of probability on $\mathbb{R}$, of C^∞ class, with compact support contained in $[0,1]$. We set, for $\varepsilon \in (0,1)$ and $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,
$$p_\varepsilon(t,x) = \frac{1-\varepsilon}{\varepsilon} \int \alpha(u) \left[\int \alpha\left(\frac{y-x}{\varepsilon}\right)\ \mu_{t+\varepsilon u}(\mathrm{d}y)\right]\ \mathrm{d}u + \varepsilon\, g(t,x)$$
with
$$g(t,x) = \frac{1}{\sqrt{2\,\pi\,(1+t)}}\ \exp\left(-\frac{x^2}{2\,(1+t)}\right)\cdot$$

Lemma 3.3. *The function p_ε is of C^∞ class on $\mathbb{R}_+ \times \mathbb{R}$ and $p_\varepsilon(t,x) > 0$ for any (t,x). Moreover,*
$$\int p_\varepsilon(t,x)\ \mathrm{d}x = 1 \quad \textit{and} \quad \int |x|\, p_\varepsilon(t,x)\ \mathrm{d}x < \infty.$$

The proof is straightforward.

We now set:
$$\mu_t^\varepsilon(\mathrm{d}x) = p_\varepsilon(t,x)\,\mathrm{d}x.$$
By Lemma 3.3, $\mu_t^\varepsilon \in \mathcal{M}_f$ and we set:
$$C_\varepsilon(t,x) = C_{\mu_t^\varepsilon}(x).$$

Lemma 3.4. *For any $t \geq 0$, the set $\{\mu_t^\varepsilon;\ 0 < \varepsilon < 1\}$ is uniformly integrable.*

Proof. Let $a = \int y\, \alpha(y)\ \mathrm{d}y$. A simple computation yields:
$$\int_{\{|x| \geq c\}} |x|\ \mu_t^\varepsilon(\mathrm{d}x) \leq \int \alpha(u)\ \left[\int_{\{|y| \geq c-1\}} (|y| + a)\ \mu_{t+\varepsilon u}(\mathrm{d}y)\right]\ \mathrm{d}u + \int_{\{|x| \geq c\}} |x|\, g(t,x)\ \mathrm{d}x$$
and the result follows from the uniform integrability of $\{\mu_v;\ 0 \leq v \leq t+1\}$ (property (1) in Prop. 2.5). □

Lemma 3.5. *One has:*

$$C_\varepsilon(t,x) = (1-\varepsilon)\int\int \alpha(u)\,\alpha(y)\,C(t+\varepsilon u, x+\varepsilon y)\,\mathrm{d}y\,\mathrm{d}u + \varepsilon\int_x^{+\infty}(y-x)\,g(t,y)\,\mathrm{d}y.$$

The function C_ε is of C^∞ class on $\mathbb{R}_+\times\mathbb{R}$. Moreover, for any $(t,x)\in\mathbb{R}_+\times\mathbb{R}$,

$$\frac{\partial C_\varepsilon}{\partial t}(t,x) > 0 \qquad \textit{and} \qquad \frac{\partial^2 C_\varepsilon}{\partial x^2}(t,x) = p_\varepsilon(t,x).$$

Proof. The above expression of C_ε follows directly from the definitions. We deduce therefrom that C_ε is of C^∞ class on $\mathbb{R}_+\times\mathbb{R}$. Now, by property (2) in Proposition 2.4,

$$\frac{\partial C_\varepsilon}{\partial t}(t,x) \geq \varepsilon\,\frac{\partial}{\partial t}\left[\int_x^{+\infty}(y-x)\,g(t,y)\,\mathrm{d}y\right] = \frac{\varepsilon}{2}\,g(t,x) > 0.$$

Finally, the equality:

$$\frac{\partial^2 C_\varepsilon}{\partial x^2}(t,x) = p_\varepsilon(t,x)$$

holds, since, by Proposition 2.1, it holds in the sense of distributions, and both sides are continuous. □

Lemma 3.6. *For $0\leq s\leq t$,*

$$\lim_{|x|\to\infty}\sup\{C_\varepsilon(t,x)-C_\varepsilon(s,x);\ 0<\varepsilon<1\} = 0.$$

Proof. By Lemma 3.5,

$$\sup\{C_\varepsilon(t,x)-C_\varepsilon(s,x);\ 0<\varepsilon<1\} \leq A(x)+B(x)$$

with

$$A(x) = \sup\{C(w,y)-C(v,y);\ 0\leq v\leq w\leq t+1, x\leq y\leq x+1\}$$

and

$$B(x) = \frac{1}{2}\int_s^t g(u,x)\,\mathrm{d}u.$$

By property (2) in Proposition 2.5, $\lim_{|x|\to\infty}A(x)=0$, and, obviously, $\lim_{|x|\to\infty}B(x)=0$. □

Lemma 3.7. *For $t\geq 0$,*

$$\lim_{\varepsilon\to 0}\mu_t^\varepsilon = \mu_t \qquad \textit{in } \mathcal{M}.$$

Proof. By property (i) in Proposition 2.2, property (1) in Proposition 2.5 and by Proposition 2.3,

$$\lim_{s\to t, s>t} C(s,x) = C(t,x) \qquad \text{uniformly on compact sets.}$$

Then, the expression of C_ε in Lemma 3.5 yields:

$$\lim_{s\to t, s>t} C_\varepsilon(s,x) = C_\varepsilon(t,x).$$

It then suffices to apply again Proposition 2.3, taking into account Lemma 3.4.

Note that we might also have proven this lemma directly from the definition of μ_t^ε. □

Lemma 3.8. *We set, for* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\sigma_\varepsilon(t,x) = \left(2\, \frac{\frac{\partial C_\varepsilon}{\partial t}(t,x)}{p_\varepsilon(t,x)} \right)^{1/2}.$$

Then, σ_ε *is continuous and strictly positive on* $\mathbb{R}_+ \times \mathbb{R}$. *Moreover, for* $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\frac{\partial p_\varepsilon}{\partial t}(t,x) = \frac{1}{2}\, \frac{\partial^2}{\partial x^2} \left(\sigma_\varepsilon^2(t,x)\, p_\varepsilon(t,x) \right),$$

which is the Fokker-Planck equation for p_ε.

Proof. This is a direct consequence of Lemmas 3.3 and 3.5. In particular, the Fokker-Planck equation can be written:

$$\frac{\partial}{\partial t}\, \frac{\partial^2 C_\varepsilon}{\partial x^2} = \frac{\partial^2}{\partial x^2}\, \frac{\partial C_\varepsilon}{\partial t}.$$

□

By Theorem 3.1, there exists a process $(M_t^\varepsilon,\ t \geq 0)$ which is a weak solution of the stochastic differential equation

$$Z_t = Z_0 + \int_0^t \sigma_\varepsilon(s, Z_s)\, \mathrm{d}B_s$$

with Z_0 a random variable with law μ_0^ε, independent of the Brownian motion $(B_s, s \geq 0)$, and this process $(M_t^\varepsilon,\ t \geq 0)$ is associated to $(\mu_t^\varepsilon,\ t \geq 0)$. For every $n \in \mathbb{N}$ and $\tau_n = (t_1, \ldots, t_n) \in \mathbb{R}_+^n$, we denote by $\mu_{\tau_n}^{(\varepsilon,n)}$ the law of $(M_{t_1}^\varepsilon, \ldots, M_{t_n}^\varepsilon)$, a probability on $\mathbb{R}^n$.

Lemma 3.9. *For every* $n \in \mathbb{N}$ *and* $\tau_n \in \mathbb{R}_+^n$, *the set of probability measures:* $\{\mu_{\tau_n}^{(\varepsilon,n)};\ 0 < \varepsilon < 1\}$, *is tight.*

Proof. Let $n \in \mathbb{N}$ and $\tau_n = (t_1, \ldots, t_n) \in \mathbb{R}_+^n$. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we set $|x| := \sup_{1 \leq j \leq n} |x_j|$. Then, for $c > 0$,

$$\begin{aligned}\mu_{\tau_n}^{(\varepsilon,n)}(|x| \geq c) &= \mathbb{P}\left(\sup_{1 \leq j \leq n} |M_{t_j}^\varepsilon| \geq c \right) \leq \frac{1}{c}\, \mathbb{E}\left[\sup_{1 \leq j \leq n} |M_{t_j}^\varepsilon| \right] \\ &\leq \frac{1}{c} \sum_{j=1}^n \mathbb{E}\left[|M_{t_j}^\varepsilon| \right] = \frac{1}{c} \sum_{j=1}^n \int |x|\, \mu_{t_j}^\varepsilon(\mathrm{d}x).\end{aligned}$$

Now, by Lemma 3.4, for $1 \leq j \leq n$,

$$\sup_{0<\varepsilon<1} \int |x|\, \mu_{t_j}^\varepsilon(\mathrm{d}x) < \infty.$$

Thus,

$$\lim_{c \to +\infty} \sup_{0<\varepsilon<1} \mu_{\tau_n}^{(\varepsilon,n)}(|x| \geq c) = 0,$$

which yields the tightness of $\{\mu_{\tau_n}^{(\varepsilon,n)};\ 0 < \varepsilon < 1\}$. □

As a consequence of the previous lemma, and with the help of the diagonal procedure, there exists a sequence $(\varepsilon_p,\ p \geq 0)$ tending to 0 such that, for every $n \in \mathbb{N}$ and every $\tau_n \in \mathbb{Q}_+^n$, the sequence of probabilities on $\mathbb{R}^n$: $(\mu_{\tau_n}^{(\varepsilon_p,n)},\ p \geq 0)$, weakly converges to a probability which we denote by $\mu_{\tau_n}^{(n)}$. We remark that, by Lemma 3.7, for any $t \in \mathbb{Q}_+$, $\mu_t^{(1)} = \mu_t$. There exists a process $(M_t,\ t \in \mathbb{Q}_+)$ such that, for every $n \in \mathbb{N}$ and every $\tau_n = (t_1, \ldots, t_n) \in \mathbb{Q}_+^n$, the law of $(M_{t_1}, \ldots, M_{t_n})$ is $\mu_{\tau_n}^{(n)}$.

Lemma 3.10. *The process $(M_t,\ t \in \mathbb{Q}_+)$ is a martingale.*

Proof. Let ϕ be a C^2-function on $\mathbb{R}$ such that $\phi(x) = 1$ for $|x| \leq 1$, $\phi(x) = 0$ for $|x| \geq 2$, and $0 \leq \phi(x) \leq 1$ for all $x \in \mathbb{R}$. We set, for $k > 0$, $\phi_k(x) = x\,\phi(k^{-1}x)$. Fix now $n \in \mathbb{N}$ and n continuous bounded functions $(g_1, \ldots, g_n)$ on $\mathbb{R}$, and finally $0 \leq s_1 \leq \ldots \leq s_n \leq s \leq t$ elements of $\mathbb{Q}_+$. We set:

$$\Theta(p,k) = \mathbb{E}[g_1(M^{\varepsilon_p}_{s_1})g_2(M^{\varepsilon_p}_{s_2})\ldots g_n(M^{\varepsilon_p}_{s_n})\,\phi_k(M^{\varepsilon_p}_t)] - \mathbb{E}[g_1(M^{\varepsilon_p}_{s_1})g_2(M^{\varepsilon_p}_{s_2})\ldots g_n(M^{\varepsilon_p}_{s_n})\,\phi_k(M^{\varepsilon_p}_s)].$$

From the definitions, we obtain:

$$\lim_{p\to\infty}\Theta(p,k) = \mathbb{E}[g_1(M_{s_1})g_2(M_{s_2})\ldots g_n(M_{s_n})\,\phi_k(M_t)] - \mathbb{E}[g_1(M_{s_1})g_2(M_{s_2})\ldots g_n(M_{s_n})\,\phi_k(M_s)]$$

and, by dominated convergence,

$$\lim_{k\to\infty}\lim_{p\to\infty}\Theta(p,k) = \mathbb{E}[g_1(M_{s_1})g_2(M_{s_2})\ldots g_n(M_{s_n})\,M_t] - \mathbb{E}[g_1(M_{s_1})g_2(M_{s_2})\ldots g_n(M_{s_n})\,M_s].$$

On the other hand, set:

$$m = \prod_{j=1}^{n}\sup_{x\in\mathbb{R}}|g_j(x)|.$$

Then, since the support of ϕ_k is compact, Itô's formula yields:

$$\begin{aligned}|\Theta(p,k)| &\leq \frac{m}{2}\int_s^t \mathbb{E}\left[|\phi_k''(M^{\varepsilon_p}_u)|\,\sigma^2_{\varepsilon_p}(u, M^{\varepsilon_p}_u)\right]\,\mathrm{d}u\\ &= m\int\int_s^t |\phi_k''(x)|\,\frac{\partial C_{\varepsilon_p}}{\partial u}(u,x)\,\mathrm{d}u\,\mathrm{d}x.\end{aligned}$$

Besides,

$$\int|\phi_k''(x)|\,\mathrm{d}x = \int|x\,\phi''(x) + 2\phi'(x)|\,\mathrm{d}x$$

and $\phi_k''(x) = 0$ for $|x| \notin [k, 2k]$. Therefore, there exists a constant $\tilde{m}$ such that:

$$|\Theta(p,k)| \leq \tilde{m}\,\sup\{C_{\varepsilon_p}(t,y) - C_{\varepsilon_p}(s,y);\ k \leq |y| \leq 2k\}. \tag{3.1}$$

Thus, by Lemma 3.6,

$$\lim_{k\to\infty}\Theta(p,k) = 0 \qquad \text{uniformly with respect to } p.$$

Consequently,

$$\begin{aligned}0 &= \lim_{p\to\infty}\lim_{k\to\infty}\Theta(p,k) = \lim_{k\to\infty}\lim_{p\to\infty}\Theta(p,k)\\ &= \mathbb{E}[g_1(M_{s_1})g_2(M_{s_2})\ldots g_n(M_{s_n})\,M_t] - \mathbb{E}[g_1(M_{s_1})g_2(M_{s_2})\ldots g_n(M_{s_n})\,M_s],\end{aligned}$$

which yields the desired result. ☐

By the classical theory of martingales (see, for example, [1]), almost surely, for every $t \geq 0$,

$$\widetilde{M}_t = \lim_{s\to t, s\in\mathbb{Q}, s>t} M_s$$

is well defined, and $(\widetilde{M}_t,\ t \geq 0)$ is a right-continuous martingale which, obviously, is associated to $(\mu_t,\ t \geq 0)$. ☐

Remark. By considering only the parameter k, the proof of Lemma 3.10 also shows that, for every $\varepsilon \in (0,1)$, the process $(M_t^\varepsilon,\ t \geq 0)$ is a (continuous) martingale.

In the following lemma, which will be useful in the next section, we state a property which is satisfied by the martingale $(\widetilde{M}_t,\ t \geq 0)$ constructed in the proof of Theorem 3.2.

Lemma 3.11. *Let $g_1, \dots, g_n, \phi_k$ and $\tilde{m}$ be as in the proof of Lemma 3.10. Then, for $0 \leq s_1 \leq \ldots \leq s_n \leq s \leq t$ elements of $\mathbb{R}_+$,*

$$\left|\mathbb{E}[g_1(\widetilde{M}_{s_1}) \dots g_n(\widetilde{M}_{s_n})\, \phi_k(\widetilde{M}_t)] - \mathbb{E}[g_1(\widetilde{M}_{s_1}) \dots g_n(\widetilde{M}_{s_n})\, \phi_k(\widetilde{M}_s)]\right| \leq \tilde{m} \sup\{C(t,y) - C(s,y);\ k \leq |y| \leq 2k\}.$$

Proof. We first suppose that $0 \leq s_1 \leq \ldots \leq s_n \leq s \leq t$ are elements of $\mathbb{Q}_+$, and we keep the notation in the proof of Lemma 3.10. By Lemma 3.7, Lemma 3.4 and Proposition 2.3, for any $t \geq 0$,

$$\lim_{p \to \infty} C_{\epsilon_p}(t,x) = C(t,x) \qquad \text{uniformly on compact sets.}$$

Therefore, letting p tend to ∞ in inequality (3.1), we get:

$$|\mathbb{E}[g_1(M_{s_1}) \dots g_n(M_{s_n})\, \phi_k(M_t)] - \mathbb{E}[g_1(M_{s_1}) \dots g_n(M_{s_n})\, \phi_k(M_s)]| \leq \tilde{m} \sup\{C(t,y) - C(s,y);\ k \leq |y| \leq 2k\}.$$

Suppose now that $0 \leq s_1 \leq \ldots \leq s_n \leq s \leq t$ are elements of $\mathbb{R}_+$. Using again Proposition 2.3 (and property (1) in Prop. 2.5), we obtain the desired result by approximation, from the above inequality. □

4. Kellerer's theorem: the general case

We now obtain, by approximation, a proof of Kellerer's theorem in the general case.

Theorem 4.1. *Let $(\mu_t,\ t \geq 0)$ be a family in $\mathcal{M}$. Then the following properties are equivalent:*

(1) *There exists a martingale associated to $(\mu_t,\ t \geq 0)$;*
(2) *$(\mu_t,\ t \geq 0)$ is a peacock.*

Proof. We consider a peacock $(\mu_t,\ t \geq 0)$ and we set $C(t,x) = C_{\mu_t}(x)$.

Lemma 4.2. *There exists a countable set $D \subset \mathbb{R}_+$ such that the map:*

$$t \longrightarrow \mu_t \in \mathcal{M}$$

is continuous at any $s \notin D$.

Proof. By property (2) in Proposition 2.4, there exists a countable set $D \subset \mathbb{R}_+$ such that, for every $x \in \mathbb{Q}$, the map:

$$t \longrightarrow C(t,x)$$

is continuous at any $s \notin D$. By equicontinuity (property (i) in Prop. 2.2), this continuity property holds for every $x \in \mathbb{R}$. It suffices then to apply Proposition 2.3, taking into account property (1) in Proposition 2.5. □

We may write $D = \{d_n;\ n \in \mathbb{N}\}$. For $p \in \mathbb{N}$, we denote by $(k_n^{(p)},\ n \geq 0)$ the increasing rearrangement of the set:

$$\{k\, 2^{-p};\ k \in \mathbb{N}\} \cup \{d_j;\ 0 \leq j \leq p\}.$$

We define $(\mu_t^{(p)},\ t \geq 0)$ by:

$$\mu_t^{(p)} = \frac{k_{n+1}^{(p)} - t}{k_{n+1}^{(p)} - k_n^{(p)}}\, \mu_{k_n^{(p)}} + \frac{t - k_n^{(p)}}{k_{n+1}^{(p)} - k_n^{(p)}}\, \mu_{k_{n+1}^{(p)}} \qquad \text{if} \quad t \in \left[k_n^{(p)}, k_{n+1}^{(p)}\right].$$

We also set: $C_p(t,x) = C_{\mu_t^{(p)}}(x)$.

Lemma 4.3. *The following properties hold:*

(i) $(\mu_t^{(p)},\ t \geq 0)$ *is a peacock and the map:* $t \longrightarrow \mu_t^{(p)} \in \mathcal{M}$ *is continuous;*

(ii) *for any* $t \geq 0$, *the set* $\{\mu_t^{(p)};\ p \in \mathbb{N}\}$ *is uniformly integrable;*

(iii) *for* $t \geq 0$; $\lim_{p\to\infty} \mu_t^{(p)} = \mu_t$ *in* $\mathcal{M}$;

(iv) *for* $0 \leq s \leq t$,

$$\lim_{|x|\to\infty} \sup\{C_p(t,x) - C_p(s,x);\ p \geq 0\} = 0.$$

Proof. Properties (i) and (iii) are clear by construction. Property (ii) (resp. property (iv)) follows directly from property (1) (resp. property (2)) in Proposition 2.5. □

By Theorem 3.2, there exists, for each p, a right-continuous martingale $(M_t^{(p)},\ t \geq 0)$ which is associated to $(\mu_t^{(p)},\ t \geq 0)$ and satisfies the property stated in Lemma 3.11. For any $n \in \mathbb{N}$ and $\tau_n = (t_1, \dots, t_n) \in \mathbb{R}_+^n$, we denote by $\mu_{\tau_n}^{(p,n)}$ the law of $(M_{t_1}^{(p)}, \dots, M_{t_n}^{(p)})$, a probability measure on $\mathbb{R}^n$. The proof of the following lemma is quite similar to that of Lemma 3.9, hence we omit this proof.

Lemma 4.4. *For every* $n \in \mathbb{N}$ *and* $\tau_n \in \mathbb{R}_+^n$, *the set of probability measures* $\{\mu_{\tau_n}^{(p,n)};\ p \geq 0\}$, *is tight.*

Let now $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$, which refines Fréchet's filter. As a consequence of the previous lemma, for every $n \in \mathbb{N}$ and every $\tau_n \in \mathbb{R}_+^n$, $\lim_{\mathcal{U}} \mu_{\tau_n}^{(p,n)}$ exists in $\mathcal{M}$ and we denote this limit by $\mu_{\tau_n}^{(\infty,n)}$. By property (iii) in Lemma 4.3, $\mu_t^{(\infty,1)} = \mu_t$. There exists a process $(M_t,\ t \geq 0)$ such that, for every $n \in \mathbb{N}$ and every $\tau_n = (t_1, \dots, t_n) \in \mathbb{R}_+^n$, the law of $(M_{t_1}, \dots, M_{t_n})$ is $\mu_{\tau_n}^{(\infty,n)}$. In particular, this process $(M_t,\ t \geq 0)$ is associated to $(\mu_t,\ t \geq 0)$.

Lemma 4.5. *The process* $(M_t,\ t \geq 0)$ *is a martingale.*

Proof. The proof is similar to that of Lemma 3.10, but we give the details for the sake of completeness.

Let ϕ be a C^2-function on $\mathbb{R}$ such that $\phi(x) = 1$ for $|x| \leq 1$, $\phi(x) = 0$ for $|x| \geq 2$, and $0 \leq \phi(x) \leq 1$ for all $x \in \mathbb{R}$. We set, for $k > 0$, $\phi_k(x) = x\,\phi(k^{-1}x)$. Fix now $n \in \mathbb{N}$ and n continuous bounded functions $(g_1, \dots, g_n)$ on $\mathbb{R}$, and finally $0 \leq s_1 \leq \ldots \leq s_n \leq s \leq t$ elements of $\mathbb{R}_+$. We set:

$$\Lambda(p,k) = \mathbb{E}[g_1(M_{s_1}^{(p)}) g_2(M_{s_2}^{(p)}) \dots g_n(M_{s_n}^{(p)})\, \phi_k(M_t^{(p)})] - \mathbb{E}[g_1(M_{s_1}^{(p)}) g_2(M_{s_2}^{(p)}) \dots g_n(M_{s_n}^{(p)})\, \phi_k(M_s^{(p)})].$$

From the definitions, we obtain, for every k,

$$\lim_{\mathcal{U}} \Lambda(p,k) = \mathbb{E}[g_1(M_{s_1}) g_2(M_{s_2}) \dots g_n(M_{s_n})\, \phi_k(M_t)] - \mathbb{E}[g_1(M_{s_1}) g_2(M_{s_2}) \dots g_n(M_{s_n})\, \phi_k(M_s)]$$

and, by dominated convergence,

$$\lim_{k\to\infty} \lim_{\mathcal{U}} \Lambda(p,k) = \mathbb{E}[g_1(M_{s_1}) g_2(M_{s_2}) \dots g_n(M_{s_n})\, M_t] - \mathbb{E}[g_1(M_{s_1}) g_2(M_{s_2}) \dots g_n(M_{s_n})\, M_s].$$

On the other hand, since $(M_t^{(p)},\ t \geq 0)$ satisfies the property stated in Lemma 3.11, there exists a constant $\tilde{m}$ such that:

$$|\Lambda(p,k)| \leq \tilde{m}\, \sup\{C_p(t,y) - C_p(s,y);\ k \leq |y| \leq 2k\}.$$

Thus, by property (iv) in Lemma 4.3,

$$\lim_{k\to\infty} \Lambda(p,k) = 0 \qquad \text{uniformly with respect to } p.$$

Consequently,

$$\begin{aligned} 0 &= \lim_{\mathcal{U}} \lim_{k\to\infty} \Lambda(p,k) = \lim_{k\to\infty} \lim_{\mathcal{U}} \Lambda(p,k) \\ &= \mathbb{E}[g_1(M_{s_1}) g_2(M_{s_2}) \dots g_n(M_{s_n})\, M_t] - \mathbb{E}[g_1(M_{s_1}) g_2(M_{s_2}) \dots g_n(M_{s_n})\, M_s], \end{aligned}$$

which yields the desired result. □

This lemma completes the proof of Theorem 4.1. □

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