RESIDUAL *A POSTERIORI* ERROR ESTIMATORS FOR CONTACT PROBLEMS IN ELASTICITY

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Abstract. This paper is concerned with the unilateral contact problem in linear elasticity. We define two *a posteriori* error estimators of residual type to evaluate the accuracy of the mixed finite element approximation of the contact problem. Upper and lower bounds of the discretization error are proved for both estimators and several computations are performed to illustrate the theoretical results.

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1. INTRODUCTION

The finite element method is currently used in the numerical realization of contact problems occurring in several engineering applications (see [18, 19, 24, 25, 30]). An important task consists of evaluating numerically the quality of the finite element computations by using *a posteriori* error estimators. There are two important difficulties in developing such tools for contact problems in elasticity: the first one comes from the inequality (unilateral) conditions in the model and the second one is due to the location of these inequality conditions which hold on (a part of) the boundary. For the linear elasticity system with standard boundary conditions (leading to a variational identity), many different approaches leading to various error estimators have been developed and a review of the different *a posteriori* error estimators can be found in [29]. Some of these approaches have been chosen and studied for frictionless or frictional unilateral contact problems, in particular in [7, 26, 31] (residual approach using a penalization of the contact condition or the normal compliance law), in [10, 11] (equilibrated residual method) and finally in [14] (residual approach for BEM-discretizations).

Besides, let us mention that many studies dealing with residual estimators for scalar variational inequality problems of the first kind have been achieved in other contexts than elasticity. In particular a great effort was devoted to the obstacle problem (see *e.g.*, [6,8,27] and the references therein). Moreover a residual type estimator for the Signorini problem in its standard formulation can be found in [21] (note that the Signorini problem could be seen as a simplification in the scalar case of the unilateral contact model). For residual estimators dealing with variational inequalities of the second kind we refer the reader to [5] and the references therein. We recall that a variational inequality of the first (resp. second) kind is of the form: $u \in C, a(u, v - u) \geq L(v - u), \forall v \in C$ (resp. $u \in X, a(u, v - u) + j(v) - j(u) \geq L(v - u), \forall v \in X$) where X is an Hilbert space, $C \subset X$ is a nonempty closed convex set, $a(.,.)$ is bilinear, X-elliptic and continuous on $X \times X$, $L(.)$ is linear and continuous on X, $j(.)$ is proper convex and lower semi continuous on X (with values in $\mathbb{R} \cup \{+\infty\}$). More details concerning variational inequalities of the first or second kind can be found in *e.g.*, [2,17].

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In the present work we are interested in developing residual estimators for the two-dimensional unilateral contact model in linear elasticity. This problem can be written as a variational inequality of the first kind but also (among others) as a mixed formulation where the unknowns are the displacement field and the contact pressure.

The paper is organized as follows. In Section 2 we introduce the equations modelling the frictionless unilateral contact problem between an elastic body and a rigid foundation. We write the problem using a mixed formulation where the unknowns are the displacement field in the body and the pressure on the contact area. In the third section, we choose a classical discretization involving continuous finite elements of degree one and continuous piecewise affine multipliers on the contact zone. Section 4 is concerned with the study of a first residual estimator which can be seen as the natural one arising from the discrete problem. We obtain both global upper and local lower bounds of the error. In Section 5 we consider a second estimator resulting from another discrete model where the displacement field is the same as in the first model but where the multiplier is modified. The main novelty of the second discrete model is that the multipliers have a constant sign. As in Section 4, we obtain global upper and local lower bounds of the error. Finally in Section 6 we implement both estimators and we compare them on several examples.

Finally we introduce some useful notation and several functional spaces. In what follows, bold letters like $\mathbf{u}, \mathbf{v}$, indicate vector valued quantities, while the capital ones (*e.g.*, $\mathbf{V}, \mathbf{K}, \ldots$) represent functional sets involving vector fields.

As usual, we denote by $(L^2(.))^d$ and by $(H^s(.))^d$, $s \geq 0, d = 1, 2$ the Lebesgue and Sobolev spaces in one and two space dimensions (see [1]). The usual norm of $(H^s(D))^d$ is denoted by $\| \cdot \|_{s,D}$ and we keep the same notation when $d = 1$ or $d = 2$. For shortness the $(L^2(D))^d$-norm will be denoted by $\| \cdot \|_D$ when $d = 1$ or $d = 2$. In the sequel the symbol $| \cdot |$ will denote either the Euclidean norm in $\mathbb{R}^2$, or the length of a line segment, or the area of a plane domain. Finally the notation $a \lesssim b$ means here and below that there exists a positive constant C independent of a and b (and of the meshsize of the triangulation) such that $a \leq C\, b$. The notation $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$ hold simultaneously.

2. The unilateral contact problem in elasticity

Let Ω represent an elastic body in $\mathbb{R}^2$ where plane strain assumptions are assumed. The boundary $\partial\Omega$ is supposed to be polygonal, *i.e.*, it is the union of a finite number of linear segments. Moreover we suppose that the boundary consists in three nonoverlapping parts Γ_D, Γ_N and Γ_C with meas$(\Gamma_D) > 0$ and meas$(\Gamma_C) > 0$. The normal unit outward vector on $\partial\Omega$ is denoted $\mathbf{n} = (n_1, n_2)$ and we choose as unit tangential vector $\mathbf{t} = (-n_2, n_1)$. In its initial configuration, the body is in contact on Γ_C and we suppose that the unknown final contact zone after deformation will be included in Γ_C. The body is clamped on Γ_D for the sake of simplicity. It is subjected to volume forces $\mathbf{f} = (f_1, f_2) \in (L^2(\Omega))^2$ and to surface forces $\mathbf{g} = (g_1, g_2) \in (L^2(\Gamma_N))^2$.

The unilateral contact problem in elasticity consists in finding the displacement field $\mathbf{u} : \Omega \to \mathbb{R}^2$ verifying the equations and conditions (1)–(6):

$$\mathbf{div}\, \boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f} = \mathbf{0} \qquad \text{in } \Omega, \tag{1}$$

where $\mathbf{div}$ denotes the divergence operator of tensor valued functions and $\boldsymbol{\sigma} = (\sigma_{ij})$, $1 \leq i,\, j \leq 2$, stands for the stress tensor field. The latter is obtained from the displacement field by the constitutive law of linear elasticity

$$\boldsymbol{\sigma}(\mathbf{u}) = \mathbf{A}\boldsymbol{\varepsilon}(\mathbf{u}) \qquad \text{in } \Omega, \tag{2}$$

where $\mathbf{A}$ is a fourth order symmetric and elliptic tensor and $\boldsymbol{\varepsilon}(\mathbf{v}) = (\nabla\mathbf{v} +^t \nabla\mathbf{v})/2$ represents the linearized strain tensor field. On Γ_D and Γ_N, the conditions are as follows:

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma_D, \tag{3}$$

$$\boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \mathbf{g} \qquad \text{on } \Gamma_N. \tag{4}$$

Afterwards we choose the following notation for any displacement field $\mathbf{v}$ and for any density of surface forces $\boldsymbol{\sigma}(\mathbf{v})\mathbf{n}$ defined on $\partial\Omega$:

$$\mathbf{v} = v_n\mathbf{n} + v_t\mathbf{t} \quad \text{and} \quad \boldsymbol{\sigma}(\mathbf{v})\mathbf{n} = \sigma_n(\mathbf{v})\mathbf{n} + \sigma_t(\mathbf{v})\mathbf{t}.$$

The conditions modelling unilateral contact on Γ_C are (see *e.g.*, [13, 15, 16]):

$$u_n \leq 0, \qquad \sigma_n(\mathbf{u}) \leq 0, \qquad \sigma_n(\mathbf{u})\, u_n = 0. \tag{5}$$

Finally the condition

$$\sigma_t(\mathbf{u}) = 0 \tag{6}$$

on Γ_C means that friction is omitted.

In order to derive the variational formulation of (1)–(6), we consider the Hilbert space

$$H^1_{\Gamma_D}(\Omega) = \left\{ v \in H^1(\Omega) : \quad v = 0 \text{ on } \Gamma_D \right\},$$

equipped with the $H^1(\Omega)$-norm. We further use the Hilbert space

$$\mathbf{V} = (H^1_{\Gamma_D}(\Omega))^2.$$

For our next uses, we introduce the trace space $H^{\frac{1}{2}}(\Gamma_C)$ as follows:

$$H^{\frac{1}{2}}(\Gamma_C) = \left\{ \phi \in L^2(\Gamma_C) : \exists u \in H^1_{\Gamma_D}(\Omega) \text{ such that } \phi = \gamma u \text{ on } \Gamma_C \right\},$$

equipped with the norm

$$\|\phi\|_{\frac{1}{2},\Gamma_C} = \inf_{u \in H^1_{\Gamma_D}(\Omega):\phi=\gamma u} \|u\|_{1,\Omega},$$

where γ is the standard trace operator from $H^1(\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$ (see [1]). The topological dual space of $H^{\frac{1}{2}}(\Gamma_C)$ will be denoted by $H^{-\frac{1}{2}}(\Gamma_C)$, whose norm is

$$\|\psi\|_{-\frac{1}{2},\Gamma_C} = \sup_{\phi \in H^{\frac{1}{2}}(\Gamma_C)} \frac{\langle \psi, \phi \rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_C}}{\|\phi\|_{\frac{1}{2},\Gamma_C}},$$

where the notation $\langle .,. \rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_C}$ represents the duality pairing between $H^{-\frac{1}{2}}(\Gamma_C)$ and $H^{\frac{1}{2}}(\Gamma_C)$.

The forthcoming mixed variational formulation uses the following convex cone of multipliers on Γ_C

$$M = \left\{ \mu \in H^{-\frac{1}{2}}(\Gamma_C) : \ \langle \mu, \psi \rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_C} \geq 0 \ \text{ for all } \psi \in H^{\frac{1}{2}}(\Gamma_C), \psi \geq 0 \text{ a.e. on } \Gamma_C \right\}.$$

Define

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \varepsilon(\mathbf{v}) \, \mathrm{d}\Omega, \qquad b(\mu, \mathbf{v}) = \langle \mu, v_n \rangle_{-\frac{1}{2},\frac{1}{2},\Gamma_C}, \\ L(\mathbf{v}) &= \int_\Omega \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}\Omega + \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{v} \, \mathrm{d}\Gamma, \end{aligned}$$

for any $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{V}$ and μ in $H^{-\frac{1}{2}}(\Gamma_C)$.

The mixed formulation of the unilateral contact problem without friction (1)–(6) consists then in finding $\mathbf{u} \in \mathbf{V}$ and $\lambda \in M$ such that:

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\lambda, \mathbf{v}) = L(\mathbf{v}), & \forall \mathbf{v} \in \mathbf{V}, \\ b(\mu - \lambda, \mathbf{u}) \leq 0, & \forall \mu \in M. \end{cases} \tag{7}$$

An equivalent formulation of (7) consists in finding $(\lambda, \mathbf{u}) \in M \times \mathbf{V}$ satisfying

$$\mathcal{L}(\mu, \mathbf{u}) \leq \mathcal{L}(\lambda, \mathbf{u}) \leq \mathcal{L}(\lambda, \mathbf{v}), \qquad \forall \mathbf{v} \in \mathbf{V}, \ \forall \mu \in M,$$

where $\mathcal{L}(\mu, \mathbf{v}) = \frac{1}{2} a(\mathbf{v}, \mathbf{v}) - L(\mathbf{v}) + b(\mu, \mathbf{v})$. Another classical weak formulation of problem (1)–(6) is a variational inequality: find $\mathbf{u}$ such that

$$\mathbf{u} \in \mathbf{K}, \qquad a(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq L(\mathbf{v} - \mathbf{u}), \quad \forall \mathbf{v} \in \mathbf{K}, \tag{8}$$

where $\mathbf{K}$ denotes the closed convex cone of admissible displacement fields satisfying the non-penetration conditions:

$$\mathbf{K} = \big\{ \mathbf{v} \in \mathbf{V} : \quad v_n \leq 0 \text{ on } \Gamma_C \big\}.$$

The existence and uniqueness of $(\lambda, \mathbf{u})$ solution to (7) has been stated in [19]. Moreover, the first argument $\mathbf{u}$ solution to (7) is also the unique solution of problem (8) and $\lambda = -\sigma_n(\mathbf{u})$.

3. MIXED FINITE ELEMENT APPROXIMATION

We approximate this problem by a standard finite element method. Namely we fix a family of meshes $T_h, h > 0$, regular in Ciarlet's sense [9], made of closed triangles. For $K \in T_h$ we recall that h_K is the diameter of K and $h = \max_{K \in T_h} h_K$. The regularity of the mesh implies in particular that for any edge E of K one has $h_E = |E| \sim h_K$.

Let us define E_h (resp. $\mathcal{N}_h$) as the set of edges (resp. nodes) of the triangulation and set $E_h^{int} = \{E \in E_h : E \subset \Omega\}$ the set of interior edges of T_h (the edges are supposed to be relatively open). We denote by $E_h^N = \{E \in E_h : E \subset \Gamma_N\}$ the set of exterior edges included into the part of the boundary where we impose Neumann conditions, and similarly $E_h^C = \{E \in E_h : E \subset \Gamma_C\}$ is the set of exterior edges included into the part of the boundary where we impose unilateral contact conditions. Set $\mathcal{N}_h^D = \mathcal{N}_h \cap \overline{\Gamma_D}$ (note that the extreme nodes of $\overline{\Gamma_D}$ belong to $\mathcal{N}_h^D$). Let $\mathcal{S}$ denote the set of vertices of Ω and denote by $\mathcal{N}_h^{NC}$ the set of nodes which belong to $\overline{\Gamma_C} \cap \overline{\Gamma_N}$ and by $\mathcal{N}_h^{CC}$ the nodes belonging to $\Gamma_C \cap \mathcal{S}$. Set finally $\mathcal{N}_h^C = (\mathcal{N}_h \setminus \mathcal{N}_h^{CC}) \cap \Gamma_C$ ($\mathcal{N}_h^C$ contains the nodes in Γ_C which are not vertices of Ω). For an element K, we will denote by E_K the set of edges of K and according to the above notation, we set $E_K^{int} = E_K \cap E_h^{int}$, $E_K^N = E_K \cap E_h^N$, $E_K^C = E_K \cap E_h^C$.

For an edge E of an element K, introduce $\mathbf{n}_{K,E} = (n_1, n_2)$ the unit outward normal vector to K along E and the tangent vector $\mathbf{t}_{K,E} = \mathbf{n}_{K,E}^{\perp} = (-n_2, n_1)$. Furthermore for each edge E we fix one of the two normal vectors and denote it by $\mathbf{n}_E$ and we set $\mathbf{t}_E = \mathbf{n}_E^{\perp}$. The jump of some vector valued function $\mathbf{v}$ across an edge $E \in E_h^{int}$ at a point $\mathbf{y} \in E$ is defined as

$$[\![\mathbf{v}]\!]_E(\mathbf{y}) = \lim_{\alpha \to +0} \mathbf{v}(\mathbf{y} + \alpha \mathbf{n}_E) - \mathbf{v}(\mathbf{y} - \alpha \mathbf{n}_E), \quad \forall E \in E_h^{int}.$$

Note that the sign of $[\![\mathbf{v}]\!]_E$ depends on the orientation of $\mathbf{n}_E$. Finally we will need local subdomains (also called patches). As usual, let ω_K be the union of all elements having a nonempty intersection with K. Similarly for a node $\mathbf{x}$ and an edge E, let $\omega_{\mathbf{x}} = \cup_{K : \mathbf{x} \in K} K$ and $\omega_E = \cup_{\mathbf{x} \in \overline{E}} \omega_{\mathbf{x}}$.

The finite element space used in Ω is then defined by

$$\mathbf{V}_h = \Big\{ \mathbf{v}_h \in (\mathcal{C}(\overline{\Omega}))^2 : \quad \forall \kappa \in T_h, \quad \mathbf{v}_h|_\kappa \in (\mathbb{P}_1(\kappa))^2, \quad \mathbf{v}_h|_{\Gamma_D} = \mathbf{0} \Big\}.$$

We suppose that the contact area consists in several straight line segments, that we denote by $\overline{\Gamma_C^i}, 1 \leq i \leq q$ such that $\overline{\Gamma_C} = \cup_i \overline{\Gamma_C^i}$. In order to express the contact constraints by using Lagrange multipliers on the contact zone, we have to introduce the space

$$W_h = \left\{ \mu_h : \cup_i \overline{\Gamma_C^i} \to \mathbb{R}, \mu_h|_{\overline{\Gamma_C^i}} \in \mathcal{C}(\overline{\Gamma_C^i}), \exists \mathbf{v}_h \in \mathbf{V}_h \text{ s.t. } \mathbf{v}_h \cdot \mathbf{n} = \mu_h \text{ on } \cup_i \overline{\Gamma_C^i} \right\}. \tag{9}$$

The choice of the space W_h allows us to define the following closed convex cone:

$$M_h = \left\{ \mu_h \in W_h : \int_{\Gamma_C} \mu_h \psi_h \, d\Gamma \geq 0, \ \forall \psi_h \in W_h, \psi_h \geq 0 \right\}.$$

Remark 3.1. It is easy to check that $M_h \not\subset M$.

The discretized mixed formulation of the unilateral contact problem without friction is to find $\mathbf{u}_h \in \mathbf{V}_h$ and $\lambda_h \in M_h$ satisfying:

$$\begin{cases} a(\mathbf{u}_h, \mathbf{v}_h) + b(\lambda_h, \mathbf{v}_h) = L(\mathbf{v}_h), & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mu_h - \lambda_h, \mathbf{u}_h) \leq 0, & \forall \mu_h \in M_h. \end{cases} \tag{10}$$

Problem (10) consists in finding $(\lambda_h, \mathbf{u}_h) \in M_h \times \mathbf{V}_h$ satisfying

$$\mathcal{L}(\mu_h, \mathbf{u}_h) \leq \mathcal{L}(\lambda_h, \mathbf{u}_h) \leq \mathcal{L}(\lambda_h, \mathbf{v}_h), \qquad \forall \mathbf{v}_h \in \mathbf{V}_h, \ \forall \mu_h \in M_h,$$

where $\mathcal{L}(\mu_h, \mathbf{v}_h) = \frac{1}{2}a(\mathbf{v}_h, \mathbf{v}_h) - L(\mathbf{v}_h) + b(\mu_h, \mathbf{v}_h)$. In order to prove that there is a unique solution to problem (10), and since we are in the finite dimensional case, we only have to check (see [19], Thm. 3.9 and Ex. 3.8) that

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h, \mathbf{v}_h \neq \mathbf{0}} \frac{b(\mu_h, \mathbf{v}_h)}{\|\mathbf{v}_h\|_{1,\Omega}}$$

is a norm on W_h. So we have to verify that

$$\{\mu_h \in W_h : \ b(\mu_h, \mathbf{v}_h) = 0, \ \forall \mathbf{v}_h \in \mathbf{V}_h\} = \{0\},$$

which is satisfied according to the definition of W_h in (9). As a consequence, we obtain the following statement:

Proposition 3.2. *Problem* (10) *admits a unique solution* $(\lambda_h, \mathbf{u}_h) \in M_h \times \mathbf{V}_h$.

Proposition 3.3. *If* $(\lambda_h, \mathbf{u}_h)$ *is the solution of* (10), *then* $\mathbf{u}_h$ *is also the unique solution of the variational inequality: find* $\mathbf{u}_h$ *such that*

$$\mathbf{u}_h \in \mathbf{K}_h, \qquad a(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) \geq L(\mathbf{v}_h - \mathbf{u}_h), \quad \forall \mathbf{v}_h \in \mathbf{K}_h, \tag{11}$$

where $\mathbf{K}_h$ *denotes the discrete closed convex cone of admissible displacement fields satisfying the non-penetration conditions, i.e.,*

$$\mathbf{K}_h = \left\{ \mathbf{v}_h \in \mathbf{V}_h : \quad v_{hn} \leq 0 \text{ on } \Gamma_C \right\}.$$

Proof. Taking $\mu_h = 0$ and $\mu_h = 2\lambda_h$ in (10) leads to $b(\lambda_h, \mathbf{u}_h) = 0$ and to

$$b(\mu_h, \mathbf{u}_h) = \int_{\Gamma_C} \mu_h u_{hn} \, d\Gamma \leq 0, \quad \forall \mu_h \in M_h.$$

The latter inequality implies by polarity that $u_{hn} \in -M_h^*$ (the notation X^* stands for the positive polar cone of X for the inner product on W_h induced by $b(.,.)$, see [22], p. 119). Let $Q_h = \{\mu_h \in W_h : \mu_h \geq 0\}$.

We have $M_h^* = (Q_h^*)^* = Q_h$ since Q_h is a closed convex cone. Hence $u_{hn} \in -Q_h$ and $\mathbf{u}_h \in \mathbf{K}_h$. Besides (10) and $b(\lambda_h, \mathbf{u}_h) = 0$ lead to

$$a(\mathbf{u}_h, \mathbf{u}_h) = L(\mathbf{u}_h) \tag{12}$$

and for any $\mathbf{v}_h \in \mathbf{K}_h$, we get

$$a(\mathbf{u}_h, \mathbf{v}_h) - L(\mathbf{v}_h) = -b(\lambda_h, \mathbf{v}_h) \quad = \quad -\int_{\Gamma_C} \lambda_h v_{hn} \, \mathrm{d}\Gamma \geq 0, \tag{13}$$

owing to $\lambda_h \in Q_h^*$.

Putting together (12) and (13) implies that $\mathbf{u}_h$ is solution of the variational inequality (11) which admits a unique solution according to Stampacchia's theorem. $\qquad \square$

Remark 3.4. *A priori* error analyses for this discretization of the unilateral contact problem can be found in [4,12,23]. The *a priori* error estimates are of order $h^{(1+\nu)/2}$ if $\mathbf{u} \in (H^{3/2+\nu}(\Omega))^2$, $0 < \nu \leq 1/2$ (see [4,12]). If an additional assumption dealing with the finiteness of transition points between contact and separation is added then an optimal error estimate of order $h^{1/2+\nu}$ is obtained (see [23]).

We consider the quasi-interpolation operator π_h: for any $v \in L^1(\Omega)$, we define $\pi_h v$ as the unique element in $V_h = \{v_h \in \mathcal{C}(\overline{\Omega}) : \forall \kappa \in T_h, \ v_h|_\kappa \in \mathbb{P}_1(\kappa), \ v_h|_{\Gamma_D} = 0\}$ such that:

$$\pi_h v = \sum_{\mathbf{x} \in \mathcal{N}_h \setminus \mathcal{N}_h^D} \alpha_{\mathbf{x}}(v) \lambda_{\mathbf{x}}, \tag{14}$$

where for any $\mathbf{x} \in \mathcal{N}_h \setminus \mathcal{N}_h^D$, $\lambda_{\mathbf{x}}$ is the standard basis function in V_h satisfying $\lambda_{\mathbf{x}}(\mathbf{x}') = \delta_{\mathbf{x},\mathbf{x}'}$, for all $\mathbf{x}' \in \mathcal{N}_h \setminus \mathcal{N}_h^D$ and $\alpha_{\mathbf{x}}(v)$ is defined as follows:

$$\alpha_{\mathbf{x}}(v) \quad = \quad \frac{1}{|\omega_{\mathbf{x}}|} \int_{\omega_{\mathbf{x}}} v(\mathbf{y}) \, \mathrm{d}\mathbf{y}, \quad \forall \mathbf{x} \in \mathcal{N}_h \setminus \mathcal{N}_h^D.$$

The following estimates hold (see, *e.g.*, [29]):

Lemma 3.5. *For any $v \in H_{\Gamma_D}^1(\Omega)$ we have*

$$\begin{aligned}
\|v - \pi_h v\|_K &\lesssim& h_K \|\nabla v\|_{\omega_K}, \forall K \in T_h, \\
\|v - \pi_h v\|_E &\lesssim& h_E^{1/2} \|\nabla v\|_{\omega_E}, \forall E \in E_h.
\end{aligned}$$

Since we deal with vector valued functions we can define a vector valued operator (which we denote again by π_h for the sake of simplicity) whose components are defined above. Consequently we can directly state the

Lemma 3.6. *For any $\mathbf{v} \in \mathbf{V}$ we have*

$$\begin{aligned}
\|\mathbf{v} - \pi_h \mathbf{v}\|_K &\lesssim& h_K \|\mathbf{v}\|_{1,\omega_K}, \forall K \in T_h, \tag{15} \\
\|\mathbf{v} - \pi_h \mathbf{v}\|_E &\lesssim& h_E^{1/2} \|\mathbf{v}\|_{1,\omega_E}, \forall E \in E_h. \tag{16}
\end{aligned}$$

4. A FIRST ERROR ESTIMATOR: η

4.1. Definition of the residual error estimators

The element residual of the equilibrium equation (1) is defined by

$$\mathbf{div}\boldsymbol{\sigma}(\mathbf{u}_h) + \mathbf{f} = \mathbf{f} \text{ on } K.$$

As usual this element residual is replaced by some computable finite dimensional approximation called approximate element residual

$$\mathbf{f}_K \in (\mathbb{P}_k(K))^2.$$

A current choice is to take $\mathbf{f}_K = \int_K \mathbf{f}(\mathbf{x})/|K|$ since for $\mathbf{f} \in (H^1(\Omega))^2$, scaling arguments yield $\|\mathbf{f} - \mathbf{f}_K\|_K \lesssim h_K\|\mathbf{f}\|_{1,K}$ and is then negligible with respect to the estimator η defined hereafter. In the same way $\mathbf{g}$ is approximated by a computable quantity denoted $\mathbf{g}_E$ on any $E \in E_h^N$.

Definition 4.1 (First residual error estimator). The local and global residual error estimators are defined by

$$\eta_K = \left(\sum_{i=1}^{6} \eta_{iK}^2\right)^{1/2},$$

$$\eta_{1K} = h_K\|\mathbf{f}_K\|_K,$$

$$\eta_{2K} = h_K^{1/2}\left(\sum_{E \in E_K^{int} \cup E_K^N} \|J_{E,n}(\mathbf{u}_h)\|_E^2\right)^{1/2},$$

$$\eta_{3K} = h_K^{1/2}\left(\sum_{E \in E_K^C} \|\lambda_h + \sigma_n(\mathbf{u}_h)\|_E^2\right)^{1/2},$$

$$\eta_{4K} = h_K^{1/2}\left(\sum_{E \in E_K^C} \|\sigma_t(\mathbf{u}_h)\|_E^2\right)^{1/2},$$

$$\eta_{5K} = \left(\sum_{E \in E_K^C} \int_E -\lambda_{h+} u_{hn}\right)^{1/2},$$

$$\eta_{6K} = \left(\sum_{E \in E_K^C} \|\lambda_{h-}\|_E^2\right)^{1/2},$$

$$\eta = \left(\sum_{K \in T_h} \eta_K^2\right)^{1/2},$$

where the notations λ_{h+} and λ_{h-} denote the positive and negative parts of λ_h, respectively; $J_{E,n}(\mathbf{u}_h)$ means the constraint jump of $\mathbf{u}_h$ in normal direction, *i.e.*,

$$J_{E,n}(\mathbf{u}_h) = \begin{cases} [\![\boldsymbol{\sigma}(\mathbf{u}_h)\mathbf{n}_E]\!]_E, \forall E \in E_h^{int}, \\ \boldsymbol{\sigma}(\mathbf{u}_h)\mathbf{n}_E - \mathbf{g}_E, \forall E \in E_h^N. \end{cases} \tag{17}$$

The local and global approximation terms are defined by

$$
\zeta_K = \left(h_K^2 \sum_{K' \subset \omega_K} \|\mathbf{f} - \mathbf{f}_{K'}\|_{K'}^2 + h_E \sum_{E \subset E_K^N} \|\mathbf{g} - \mathbf{g}_E\|_E^2 \right)^{1/2} , \zeta = \left(\sum_{K \in T_h} \zeta_K^2 \right)^{1/2} .
\tag{18}
$$

Remark 4.2. In the Definition 4.1, we could also set instead of η_{5K}:

$$
\hat{\eta}_{5K} = \left(\sum_{E \in E_K^C} \int_E -\lambda_{h-} u_{hn} \right)^{1/2}
$$

since $\int_{\Gamma_C} \lambda_{h+} u_{hn} = \int_{\Gamma_C} \lambda_{h-} u_{hn}$. Note that $\hat{\eta}_{5K} \neq \eta_{5K}$ although $\sum_{K \in T_h} \hat{\eta}_{5K}^2 = \sum_{K \in T_h} \eta_{5K}^2$.

4.2. Upper error bound

Theorem 4.3. *Let* $(\lambda, \mathbf{u})$ *be the solution of* (7) *and let* $(\lambda_h, \mathbf{u}_h)$ *be the solution of* (10). *Then we have*

$$
\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|\lambda - \lambda_h\|_{-\frac{1}{2},\Gamma_C} \lesssim \eta + \zeta.
$$

Proof. Afterwards we adopt the following notation for the displacement error term:

$$
\mathbf{e_u} = \mathbf{u} - \mathbf{u}_h.
$$

Let $\mathbf{v}_h \in \mathbf{V}_h$. From the $\mathbf{V}$-ellipticity of $a(.,.)$ and the equilibrium equations in (7) and (10) we obtain:

$$
\begin{aligned}
\|\mathbf{e_u}\|_{1,\Omega}^2 & \lesssim & a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h) \\
& = & a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{v}_h) + a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) \\
& = & L(\mathbf{u} - \mathbf{v}_h) - b(\lambda, \mathbf{u} - \mathbf{v}_h) - a(\mathbf{u}_h, \mathbf{u} - \mathbf{v}_h) + b(\lambda_h - \lambda, \mathbf{v}_h - \mathbf{u}_h).
\end{aligned}
$$

Integrating by parts on each triangle K, using the definition of $J_{E,n}(\mathbf{u}_h)$ in (17) and the complementarity conditions $b(\lambda, \mathbf{u}) = b(\lambda_h, \mathbf{u}_h) = 0$ yields:

$$
\begin{aligned}
\|\mathbf{e_u}\|_{1,\Omega}^2 \lesssim & \int_\Omega \mathbf{f} \cdot (\mathbf{u} - \mathbf{v}_h) + b(\lambda_h, \mathbf{v}_h) + b(\lambda, \mathbf{u}_h) + \sum_{E \in E_h^N} \int_E (\mathbf{g} - \mathbf{g}_E) \cdot (\mathbf{u} - \mathbf{v}_h) \\
& - \sum_{E \in E_h^C} \int_E (\boldsymbol{\sigma}(\mathbf{u}_h)\mathbf{n}) \cdot (\mathbf{u} - \mathbf{v}_h) - \sum_{E \in E_h^{int} \cup E_h^N} \int_E J_{E,n}(\mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{v}_h).
\end{aligned}
\tag{19}
$$

Splitting up the integrals on Γ_C into normal and tangential components gives:

$$
\begin{aligned}
\|\mathbf{e_u}\|_{1,\Omega}^2 \lesssim & \int_\Omega \mathbf{f} \cdot (\mathbf{u} - \mathbf{v}_h) + b(\lambda_h, \mathbf{u}) + b(\lambda, \mathbf{u}_h) \\
& + \sum_{E \in E_h^C} \int_E (\lambda_h + \sigma_n(\mathbf{u}_h))(v_{hn} - u_n) + \sum_{E \in E_h^C} \int_E \sigma_t(\mathbf{u}_h)(v_{ht} - u_t) \\
& - \sum_{E \in E_h^{int} \cup E_h^N} \int_E J_{E,n}(\mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{v}_h) + \sum_{E \in E_h^N} \int_E (\mathbf{g} - \mathbf{g}_E) \cdot (\mathbf{u} - \mathbf{v}_h).
\end{aligned}
\tag{20}
$$

We now need to estimate each term of this right-hand side. For that purpose, we take

$$\mathbf{v}_h = \mathbf{u}_h + \pi_h(\mathbf{u} - \mathbf{u}_h) \tag{21}$$

where π_h is the quasi-interpolation operator defined in Lemma 3.6.

We start with the integral term. Cauchy-Schwarz's inequality implies

$$\left| \int_\Omega \mathbf{f} \cdot (\mathbf{u} - \mathbf{v}_h) \right| \leq \sum_{K \in T_h} \|\mathbf{f}\|_K \|\mathbf{u} - \mathbf{v}_h\|_K,$$

and it suffices to estimate $\|\mathbf{u} - \mathbf{v}_h\|_K$ for any triangle K. From the definition of $\mathbf{v}_h$ and (15) we get:

$$\|\mathbf{u} - \mathbf{v}_h\|_K = \|\mathbf{e_u} - \pi_h \mathbf{e_u}\|_K \lesssim h_K \|\mathbf{e_u}\|_{1,\omega_K}.$$

As a consequence

$$\left| \int_\Omega \mathbf{f} \cdot (\mathbf{u} - \mathbf{v}_h) \right| \lesssim (\eta + \zeta) \|\mathbf{e_u}\|_{1,\Omega}. \tag{22}$$

We now consider the interior and Neumann boundary terms in (20): as previously the application of Cauchy-Schwarz's inequality leads to

$$\left| \sum_{E \in E_h^{int} \cup E_h^N} \int_E J_{E,n}(\mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{v}_h) \right| \leq \sum_{E \in E_h^{int} \cup E_h^N} \|J_{E,n}(\mathbf{u}_h)\|_E \|\mathbf{u} - \mathbf{v}_h\|_E.$$

Therefore using the expression (21) and estimate (16), we obtain

$$\|\mathbf{u} - \mathbf{v}_h\|_E = \|\mathbf{e_u} - \pi_h \mathbf{e_u}\|_E \lesssim h_E^{1/2} \|\mathbf{e_u}\|_{1,\omega_E}.$$

Inserting this estimate in the previous one we deduce that

$$\left| \sum_{E \in E_h^{int} \cup E_h^N} \int_E J_{E,n}(\mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{v}_h) \right| \lesssim \eta \|\mathbf{e_u}\|_{1,\Omega}. \tag{23}$$

Moreover

$$\left| \sum_{E \in E_h^N} \int_E (\mathbf{g} - \mathbf{g}_E) \cdot (\mathbf{u} - \mathbf{v}_h) \right| \lesssim \zeta \|\mathbf{e_u}\|_{1,\Omega}. \tag{24}$$

The two following terms are handled in a similar way as the previous ones so that

$$\left| \sum_{E \in E_h^C} \int_E (\lambda_h + \sigma_n(\mathbf{u}_h))(v_{hn} - u_n) \right| \lesssim \eta \|\mathbf{e_u}\|_{1,\Omega} \tag{25}$$

and

$$\left| \sum_{E \in E_h^C} \int_E \sigma_t(\mathbf{u}_h)(v_{ht} - u_t) \right| \lesssim \eta \|\mathbf{e_u}\|_{1,\Omega}. \tag{26}$$

Noting that $u_{hn} \leq 0$ on Γ_C, we have

$$b(\lambda, \mathbf{u}_h) \leq 0, \tag{27}$$

and it remains to estimate one term in (20). Using the discrete complementarity condition $b(\lambda_h, \mathbf{u}_h) = 0$ implies

$$
\begin{aligned}
b(\lambda_h, \mathbf{u}) = \int_{\Gamma_C} \lambda_h u_n & = \int_{\Gamma_C} \lambda_h (u_n - u_{hn}) \\
& = \int_{\Gamma_C} (\lambda_{h+} - \lambda_{h-})(u_n - u_{hn}) \\
& \leq -\int_{\Gamma_C} \lambda_{h+} u_{hn} - \int_{\Gamma_C} \lambda_{h-}(u_n - u_{hn}) \\
& \leq \eta^2 - \int_{\Gamma_C} \lambda_{h-}(u_n - u_{hn}).
\end{aligned}
\tag{28}
$$

The last term in the previous expression is estimated using Cauchy-Schwarz's and Young's inequalities:

$$
\begin{aligned}
\left| \int_{\Gamma_C} \lambda_{h-}(u_n - u_{hn}) \right| & = \left| \sum_{E \in E_h^C} \int_E \lambda_{h-}(u_n - u_{hn}) \right| \\
& \leq \sum_{E \in E_h^C} \|\lambda_{h-}\|_E \|u_n - u_{hn}\|_E \\
& \leq \sum_{E \in E_h^C} \left(\alpha \|u_n - u_{hn}\|_E^2 + \frac{1}{4\alpha} \|\lambda_{h-}\|_E^2 \right),
\end{aligned}
$$

for any $\alpha > 0$. A standard trace theorem implies the existence of $C > 0$ such that

$$
\begin{aligned}
\left| \int_{\Gamma_C} \lambda_{h-}(u_n - u_{hn}) \right| & \leq \alpha \|u_n - u_{hn}\|_{\Gamma_C}^2 + \frac{1}{4\alpha} \sum_{E \in E_h^C} \|\lambda_{h-}\|_E^2 \\
& \leq C\alpha \|\mathbf{e_u}\|_{1,\Omega}^2 + \frac{\eta^2}{4\alpha}.
\end{aligned}
\tag{29}
$$

Estimates (28) and (29) give

$$
b(\lambda_h, \mathbf{u}) \leq C\alpha \|\mathbf{e_u}\|_{1,\Omega}^2 + \eta^2 \left(1 + \frac{1}{4\alpha} \right)
\tag{30}
$$

for any $\alpha > 0$.

Putting together the estimates (22)–(27) and (30) with α small enough in (20), and using Young's inequality, we come to the conclusion that

$$
\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \lesssim \eta + \zeta.
\tag{31}
$$

We now search for an upper bound on the discretization error $\lambda - \lambda_h$ corresponding to the multipliers. Let $\mathbf{v} \in \mathbf{V}$ and $\mathbf{v}_h \in \mathbf{V}_h$. From the equilibrium equations in (7) and (10) we get:

$$
\begin{aligned}
b(\lambda - \lambda_h, \mathbf{v}) & = b(\lambda, \mathbf{v} - \mathbf{v}_h) - b(\lambda_h, \mathbf{v} - \mathbf{v}_h) + b(\lambda - \lambda_h, \mathbf{v}_h) \\
& = L(\mathbf{v} - \mathbf{v}_h) - a(\mathbf{u}, \mathbf{v} - \mathbf{v}_h) - b(\lambda_h, \mathbf{v} - \mathbf{v}_h) + a(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h) \\
& = L(\mathbf{v} - \mathbf{v}_h) - a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) - a(\mathbf{u}_h, \mathbf{v} - \mathbf{v}_h) - b(\lambda_h, \mathbf{v} - \mathbf{v}_h).
\end{aligned}
$$

An integration by parts on each element K gives

$$
\begin{aligned}
b(\lambda - \lambda_h, \mathbf{v}) \;=\; & \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{v}_h) - a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) - \sum_{E \in E_h^{int} \cup E_h^N} \int_E J_{E,n}(\mathbf{u}_h) \cdot (\mathbf{v} - \mathbf{v}_h) \\
& - \sum_{E \in E_h^C} \int_E (\lambda_h + \sigma_n(\mathbf{u}_h))(v_n - v_{hn}) - \sum_{E \in E_h^C} \int_E \sigma_t(\mathbf{u}_h)(v_t - v_{ht}) \\
& + \sum_{E \in E_h^N} \int_E (\mathbf{g} - \mathbf{g}_E) \cdot (\mathbf{v} - \mathbf{v}_h).
\end{aligned}
$$

Choosing $\mathbf{v}_h = \pi_h \mathbf{v}$ where π_h is the quasi-interpolation operator defined in Lemma 3.6 and achieving a similar calculation as in (22)–(26) we deduce that

$$
|b(\lambda - \lambda_h, \mathbf{v})| \lesssim (\eta + \zeta + \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}) \|\mathbf{v}\|_{1,\Omega}
$$

for any $\mathbf{v} \in \mathbf{V}$. As a consequence

$$
\|\lambda - \lambda_h\|_{-\frac{1}{2},\Gamma_C} \lesssim \eta + \zeta + \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}. \tag{32}
$$

Putting together the two estimates (31) and (32) ends the proof of the theorem. $\qquad \square$

4.3. Lower error bound

Theorem 4.4. *For all elements K, the following local lower error bounds hold:*

$$
\eta_{1K} \lesssim \|\mathbf{u} - \mathbf{u}_h\|_{1,K} + \zeta_K, \tag{33}
$$

$$
\eta_{2K} \lesssim \|\mathbf{u} - \mathbf{u}_h\|_{1,\omega_K} + \zeta_K. \tag{34}
$$

Assume that $\lambda \in L^2(\Gamma_C)$. For all elements K such that $K \cap E_h^C \neq \emptyset$, the following local lower error bounds hold:

$$
\eta_{3K} \;\lesssim\; \sum_{E \in E_K^C} h^{1/2} \|\lambda - \lambda_h\|_E + \|\mathbf{u} - \mathbf{u}_h\|_{1,K} + \zeta_K, \tag{35}
$$

$$
\eta_{4K} \;\lesssim\; \|\mathbf{u} - \mathbf{u}_h\|_{1,K} + \zeta_K, \tag{36}
$$

$$
\eta_{5K} \;\lesssim\; \sum_{E \in E_K^C} \left(\|\lambda - \lambda_h\|_E + \|\mathbf{u} - \mathbf{u}_h\|_E + \|\lambda - \lambda_h\|_E^{1/2} \|u_n\|_E^{1/2} + \|\mathbf{u} - \mathbf{u}_h\|_E^{1/2} \|\lambda\|_E^{1/2} \right), \tag{37}
$$

$$
\eta_{6K} \;\lesssim\; \sum_{E \in E_K^C} \|\lambda - \lambda_h\|_E. \tag{38}
$$

Proof. The estimates of η_{1K} and η_{2K} in (33) and (34) are standard (see, *e.g.*, [28]).

We now estimate η_{3K}. Writing $\mathbf{w}_E = w_{En}\mathbf{n} + w_{Et}\mathbf{t}$ on $E \in E_K^C$ and denoting by b_E the edge bubble function associated with E (*i.e.*, $b_E = 4\lambda_{a_1}\lambda_{a_2}$, when a_1, a_2 are the two extremities of E; we recall that $\lambda_\mathbf{x}$ is the standard basis function at node $\mathbf{x}$ in V_h satisfying $\lambda_\mathbf{x}(\mathbf{x}') = \delta_{\mathbf{x},\mathbf{x}'}$ for any node $\mathbf{x}'$, see (14)), we choose $w_{En} = (\lambda_h + \sigma_n(\mathbf{u}_h))b_E$ and $w_{Et} = 0$ in the element K containing E (here we made a slight abuse of notation

to simplify) and $\mathbf{w}_E = \mathbf{0}$ in $\overline{\Omega} \setminus K$. Therefore

$$
\begin{aligned}
\|\lambda_h + \sigma_n(\mathbf{u}_h)\|_E^2 \quad &\sim \quad \int_E (\lambda_h + \sigma_n(\mathbf{u}_h)) w_{En} \\
&= \quad b(\lambda_h, \mathbf{w}_E) + \int_K \boldsymbol{\sigma}(\mathbf{u}_h) : \varepsilon(\mathbf{w}_E) \\
&= \quad b(\lambda_h, \mathbf{w}_E) - \int_K \boldsymbol{\sigma}(\mathbf{u} - \mathbf{u}_h) : \varepsilon(\mathbf{w}_E) + \int_K \boldsymbol{\sigma}(\mathbf{u}) : \varepsilon(\mathbf{w}_E) \\
&= \quad b(\lambda_h - \lambda, \mathbf{w}_E) + L(\mathbf{w}_E) - \int_K \boldsymbol{\sigma}(\mathbf{u} - \mathbf{u}_h) : \varepsilon(\mathbf{w}_E) \\
&\lesssim \quad \|\lambda - \lambda_h\|_E \|\mathbf{w}_E\|_E + \|\mathbf{f}\|_K \|\mathbf{w}_E\|_K + \|\mathbf{u} - \mathbf{u}_h\|_{1,K} \|\mathbf{w}_E\|_{1,K}.
\end{aligned}
$$

An inverse inequality and estimate (33) imply

$$
\begin{aligned}
h_K^{1/2} \|\lambda_h + \sigma_n(\mathbf{u}_h)\|_E \quad &\lesssim \quad h^{1/2} \|\lambda - \lambda_h\|_E + \|\mathbf{u} - \mathbf{u}_h\|_{1,K} + h\|\mathbf{f}\|_K \\
&\lesssim \quad h^{1/2} \|\lambda - \lambda_h\|_E + \|\mathbf{u} - \mathbf{u}_h\|_{1,K} + \zeta_K.
\end{aligned}
$$

This estimate gives the estimate of η_{3K} in (35). The bound of η_{4K} in (36) is obtained as previously by choosing $w_{En} = 0$ and $w_{Et} = \sigma_t(\mathbf{u}_h) b_E$.

We now consider η_{5K}. If $E \in E_K^C$, let $F \subset E$ be the part of the edge where $\lambda_h = \lambda_{h+}$. So

$$
\begin{aligned}
\int_E -\lambda_{h+} u_{hn} \quad &= \quad \int_F -\lambda_h u_{hn} \\
&= \quad \int_F (\lambda_h - \lambda)(u_n - u_{hn}) - \int_F \lambda_h u_n - \int_F \lambda u_{hn} \\
&= \quad \int_F (\lambda_h - \lambda)(u_n - u_{hn}) - \int_F (\lambda_h - \lambda) u_n - \int_F \lambda(u_{hn} - u_n) \\
&\lesssim \quad \|\lambda - \lambda_h\|_E \|\mathbf{u} - \mathbf{u}_h\|_E + \|\lambda - \lambda_h\|_E \|u_n\|_E + \|\mathbf{u} - \mathbf{u}_h\|_E \|\lambda\|_E.
\end{aligned}
$$

The last estimate implies (37) by taking the square root.

The estimate on η_{6K} is obvious. Since $\lambda \geq 0$ then we have $0 \leq \lambda_{h-} \leq |\lambda - \lambda_h|$ on Γ_C. So

$$
\|\lambda_{h-}\|_E \leq \|\lambda - \lambda_h\|_E
$$

and (38) is proved. $\square$

Remark 4.5. Assume that $\mathbf{u} \in (H^2(\Omega))^2$ (so $\lambda \in H^{\frac{1}{2}}(\Gamma_C)$), and that optimal *a priori* error estimates hold (note that the question of optimality remains open in the *a priori* error analysis since the known *a priori* error estimates are not optimal) and define:

$$
\eta_i = \Big(\sum_{K \in T_h} \eta_{iK}^2 \Big)^{1/2}, \quad 1 \leq i \leq 6.
$$

Then one would have $\eta_1 \lesssim h$, $\eta_2 \lesssim h$, $\eta_3 \lesssim h$, $\eta_4 \lesssim h$, $\eta_5 \lesssim h^{1/4}$, $\eta_6 \lesssim h^{1/2}$. So $\eta \lesssim h^{1/4}$.

5. A SECOND ERROR ESTIMATOR: $\tilde{\eta}$

The analysis of this error estimator requires a nonstandard definition of the error (in comparison with the already known results in the literature dealing with contact problems). We begin with some preliminaries.

5.1. **Preliminaries**

For any $\mu_h \in W_h$ and $\mathbf{v}_h \in \mathbf{V}_h$ we define the bilinear form $c(.,.)$ such that

$$c(\mu_h, \mathbf{v}_h) = \sum_{i=1}^{q} \int_{\Gamma_C^i} I_h(\mu_h v_{hn})$$

where I_h is the linear Lagrange interpolation operator at the nodes of $\overline{\Gamma_C^i}$ (to simplify the notations we write I_h instead of I_h^i). Let W_h^+ be the closed convex cone of nonnegative functions in W_h. We define the following discrete problem issued from (7): find $\mathbf{u}_h \in \mathbf{V}_h$ and $\tilde{\lambda}_h \in W_h^+$ such that

$$
\begin{cases}
a(\mathbf{u}_h, \mathbf{v}_h) + c(\tilde{\lambda}_h, \mathbf{v}_h) = L(\mathbf{v}_h), & \forall \mathbf{v}_h \in \mathbf{V}_h, \\
c(\mu_h - \tilde{\lambda}_h, \mathbf{u}_h) \leq 0, & \forall \mu_h \in W_h^+.
\end{cases}
\tag{39}
$$

The following proposition establishes the link between problems (39) and (10).

Proposition 5.1. *(i) Problem* (39) *admits a unique solution* $(\tilde{\lambda}_h, \mathbf{u}_h) \in W_h^+ \times \mathbf{V}_h$.
(ii) The displacement field $\mathbf{u}_h$ *in* (39) *coincides with the displacement field solving* (10) *(and also with the one solving* (11)*).*
(iii) The link between the contact pressures $\tilde{\lambda}_h$ *and* λ_h *solving* (39) *and* (10) *is:*

$$c(\tilde{\lambda}_h, \mathbf{v}_h) = b(\lambda_h, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{40}$$

More precisely, if ψ_j^i *denotes the (scalar) canonical basis function at node* $\mathbf{x}_j$ *in* $\overline{\Gamma_C^i} \setminus \overline{\Gamma_D}$*, we have*

$$\tilde{\lambda}_h(\mathbf{x}_j) = \frac{\displaystyle\int_{\Gamma_C^i} \lambda_h \psi_j^i}{\displaystyle\int_{\Gamma_C^i} \psi_j^i}, \ \forall i, \forall j. \tag{41}$$

Proof. (i). As for problem (10) and according to [19] (Thm. 3.9 and Ex. 3.8), it suffices to verify that

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h, \mathbf{v}_h \neq \mathbf{0}} \frac{c(\mu_h, \mathbf{v}_h)}{\|\mathbf{v}_h\|_{1,\Omega}}$$

is a norm on W_h, which reduces to the condition $\{\mu_h \in W_h : c(\mu_h, \mathbf{v}_h) = 0, \ \forall \mathbf{v}_h \in \mathbf{V}_h\} = \{0\}$. The latter condition is fulfilled according to the definition of W_h in (9).

(ii) The discussion is the same as in Proposition 3.3 noting that $c(.,.)$ induces an inner product on W_h and that W_h^+ is a closed convex cone.

(iii) Equality (40) is straightforward. Let us show that $\dim(W_h) = \#\mathcal{N}_h^C + \#\mathcal{N}_h^{NC} + 2\#\mathcal{N}_h^{CC}$ where the notation $\#$ stands for the cardinal of a set. We denote by ψ_j^i the "canonical basis function" at node $\mathbf{x}_j$ in $\overline{\Gamma_C^i} \setminus \overline{\Gamma_D}$ (*i.e.*, ψ_j^i is defined on $\cup_\ell \overline{\Gamma_C^\ell}$, the support of ψ_j^i lies in $\overline{\Gamma_C^i}$, ψ_j^i is continuous and piecewise of degree one on $\overline{\Gamma_C^i}$, and $\psi_j^i(\mathbf{x}_k) = \delta_{k,j}, \forall \mathbf{x}_k \in \overline{\Gamma_C^i} \setminus \overline{\Gamma_D}$). Note that ψ_j^i is not continuous if $\mathbf{x}_j \in \mathcal{N}_h^{CC}$.

If $\mathbf{x}_j \in \mathcal{N}_h^C \cup \mathcal{N}_h^{NC}$ it is straightforward that $\psi_j^i \in W_h$.

If $\mathbf{x}_j \in \mathcal{N}_h^{CC}$ then $\mathbf{x}_j \in \overline{\Gamma_C^i} \cup \overline{\Gamma_C^{i+1}}$. It suffices to show that there exists $\mathbf{v}_h = (v_{h1}, v_{h2}) \in \mathbf{V}_h$ such that $\psi_j^i = \mathbf{v}_h \cdot \mathbf{n}$ on $\cup_\ell \overline{\Gamma_C^\ell}$ and $\mathbf{w}_h = (w_{h1}, w_{h2}) \in \mathbf{V}_h$ such that $\psi_j^{i+1} = \mathbf{w}_h \cdot \mathbf{n}$ on $\cup_\ell \overline{\Gamma_C^\ell}$. Without loss of generality we can assume that the unit outward normal vector along Γ_C^i is equal to $(0, -1)$ and then the unit outward normal vector along Γ_C^{i+1} is equal to $(-\sin\theta, \cos\theta)$, for some $\theta \in]0, 2\pi[$ such that $\theta \neq \pi$ (θ being the interior

angle between Γ_C^i and Γ_C^{i+1}). Since $\mathbf{v}_h \cdot \mathbf{n}$ is linear on Γ_C^i and Γ_C^{i+1}, it suffices to show that $\mathbf{v}_h \cdot \mathbf{n}$ coincides with ψ_j^i at the nodes. Therefore we take $\mathbf{v}_h$ equal to zero at each node except $\mathbf{x}_j$ and for the values at $\mathbf{x}_j$ we get

$$
\begin{aligned}
\mathbf{v}_h(\mathbf{x}_j) \cdot (0, -1) &= -v_{h2}(\mathbf{x}_j) = 1, \\
\mathbf{v}_h(\mathbf{x}_j) \cdot (-\sin\theta, \cos\theta) &= -v_{h1}(\mathbf{x}_j)\sin\theta + v_{h2}(\mathbf{x}_j)\cos\theta = 0.
\end{aligned}
$$

As $\sin\theta \neq 0$, this system has the unique solution: $v_{h1}(\mathbf{x}_j) = -\cos\theta/\sin\theta$ and $v_{h2}(\mathbf{x}_j) = -1$. The solution $\mathbf{w}_h$ is obtained similarly and is characterized by

$$
w_{h1}(\mathbf{x}_j) = -\frac{1}{\sin\theta}, \quad w_{h2}(\mathbf{x}_j) = 0,
$$

$\mathbf{w}_h$ being zero at the other vertices.

Equality (41) follows from (40) and the previous discussion. $\qquad\square$

Remark 5.2. The *a priori* error estimates for the discretization (39) of the unilateral contact problem are given in [20]. The obtained estimates are the same as for the discretization (10) (see Rem. 3.4).

5.2. **Definition of the residual error estimators**

As for the first estimator the element residual is defined by $\mathbf{div}\boldsymbol{\sigma}(\mathbf{u}_h) + \mathbf{f} = \mathbf{f}$ on K and this element residual is replaced by some computable finite dimensional approximation called approximate element residual: $\mathbf{f}_K \in (\mathbb{P}_k(K))^2$. Similarly $\mathbf{g}$ is approximated by a computable quantity denoted $\mathbf{g}_E$ on any $E \in E_h^N$.

Definition 5.3 (Second residual error estimator)**.** The local and global residual error estimators are defined by

$$
\begin{aligned}
\tilde{\eta}_K &= \left(\sum_{i=1}^{5} \tilde{\eta}_{iK}^2\right)^{1/2}, \\
\tilde{\eta}_{1K} &= h_K \|\mathbf{f}_K\|_K, \\
\tilde{\eta}_{2K} &= h_K^{1/2}\left(\sum_{E \in E_K^{int}\cup E_K^N} \|J_{E,n}(\mathbf{u}_h)\|_E^2\right)^{1/2}, \\
\tilde{\eta}_{3K} &= h_K^{1/2}\left(\sum_{E \in E_K^C} \|\tilde{\lambda}_h + \sigma_n(\mathbf{u}_h)\|_E^2\right)^{1/2}, \\
\tilde{\eta}_{4K} &= h_K^{1/2}\left(\sum_{E \in E_K^C} \|\sigma_t(\mathbf{u}_h)\|_E^2\right)^{1/2}, \\
\tilde{\eta}_{5K} &= \left(\sum_{E \in E_K^C} \int_E -\tilde{\lambda}_h u_{hn}\right)^{1/2}, \\
\tilde{\eta} &= \left(\sum_{K \in T_h} \tilde{\eta}_K^2\right)^{1/2},
\end{aligned}
$$

where we recall that $J_{E,n}(\mathbf{u}_h)$ is the constraint jump of $\mathbf{u}_h$ in normal direction defined by (17).

As in the previous section, the local and global approximation terms ζ_K and ζ are defined by (18).

Remark 5.4. From the previous definitions we have $\tilde{\eta}_{1K} = \eta_{1K}$, $\tilde{\eta}_{2K} = \eta_{2K}$ and $\tilde{\eta}_{4K} = \eta_{4K}$. We mention that there is no term as η_6 in the second estimator since the multipliers $\tilde{\lambda}_h$ are of constant sign (note that a similar approach was adopted in [27] for the obstacle problem).

5.3. Upper error bound

Theorem 5.5. *Let $(\lambda, \mathbf{u})$ be the solution of (7) and let $(\tilde{\lambda}_h, \mathbf{u}_h)$ be the solution of (39). We have*

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|\lambda - \tilde{\lambda}_h\|_{-\frac{1}{2},\Gamma_C} \lesssim \tilde{\eta} + \zeta.$$

Proof. We adopt the following notations for the error term in the displacement: $\mathbf{e_u} = \mathbf{u} - \mathbf{u}_h$. Let $\mathbf{v}_h \in \mathbf{V}_h$. According to the **V**-ellipticity of $a(.,.)$ and the equilibrium equations in (7) and (39) we obtain:

$$
\begin{aligned}
\|\mathbf{e_u}\|_{1,\Omega}^2 &\lesssim a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h) \\
&= a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{v}_h) + a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) \\
&= L(\mathbf{u} - \mathbf{v}_h) - b(\lambda, \mathbf{u} - \mathbf{v}_h) - a(\mathbf{u}_h, \mathbf{u} - \mathbf{v}_h) + b(\tilde{\lambda}_h - \lambda, \mathbf{v}_h - \mathbf{u}_h) \\
&\quad + c(\tilde{\lambda}_h, \mathbf{v}_h - \mathbf{u}_h) - b(\tilde{\lambda}_h, \mathbf{v}_h - \mathbf{u}_h).
\end{aligned}
$$

Integrating by parts on each triangle K, using the definition of $J_{E,n}(\mathbf{u}_h)$ in (17) and the complementarity conditions $b(\lambda, \mathbf{u}) = b(\lambda_h, \mathbf{u}_h) = c(\tilde{\lambda}_h, \mathbf{u}_h) = 0$ gives:

$$
\begin{aligned}
\|\mathbf{e_u}\|_{1,\Omega}^2 &\lesssim \int_\Omega \mathbf{f} \cdot (\mathbf{u} - \mathbf{v}_h) + b(\lambda, \mathbf{u}_h) + c(\tilde{\lambda}_h, \mathbf{v}_h) + \sum_{E \in E_h^N} \int_E (\mathbf{g} - \mathbf{g}_E) \cdot (\mathbf{u} - \mathbf{v}_h) \\
&\quad - \sum_{E \in E_h^C} \int_E (\boldsymbol{\sigma}(\mathbf{u}_h)\mathbf{n}) \cdot (\mathbf{u} - \mathbf{v}_h) - \sum_{E \in E_h^{int} \cup E_h^N} \int_E J_{E,n}(\mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{v}_h),
\end{aligned}
$$

which is the same inequality as in (19) according to (40). Splitting up the integrals on Γ_C into normal and tangential components yields:

$$
\begin{aligned}
\|\mathbf{e_u}\|_{1,\Omega}^2 &\lesssim \int_\Omega \mathbf{f} \cdot (\mathbf{u} - \mathbf{v}_h) + b(\lambda, \mathbf{u}_h) + b(\tilde{\lambda}_h, \mathbf{u}) - b(\tilde{\lambda}_h, \mathbf{v}_h) + c(\tilde{\lambda}_h, \mathbf{v}_h) \\
&\quad + \sum_{E \in E_h^C} \int_E (\tilde{\lambda}_h + \sigma_n(\mathbf{u}_h))(v_{hn} - u_n) + \sum_{E \in E_h^C} \int_E \sigma_t(\mathbf{u}_h)(v_{ht} - u_t) \\
&\quad - \sum_{E \in E_h^{int} \cup E_h^N} \int_E J_{E,n}(\mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{v}_h) + \sum_{E \in E_h^N} \int_E (\mathbf{g} - \mathbf{g}_E) \cdot (\mathbf{u} - \mathbf{v}_h).
\end{aligned}
\tag{42}
$$

As before to estimate each term of this right-hand side, we take $\mathbf{v}_h$ of the form (21).

We start with the integral term. As in the case of the first estimator we deduce from (21) and (15) that

$$
\begin{aligned}
\left| \int_\Omega \mathbf{f} \cdot (\mathbf{u} - \mathbf{v}_h) \right|
&\leq \sum_{K \in T_h} \|\mathbf{f}\|_K \|\mathbf{u} - \mathbf{v}_h\|_K \\
&\lesssim \sum_{K \in T_h} \|\mathbf{f}\|_K h_K \|\mathbf{e_u}\|_{1,\omega_K} \\
&\lesssim (\tilde\eta + \zeta) \|\mathbf{e_u}\|_{1,\Omega}.
\end{aligned}
\tag{43}
$$

We now consider the interior and Neumann boundary terms in (42): as previously the application of Cauchy-Schwarz's inequality and (16) lead to

$$
\begin{aligned}
\left| \sum_{E \in E_h^{int} \cup E_h^N} \int_E J_{E,n}(\mathbf{u}_h) \cdot (\mathbf{u} - \mathbf{v}_h) \right|
&\leq \sum_{E \in E_h^{int} \cup E_h^N} \|J_{E,n}(\mathbf{u}_h)\|_E \|\mathbf{u} - \mathbf{v}_h\|_E \\
&\lesssim \sum_{E \in E_h^{int} \cup E_h^N} \|J_{E,n}(\mathbf{u}_h)\|_E h_E^{1/2} \|\mathbf{e_u}\|_{1,\omega_E} \\
&\lesssim \tilde\eta \|\mathbf{e_u}\|_{1,\Omega}.
\end{aligned}
\tag{44}
$$

Besides we get

$$
\left| \sum_{E \in E_h^N} \int_E (\mathbf{g} - \mathbf{g}_E) \cdot (\mathbf{u} - \mathbf{v}_h) \right| \lesssim \zeta \|\mathbf{e_u}\|_{1,\Omega}.
\tag{45}
$$

The two following terms are handled in a similar way as the previous ones so that

$$
\left| \sum_{E \in E_h^C} \int_E (\tilde\lambda_h + \sigma_n(\mathbf{u}_h))(v_{hn} - u_n) \right| \lesssim \tilde\eta \|\mathbf{e_u}\|_{1,\Omega}
\tag{46}
$$

and

$$
\left| \sum_{E \in E_h^C} \int_E \sigma_t(\mathbf{u}_h)(v_{ht} - u_t) \right| \lesssim \tilde\eta \|\mathbf{e_u}\|_{1,\Omega}.
\tag{47}
$$

Noting that $u_{hn} \leq 0$ and $\tilde\lambda_h \geq 0$ on Γ_C, we have

$$
b(\lambda, \mathbf{u}_h) \leq 0, \qquad b(\tilde\lambda_h, \mathbf{u}) \leq 0
\tag{48}
$$

and it remains to estimate two terms in (42), namely

$$
\begin{aligned}
c(\tilde\lambda_h, \mathbf{v}_h) - b(\tilde\lambda_h, \mathbf{v}_h)
&= -b(\tilde\lambda_h, \mathbf{u}_h) + c(\tilde\lambda_h, \pi_h \mathbf{e_u}) - b(\tilde\lambda_h, \pi_h \mathbf{e_u}) \\
&\leq \tilde\eta^2 + \int_{\Gamma_C} I_h(\tilde\lambda_h(\pi_h \mathbf{e_u})_n) - \tilde\lambda_h(\pi_h \mathbf{e_u})_n.
\end{aligned}
\tag{49}
$$

The integral term in the previous expression is estimated as follows using a basic error estimate of numerical integration (trapezoidal formula):

$$
\begin{aligned}
\left| \int_{\Gamma_C} I_h(\tilde{\lambda}_h(\pi_h \mathbf{e_u})_n) - \tilde{\lambda}_h(\pi_h \mathbf{e_u})_n \right| &= \left| \sum_{E \in E_h^C} \int_E I_h(\tilde{\lambda}_h(\pi_h \mathbf{e_u})_n) - \tilde{\lambda}_h(\pi_h \mathbf{e_u})_n \right| \\
&\lesssim \sum_{E \in E_h^C} h_E^3 |(\tilde{\lambda}_h(\pi_h \mathbf{e_u})_n)''| \\
&\lesssim \sum_{E \in E_h^C} h_E^3 |\tilde{\lambda}_h'((\pi_h \mathbf{e_u})_n)'| \\
&\lesssim \sum_{E \in E_h^C} h_E^2 \|\tilde{\lambda}_h'\|_E \|((\pi_h \mathbf{e_u})_n)'\|_E \\
&\lesssim \sum_{E \in E_h^C} h_E^{3/2} \|\tilde{\lambda}_h'\|_E \|\pi_h \mathbf{e_u}\|_{1,\omega_E} \\
&\lesssim \sum_{E \in E_h^C} h_E^{3/2} \|\tilde{\lambda}_h'\|_E \|\mathbf{e_u}\|_{1,\omega_E} \\
&= \sum_{E \in E_h^C} h_E^{3/2} \|(\tilde{\lambda}_h + \sigma_n(\mathbf{u}_h))'\|_E \|\mathbf{e_u}\|_{1,\omega_E} \\
&\lesssim \sum_{E \in E_h^C} h_E^{1/2} \|\tilde{\lambda}_h + \sigma_n(\mathbf{u}_h)\|_E \|\mathbf{e_u}\|_{1,\omega_E} \\
&\leq \tilde{\eta} \|\mathbf{e_u}\|_{1,\Omega}.
\end{aligned}
\tag{50}
$$

Above we have used the H^1 stability of π_h, proved in Lemma 3.1 of [8] (see also [28]) and the trace inequality on an element (see [28]).

Putting together the estimates (43)–(50) in (42) and using Young's inequality, we come to the conclusion that

$$
\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \lesssim \tilde{\eta} + \zeta.
\tag{51}
$$

Next we search for an upper bound on the discretization error $\lambda - \lambda_h$ corresponding to the multipliers. Let $\mathbf{v} \in \mathbf{V}$ and $\mathbf{v}_h \in \mathbf{V}_h$. From the equilibrium equations in (7) and (39) we get:

$$
\begin{aligned}
b(\lambda - \tilde{\lambda}_h, \mathbf{v}) &= b(\lambda, \mathbf{v} - \mathbf{v}_h) - b(\tilde{\lambda}_h, \mathbf{v} - \mathbf{v}_h) + b(\lambda - \lambda_h, \mathbf{v}_h) + b(\lambda_h - \tilde{\lambda}_h, \mathbf{v}_h) \\
&= L(\mathbf{v} - \mathbf{v}_h) - a(\mathbf{u}, \mathbf{v} - \mathbf{v}_h) - b(\tilde{\lambda}_h, \mathbf{v} - \mathbf{v}_h) + a(\mathbf{u}_h - \mathbf{u}, \mathbf{v}_h) + b(\lambda_h - \tilde{\lambda}_h, \mathbf{v}_h) \\
&= L(\mathbf{v} - \mathbf{v}_h) - a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) - a(\mathbf{u}_h, \mathbf{v} - \mathbf{v}_h) - b(\tilde{\lambda}_h, \mathbf{v} - \mathbf{v}_h) + b(\lambda_h - \tilde{\lambda}_h, \mathbf{v}_h).
\end{aligned}
$$

An integration by parts on each element K yields

$$
\begin{aligned}
b(\lambda - \tilde{\lambda}_h, \mathbf{v}) &= \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{v}_h) - a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) - \sum_{E \in E_h^{int} \cup E_h^N} \int_E J_{E,n}(\mathbf{u}_h) \cdot (\mathbf{v} - \mathbf{v}_h) \\
&\quad - \sum_{E \in E_h^C} \int_E (\tilde{\lambda}_h + \sigma_n(\mathbf{u}_h))(v_n - v_{hn}) - \sum_{E \in E_h^C} \int_E \sigma_t(\mathbf{u}_h)(v_t - v_{ht}) \\
&\quad + c(\tilde{\lambda}_h, \mathbf{v}_h) - b(\tilde{\lambda}_h, \mathbf{v}_h) + \sum_{E \in E_h^N} \int_E (\mathbf{g} - \mathbf{g}_E) \cdot (\mathbf{v} - \mathbf{v}_h).
\end{aligned}
$$

Choosing $\mathbf{v}_h = \pi_h \mathbf{v}$ where π_h is the quasi-interpolation operator defined Lemma 3.6 and achieving a similar calculation as in (43)–(47) and (50) we deduce that

$$\left| b(\lambda - \tilde{\lambda}_h, \mathbf{v}) \right| \lesssim (\tilde{\eta} + \zeta + \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}) \|\mathbf{v}\|_{1,\Omega}$$

for any $\mathbf{v} \in \mathbf{V}$. As a consequence

$$\|\lambda - \tilde{\lambda}_h\|_{-\frac{1}{2},\Gamma_C} \lesssim \tilde{\eta} + \zeta + \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}. \tag{52}$$

Putting together the two estimates (51) and (52) ends the proof of the theorem. □

5.4. Lower error bound

Theorem 5.6. *For all elements K, the following local lower error bounds hold:*

$$\tilde{\eta}_{1K} \lesssim \|\mathbf{u} - \mathbf{u}_h\|_{1,K} + \zeta_K, \tag{53}$$

$$\tilde{\eta}_{2K} \lesssim \|\mathbf{u} - \mathbf{u}_h\|_{1,\omega_K} + \zeta_K. $$

Assume that $\lambda \in L^2(\Gamma_C)$. For all elements K such that $K \cap E_h^C \neq \emptyset$, the following local lower error bounds hold:

$$\tilde{\eta}_{3K} \quad \lesssim \quad \sum_{E \in E_K^C} h_E^{1/2} \|\lambda - \tilde{\lambda}_h\|_E + \|\mathbf{u} - \mathbf{u}_h\|_{1,K} + \zeta_K, \tag{54}$$

$$\tilde{\eta}_{4K} \quad \lesssim \quad \|\mathbf{u} - \mathbf{u}_h\|_{1,K} + \zeta_K,$$

$$\tilde{\eta}_{5K} \quad \lesssim \quad \tilde{\eta}_{3K}^{1/2} \|\mathbf{u}_h\|_{1,K}^{1/2}. \tag{55}$$

Proof. According to Remark 5.4 and Theorem 4.4 we only need to estimate $\tilde{\eta}_{3K}$ and $\tilde{\eta}_{5K}$.

We first estimate $\tilde{\eta}_{3K}$. Writing $\mathbf{w}_E = w_{En}\mathbf{n} + w_{Et}\mathbf{t}$ on $E \in E_K^C$ and denoting by b_E the edge bubble function associated with E (*i.e.*, $b_E = 4\lambda_{a_1}\lambda_{a_2}$, when a_1, a_2 are the two extremities of E), we choose $w_{En} = (\tilde{\lambda}_h + \sigma_n(\mathbf{u}_h))b_E$ and $w_{Et} = 0$ in the element K containing E (here we made a slight abuse of notation to simplify) and $\mathbf{w}_E = \mathbf{0}$ in $\overline{\Omega} \setminus K$. So

$$\begin{aligned}
\|\tilde{\lambda}_h + \sigma_n(\mathbf{u}_h)\|_E^2 \quad &\sim \quad \int_E (\tilde{\lambda}_h + \sigma_n(\mathbf{u}_h)) w_{En} \\
&= \quad b(\tilde{\lambda}_h, \mathbf{w}_E) + \int_K \boldsymbol{\sigma}(\mathbf{u}_h) : \boldsymbol{\varepsilon}(\mathbf{w}_E) \\
&= \quad b(\tilde{\lambda}_h, \mathbf{w}_E) - \int_K \boldsymbol{\sigma}(\mathbf{u} - \mathbf{u}_h) : \boldsymbol{\varepsilon}(\mathbf{w}_E) + \int_K \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{w}_E) \\
&= \quad b(\tilde{\lambda}_h - \lambda, \mathbf{w}_E) + L(\mathbf{w}_E) - \int_K \boldsymbol{\sigma}(\mathbf{u} - \mathbf{u}_h) : \boldsymbol{\varepsilon}(\mathbf{w}_E) \\
&\lesssim \quad \|\lambda - \tilde{\lambda}_h\|_E \|\mathbf{w}_E\|_E + \|\mathbf{f}\|_K \|\mathbf{w}_E\|_K + \|\mathbf{u} - \mathbf{u}_h\|_{1,K} \|\mathbf{w}_E\|_{1,K}.
\end{aligned}$$

An inverse inequality and estimate (53) imply

$$\begin{aligned}
h_K^{1/2} \|\tilde{\lambda}_h + \sigma_n(\mathbf{u}_h)\|_E \quad &\lesssim \quad h^{1/2} \|\lambda - \tilde{\lambda}_h\|_E + \|\mathbf{u} - \mathbf{u}_h\|_{1,K} + h\|\mathbf{f}\|_K \\
&\lesssim \quad h^{1/2} \|\lambda - \tilde{\lambda}_h\|_E + \|\mathbf{u} - \mathbf{u}_h\|_{1,K} + \zeta_K.
\end{aligned}$$

This bound gives the estimate of $\tilde{\eta}_{3K}$ in (54).

We finally consider $\tilde{\eta}_{5K}$. If $E \in E_K^C$, one has

$$
\begin{aligned}
\int_E -\tilde{\lambda}_h u_{hn} &= \int_E I_h(\tilde{\lambda}_h u_{hn}) - \tilde{\lambda}_h u_{hn} \\
&\lesssim h_E^3 |(\tilde{\lambda}_h u_{hn})''| \\
&\lesssim h_E^3 |\tilde{\lambda}_h' u_{hn}'| \\
&\lesssim h_E^2 \|\tilde{\lambda}_h'\|_E \|u_{hn}'\|_E \\
&= h_E^2 \|(\tilde{\lambda}_h + \sigma_n(\mathbf{u}_h))'\|_E \|u_{hn}'\|_E \\
&\lesssim h_E \|\tilde{\lambda}_h + \sigma_n(\mathbf{u}_h)\|_E \|u_{hn}'\|_E \\
&\lesssim h_E^{1/2} \tilde{\eta}_{3K} \|u_{hn}'\|_E \\
&\lesssim \tilde{\eta}_{3K} \|\mathbf{u}_h\|_{1,K}.
\end{aligned}
$$

The last estimate implies (55) by taking the square root.   □

Remark 5.7. Assume that $\mathbf{u} \in (H^2(\Omega))^2$ (so that $\lambda \in H^{\frac{1}{2}}(\Gamma_C)$). Then the integral term in $\tilde{\eta}_{5K}$ can be bounded as follows:

$$
\begin{aligned}
\int_E -\tilde{\lambda}_h u_{hn} &= \int_E I_h(\tilde{\lambda}_h u_{hn}) - \tilde{\lambda}_h u_{hn} \\
&\lesssim h_E^3 |(\tilde{\lambda}_h u_{hn})''| \\
&\lesssim h_E^3 |\tilde{\lambda}_h' u_{hn}'| \\
&= h_E^2 \|\tilde{\lambda}_h' u_{hn}'\|_{L^1(E)} \\
&\leq h_E^2 \|\tilde{\lambda}_h'(u_{hn}' - u_n')\|_{L^1(E)} + h_E^2 \|\tilde{\lambda}_h' u_n'\|_{L^1(E)} \\
&\lesssim h_E^2 \|\tilde{\lambda}_h'\|_E \|(u_{hn} - u_n)'\|_E + h_E^2 \|\tilde{\lambda}_h'\|_{L^{\frac{q}{q-1}}(E)} \|u_n'\|_{L^q(\Gamma_C)} \\
&\lesssim h_E^2 \|(\tilde{\lambda}_h + \sigma_n(\mathbf{u}_h))'\|_E \|(u_{hn} - u_n)'\|_E + h_E^2 \sqrt{q} \|(\tilde{\lambda}_h + \sigma_n(\mathbf{u}_h))'\|_{L^{\frac{q}{q-1}}(E)} \|\mathbf{u}\|_{2,\Omega} \\
&\lesssim h_E \|\tilde{\lambda}_h + \sigma_n(\mathbf{u}_h)\|_E \|(u_{hn} - u_n)'\|_E + h_E^2 h_E^{\frac{q-2}{2q}} \sqrt{q} \|(\tilde{\lambda}_h + \sigma_n(\mathbf{u}_h))'\|_E \|\mathbf{u}\|_{2,\Omega} \\
&\lesssim h_E^{1/2} \tilde{\eta}_{3K} \|u_n - u_{hn}\|_{1,E} + h_E \sqrt{-\ln(h_E)} \tilde{\eta}_{3K} \|\mathbf{u}\|_{2,\Omega},
\end{aligned}
$$

where $1 < q < \infty$ (we choose $q = -\ln(h_E)$, h_E is supposed small enough) and we have used the following embedding (see [3]): for any real number $p \in [1, \infty[$,

$$
\|v\|_{L^p(\Gamma_C)} \leq C\sqrt{p} \|v\|_{H^{\frac{1}{2}}(\Gamma_C)}, \quad \forall v \in H^{\frac{1}{2}}(\Gamma_C),
$$

where C is independent of p. Define

$$
\tilde{\eta}_i = \left(\sum_{K \in T_h} \tilde{\eta}_{iK}^2 \right)^{1/2}, \quad 1 \leq i \leq 5.
$$

Assume that optimal *a priori* error estimates hold (this requires additional assumptions, see Rem. (5.2)). Then the previous result leads to the following bounds: $\tilde{\eta}_i \lesssim h, 1 \leq i \leq 4$ and $\tilde{\eta}_5 \lesssim h^{3/4}(-\ln(h))^{1/4}$. Therefore $\tilde{\eta} \lesssim h^{3/4}(-\ln(h))^{1/4}$. If we use only the known *a priori* error estimates (see Rem. (5.2)), we get: $\tilde{\eta}_i \lesssim h^{3/4}, 1 \leq i \leq 4$ and $\tilde{\eta}_5 \lesssim h^{5/8}(-\ln(h))^{1/4}$. Therefore $\tilde{\eta} \lesssim h^{5/8}(-\ln(h))^{1/4}$ and our result is not far away from "optimality" since we have a loss of convergence of only $h^{1/8}(-\ln(h))^{-1/4}$.

The second error estimator allows us to obtain improved upper and lower bounds of the error in comparison with the one in the previous section. Note that the definition of the discretization error was modified in this section where we compare $\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|\lambda - \tilde{\lambda}_h\|_{-\frac{1}{2},\Gamma_C}$ with $\tilde{\eta}$ (whereas in the previous section we compare $\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|\lambda - \lambda_h\|_{-\frac{1}{2},\Gamma_C}$ with η). The forthcoming numerical experiments will help us to compare the performances of both estimators and suggest us that $\tilde{\eta}$ is more appropriate than η for the unilateral contact problem.

6. NUMERICAL EXPERIMENTS

This section is concerned with the numerical implementation of both estimators. We suppose that the bodies are homogeneous isotropic materials so that Hooke's law (2) becomes:

$$\boldsymbol{\sigma}(\mathbf{u}) = \frac{E\nu}{(1-2\nu)(1+\nu)}\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u}))\mathbf{I} + \frac{E}{1+\nu}\boldsymbol{\varepsilon}(\mathbf{u}) \tag{56}$$

where $\mathbf{I}$ represents the identity matrix, tr is the trace operator, E and ν denote Young's modulus and Poisson's ratio, respectively with $E > 0$ and $0 \leq \nu < 1/2$.

Our main aim is to validate our theoretical results by computing the different contributions of the estimators η and $\tilde{\eta}$ and their orders of convergence for different meshes. We also compute some effectivity indices and show that the estimator can be determined in more general cases than the theoretical framework. In our numerical experiments we do not consider optimized computations obtained from the estimators and a mesh adaptivity procedure which are beyond the scope of this paper. In the following we denote by N_C, the number of elements of the mesh on Γ_C. Since we use uniform meshes (made of triangular elements), this parameter measures the size of the mesh.

6.1. First example: comparison of the error terms in both estimators

We consider the domain $\Omega =]0,1[\times]0,1[$. We choose a realistic physical example. Namely we suppose that the body is an iron square of 1 m^2 whose material characteristics are $E = 2.1 \times 10^{11}$ Pa, $\nu = 0.3$ and $\rho = 7800$ kg $\cdot$ m^{-3}. The body is clamped on $\Gamma_D = \{1\}\times]0,1[$, it is initially in contact with $\Gamma_C = \{0\}\times]0,1[$ and it is acted on by its own weight only (with $g = 9.81$ m $\cdot$ s^{-2}). Moreover $\Gamma_N =]0,1[\times(\{0\}\cup\{1\})$. We begin with achieving computations involving criss-cross meshes (this means that the body is divided into identical squares, each of them being divided into four identical triangles).

We observe (see Fig. 1) that Γ_C is divided into two parts: an upper part where the body remains in contact with the axis $x = 0$ and the lower part of Γ_C where it separates from this axis. We determine the convergence of all the terms involved in both estimators η and $\tilde{\eta}$ and we report the results in Tables 1 and 2 where we adopt the notations of Remarks 4.5 and 5.7:

$$\eta_i = \left(\sum_{K \in T_h} \eta_{iK}^2\right)^{1/2}, 1 \leq i \leq 6, \qquad \tilde{\eta}_j = \left(\sum_{K \in T_h} \tilde{\eta}_{jK}^2\right)^{1/2}, 1 \leq j \leq 5.$$

In these tables we compute the average convergence rates by averaging the rates between $N_C = 4$ and $N_C = 128$ and we give the limit rates if we observe that the rates seem to converge (this is not always the case). Note that the convergence rate of the terms: $\eta_1 = \tilde{\eta}_1 = h(\sum_{K \in T_h} \|\mathbf{f}_K\|_K^2)^{1/2} \sim h$ is 1.

From the experiments we see that the terms $\eta_2 = \tilde{\eta}_2$, η_3, $\tilde{\eta}_3$ and $\eta_4 = \tilde{\eta}_4$ converge towards zero and that η_3 is close to $\tilde{\eta}_3$ (η_2 is the term converging the slowest towards zero and we observe that the main part of the error in η and $\tilde{\eta}$ is located near the singular points $(1,0)$ and $(1,1)$). The error terms measuring the non fulfillment of the complementary condition (i.e., η_5 and $\tilde{\eta}_5$) show a convergence rate close to 1.5 which is much higher than the ones expected from the theoretical part (see Rems. 4.5 and 5.7). Moreover η_5 and $\tilde{\eta}_5$ are small in comparison with the other terms: this is not surprising since these terms are the only ones which depend on the Young modulus E. More precisely, if $\mathbf{u}(E)$ solves the elasticity problem in Figure 1 with a Young modulus E
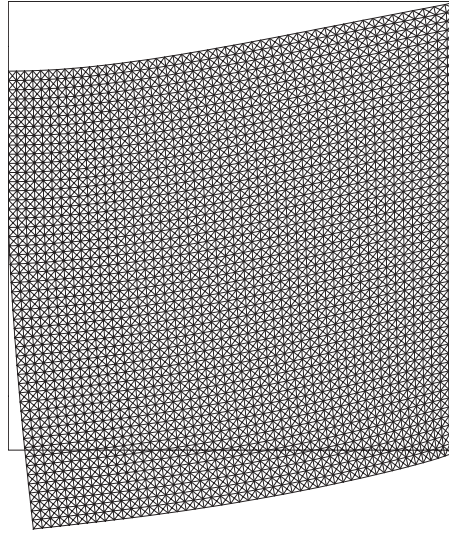
FIGURE 1. Initial and deformed configuration with $N_C = 50$ (deformation is amplified by a factor 2×10^5).

TABLE 1. Contributions in the estimator η (criss-cross mesh).

Estimator η	η_2	η_3	η_4	η_5	η_6
$N_C = 1$	94 716	35 788	29 586	3.20112×10^{-2}	13 771
$N_C = 2$	102 670	23 307	14 128	1.96822×10^{-2}	12 322
$N_C = 4$	78 460	9240.6	5989.7	5.02411×10^{-3}	820.96
$N_C = 8$	50 672	3150.8	2429.3	3.32207×10^{-3}	1562.1
$N_C = 16$	30 376	1108.1	938.10	4.31419×10^{-4}	53.836
$N_C = 32$	17 806	389.83	356.38	1.67433×10^{-4}	31.015
$N_C = 64$	10 424	138.49	134.19	6.30827×10^{-5}	17.193
$N_C = 128$	6142.2	49.502	50.194	2.39545×10^{-5}	9.8331
Convergence:					
Average rate	0.74	1.51	1.38	1.54	1.28
Limit rate	0.76	1.48	1.42	1.40	0.81

then it is easy to check from (56) that $\mathbf{u}(E)/k$ solves the same problem with a Young modulus kE (if we had nonhomogeneous Dirichlet conditions, this would not be true) whereas $\boldsymbol{\sigma}(\mathbf{u}(E)) = \boldsymbol{\sigma}(\mathbf{u}(kE))$. This implies that η_5 and $\tilde{\eta}_5$ behave as $E^{-1/2}$ and maybe a normalization of η_5 and $\tilde{\eta}_5$ would be necessary to avoid this phenomenon. The term η_6 whose theoretical convergence rate is also not optimal shows a non uniform decay towards zero, but faster than η_2. So we can reasonably expect that η_2 is the greatest term when $N_C \to +\infty$.

If we choose more general unstructured quasi-uniform meshes (instead of criss-cross meshes) on Ω we obtain the results reported in Tables 3 and 4.

The main conclusions are similar to the ones when criss-cross meshes are used and we notice that the terms η_5 and η_6 converge rapidly but with a non uniform rate towards zero.

From this test we conclude that the implementation of $\tilde{\eta}$ is simpler than for η: there is one term less in the computation of $\tilde{\eta}$ and the terms η_5 and η_6 involve negative parts of λ_h (this would be more difficult to compute,

TABLE 2. Contributions in the estimator $\tilde{\eta}$ (criss-cross mesh).

Estimator $\tilde{\eta}$	$\tilde{\eta}_2$	$\tilde{\eta}_3$	$\tilde{\eta}_4$	$\tilde{\eta}_5$
$N_C = 1$	94 716	11 959	29 586	3.39530×10^{-2}
$N_C = 2$	102 670	9536.6	14 128	1.04538×10^{-2}
$N_C = 4$	78 460	5014.3	5989.7	6.93578×10^{-3}
$N_C = 8$	50 672	2270.7	2429.3	2.16306×10^{-3}
$N_C = 16$	30 376	900.17	938.10	7.61047×10^{-4}
$N_C = 32$	17 806	339.23	356.38	2.71989×10^{-4}
$N_C = 64$	10 424	125.04	134.19	9.69379×10^{-5}
$N_C = 128$	6142.2	45.618	50.194	3.46036×10^{-5}
Convergence:				
Average rate	0.74	1.36	1.38	1.53
Limit rate	0.76	1.45	1.42	1.49

TABLE 3. Contributions in the estimator η (unstructured mesh)

| Estimator η	η_2	η_3	η_4	η_5	η_6
$N_C = 1$	88 542	31 184	32 591	2.24674×10^{-2}	9851.5
$N_C = 2$	99 918	21 238	13 833	1.70126×10^{-2}	10 315
$N_C = 4$	95 846	10 861	5703.5	2.50567×10^{-3}	737.97
$N_C = 8$	69 987	4082.7	2448.9	3.27411×10^{-3}	1502.8
$N_C = 16$	44 123	1494.5	981.63	3.83538×10^{-4}	144.18
$N_C = 32$	26 776	505.22	373.78	1.0720×10^{-4}	42.542
$N_C = 64$	15 548	158.77	145.82	2.53311×10^{-5}	9.1921
$N_C = 128$	9296.4	54.193	53.902	3.58675×10^{-6}	0.72587
Convergence:					
Average rate	0.67	1.53	1.35	1.89	2.00
Limit rate	0.74	1.55	1.44	no limit rate	no limit rate

especially in the three-dimensional case). Besides it seems that the convergence rate of $\tilde{\eta}_5$ is more uniform than the ones of η_5 and η_6 and that there are very few elements (near the transition points from contact to separation) where the error of $\tilde{\eta}_5$ is located and this is not the case for η_5 and η_6.

6.2. Second example: a more regular case

We consider the geometry $\hat{\Omega} =]0, 2[\times]0, 1[$ of area 2 square meters. We adopt symmetry conditions (*i.e.*, $u_n = 0, \sigma_t(\mathbf{u}) = 0$) on $\Gamma_S = \{1\}\times]0, 1[$ and we achieve the computations on the square $\Omega =]0, 1[\times]0, 1[$. We set $\Gamma_C =]0, 1[\times\{0\}$ and Γ_N is the remaining part of the boundary of Ω. A Poisson ratio of $\nu = 0.2$ and a Young modulus of $E = 1$ Pa are chosen (the latter value is of course not realistic from a physical point of view). A density of surface forces $\mathbf{g}$ of magnitude 1 N·m^{-2} oriented inwards Ω is applied on $\{0\}\times]1/2, 1[$ and $]1/2, 1[\times\{1\}$. Such a configuration corresponds to a $\mathbf{K}$-elliptic case (see [19], Thm. 6.3) and the problem admits a unique solution. We use criss-cross meshes in this example. Figure 2 depicts the initial and deformed configurations of the body. Here again Γ_C shows a contact and a separation part.

TABLE 4. Contributions in the estimator $\tilde{\eta}$ (unstructured mesh).

Estimator $\tilde{\eta}$	$\tilde{\eta}_2$	$\tilde{\eta}_3$	$\tilde{\eta}_4$	$\tilde{\eta}_5$
$N_C = 1$	88 542	19 751	32 591	2.38302×10^{-2}
$N_C = 2$	99 918	10 687	13 833	1.06738×10^{-2}
$N_C = 4$	95 846	7155.9	5703.5	5.95065×10^{-3}
$N_C = 8$	69 987	3387.7	2448.9	2.20742×10^{-3}
$N_C = 16$	44 123	1288.3	981.63	6.76565×10^{-4}
$N_C = 32$	26 776	453.06	373.78	2.45212×10^{-4}
$N_C = 64$	15 548	143.92	145.82	8.87555×10^{-5}
$N_C = 128$	9296.4	49.590	53.902	3.17811×10^{-5}
Convergence Average rate	0.67	1.43	1.35	1.51
Limit rate	0.74	1.54	1.44	1.48
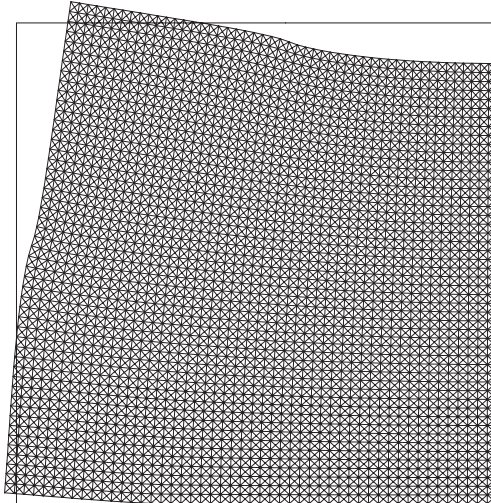
FIGURE 2. Initial and deformed configuration with $N_C = 50$ (deformation is amplified by a factor 0.1).

It is easy to see that the symmetry conditions on Γ_S lead to supplementary error terms similar to the ones in $\eta_4 = \tilde{\eta}_4$ and we add these terms to η_2 and $\tilde{\eta}_2$. Moreover we have $\eta_1 = \tilde{\eta}_1 = 0$. The results concerning both estimators are reported in Tables 5 and 6.

As in the previous example, η_2 is the main term in the estimators with the lowest (but greater then in the previous example) convergence rate. We observe that the error is mainly located near the transition point between contact and separation and near the singularities $(0, 1/2)$ and $(1/2,1)$ due to the jumps of the density of surface forces at these points and also, due to the very small Young modulus. As in the previous example the error terms η_5 and $\tilde{\eta}_5$ measuring the non fulfillment of the complementary condition converge with an higher rate than theoretically expected. The term η_6 shows a slow convergence rate in comparison with η_5 and $\tilde{\eta}_5$.

If we denote by $(a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h))^{1/2}$ the energy norm of the discretization error (which is equivalent to the $(H^1(\Omega))^2$- norm of the error), we compute the convergence rates α, $\tilde{\alpha}$ and β of η, $\tilde{\eta}$ and $(a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h))^{1/2}$

TABLE 5. Contributions in the estimator η.

Estimator η	η_2	η_3	η_4	η_5	η_6
$N_C = 2$	0.93715	0.16483	5.46634×10^{-2}	5.77944×10^{-2}	7.91410×10^{-2}
$N_C = 4$	0.56976	7.17157×10^{-2}	3.63730×10^{-2}	2.32941×10^{-2}	4.26924×10^{-2}
$N_C = 8$	0.33391	2.27644×10^{-2}	1.53814×10^{-2}	1.67857×10^{-2}	1.62697×10^{-2}
$N_C = 16$	0.19152	1.00065×10^{-2}	7.79360×10^{-3}	4.73869×10^{-3}	7.69676×10^{-3}
$N_C = 32$	0.10682	3.66909×10^{-3}	3.24986×10^{-3}	2.44547×10^{-3}	5.90999×10^{-3}
$N_C = 64$	5.84672×10^{-2}	1.17072×10^{-3}	1.24725×10^{-3}	8.11375×10^{-4}	1.44793×10^{-3}
$N_C = 128$	3.15747×10^{-2}	4.85373×10^{-4}	5.05263×10^{-4}	3.12555×10^{-4}	1.38974×10^{-3}
Convergence: Average rate	0.83	1.44	1.23	1.24	0.99
Limit rate	0.89	1.27	1.30	1.38	no limit rate

TABLE 6. Contributions in the estimator $\tilde{\eta}$.

Estimator $\tilde{\eta}$	$\tilde{\eta}_2$	$\tilde{\eta}_3$	$\tilde{\eta}_4$	$\tilde{\eta}_5$
$N_C = 2$	0.93715	0.10459	5.46634×10^{-2}	7.89660×10^{-2}
$N_C = 4$	0.56976	5.01141×10^{-2}	3.63730×10^{-2}	1.41282×10^{-2}
$N_C = 8$	0.33391	1.84373×10^{-2}	1.53814×10^{-2}	1.30765×10^{-2}
$N_C = 16$	0.19152	7.52496×10^{-3}	7.79360×10^{-3}	2.79887×10^{-3}
$N_C = 32$	0.10682	2.85827×10^{-3}	3.24986×10^{-3}	1.04594×10^{-3}
$N_C = 64$	5.84672×10^{-2}	1.03522×10^{-3}	1.24725×10^{-3}	6.23388×10^{-4}
$N_C = 128$	3.15747×10^{-2}	4.01120×10^{-4}	5.05263×10^{-4}	1.50507×10^{-4}
Convergence: Average rate	0.83	1.39	1.23	1.31
Limit rate	0.89	1.37	1.30	no limit rate

respectively. Moreover we are interested in determining the effectivity indices:

$$\gamma = \frac{\eta}{(a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h))^{1/2}} \quad \text{and} \quad \tilde{\gamma} = \frac{\tilde{\eta}}{(a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h))^{1/2}}.$$

These ratios measure the reliability of our estimators.

To our knowledge this problem does not admit an explicit solution $\mathbf{u}$. So, in order to determine $(a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h))^{1/2}$, we need to compute a reference solution denoted by $\mathbf{u}_{ref}$ corresponding to a mesh which is as fine as possible. The most refined mesh corresponds to $N_C = 128$ and it furnishes the reference solution $\mathbf{u}_{ref}$ which is the chosen approximation for $\mathbf{u}$.

The results are reported in Table 7 where the errors are computed from $N_C = 2$ to $N_C = 32$ (the value $N_C = 64$ would give a underestimated error in the energy norm since the field $\mathbf{u}_h$ is then too close to the reference solution). The average convergence rates (between $N_C = 2$ and $N_C = 32$) are the following: $\alpha = 0.79$, $\tilde{\alpha} = 0.79$ and $\beta = 0.84$ and are therefore close. We also observe that the effectivity indices vary between 3.24 and 3.80 which correspond to reasonable values.

TABLE 7. Estimators, error in the energy norm and effectivity indices.

	η	$\tilde{\eta}$	$(a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h))^{1/2}$	γ	$\tilde{\gamma}$
$N_C = 2$	0.95823	0.94784	0.29249	3.28	3.24
$N_C = 4$	0.57746	0.57329	0.17068	3.38	3.36
$N_C = 8$	0.33585	0.33503	9.66688×10^{-2}	3.47	3.47
$N_C = 16$	0.19215	0.19185	5.32324×10^{-2}	3.61	3.61
$N_C = 32$	0.10712	0.10691	2.81764×10^{-2}	3.80	3.79

TABLE 8. Contributions in the estimator $\tilde{\eta}$.

Estimator $\tilde{\eta}$	$\tilde{\eta}_2$	$\tilde{\eta}_3$	$\tilde{\eta}_4$	$\tilde{\eta}_5$
$h = 1/10$	236.36	48.899	41.727	0.20734
$h = 1/20$	138.60	17.341	20.835	6.25309×10^{-2}
$h = 1/40$	89.959	10.645	10.912	2.98565×10^{-2}
$h = 1/80$	42.857	2.2564	4.3787	1.68111×10^{-2}
$h = 1/160$	22.865	1.1549	2.5559	4.50074×10^{-3}
Convergence: Average rate	0.84	1.35	1.00	1.38

In this example the term η_6 converges slowly whereas the terms η_3 and $\tilde{\eta}_3$, η_5 and $\tilde{\eta}_5$ are similar. Concerning the convergence rates of η and $\tilde{\eta}$ we find that there are similar and that the effectivity indices are also very close. So we conclude that there is no reason to choose η instead of $\tilde{\eta}$ which is simpler to implement.

6.3. **Third example: a Hertz type problem**

In this test we consider the contact problem between an elastic disc (of 1 m in diameter) and a rigid half plane which corresponds to a Hertz type problem. A Poisson ratio of 0.4 and a Young modulus $E = 10\,000$ Pa are chosen. The aim of this example is to extend the range of applicability of the estimator $\tilde{\eta}$ to a more general case involving a curved contact zone with an initial gap between the foundation and the elastic body and an increasing contact area. Initially in the unconstrained configuration, the contact part between the disc and the half-plane is a single point. A density of surface loads $\mathbf{g} = (0, -200)$ N $\cdot$ m^{-2} is applied on the upper quarter part of the boundary so that the problem becomes symmetric. We use quasi-uniform unstructured meshes. Note that the unilateral contact conditions in (5) have to be changed to take into account the gap between the contacting bodies. The conditions modelling unilateral contact on Γ_C become:

$$u_n - \xi \le 0, \qquad \sigma_n(\mathbf{u}) \le 0, \qquad \sigma_n(\mathbf{u})\,(u_n - \xi) = 0,$$

where $\xi = \xi(\mathbf{x})$ is the distance from $\mathbf{x} \in \Gamma_C$ to the rigid foundation. As a consequence the definition of $\tilde{\eta}_5$ in Definition 5.3 has to be changed into

$$\tilde{\eta}_{5K} = \left(\sum_{E \in E_K^C} \int_E -\tilde{\lambda}_h (u_{hn} - \xi) \right)^{1/2}.$$

The initial and a deformed configuration are depicted in Figure 3. The results concerning the implementation of $\tilde{\eta}$ are reported in Table 8.
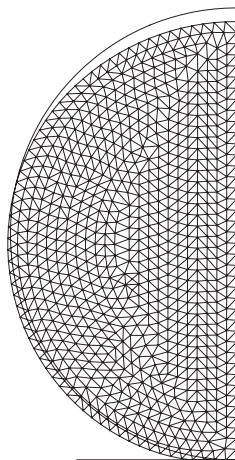
FIGURE 3. Initial and deformed configuration (deformation is not amplified).

As previously explained and observed in the first example, the term $\tilde{\eta}_5$ is small and it admits a convergence rate which is really more important than theoretically expected. We obtain for this example results which are similar to the previous ones.

7. CONCLUSION

In this work we propose and analyze two estimators (η and $\tilde{\eta}$) of residual type associated with two mixed finite element approximations of the two-dimensional frictionless unilateral contact problem in linear elasticity. For both estimators we obtain upper and lower bounds of the discretization error. From the numerical experiments we come to the conclusion that the results given by the two estimators are roughly speaking similar and that all the terms converge towards zero with satisfactory convergence rates (the slowest convergence is observed for the classical terms denoted by η_2 and $\tilde{\eta}_2$ which measure in particular the constraint jumps across the interior edges, and all the terms coming from the contact approximation converge better). Nevertheless $\tilde{\eta}$ has one term less to evaluate in comparison with η and its numerical implementation is (a bit) simpler. We also observe that the supplementary term in η (*i.e.*, η_6) does not admit in the general case a uniform convergence rate. Our conclusion concerning the comparison of both estimators is that $\tilde{\eta}$ could be more promising than η. Besides we see that the error terms measuring the non fulfillment of the complementarity condition (*i.e.*, η_5 and $\tilde{\eta}_5$) converge much faster than theoretically expected; this allows us to expect that some improved theoretical estimates for these terms could be obtained.

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