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HIGH ORDER EDGE ELEMENTS ON SIMPLICIAL MESHES

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Abstract. Low order edge elements are widely used for electromagnetic field problems. Higher order edge approximations are receiving increasing interest but their definition become rather complex. In this paper we propose a simple definition for Whitney edge elements of polynomial degree higher than one. We give a geometrical localization of all degrees of freedom over particular edges and provide a basis for these elements on simplicial meshes. As for Whitney edge elements of degree one, the basis is expressed only in terms of the barycentric coordinates of the simplex.

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Introduction

Whitney elements on simplices [7, 18] are perhaps the most widely used finite elements in computational electromagnetics. They offer the simplest construction of polynomial discrete differential forms on simplicial complexes. Their associated degrees of freedom (dofs) have a very clear meaning as cochains and, thus, give a recipe for discretizing physical balance laws, *e.g.*, Maxwell's equations. High order extensions of Whitney forms are known and have become an important computational tool for their better convergence and accuracy properties. However, in addition to the complexity in generating element basis functions, it has remained unclear what kind of cochains they should be associated with, namely, the localization of the corresponding dofs on the mesh volumes (here, tetrahedra). The current paper settles this issue for edge elements (see also [19] for a short presentation of high order Whitney forms).

The interest in the use of high order schemes, such as hk-finite element or spectral element methods, has become increasingly widespread (see [15] for a presentation of these methods). In the framework of curl-conforming finite elements, *i.e.*, vector approximations whose tangential components are continuous across element interfaces, the approach proposed in [18] is efficient in terms of number of dofs but difficult to follow. Consequently, the construction of basis functions which corresponds to the dofs defined in [18] has been the subject of numerous papers in the literature, see, *e.g.*, [11, 12, 14, 22].

Viable sets of basis functions for high order Whitney edge forms in dimension three have been proposed in [2], with resulting well-conditioned Galerkin matrices (see also [1, 3]). In [6], the authors have developed an alternative technique relying on projection based interpolation where the high order space is built by using a hierarchical basis, with resulting optimal interpolation error estimates. A parallel approach using the Koszul

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differential complex has been developed in [5] and a general construction of high order discrete differential forms can be found in [13].

The existing constructions of high order extensions of Whitney edge elements follow the traditional FEM path of using higher and higher moments to define the needed dofs. As a result, such high order finite elements include non-physical dofs associated to faces and tetrahedra that are not easy to interpret as field circulations along edges. The present paper offers an approach based on using "small simplices", a set of simplices obtained through affine contractions of a mesh simplex, associated with the principal lattice of the simplex. The small elements are used in conjunction with physical (natural) dofs only associated to edges. The present approach can be seen as a high order reconstruction formula based on subdivision of the master simplex into an auxiliary mesh of sub-simplices. By means of this subdivision, we provide a basis for these elements: at each tetrahedron, this basis is obtained as the product of Whitney forms of degree one by suitable homogeneous monomials in the barycentric coordinate functions of the simplex. Note that the adopted approach is consistent with the classical technique to generate curl-conforming bases $\mathcal{B}$ complete to order k by considering the product of zeroth-order curl-conforming bases with complete polynomial factors of order k (see [12] for an overview). In the present paper, we justify from a geometric point of view the construction of curl-conforming basis functions from products of barycentric coordinate monomials and Whitney 1-forms. The key heuristic points underlying this construction are three: (i) these high order forms satisfy a partition of unity property; (ii) being a larger number with respect to those of degree one, they are associated to a finer partition in each tetrahedron, the so-called "small simplices"; (iii) the spaces spanned by high order p-forms constitute an exact sequence (see [9] for details).

Handy local bases for high order curl-conforming elements are already known and widely used and the basis we propose is not as conveniently implementable as those ones. Indeed, despite the element basis functions are very simple to generate, the high order Whitney edge forms that we are going to describe are not linearly independent: a selection procedure has to be specified to obtain a valid set of unisolvent local shape functions. In addition, the resulting Galerkin matrices are badly-conditioned. In this paper, we aim at providing an insight into the "geometrical nature" of high order edge elements, in the language of differential forms. Whitney forms are differential geometric objects created long ago for other purposes and whose main characteristic is the interpretation they suggest of dofs as integrals over geometric elements (edges, faces, ...) of the discretization mesh.

The paper is focused on edge elements and is organized as follows. In Section 1, some notations are introduced and the chain/cochain concepts are briefly recalled. In Section 2, the definition and some properties of Whitney edge elements of polynomial degree one together with their connection with the lowest-order Nédélec elements are revisited. Section 3 is devoted to Whitney edge elements of higher order. Preliminary numerical tests in two dimensions concerning the convergence rate and conditioning behavior with respect to the maximal diameter h of the mesh triangles and to the polynomial degree k of these elements are presented in Section 4.

1. Algebraic tools

In this section, we recall some basic notions in algebraic topology (see, *e.g.*, [4,21]) and explain the adopted notation. We consider a three-dimensional domain Ω but notions and proofs have general validity. For all integrals we omit specifying the integration variable when this can be done without ambiguity. We shall denote by $\int_\gamma u$ the circulation of a vector field u along the curve γ. Moreover, we shall put emphasis on the map $\gamma \to \int_\gamma u$, that is to say, the differential form of degree 1 which one can associate with a given vector field u, and we use notations specific to exterior calculus, such as the exterior derivative d, as in the Stokes theorem.

1.1. **Simplicial mesh**

Let d be the ambient dimension. Given a domain $\Omega \subset \mathbb{R}^d$, a *simplicial mesh* $\mathtt{m}$ in Ω is a tessellation of $\overline{\Omega}$ by d-simplices, under the condition that any two of them may intersect along a common face, *i.e.*, a common subsimplex of dimension $0 \leq p \leq (d-1)$. In dimension $d = 3$, which we shall assume when giving examples, this means along a common face, edge or node, but in no other way. Labels n, e, f, t are used for nodes, edges, *etc.*, each with its own orientation, and w^n, w^e, *etc.*, refer to the corresponding Whitney forms of degree one [7]. Note that e (resp., f, t) is by definition an ordered couple (resp. triplet, quadruplet) of vertices, not merely a collection. The forms w^e (resp., w^f, w^t) are indexed over the set of these couples (resp. triplets, quadruplets), thus we use e (resp., f, t) also as a label since it points to the same object in both cases. The sets of nodes, edges, faces, volumes (*i.e.*, tetrahedra) of the mesh $\mathtt{m}$, are denoted by $\mathcal{N}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{T}$ and the sets of nodes, edges, faces of a tetrahedron t are denoted by $\mathcal{N}(t)$, $\mathcal{E}(t)$, $\mathcal{F}(t)$. In short, we denote by $\mathcal{S}^p$ the set of p-simplices of $\mathtt{m}$, by $\#\mathcal{S}^p$ its cardinality, with similar notations when restricted to a given tetrahedron t.

The sets of p-simplices are linked, as in [7], by incidence matrices for which the generic notation $\mathbf{d}$ is used, the symbol $\mathbf{d}^s_\sigma$ stands for the incidence matrix entry linking the p-simplex s to the $(p+1)$-simplex σ, $0 \le p \le d-1$. In three dimensions, incidence matrices are usually denoted $\mathbf{G}$ for $p = 0$, $\mathbf{R}$ for $p = 1$ and $\mathbf{D}$ for $p = 2$. In detail, let $e = \{\ell, n\}$ be an edge of the mesh oriented from the node ℓ to n. We can define the incidence numbers $\mathbf{G}^n_e = 1$, $\mathbf{G}^\ell_e = -1$ and $\mathbf{G}^k_e = 0$ for all nodes k other than ℓ and n. These numbers form the matrix $\mathbf{G}$, which describes how edges connect to nodes. A face $f = \{\ell, n, k\}$ has three vertices which are the nodes ℓ, n, k. Note that $\{n, k, \ell\}$ and $\{k, \ell, n\}$ denote the same face f whereas $\{n, \ell, k\}$ denotes an oppositely oriented face, which is not supposed to belong to $\mathcal{F}$ if f does. An orientation of f induces an orientation of its boundary. So, with respect to the orientation of the face f, the one of the edge $\{\ell, n\}$ is positive and that of $\{k, n\}$ is negative. So we can define the incidence number $\mathbf{R}^e_f = 1$ (resp. -1) if the orientation of e matches (resp. does not match) the one on the boundary of f and $\mathbf{R}^e_f = 0$ if e is not an edge of f. Finally, let us consider the tetrahedron $t = \{k, \ell, m, n\}$, positively oriented if the three vectors $\{k, \ell\}$, $\{k, m\}$ and $\{k, n\}$ define a positive frame ($t' = \{\ell, m, n, k\}$ has a negative orientation and does not belong to $\mathcal{T}_m$ if t does). The matrix $\mathbf{D}$ can be defined by setting $\mathbf{D}^f_t = \pm 1$ if face f bounds the tetrahedron t, with the sign depending on whether the orientation of f and of the boundary of t match or not, and $\mathbf{D}^f_t = 0$ in case f does not bound t. For consistency, we may attribute an orientation to nodes as well, a sign ± 1. Implicitly, we have been orienting all nodes the same way $(+1)$ up to now. Note that a sign (-1) to node n changes the sign of all entries of column n in the above $\mathbf{G}$. It can easily be proved that $\mathbf{RG} = \mathbf{0}$ and $\mathbf{DR} = \mathbf{0}$.

1.2. Chains

We now introduce *chains* over the mesh $\mathtt{m}$. A p-chain c, $0 \le p \le d$, is an assignment to each p-simplex s of an integer α_s. This can be denoted by $c = \sum_{s \in \mathcal{S}^p} \alpha_s\, s$. Let C_p be the set of all p-chains. This set has a structure of Abelian group with respect to the addition of p-chains: two p-chains are added by adding the corresponding coefficients. If s is an oriented simplex, the *elementary chain* corresponding to s is the assignment $\alpha_s = 1$ and $\alpha'_s = 0$ for all $s' \neq s$. In the sequel, we will use the same symbol s (or n, e, *etc.*) to denote the oriented simplex and the associated elementary chain. Note how this is consistent with the above expansion of c as a formal weighted sum of simplices.

The *boundary* of an oriented p-simplex of $\mathtt{m}$ is a $(p-1)$-chain determined by the sum of its $(p-1)$-dimensional faces, each taken with the orientation induced from that of the whole simplex. For instance, $\partial e = \sum_{n \in \mathcal{N}} \mathbf{G}^n_e\, n$ expresses the boundary of edge e as a formal linear combination of nodes (such a thing is a p-chain, with $p = 0$ here). Symbol $\boldsymbol{\partial}$ will serve for $\mathbf{d}^t$, *i.e.*, as a generic notation for the transposed $\mathbf{D}^t$, $\mathbf{R}^t$, $\mathbf{G}^t$. Given $\mathbf{c} = \{c^e : e \in \mathcal{E}\}$, we have $\partial(\sum_{e \in \mathcal{E}} c^e\, e) = \sum_{n \in \mathcal{N}} (\boldsymbol{\partial}\, \mathbf{c})^n n$, with $\boldsymbol{\partial} = \mathbf{G}^t$ in this case.

1.3. Cochains

We introduce the dual concept of *p-cochain.* A p-cochain is a linear functional on the vector space of p-chains. For instance, given an array $\mathbf{b} = \{b^s : s \in \mathcal{S}^p\}$ of real numbers, we can define the p-cochain $c \to \sum_{s \in \mathcal{S}^p} b^s\, c_s$ acting on p-chains c with coefficients c_s. Given a differential form w, the mapping $c \to \int_c w$ defines a p-cochain. More generally, the p-cochain coefficients are obtained by integrating the differential form w on the elements of the p-chain c, *i.e.*, the map $c \to \sum_{s \in \mathcal{S}^p_m} c_s \int_s w$ is a cochain.

Once a metric is introduced on the ambient affine space, differential forms are in correspondence with scalar and vector fields (called "proxy fields", metric dependent, of course). The coefficients of p-cochains become the dofs of scalar and vector fields (and this is exactly what occurs with Whitney finite elements). Let W^p denote the set of p-cochains (or p-forms) defined on Ω when triangulated by $\mathtt{m}$. Then, C_p and W^p are *in duality via* the bilinear bicontinuous map $\langle \cdot \ ; \ \cdot \rangle : W^p \times C_p \to \mathbb{R}$ defined as $\langle w \ ; \ c \rangle = \int_c w$.

For $p > 0$, the *exterior derivative* of the $(p-1)$-form w is the p-form $\mathrm{d}w$. Note that the matrices $\mathbf{G}$, $\mathbf{R}$, $\mathbf{D}$ are the discrete representation of the exterior derivative operator d when applied to a 0-, 1- and 2-form, respectively. The integral $\int_c w$ is treated in two ways: if $c = \partial\tau$ and w is smooth, one may go forward and integrate $\mathrm{d}w$ over τ. Alternatively, if the form $w = \mathrm{d}v$, one may go backward and integrate v over ∂c. In particular, we have $\int_{\partial c} w = \int_c \mathrm{d}w$, which is the common form of Stokes' theorem, or equivalently,

$$\langle w \ ; \ \partial c \rangle = \langle \mathrm{d}w \ ; \ c \rangle \qquad \forall c \in C_p \quad \text{and} \quad \forall w \in W^{p-1}. \tag{1}$$

Equation (1) reveals that d is the *dual* of ∂. As a corollary of the boundary operator property $\partial \circ \partial = 0$, we have that $\mathrm{d} \circ \mathrm{d} = 0$.

2. Whitney elements of polynomial degree one

Fields, in electromagnetism, are observed *via* quantities, such as electromotive forces, intensities, *etc.*, which correspond to line integrals (*circulations*), surface integrals (*fluxes*), *etc.* A field (say, for example, the vector potential a) then maps a p-manifold γ ($p = 0$ for points, 1 in our example where γ is a line, 2 for surfaces, and so on) to a number, here $\int_\gamma a$. If w^e are edge elements, then a is represented by $\sum_{e\in\mathcal{E}} a_e w^e$ which we shall denote by $p_{\mathtt{m}} a$, being $p_{\mathtt{m}}$ the interpolation operator of a field on the Whitney forms. Suppose that we replace γ by a p-chain $p^t_{\mathtt{m}}\gamma = \sum_{e\in\mathcal{E}} c^e e$, being $p^t_{\mathtt{m}}$ the operator mapping a p-manifold in its "finite" representation, and let us interpret the scalars a_e as the elementary values $\int_e a$ (circulations, here). Then a natural approximation of $\int_\gamma a$ is obtained by substituting $p^t_{\mathtt{m}}\gamma$ for γ. Hence an approximate knowledge of the field a, *i.e.*, of all its measurable attributes, from the array $\mathbf{a} = \{a_e : e \in \mathcal{E}\}$. The problem is then: "how best to represent γ by a chain?" (*cf.* Fig. 1). Solving it yields, by duality, a *definition* of Whitney forms [23]: w^e, for instance, is, like the field a itself, a cochain, a map from lines γ to real numbers c^e, whose value we denote by $\int_\gamma w^e$ or by $\langle w^e, \gamma\rangle$. Note that, with this convention, $\langle a, p^t_{\mathtt{m}}\gamma\rangle = \langle a, \sum_{e\in\mathcal{E}}(\int_\gamma w^e)e\rangle = \sum_{e\in\mathcal{E}} \int_\gamma w^e \langle a, e\rangle \equiv \langle p_{\mathtt{m}} a, \gamma\rangle$. So, w^e is the Whitney form of polynomial degree one associated to e and the weight (or *moment*) of γ in the chain $p^t_{\mathtt{m}}\gamma$ is $\int_\gamma w^e \equiv \langle w^e, \gamma\rangle$. Note how this justifies the " $p^t_{\mathtt{m}}$ " notation. In the next sections we define w^e in the case of polynomial degree one and higher.

2.1. Generative formula for Whitney edge elements

Our starting point will be the rationale for Whitney forms given in [8]. We wish to represent a p-manifold by a p-chain, with special attention to the case $p = 1$. To do this, we give a look to what is usually done for $p = 0$.

Let's first recall the notion of barycentric coordinates. Let $t = \{n_1, n_2, n_3, n_4\}$ be a tetrahedron of $\mathtt{m}$. Four real numbers $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that $\sum_i \lambda_i = 1$ determine a point x, the barycenter of the n_is for these weights, uniquely defined by $x - n_0 = \sum_i \lambda_i(n_i - n_0)$, where n_0 is any origin (for example one of the n_is). Conversely, any point x has a unique representation of the form $x - n_0 = \sum_i \lambda_i(n_i - n_0)$, and the weights λ_i, considered as functions of x, are the barycentric coordinates of x in the affine basis provided by the four vertices n_is. Note that x belongs to the tetrahedron t if $\lambda_i(x) \geq 0$ for all i. The λ_is are affine functions of x. Now, consider the mesh $\mathtt{m}$ of tetrahedra over $\overline{\Omega}$. To each node n of the mesh, we attribute a function whose value at point x is 0, if none of the tetrahedra with a vertex in n contains x, otherwise, it is the barycentric coordinate of x with respect to n, in the affine basis provided by the vertices of the tetrahedron to which x belongs. We attribute to this nodal function the symbol w^n. Note that by construction, $w^n(x) \geq 0$ and $\sum_{n\in\mathcal{N}} w^n(x) = 1$ for all $x \in \overline{\Omega}$. The w^ns themselves are often called "hat functions". Note that working by restriction to the master d-simplex t, w^n and λ_n coincide. Any point x of the meshed domain can be represented as $x = \sum_{n\in\mathcal{N}} w^n(x) n$, where w^n is the only piecewise affine (affine by restriction to each tetrahedron) function that takes value 1 at node n and 0

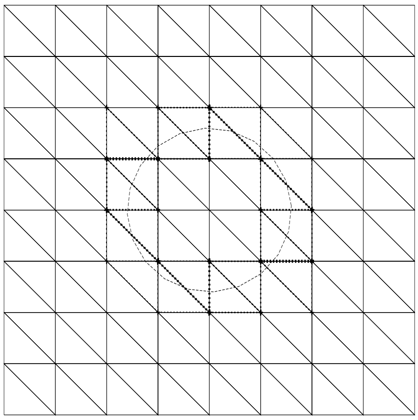

FIGURE 1. Representing the circle γ by a weighted sum of mesh edges, *i.e.*, by a 1-chain. Graded thickness of the edges are meant to suggest the respective weights assigned to them. Edges whose "control domain" (two neighboring triangles) doesn't intersect γ, have zero weight. (A weight can be negative, if the edge is oriented backward with respect to c.) Which weights thus to assign is the central issue in the approach proposed in [8] to Whitney forms.

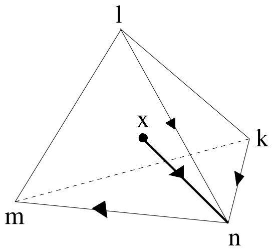

FIGURE 2. The 1-chain associated with the oriented segment xn is $-\lambda_m(x)nm + \lambda_k(x)kn + \lambda_l(x)ln$. The minus sign in front of $\lambda_m(x)$ is due to the fact that the oriented edge nm starts in n, and ends in m. In terms of incidence numbers, $\mathbf{G}_m^{nm} = 1$ and $\mathbf{G}_n^{nm} = -1$. Edges e of the volume $\{m, k, n, l\}$ which do not have n as vertex make no contribution to the 1-chain.

at all other nodes. So, the weight of x with respect to node n is $w^n(x)$. In the following, when $e = \{m, n\}$ and $f = \{l, m, n\}$, we denote the node l by $f - e$. Thus λ_{f-e} refers, in that case, to λ_l.

The Whitney 0-form w^n is then λ_n, the hat function of the finite element method. The definition of $p_{\mathtt{m}}q = \sum_{n\in\mathcal{N}} q_n w^n$ for a scalar field q is obtained by transposition:

$$\langle q, p_{\mathtt{m}}^t x\rangle = \langle q, \sum_{n\in\mathcal{N}} \lambda_n(x) n\rangle = \sum_{n\in\mathcal{N}} \lambda_n(x)\langle q, n\rangle = \sum_{n\in\mathcal{N}} q_n w^n(x) = \langle \sum_{n\in\mathcal{N}} q_n w^n, x\rangle \equiv \langle p_{\mathtt{m}} q, x\rangle.$$

Hat functions have a double feature: they are the weights that represent a generic point as a linear combination of the mesh nodes, as well as the interpolants that allow to define scalar functions from their nodal values at the mesh nodes.

For $p = 1$, let xy be the oriented segment going from point x to point y. We know that $p_{\mathtt{m}}^t x = \sum_{n\in\mathcal{N}}\langle w^n, x\rangle n$, and we figure out $p_{\mathtt{m}}^t xy$ by linearity: $p_{\mathtt{m}}^t xy = \sum_{n\in\mathcal{N}}\langle w^n, y\rangle p_{\mathtt{m}}^t(xn)$. As suggested in Figure 2, $p_{\mathtt{m}}^t(xn) =$

$\sum_{e\in\mathcal{E}} \mathbf{G}_n^e \lambda_{e-n}(x)e$. Thus $p_{\mathtt{m}}^t xy = \sum_{n\in\mathcal{N},e\in\mathcal{E}} \mathbf{G}_n^e \lambda_{e-n}(x)\langle w^n, y\rangle e \equiv \sum_{e\in\mathcal{E}} \langle w^e, xy\rangle e$. Hence

$$\langle w^e, xy\rangle = \sum_{n\in\mathcal{N}} \mathbf{G}_n^e \lambda_{e-n}(x)\langle w^n, y\rangle.$$

Being $0 = \langle w^e, xx\rangle = \sum_{n\in\mathcal{N}} \mathbf{G}_n^e \lambda_{e-n}(x)\langle w^n, x\rangle$ and d the dual of ∂, we get that

$$\langle w^e, xy\rangle = \sum_{n\in\mathcal{N}} \mathbf{G}_n^e \lambda_{e-n}(x)\langle w^n, y-x\rangle = \sum_{n\in\mathcal{N}} \mathbf{G}_n^e \lambda_{e-n}(x)\langle w^n, \partial(xy)\rangle = \sum_{n\in\mathcal{N}} \mathbf{G}_n^e \lambda_{e-n}(x)\langle \mathrm{d}w^n, xy\rangle$$

for any "small edge" xy, *i.e.*, a segment xy entirely contained in the cluster of tetrahedra around e. We then obtain the following definition.

Definition 2.1. The Whitney 1-form of polynomial degree 1 associated to the 1-simplex e is

$$w^e = \sum_{n\in\mathcal{N}} \mathbf{G}_e^n \, \lambda_{e-n} \mathrm{d}w^n. \tag{2}$$

The space of Whitney edge elements of polynomial degree 1 over $\mathtt{m}$ is $W_1^1 = \mathrm{span}\{w^e,\, e\in\mathcal{E}\}$ $(\equiv W^1)$.

As remarked in Section 1.3, differential forms such as w^n, w^e, *etc.*, are in correspondence with scalar and vector fields. For instance, the vector $w^e = \lambda_\ell \,\mathbf{grad}\, \lambda_m - \lambda_m \mathbf{grad}\, \lambda_\ell$, whose expression is recovered form (2) by replacing d with **grad**, is the vector field associated to the edge $e = \{\ell, m\}$. Its weight with respect to e (or *circulation* along e) is 1 and 0 on other simplices in $\mathtt{m}$ of matching dimension.

Let xy be the oriented edge with vertices x, y. In a code conceived in terms of proxy vector fields, with an underlying metric, instead of differential forms, the evaluation of circulations along edges of w^e is done according to the following well known result.

Proposition 2.2. *Let t be a given tetrahedron. Then*

$$\langle w^e, xy\rangle = |xy|(w^e(b_{xy})\cdot \mathbf{t}_{xy}), \qquad xy\subset t, \qquad e\in\mathcal{E}(t) \tag{3}$$

where b_{xy} is the barycenter of xy, $\mathbf{t}_{xy}$ is the unit vector along xy, $|xy|$ is the length of xy.

Note that Proposition 2.2 relies on *metric tools*, such as dot product, segment lengths, *etc.*, to compute *metric-free* quantities. The weight $\langle w^e, xy\rangle$ does not depend, in fact, on the shape of e and xy but on their relative position and orientation.

Thanks to formula (2), in the following proposition we state an equivalent but *affine* way to compute the weights.

Proposition 2.3. *Let $t = \{k, l, m, n\}$ be a given tetrahedron. Then*

$$\langle w^e, xy\rangle \;=\; \det\begin{pmatrix} \lambda_m(x) & \lambda_n(x) \\ \lambda_m(y) & \lambda_n(y)\end{pmatrix}, \qquad xy\subset t, \qquad e=\{m,n\}\in\mathcal{E}(t),$$

Proof. Thanks to Definition 2.1, we can write $\langle w^e, xy\rangle = \sum_{n\in\mathcal{N}} \mathbf{G}_e^n \lambda_{e-n}(x)\langle \mathrm{d}w^n, xy\rangle$. By duality between d and ∂, the equality $\langle \mathrm{d}w^n, xy\rangle = \langle w^n, \partial(xy)\rangle$ holds for any node $n\in\mathcal{N}$ and for any segment $xy\subset v$. We remark that $\langle w^n, \partial(xy)\rangle = \langle w^n, y-x\rangle$. Since $0 = \langle w^e, xx\rangle$, we have

$$\langle w^e, xy\rangle = \sum_{n\in\mathcal{N}} \mathbf{G}_e^n \lambda_{e-n}(x)\langle w^n, y\rangle = -\lambda_n(x)\lambda_m(y) + \lambda_m(x)\lambda_n(y) = \det\begin{pmatrix} \lambda_m(x) & \lambda_n(x) \\ \lambda_m(y) & \lambda_n(y)\end{pmatrix}. \qquad \square$$

Corollary 2.4. *Let $t = \{k, l, m, n\}$ be a given tetrahedron with unit volume $|klmn|$. Then*

$$\begin{aligned} &\langle w^n, x\rangle = |xklm|, \quad x \in t, \\ &\langle w^e, xy\rangle = |xykl|, \qquad xy \subset t, \qquad e = \{m, n\} \in \mathcal{E}(t). \end{aligned} \tag{4}$$

Proof. By definition of barycentric coordinates of a point $x \in v$ with respect to the vertices k, l, m, n of t, we can write

$$x = \lambda_k(x)\, k + \lambda_l(x)\, l + \lambda_m(x)\, m + \lambda_n(x)\, n, \quad \text{with} \qquad 1 = \lambda_k(x) + \lambda_l(x) + \lambda_m(x) + \lambda_n(x). \tag{5}$$

By subtracting k from both sides, we get $x - k = \lambda_l(x)\,(l-k) + \lambda_m(x)\,(m-k) + \lambda_n(x)\,(n-k)$. So, $\lambda_n(x) = \det(l-k, m-k, x-k)/\det(l-k, m-k, n-k) = |xklm|/|nklm|$, being the mixed product $(l-k)\cdot[(m-k)\times(x-k)] = 6|xklm|$ equal to $\det(l-k, m-k, x-k)$. Concerning the second statement, we write

$$\det\begin{pmatrix} \lambda_m(x) & \lambda_n(x) \\ \lambda_m(y) & \lambda_n(y) \end{pmatrix} = \det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \lambda_k(x) & \lambda_l(x) & \lambda_m(x) & \lambda_n(x) \\ \lambda_k(y) & \lambda_l(y) & \lambda_m(y) & \lambda_n(y) \end{pmatrix} = \det\begin{pmatrix} \lambda_k(k) & \lambda_l(k) & \lambda_m(k) & \lambda_n(k) \\ \lambda_k(l) & \lambda_l(l) & \lambda_m(l) & \lambda_n(l) \\ \lambda_k(x) & \lambda_l(x) & \lambda_m(x) & \lambda_n(x) \\ \lambda_k(y) & \lambda_l(y) & \lambda_m(y) & \lambda_n(y) \end{pmatrix}.$$

Thanks to Proposition 2.3 and the change of basis (5) from barycentric coordinates to Cartesian ones, we have $\langle w^e, xy\rangle = |xykl|$. $\square$

2.2. Properties

In this subsection, we recall some properties of the forms defined in (2).

Proposition 2.5. *For any edge e and face f we have*

$$\sum_{n\in\mathcal{N}} \mathbf{G}_e^n\, \lambda_{e-n}\, w^n = 0, \qquad \sum_{e\in\mathcal{E}} \mathbf{R}_f^e\, \lambda_{f-e}\, w^e = 0. \tag{6}$$

Proof. The first identity is evident: we get $-\lambda_m\,\lambda_n + \lambda_n\,\lambda_m = 0$ for the edge $e = \{n, m\}$. To prove the second identity, we replace w^e by its expression given in (2) and we get $\sum_e \mathbf{R}_f^e\, \lambda_{f-e}\, w^e = \sum_{n,e} \lambda_{f-e}\lambda_{e-n}\, \mathbf{R}_f^e\, \mathbf{G}_e^n\, \mathrm{d}w^n = 0$ since $\mathbf{RG} = \mathbf{0}$ and $\lambda_{f-e}\lambda_{e-n}$ is the same for all e in ∂f. $\square$

Recall that barycentric functions sum to 1, thus forming a "partition of unity": $\sum_{n\in\mathcal{N}} w^n = 1$. For 1-forms, going back to the standard vector formalism, let us denote by $\mathbf{w}^e$ the vector field associated to w^e.

Proposition 2.6. *At all points x, for all vectors $\mathbf{v}$,*

$$\sum_{e\in\mathcal{E}} (\mathbf{w}^e(x)\cdot\mathbf{v})\,\mathbf{e} = \mathbf{v}. \tag{7}$$

Proof. A vector length associated to an edge e is a vector $\mathbf{e}$ of modulus length(e) parallel to the edge e. Relation (7) results from the identity $xy = \sum_{e\in\mathcal{E}}\langle w^e, xy\rangle e$. We replace w^e by $\mathbf{w}^e$, then xy by its vector length $\mathbf{v}$ and e by its vector length $\mathbf{e}$. $\square$

The connection between the lowest order Nédélec elements (see [18], Defs. 2 and 4 with $d = 3$ and $k = 1$) and Whitney forms is well known and revisited in the following proposition.

Proposition 2.7. *In a given tetrahedron $t = \{m, n, k, l\}$, the vector field $w^{\{m,n\}}$ is of the form $a \times x + b$, where $a, b \in \mathbb{R}^d$ and a is parallel to the edge $\{k, l\}$ opposite to $\{m, n\}$.*

Proof. Let $|t|$ denote the volume of t and $(l-k)$ the vector starting in k and ending in l. We can write that

$$w^{\{m,n\}} = \frac{(l-k)\times(x-k)}{6|t|}$$

since $(l-k)\times(x-k)\cdot(m-n) = 6|t|$ for a point x lying on the edge $\{m,n\}$ and $\langle w^{\{m,n\}}, \{m,n\}\rangle = 1$. This yields,

$$w^{\{n,m\}} = \frac{(l-k)\times[(x-o)+(k-o)]}{6|t|} = \frac{(l-k)}{6|t|}\times x + b = a\times x + b$$

with $a = \frac{(l-k)}{6|t|}$ parallel to $\{k,l\}$ (being o the origin in the Cartesian coordinates). □

Next proposition states the curl-conformity (see [18] for a definition of this conformity) of the space W_1^1.

Proposition 2.8. *Vector fields in W_1^1 have tangential part continuous across faces.*

Proof. Let us consider two tetrahedra $t_1 = \{n,m,l,k\}$ and $t_2 = \{m,l,k,q\}$ with face $f = \{m,l,k\}$ in common and let x be a point of f. The vector $\mathbf{grad}\,\lambda_k$ is not continuous at x. We have in fact that $(l-m)\times(n-m) = 2\,\underline{nml}$, where $\underline{nml}$ denotes the vector area associated to $\{n,m,l\}$, and that $1/|\mathbf{grad}\,\lambda_k|$ is the height of t_1 with respect to $\{n,m,l\}$. This yields $(\mathbf{grad}\,\lambda_k)_{|t_1} = \underline{nml}/(3\,vol(t_1))$. Similarly, $(\mathbf{grad}\,\lambda_k)_{|t_2} = \underline{mlq}/3vol(t_2)$. But the tangential part of $\mathbf{grad}\,\lambda_k$ on face f changes in a continuous way when one crosses the face from one tetrahedron to its neighbor; indeed, it only depends on the values of λ_k on this face, whatever the tetrahedron one considers. As it goes the same for $\mathbf{grad}\,\lambda_m$, and for all the faces of the mesh, one may conclude that the tangential part of $w^{\{k,m\}}$ is continuous across faces. □

Thanks to this property, the set $W_1^1 = \mathrm{span}\,\{w^e,\, e\in\mathcal{E}\}$ plays the role of internal Galerkin approximation space for the Sobolev space $H(\mathbf{curl},\Omega)$ (see [10] for a definition of $H(\mathbf{curl},\Omega)$). Therefore, a vector field $h\in H(\mathbf{curl},\Omega)$ can be represented in W_1^1 by $p_{\mathtt{m}}h = \sum_{e\in\mathcal{E}} h_e\, w^e$ where the scalar h_e is the circulation of h along the mesh edge $e\in\mathcal{E}$, *i.e.*, the weight $\langle p_{\mathtt{m}}h, e\rangle$.

3. Whitney edge elements of higher degree

We set out to construct edge elements of higher degree and from now on multi-index notations are used. Note that the integer k will be no more a vertex label but a multi-index weight. Let $\mathbf{k}$, boldface, be the array $(k_0,\dots,k_d)$ of $d+1$ integers $k_i\geq 0$, and denote by k its weight $\sum_{i=0}^d k_i$. The set of multi-indices $\mathbf{k}$ with $d+1$ components and of weight k is denoted $\mathcal{I}(d+1,k)$ and its cardinality $\#\mathcal{I}(d+1,k)$ is the binomial coefficient $\begin{pmatrix} k+d \\ d\end{pmatrix} = \frac{(k+d)!}{d!\,k!}$. We then adopt the following definition.

Definition 3.1. Let $\mathbf{k}\in\mathcal{I}(d+1,k)$. Then $\lambda^{\mathbf{k}}$ means the homogeneous monomial $\Pi_{i=0}^d\,(\lambda_i)^{k_i}$.

Let us denote by $\mathbb{P}_k(\Sigma)$ the vector space of polynomials defined on a domain $\Sigma\subset\mathbb{R}^d$ in d variables of degree $\leq k$ and by $\widetilde{\mathbb{P}}_k(\Sigma)$ the subspace of $\mathbb{P}_k(\Sigma)$ of homogeneous polynomials of degree k. A well-known result in algebra states that the dimension of $\mathbb{P}_k(\Sigma)$ is $\begin{pmatrix} k+d \\ k\end{pmatrix}$ and that of $\widetilde{\mathbb{P}}_k(\Sigma)$ is $\begin{pmatrix} k+d-1 \\ k\end{pmatrix}$. Homogeneous polynomials of degree k in the $d+1$ barycentric coordinates are in 1-to-1 correspondence with polynomials of degree $\leq k$ in the d Cartesian ones. For this reason, we can say that $\mathbb{P}_k(t) = \mathrm{span}(\lambda^{\mathbf{k}})_{\mathbf{k}\in\mathcal{I}(d+1,k)}$ on each tetrahedron t.

In each tetrahedron t of vertices a_i, $i=0,d$, and for each integer $k\geq 0$, the principal lattice of order $k+1$ in t is the set of points T_{k+1} defined by their barycentric coordinates with respect to the vertices a_i as follows

$$T_{k+1} = \{x\in\mathbb{R}^d,\ \lambda_j(x)\in\{0,\frac{1}{k+1},\dots,\frac{k}{k+1},1\},\ 1\leqslant j\leqslant d+1\}$$

(*cf.* Fig. 3 for T_3).

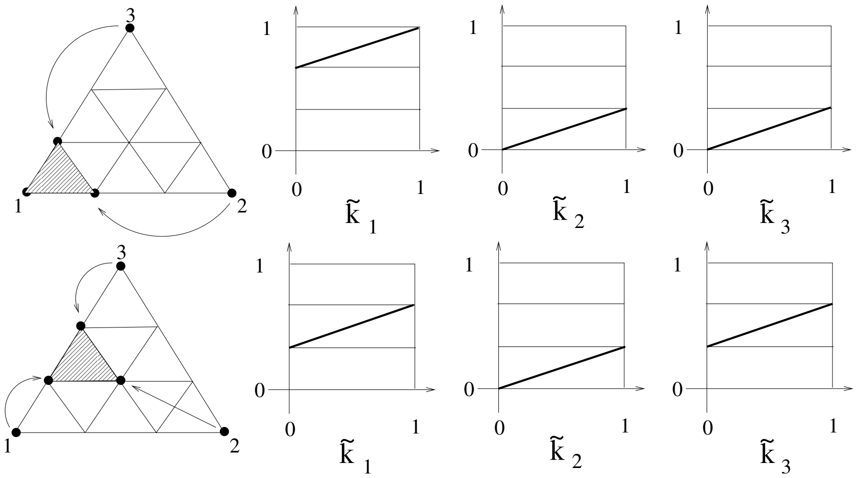

FIGURE 3. The mapping $\tilde{\mathbf{k}}$ associated to $\mathbf{k} = (2, 0, 0)$ (top) and $\mathbf{k} = (1, 0, 1)$ (bottom) with $k = 2$, and the (dashed) triangle $\tilde{\mathbf{k}}(t)$.

In order to define higher order Whitney edge elements, we do not follow the traditional FEM path of using higher and higher moments to define the needed dofs to avoid including non-physical quantities that are not easy to interpret. We follow a new approach based on a set of edges, called "small edges", associated to the principal lattice of the simplex and defined by means of a particular application, the $\tilde{\mathbf{k}}$ map, explained in the next section.

3.1. The $\tilde{\mathbf{k}}$ map

We start by defining a geometrical partition within each mesh tetrahedron t: this partition is the key point in the construction Whitney forms of any degree k.

Definition 3.2. To each multi-integer $\mathbf{k} \in \mathcal{I}(d+1, k)$ corresponds a map, denoted by $\tilde{\mathbf{k}}$, from t into itself. Let $\tilde{k}_i$ denote the affine function that maps $[0, 1]$ onto $[\frac{k_i}{k+1}, \frac{1+k_i}{k+1}]$. If $\lambda_i(x)$, $0 \le i \le d$, are the barycentric coordinates of point $x \in t$, its image $\tilde{\mathbf{k}}(x)$ has barycentric coordinates $\tilde{k}_i(\lambda_i(x))$, with $\tilde{k}_i(\lambda_i(x)) = \frac{\lambda_i(x)+k_i}{k+1}$.

Geometrically, this map is a homothety, more precisely a transformation of space which dilates distances of a factor $\frac{1}{k+1}$ with respect to a fixed point of barycentric coordinates $\frac{k_i}{k}$ (*cf.* Fig. 3 for two examples). Note that $\tilde{\mathbf{k}}(t)$ for all possible $\mathbf{k} \in \mathcal{I}(d+1, k)$, are congruent by translation and homothetic to t. They don't pave t, and the holes left are not necessarily homothetic to t. As an example, take $k = 2$: for $d = 2$, *cf.* Figure 4 (left), the holes left are 3 small triangles not homothetic to t; for $d = 3$, *cf.* Figure 4 (center and right), the holes left are 1 central small tetrahedron and 4 octahedra. The $\tilde{\mathbf{k}}(x_i)$, for all possible multi-indices $\mathbf{k} \in \mathcal{I}(d+1, k)$ and nodes i of t, make T_{k+1}, the principal lattice of order $k+1$ in t.

Definition 3.3. We call small edges of a mesh tetrahedron t the images $\tilde{\mathbf{k}}(E)$ for all (big) edge $E \in \mathcal{E}(t)$ and all $\mathbf{k} \in \mathcal{I}(d+1, k)$, and denote them by $e = \{\mathbf{k}, E\}$.

In short, all the edges of $\tilde{\mathbf{k}}(t)$, for $\mathbf{k} \in \mathcal{I}(d+1, k)$, are small edges. Looking again to Figure 4, one has 6 small triangles $\tilde{\mathbf{k}}(t)$ in 2D and 10 small tetrahedra $\tilde{\mathbf{k}}(t)$ in 3D. This yields 18 small edges in 2D and 60 in 3D for each mesh element t.

3.2. High order edge elements

Edge elements of higher degree in a tetrahedron t are associated to the geometrical partition in t defined by the $\tilde{\mathbf{k}}$ map for all possible multi-indices $\mathbf{k} \in \mathcal{I}(d+1, k)$.

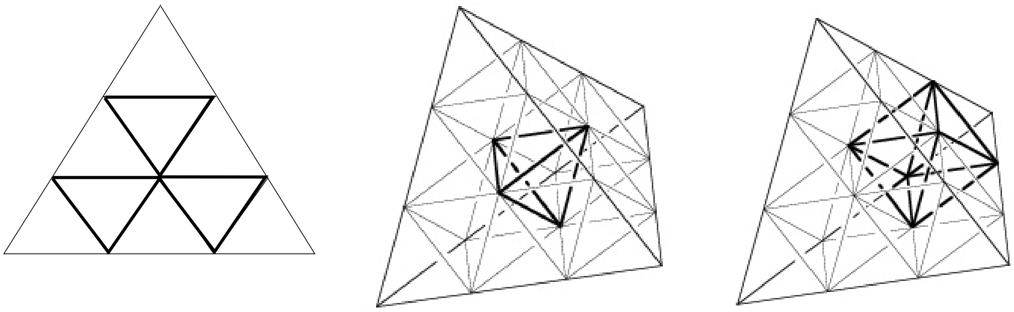

FIGURE 4. The images $\tilde{\mathbf{k}}(t)$ for a mesh triangle and tetrahedron t, for $k = 2$ and (in thick line) examples of "holes", *i.e.*, triangles, tetrahedra or octahedra not congruent to t.

Definition 3.4. Whitney 1-forms of polynomial degree $k+1$ over $\mathtt{m}$ are the $w^e = \lambda^{\mathbf{k}} w^E$, for all multi-index $\mathbf{k} \in \mathcal{I}(d+1, k)$, for all (big) edges $E \in \mathcal{E}$, being $e = \{\mathbf{k}, E\}$ and w^E defined in (2). The space of Whitney edge elements of polynomial degree $k+1$ over $\mathtt{m}$ is $W^1_{k+1} = \text{span}\{w^e,\ e = \{\mathbf{k}, E\},\ \mathbf{k} \in \mathcal{I}(d+1, k),\ E \in \mathcal{E}\}$.

Examples. In two dimensions ($d = 2$), let us consider the triangle $t = \{n, l, m\}$. The set of its (big) edges E is $\mathcal{E}(t) = \{\{n, l\}, \{l, m\}, \{m, n\}\}$ and $\lambda^{\mathbf{k}} = \lambda_n^{k_0} \lambda_l^{k_1} \lambda_m^{k_2}$.

(1) For $k = 0$, the set of multi-indices with 3 components and of weight 0 is $\mathcal{I}(3, 0) = \{(0, 0, 0)\}$ and the $\tilde{\mathbf{k}}$ map is the identity over the considered triangle. The space of Whitney edge elements of polynomial degree 1 over t is thus

$$W^1_1(t) = \text{span}\{w^{\{n,l\}},\ w^{\{l,m\}},\ w^{\{m,n\}}\}, \tag{8}$$

where, *e.g.*, $w^{\{n,l\}} = \lambda_n \mathbf{grad}\lambda_l - \lambda_l \mathbf{grad}\lambda_n$.

(2) For $k = 1$, the set of multi-indices with 3 components and of weight 1 is

$$\mathcal{I}(3, 1) = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

and the space of Whitney edge elements of polynomial degree 2 over t is

$$\begin{aligned} W^1_2(t) = \text{span}\{&\lambda_n w^{\{n,l\}},\ \lambda_n w^{\{l,m\}},\ \lambda_n w^{\{m,n\}}, \\ &\lambda_l w^{\{n,l\}},\ \lambda_l w^{\{l,m\}},\ \lambda_l w^{\{m,n\}}, \\ &\lambda_m w^{\{n,l\}},\ \lambda_m w^{\{l,m\}},\ \lambda_m w^{\{m,n\}}\}. \end{aligned} \tag{9}$$

Note that in (9), the forms $\lambda_n w^{\{n,l\}}$, $\lambda_n w^{\{l,m\}}$, $\lambda_n w^{\{m,n\}}$ are obtained from Definition 3.4 for $\mathbf{k} = (1, 0, 0)$, the forms $\lambda_l w^{\{n,l\}}$, $\lambda_l w^{\{l,m\}}$, $\lambda_l w^{\{m,n\}}$ for $\mathbf{k} = (0, 1, 0)$ and $\lambda_m w^{\{n,l\}}$, $\lambda_m w^{\{l,m\}}$, $\lambda_m w^{\{m,n\}}$ for $\mathbf{k} = (0, 0, 1)$, respectively (see Fig. 5 left).

(3) For $k = 2$, the set of multi-indices with 3 components and of weight 2 is

$$\mathcal{I}(3, 2) = \{(2, 0, 0), (0, 2, 0), (0, 0, 2), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}$$

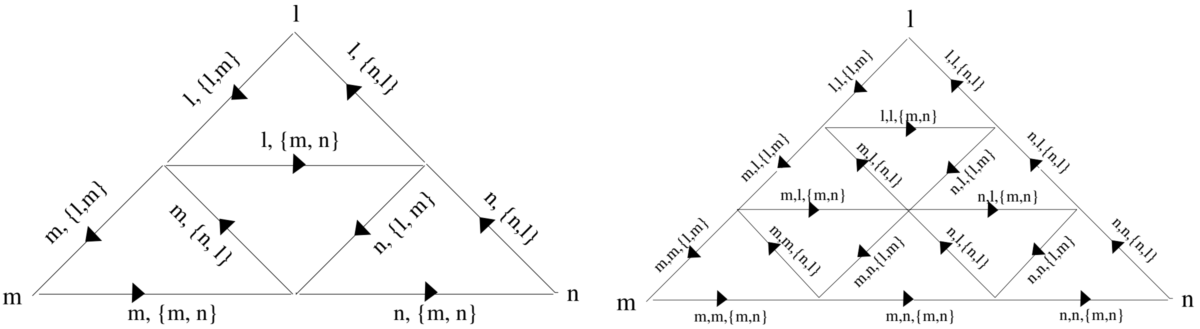

FIGURE 5. Indexing and orientation of the small edges in the triangle $t = \{n, l, m\}$ for the forms spanning $W_2^1(t)$ (left) and $W_3^1(t)$ (right) respectively. The index $n, \{m, n\}$ denotes the small edge $\{\mathbf{k}, \{m, n\}\}$ where $\mathbf{k} = (1, 0, 0)$ and is linked to the form $\lambda_n w^{\{m,n\}}$, *etc.* The index $m, m, \{l, m\}$ denotes the small edge $\{\mathbf{k}, \{l, m\}\}$ with $\mathbf{k} = (2, 0, 0)$ and is linked to the form $\lambda_m^2 w^{\{l,m\}}$, *etc.*

and the space of Whitney edge elements of polynomial degree 3 over t is

$$
\begin{aligned}
W_3^1(t) = \operatorname{span}\{&\lambda_n^2 w^{\{n,l\}},\ \lambda_n^2 w^{\{l,m\}},\ \lambda_n^2 w^{\{m,n\}},\\
&\lambda_l^2 w^{\{n,l\}},\ \lambda_l^2 w^{\{l,m\}},\ \lambda_l^2 w^{\{m,n\}},\\
&\lambda_m^2 w^{\{n,l\}},\ \lambda_m^2 w^{\{l,m\}},\ \lambda_m^2 w^{\{m,n\}},\\
&\lambda_n\lambda_l w^{\{n,l\}},\ \lambda_n\lambda_l w^{\{l,m\}},\ \lambda_n\lambda_l w^{\{m,n\}},\\
&\lambda_n\lambda_m w^{\{n,l\}},\ \lambda_n\lambda_m w^{\{l,m\}},\ \lambda_n\lambda_m w^{\{m,n\}},\\
&\lambda_l\lambda_m w^{\{n,l\}},\ \lambda_l\lambda_m w^{\{l,m\}},\ \lambda_l\lambda_m w^{\{m,n\}}\}.
\end{aligned} \tag{10}
$$

In (10), the forms $\lambda_n^2 w^{\{n,l\}}$, $\lambda_n^2 w^{\{l,m\}}$, $\lambda_n^2 w^{\{m,n\}}$ are obtained from Definition 3.4 for $\mathbf{k} = (2, 0, 0)$, *etc.* (see Fig. 5 right).

The recipe given in Definition 3.4 for Whitney 1-forms of higher polynomial degree over `m` is simple: for W_2^1, attach to edges $E \in \mathcal{E}$ products $\lambda_n w^E$, where n spans $\mathcal{N}$ and $w^E \in W_1^1$. For W_3^1, attach to edges $E \in \mathcal{E}$ products $\lambda_n\lambda_m w^E$, where n, m span $\mathcal{N}$ and $w^E \in W_1^1$, *etc.*

The quantities $\langle \lambda^{\mathbf{k}} w^E, \{\mathbf{k}', E'\}\rangle$, *i.e.*, the circulations of w^e defined in Definition 3.4 along oriented small edges, can be interpreted again as volumes of suitable tetrahedra, as done for $k = 0$. The scaling factor $\int_{\{\mathbf{k}', E'\}} \lambda^{\mathbf{k}}$ is involved, whose dependency on the homothetic parameters of the maps $\tilde{\mathbf{k}}$ and $\tilde{\mathbf{k}}'$ is given in the following.

Let $xy \subset t$ be the oriented segment with vertices x, y, contained in the tetrahedron t, $E \in \mathcal{E}(t)$ and $\mathbf{k} \in \mathcal{I}(d+1, k)$. Then

$$
\langle \lambda^{\mathbf{k}} w^E, xy\rangle = \langle w^E, xy\rangle \int_{xy} \lambda^{\mathbf{k}}. \tag{11}
$$

Identity (11) results from the fact that $\langle w^E, xy\rangle$ represents the weight of the oriented segment xy with respect to the oriented edge $E \in \mathcal{E}(t)$ and does not depend on the point over the line xy (see the second line of (4) in Cor. 2.4). Given the line xy and the edge E, the weight $\langle w^E, xy\rangle$ is a fixed quantity, namely, here comes again the definition of Whitney 1-forms as a tool to describe segments as 1-chains. This is a peculiarity of Whitney forms. Thank to the linearity of the map $\langle \cdot, \cdot\rangle$ and identity (11), we have that if xy is a 1-chain of small or big edges e_i of t, *i.e.*, $xy = \sum c_i e_i$, then

$$
\langle \lambda^{\mathbf{k}} w^E, xy\rangle = \langle \lambda^{\mathbf{k}} w^E, \sum c_i e_i\rangle = \sum c_i \langle \lambda^{\mathbf{k}} w^E, e_i\rangle = \sum c_i \langle w^E, e_i\rangle \int_{e_i} \lambda^{\mathbf{k}}.
$$

The quantity $\int_{e_i} \lambda^{\mathbf{k}}$ can be computed by applying Proposition 3.5 with $d=1$, for any big edge e_i, and Proposition 3.6 with $d=1$, for any small one.

Let us start by Proposition 3.5 that represents a classical result in nodal finite elements.

Proposition 3.5. *The average* $A(d,\mathbf{k})$ *of* $\lambda^{\mathbf{k}}$ *over the tetrahedron* t *is* $\int_t \lambda^{\mathbf{k}} = |t| \frac{k_0!\,k_1!\ldots k_d!\,d!}{(k+d)!}$.

Proof. This result is normally stated on the unit simplex and proved by recurrence of the simplex dimension d, as we recall here. Let t be the unit d-simplex $\{(\lambda_i)_{i=0,d},\, 0 \le \lambda_i \le 1,\, \sum_{i=0}^d \lambda_i = 1\}$. Moreover, $\int_t \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_d = \frac{1}{d!}$. We have that

$$\begin{aligned}
\int_t \lambda^{\mathbf{k}} \mathrm{d}x &= d! \int_{\sum_{i=0}^d \lambda_i \le 1} \Pi_{i=0}^d \lambda_i^{k_i} \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_d \\
&= d \int_{\lambda_d=0}^1 \lambda_d^{k_d} \left((d-1)! \int_{\sum_{i=0}^{d-1} \lambda_i \le 1-\lambda_d} \lambda_0^{k_0} \ldots \lambda_{d-1}^{k_{d-1}} \mathrm{d}\lambda' \right) \mathrm{d}\lambda_d \\
&= d \int_{\lambda_d=0}^1 \lambda_d^{k_d} \left((d-1)! \int_{\sum_{i=0}^{d-1} \lambda_i \le 1-\lambda_d} (1-\lambda_d - \textstyle\sum_{i=1}^{d-1} \lambda_i)^{k_0} \Pi_{i=1}^{d-1} \lambda_i^{k_i} \mathrm{d}\lambda' \right) \mathrm{d}\lambda_d.
\end{aligned}$$

By the change of variable $\lambda' = (\lambda_1, \ldots, \lambda_{d-1}) = (1-\lambda_d)\mu$, being $r = \sum_{i=1}^{d-1} k_i$, we have

$$\begin{aligned}
\int_T \lambda^{\mathbf{k}} \mathrm{d}x &= d \int_{\lambda_d=0}^1 \lambda_d^{k_d} (1-\lambda_d)^{r+k_0+d-1} \mathrm{d}\lambda_d \left((d-1)! \int_{\sum_{i=0}^{d-1} \mu_i \le 1} (1 - \textstyle\sum_{i=1}^{d-1} \mu_i)^{k_0} \Pi_{i=1}^{d-1} \mu_i^{k_i} \mathrm{d}\mu \right) \\
&= d \int_{\lambda_d=0}^1 \lambda_d^{k_d} (1-\lambda_d)^{r+k_0+d-1} \mathrm{d}\lambda_d \left((d-1)! \int_{\sum_{i=0}^{d-1} \mu_i \le 1} \Pi_{i=0}^{d-1} \mu_i^{k_i} \mathrm{d}\mu \right).
\end{aligned}$$

This means that

$$A(d,\mathbf{k}) = d\, B(k_d+1, k'+d-1+1)\, A(d-1,\mathbf{k}')$$

where $\mathbf{k}' = (k_0, \ldots, k_{d-1})$ and B denotes Euler's Beta function, defined by

$$B(m+1,n+1) = \int_0^1 x^m (1-x)^n \mathrm{d}x = m!\,n!/(m+n+1)!.$$

For $k=1$, then $\lambda^{\mathbf{k}}$ is one of the λ_i, $i=0,d$, and $A(d,\mathbf{k}) = \frac{1}{1+d}$. Suppose that the formula is correct for $A(d-1,\mathbf{k}')$ (as recursion hypothesis), then

$$A(d,\mathbf{k}) = d\, \frac{k_d!(k'+d-1)!}{(k+d)!}\, A(d-1,\mathbf{k}') = d\, \frac{k_d!(k'+d-1)!}{(k+d)!} \frac{k_0!\ldots k_{d-1}!(d-1)!}{(k'+d-1)!} = \frac{k_0!\ldots k_d!\,d!}{(k+d)!}$$

being $k'+k_d = k$. □

For Proposition 3.6, we adopt the following notation that generalizes the binomial expansion formula to arrays. Let $\mathbf{x}$, $\mathbf{y}$ be two real arrays and $\mathbf{k}$ an integer array indexed over the set $\mathcal{J} = \{0,1,\ldots,d\}$. Then $(\mathbf{x}+\mathbf{y})^{\mathbf{k}} = \sum_{\mathbf{r}=\mathbf{0}}^{\mathbf{k}} \begin{pmatrix} \mathbf{k} \\ \mathbf{r} \end{pmatrix} \mathbf{x}^{\mathbf{k}-\mathbf{r}} \mathbf{y}^{\mathbf{r}}$. Indeed, when both x and y are real, we have $(x+y)^k = \sum_{r=0}^k \begin{pmatrix} k \\ r \end{pmatrix} x^{k-r} y^r$ for all integers k. We now consider two real arrays $\mathbf{x}$, $\mathbf{y}$, and an integer array $\mathbf{k}$ indexed over the set $\mathcal{J} = \{0,1,\ldots,d\}$. With our conventions,

$$(\mathbf{x}+\mathbf{y})^{\mathbf{k}} = \Pi_{j\in\mathcal{J}} (x_j+y_j)^{k_j} = \Pi_{j\in\mathcal{J}} \sum_{r_j=0}^{k_j} \begin{pmatrix} k_j \\ r_j \end{pmatrix} x_j^{k_j-r_j} y_j^{r_j},$$

which prompts us to introduce the notation

$$\begin{pmatrix} \mathbf{k} \\ \mathbf{r} \end{pmatrix} = \Pi_{j\in\mathcal{J}} \begin{pmatrix} k_j \\ r_j \end{pmatrix} = \Pi_{j\in\mathcal{J}} \frac{k_j!}{(k_j-r_j)!\,r_j!} = \frac{\Pi_{j\in\mathcal{J}} k_j!}{\Pi_{j\in\mathcal{J}} (k_j-r_j)!\,\Pi_{j\in\mathcal{J}} r_j!}$$

which we may as well write $\left(\begin{array}{c}\mathbf{k}\\ \mathbf{r}\end{array}\right) = \frac{\mathbf{k}!}{(\mathbf{k}-\mathbf{r})!\,\mathbf{r}!}$ by introducing the natural convention $\mathbf{k}! = \Pi_{j\in\mathcal{J}} k_j!$ for an $\mathcal{J}$-indexed multi-integer. By distributivity, we get the generalized formula.

Proposition 3.6. *The average of* $\lambda^{\mathbf{k}}$, $\mathbf{k}\in\mathcal{I}(d+1,k)$, *over a small volume* $\{\mathbf{k}',t\}$, $\mathbf{k}'\in\mathcal{I}(d+1,k')$, *is*

$$I = \int_{\{\mathbf{k}',t\}} \lambda^{\mathbf{k}} = \frac{1}{(1+k')^{d+k}} \sum_{\mathbf{r}=\mathbf{0}}^{\mathbf{k}} \left(\begin{array}{c}\mathbf{k}\\ \mathbf{r}\end{array}\right) (\mathbf{k}')^{\mathbf{k}-\mathbf{r}} A(d,\mathbf{r}).$$

Proof. Let us suppose that t is the unit d-simplex. We recall that the small volume $\{\mathbf{k}',t\}$ is homothetic to t through the homothety $s=h(x)$ with ration $\frac{1}{1+k'}$ and center the point $\frac{\mathbf{k}'}{k'}$. By definition of homothety, we have

$$(h(x))_i = \frac{k_i'}{k'} + \frac{1}{1+k'}\left(x_i - \frac{k_i'}{k'}\right) = \frac{1}{1+k'}\left(x_i + k_i'\right), \qquad 1\le i\le d.$$

So that

$$I = \int_{\{\mathbf{k}',t\}} \lambda^{\mathbf{k}}(s)\mathrm{d}s = \int_t \lambda^{\mathbf{k}}(h(x))|jac(h(x))|\mathrm{d}x = \frac{1}{(1+k')^d}\int_t \lambda^{\mathbf{k}}(h(x))\mathrm{d}x.$$

In terms of barycentric coordinates (we are in the unit d-simplex), we can write

$$\lambda_i(h(x)) = \frac{1}{1+k'}(x_i+k_i'), \qquad 1\le i\le d, \qquad\qquad \lambda_0(h(x)) = 1 - \sum_{i=1}^d \lambda_i(h(x)) = \frac{1}{1+k'}\left(1-\sum_{i=1}^d x_i\right).$$

Then

$$I = \frac{1}{(1+k')^d}\int_t \Pi_{i=0}^d \lambda_i^{k_i}(h(x))\mathrm{d}x = \frac{1}{(1+k')^{d+k}}\int_t \left(1-\sum_{i=1}^d x_i\right)^{k_0} \Pi_{i=1}^d (x_i+k_i')^{k_i}\mathrm{d}x.$$

We can write

$$\begin{aligned} I &= \frac{1}{(1+k')^{d+k}}\int_t \left(1-\sum_{i=1}^d x_i\right)^{k_0} (\mathbf{k}'+\mathbf{x})^{\mathbf{k}}\mathrm{d}x = \frac{1}{(1+k')^{d+k}}\int_t \left(1-\sum_{i=1}^d x_i\right)^{k_0} \sum_{\mathbf{r}=\mathbf{0}}^{\mathbf{k}} \left(\begin{array}{c}\mathbf{k}\\ \mathbf{r}\end{array}\right)(\mathbf{k}')^{\mathbf{k}-\mathbf{r}}\mathbf{x}^{\mathbf{r}}\mathrm{d}x \\ &= \frac{1}{(1+k')^{d+k}} \sum_{\mathbf{r}=\mathbf{0}}^{\mathbf{k}} \left(\begin{array}{c}\mathbf{k}\\ \mathbf{r}\end{array}\right)(\mathbf{k}')^{\mathbf{k}-\mathbf{r}} \int_t (1-\sum_{i=1}^d x_i)^{k_0}\mathbf{x}^{\mathbf{r}}\mathrm{d}x = \frac{1}{(1+k')^{d+k}} \sum_{\mathbf{r}=\mathbf{0}}^{\mathbf{k}} \left(\begin{array}{c}\mathbf{k}\\ \mathbf{r}\end{array}\right)(\mathbf{k}')^{\mathbf{k}-\mathbf{r}} \int_t \lambda^{\mathbf{r}}\mathrm{d}x. \end{aligned}$$ □

3.3. Properties and remarks

We now state some properties of the 1-forms of Definition 3.4. A straightforward generalization of Proposition 2.6 for Whitney edge elements of degree one, yields the following property.

Proposition 3.7. *The* 1*-forms of Definition* 3.4 *constitute a partition of unity.*

Thank to Proposition 2.8, we have the following property.

Proposition 3.8. *Functions in* W_{k+1}^1 *are curl-conforming.*

Proof. Elements in W_{k+1}^1 are defined as products between an element in W_1^1 and the continuous function $\lambda^{\mathbf{k}}$, $\mathbf{k}\in\mathcal{I}(d+1,k)$. □

The main difficulty with the 1-forms of Definition 3.4 is recalled in the following proposition.

Proposition 3.9. *The forms* w^e *of Definition* 3.4 *are generators of* W_{k+1}^1 *but not linearly independent.*

Table 1. The elements of the matrix $A_e^{e'}$ over the triangle $\{n, l, m\}$ for $k = 0$.

$A_e^{e'}$	$\{n,l\}$	$\{l,m\}$	$\{m,n\}$
$w^{\{n,l\}}$	1	0	0
$w^{\{l,m\}}$	0	1	0
$w^{\{m,n\}}$	0	0	1

Table 2. The elements of the matrix $A_{\{\mathbf{k},E\}}^{\{\mathbf{k}',E'\}}$ multiplied by 16 over the triangle $\{n, l, m\}$ for $k = 1$ (*cf.* Fig. 5 left for the small edge indexing and orientation). This 9×9 matrix is not regular, it has rank 8 and the vector $(4, 1, 1, 1, 4, 1, 1, 1, 4)$ generates its kernel.

$A_{\{\mathbf{k},E\}}^{\{\mathbf{k}',E'\}}$	$m, \{n,l\}$	$n, \{n,l\}$	$l, \{n,l\}$	$m, \{l,m\}$	$n, \{l,m\}$	$l, \{l,m\}$	$m, \{m,n\}!$	$n, \{m,n\}$	$l, \{m,n\}$
$\lambda^m w^{\{n,l\}}$	2	0	0	0	−1	0	0	0	−1
$\lambda^n w^{\{n,l\}}$	1	6	2	0	−2	0	0	0	−1
$\lambda^l w^{\{n,l\}}$	1	2	6	0	−1	0	0	0	−2
$\lambda^m w^{\{l,m\}}$	−2	0	0	6	1	2	0	0	−1
$\lambda^n w^{\{l,m\}}$	−1	0	0	0	2	0	0	0	−1
$\lambda^l w^{\{l,m\}}$	−1	0	0	2	1	6	0	0	−2
$\lambda^m w^{\{m,n\}}$	−2	0	0	0	−1	0	6	2	1
$\lambda^n w^{\{m,n\}}$	−1	0	0	0	−2	0	2	6	1
$\lambda^l w^{\{m,n\}}$	−1	0	0	0	−1	0	0	0	2

With standard edge elements, the dof $\mathbf{v}_e$ was the integral of the 1-form $\sum_{e'} \mathbf{v}_{e'} w^{e'}$ over edge e. In other words, the square matrix $A_e^{e'} = \langle w^e, e' \rangle$ is the identity (*cf.* Tab. 1 for an example in two dimensions). This means that edges and 1-forms are in duality. To sum up, with edge elements of the lowest degree, dofs are localized over mesh edges and represent field circulations along them.

With the higher order forms defined in Definition 3.4, we cannot expect to find a family of edges, simple 1-chains, such that each $\mathbf{v}_{\mathbf{k}E}$ would be the integral of $\sum_{\mathbf{k},E} \mathbf{v}_{\mathbf{k}E} \lambda^{\mathbf{k}}\, w^E$ over one of them, *and have a null integral over all other chains* of the family, *i.e.*, a family of 1-chains in duality with the $\lambda^{\mathbf{k}}\, w^E$. This fact makes the interpretation of dofs difficult with such forms. The most natural candidates, *i.e.*, the "small edges" $\{\mathbf{k}, E\}$, fail because the square matrix $A_{\{\mathbf{k},E\}}^{\{\mathbf{k}',E'\}} = \langle \lambda^{\mathbf{k}}\, w^E, \{\mathbf{k}', E'\} \rangle$ is not the identity (*cf.* Tab. 2 for an example in two dimensions with $k = 1$). Moreover, the matrix $A_{\{\mathbf{k},E\}}^{\{\mathbf{k}',E'\}}$ is not regular, we cannot invert it to find out another family of chains, linear combination of the $\{\mathbf{k}, E\}$ in duality with $\lambda^{\mathbf{k}}\, w^E$. We must be content with less: 1-cells such that integrals over them of $\sum_{\mathbf{k},E} \mathbf{v}_{\mathbf{k}E} \lambda^{\mathbf{k}}\, w^E$ determine the $\mathbf{v}_{\mathbf{k}E}$ and in clear 1-to-1 correspondence with the forms $\lambda^{\mathbf{k}}\, w^E$. To sum up, with edge elements of the higher degree, dofs are localized over 1-chains composed of several (small) edges and represent field circulations along them.

Actually, to obtain a set of linearly independent forms for the space $W^1_{k+1}(t)$, one can eliminate the redundant ones, *i.e.*, one form for each $\tilde{\mathbf{k}}(t)$ not homothetic to t (and the corresponding small edge). As an example, for the space defined in (9), we can neglect the form $\lambda_l w^{\{m,n\}}$, and for that defined in (10), the three forms $\lambda_n^2 w^{\{l,m\}}$, $\lambda_n \lambda_l w^{\{l,m\}}$, $\lambda_n \lambda_m w^{\{l,m\}}$. Numerical results are independent of the selection procedure.

Remark 3.10 (*Geometrical interpretation).* In this paper we have shown that high order 1-forms are indeed associated with geometric figures of dimension 1, even though this association is not as obvious as in the case of Whitney 1-forms of degree one. Whatever the polynomial degree, dofs are paired with 1-chains composed of small edges. Let us analyze in detail the basis we found out for $W^1_{k+1}(t)$ giving a geometrical interpretation with respect to the geometry of the tetrahedron $t = \{n, l, m, i\}$ we consider.

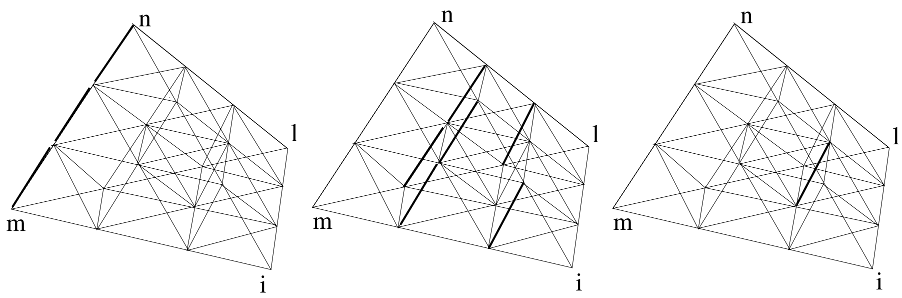

FIGURE 6. In light line, the refinement of t associated to $k=2$ and, in thick line, the (three, six and one) small edges to list the functions $\lambda_r\lambda_s w^{\{m,n\}}$ in $\mathcal{B}_e$ (left), $\mathcal{B}_f$ (center) and $\mathcal{B}_v$ (right), respectively.

The set $\mathcal{B}$ of basis functions for $W^1_{k+1}(t)$ can be partitioned in three subsets, namely $\mathcal{B}_e$, $\mathcal{B}_f$ and $\mathcal{B}_v$. Note that $\mathcal{B}_f$ is empty for $k=0$ and $\mathcal{B}_v$ is empty for $k=0,1$. So, we assume that $k=2$, *i.e.*, the minimum value for k for which the three subsets $\mathcal{B}_e$, $\mathcal{B}_f$ and $\mathcal{B}_v$ are not empty.

$$\mathcal{B}_e = \bigcup_{\{n,m\}\in\mathcal{E}(t)}\{\lambda_r\,\lambda_s\,w^{\{n,m\}},\quad \text{both } r \text{ and } s \text{ equal to } n \text{ or } m\},\qquad \#\mathcal{B}_e = 18$$
$$\mathcal{B}_v = \bigcup_{\{n,m\}\in\mathcal{E}(t)}\{\lambda_r\,\lambda_s\,w^{\{n,m\}},\quad \text{both } r \text{ and } s \text{ different from } n \text{ and } m\},\qquad \#\mathcal{B}_v = 6,$$
$$\mathcal{B}_f = \bigcup_{\{n,m\}\in\mathcal{E}(t)}\{\lambda_r\,\lambda_s\,w^{\{n,m\}} \notin \mathcal{B}_e\cup\mathcal{B}_v\},\qquad \#\mathcal{B}_f = 36.$$

Functions in $\mathcal{B}$ verify 15 relations, three for face plus three for the internal tetrahedron not similar to t (*cf.* Fig. 4 center).

The set $\mathcal{B}_v$ gives "interior" basis functions in the sense that the moments on the (big) edges and faces (see [18], Def. 4) are zero for these functions. Indeed, functions in $\mathcal{B}_v$ can be expressed as a difference of two terms each of which has a factor of the form $\lambda_k\lambda_l\lambda_n\mathbf{grad}\lambda_m$. This factor is zero on faces containing the vertex m, while on face $\{k,l,n\}$, its tangential component is zero due to the presence of $\mathbf{grad}\lambda_m$. Hence, functions in $\mathcal{B}_v$ have zero face and edge moments.

The set $\mathcal{B}_f$ gives "face" basis functions in the sense that these functions have edge dofs equal to zero. Indeed, these functions are of the form $\lambda_n^\alpha\lambda_m^\beta\lambda_l^\gamma(\lambda_l\,w^{\{m,n\}})$ for some nonnegative integer powers α, β and γ with $\alpha+\beta+\gamma=1$. It is easily verified that the tangential components of the function $\lambda_l\,w^{\{m,n\}}$ is zero on all faces except face $\{l,m,n\}$. Its tangential component is zero on all edges of the tetrahedron.

The remaining functions, those in $\mathcal{B}_e$, are "edge" basis functions. These functions take the form $\lambda_n^\alpha\lambda_m^\beta w^{\{m,n\}}$. Their tangential component is nonzero on the edge $\{m,n\}$ but vanishes on every other edge of the tetrahedron.

The small edges to build up functions $\lambda_r\lambda_s w^{\{m,n\}}$ in $\mathcal{B}_e$, $\mathcal{B}_f$ and $\mathcal{B}_v$ respectively, are drawn in Figure 6. We recall that the edge basis functions we proposed may not have zero face dofs, and the face basis functions may not have zero interior moments. Figure 6 is just a visualization help to list all the basis functions contained in $\mathcal{B}$. Thank to Figure 6, it is easy to see that, for $k=2$, three functions in $\mathcal{B}_e$ are $\lambda_n^2 w^{\{m,n\}}$, $\lambda_n\lambda_m w^{\{m,n\}}$, $\lambda_m^2 w^{\{m,n\}}$, six functions in $\mathcal{B}_f$ are $\lambda_n\lambda_l w^{\{m,n\}}$, $\lambda_m\lambda_l w^{\{m,n\}}$, $\lambda_l^2 w^{\{m,n\}}$, $\lambda_n\lambda_k w^{\{m,n\}}$, $\lambda_m\lambda_k w^{\{m,n\}}$, $\lambda_k^2 w^{\{m,n\}}$ and one function in $\mathcal{B}_v$ is $\lambda_k\lambda_l w^{\{m,n\}}$ (*cf.* Tab. 3 for the list of basis functions in affine coordinates for $W^1_{k+1}(t)$, $k=0,1,2$).

Remark 3.11 (*Other bases)***.** Edge element basis functions of Nédélec type are distinguished by two important properties, namely, their curl-conforming nature and their reduced number of dofs for a fixed order of approximation. Given a basis that is complete to polynomial order $k+1$, the accuracy of the overall solution is limited by the terms involving the curl of the basis, which can only be of complete polynomial order k. Hence, those terms of exact order $k+1$, and whose curls vanish, neither contribute to the modeling of the curl nor to the overall accuracy of the solution. Hence, these unnecessary dofs can be discarded in forming curl-conforming bases.

TABLE 3. Basis for Whitney edge element spaces on a tetrahedron $\{n, l, m, i\}$ in barycentric coordinates for some degrees.

Degree	Interior	Basis functions face $\{n, l, m\}$	edge $\{l, m\}$
1	none	none	$w^{\{l,m\}}$
2	none	$\lambda_m w^{\{n,l\}}$ $\lambda_n w^{\{l,m\}}$	$\lambda_l w^{\{l,m\}}$ $\lambda_m w^{\{l,m\}}$
3	$\lambda_m \lambda_l w^{\{n,i\}}$ $\lambda_m \lambda_i w^{\{n,l\}}$ $\lambda_i \lambda_l w^{\{m,n\}}$	$\lambda_m \lambda_n w^{\{n,l\}}$ $\lambda_m^2 w^{\{n,l\}}$ $\lambda_m \lambda_l w^{\{n,l\}}$ $\lambda_l^2 w^{\{m,n\}}$ $\lambda_m \lambda_l w^{\{m,n\}}$ $\lambda_n \lambda_l w^{\{m,n\}}$	$\lambda_l^2 w^{\{l,m\}}$ $\lambda_m \lambda_l w^{\{l,m\}}$ $\lambda_m^2 w^{\{l,m\}}$

As their construction makes clear, a feature of Nédélec curl-conforming bases is that the bases of polynomial order $k+1$ are complete only to the same order as their curl, that is k. Then, a Nédélec family of order k can be simply generated as the product of bases of the Nédélec type of order zero and a complete scalar polynomial of degree k. The set of polynomial factors used may take one of several different forms chosen for convenience (*cf.* [12,20] for an overview); since all are complete, they span the same space and are merely linear combination of one another. In this sense they are equivalent. Among several possibilities, the polynomial factor of degree k may be of homogeneous form, as in our case. A numerical comparison with respect to other bases presented in the literature has still to be done (see [20] where computationally effective high order edge elements are presented).

Remark 3.12 (*Error estimates*)**.** Let h be the maximal diameter of the elements in the mesh, $I_{h,k}$ the interpolation operator over W_k^1 and $||\cdot||$ the norm defined in W_k^1. While we can analyze the interpolation error $||u - I_{h,k}u||$ in the mesh elements for a fixed polynomial degree k as the mesh is refined, *i.e.*, $h \to 0$ (*cf.* [17,18]), there is no analysis yet that provides error estimates including the influence of the polynomial degree k. We would like that a result such as

$$||u - I_{h,k}u|| = O(h^{\mu-1}k^{1-r}) \tag{12}$$

where $\mu = \min(r, k+1)$ and r is the regularity of the function u, which is valid for H^1-conforming hk elements also holds for hk curl-conforming elements. If the function u is smooth enough to have bounded derivatives such that $r \geq k+1$, then estimate (12) states that we can achieve super-algebraic convergence as we increase the polynomial order k and algebraic convergence as we decrease the mesh element size h. The results of preliminary numerical tests done in 2D are presented in the following section and are in agreement with estimate (12).

There is also another way to understand the dependency on h. Whitney forms are best viewed as a device to represent manifolds by simplicial chains, here the manifolds being the mesh edges and the simplicial chains being the "small edges". The representation gets better and better as the "small edges" get smaller and smaller, and, by duality, we improve the approximation of the differential form associated to the manifold.

4. Numerical results and conclusions

We wish to give a simple concrete application of the notions presented in the previous sections. In particular, we shall solve the model problem

$$a\,\mathbf{u} + \operatorname{curl}(b \operatorname{curl} \mathbf{u}) = \mathbf{f} \quad \text{in } \Omega, \quad \mathbf{u} \cdot \mathbf{t} = 0 \quad \text{on } \partial\Omega, \tag{13}$$

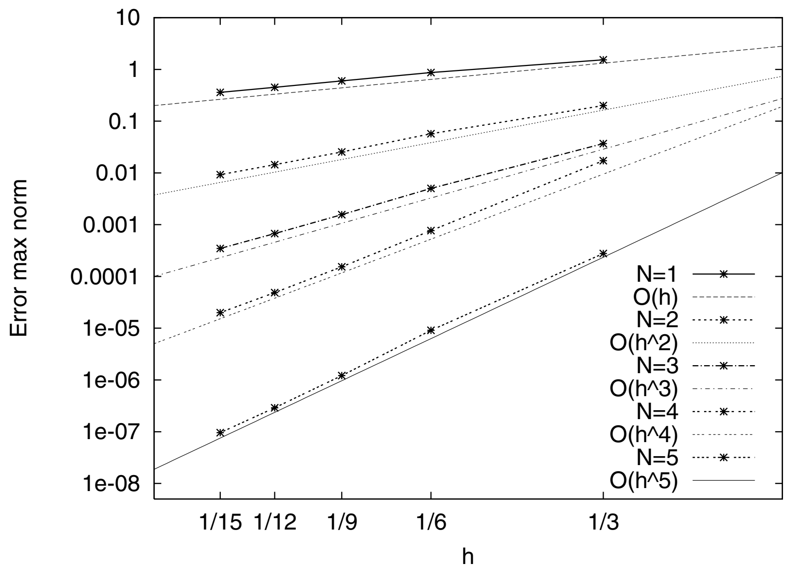

FIGURE 7. h-convergence of the approximation error.

for some parameters $a, b > 0$ and Ω a bounded polygonal domain in $\mathbb{R}^2$.

We report the results of numerical experiments for the conjugate gradient applied to the linear system obtained by discretizing the model problem (13) with triangular Whitney edge elements of higher order. The computational domain is the rectangle $\Omega = [0.5, 1.5] \times [0.25, 0.75]$ and the body force $\mathbf{f}$ is consistent with

$$\mathbf{u} = \begin{pmatrix} 2\,\pi\,\sin(\pi\,x)\,\cos(2\,\pi\,y) \\ -\pi\,\cos(\pi\,x)\,\sin(2\,\pi\,y) \end{pmatrix}$$

as exact solution of (13). The triangular mesh is simply obtained by first dividing Ω into J^2 rectangles and second by dividing each of them in two triangles. As a results, the number of triangles is $2J^2$. Computations have been made for different values of the total polynomial approximation degree, *i.e.*, $k+1 = N = \{1, 2, 3, 4, 5\}$ and for a number of triangles obtained with $J = \{3, 6, 9, 12, 15\}$.

As stated by Proposition 3.9, the high order edge element functions we consider are not linearly independent and a selection procedure must be specified in order to get a basis for W^1_{k+1}. For the numerical results presented in this section, given the mesh triangle $t = \{n, l, m\}$, we have neglected all functions associated to internal small edges parallel to $\{l, m\}$. For the degrees $\{1, 2, 3, 4, 5\}$, we have discarded, respectively, $0, 1, 3, 6, 10$ functions in each mesh triangle, one function for each $\tilde{\mathbf{k}}(t)$ not homothetic to t, $\mathbf{k} \in \mathcal{I}(3, k)$. Numerical results have shown to be independent of the selection procedure. The evaluation of the quantities $\langle \lambda^{\mathbf{k}}\, w^E, \{\mathbf{k}', E'\}\rangle$, for $\mathbf{k}, \mathbf{k}' \in \mathcal{I}(d+1, k)$ and $E, E' \in \mathcal{E}$, *i.e.*, the circulations of w^e defined in Definition 3.4 along oriented small edges, is done numerically by suitable high order Gauss-Legendre quadrature formulas. For the adopted non-preconditioned conjugate gradient (CG) method, the initial guess is zero and the stopping criterion is $||\mathbf{r}^{(\nu)}||/||\mathbf{r}^{(0)}|| \le 10^{-6}$, where $\mathbf{r}^{(\nu)}$ is the νth residual.

Figure 7 shows the log-plots of the error for the considered choices of $N = k+1$. As expected the convergence to the exact solution is of algebraic type and achieved with an order of accuracy equal to N with respect to h. Figure 8 shows the semi log-plots of the error for the considered choices of h. A super-algebraic convergence is achieved with respect to N.

One important aspect to consider is the condition number of the system matrix which results from the edge element approximation of problem (13). Figure 9 shows the condition number in log-scaling (as computed by

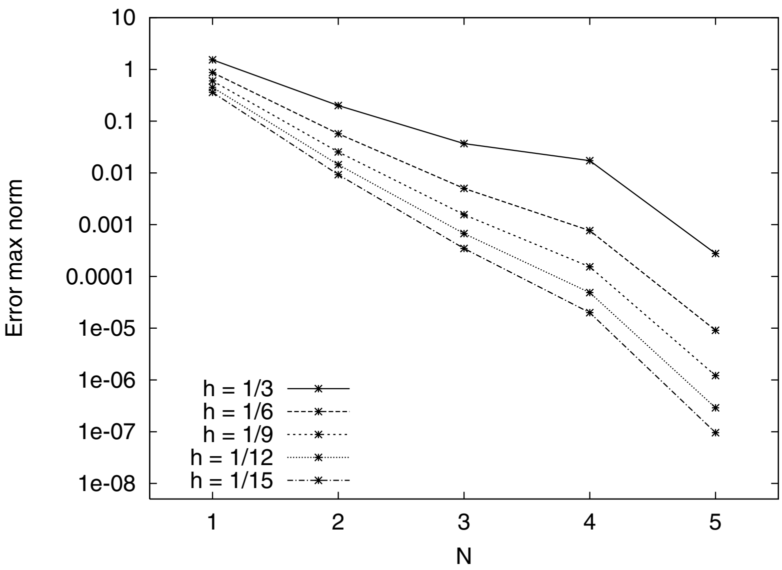

FIGURE 8. N-convergence of the approximation error.

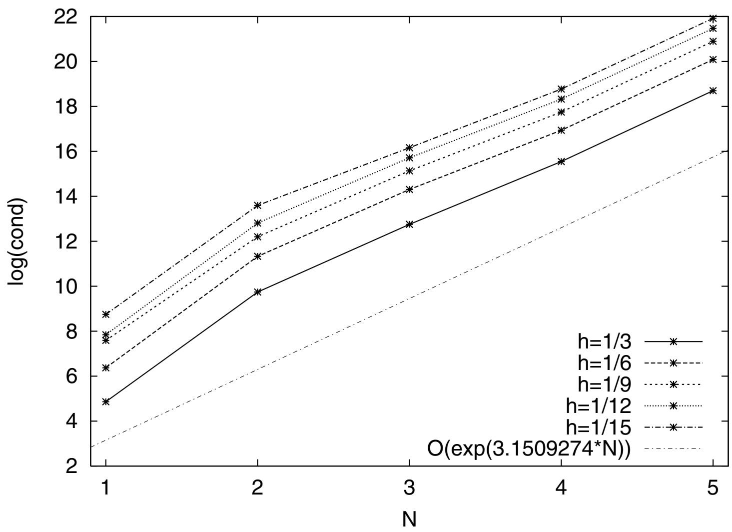

FIGURE 9. N-convergence of the condition number of the system matrix obtained after discretization of problem (13) by Whitney high order edge elements.

the routine DGESVX of the LAPACK library) *versus* the adopted degree N for different values of h. Unluckily, we find an exponential dependency of the condition number with respect to N. We have an asymptotic behavior of the form $c\,e^{\alpha\,N}$ for the condition number presented in Figure 9, with, *e.g.*, $c = 6.3873278$ and $\alpha = 3.1509274$ for $h = 1/15$. The proposed basis does not prove to be well conditioned with respect to the polynomial degree N and some preconditioning should be attempted. Figure 10 shows the log-plot of the inverse of the condition number *versus* the mesh size for different values of N. Clearly the condition number shows a $O(1/h^2)$ behavior, in agreement with classical theoretical results [16].

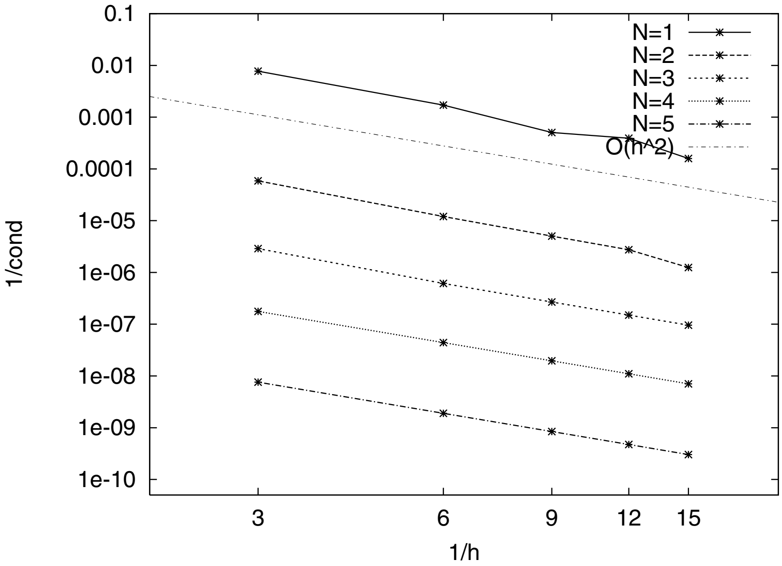

FIGURE 10. h-convergence of the condition number of the system matrix obtained after discretization of problem (13) by Whitney high order edge elements.

To conclude, this work deals with the definition of shape functions for high order edge finite element spaces. The case of lowest order is well understood in the literature and dofs are generally associated with suitable moments along edges of the triangulation. This is where the name "edge" elements originates. When moving to higher orders, the situation is somehow more complicated and various possible definitions of shape functions have been introduced in the literature. Such definitions generally involve dofs which do not correspond to moments along edges (like face or volume moments). Here, the main goal consists in designing shape functions which use only edge dofs. This task is performed with the introduction of so-called "small edges" which are defined by means of a particular homothety. Simple numerical tests have been presented which aim at showing that the proposed shape functions can be used in practice. Optimal error estimates are numerically proved and the condition number of the resulting algebraic system is computed as a function of h and N.

As a final remark, we remind how important it was, back in the 80 s when edge elements began to be used, to realize that gradients of nodal scalar functions were included in the span of edge elements. But this all important inclusion property is only a part of a larger frame: the "exact sequence" property of Whitney forms. Hence higher degree Whitney forms, whatever they are, must keep this property (we refer to [9] for more details). Here we have shown that this can be achieved without forfeiting another property of Whitney 1-forms, which also contributed to the popularity of edge elements: the natural association of dofs with geometric mesh elements of dimension 1, namely the "small edges" discussed here.

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