ABOUT ASYMPTOTIC APPROXIMATIONS IN THIN WAVEGUIDES

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Abstract. We study the propagation of electromagnetic waves in a guide the section of which is a thin annulus. Owing to the presence of a small parameter, explicit approximations of the TM and TE eigenmodes are obtained. The cases of smooth and non smooth boundaries are presented.

1. INTRODUCTION

Physical studies often give rise to problems of applied mathematics. To obtain solutions of these problems, numerical methods are generally used. Beside this, an interesting field is that where asymptotic expansions can be operated. This work enters this field. The propagation in closed waveguides is studied. We assume that one of the two characteristic lengths of the section is small compared to the other: precisely, the transverse section of the guide is a thin annulus. Its thickness is the small parameter of the problem. We consider an electromagnetic waveguide and we formulate two assumptions: the electromagnetic material is homogeneous and the distance between the two curves of the boundary of the section is constant. Then, explicit expressions result from the Laplacian operator of the problem.

The paper is mainly divided into two parts. The first one is devoted to the study for a smooth boundary and the other to a boundary with angular points. Asymptotic results are compared, in both cases, with computed ones.

2. STATEMENT OF THE PROBLEM

Let us consider an electromagnetic cylindrical waveguide $\Omega \times \mathbb{R}$. The bounded section has a boundary $\partial\Omega$, constituted by two curves, Γ_0 and Γ_1 (Fig. 1). The distance between them is small compared to the diameter of the section Ω.
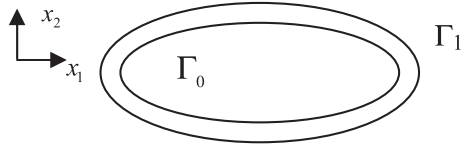
FIGURE 1. The problem.

Along $\Gamma_0 \times \mathbb{R}$ and $\Gamma_1 \times \mathbb{R}$, the waveguide is in contact with a perfect infinite conductor.

Let (x_1, x_2, x_3) represents the Cartesian coordinates of a point of $\mathbb{R}^3$ and (x_1, x_2) the transverse coordinates. An electromagnetic wave $(\vec{E}(x), \vec{H}(x)) \exp[i(\omega t - \beta x_3)]$ propagates inside the guide. The real positive given parameter β is the wave number in the direction x_3 of the propagation. Maxwell's harmonic equations [1] govern the problem:

$$i \omega \varepsilon \vec{E} - \mathrm{Rot}_\beta \vec{H} = 0, \quad i \omega \mu \vec{H} + \mathrm{Rot}_\beta \vec{E} = 0 \qquad \text{in } \Omega$$

$$\vec{E} \wedge \vec{N} = 0, \quad \vec{H} \wedge \vec{N} = 0 \qquad \text{on } \partial\Omega,$$

$\vec{N}$ is a normal vector to $\partial\Omega$ and $\mathrm{Rot}_\beta \vec{u}$ is the vector with the following components:

$$(u_{3,2} + i\beta\, u_2, \; -u_{3,1} - i\beta\, u_1, \; u_{2,1} - u_{1,2}) \quad \text{(the notation } u_{i,j} = \partial u_i / \partial x_j \text{ is used)}.$$

Assumption 1. We assume that the dielectric constant ε and the magnetic permeability μ of the medium are real, positive constants.

We introduce real valued unknown fields [2]:

$$e_1 = E_1, \quad e_2 = E_2, \quad e_3 = i\, E_3, \quad h_1 = H_1, \quad h_2 = H_2, \quad h_3 = -i\, H_3 \quad (i^2 = -1).$$

The equations now can be written, with $\mathrm{rot}_\beta = \mathrm{Rot}_{i\beta}$:

$$\mathrm{rot}_\beta \vec{u} = (u_{3,2} - \beta\, u_2, \; -u_{3,1} + \beta\, u_1, \; u_{2,1} - u_{1,2}) \tag{2.1}$$

$$\omega \varepsilon \vec{e} = \mathrm{rot}_{-\beta} \vec{h}, \quad \omega \mu \vec{h} = \mathrm{rot}_\beta \vec{e}. \tag{2.2}$$

Eliminating the magnetic field $\vec{h}$, the problem in the electric field $\vec{e}$ follows:

$$\mathrm{rot}_{-\beta} (\mathrm{rot}_\beta \vec{e}) = \omega^2 \varepsilon \mu \vec{e} \qquad \text{in } \Omega \tag{2.3}$$

$$\vec{e} \wedge \vec{N} = 0 \qquad \text{on } \partial\Omega. \tag{2.4}$$

In a closed guide, for a given wave number β, there exists an infinite countable sequence of eigenfrequencies. The longitudinal and transversal fields problems provide them. Let us denote by e_3 and e the longitudinal and transversal components of the electric field $\vec{e}$. From (2.2), the relation $\mathrm{div}\, e = \beta\, e_3$ results, with $\mathrm{div}\, e = e_{1,1} + e_{2,2}$ and problem (2.3), (2.4) leads to a Dirichlet problem for e_3 and then to a problem for e:

$$\Delta e_3 + (\omega^2 \varepsilon \mu - \beta^2)\, e_3 = 0 \qquad \text{in} \quad \Omega \tag{2.5}$$

$$e_3 = 0 \qquad \text{on} \quad \partial\Omega \tag{2.6}$$

$$\mathrm{rot}\,\mathrm{rot}\, e - (\omega^2 \varepsilon \mu - \beta^2)\, e = \beta\, \mathrm{grad}\, e_3 \qquad \text{in} \quad \Omega \tag{2.7}$$

$$e \wedge \vec{N} \qquad \text{on} \quad \partial\Omega. \tag{2.8}$$

This last boundary condition is associated with the operator $\mathrm{rot}\,\mathrm{rot}$, which is defined by its two components:

$$\mathrm{rot}\,\mathrm{rot}\, u = (u_{2,12} - u_{1,22}, \; u_{1,21} - u_{2,11}).$$

Related to the distance between Γ_0 and Γ_1, a small parameter η allows approximations of the eigenfrequencies and the eigenfunctions.

3. APPROXIMATIONS FOR A C^1 BOUNDARY

3.1. Local coordinates

We introduce local non dimensional coordinates. Two characteristic lengths are to be considered: the diameter l and the thickness ηl of Ω, η being a small parameter ($\eta \ll 1$). We denote by $s\,l$ the arc length parameter along the boundary Γ_0. And a point P is located in Ω from a point M on Γ_0 by: $\overrightarrow{MP} = \eta\,l\,\nu\,\vec{n}$, where $\vec{n} = \vec{n}(s)$ is the unitary normal vector to Γ_0 in M, oriented to the interior of Ω (Fig. 2).
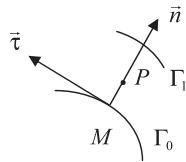

FIGURE 2. Local basis.

Assumption 2. We assume that the equation of Γ_0 (resp. Γ_1) is $\nu = 0$ (resp. $\nu = 1$).

Let $\vec{\tau}$ be the tangential vector to Γ_0 in M: $\vec{\tau} = \vec{x}_3 \wedge \vec{n}$. For this plane curve, the Frenet-Serret formulas hold:

$$\mathrm{d}\vec{n} = R^{-1}\mathrm{d}s\,\vec{\tau}, \quad \mathrm{d}\vec{\tau} = -R^{-1}\mathrm{d}s\,\vec{n} \tag{3.1}$$

where $R(s)$ is the non dimensional radius of curvature of Γ_0 in M. Then, from $\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP}$, we have:

$$\mathrm{d}\vec{P} = \eta\,l\,\mathrm{d}\nu\,\vec{n} + \sigma\,l\,\mathrm{d}s\,\vec{\tau}; \quad \sigma = 1 + \eta\,\nu\,R^{-1}. \tag{3.2}$$

Owing to relations (3.1) and (3.2), we can express, on the local basis, the operators $\operatorname{grad} p$ and $\Delta\,p$, for a scalar function $p(\nu, s)$. The gradient of p is defined by: $\operatorname{grad} p\,.\mathrm{d}\vec{P} = \mathrm{d}p$. The Laplacian $\Delta\,p$ is nothing other than the trace of the gradient $\operatorname{grad} V$, with $V = \operatorname{grad} p$, and the vector gradient operator defined by: $(\operatorname{grad} V)(\mathrm{d}\vec{P}) = \mathrm{d}V$. We obtain:

$$\operatorname{grad} p = \frac{1}{\eta\,l}\frac{\partial p}{\partial \nu}\,\vec{n} + \frac{1}{\sigma\,l}\frac{\partial p}{\partial s}\,\vec{\tau}; \quad \Delta\,p = \frac{1}{\eta^2 l^2}\left(\frac{\partial^2 p}{\partial \nu^2} + \frac{\eta}{\sigma\,R}\frac{\partial p}{\partial \nu} + \frac{\eta^2}{\sigma^2}\frac{\partial^2 p}{\partial s^2} + \eta^3\frac{\nu\,R'}{\sigma^3 R^2}\frac{\partial p}{\partial \nu}\right). \tag{3.3}$$

In (3.3-2), R' denotes the derivative $\mathrm{d}R/\mathrm{d}s$.

3.2. Transverse magnetic (TM) wave: $e_3 \neq 0$, $h_3 = 0$

With the new variables, the equation for the longitudinal electric field problem becomes:

$$\frac{\partial^2 e_3}{\partial \nu^2} + \frac{\eta}{\sigma\,R}\frac{\partial e_3}{\partial \nu} + \frac{\eta^2}{\sigma^2}\frac{\partial^2 e_3}{\partial s^2} + \eta^3\frac{\nu\,R'}{\sigma^3 R^2}\frac{\partial e_3}{\partial \nu} + \eta^2 l^2(\omega^2\varepsilon\,\mu - \beta^2)\,e_3 = 0 \quad \text{in} \quad \Omega \tag{3.4}$$

$$e_3 = 0 \quad \text{on} \quad \partial\Omega. \tag{3.5}$$

The presence of the parameter η involves that the eigenelements depend on the parameter η. Let us denote: $e_3 = w_\eta$, $\eta\,l\sqrt{\omega^2\varepsilon\,\mu - \beta^2} = \alpha_\eta$. For small η, we search expansions of the solution:

$$w_\eta = w_0 + \eta\,w_1 + \eta^2\,w_2 + \ldots; \quad \alpha_\eta = \alpha_0 + \eta\,\alpha_1 + \eta^2\alpha_2 + \ldots \tag{3.6}$$

We insert these expressions in (3.4), (3.5) (with Assumption 2) and we identify the successive terms, powers of η, assuming that is R is $O(1)$ compared to η.

From the term η^0, we obtain the homogeneous problem for w_0:

$$\frac{\partial^2 w_0}{\partial \nu^2} + \alpha_0^2 w_0 = 0 \quad \text{for} \quad 0 < \nu < 1 \tag{3.7}$$

$$w_0 = 0 \quad \text{on} \quad \nu = 0, \; \nu = 1. \tag{3.8}$$

Its solution is:

$$\alpha_0 = \alpha_0(n) = n\pi \quad (n \in \mathbf{N}^*); \quad w_0(\nu, s) = C_0(s) \sin(\alpha_0 \nu). \tag{3.9}$$

To determine the function C_0, we must go on.

From the term η^1, we obtain a non homogeneous problem for w_1.

$$\frac{\partial^2 w_1}{\partial \nu^2} + \alpha_0^2 w_1 = -\frac{1}{R}\frac{\partial w_0}{\partial \nu} - 2\alpha_0\alpha_1 w_0 \quad \text{for} \quad 0 < \nu < 1 \tag{3.10}$$

$$w_1 = 0 \quad \text{on} \quad \nu = 0, \; \nu = 1. \tag{3.11}$$

The scalar product in $L^2(0,1)$ of equation (3.10) by $\sin(\alpha_0 \nu)$ leads, after integrations by parts, to:

$$\alpha_1 = 0. \tag{3.12}$$

Then, we obtain:

$$w_1(\nu, s) = C_1(s) \sin(\alpha_0 \nu) - \frac{C_0(s)}{2R}\nu \sin(\alpha_0 \nu). \tag{3.13}$$

The term η^2 gives α_2 and the function C_0. The problem to be solved is, for $0 < \nu < 1$:

$$\frac{\partial^2 w_2}{\partial \nu^2} + \alpha_0^2 w_2 = -\frac{1}{R}\frac{\partial w_1}{\partial \nu} + \frac{\nu}{R^2}\frac{\partial w_0}{\partial \nu} - \frac{\partial^2 w_0}{\partial s^2} - (\alpha_1^2 + 2\alpha_0\alpha_2)w_0 - 2\alpha_0\alpha_1 w_1 \tag{3.14}$$

$$w_2 = 0 \qquad\qquad\qquad \text{on} \quad \nu = 0, \; \nu = 1. \tag{3.15}$$

With (3.12), from the scalar product of equation (3.12) by $\sin(\alpha_0 \nu)$ and integrations by parts, we obtain the differential equation:

$$\frac{\mathrm{d}^2 C_0}{\mathrm{d}s^2} + \left(\frac{1}{4R^2} + 2\alpha_0\alpha_2\right) C_0 = 0. \tag{3.16}$$

Then, we get the term w_2:

$$w_2(\nu, s) = C_2(s) \sin(\alpha_0 \nu) - \frac{C_1(s)}{2R}\nu \sin(\alpha_0 \nu) + \frac{3\,C_0(s)}{8R^2}\nu^2 \sin(\alpha_0 \nu). \tag{3.17}$$

Equation (3.16) is a Hill's equation which has periodic solutions for an infinite countable sequence of the parameter α_2 [3]. We can observe that C_0 depends on two independent fundamental solutions of (3.16).

So, we obtain the leading term w_0 of the longitudinal field and the approximation of the dispersion relation:

$$\omega^2 \varepsilon\,\mu = \beta^2 + \eta^{-2}l^{-2}\alpha_0^2 + 2l^{-2}\alpha_0\alpha_2 + O(\eta). \tag{3.18}$$

As for the transversal electric field, from $h_3 = 0$ and equation (2.2-2), we get $\operatorname{rot} \operatorname{rot} e = 0$. Thus, from (2.7), we have approximation of e on the local basis. We introduce the decomposition on the local basis:

$$e = u\,\vec{n} + v\vec{\tau}. \tag{3.19}$$

TABLE 1. Computed eigenvalues and approximations for TM waves.

	$n=1$ $m=0$	$n=1$ $m=1$	$n=1$ $m=2$	$n=2$ $m=0$	$n=2$ $m=1$	$n=2$ $m=2$
$\eta^{-1}\alpha$	62.829	62.837	62.860	125.663	125.667	125.678
$\eta^{-1}\alpha_{\mathrm{app}}$	62.829	62.837	62.861	125.662	125.667	125.678
$\eta^{-1}\alpha_{\mathrm{fem}}$	62.846	62.853	62.876	125.791	125.795	125.806

The leading term of each component is (with the similar notations as in (3.6-1)):

$$u_1(\nu,\, s) = -\beta\, l\alpha_0^{-1}\, C_0(s)\cos(\alpha_0\nu), \quad v_2(\nu,\, s) = -\beta\, l\alpha_0^{-2}\, C_0'(s)\sin(\alpha_0\nu).$$

We observe therefore that, in this case, the longitudinal component is the significant term.

For a circular section, the characteristic length of Γ_0 is the radius, $R = 1$, and an approximation of $\sqrt{\omega^2\varepsilon\,\mu - \beta^2}$ is:

$$\eta^{-1}\alpha_{\mathrm{app}} = \eta^{-1}n\pi + \frac{\eta}{2n\pi}\left(m^2 - \frac{1}{4}\right)\ (n \in \mathbf{N}^*,\, m \in \mathbf{N})\,.$$

Some computed values $\eta^{-1}\alpha$ and $\eta^{-1}\alpha_{\mathrm{fem}}$ are compared to $\eta^{-1}\alpha_{\mathrm{app}}$, for $\eta = 0.05$ (Tab. 1). The value $\eta^{-1}\alpha$ is obtained from the use of *Mathematica*® and $\eta^{-1}\alpha_{\mathrm{fem}}$ from a finite elements method (code ASTER for a quarter circle with $32\,841$ nodes in the radial direction, with linear element approximation). The parameter m is related to the index of Bessel's functions.

For $m \neq 0$, we observe that the order of multiplicity of the eigenvalues $\eta^{-1}\alpha_{\mathrm{fem}}$ is 2.

The difference between the *fem* solution and the approximation comes from the error of finite elements approximation. It should be reduced by increasing the number of nodes.

3.3. Transverse electric (TE) wave: $h_3 \neq 0,\ e_3 = 0$

In this case, from equation (2.7), we have $\mathrm{div}\, e = 0$ and the Hodge's decomposition [1] of e is: $e_1 = p_{,2}$, $e_2 = -p_{,1}$, i.e. $e = \mathrm{grad}\, p \wedge \vec{x}_3$. The unknown scalar function p can be determined from the Neumann problem:

$$\Delta\, p + (\omega^2\,\varepsilon\,\mu - \beta^2)\, p = 0 \quad \mathrm{in} \quad \Omega \tag{3.20}$$

$$\frac{\partial p}{\partial n} = 0 \quad \mathrm{on} \quad \partial\Omega. \tag{3.21}$$

For small values of the parameter η, in a similar way as for the TM wave, we search expansions for a solution of this eigenvalue problem:

$$p = p_\eta = p_0 + \eta\, p_1 + \eta^2\, p_2 + \ldots;\quad \eta\, l\sqrt{\omega^2\varepsilon\,\mu - \beta^2} = a_\eta = a_0 + \eta\, a_1 + \eta^2 a_2 + \ldots \tag{3.22}$$

We insert these expressions in (3.20), (3.21) and, using (3.3-2), we identify powers of η. Various problems are obtained whose equations are similar to the preceding ones, (3.7), (3.10), (3.14), with a_i instead of α_i $(i = 0,\, 1,\, 2\,\ldots)$. The boundary conditions are different: $\frac{\partial p_i}{\partial\nu} = 0$.

At order η^0, a first case emerges:

$$a_0 = 0. \tag{3.23}$$

The first approximation of p is: $p_0(\nu, s) = D_0(s)$. At order η, we get: $p_1(\nu, s) = D_1(s)$. The term η^2 (see (3.14)) provides to the function D_0. As to respect the periodicity, if T denotes the non dimensional length of the curve Γ_0, we obtain:

$$a_1 = 2\pi\, m/\mathrm{T}\quad (m \in \mathbf{N}^*);\quad D_0(s) = A_0\cos(a_1 s) + B_0\sin(a_1 s). \tag{3.24}$$

TABLE 2. Computed eigenvalues and approximations for TE waves

	$n = 1$ $m = 0$	$n = 1$ $m = 1$	$n = 1$ $m = 2$	$n = 2$ $m = 0$	$n = 2$ $m = 1$	$n = 2$ $m = 2$
$\eta^{-1}a$	62.837	62.845	62.867	125.663	125.671	125.682
$\eta^{-1}a_{app}$	62.837	62.845	62.869	125.666	125.671	125.682
$\eta^{-1}a_{fem}$	62.853	62.861	62.884	125.796	125.799	125.810

A_0 and B_0 are arbitrary constants. An approximation of the solution of the problem is then, using (3.3-1) and the notation (3.19):

$$\omega^2 \varepsilon \, \mu = \beta^2 + l^{-2}a_1^2 + O(\eta); \quad u_0(\nu, s) = 0, \quad v_0(\nu, s) = l^{-1}a_1[-A_0 \sin(a_1 s) + B_0 \cos(a_1 s)]. \tag{3.25}$$

In the general case ($a_0 \neq 0$), at order η^0, we obtain:

$$a_0 = a_0(n) = n\pi \ (n \in \mathbf{N}^*); \quad p_0(\nu, s) = D_0(s) \ \cos(a_0\nu). \tag{3.26}$$

We use the same step as in Section 3.2. At order η, after a projection on $\cos(a_0\nu)$, we obtain a_1 and then the function p_1.

$$a_1 = 0; \quad p_1(\nu, s) = D_1(s) \cos(a_0\nu) + \frac{D_0(s)}{2a_0^2 R}[\sin(a_0\nu) - \nu a_0 \cos(a_0\nu)]. \tag{3.27}$$

At order η^2, the following Hill's equation supplies the term a_2 and the function $D_0(s)$:

$$\frac{\mathrm{d}^2 D_0}{\mathrm{d}s^2} + \left(2a_0 a_2 - \frac{3}{4R^2}\right) D_0 = 0. \tag{3.28}$$

The transversal electric field is deduced from p, and the leading term of each component on the local basis is, for $a_0 \neq 0$:

$$u_1(\nu, \ s) = l^{-1} D_0'(s) \ \cos(a_0\nu); \quad v_0(\nu, \ s) = l^{-1} D_0(s) a_0 \sin(a_0\nu). \tag{3.29}$$

The tangential component here is the significant term. An approximation of the dispersion relation is, in this case:

$$\omega^2 \varepsilon \, \mu = \beta^2 + \ \eta^{-2}l^{-2}a_0^2 + 2l^{-2}a_0 a_2 + O(\eta). \tag{3.30}$$

For a circular section, with $R = 1$, some computed values $\eta^{-1}a$, $\eta^{-1}a_{\mathrm{fem}}$, and approximations $\eta^{-1}a_{\mathrm{app}}$, $\eta^{-1}a_{\mathrm{app}} = \eta^{-1}n\pi + \frac{\eta}{2n\pi}(m^2 + \frac{3}{4})$ ($n \in \mathbf{N}^*$, $m \in \mathbf{N}$), are compared for $\eta = 0.05$ (Tab. 2).

A good agreement between these some values $\eta^{-1}a$ and $\eta^{-1}a_{\mathrm{app}}$ is observed.

3.4. Transverse electric magnetic (TEM) wave: $e_3 = 0$, $h_3 = 0$

From $h_3 = 0$ and (2.2-2), we have $\mathrm{rot} \, \mathrm{rot} \, e = 0$. So, from $e_3 = 0$ and (2.7), we get the well known result: $\omega^2 \varepsilon \mu = \beta^2$. The relation between the frequency and the wave number does not depend on the geometry of the section.

As for the electric field, it is given, as in Section 3.3, from a scalar function p which satisfies:

$$\Delta p = 0 \quad \text{in} \quad \Omega \tag{3.31}$$

$$\frac{\partial p}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \tag{3.32}$$

We identify the successive powers of η in the expansions of (3.31) and (3.32). An approximation of p is $p_0(\nu, s) = D_0 s + E_0$, ($D_0$, E_0 constant). And the leading term of e is $u_0\vec{n}$, with u_0 constant.

We can mention that, for TM and TE waves, the first approximation of $\omega^2 \varepsilon \, \mu - \beta^2$ is the same.

4. Justification of the approximations

Let us consider a TM wave. The justification of the results of Section 3.2 comes from a variational form P_η of the eigenvalue problem (2.5), (2.6) for the longitudinal electric field. We introduce a test function φ of the Sobolev space $H_0^1(\Omega)$. Owing to a Green's formula, we obtain:

$$w_\eta \in H_0^1(\Omega); \quad \int_\Omega \operatorname{grad} w_\eta.\operatorname{grad}\varphi \, \mathrm{d}x = (\omega^2 \varepsilon\,\mu - \beta^2) \int_\Omega w_\eta \varphi \, \mathrm{d}x, \quad \forall \varphi \in H_0^1(\Omega).$$

In the variables (η, s), the problem P_η follows:

$$w_\eta \in H_0^1(\Omega); \quad a_\eta(w_\eta, \varphi) = \alpha_\eta^2\, b_\eta(w_\eta, \varphi), \quad \forall \varphi \in H_0^1(\Omega) \tag{4.1}$$

where a_η et b_η are the bilinear forms

$$a_\eta(w, \varphi) = \int_\Omega \left(\frac{\partial w}{\partial \nu}\frac{\partial \varphi}{\partial \nu} + \frac{\eta^2}{\sigma^2}\frac{\partial w}{\partial s}\frac{\partial \varphi}{\partial s} \right) \sigma \, \mathrm{d}\nu \, \mathrm{d}s, \quad b_\eta(w, \varphi) = \int_\Omega w\,\varphi\,\sigma \, \mathrm{d}\nu \, \mathrm{d}s.$$

As Ω can be identified, in the variables (η, s), to the set $[0,1] \times [0,\mathrm{T}]$ (T is the non dimensional length of the interior closed curve Γ_0), the accurate meaning of $\varphi \in H_0^1(\Omega)$ is that φ belongs to the Sobolev space $H^1([0,1] \times [0,\mathrm{T}])$ and satisfies:

$$\varphi(0, s) = \varphi(1, s) = 0, \quad \varphi(\nu, 0) = \varphi(\nu, \mathrm{T}). \tag{4.2}$$

In other words, the test function is an element of $H_{0,p}^1([0,1] \times [0,\mathrm{T}])$ (0 for the boundary conditions on η, p for the periodicity on s).

According to the results of Section 3.2, we define new eigenfunction and test function: $w_\eta(\nu, s) = \sigma^{-\frac{1}{2}}\, f_\eta(\nu, s)$, $\varphi(\nu, s) = \sigma^{-\frac{1}{2}}\, \psi(\nu, s)$, with $\sigma = 1 + \eta\,\nu\,R^{-1}$.

We obtain for P_η:

$$f_\eta \in H_{0,p}^1([0,1] \times [0,\mathrm{T}]); \quad \int_0^1 \int_0^\mathrm{T} \left[\frac{\partial f_\eta}{\partial \nu}\frac{\partial \psi}{\partial \nu} - \frac{\eta}{2\sigma R}\left(f_\eta \frac{\partial \psi}{\partial \nu} + \psi\frac{\partial f_\eta}{\partial \nu} \right) + \frac{\eta^2}{4\sigma^2 R^2}f_\eta\psi + \frac{\eta^2}{\sigma^2}\frac{\partial f_\eta}{\partial s}\frac{\partial \psi}{\partial s} \right.$$

$$\left. - \frac{\eta^3\nu R'}{2\sigma^3 R^2}\left(f_\eta\frac{\partial \psi}{\partial s} + \psi\frac{\partial f_\eta}{\partial s} \right) + \frac{\eta^4\nu^2 R'^2}{4\sigma^4 R^4}f_\eta\psi \right] \mathrm{d}\nu \, \mathrm{d}s = \alpha_\eta^2 \int_0^1 \int_0^\mathrm{T} f_\eta\psi \, \mathrm{d}\nu \, \mathrm{d}s, \quad \forall \psi \in H_{0,p}^1([0,1] \times [0,\mathrm{T}]).$$

Let us pay attention to the first terms of the integral in the first member and, specially, to the second one. Using an expansion of the expression (3.2-2) of σ, we transform it. After integrations by parts and thanks to (4.2-1), we obtain that problem P_η appears to be a perturbation of the eigenproblem:

$$\tilde{w}_\eta \in H_{0,p}^1([0,1] \times [0,\mathrm{T}]);$$

$$\int_0^1 \int_0^\mathrm{T} \left[\frac{\partial \tilde{w}_\eta}{\partial \nu}\frac{\partial \psi}{\partial \nu} + \eta^2\frac{\partial \tilde{w}_\eta}{\partial s}\frac{\partial \psi}{\partial s} - \frac{\eta^2}{4R^2}\tilde{w}_\eta\psi \right] \mathrm{d}\nu \, \mathrm{d}s = \gamma_\eta^2 \int_0^1 \int_0^\mathrm{T} \tilde{w}_\eta\psi \, \mathrm{d}\nu \, \mathrm{d}s, \quad \forall \psi \in H_{0,p}^1([0,1] \times [0,\mathrm{T}]).$$

We can verify that the function $\tilde{w}_\eta(\nu, s) = C_0(s)\,\sin(\alpha_0\nu)$, with C_0 being a T-periodic solution of (3.16), is an eigenfunction of this problem, associated to the eigenvalue $\alpha_0^2 + 2\eta^2\alpha_0\alpha_2$. And Kato's perturbation theorem [4] gives a justification of the approximations of Section 3.2.

We can present here a variational form of the eigenvalue problem issued from the Hill's equation (3.16). Denoting by $H_p^1[0, \mathrm{T}]$ the set of the T-periodic elements of $H^1[0, \mathrm{T}]$, we have:

$$C_0 \in H_p^1[0, \mathrm{T}]; \quad \int_0^{\mathrm{T}} \left(\frac{\mathrm{d}C_0}{\mathrm{d}s} \frac{\mathrm{d}c}{\mathrm{d}s} - \frac{1}{4R^2} C_0 c \right) \mathrm{d}s = 2\alpha_0 \alpha_2 \int_0^{\mathrm{T}} C_0 c \, \mathrm{d}s, \quad \forall c \in H_p^1[0, \mathrm{T}]. \tag{4.3}$$

As the boundary is regular, $R(s)$ is continuous on the interval [0,T]. After the shift of the spectral parameter: $2\alpha_0\alpha_2 \to 2\alpha_0\alpha_2 + m^2$, $m^2 = \max(1/R^2(s))$ for $s \in [0, \mathrm{T}]$, we obtain a standard eigenvalue problem on $H_p^1[0, \mathrm{T}]$, from which computational values of C_0 and α_2 can be derived.

For the TE wave, the same approach leads to the justification.

5. Approximations for a piecewise C^1 boundary

5.1. Equations near a corner

The presence of an angular point has two effects. First, it involves a modification in the preceding approximations. And, in addition, it introduces other modes, we can call them "corner modes".

The TM wave problem is considered. Our study is mainly devoted to the perturbation brought by a corner, in the previous results. Let O be an angular point of Γ_0. Near O three regions are to be considered (Fig. 3).
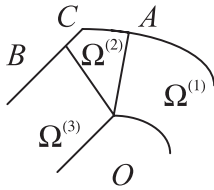
FIGURE 3. The geometry near a corner.

$\Omega^{(1)}$ and $\Omega^{(3)}$ are strips with thickness ηl. We assume that in $\Omega^{(1)}$, $s \leq \mathrm{T}$ and in $\Omega^{(3)}$, $s \geq 0$. $\Omega^{(2)}$ is a curvilinear quadrilateral $OACB$, with an angle ϕ in O. The geometry of this quadrilateral should be specified but it does not interfere in the first approximation of the field.

For the study in $\Omega^{(2)}$, a suitable local basis is $(O; \vec{n}(\mathrm{T}^-), \vec{n}(0^+))$ with $\overrightarrow{OA} = \eta l \, \vec{n}(\mathrm{T}^-)$ and $\overrightarrow{OB} = \eta \, l \, \vec{n}(0^+)$. A point P in $\Omega^{(2)}$ is located by: $\overrightarrow{OP} = x\eta l \, \vec{n}(\mathrm{T}^-) + y\eta l \, \vec{n}(0^+)$.

We consider the case of only one angular point, so $\Omega^{(1)} = \Omega^{(3)}$. In $\Omega^{(1)}$, the field $w_\eta^{(1)}(\nu, s)$ satisfies the equation and boundary conditions:

$$\frac{\partial^2 w_\eta^{(1)}}{\partial \nu^2} + \frac{\eta}{\sigma R} \frac{\partial w_\eta^{(1)}}{\partial \nu} + \frac{\eta^2}{\sigma^2} \frac{\partial^2 w_\eta^{(1)}}{\partial s^2} + \eta^3 \frac{\nu R'}{\sigma^3 R^2} \frac{\partial w_\eta^{(1)}}{\partial \nu} + \alpha^2 w_\eta^{(1)} = 0 \tag{5.1}$$

$$w_\eta^{(1)}(0, s) = w_\eta^{(1)}(1, s) = 0. \tag{5.2}$$

In $\Omega^{(2)}$, the field $w_\eta^{(2)}(x, y)$ satisfies the equation and boundary conditions:

$$\frac{\partial^2 w_\eta^{(2)}}{\partial x^2} + \frac{\partial^2 w_\eta^{(2)}}{\partial x^2} + \alpha^2 w_\eta^{(2)} = 0 \tag{5.3}$$

$$w_\eta^{(2)} = 0 \qquad \qquad \text{on} \quad \widehat{AC} \text{ and } \widehat{BC}. \tag{5.4}$$

These problems are to be completed by conditions of continuity on $O\widehat{A}$ and $O\widehat{B}$. On $O\widehat{A}$, using (3.3-1), we have:

$$w_\eta^{(1)}(\nu, \mathrm{T}) = w_\eta^{(2)}(x, 0); \quad \frac{\eta}{\sigma(\mathrm{T})} \frac{\partial w_\eta^{(1)}}{\partial s}(\nu, \mathrm{T}) = -\frac{\partial w_\eta^{(2)}}{\partial n}(x, 0), \quad \text{for} \quad 0 \le \nu = x \le 1 \tag{5.5}$$

with $\frac{\partial w_\eta^{(2)}}{\partial n}(x, 0) = \cot \mathrm{an}\phi \frac{\partial w_\eta^{(2)}}{\partial x}(x, 0) - (\sin \phi)^{-1} \frac{\partial w_\eta^{(2)}}{\partial y}(x, 0)$. Similar conditions exist on $O\widehat{B}$.

We assume expansions of the eigenelements, similar to (3.6). We study how the presence of the angular point modifies the first approximation, *i.e.* the terms $w_0^{(1)}$ and α_2.

5.2. **First approximation**

As no transversal boundary conditions is prescribed in the problem (5.1), (5.2), in $\Omega^{(1)}$ some results obtained section 3 still remain, and specially:

$$\alpha_0 = \alpha_0(n) = n\pi, \quad \alpha_1 = 0, \quad w_0^{(1)}(\nu, s) = C_0^{(1)}(s) \sin(\alpha_0 \nu).$$

The problem in $\Omega^{(2)}$, at order η^0, is a homogeneous problem, with Dirichlet-Neumann conditions deduced from (5.4) and (5.5-2). It has only one solution: $w_0^{(2)}(x, y) = 0$, except for special cases we do not study. According to the continuity of the field on $O\widehat{A}$ and $O\widehat{B}$, we obtain:

$$C_0^{(1)}(\mathrm{T}) = C_0^{(1)}(0) = 0. \tag{5.6}$$

So, a restriction has to be done among the previous solutions of the Hill's equation (3.16). From this supplementary condition, we can remark that $C_0^{(1)}(s)$ is generated by only one fundamental solution of (3.16).

Such result is valid for more than one angular point. But, for special geometries, indeterminations can occur and to go on the next terms, we have to deal with stiff problems. Both conditions of continuity on $O\widehat{A} \cup O\widehat{B}$ cannot be satisfied locally. For one of them, we adopt a global condition. An example is given in Section 6.

To justify the first approximation, we can use the arguments developed in [5] (Chap. VII). We transform the problem in one settled on $\Omega^{(1)}$, thank to a frontier operator defined from the problem in $\Omega^{(2)}$. The problem obtained is in the general framework of holomorphic perturbation theory.

5.3. **Corner modes**

Another kind of waves which can propagate is those due mainly to the angular point. In this case, the field is prevalent in the corner. But, in general, the eigenvalue is different from $n\pi$, so elsewhere than in the corner, the field is null. Another partition of the domain Ω is made, insulating the corner (Fig. 4).
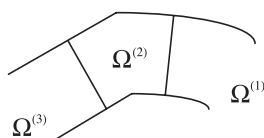

FIGURE 4. Geometry for the corner modes.

In such a geometry, the singularity that can occur in O can be taken in account.

The continuity of the field implies that α is an eigenvalue of the Dirichlet problem in $\Omega^{(2)}$:

$$\frac{\partial^2 w^{(2)}}{\partial x^2} + \frac{\partial^2 w^{(2)}}{\partial x^2} + \alpha^2 w^{(2)} = 0 \quad \text{in} \quad \Omega^{(2)} \tag{5.7}$$

$$w^{(2)} = 0 \quad \text{on} \quad \partial\Omega^{(2)}. \tag{5.8}$$

Of course, the approximation which is obtained depends on the choice of the domain $\Omega^{(2)}$. A study would be done to optimise the results.

6. THIN SQUARE WAVEGUIDER

6.1. **Analytical results**

To illustrate the case of more than one angular point, we present explicit expressions of approximations for the TM waves in a thin square guide. Thanks to the symmetries of the section, we can reduce the domain. Also, we get simplifications in the equations, coming from the fact that the curvature is null in every smooth point of Γ_0.

The section has a centre and three independent axis of symmetry: two are parallel to the sides of the interior square and the other has the direction of one of the bisecting lines. So the domain can be reduced to the eighth of the section. Figure 5 shows the general shape of the selected eighth of the section and the exact shape in variables (ν, s) and (x, y).
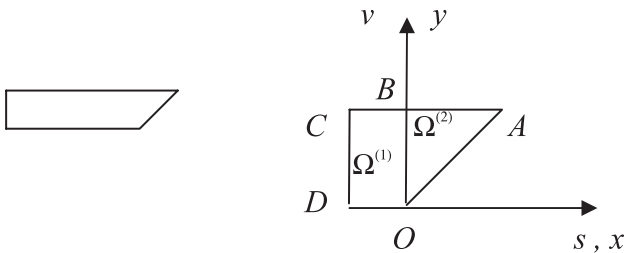

FIGURE 5. The eighth of the section.

The domain is divided in $\Omega^{(1)}$, part of the side, $\Omega^{(1)} = \{(\nu, s), 0 \le \nu \le 1, -1/2 \le s \le 0\}$ and in $\Omega^{(2)}$, part of the corner, $\Omega^{(2)} = \{(x, y), 0 \le x \le 1, 0 \le y \le 1, x \le y\}$ (in the non dimensional variables).

The boundary conditions on the sides $[CD]$ and $[OA]$ depend on the properties of the field. On an axis of symmetry, the condition to be taken is:

$$\frac{\partial w}{\partial n} = 0 \text{ for a symmetrical field; } w = 0 \text{ for an antisymmetrical field.}$$

a) *Case of a symmetrical field compared to the lines (CD) and (OA).*

Taking into account simplifications, the equations of the problem are:

$$\text{In } \Omega^{(1)} \qquad \frac{\partial^2 w_\eta^{(1)}}{\partial \nu^2} + \eta^2 \frac{\partial^2 w_\eta^{(1)}}{\partial s^2} + \alpha_\eta^2 \, w_\eta^{(1)} = 0 \tag{6.1}$$

$$w_\eta^{(1)}(0, s) = w_\eta^{(1)}(1, s) = 0; \quad \frac{\partial w_\eta^{(1)}}{\partial s}\left(\nu, -\frac{1}{2}\right) = 0. \tag{6.2}$$

$$\text{In } \Omega^{(2)} \qquad \frac{\partial^2 w_\eta^{(2)}}{\partial x^2} + \frac{\partial^2 w_\eta^{(2)}}{\partial y^2} + \alpha_\eta^2 \, w_\eta^{(2)} = 0 \tag{6.3}$$

$$w_\eta^{(2)}(x, 1) = 0; \quad \frac{\partial w_\eta^{(2)}}{\partial n}(x, x) = 0. \tag{6.4}$$

To these equations, we have to add the conditions of continuity on $[OB]$:

$$w_\eta^{(1)}(\nu, 0) = w_\eta^{(2)}(0, y); \quad \eta \frac{\partial w_\eta^{(1)}}{\partial s}(\nu, \, 0) = \frac{\partial w_\eta^{(2)}}{\partial x}(0, y) \quad (\nu = y). \tag{6.5}$$

For $\eta \ll 1$, the results obtained in Section 3 are valid and they provide general expressions of the different terms of the approximations. In $\Omega^{(1)}$, from (6.1) and (6.2-1), we get:

$$w_j^{(1)}(\nu, s) = C_j^{(1)}(s) \sin(n\pi\nu) \quad (j = 0, 1, 2, ...); \quad \alpha_0 = n\pi, \quad \alpha_1 = 0 \tag{6.6}$$

$$\frac{\mathrm{d}^2 C_0^{(1)}}{\mathrm{d}s^2} + 2\alpha_0\alpha_2 \, C_0^{(1)} = 0 \quad \left(0 < s < \frac{1}{2}\right). \tag{6.7}$$

And from (6.2-2), we have the condition:

$$\frac{\mathrm{d}C_j^{(1)}}{\mathrm{d}s}\left(-\frac{1}{2}\right) = 0 \quad (j = 0, 1, 2). \tag{6.8}$$

This relation provides one of the boundary condition for equation (6.7), the other comes from the solution in $\Omega^{(2)}$ and the continuity on $[OB]$. For the problem in $\Omega^{(2)}$, we have to deal with two conditions on $[OB]$, for $j = 0, 1, 2$ and $\nu = y$:

$$w_j^{(2)}(0, y) = C_j^{(1)}(0) \sin(n\pi\nu); \quad \frac{\partial w_0^{(2)}}{\partial x}(0, y) = 0, \quad \frac{\partial w_{j+1}^{(2)}}{\partial x}(0, y) = \frac{\mathrm{d}C_j^{(1)}}{\mathrm{d}s}(0) \sin(n\pi\nu). \tag{6.9}$$

Both cannot be satisfied locally. We use global properties and from a variational form of the problem we determine relations between the unknown coefficients $C_j^{(1)}(0)$.

Let us consider the function: $\varphi_n(x, y) = (1 - y)\sin(n\pi x) + (1 - x)\sin(n\pi y)$. It satisfies equation (6.3), with $\alpha_\eta^2 = n^2\pi^2$, and the boundary conditions (6.4). Thanks to a Green's formula, from (6.3), (6.4) and (6.5), we obtain:

$$\int_0^1 \eta\frac{\partial w_\eta^{(1)}}{\partial s}(y, 0)\sin(n\pi y) - w_\eta^{(1)}(y, 0)[n\pi(1 - y) - \sin(n\pi y)]\mathrm{d}y = (\alpha_\eta^2 - n^2\pi^2)\int_{\Omega^{(2)}} w_\eta^{(2)}\varphi_n \, \mathrm{d}x\mathrm{d}y. \tag{6.10}$$

The identification in (6.10) of the successive terms, powers of η, provides relations between the coefficients $C_j^{(1)}(0)$.

At order η^0, we get, in agreement with Section 5: $C_0^{(1)}(0) = 0$. So, the function $C_0^{(1)}(s)$ satisfies (6.7) and:

$$\frac{\mathrm{d}C_0^{(1)}}{\mathrm{d}s}\left(-\frac{1}{2}\right) = 0, \quad C_0^{(1)}(0) = 0. \tag{6.11}$$

The solution of this differential problem gives the approximations:

$$2\alpha_0\alpha_2 = (2p + 1)^2\pi^2 \quad (p \in \mathbf{N}); \quad C_0^{(1)}(s) = A_0^{(1)} \sin[(2p + 1)\pi \, s]. \tag{6.12}$$

As for the field $w_0^{(2)}$, it satisfies:

$$\frac{\partial^2 w_0^{(2)}}{\partial x^2} + \frac{\partial^2 w_0^{(2)}}{\partial y^2} + \alpha_0^2 \, w_0^{(2)} = 0 \tag{6.13}$$

$$w_0^{(2)}(0, y) = 0, \quad \frac{\partial w_0^{(2)}}{\partial x}(0, y) = 0 \tag{6.14}$$

$$w_0^{(2)}(x, 1) = 0, \quad \frac{\partial w_0^{(2)}}{\partial n}(x, x) = 0. \tag{6.15}$$

Let us consider the problem (6.13), (6.14-1) and (6.15). Its solution depends on the eigenelements of the associated eigenvalue problem. These eigenvalues and eigenfunctions are:

$$\omega_{q,r}^2 = \pi^2(q^2 + r^2) \quad (q, r \in \mathbf{N}^*), \ w_{q,r}(x,y) = \sin(q\pi x)\sin(r\pi y) + \sin(r\pi x)\sin(q\pi y). \tag{6.16}$$

• For $n^2 \neq q^2 + r^2$, i.e. when (q, r, n) is not a pythagorician triplet, the only solution of (6.13), (6.14-1) and (6.15) is $w_0^{(2)}(x,y) = 0$. This function which also satisfies (6.14-2) is the solution of (6.13), (6.14) and (6.15). In this case, we get the following approximations of the eigenvalue and the normalized eigenfunction (in $L^2([0,1] \times [-1/2, 0])$):

$$\alpha_{\text{app}} = n\,\pi + \eta^2\,\frac{(2p+1)^2}{2n}\,\pi + O(\eta^3) \quad (n \in \mathbf{N}^*, \ p \in \mathbf{N}) \tag{6.17}$$

$$w_0^{(1)}(\nu, s) = 2\sqrt{2}\sin(n\pi\nu)\sin[(2p+1)\pi s], \quad w_0^{(2)}(x,y) = 0. \tag{6.18}$$

These approximations are periodic and depend on two integers.

At order η^1, (6.10) involves that, for $s = 0$, $C_1^{(1)}$ and $\mathrm{d}C_0^{(1)}/\mathrm{d}s$ are equal and we get:

$$C_1^{(1)}(0) = 2\sqrt{2}(2p+1)\pi. \tag{6.19}$$

We can specify the function $C_1^{(1)}(s)$ if we continue the research of the approximations in $\Omega^{(1)}$. From the term η^3 in (6.1), we obtain, after a projection on $\sin(n\pi\nu)$ as in Section 3:

$$\frac{d^2 C_1^{(1)}}{ds^2} + 2\alpha_0\alpha_2\,C_1^{(1)} = -2\alpha_0\alpha_3\,C_0^{(1)}. \tag{6.20}$$

The solution of (6.20), (6.8) is, with (6.19):

$$\alpha_3 = 4\alpha_2; \quad C_1^{(1)}(s) = A_1^{(1)}\sin[(2p+1)\pi\,s] + C_1^{(1)}(0)\,(1+2s)\,\cos[(2p+1)\pi\,s]. \tag{6.21}$$

The constant $A_1^{(1)}$ is determined from the normalization of the eigenfunction, what amounts choosing $w_1^{(1)}$ orthogonal to $w_0^{(1)}$: $A_1^{(1)} = 2\sqrt{2}$.

To exhibit an explicit approximation of $w_1^{(2)}$, we make the assumption that (6.9-1) is satisfied, whereas (6.9-2) is globally satisfied thanks to the fundamental relation (6.10). And we have:

$$w_1^{(2)}(x,y) = C_1^{(1)}(0)\,[(1-y)\,\sin(n\pi x) + (1-x)\,\sin(n\pi y)]. \tag{6.22}$$

The next approximation can be obtained in a similar way.

• For $n^2 = q^2 + r^2$, $w_0^{(2)}$ is one of the eigenfunction $k_0 w_{q,r}$ of (6.16) (k_0 constant). We choose $w_{q,r}$ for test function in a variational form of the problem in $\Omega^{(2)}$. From the structure (6.6-1) of the solution in $\Omega^{(1)}$, we obtain that $w_\eta^{(2)}$ is orthogonal to $w_{q,r}$. So $w_0^{(2)} = 0$ and the previous approximations are still valid.

b) *Case of a symmetrical field compared to (CD) and antisymmetrical compared to (OA)*

For the problem in $\Omega^{(2)}$, to the condition (6.4-2) we have to substitute: $w_\eta^{(2)}(x,x) = 0$.

With a method similar to that of *a*), we consider here the test function: $\psi_n(x,y) = -(1-y)\sin(n\pi x) + (1-x)\sin(n\pi y)$. We obtain that the approximations (6.17), (6.18) and (6.21-2) are still valid. But $C_1^{(1)}(0)$ and α_3 are different:

$$C_1^{(1)}(0) = -\frac{2}{3}\sqrt{2}\,(2p+1)\pi; \quad \alpha_3 = -\frac{4}{3}\alpha_2.$$

Table 3 shows the agreement of these results with the computations.

TABLE 3. Approximations and computed eigenvalues.

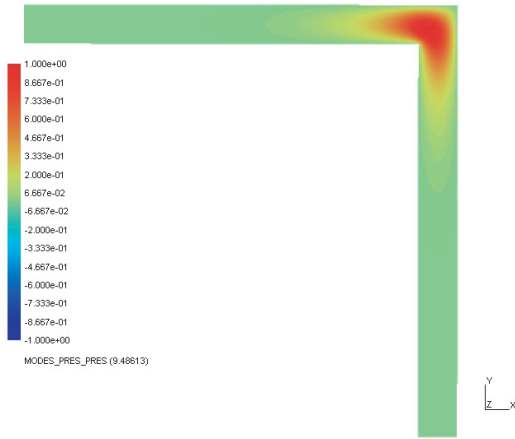|	$n = 1$ $m = 1$	$n = 1$ $m = 2$	$n = 1$ $m = 3$	$n = 1$ $m = 4$	$n = 1$ $m = 5$	$n = 2$ $m = 1$	$n = 2$ $m = 2$	$n = 2$ $m = 3$
$\eta^{-1}\alpha_{\text{app}}$	62.91	63.15	63.54	64.09	64.79	125.70	125.82	126.02
$\eta^{-1}\alpha_{\text{fem}}^{(a)}$	62.92	63.15	63.52	64.03	64.69	125.81	125.94	125.94
$\eta^{-1}\alpha_{\text{fem}}^{(s)}$	62.93	63.20	63.62	64.22	64.95	125.83	125.95	126.13

FIGURE 6. Corner mode $\eta^{-1}\alpha_{\text{fem}} = 59.60$.
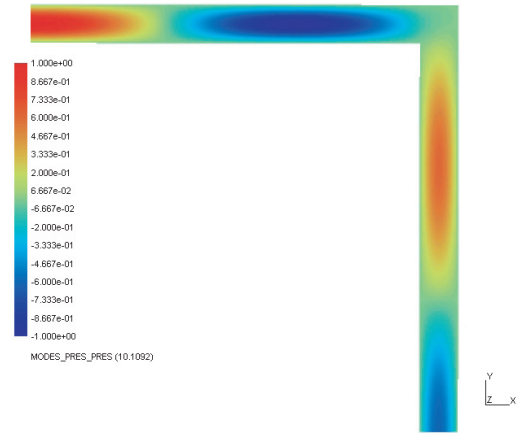

FIGURE 7. Symmetrical field / (CD) ($n = 1$, $m = 3$), antisymmetrical field / (OA) $\eta^{-1}\alpha_{\text{fem}}^{(a)} = 63.52$.

c) *Case of an antisymmetrical field compared to the lines (CD) and (OA)*

The boundary conditions along these lines are now respectively:

$$w_\eta^{(1)}\left(\nu, -\frac{1}{2}\right) = 0; \quad w_\eta^{(2)}(x, x) = 0.$$

In the same way as previously, we obtain the system for the determination of $C_0^{(1)}(s)$:

$$\frac{\mathrm{d}^2 C_0^{(1)}}{\mathrm{d}s^2} + 2\alpha_0\alpha_2\, C_0^{(1)} = 0 \quad \left(0 < s < \frac{1}{2}\right); \quad C_0^{(1)}\left(-\frac{1}{2}\right) = 0, \quad C_0^{(1)}(0) = 0.$$

So, an approximation of the eigenelements is:

$$\alpha_{\text{app}} = n\,\pi + \eta^2\, \frac{(2p)^2}{2n}\, \pi + O(\eta^3) \quad (n \in \mathbf{N}^*, p \in \mathbf{N}^*) \tag{6.23}$$

$$w_0^{(1)}(\nu, s) = 2\sqrt{2}\, \sin(n\pi\nu)\, \sin(2p\pi s), \quad w_0^{(2)}(x, y) = 0. \tag{6.24}$$

These results (6.23) and (6.24) still hold true in the *case of an antisymmetrical field compared to (CD) and symmetrical to (OA)*.
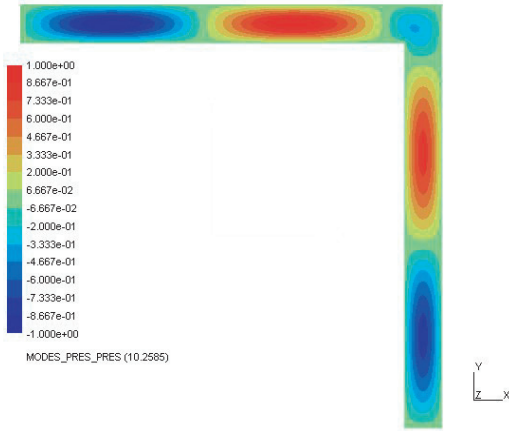
FIGURE 8. Antisymmetrical field / (CD) $(n = 1, m = 4)$, symmetrical field / (OA) $\eta^{-1}\alpha_{\mathrm{fem}}^{(s)} = 64.22$.
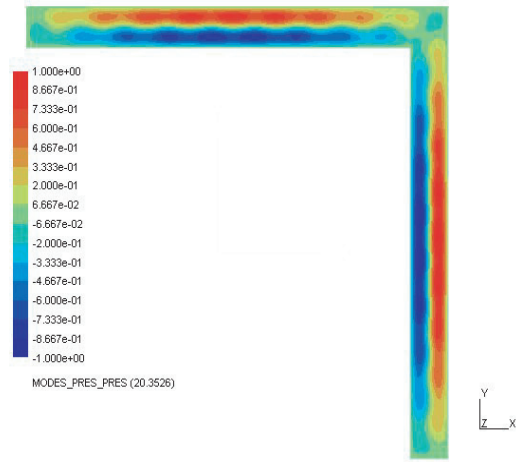

FIGURE 9. Antisymmetrical field / (CD) $(n = 2, m = 2)$, symmetrical field / (OA) $\eta^{-1}\alpha_{\mathrm{fem}}^{(s)} = 125.95$.

To summarize, the first approximation which is found for a TM wave is:

$$\eta^{-1}\alpha_{\mathrm{app}} = \frac{n\pi}{\eta} + \eta\frac{m^2}{2n}\pi \quad (n \in \mathbf{N}^*, m \in \mathbf{N}^*) \tag{6.25}$$

$$w_0^{(1)}(\nu, s) = 2\sqrt{2}\,\sin(n\pi\nu)\,\sin(m\pi s), \quad w_0^{(2)}(x, y) = 0. \tag{6.26}$$

6.2. **Comparison with computed results**

Table 3 shows results which are obtained, with $\eta = 0.05$, for fields which propagate in the sides of the section. $\eta^{-1}\alpha_{\mathrm{app}}$ denotes the analytical approximation (6.25) and $\eta^{-1}\alpha_{\mathrm{fem}}$ is the value obtained from a finite elements method (code ASTER, 67281 nodes, with linear finite elements approximation, for a quarter of the section). The two values $\eta^{-1}\alpha_{\mathrm{fem\,(a)}}^{(a)}$, $\eta^{-1}\alpha_{\mathrm{fem\,(s)}}^{(a)}$ are respectively related to antisymmetrical and symmetrical modes in the corners, *i.e.* along (OA).

The finite elements method also provide the corner modes (Fig. 6). The first eigenvalue of these modes is lower than all the preceding ones: $\eta^{-1}\alpha_{\mathrm{fem}} = 59.603$.

A visualisation of other modes shows the agreement with the expression (6.26), both for symmetrical modes (Fig. 7) and for antisymmetrical ones (Figs. 8 and 9).

One can observe that the values are close, the difference between them being less than 1%. The asymptotic approximations seem to be less expensive. They provide values which are obtained by finite elements method with fine meshes. Moreover, the asymptotic method proposes continuous expression of the field.

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