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hp-FEM FOR THREE-DIMENSIONAL ELASTIC PLATES

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Abstract. In this work, we analyze hierarchic *hp*-finite element discretizations of the full, three-dimensional plate problem. Based on two-scale asymptotic expansion of the three-dimensional solution, we give specific mesh design principles for the *hp*-FEM which allow to resolve the three-dimensional boundary layer profiles at robust, exponential rate. We prove that, as the plate half-thickness ε tends to zero, the *hp*-discretization is consistent with the three-dimensional solution to any power of ε in the energy norm for the degree $p = \mathcal{O}(|\log \varepsilon|)$ and with $\mathcal{O}(p^4)$ degrees of freedom.

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1. Introduction

The numerical analysis of thin three-dimensional structures such as beams, plates and shells is a basic problem in engineering. It amounts to solving numerically a problem of three-dimensional elasticity in a 'thin' domain. The classical engineering approach to these problems has been to replace the three-dimensional problem by simplified, lower-dimensional models which are in turn solved numerically.

Lower dimensional models have been derived roughly speaking in three ways: by kinematical hypothesis, by asymptotic analysis or by energy projection. We refer to [4] for a survey and references. Alternatively, in recent years, it has become possible to solve the three-dimensional problems directly by high order finite element methods which afford anisotropic mesh refinement [1, 2].

In the dimension reduction process, information is necessarily lost and the question arises what the relation of the dimensionally reduced models to the original, three-dimensional problem is. Numerous models have been found to be consistent with the three-dimensional problem in the limit of vanishing thickness ε. In the case of plate models, the order of consistency is, however, only $\sqrt{\varepsilon}$, due to the boundary layers of the three-dimensional problem not being accurately resolved by the plate model. This state of affairs cannot be improved by incorporation of 'higher-order' kinematical hypotheses into the plate model, since near the edge region, the deformation states are generically three-dimensional, as was shown in full asymptotic analyses of the three-dimensional plate problem in [20]. The coupling of the limiting plate models in the plate's interior with fully three-dimensional finite element methods near the edge region has been proposed in an engineering framework by [21].

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To achieve higher order asymptotic consistency in plate models, higher order kinematical hypotheses in the interior of the plate must thus be coupled with full resolution of the three-dimensional effects near the edge of the plate. This can be done by *hp*-Finite Element (FE) discretization and was proposed first in [18]. To analyze the design of a *hp*-Finite Element Discretization of the three-dimensional plate problem is the purpose of the present paper.

Relying on the full asymptotics of the three-dimensional solution of the plate problem [6,7,9], we show that it is possible to achieve consistency of the FE-approximation with the three-dimensional solution to any order of ε with a properly designed *hp*-FE discretization. It involves hierarchical plate models which are refined inside the boundary layer in a vicinity of the edge to resolve the singularities, thereby abandoning the dimensional reduction point of view. The degree p to achieve this is $\mathcal{O}(|\log\varepsilon|)$ in general, with $\mathcal{O}(p)$ elements corresponding to a number N of degrees of freedom which is bounded by $\mathcal{O}(p^4)$.

Let us describe our results in more detail. On the family of thin plates $\omega \times (-\varepsilon, \varepsilon)$, $\varepsilon \in (0, \varepsilon_0)$, such *hp*-FE spaces are defined as follows: a fixed mesh τ_ω is designed on the mid-surface ω, so that there exists a layer of quadrilateral elements along its boundary $\partial\omega$. The tensor three-dimensional mesh $\mathcal{T}^0_\varepsilon := \tau_\omega \times (-\varepsilon, \varepsilon)$ is geometrically refined anisotropicly to the edges $\partial\omega \times \{-\varepsilon, \varepsilon\}$ to obtain the new mesh $\mathcal{T}^p_\varepsilon$ with p layers of elements. Polynomials of degree p on $\mathcal{T}^p_\varepsilon$ form the discrete space.

If the boundary of ω is analytic, and if an analytic load is fixed, we prove in this paper that the relative energy error between the three-dimensional solution and its Galerkin approximation in the above described space is bounded for all $K \geq 1$ by

$$C\left(\varepsilon^K + \varepsilon^{-1}\mathrm{e}^{-bp}\right) \tag{1.1}$$

with positive constant C and b independent of ε and p (but depending on K).

We also prove that in certain cases (existence of underlying C^1 discrete spaces on ω, or membrane load), the factor ε^{-1} in the bound (1.1) can be omitted, which means that, to achieve a given bound to the relative error, a certain polynomial degree p, corresponding to a certain number $N = \mathcal{O}(p^4)$ of degrees of freedom *fixed independently of* ε are sufficient.

These results are based on the *hp* FE-approximation of each piece of the two-scale expansion of the solution displacement $\boldsymbol{u}(\varepsilon)$: this expansion has two parts, the outer expansion part $\sum_k \varepsilon^k \boldsymbol{v}^k$ (regular profiles), and the inner expansion part $\sum_k \varepsilon^k \boldsymbol{w}^k$ (boundary layer profiles).

In this paper, we also pay much attention to the transverse degrees of the polynomials involved in the outer expansion part, which allows in particular to show that $(3,3,2)$ transverse degree outside the support of the load and away from the boundary layer is sufficient to obtain estimate (1.1).

The outline of the paper is as follows: in Section 2, we set the problem and give a rough description of the inner-outer expansion. Sections 3–5 are devoted to the outer part, whereas Sections 6 and 7 are devoted to the inner part. We explain in more detail the structure of the outer part study at the beginning of Section 3, and for the inner part, at the beginning of Section 6. The synthesis and the conclusions are drawn in Section 8.

2. The three-dimensional plate problem

2.1. Domains and coordinates

The plate problem under consideration here is a boundary value problem of three-dimensional elastostatics which is set in the family of domains

$$\Omega^\varepsilon = \omega \times (-\varepsilon, +\varepsilon),$$

where the midsurface ω is open, bounded and has an analytic boundary $\partial\omega$. Let $\Gamma^\varepsilon_\pm$ be their upper and lower faces $\omega \times \{\pm\varepsilon\}$ and Γ^ε_0 be their lateral faces $\partial\omega \times (-\varepsilon, +\varepsilon)$. If $x = (x_1, x_2, x_3)$ are the cartesian coordinates in the plates Ω^ε, we will often denote by x_* the in-plane coordinates $(x_1, x_2) \in \omega$ and by α or β the indices in $\{1,2\}$ corresponding to the in-plane variables. The dilatation along the vertical axis ($X_3 = \varepsilon^{-1}x_3$) transforms Ω^ε into the fixed reference configuration $\Omega = \omega \times (-1,+1)$:

$$x = (x_*,\, x_3) \in \Omega^\varepsilon = \omega \times (-\varepsilon, +\varepsilon) \longmapsto X = (x_*,\, X_3) \in \Omega = \omega \times (-1,+1). \tag{2.1}$$

In general, we will distinguish by a superscript $\widetilde{}$ the vector fields defined in the "physical" domains Ω^ε, from the scaled fields defined on Ω.

2.2. Governing equations

We consider linearly elastic deformations of the plate Ω^ε. The displacement $\widetilde{\boldsymbol{u}} : \Omega^\varepsilon \to \mathbb{R}^3$ of the plate satisfies the equilibrium equations

$$B\widetilde{\boldsymbol{u}} = -\operatorname{div}\sigma(\widetilde{\boldsymbol{u}}) = \widetilde{\boldsymbol{f}} \quad \text{in} \quad \Omega^\varepsilon, \tag{2.2}$$

where $\widetilde{\boldsymbol{f}}$ are volume forces. We assume here that $\widetilde{\boldsymbol{f}}$ is the restriction to $\omega \times (-\varepsilon, +\varepsilon)$ of a function $\overline{\boldsymbol{f}}$ which is analytic in $\overline{\omega} \times (-\varepsilon_0, +\varepsilon_0)$ for a fixed $\varepsilon_0 > \varepsilon$. Furthermore, $\sigma(\widetilde{\boldsymbol{u}})$ is the stress tensor. It is expressed in terms of the infinitesimal strain tensor $e(\widetilde{\boldsymbol{u}})$ by Hooke's law (here summation convention over repeated indices is used)

$$\sigma_{ij}(\widetilde{\boldsymbol{u}}) = A_{ijkl}e_{kl}(\widetilde{\boldsymbol{u}}). \tag{2.3}$$

We assume homogeneous and isotropic material, *i.e.* $A_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ with $\lambda \geq 0$ and $\mu > 0$ denoting the Lamé-constants. On the faces $\Gamma^\varepsilon_\pm = \omega \times \{\pm\varepsilon\}$ of the plate, zero traction boundary conditions are given:

$$G\widetilde{\boldsymbol{u}} = \sigma(\widetilde{\boldsymbol{u}})\boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \Gamma^\varepsilon_\pm \tag{2.4}$$

where $\boldsymbol{n}$ denotes the exterior unit normal vector on $\Gamma^\varepsilon_\pm$.

Problem (2.2–2.4) is completed by boundary conditions on the lateral edge Γ^ε_0. We consider here for simplicity only Dirichlet boundary conditions, *i.e.* the plate is hard clamped,

$$\widetilde{\boldsymbol{u}}|_{\Gamma^\varepsilon_0} = \boldsymbol{0} \tag{2.5}$$

and give the proofs of the results in this case. We emphasize, however, that our results will also hold for all other sets of boundary conditions which lead to a meaningful variational formulation of (2.2–2.4) *cf.* [9].

2.3. Finite Element Approximation

The variational form of (2.2–2.5) is: *Find* $\widetilde{\boldsymbol{u}}$ *such that*

$$\widetilde{\boldsymbol{u}} \in \boldsymbol{H}: \qquad a(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{v}}) = L(\widetilde{\boldsymbol{v}}) \quad \forall \widetilde{\boldsymbol{v}} \in \boldsymbol{H}. \tag{2.6}$$

Here, the bilinear form $a(\cdot,\cdot)$ and the loading $L(\cdot)$ are given by

$$a(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{v}}) = \int_{\Omega^\varepsilon} Ae(\widetilde{\boldsymbol{u}}) : e(\widetilde{\boldsymbol{v}})\,\mathrm{d}x, \quad L(\widetilde{\boldsymbol{v}}) = \int_{\Omega^\varepsilon} \widetilde{\boldsymbol{f}} \cdot \widetilde{\boldsymbol{v}}\,\mathrm{d}x.$$

The proper choice of $\boldsymbol{H}$ incorporates the homogeneous essential boundary conditions on Γ^ε_0:

$$\boldsymbol{H} = \left\{ \widetilde{\boldsymbol{u}} \in H^1(\Omega^\varepsilon)^3 : \widetilde{\boldsymbol{u}}|_{\Gamma^\varepsilon_0} = \boldsymbol{0} \right\} \cdot$$

Korn's inequality implies that the bilinear form $a(\cdot,\cdot)$ in (2.6) is H^1-coercive on $\boldsymbol{H}$, and hence for every smooth volume loading $\widetilde{\boldsymbol{f}}$ exists a unique weak solution $\widetilde{\boldsymbol{u}} \in \boldsymbol{H}$ of (2.6).

Finite Element approximations $\widetilde{\boldsymbol{u}}_N$ of $\widetilde{\boldsymbol{u}}$ are obtained by energy projection: for any finite dimensional subspace $\boldsymbol{H}_N \subset \boldsymbol{H}$, we define

$$\widetilde{\boldsymbol{u}}_N \in \boldsymbol{H}_N : \quad a(\widetilde{\boldsymbol{u}}_N, \widetilde{\boldsymbol{v}}_N) = L(\widetilde{\boldsymbol{v}}_N) \quad \forall \widetilde{\boldsymbol{v}}_N \in \boldsymbol{H}_N. \tag{2.7}$$

There exists a unique solution $\widetilde{\boldsymbol{u}}_N$ of (2.7), and this solution satisfies

$$\forall \widetilde{\boldsymbol{v}}_N \in \boldsymbol{H}_N : \quad \|\widetilde{\boldsymbol{u}} - \widetilde{\boldsymbol{u}}_N\|_{E(\Omega^\varepsilon)} \leq \|\widetilde{\boldsymbol{u}} - \widetilde{\boldsymbol{v}}_N\|_{E(\Omega^\varepsilon)} \tag{2.8}$$

where the energy norm is defined by $\|\widetilde{\boldsymbol{u}}\|^2_{E(\Omega^\varepsilon)} := a(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}})$. Note that for all $\widetilde{\boldsymbol{u}} \in \boldsymbol{H}$ we have the bound $\|\widetilde{\boldsymbol{u}}\|_{E(\Omega^\varepsilon)} \leq C \|\widetilde{\boldsymbol{u}}\|_{H^1(\Omega^\varepsilon)}$ with a constant $C > 0$ independent of ε.

In this paper, we propose a *hp* design for the FE subspace $\boldsymbol{H}_N$ and estimate the approximation error (2.8) in dependence on ε. This is based on a detailed asymptotic analysis of the three-dimensional solution $\widetilde{\boldsymbol{u}}$ in dependence on ε.

2.4. **Asymptotics of the solution**

The complete asymptotics of the solution $\widetilde{\boldsymbol{u}}$ is easier to describe on the reference configuration Ω and using the scaled displacement $\boldsymbol{u}(\varepsilon)$ and the scaled load $\boldsymbol{f}(\varepsilon)$ according to

$$\boldsymbol{u}(\varepsilon)(X) = (\widetilde{u}_1, \widetilde{u}_2, \varepsilon\widetilde{u}_3)(x) \quad \text{and} \quad \boldsymbol{f}(\varepsilon)(X) = \left(\widetilde{f}_1, \widetilde{f}_2, \varepsilon^{-1}\widetilde{f}_3\right)(x).$$

Due to our analyticity assumption on the loading $\widetilde{\boldsymbol{f}}$, we have the (convergent) expansion

$$\boldsymbol{f}(\varepsilon) = \sum_{k=-1}^{\infty} \varepsilon^k \boldsymbol{f}^k(X) \ \text{ with } \begin{cases} \boldsymbol{f}^{-1} = \left(0, 0, \overline{f}_3(x_*, 0)\right) \\ \boldsymbol{f}^k \ = \left(\dfrac{X_3^k}{k!}\partial_3^k f_*(x_*, 0), \dfrac{X_3^{k+1}}{(k+1)!}\partial_3^k f_3(x_*, 0)\right), \ k \geq 0. \end{cases} \tag{2.9}$$

It may be deduced from the results in [9] that $\boldsymbol{u}(\varepsilon)$ admits the asymptotic expansion at any order

$$\boldsymbol{u}(\varepsilon) \sim \sum_{k\geq -1} \varepsilon^k \boldsymbol{u}^k = \sum_{k\geq -1} \varepsilon^k \left(\boldsymbol{v}^k + \chi \boldsymbol{w}^k\right). \tag{2.10}$$

The terms $\boldsymbol{v}^k$ constitute the *outer expansion part.* They essentially satisfy the three-dimensional equilibrium conditions (2.2–2.4). The complementing terms $\boldsymbol{w}^k$ are the *boundary layer terms* which constitute the *inner expansion part* — the function $\chi(x_*)$ is a smooth cut-off function which is identically equal to one in a vicinity of $\partial\omega$. The terms $\boldsymbol{w}^k$ compensate nonhomogeneous edge-conditions in (2.5) due to the $\boldsymbol{v}^k$.

To state the estimates satisfied by the expansion (2.10), we introduce the unscaled sequence of displacement fields on the physical domain Ω^ε (note that it starts with power ε^{-2})

$$\widetilde{\boldsymbol{u}}^{-2}(x) = \left(0, 0, u_3^{-1}\right)(X), \quad \widetilde{\boldsymbol{u}}^k(x) = \left(u_1^k, u_2^k, u_3^{k+1}\right)(X), \quad k \geq -1. \tag{2.11}$$

Theorem 2.1. *For every $\varepsilon > 0$ let $\widetilde{\boldsymbol{u}}(x) \in \boldsymbol{H}$ be the unique solution of problem* (2.6). *Then for every integer $K \geq 0$ there holds for expansion* (2.10) *the error estimate in energy norm*

$$\left\| \widetilde{\boldsymbol{u}} - \sum_{k=-2}^{K-1} \varepsilon^k \widetilde{\boldsymbol{u}}^k \right\|_{E(\Omega^\varepsilon)} \leq C\,\varepsilon^{K-1/2} \tag{2.12}$$

where $C > 0$ is independent of ε, but depends on K.

Information about the first non-vanishing term in expansion (2.9) yields the behavior as $\varepsilon \to 0$ of the energy $\|\widetilde{\boldsymbol{u}}\|_{E(\Omega^\varepsilon)}$ and allows *relative energy error estimates*:

Theorem 2.2. (A) *If $\overline{f}_3(x_*, 0) \not\equiv 0$, $\widetilde{\boldsymbol{u}}$ is bending dominated, its principal term is a* Kirchhoff-Love displacement, *and $\|\widetilde{\boldsymbol{u}}\|_{E(\Omega^\varepsilon)} \simeq \varepsilon^{-1/2}$, therefore*

$$\left\| \widetilde{\boldsymbol{u}} - \sum_{k=-2}^{K-1} \varepsilon^k \widetilde{\boldsymbol{u}}^k \right\|_{E(\Omega^\varepsilon)} \leq C\,\varepsilon^{K} \left\| \widetilde{\boldsymbol{u}} \right\|_{E(\Omega^\varepsilon)}. \tag{2.13}$$

(B) *If $\overline{f}_3(x_*, 0) \equiv 0$ and $\overline{\boldsymbol{f}}_*(x_*, 0) \not\equiv 0$, the principal term of $\widetilde{\boldsymbol{u}}$ is $\widetilde{\boldsymbol{u}}^0$, which contains a membrane part and $\|\widetilde{\boldsymbol{u}}\|_{E(\Omega^\varepsilon)} \simeq \varepsilon^{1/2}$, therefore*

$$\left\| \widetilde{\boldsymbol{u}} - \sum_{k=0}^{K} \varepsilon^k \widetilde{\boldsymbol{u}}^k \right\|_{E(\Omega^\varepsilon)} \leq C\,\varepsilon^{K} \left\| \widetilde{\boldsymbol{u}} \right\|_{E(\Omega^\varepsilon)}. \tag{2.14}$$

Based on (2.8), we will obtain an upper bound for the Finite Element error $\|\widetilde{\boldsymbol{u}} - \widetilde{\boldsymbol{u}}_N\|_{E(\Omega^\varepsilon)}$ by the triangle inequality:

$$\|\widetilde{\boldsymbol{u}} - \widetilde{\boldsymbol{u}}_N\|_{E(\Omega^\varepsilon)} \leq \left\| \widetilde{\boldsymbol{u}} - \sum_{k=-2}^{K} \varepsilon^k \widetilde{\boldsymbol{u}}^k \right\|_{E(\Omega^\varepsilon)} + \min_{\widetilde{\boldsymbol{v}}_N \in \boldsymbol{H}_N} \left\| \sum_{k=-2}^{K} \varepsilon^k \widetilde{\boldsymbol{u}}^k - \widetilde{\boldsymbol{v}}_N \right\|_{E(\Omega^\varepsilon)}. \tag{2.15}$$

Thus bounding the Finite Element error will be achieved by approximating the asymptotic terms $\widetilde{\boldsymbol{u}}^k$ from the Finite Element space $\boldsymbol{H}_N$.

3. Structure of the outer expansion part

In this section, we essentially reformulate the results of Section 3 in [9] providing a solution of equations (2.2–2.4), *i.e.* without lateral boundary conditions, in formal series algebras. This yields a general description of the terms $\boldsymbol{v}^k$ in (2.10) as coefficients of a formal series $\boldsymbol{v}[\varepsilon]$ satisfying functional equations involving the formal series $\boldsymbol{f}[\varepsilon]$ with the coefficients $\boldsymbol{f}^k$ of (2.9) and a formal series $\boldsymbol{\zeta}[\varepsilon]$ of two-dimensional generators. This "generating series" $\boldsymbol{\zeta}[\varepsilon]$ satisfies itself a functional equation inside ω. We describe the four series of operators $\mathbf{V}[\varepsilon]$, $\mathbf{Q}[\varepsilon]$, $\mathbf{A}[\varepsilon]$ and $\mathbf{R}[\varepsilon]$ involved in these functional equations.

In Section 4, we will deduce from the formulas stated in Section 3 new results about the properties of the operators entering the formal series equations, concerning their action on analytic functions in in-plane variables and polynomials in the transverse variable.

In Section 5, we recall from [9] the series of *boundary conditions* on $\partial\omega$ satisfied by the formal series $\boldsymbol{\zeta}[\varepsilon]$ of two-dimensional generators. These boundary conditions complement the functional equation inside ω. We show that it has a unique *analytic* solution, which yields that the $\boldsymbol{v}^k$ are uniquely determined polynomial functions in X_3 with coefficients in analytic fields on $\overline{\omega}$. We deduce from this tensorial structure the approximation properties of a simple p-version FEM on Ω^ε for the outer part.

3.1. General structure of the asymptotics

A comprehensive way of solving equations (2.2–2.4) on the reference domain Ω is the use of formal series of operators and vector functions, as initiated in [11]. The basic notion is the following: if $A[\varepsilon]$ is a formal series with operator coefficients

$$A[\varepsilon] = \sum_k \varepsilon^k A^k \quad \text{with} \quad A_k \in \mathcal{L}(E,F),$$

with E, F functional spaces, and if $b[\varepsilon]$ and $c[\varepsilon]$ are formal series in E and F

$$b[\varepsilon] = \sum_k \varepsilon^k b^k, \quad b^k \in E, \quad \text{and} \quad c[\varepsilon] = \sum_k \varepsilon^k c^k, \quad c^k \in F,$$

the equation $A[\varepsilon]b[\varepsilon] = c[\varepsilon]$ means that

$$\forall k \in \mathbb{N}, \quad \sum_{\ell=0}^k A^{k-\ell} b^\ell = c^k.$$

As prerequisite, we first expand the operators B and G in equations (2.2–2.4) corresponding to the scaled problem on Ω and we obtain the following problem without lateral boundary conditions that we write in the form

$$\begin{cases} B[\varepsilon]\boldsymbol{v}[\varepsilon] = \boldsymbol{f}[\varepsilon] := \sum_{k\geq -1} \varepsilon^k \boldsymbol{f}^k & \text{in} \quad \Omega, \\ G[\varepsilon]\boldsymbol{v}[\varepsilon] = \mathbf{0} & \text{on} \quad \Gamma_\pm. \end{cases} \tag{3.1}$$

Then the results in Section 3 of [9] can be reformulated following the lines of [5, 10, 11]:

Theorem 3.1. (i) *There exist a formal series of surface operators $\mathbf{A}[\varepsilon]$ with coefficients $\mathbf{A}^k$ continuous from $\mathcal{C}^\infty(\overline{\varpi})^3$ into itself and a formal series of reconstruction operators $\mathbf{V}[\varepsilon]$ with coefficients $\mathbf{V}^k$ continuous from $\mathcal{C}^\infty(\overline{\varpi})^3$ into $\mathcal{C}^\infty(\overline{\Omega})^3$ such that for any formal series $\boldsymbol{\zeta}[\varepsilon]$ of two-dimensional generators $\boldsymbol{\zeta}^k \in \mathcal{C}^\infty(\overline{\varpi})^3$ satisfying the equation*

$$\mathbf{A}[\varepsilon]\boldsymbol{\zeta}[\varepsilon] = 0,$$

we obtain a solution $\boldsymbol{v}[\varepsilon]$ of problem (3.1) *with $\boldsymbol{f}[\varepsilon] \equiv \mathbf{0}$ by setting*

$$\boldsymbol{v}[\varepsilon] = \mathbf{V}[\varepsilon]\boldsymbol{\zeta}[\varepsilon].$$

(ii) *There exist a formal series of reduction operators $\mathbf{R}[\varepsilon]$ with coefficients $\mathbf{R}^k$ continuous from $\mathcal{C}^\infty(\overline{\Omega})^3$ into $\mathcal{C}^\infty(\overline{\varpi})^3$ and a formal series of solution operators $\mathbf{Q}[\varepsilon]$ with coefficients $\mathbf{Q}^k$ continuous from $\mathcal{C}^\infty(\overline{\Omega})^3$ into itself such that for any formal series $\boldsymbol{\zeta}[\varepsilon]$ of two-dimensional generators satisfying the equation*

$$\mathbf{A}[\varepsilon]\boldsymbol{\zeta}[\varepsilon] = \mathbf{R}[\varepsilon]\boldsymbol{f}[\varepsilon], \tag{3.2}$$

we obtain a solution $\boldsymbol{v}[\varepsilon]$ of problem (3.1) *by setting*

$$\boldsymbol{v}[\varepsilon] = \mathbf{V}[\varepsilon]\boldsymbol{\zeta}[\varepsilon] + \mathbf{Q}[\varepsilon]\boldsymbol{f}[\varepsilon]. \tag{3.3}$$

3.2. Series $\mathbf{V}[\varepsilon]$

This series has only even terms: for all $\ell \in \mathbb{N}$, $\mathbf{V}^{2\ell+1} \equiv 0$.

The first term $\mathbf{V}^0$ of $\mathbf{V}[\varepsilon]$ is the Kirchhoff-Love operator: for $\boldsymbol{\zeta} = (\boldsymbol{\zeta}_*, \zeta_3) \in \mathcal{C}^\infty(\overline{\varpi})^3$ there holds

$$\mathbf{V}^0\boldsymbol{\zeta} = (\boldsymbol{\zeta}_* - X_3\nabla_*\zeta_3, \zeta_3) \tag{3.4}$$

and the second non-zero term has the explicit form

$$
\begin{aligned}
(\mathbf{V}^2\boldsymbol{\zeta})_\alpha &= \bar{p}_2(X_3)\,\partial_\alpha \operatorname{div}_* \boldsymbol{\zeta}_* + \bar{p}_3(X_3)\,\partial_\alpha \Delta_* \zeta_3 \\
(\mathbf{V}^2\boldsymbol{\zeta})_3 &= \bar{p}_1(X_3) \operatorname{div}_* \boldsymbol{\zeta}_* + \bar{p}_2(X_3)\,\Delta_* \zeta_3
\end{aligned}
\tag{3.5}
$$

with $\bar{p}_j$ for $j = 1, 2, 3$ the polynomials in the variable X_3 of degrees j defined as

$$
\bar{p}_1 = -\mathfrak{p}\, X_3, \qquad \bar{p}_2 = \frac{\mathfrak{p}}{6}\left(3X_3^2 - 1\right), \qquad \bar{p}_3 = \frac{\mathfrak{p}+2}{6}\, X_3^3 - \frac{5\mathfrak{p}+1}{6}\, X_3,
$$

where $\mathfrak{p} := \lambda/(\lambda + 2\mu)$. The next ones have the general form, for $\ell = 2, 3, \ldots$

$$
\begin{aligned}
(\mathbf{V}^{2\ell}\boldsymbol{\zeta})_\alpha &= \bar{s}_{2\ell}(X_3)\,\partial_\alpha \Delta_*^{\ell-1} \operatorname{div}_* \boldsymbol{\zeta}_* + \bar{t}_{2\ell}(X_3)\,\Delta_*^\ell \zeta_\alpha + \bar{s}_{2\ell+1}(X_3)\,\partial_\alpha \Delta_*^\ell \zeta_3 \\
(\mathbf{V}^{2\ell}\boldsymbol{\zeta})_3 &= \bar{q}_{2\ell-1}(X_3)\,\Delta_*^{\ell-1} \operatorname{div}_* \boldsymbol{\zeta}_* + \bar{q}_{2\ell}(X_3)\,\Delta_*^\ell \zeta_3,
\end{aligned}
\tag{3.6}
$$

with $\bar{s}_j$, $\bar{t}_j$ and $\bar{q}_j$ polynomials of degree j (note that the definition of these polynomials differs slightly from those introduced in Sect. 3 of [9], but they play a quite similar role).

3.3. Series $\mathbf{Q}[\varepsilon]$

Again, this series has only even terms: for all $\ell \in \mathbb{N}$, $\mathbf{Q}^{2\ell+1} \equiv 0$.

The first term $\mathbf{Q}^0$ of $\mathbf{Q}[\varepsilon]$ is zero. The first non-zero term is $\mathbf{Q}^2$ and it coincides with the operator G introduced in Section 3 of [9], that we recall now. For doing this we need two sorts of primitive of an integrable function u on the interval $(-1, +1)$:

Notation 3.2. *Let us introduce:*

- *The primitive of u with zero mean value on $(-1, +1)$*

$$
\oint^{x_3} u \,\mathrm{d}y_3 := \int_{-1}^{x_3} u(y_3)\,\mathrm{d}y_3 - \frac{1}{2}\int_{-1}^{+1}\int_{-1}^{z_3} u(y_3)\,\mathrm{d}y_3\,\mathrm{d}z_3.
$$

- *The primitive of u which vanishes in -1 and 1 if u has a zero mean value on $(-1, +1)$ and which is even, resp. odd, if u is odd, resp. even*

$$
\fint^{y_3} u \,\mathrm{d}z_3 := \frac{1}{2}\left(\int_{-1}^{y_3} u(z_3)\,\mathrm{d}z_3 - \int_{y_3}^{+1} u(z_3)\,\mathrm{d}z_3\right).
$$

Then for $\boldsymbol{f} \in \mathcal{C}^\infty(\overline{\Omega})^3$, we define $G\boldsymbol{f}$ as

$$
\begin{aligned}
(G\boldsymbol{f})_3 &= 0 \\
(G\boldsymbol{f})_\alpha &= \frac{1}{2\mu}\oint^{x_3}\left[-2\fint^{y_3} f_\alpha + \Big(\int_{-1}^{+1} f_\alpha\Big)\, y_3\right]\mathrm{d}y_3\,.
\end{aligned}
\tag{3.7}
$$

Next $\mathbf{Q}^4 = WG + H$, where the operator $W : \boldsymbol{v} \mapsto W\boldsymbol{v}$ is defined from $\mathcal{C}^\infty(\overline{\Omega})^3$ into itself by

$$
\begin{aligned}
(W\boldsymbol{v})_3 &= -\oint^{x_3}\left(\frac{\tilde{\lambda}}{2\mu}\operatorname{div}_* \boldsymbol{v}_* + \frac{\tilde{\lambda}}{\lambda}\fint^{y_3} \partial_\beta e_{\beta 3}(\boldsymbol{v})\right)\mathrm{d}y_3 \\
(W\boldsymbol{v})_\alpha &= -\oint^{x_3}\left(\partial_\alpha W_3\boldsymbol{v} + \fint^{y_3}\Big(\frac{\lambda}{\mu}\,\partial_{\alpha 3} W_3\boldsymbol{v} + \frac{\lambda+\mu}{\mu}\,\partial_\alpha \operatorname{div}_* \boldsymbol{v}_* + \Delta_* v_\alpha\Big)\right)\mathrm{d}y_3.
\end{aligned}
\tag{3.8}
$$

In order to define H, we need the auxiliary operator F from $\mathcal{C}^\infty(\overline{\Omega})^3$ into $\mathcal{C}^\infty(\overline{\varpi})^3$:

$$\begin{aligned}(F\boldsymbol{v})_3 &= \mu \int_{-1}^{+1} \partial_\beta e_{\beta 3}(\boldsymbol{v})\,\mathrm{d}y_3\,,\\ (F\boldsymbol{v})_\alpha &= \frac{\tilde\lambda}{2}\int_{-1}^{+1}\fint^{y_3} \partial_{\alpha\beta} e_{\beta 3}(\boldsymbol{v})\,\mathrm{d}z_3\,\mathrm{d}y_3\;.\end{aligned}\tag{3.9}$$

Then for $\boldsymbol{f} \in \mathcal{C}^\infty(\overline{\Omega})^3$, we define $H\boldsymbol{f}$ as

$$\begin{aligned}(H\boldsymbol{f})_3 &= \frac{1}{2(\lambda+2\mu)}\oint^{x_3}\Big[\Big(-2\fint^{y_3} f_3\Big)\Big]\mathrm{d}y_3\\ (H\boldsymbol{f})_\alpha &= -\oint^{x_3}\Big[\partial_\alpha H_3 + \frac{1}{\mu}\,y_3\,(FG)_\alpha + \frac{\lambda}{\mu}\fint^{y_3}\Big\{\partial_{\alpha 3}H_3 - \frac{1}{2}\int_{-1}^{+1}\partial_{\alpha 3}H_3\,\mathrm{d}z_3\Big\}\Big]\mathrm{d}y_3\;,\end{aligned}\tag{3.10}$$

where, in the last line, H_3 denotes $(H\boldsymbol{f})_3$ and $(FG)_\alpha$ denotes $\big(F(G\boldsymbol{f})\big)_\alpha$.

With the convention that W^0 is the identity and for any $k>0$, W^{-k} is zero, we have the general formula for $\mathbf{Q}^{2\ell}$

$$\mathbf{Q}^{2\ell} = W^{2\ell-2}G + W^{2\ell-4}H.\tag{3.11}$$

3.4. Series $\mathbf{A}[\varepsilon]$

The first term $\mathbf{A}^0$ of the formal operator series $\mathbf{A}[\varepsilon]$ is the block diagonal membrane-flexion operator

$$\mathbf{A}^0 = \begin{pmatrix} -L^{\mathrm{m}} & 0\\ 0 & \frac{1}{3}L^{\mathrm{b}}\end{pmatrix}\tag{3.12}$$

where – we recall that $\mathfrak{p} = \lambda/(\lambda+2\mu)$:

$$L^{\mathrm{m}} = \mu\begin{pmatrix}\Delta_* & 0\\ 0 & \Delta_*\end{pmatrix} + \mu(2\mathfrak{p}+1)\begin{pmatrix}\partial_1\\ \partial_2\end{pmatrix}\mathrm{div}_* \qquad\text{and}\qquad L^{\mathrm{b}} = 2\mu(\mathfrak{p}+1)\Delta_*^2\,.$$

The terms of odd rank of $\mathbf{A}[\varepsilon]$ are zero. The next non-zero term $\mathbf{A}^2$ is given by (using the operators W and F introduced above, and denoting the triple $(F_1,F_2,0)$ by F_*)

$$\begin{aligned}(\mathbf{A}^2\boldsymbol{\zeta})_\alpha &= 0\,,\\ (\mathbf{A}^2\boldsymbol{\zeta})_3 &= -F_3(W\mathbf{V}^2\boldsymbol{\zeta})\end{aligned}$$

and the generic terms of even order are, *cf.* Table 3.1 in [9]:

$$\begin{aligned}(\mathbf{A}^{2\ell}\boldsymbol{\zeta})_\alpha &= F_\alpha(\mathbf{V}^{2\ell}\boldsymbol{\zeta})\,,\\ (\mathbf{A}^{2\ell}\boldsymbol{\zeta})_3 &= -F_3\Big(W\mathbf{V}^{2\ell}\boldsymbol{\zeta} - \tfrac{1}{6\mu}\,(3X_3^2-1)\,F_*(\mathbf{V}^{2\ell}\boldsymbol{\zeta})\Big)\,.\end{aligned}$$

3.5. Series $\mathbf{R}[\varepsilon]$

The first term $\mathbf{R}^0$ is given by

$$
\begin{aligned}
(\mathbf{R}^0\boldsymbol{f})_\alpha &= \frac{1}{2}\int_{-1}^{+1} \boldsymbol{f}_\alpha(x_*,X_3)\,\mathrm{d}X_3\\
(\mathbf{R}^0\boldsymbol{f})_3 &= \frac{1}{2}\int_{-1}^{+1} \big(f_3 + X_3\ \mathrm{div}_*\,\boldsymbol{f}_*\big)(x_*,X_3)\,\mathrm{d}X_3\ .
\end{aligned}
$$

The terms of odd rank of $\mathbf{R}[\varepsilon]$ are zero. Relying again on Table 3.1 in [9], we find successively

$$
\begin{aligned}
(\mathbf{R}^2\boldsymbol{f})_\alpha &= -F_\alpha(\mathbf{Q}^2\boldsymbol{f}) + \frac{\mathfrak{p}}{2}\int_{-1}^{+1} X_3\,\partial_\alpha f_3(x_*,X_3)\,\mathrm{d}X_3\,,\\
(\mathbf{R}^2\boldsymbol{f})_3 &= F_3(\mathbf{Q}^4\boldsymbol{f})
\end{aligned}
$$

and for $\ell \geq 2$

$$
\begin{aligned}
(\mathbf{R}^{2\ell}\boldsymbol{f})_\alpha &= -F_\alpha(\mathbf{Q}^{2\ell}\boldsymbol{f})\,,\\
(\mathbf{R}^{2\ell}\boldsymbol{f})_3 &= F_3\Big(\mathbf{Q}^{2\ell+2}\boldsymbol{f} - \tfrac{1}{6\mu}\,(3X_3^2-1)\,F_*(\mathbf{Q}^{2\ell}\boldsymbol{f})\Big).
\end{aligned}
$$

4. Transverse degree and analyticity of the outer expansion operators

The aim of this section is to deduce from the above formulas for the series $\mathbf{V}[\varepsilon]$, $\mathbf{Q}[\varepsilon]$, $\mathbf{A}[\varepsilon]$ and $\mathbf{R}[\varepsilon]$ information on the way they act on polynomials in the transverse variable X_3 and analytic functions in the in-plane variables $x_* = (x_\alpha)$. Moreover, by a factorization of certain coefficients of these series, we will exhibit a simpler equivalent expression for equations (3.2) and (3.3), where properties on polynomials and analytic functions are easier to deduce.

4.1. In-plane and transverse degrees

We first remark that all the operators $\mathbf{V}^k$ and $\mathbf{Q}^k$ have the generic block form (obtained by spitting the inplane and transverse components)

$$
C = \begin{pmatrix} C_{**} & C_{*3}\\ C_{3*} & C_{33}\end{pmatrix}
$$

and that each operator C_{ij} in the above matrix is a linear combination of operators of the form $J\circ D$ where D is a partial derivative operator in the in-plane variables x_* with constant coefficients and J is a combination of derivations, integrations in X_3 and multiplication by polynomials in X_3. We adopt the following notation: $\deg_*(J\circ D)$ denotes the degree of the operator D, whereas $\deg_3(J\circ D)$ denotes the degree of J acting on polynomials in X_3: if the degree of J is d, then J transforms a polynomial of degree n into a polynomial of degree $n-d$. From these definitions, we deduce the natural notion of block degrees for an operator C as above. Inspecting formulas (3.6)–(3.11), we obtain

Lemma 4.1. *For any even number k the block degrees of $\mathbf{V}^k$ and $\mathbf{Q}^k$ are the following*

$$
\deg_*(\mathbf{V}^k) = \begin{pmatrix} k & k+1\\ k-1 & k\end{pmatrix} \quad \textit{and} \quad \deg_3(\mathbf{V}^k) = -\begin{pmatrix} k & k+1\\ k-1 & k\end{pmatrix}
$$

and

$$
\deg_*(\mathbf{Q}^k) = \begin{pmatrix} k-2 & k-3\\ k-3 & k-4\end{pmatrix} \quad \textit{and} \quad \deg_3(\mathbf{Q}^k) = -\begin{pmatrix} k & k-1\\ k-1 & k-2\end{pmatrix}.
$$

In particular $\mathbf{V}^k$ *is a continuous operator from* $\mathcal{A}(\overline{\omega})^3$ *into* $\mathcal{A}(\overline{\Omega})^3$ *and* $\mathbf{Q}^k$ *is continuous from* $\mathcal{A}(\overline{\Omega})^3$ *into itself. Moreover* $\mathbf{V}^k$ *and* $\mathbf{Q}^k$ *are* block-homogeneous *in* x_*.

4.2. Factorization by $\mathbf{A^0}$

For $\ell \geq 2$, the operators $\mathbf{V}^{2\ell}$ can be factorized through $\mathbf{A}^0$: concerning the action on the transverse component ζ_3 this is clear from formulas (3.6); concerning the action on the in-plane components $\boldsymbol{\zeta}_*$ we note that the following formulas hold

$$\Delta_* \operatorname{div}_* \boldsymbol{\zeta}_* = \tfrac{1}{2\mu(\mathfrak{p}+1)} \operatorname{div}_* L^{\mathrm{m}}\boldsymbol{\zeta}_*$$

and

$$\Delta_*^2 \zeta_\alpha = \tfrac{1}{\mu}\, \Delta_*(L^{\mathrm{m}}\boldsymbol{\zeta}_*)_\alpha - \tfrac{2\mathfrak{p}+1}{2\mu(\mathfrak{p}+1)}\, \partial_\alpha \operatorname{div}_*(L^{\mathrm{m}}\boldsymbol{\zeta}_*)\ .$$

As a result we find that each operator $V := \mathbf{V}^{2\ell}$ for $\ell \geq 2$, and also $V := W\mathbf{V}^2$, can be factorized by $\mathbf{A}^0$, *i.e.* that there exists a matrix partial differential operator T such that $T\mathbf{A}^0$ coincides with V. Combining with formulas giving $\mathbf{A}^{2\ell}$ for $\ell \geq 1$, we obtain:

Lemma 4.2. *For any* $\ell \geq 1$*, there exists a partial differential operator* $\boldsymbol{T}^{2\ell}$ *in* x_* *such that* $\mathbf{A}^{2\ell} = \boldsymbol{T}^{2\ell}\mathbf{A}^0$*. The operator* $\boldsymbol{T}^{2\ell}$ *is block-homogeneous of degree*

$$\deg_* \boldsymbol{T}^{2\ell} = \begin{pmatrix} 2\ell & 2\ell-1 \\ 2\ell+1 & 2\ell \end{pmatrix}.$$

Therefore the equation (3.2): $\mathbf{A}[\varepsilon]\boldsymbol{\zeta}[\varepsilon] = \mathbf{R}[\varepsilon]\boldsymbol{f}[\varepsilon]$, can be written

$$\boldsymbol{T}[\varepsilon]\mathbf{A}^0\boldsymbol{\zeta}[\varepsilon] = \mathbf{R}[\varepsilon]\boldsymbol{f}[\varepsilon] \quad \text{where} \quad \boldsymbol{T}[\varepsilon] = \mathrm{Id} + \textstyle\sum_{\ell\geq 1} \varepsilon^{2\ell}\boldsymbol{T}^{2\ell}.$$

The series $\boldsymbol{T}[\varepsilon]$, since starting by $\boldsymbol{T}^0 = \mathrm{Id}$, is invertible in the formal series algebra and the 2ℓ-th rank operator of $\boldsymbol{T}[\varepsilon]^{-1}$ is a block homogeneous partial differential operator in x_* of the same degrees as $\boldsymbol{T}^{2\ell}$. As $\mathbf{R}^{2\ell}$ is also a block-homogeneous partial differential operator in x_* of the same degrees as $\boldsymbol{T}^{2\ell}$, setting $\breve{\mathbf{R}}[\varepsilon] := \boldsymbol{T}[\varepsilon]^{-1}\mathbf{R}[\varepsilon]$, we obtain:

Lemma 4.3. *There is a series* $\breve{\mathbf{R}}[\varepsilon] = \sum_{\ell\geq 0} \varepsilon^{2\ell}\breve{\mathbf{R}}^{2\ell}$ *such that equation* (3.2) *is equivalent to*

$$\mathbf{A}^0\boldsymbol{\zeta}[\varepsilon] = \breve{\mathbf{R}}[\varepsilon]\boldsymbol{f}[\varepsilon] \tag{4.1}$$

The operator $\breve{\mathbf{R}}^{2\ell}$ *is block-homogeneous and its block degree* $\deg_* \breve{\mathbf{R}}^{2\ell}$ *is equal to* $\deg_* \boldsymbol{T}^{2\ell}$*, cf. Lemma* 4.2.

Moreover, since each $\mathbf{V}^{2\ell}$ for $\ell \geq 2$ can be factorized by $\mathbf{A}^0$, there exists a formal series of operators, $\boldsymbol{T}'[\varepsilon] = \sum_{\ell\geq 2} \varepsilon^{2\ell}\boldsymbol{T}'^{\,2\ell}$ such that

$$\mathbf{V}[\varepsilon] = \mathbf{V}^0 + \varepsilon^2\mathbf{V}^2 + \boldsymbol{T}'[\varepsilon]\mathbf{A}^0.$$

We check that $\boldsymbol{T}'^{\,2\ell}$ is block-homogeneous of degree

$$\deg_* \boldsymbol{T}'^{\,2\ell} = \begin{pmatrix} 2\ell-2 & 2\ell-3 \\ 2\ell-3 & 2\ell-4 \end{pmatrix}.$$

Combining with the equation (4.1): $\mathbf{A}^0\boldsymbol{\zeta}[\varepsilon] = \breve{\mathbf{R}}[\varepsilon]\boldsymbol{f}[\varepsilon]$, we obtain that $\mathbf{V}[\varepsilon]\boldsymbol{\zeta}[\varepsilon]$ is equal to $(\mathbf{V}^0 + \varepsilon^2\mathbf{V}^2)\boldsymbol{\zeta}[\varepsilon] + \boldsymbol{T}'[\varepsilon]\breve{\mathbf{R}}[\varepsilon]\boldsymbol{f}[\varepsilon]$. Setting $\breve{\mathbf{Q}}[\varepsilon] := \boldsymbol{T}'[\varepsilon]\breve{\mathbf{R}}[\varepsilon] + \mathbf{Q}[\varepsilon]$, we have obtained:

Lemma 4.4. *There is a series* $\breve{\mathbf{Q}}[\varepsilon] = \sum_{\ell\geq 1} \varepsilon^{2\ell}\breve{\mathbf{Q}}^{2\ell}$ *such that identity* (3.3) *can be written in the new form*

$$\boldsymbol{v}[\varepsilon] = \breve{\mathbf{V}}[\varepsilon]\boldsymbol{\zeta}[\varepsilon] + \breve{\mathbf{Q}}[\varepsilon]\boldsymbol{f}[\varepsilon], \quad \textit{where} \quad \breve{\mathbf{V}}[\varepsilon] = \mathbf{V}^0 + \varepsilon^2\mathbf{V}^2, \tag{4.2}$$

cf. (3.4, 3.5). The operator $\breve{\mathbf{Q}}^{2\ell}$ *is block-homogeneous in* x_* *and its block degrees are such that* $\deg_* \breve{\mathbf{Q}}^{2\ell} = \deg_* \boldsymbol{T}'^{\,2\ell}$*, see above, and* $\deg_3 \breve{\mathbf{Q}}^{2\ell} = \deg_3 \mathbf{V}^{2\ell}$*, cf. Lemma* 4.1.

Using operators G, W and H introduced in (3.7, 3.8, 3.10) we have for the first terms

$$\breve{\mathbf{Q}}^2 = G \quad \text{and} \quad \breve{\mathbf{Q}}^4 = \frac{1}{\tilde{\lambda} + 2\mu}\begin{pmatrix} -\bar{r}_4\partial_\alpha \operatorname{div}_* & -\bar{r}_5\partial_\alpha \\ 3\bar{q}_3 \operatorname{div}_* & 3\bar{q}_4 \end{pmatrix}\mathbf{R}^0 + WG + H, \tag{4.3}$$

where $\bar{q}_3$, $\bar{q}_4$, $\bar{r}_4$ and $\bar{r}_5$ are the polynomials of X_3 appearing in (3.6).

5. The outer expansion and its p-approximation

In this section we prove that the formal generating series $\boldsymbol{\zeta}[\varepsilon]$ has analytic coefficients, and we deduce that the terms $\boldsymbol{v}^k$ of the outer expansion part are polynomial in X_3 and analytic in x_*. Such a structure allows approximation by tensor p-version FE at an exponential rate.

5.1. The analyticity of the generators

So far, the formal generating series $\boldsymbol{\zeta}[\varepsilon]$ is still not completely determined. We know that it has to solve equation (3.2) which we proved to be equivalent to (4.1): $\mathbf{A}^0\boldsymbol{\zeta}[\varepsilon] = \breve{\mathbf{R}}[\varepsilon]\boldsymbol{f}[\varepsilon]$. Up to now, every equality was deduced from the elasticity equations on Ω^ε *without lateral boundary conditions.* Taking the lateral clamped condition into account and introducing the two-scale Ansatz of the inner-outer expansion, it can be proved, see [6, 7, 9], that the series $\boldsymbol{\zeta}[\varepsilon]$ has also to satisfy *boundary conditions on* $\partial\omega$.

Let $s \mapsto x_*(s)$ be an arclength coordinate along $\partial\omega$. By extension, we will also often write $s \in \partial\omega$. Translating the results of [9] with the formalism of [5, 10] we obtain that there exists a formal series of trace operators $\boldsymbol{\delta}[\varepsilon]$ with coefficients $\boldsymbol{\delta}^k(s; \partial_s, \partial_r)$ continuous from $\mathcal{C}^\infty(\overline{\omega})^3$ into $\mathcal{C}^\infty(\partial\omega)^4$ and a formal series of trace operators $\boldsymbol{\gamma}[\varepsilon]$ with coefficients $\boldsymbol{\gamma}^k(s; \partial_s, \partial_r, \partial_3)$ continuous from $\mathcal{C}^\infty(\overline{\Omega})^3$ into $\mathcal{C}^\infty(\partial\omega)^4$ such that the 2D-generator formal series $\boldsymbol{\zeta}[\varepsilon]$ has to solve $\boldsymbol{\delta}[\varepsilon]\boldsymbol{\zeta}[\varepsilon] = \boldsymbol{\gamma}[\varepsilon]\boldsymbol{f}[\varepsilon]$ on $\partial\omega$.

The dependence on $s \in \partial\omega$ of the operators $\boldsymbol{\delta}^k$ and $\boldsymbol{\gamma}^k$ is only governed by the equation of $\partial\omega$ which is an analytic curve. Therefore $\boldsymbol{\delta}^k$ is continuous from $\mathcal{A}(\overline{\omega})^3$ into $\mathcal{A}(\partial\omega)^4$ and $\boldsymbol{\gamma}^k$ is continuous from $\mathcal{A}(\overline{\Omega})^3$ into $\mathcal{A}(\partial\omega)^4$. The trace operator $\boldsymbol{\delta}^0$ is the Dirichlet trace of $\mathbf{A}^0$, *cf.* (3.12):

$$\boldsymbol{\delta}^0(\boldsymbol{\zeta}) = \big(\zeta_1, \zeta_2, \zeta_3, \partial_n\zeta_3\big)\big|_{\partial\omega}.$$

and the first terms $\boldsymbol{\gamma}^0 = \boldsymbol{\gamma}^1 = \mathbf{0}$.

Combining the results of [9] with our Lemmas 4.3 and 4.4 we obtain:

Proposition 5.1. *The outer expansion part* $\boldsymbol{v}[\varepsilon]$ *is given by*

$$\boldsymbol{v}[\varepsilon] = \big(\mathbf{V}^0 + \varepsilon^2\mathbf{V}^2\big)\boldsymbol{\zeta}[\varepsilon] + \breve{\mathbf{Q}}[\varepsilon]\boldsymbol{f}[\varepsilon], \tag{5.1}$$

where $\boldsymbol{\zeta}[\varepsilon]$ *is solution of the series of boundary value problems on* ω

$$\begin{cases} \mathbf{A}^0\boldsymbol{\zeta}[\varepsilon] = \breve{\mathbf{R}}[\varepsilon]\boldsymbol{f}[\varepsilon] & \textit{in} \quad \omega \\ \boldsymbol{\delta}[\varepsilon]\boldsymbol{\zeta}[\varepsilon] = \gamma[\varepsilon]\boldsymbol{f}[\varepsilon] & \textit{on} \quad \partial\omega. \end{cases} \tag{5.2}$$

The ellipticity and coercivity of the leading part $(\mathbf{A}^0, \boldsymbol{\delta}^0)$ in (5.2) implies, [17], that $(\mathbf{A}^0, \boldsymbol{\delta}^0)$ is an isomorphism from $\mathcal{A}(\overline{\omega})^3$ onto $\mathcal{A}(\overline{\omega})^3 \times \mathcal{A}(\partial\omega)^4$. Combining this with the analyticity of $\overline{\boldsymbol{f}}$ gives the analyticity of the $\boldsymbol{\zeta}^k$ in $\overline{\omega}$. Moreover the expression (2.9) of the coefficients of $\boldsymbol{f}[\varepsilon]$ and the bound on the transverse degrees of the operators $\check{\mathbf{Q}}^k$, *cf.* Lemma 4.4, yields that the $\boldsymbol{v}^k$ are polynomial in X_3 with a bound for their degrees:

Theorem 5.2. *The series $\boldsymbol{\zeta}[\varepsilon]$ of two-dimensional generators solution of* (5.2) *has coefficients $\boldsymbol{\zeta}^k \in \mathcal{A}(\overline{\omega})^3$. The series $\boldsymbol{v}[\varepsilon]$ giving the outer expansion part is determined by formula* (5.1)*. The term $\boldsymbol{v}^k$ is polynomial in X_3 of degree $\deg_3 \boldsymbol{v}^k \le k+1$, with analytic coefficients:*

$$\boldsymbol{v}^k(X) = \sum_{\ell=0}^{k+1} X_3^\ell\, \boldsymbol{\eta}^{k,\ell}(x_*), \quad \boldsymbol{\eta}^{k,\ell} \in \mathcal{A}(\overline{\omega})^3. \tag{5.3}$$

5.2. Finite transverse degree

If $\overline{\boldsymbol{f}}$ is a polynomial in all three variables x, the transverse polynomial degree of the profiles $\boldsymbol{v}^k$ is in fact bounded *indepently of k*. This result is formalized by the notion of finite transverse degree. We say that the outer expansion part $\boldsymbol{v}[\varepsilon]$ has *a finite transverse degree* if there exists an integer $d \ge 0$ such that each term $\boldsymbol{v}^k$ is polynomial in X_3 with a degree $\le d$. From formulas (3.7)–(3.11), it is clear that a necessary condition for $\boldsymbol{v}[\varepsilon]$ to have a finite transverse degree is that the series $\boldsymbol{f}[\varepsilon]$ defining the right hand side has itself a finite transverse degree.

For a field $\boldsymbol{f}$ depending polynomially on all variables x_1, x_2 and X_3 we denote by $\deg_* \boldsymbol{f}$ the two component vector of the in-plane degrees of $\boldsymbol{f}_*$ and f_3, and by $\deg_3 \boldsymbol{f}$ the two component vector of the transverse degrees. For a formal series $\boldsymbol{f}[\varepsilon]$, $\deg_* \boldsymbol{f}[\varepsilon]$ is defined as $\sup_k \deg_* \boldsymbol{f}^k$, and similarly for $\deg_3 \boldsymbol{f}[\varepsilon]$. As a consequence of Lemma 4.2 we obtain:

Theorem 5.3. *Let q, $p' \in \mathbb{N}$ be two integers with $q \ge 1$. If*

$$\deg_* \boldsymbol{f}[\varepsilon] \le \big(2p' \ 2p'-1\big)^\top \quad \textit{and} \quad \deg_3 \boldsymbol{f}[\varepsilon] \le \big(q \ q-1\big)^\top \tag{5.4}$$

then the outer expansion part $\boldsymbol{v}[\varepsilon]$ has finite transverse degree and satisfies

$$\deg_3 \boldsymbol{v}[\varepsilon] \le \big(2p'+q+2 \ 2p'+q+1\big)^\top. \tag{5.5}$$

Proof. If $\deg_* \boldsymbol{f} \le (2p' \ 2p'-1)^\top$, then for any integer $\ell > p'+1$, Lemma 4.4 yields that there holds $\check{\mathbf{Q}}^{2\ell}\boldsymbol{f} = 0$, since $\check{\mathbf{Q}}^{2\ell}$ is block-homogeneous of sufficient high degree. Therefore, the transverse degree is provided by the action of $\sum_{k\le 2(p'+1)} \varepsilon^k \check{\mathbf{Q}}^k$ on $\boldsymbol{f}[\varepsilon]$, whence (5.5). □

Remark 5.4.

(a) As all operators $\mathbf{V}^k$ and $\mathbf{Q}^k$ are differential in x_*, therefore *local* in x_*, the result of Theorem 5.3 can be localized in x_*: if for a subdomain $\omega' \subset \omega$, the series $\boldsymbol{f}[\varepsilon]\big|_{\omega'\times(-1,1)}$ depends polynomially on x_1, x_2, X_3 and satisfies (5.4) on $\omega' \times (-1,1)$, then (5.5) holds on $\omega' \times (-1,1)$.

(b) In particular, if $\boldsymbol{f}[\varepsilon]\big|_{\omega'\times(-1,1)} \equiv 0$, then $\deg_3 \boldsymbol{v}[\varepsilon] \le \big(3 2\big)^\top$ in $\omega' \times (-1,1)$, and for the special value $\nu = 0$ of the Poisson ratio, $\deg_3 \boldsymbol{v}[\varepsilon] \le \big(3 0\big)^\top$ in $\omega' \times (-1,1)$.

(c) If moreover, $\boldsymbol{f}[\varepsilon]$ represents a membrane volume force, then $\deg_3 \boldsymbol{v}[\varepsilon] \le \big(2 1\big)^\top$ in $\omega' \times (-1,1)$.

Remark 5.5.

(a) For a constant bending volume force $(0,0,1)$ there holds $\deg_3 \boldsymbol{v}[\varepsilon] \le \big(34\big)^\top$.
The same is valid in the case when a constant bending load is applied on the upper and lower faces, *cf.* the formulas in [9].

(b) For a constant membrane volume force $(a,b,0)$ there holds $\deg_3 \boldsymbol{v}[\varepsilon] \le \big(21\big)^\top$.
This is still valid in the case of a constant membrane load on the upper and lower faces.

(c) Localized versions of all the above statements hold too.

5.3. *p*-version approximation of the outer expansion

We now discuss the approximation of the *unscaled* outer expansion part $\widetilde{\boldsymbol{v}}[\varepsilon]$ in the framework of the p-version of Finite Elements. Since the series $\boldsymbol{f}[\varepsilon]$ starts with the degree $k=-1$, see (2.9), such is also the case for the series of generators $\boldsymbol{\zeta}[\varepsilon]$ solution of problem (5.2). Therefore the outer expansion $\boldsymbol{v}[\varepsilon]$ also starts with $k=-1$, and the unscaled outer expansion is defined as

$$\widetilde{\boldsymbol{v}}(\varepsilon) = \sum_{k\ge -2}^{\infty} \varepsilon^k \widetilde{\boldsymbol{v}}^k(X) \quad \text{with} \quad \begin{cases} \widetilde{\boldsymbol{v}}^{-2}(x) = \big(0,0,v_3^{-1}\big)\,(X) \\ \widetilde{\boldsymbol{v}}^{k}(x) = \big(v_1^k, v_2^k, v_3^{k+1}\big)\,(X), \quad k\ge -1. \end{cases} \tag{5.6}$$

By superposition, it suffices to investigate the approximation of the generic term $\widetilde{\boldsymbol{v}}^k$ from a suitable FE-space. This will rely on Theorem 5.2 above, which gives the structure of the $\boldsymbol{v}^k$.

Our approximation shall be based on an analytic regular partition τ_ω of ω, which is *fixed independently of* ε *and* k: the mid-surface ω is covered by a curvilinear partition τ_ω of triangular or quadrilateral elements κ, which are images of a reference element $\widehat{\kappa}$ under analytic element maps m_κ: $\widehat{\kappa} \to \kappa \in \tau_\omega$ which are diffeomorphisms. Two different reference elements may be used in the design of τ_ω: a triangular reference element $\widehat{\kappa}_{\mathrm{T}}$ and a square one $\widehat{\kappa}_{\mathrm{Q}}$.

The mesh τ_ω is assumed to be regular, *i.e.* the intersection of two elements κ, $\kappa' \in \tau_\omega$ is either empty, a vertex or an entire side and in the latter case, the common side γ has the same parametrization from both sides, *i.e.* for a common edge $\gamma = \overline{\kappa} \cap \overline{\kappa'}$ holds: $m_{\kappa'} \circ m_\kappa(\gamma) = \gamma$.

Proposition 5.6. *Let $\omega \subset \mathbb{R}^2$ be a bounded domain with analytic boundary curve $\partial\omega$. For any polynomial degree p, define the FE-space*

$$S^p(\omega,\tau_\omega) = \big\{ v \in C^0(\overline{\omega}) : \, v|_\kappa \circ m_\kappa \in Q_p(\widehat{\kappa}),\, \kappa \in \tau_\omega \big\} \tag{5.7}$$

where Q_p denotes the polynomials of total degree p if $\widehat{\kappa}$ is the triangle $\widehat{\kappa}_{\mathrm{T}}$ and of separate degree p if $\widehat{\kappa}$ is the square $\widehat{\kappa}_{\mathrm{Q}}$.

Then for any $p \ge 1$, there exists an interpolation operator $i_p : \mathcal{A}(\overline{\omega}) \to S^p(\omega,\tau_\omega)$, $v \mapsto i_p v$ such that if $v|_{\partial\omega} = 0$, also $i_p v|_{\partial\omega} = 0$, and satisfying the uniform estimates

$$\|v - i_p v\|_{L^\infty(\omega)} + \|\nabla(v - i_p v)\|_{L^\infty(\omega)} \le C\mathrm{e}^{-bp} \tag{5.8}$$

where b, $C > 0$ are independent of p, and b depends only on the domain of analyticity of v.

For a proof of this assertion, we refer to [16], for example.

We define next the FE space for the approximation of the $\widetilde{\boldsymbol{v}}^k$. To this end, denote by $\mathcal{T}_\varepsilon^0$ the three dimensional mesh family in Ω_ε which corresponds to τ_ω, *i.e.*

$$\mathcal{T}_\varepsilon^0 := \big\{ K = \kappa \times (-\varepsilon,\varepsilon) : \kappa \in \tau_\omega \big\}\,, \tag{5.9}$$

where $K = M_K(\widehat{K})$ with the reference element $\widehat{K} = \widehat{\kappa} \times (-1,1)$ and the element map

$$x = M_K(\widehat{x}_*, X_3) = (m_\kappa(\widehat{x}_*),\, \varepsilon X_3) : \, \widehat{K} \to K\,. \tag{5.10}$$

On $\mathcal{T}_\varepsilon^0$, we introduce the anisotropic tensor product FE-space

$$V^{p,\boldsymbol{q}}(\Omega^\varepsilon,\mathcal{T}_\varepsilon^0) = S^p(\omega,\tau_\omega)\otimes\Big(\mathbb{P}_{q_*}(-\varepsilon,\varepsilon)^2\times\mathbb{P}_{q_3}(-\varepsilon,\varepsilon)\Big) \tag{5.11}$$

of transverse degree $\boldsymbol{q} = \big(q_* q_3\big)^\top$. If $q_* = q_3 = q$, we write $V^{p,q}(\Omega^\varepsilon,\mathcal{T}_\varepsilon^0)$ instead.

Here follows the approximation result for the unscaled outer expansion:

Lemma 5.7. (i) *Under the general assumption made in Section 2.2 — the volume loading is the restriction of an analytic field $\overline{\boldsymbol{f}}$, the generic term $\widetilde{\boldsymbol{v}}^k$ in the unscaled outer expansion* (5.6) *can be approximated from $V^{p,\boldsymbol{q}}(\Omega^\varepsilon,\mathcal{T}_\varepsilon^0)$ at an exponential rate in energy norm:*

$$\exists\widetilde{\boldsymbol{v}}_N^k\in V^{p,\boldsymbol{q}}(\Omega^\varepsilon,\mathcal{T}_\varepsilon^0)\quad\textit{such that}\quad\Big\|\widetilde{\boldsymbol{v}}^k-\widetilde{\boldsymbol{v}}_N^k\Big\|_{E(\Omega^\varepsilon)}\le C\varepsilon^{-1/2}\mathrm{e}^{-b\min\{p,\boldsymbol{q}\}}, \tag{5.12}$$

where $C>0$ is independent of ε, p and $\boldsymbol{q}$, but depends on k and $\overline{\boldsymbol{f}}$.

If $\boldsymbol{v}^k$ depends only on x_*, the factor $\varepsilon^{-1/2}$ in (5.12) *can be replaced by $\varepsilon^{1/2}$.*

(ii) *If, moreover, the load $\overline{\boldsymbol{f}}$ satisfies, with even $p_{\boldsymbol{f}}\ge 0$ and with $q_{\boldsymbol{f}}\ge 1$*

$$\deg_*\boldsymbol{f}\le\big(p_{\boldsymbol{f}}\ p_{\boldsymbol{f}}-1\big)^\top\quad\textit{and}\quad\deg_3\boldsymbol{f}\le\big(q_{\boldsymbol{f}}\ q_{\boldsymbol{f}}-1\big)^\top \tag{5.13}$$

then, provided the transverse degree $\boldsymbol{q}$ satisfies $\boldsymbol{q}\ge\big(p_{\boldsymbol{f}}+q_{\boldsymbol{f}}+2\ p_{\boldsymbol{f}}+q_{\boldsymbol{f}}+1\big)^\top$, we have

$$\exists\widetilde{\boldsymbol{v}}_N^k\in V^{p,\boldsymbol{q}}(\Omega^\varepsilon,\mathcal{T}_\varepsilon^0)\quad\textit{such that}\quad\Big\|\widetilde{\boldsymbol{v}}^k-\widetilde{\boldsymbol{v}}_N^k\Big\|_{E(\Omega^\varepsilon)}\le C\varepsilon^{-1/2}\mathrm{e}^{-bp}, \tag{5.14}$$

with the same improvement as above if $\boldsymbol{v}^k$ depends only on x_*.

Proof. Let us fix $k\ge -2$ and a component $\widetilde{v}_i^k$ of $\widetilde{\boldsymbol{v}}^k$. Let us denote $q:=q_*$ if $i=1,2$ and $q:=q_3$ if $i=3$.

(i) From Theorem 5.2 and in particular representation (5.3), we have the existence of $k+3$ functions $\eta^\ell\in\mathcal{A}(\overline{\omega})$, $\ell=0,\dots,k+2$ such that

$$\widetilde{v}_i^k(x)=\sum_{\ell=0}^{k+2}\Big(\frac{x_3}{\varepsilon}\Big)^\ell\eta^\ell(x_*).$$

Let us denote by j_q an approximation of analytic functions on $[-1,1]$ by polynomials of degree $\le q$ at exponential rate. We denote by π^ℓ the monomial $X_3\mapsto X_3^\ell$ and we set

$$\big(\mathcal{I}_{p,q}\widetilde{v}_i^k\big)(x)=\sum_{\ell=0}^{k+2}\big(j_q\pi^\ell\big)\Big(\frac{x_3}{\varepsilon}\Big)\ \big(i_p\eta^\ell\big)(x_*),$$

with i_p the interpolation operator of Proposition 5.6. It is then obvious that $\big(\mathcal{I}_{p,q}\widetilde{v}_i^k\big)$ belongs to $S^p(\omega,\tau_\omega)\otimes\mathbb{P}_q(-\varepsilon,\varepsilon)$ and that there holds

$$\big\|v_i^k-\big(\mathcal{I}_{p,q}\widetilde{v}_i^k\big)(X)\big\|_{H^1(\Omega)}\le C\mathrm{e}^{-b\min\{p,q\}}.$$

Going back to Ω^ε, we find, with the same constant C

$$\big\|\widetilde{v}_i^k-\mathcal{I}_{p,q}\widetilde{v}_i^k\big\|_{L^2(\omega,H^1(-\varepsilon,\varepsilon))}+\varepsilon^{-1}\big\|\widetilde{v}_i^k-\mathcal{I}_{p,q}\widetilde{v}_i^k\big\|_{H^1(\omega,L^2(-\varepsilon,\varepsilon))}\le C\varepsilon^{-1/2}\mathrm{e}^{-b\min\{p,q\}}.$$

Estimate (5.12) is easily deduced (with the improvement if $\boldsymbol{v}^k$ depends only on x_*).

(ii) Under the assumption (5.13) on $\overline{\boldsymbol{f}}$, Theorem 5.3 yields that

$$\deg_3 \boldsymbol{v}[\varepsilon] \le \big(p_{\boldsymbol{f}} + q_{\boldsymbol{f}} + 2 \;\; p_{\boldsymbol{f}} + q_{\boldsymbol{f}} + 1\big)^\top .$$

The assumption over $\boldsymbol{q}$ gives that the transverse degree of $\widetilde{v}_i^k$ is less than q. Therefore, it suffices to set

$$\big(\mathcal{I}_p \widetilde{v}_i^k\big)(x) = \sum_{\ell=0}^{q} \Big(\frac{x_3}{\varepsilon}\Big)^\ell \, \big(i_p \eta^\ell\big)(x_*),$$

to obtain the interpolant satisfying estimate (5.14). □

For $K \ge 0$, let us denote by $\widetilde{\boldsymbol{v}}^{[K]}$ the truncated series of the outer expansion

$$\widetilde{\boldsymbol{v}}^{[K]} = \sum_{k=-2}^{K} \varepsilon^k \widetilde{\boldsymbol{v}}^k. \tag{5.15}$$

As a consequence of Lemma 5.7, and taking into account that $\widetilde{\boldsymbol{v}}^{-2} = (0, 0, \zeta_3^{-1})$ only depends on x_*, we obtain immediately the estimate for any $K \ge 0$:

$$\exists \widetilde{\boldsymbol{v}}_N^{[K]} \in V^{p,\boldsymbol{q}}(\Omega^\varepsilon, \mathcal{T}_\varepsilon^0) \quad \text{such that} \quad \Big\| \widetilde{\boldsymbol{v}}^{[K]} - \widetilde{\boldsymbol{v}}_N^{[K]} \Big\|_{E(\Omega^\varepsilon)} \le C \varepsilon^{-3/2} \mathrm{e}^{-b \min\{p,\boldsymbol{q}\}}. \tag{5.16}$$

Relying on Theorem 2.2, we can now deduce from (5.16) *relative error estimates*:

Theorem 5.8. *Let $K \ge 0$. Let the volume load $\overline{\boldsymbol{f}}$ be such that $\overline{\boldsymbol{f}}(x_*, 0) \not\equiv 0$.*
(A.I) *If $\overline{f}_3(x_*, 0) \not\equiv 0$, we have, with constants b, $C > 0$ independent of ε and $(p, \boldsymbol{q})$ but depending on K:*

$$\exists \widetilde{\boldsymbol{v}}_N^{[K]} \in V^{p,\boldsymbol{q}}(\Omega^\varepsilon, \mathcal{T}_\varepsilon^0) \quad \textit{such that} \quad \Big\| \widetilde{\boldsymbol{v}}^{[K]} - \widetilde{\boldsymbol{v}}_N^{[K]} \Big\|_{E(\Omega^\varepsilon)} \le C \varepsilon^{-1} \mathrm{e}^{-b \min\{p,\boldsymbol{q}\}} \, \|\widetilde{\boldsymbol{u}}\|_{E(\Omega^\varepsilon)} . \tag{5.17}$$

(A.II) *If, moreover, there exists a family of interpolation operators i'_p with values in the subspace of $S^p(\omega, \tau_\omega)$ of C^1 functions, and still satisfying exponential estimates* (5.8)*, then the approximation bound in* (5.17) *can be replaced with $C\mathrm{e}^{-b \min\{p,\boldsymbol{q}\}} \, \|\widetilde{\boldsymbol{u}}\|_{E(\Omega^\varepsilon)}$.*
(B) *If $\overline{f}_3(x_*, 0) \equiv 0$ and $\overline{\boldsymbol{f}}_*(x_*, 0) \not\equiv 0$, then, again, the approximation bound in* (5.17) *can be replaced with $C\mathrm{e}^{-b \min\{p,\boldsymbol{q}\}} \, \|\widetilde{\boldsymbol{u}}\|_{E(\Omega^\varepsilon)}$.*
(C) *If, moreover, the conditions of Lemma* 5.7(ii) *are satisfied, then $\mathrm{e}^{-b \min\{p,\boldsymbol{q}\}}$ can be replaced by e^{-bp} everywhere.*

Proof. (A) In that case, the energy norm $\|\widetilde{\boldsymbol{u}}\|_{E(\Omega^\varepsilon)}$ is equivalent to $\varepsilon^{-1/2}$, which, together with (5.16), gives (5.17). We note that this "low" energy is due to the structure of the first terms in the outer expansion series: Indeed

$$\varepsilon^{-2} \widetilde{\boldsymbol{v}}^{-2} + \varepsilon^{-1} \widetilde{\boldsymbol{v}}_*^{-1} = \varepsilon^{-2}(-x_3 \nabla_* \zeta, \; \zeta)$$

with $\zeta = \zeta_3^{-2}(x_*)$. For non-zero ζ, its energy is $\mathcal{O}(\varepsilon^{-1/2})$, whereas, in general the energy of its interpolate $\varepsilon^{-2}(-x_3 \, i_p(\nabla_* \zeta), \; i_p \zeta)$ is $\mathcal{O}(\varepsilon^{-3/2})$, because the interpolate is not a Kirchhoff-Love displacement.

If a C^1 interpolation operator i'_p does exist, then we may choose $\varepsilon^{-2}(-x_3 \, \nabla_*(i'_p \zeta), \; i'_p \zeta)$ as interpolate and, thus, recover robustness as $\varepsilon \to 0$.

(B) In the situation of dominating membrane load, the energy norm of $\widetilde{\boldsymbol{u}}$ on Ω^ε is equivalent to $\varepsilon^{1/2}$ and the outer expansion series starts with $\widetilde{\boldsymbol{v}}^0$ the energy of which is a $\mathcal{O}(\varepsilon^{1/2})$, and, by superposition we obtain from Lemma 5.7 the bound $C\varepsilon^{1/2} \mathrm{e}^{-b \min\{p,\boldsymbol{q}\}}$ for (5.16), whence the statement of Theorem 5.8 (B). □

6. Properties of the boundary layer profiles (inner expansion)

Now, we study the inner expansion part in (2.10), that is, the sum of the boundary layer terms $\varepsilon^k \boldsymbol{w}^k$. It is in fact easier to consider unscaled terms $\boldsymbol{\varphi}^k$ defined as:

$$\boldsymbol{\varphi}^k = (\varphi_1^k, \varphi_2^k, \varphi_3^k) := (w_1^k, w_2^k, w_3^{k+1}).$$

In a similar way as for the outer expansion, the terms $\boldsymbol{\varphi}^k$ are determined as coefficients of a formal series $\boldsymbol{\varphi}[\varepsilon]$ satisfying functional equations: We first reformulate results from [5, 9]. Then we will deduce from these results, analyticity properties for the profiles in weighted spaces. Finally, in Section 7, we construct the *hp*-approximation of the profiles.

6.1. Prerequisite

We introduce in a tubular neighborhood $\mathcal{U} \subset \omega$ of $\partial\omega$ the usual boundary fitted coordinates (s, r): if $x_*(s)$ denotes a parametric representation of $\partial\omega$, any $x_* \in \mathcal{U}$ can be written in a unique way as $x_* = x_*(s) - r\boldsymbol{n}(s)$ for some $0 \leq s < \text{length}(\partial\omega)$ and $0 \leq r \leq r_0$ with r_0 sufficiently small, if $\boldsymbol{n}(s)$ denotes the exterior unit normal vector to $\partial\omega$ at s. With r, we associate further the stretched variable $R = r/\varepsilon$. The terms of the inner expansion are profiles, *i.e.* $\boldsymbol{\varphi}^k = \boldsymbol{\varphi}^k(s, R, X_3)$. To the profiles $\boldsymbol{\varphi}^k$ we associate their (s, R, X_3) component functions $(\varphi_s^k, \varphi_R^k, \varphi_3^k)$.

The boundary condition in (5.2) on the two-dimensional generator series $\boldsymbol{\zeta}[\varepsilon]$ does not ensure that the outer expansion part $\boldsymbol{v}[\varepsilon]$ satisfies the lateral boundary conditions (2.5), but that the inner-outer expansion does. There exist operator series $\boldsymbol{\Phi}[\varepsilon]$ and $\boldsymbol{\Theta}[\varepsilon]$ such that the $\boldsymbol{\varphi}^k$ are the coefficients of the series $\boldsymbol{\varphi}[\varepsilon]$ given by

$$\boldsymbol{\varphi}[\varepsilon] = \boldsymbol{\Phi}[\varepsilon]\boldsymbol{\zeta}[\varepsilon] + \boldsymbol{\Theta}[\varepsilon]\boldsymbol{f}[\varepsilon]. \tag{6.1}$$

We first give the functional equations solved by (6.1). Next, we define functional spaces of exponentially decreasing functions at infinity on Σ^+.

6.1.1. *Expansion of operators in stretched tubular coordinates*

In tubular coordinates (s, r, x_3) associated with components (u_s, u_r, u_3), the interior operator B (2.2) is transformed into an operator $\mathcal{B}(s, r; \partial_s, \partial_r, \partial_{x_3})$ and the horizontal boundary operator G (2.4) into $\mathcal{G}(s, r; \partial_s, \partial_r, \partial_{x_3})$. In the stretched tubular coordinates (s, R, X_3), these operators become

$$\mathcal{B}(s, R\varepsilon\,; \partial_s, \varepsilon^{-1}\partial_R, \varepsilon^{-1}\partial_{X_3}), \quad \text{and} \quad \mathcal{G}(s, R\varepsilon\,; \partial_s, \varepsilon^{-1}\partial_R, \varepsilon^{-1}\partial_{X_3}).$$

The Taylor expansion at $R = 0$ of the coefficients of the above operators provides the operator valued formal series

$$\mathcal{B}[\varepsilon] = \sum_k \varepsilon^k \mathcal{B}^k \quad \text{and} \quad \mathcal{G}[\varepsilon] = \sum_k \varepsilon^k \mathcal{G}^k$$

where the $\mathcal{B}^k(s, R\,; \partial_s, \partial_R, \partial_3)$ are partial differential systems of order 2 in the stretched domain $\partial\omega \times \Sigma^+$ whereas the $\mathcal{G}^k(s, R\,; \partial_s, \partial_R, \partial_3)$ are partial differential systems of order 1 on its horizontal boundaries $\partial\omega \times \gamma_\pm$, where $\gamma_\pm = \mathbb{R}^+ \times \{X_3 = \pm 1\}$ denotes the horizontal boundaries of Σ^+; all operators depend polynomially on R.

Therefore, each coefficient of the matrices $\mathcal{B}^k$ and $\mathcal{G}^k$ is a finite sum of terms of the form $a(s)R^n \partial_s^i \partial_R^j \partial_3^l$, with $i + j + \ell$ less than 2 for $\mathcal{B}^k$ and less than 1 for $\mathcal{G}^k$. As a consequence of the analyticity of the boundary of ω, the coefficients $s \mapsto a(s)$ belong to $\mathcal{A}(\partial\omega)$.

The first terms $\mathcal{B}^0$ and $\mathcal{G}^0$ are explicitly given by:

$$\begin{aligned}
&(\mathcal{B}^0\boldsymbol{\varphi})_s = \mu\,\Delta_{R,3}\varphi_s, && (\mathcal{G}^0\boldsymbol{\varphi})_s = \mu\partial_3\varphi_s\,.\\
&(\mathcal{B}^0\boldsymbol{\varphi})_R = \mu\,\Delta_{R,3}\varphi_R + (\lambda+\mu)\,\partial_R(\partial_R\varphi_R+\partial_3\varphi_3), && (\mathcal{G}^0\boldsymbol{\varphi})_R = \mu(\partial_3\varphi_R+\partial_R\varphi_3),\\
&(\mathcal{B}^0\boldsymbol{\varphi})_3 = \mu\,\Delta_{R,3}\varphi_3 + (\lambda+\mu)\,\partial_3(\partial_R\varphi_R+\partial_3\varphi_3), && (\mathcal{G}^0\boldsymbol{\varphi})_3 = (\lambda+2\mu)\partial_3\varphi_3+\lambda\,\partial_R\varphi_R\,.
\end{aligned}$$

We note the splitting into 2D-Laplace and 2D-Lamé operators in variables (R, x_3) with Neumann boundary conditions.

The series $\boldsymbol{\varphi}[\varepsilon]$ is associated with *zero volume and surface loads*, which is written as:

$$\begin{cases} \mathcal{B}[\varepsilon]\boldsymbol{\varphi}[\varepsilon] = \mathbf{0} & \text{in} \quad \partial\omega\times\Sigma^+,\\ \mathcal{G}[\varepsilon]\boldsymbol{\varphi}[\varepsilon] = \mathbf{0} & \text{on} \quad \partial\omega\times\gamma_\pm. \end{cases} \tag{6.2}$$

6.1.2. *Spaces of exponentially decreasing functions*

The profiles $\boldsymbol{\varphi}^k(s, R, X_3)$ are exponentially decreasing as $R\to\infty$ and belong to a class of weighted spaces in Σ^+. These spaces depend on two real parameters $\delta > 0$ and $\beta\in(0,1)$. The parameter δ describes the exponential decay at infinity and β the regularity near the two corners $(0,\pm1)$ of Σ^+.

We denote by $\rho_\pm$ the distance to the corners $(0,\pm1)$ and set $\rho = \min\{1, \rho_+\rho_-\}$. Let first $\mathfrak{H}^\infty_{\beta,\delta}(\Sigma^+)$ be the space of $\mathcal{C}^\infty(\Sigma^+)$ functions φ, which are smooth up to any regular point of the boundary of Σ^+, are exponentially decreasing as $R\to\infty$ and satisfy the growth estimates near $(0,\pm1)$ in the following sense

$$\mathrm{e}^{\delta R}\,\varphi\in L^2(\Sigma^+) \qquad\text{and}\qquad \forall\boldsymbol{\ell}\in\mathbb{N}^2,\ |\boldsymbol{\ell}|>0,\quad \mathrm{e}^{\delta R}\,\rho^{|\boldsymbol{\ell}|-1-\beta}\,\partial^{|\boldsymbol{\ell}|}_{R,3}\varphi\in L^2(\Sigma^+).$$

Then we define the corresponding displacement space $\boldsymbol{\mathfrak{H}}^\infty_{\beta,\delta}(\Sigma^+) := \mathfrak{H}^\infty_{\beta,\delta}(\Sigma^+)^3$.

The space for the right hand sides is defined along similar lines. Let $\mathfrak{K}^\infty_{\beta,\delta}(\Sigma^+)$ be the space of triples $(\Psi,\psi^\pm)\in\mathcal{C}^\infty(\Sigma^+)\times\mathcal{C}^\infty(\gamma_\pm)$ which satisfy

$$\forall\boldsymbol{\ell}\in\mathbb{N}^2,\qquad \mathrm{e}^{\delta R}\,\rho^{|\boldsymbol{\ell}|+1-\beta}\,\partial^{|\boldsymbol{\ell}|}_{R,3}\Psi\in L^2(\Sigma^+)\quad\text{and}\quad \mathrm{e}^{\delta R}\,\rho^{|\boldsymbol{\ell}|+1/2-\beta}\,\partial^{|\boldsymbol{\ell}|}_{R,3}\psi^\pm\in L^2(\gamma_\pm).$$

Then we define the corresponding space for right hand sides:

$$\boldsymbol{\mathfrak{K}}^\infty_{\beta,\delta}(\Sigma^+) := \big\{(\boldsymbol{\Psi},\boldsymbol{\psi})\in\mathfrak{K}^\infty_{\beta,\delta}(\Sigma^+)^3\big\}\cdot$$

These spaces are convenient to solve problem (6.2) coupled with lateral boundary conditions because there hold the two following lemmas.

Lemma 6.1. *Let $\delta>0$ and $\beta\in(0,1)$ be fixed. For any k and any $\boldsymbol{\varphi}\in\mathcal{C}^\infty(\partial\omega,\boldsymbol{\mathfrak{H}}^\infty_{\beta,\delta}(\Sigma^+))$, there holds:*

$$(\mathcal{B}^k\boldsymbol{\varphi},\mathcal{G}^k\boldsymbol{\varphi})\in\mathcal{A}(\partial\omega,\boldsymbol{\mathfrak{K}}^\infty_{\beta,\delta'}(\Sigma^+))\quad\textit{for any }\delta'<\delta.$$

This is a straightforward consequence of the structure of the coefficients of the operators $\mathcal{B}^k$ and $\mathcal{G}^k$ (analytic in s and polynomial in R).

Lemma 6.2. *Let $\delta_0>0$ be the smallest exponent arising from the Papkovich-Fadle eigenfunctions, see* [12]*. Let $\beta_0\in(0,1)$ be the smallest real part of the corner singularity exponents associateed with the corners $(0,\pm1)$ of Σ^+ for the operator $(\mathcal{B}^0,\mathcal{G}^0)$ with Dirichlet boundary conditions on $R=0$, see* [8,15]*. For any $0<\beta<\beta_0$ and $0<\delta<\delta_0$, for any $(\boldsymbol{\Psi},\boldsymbol{\psi})\in\boldsymbol{\mathfrak{K}}^\infty_{\beta,\delta}(\Sigma^+)$, and any $\mathbf{P}\in\mathcal{C}^\infty([-1,1])^3$ there exist a unique $\boldsymbol{\varphi}\in\boldsymbol{\mathfrak{H}}^\infty_{\beta,\delta}(\Sigma^+)$ and* *a unique rigid displacement $\boldsymbol{Z}$ such that*

$$\begin{cases} \mathcal{B}^0\boldsymbol{\varphi} + \boldsymbol{\Psi} = 0 & \text{in} \quad \Sigma^+ \\ \mathcal{G}^0\boldsymbol{\varphi} + \boldsymbol{\psi} = 0 & \text{in} \quad \gamma_+ \cup \gamma_- \\ (\boldsymbol{\varphi} - \boldsymbol{Z})\big|_{R=0} + \mathbf{P} = 0. \end{cases} \tag{6.3}$$

This result is proved in [5, 7, 9]. Let us denote by

$$\mathcal{R}^0(\boldsymbol{\Psi}, \boldsymbol{\psi}, \mathbf{P}) \quad \text{the solution } \boldsymbol{\varphi} \text{ of problem (6.3).}$$

If $(\boldsymbol{\Psi}, \boldsymbol{\psi})$ belongs to $\mathcal{A}(\partial\omega, \mathfrak{K}^\infty_{\beta,\delta}(\Sigma^+))$ and $\mathbf{P}$ belongs to $\mathcal{A}(\partial\omega, \mathcal{C}^\infty([-1,1])^3)$, then $s \mapsto \mathcal{R}^0(\boldsymbol{\Psi}(s), \boldsymbol{\psi}(s), \mathbf{P}(s))$ defines an element $\boldsymbol{\varphi} \in \mathcal{A}(\partial\omega, \mathfrak{H}^\infty_{\beta,\delta}(\Sigma^+))$, which is still denoted by $\mathcal{R}^0(\boldsymbol{\Psi}, \boldsymbol{\psi}, \mathbf{P})$. In particular, if the right hand side $(\boldsymbol{\Psi}, \boldsymbol{\psi}, \mathbf{P})$ has a tensor product form

$$a(s)\big(\boldsymbol{\Psi}'(R, X_3), \boldsymbol{\psi}'(R), \mathbf{P}'(X_3)\big)$$

then $\mathcal{R}^0(\boldsymbol{\Psi}, \boldsymbol{\psi}, \mathbf{P}) = a(s)\boldsymbol{\varphi}'(R, X_3)$ with $\boldsymbol{\varphi}' = \mathcal{R}^0(\boldsymbol{\Psi}', \boldsymbol{\psi}', \mathbf{P}')$ since $(\mathcal{B}^0, \mathcal{G}^0)$ does not depend on s.

6.2. Series $\boldsymbol{\Phi}[\varepsilon]$ and $\boldsymbol{\Theta}[\varepsilon]$

As the boundary layer profiles are expressed in *unscaled* components, we have to define the unscaled version of operators $\mathbf{V}[\varepsilon]$ and $\mathbf{Q}[\varepsilon]$. This only consists in dividing the transverse component by ε. This amounts to define

$$\widetilde{\mathbf{V}}[\varepsilon] = D[\varepsilon]\mathbf{V}[\varepsilon] \quad \text{and} \quad \widetilde{\mathbf{Q}}[\varepsilon] = D[\varepsilon]\mathbf{Q}[\varepsilon]$$

where we have set $D[\varepsilon] = \varepsilon^{-1}D_{-1} + D_0$, with

$$D_{-1}(\boldsymbol{u}_*, u_3) = (0, u_3) \quad \text{and} \quad D_0(\boldsymbol{u}_*, u_3) = (\boldsymbol{u}_*, 0).$$

Note that for any $k \geq 1$, $\widetilde{\mathbf{V}}^k$ and $\widetilde{\mathbf{Q}}^k$ are nonzero operators.

We have now all material for the definition of the formal operator series $\boldsymbol{\Phi}[\varepsilon]$ and $\boldsymbol{\Theta}[\varepsilon]$ present in (6.1). Beyond the equations $\mathcal{B}[\varepsilon]\boldsymbol{\Phi}[\varepsilon] = 0$, $\mathcal{G}[\varepsilon]\boldsymbol{\Phi}[\varepsilon] = 0$, $\mathcal{B}[\varepsilon]\boldsymbol{\Theta}[\varepsilon] = 0$, $\mathcal{G}[\varepsilon]\boldsymbol{\Theta}[\varepsilon] = 0$ corresponding to system (6.2), they satisfy that $\boldsymbol{\Phi}[\varepsilon] + \widetilde{\mathbf{V}}[\varepsilon]$ and $\boldsymbol{\Theta}[\varepsilon] + \widetilde{\mathbf{Q}}[\varepsilon]$ takes their values in a rigid displacement series.

The zero-order operators $\boldsymbol{\Phi}^0$ and $\boldsymbol{\Theta}^0$ vanish. For any $k \geq 1$, there holds

$$\forall \boldsymbol{\zeta} \in \mathcal{C}^\infty(\overline{\omega})^3, \quad \boldsymbol{\Phi}^k\boldsymbol{\zeta} = \mathcal{R}^0\Big(\sum_{\ell=1}^k \mathcal{B}^\ell\boldsymbol{\Phi}^{k-\ell}\boldsymbol{\zeta}\ ,\ \sum_{\ell=1}^k \mathcal{G}^\ell\boldsymbol{\Phi}^{k-\ell}\boldsymbol{\zeta}\ ,\ \widetilde{\mathbf{V}}^k\boldsymbol{\zeta}\big|_{\Gamma_0}\Big) \tag{6.4}$$

and

$$\forall \boldsymbol{f} \in \mathcal{C}^\infty(\overline{\Omega})^3, \quad \boldsymbol{\Theta}^k\boldsymbol{f} = \mathcal{R}^0\Big(\sum_{\ell=1}^k \mathcal{B}^\ell\boldsymbol{\Theta}^{k-\ell}\boldsymbol{f}\ ,\ \sum_{\ell=1}^k \mathcal{G}^\ell\boldsymbol{\Theta}^{k-\ell}\boldsymbol{f}\ ,\ \widetilde{\mathbf{Q}}^k\boldsymbol{f}\big|_{\Gamma_0}\Big). \tag{6.5}$$

Gathering all information about the structure of series $\mathbf{V}[\varepsilon]$ and $\mathbf{Q}[\varepsilon]$ the decomposition of operators $\mathcal{B}^k$ and $\mathcal{G}^k$ in tensor product terms, and of solutions of problems (6.3) we obtain:

Lemma 6.3. *Let β and δ be as in Lemma 6.2.*
(i) For any integer $k \geq 1$, there exists an integer $L = L(k)$ and for any $\ell = 1, \ldots, L$ exponentially decreasing fields $\boldsymbol{\varphi}^{k,\ell} \in \boldsymbol{\mathfrak{H}}^{\infty}_{\beta,\delta}(\Sigma^+)$ and partial differential operators $\delta^{k,\ell}(s; \partial_s, \partial_r)$ on $\partial\omega$ with analytic coefficients on $\partial\omega$ such that

$$\boldsymbol{\Phi}^k \boldsymbol{\zeta} = \sum_{\ell=1}^{L} \boldsymbol{\varphi}^{k,\ell}(R, X_3)\, \delta^{k,\ell}(s; \partial_s, \partial_r) \boldsymbol{\zeta}\,\big|_{\partial\omega}.$$

Each $\boldsymbol{\varphi}^{k,\ell}$ is the solution $\mathcal{R}^0(\boldsymbol{\Psi}^{k,\ell}, \boldsymbol{\psi}^{k,\ell}, \mathbf{P}^{k,\ell})$ of problem (6.3) where the $\mathbf{P}^{k,\ell}(X_3)$ are triples of polynomials, and $(\boldsymbol{\Psi}^{k,\ell}, \boldsymbol{\psi}^{k,\ell}) = (\mathbf{b}\,\boldsymbol{\varphi}^{k',\ell'}, \mathbf{g}\,\boldsymbol{\varphi}^{k',\ell'})$ with $k' < k$, $\ell' \leq L(k')$, and $\mathbf{b}$, $\mathbf{g}$ matrix operators with coefficients of the form $R^n \partial_{R,3}^{|\boldsymbol{m}|}$ (with $|\boldsymbol{m}| \leq 2$ for $\mathbf{b}$ and ≤ 1 for $\mathbf{g}$).

(ii) If $\boldsymbol{f}$ depends polynomially on X_3 (with degree $\deg_3 \boldsymbol{f}$), for any integer $k \geq 1$, there exists an integer $J = J(k, \deg_3 \boldsymbol{f})$ and for any $j = 1, \ldots, J$ exponentially decreasing fields $\boldsymbol{\theta}^{k,j} \in \boldsymbol{\mathfrak{H}}^{\infty}_{\beta,\delta}(\Sigma^+)$ and partial differential operators $\gamma^{k,j}(s; \partial_s, \partial_r, \partial_3)$ on $\partial\omega \times (-1, 1)$ with analytic coefficients on $\partial\omega$ such that

$$\boldsymbol{\Theta}^k \boldsymbol{f} = \sum_{j=1}^{J} \boldsymbol{\theta}^{k,j}(R, X_3)\, \gamma^{k,j}(s; \partial_s, \partial_r, \partial_3) \boldsymbol{f}\,\big|_{\partial\omega \times (-1,1)}.$$

Each $\boldsymbol{\theta}^{k,j}$ is the solution $\mathcal{R}^0(\boldsymbol{\Psi}^{k,j}, \boldsymbol{\psi}^{k,j}, \mathbf{P}^{k,j})$ of a problem (6.3) where the $\mathbf{P}^{k,j}(X_3)$ are triples of polynomials, and $(\boldsymbol{\Psi}^{k,j}, \boldsymbol{\psi}^{k,j}) = (\mathbf{b}\,\boldsymbol{\theta}^{k',j'}, \mathbf{g}\,\boldsymbol{\theta}^{k',j'})$ with $k' < k$, $j' \leq J(k', \deg_3 \boldsymbol{f})$, and $\mathbf{b}$, $\mathbf{g}$ matrix operators as above.

6.3. Analytic regularity of the boundary layer profiles

Lemma 6.3 states that the generating layer profiles $\boldsymbol{\varphi}^{k,\ell}(R, X_3)$ and $\boldsymbol{\theta}^{k,j}(R, X_3)$ are solutions of problem (6.3) with sets of data coming from generating terms of lower degree. Therefore, we obtain by recursion that they are analytic in the interior of Σ^+. To estimate the rate of convergence of *hp*-FE approximations of the boundary layer profiles, however, we quantify the analytic regularity of $\boldsymbol{\varphi}^{k,\ell}$ and $\boldsymbol{\theta}^{k,j}$ in the interior of Σ^+. We need for this an analytic version of the spaces $\boldsymbol{\mathfrak{H}}^{\infty}_{\beta,\delta}$ and $\boldsymbol{\mathfrak{K}}^{\infty}_{\beta,\delta}$.

Definition 6.4. For real parameters $0 \leq \beta \leq 1$, $\delta > 0$, define the space $\mathfrak{H}^{\mathcal{A}}_{\beta,\delta}(\Sigma^+)$ as the set of all $\varphi \in \mathfrak{H}^{\infty}_{\beta,\delta}(\Sigma^+)$ for which there exists a constant $C > 0$ such that

$$\forall \boldsymbol{\ell} \in \mathbb{N}^2,\ |\boldsymbol{\ell}| > 0 \quad \left\| \mathrm{e}^{\delta R} \rho^{|\boldsymbol{\ell}|-1-\beta} \partial^{\boldsymbol{\ell}}_{R,3} \varphi \right\|_{L^2(\Sigma^+)} \leq C^{|\boldsymbol{\ell}|+1} \boldsymbol{\ell}! \tag{6.6}$$

Analogously, we denote by $\mathfrak{K}^{\mathcal{A}}_{\beta,\delta}(\Sigma^+)$ the space of triples $(\Psi, \psi^{\pm})$ for which there exist $C > 0$ such that

$$\forall \boldsymbol{\ell} \in \mathbb{N}^2 \quad \left\| \mathrm{e}^{\delta R} \rho^{|\boldsymbol{\ell}|+1-\beta} \partial^{\boldsymbol{\ell}}_{R,3} \Psi \right\|_{L^2(\Sigma^+)} \leq C^{|\boldsymbol{\ell}|+1} \boldsymbol{\ell}! \tag{6.7}$$

and

$$\forall \boldsymbol{\ell} \in \mathbb{N}^2 \quad \left\| \mathrm{e}^{\delta R} \rho^{|\boldsymbol{\ell}|+1/2-\beta} \partial^{\boldsymbol{\ell}}_{R,3} \psi^{\pm} \right\|_{L^2(\gamma^{\pm})} \leq C^{|\boldsymbol{\ell}|+1} \boldsymbol{\ell}! \tag{6.8}$$

As before, we denote by $\boldsymbol{\mathfrak{H}}^{\mathcal{A}}_{\beta,\delta}(\Sigma^+) = \mathfrak{H}^{\mathcal{A}}_{\beta,\delta}(\Sigma^+)^3$ and likewise $\boldsymbol{\mathfrak{K}}^{\mathcal{A}}_{\beta,\delta}(\Sigma^+)$.

With these definitions we can now prove the two following lemmas, which are the analytic version of lemmas 6.1 and 6.2.

Lemma 6.5. *Let $\delta > 0$ and $\beta \in (0, 1)$ be fixed. For any k and any $\boldsymbol{\varphi} \in \mathcal{A}(\partial\omega, \boldsymbol{\mathfrak{H}}^{\mathcal{A}}_{\beta,\delta}(\Sigma^+))$, there holds: $(\mathcal{B}^k \boldsymbol{\varphi}, \mathcal{G}^k \boldsymbol{\varphi}) \in \mathcal{A}(\partial\omega, \boldsymbol{\mathfrak{K}}^{\mathcal{A}}_{\beta,\delta'}(\Sigma^+))$ for any $\delta' < \delta$.*

Lemma 6.6. *With $\delta > 0$ and $\beta \in (0,1)$ as in Lemma 6.2, for any $(\boldsymbol{\Psi}, \boldsymbol{\psi}) \in \mathfrak{K}^{\mathcal{A}}_{\beta,\delta}(\Sigma^+)$, and any $\mathbf{P} \in \mathcal{A}([-1,1])^3$ the solution $\boldsymbol{\varphi} = \mathcal{R}^0(\boldsymbol{\Psi}, \boldsymbol{\psi}, \mathbf{P})$ belongs to $\mathfrak{H}^{\mathcal{A}}_{\beta,\delta}(\Sigma^+)$.*

Proof. To this end, for $p, q \in \mathbb{N}$, we define $\Sigma_{p,q} = (p,q) \times (-1,1)$, $\Sigma_p = (p,\infty) \times (-1,1)$, $\gamma^{\pm}_{p,q} := (p,q) \times \{\pm 1\}$ and $\gamma^{\pm}_p := (p,\infty) \times \{\pm 1\}$. Then we may split for example Σ^+ in $\overline{\Sigma}_{0,2} \cup \overline{\Sigma}_2$. We establish the analytic regularity (6.6) in $\Sigma_{0,2}$ and in Σ_2 separately.

Step (i): Analytic estimates in the half-strip $\Sigma_2 = (2,\infty) \times (-1,1)$.

For any $\beta \in \mathbb{R}$ and $\mathcal{B}^0, \mathcal{G}^0$ as in (6.3), we have the equivalence

$$\begin{cases} \mathcal{B}^0 \boldsymbol{\varphi} + \boldsymbol{\Psi} = 0 & \text{in} \quad \Sigma^+ \\ \mathcal{G}^0 \boldsymbol{\varphi} + \boldsymbol{\psi} = 0 & \text{in} \quad \gamma_+ \cup \gamma_- \end{cases} \Longleftrightarrow \begin{cases} \mathcal{B}^0_\beta(\mathrm{e}^{\beta R}\boldsymbol{\varphi}) + \mathrm{e}^{\beta R}\boldsymbol{\Psi} = 0 & \text{in} \quad \Sigma^+ \\ \mathcal{G}^0_\beta(\mathrm{e}^{\beta R}\boldsymbol{\varphi}) + \mathrm{e}^{\beta R}\boldsymbol{\psi} = 0 & \text{in} \quad \gamma_+ \cup \gamma_- \end{cases}$$

where $(\mathcal{B}^0_\beta, \mathcal{G}^0_\beta)$ is an elliptic operator pencil depending on β with constant coefficients and principal part $(\mathcal{B}^0, \mathcal{G}^0)$.

By the ellipticity and analyticity of the data $\boldsymbol{\Psi}, \boldsymbol{\psi}$ in $\overline{\Sigma}_1$, we have for any $p \geq 3$ and every $\boldsymbol{\ell} = (\ell_R, \ell_3) \in \mathbb{N}^2$ for $\tilde{\boldsymbol{\varphi}} := \mathrm{e}^{\beta R}\boldsymbol{\varphi}$ the analytic regularity estimate, see [17]

$$\begin{aligned} \frac{1}{\boldsymbol{\ell}!} \left\| \partial^{\boldsymbol{\ell}}_{R,3} \tilde{\boldsymbol{\varphi}} \right\|_{L^2(\Sigma_{p-1,p+1})} \leq C^{|\boldsymbol{\ell}|+1} & \left(\sum_{|\boldsymbol{n}| \leq (|\boldsymbol{\ell}|-2)_+} \frac{1}{\boldsymbol{n}!} \left\| \partial^{\boldsymbol{n}}_{R,3} (\mathcal{B}^0_\beta \tilde{\boldsymbol{\varphi}}) \right\|_{L^2(\Sigma_{p-2,p+2})} + \|\tilde{\boldsymbol{\varphi}}\|_{L^2(\Sigma_{p-2,p+2})} \right) \\ & + C^{|\boldsymbol{\ell}|+1} \left(\sum_{|\boldsymbol{n}| \leq (|\boldsymbol{\ell}|-1)_+} \frac{1}{\boldsymbol{n}!} \left\| \partial^{n_R}_{R} (\mathcal{G}^0_\beta \tilde{\boldsymbol{\varphi}}) \right\|_{L^2(\gamma^{\pm}_{p-2,p+2})} + \|\tilde{\boldsymbol{\varphi}}\|_{L^2(\gamma^{\pm}_{p-2,p+2})} \right) \end{aligned}$$

where the constant C depends on β, but not on $p \geq 3$ or on $\boldsymbol{\ell}$. Summing up for $p \geq 3$, we get that

$$\begin{aligned} \frac{1}{\boldsymbol{\ell}!} \left\| \partial^{\boldsymbol{\ell}}_{R,3} \tilde{\boldsymbol{\varphi}} \right\|_{L^2(\Sigma_2)} \leq C^{|\boldsymbol{\ell}|+1} & \left(\sum_{|\boldsymbol{n}| \leq (|\boldsymbol{\ell}|-2)_+} \frac{1}{\boldsymbol{n}!} \left\| \partial^{\boldsymbol{n}}_{R,3} (\mathcal{B}^0_\beta \tilde{\boldsymbol{\varphi}}) \right\|_{L^2(\Sigma_1)} + \|\tilde{\boldsymbol{\varphi}}\|_{L^2(\Sigma_1)} \right) \\ & + C^{|\boldsymbol{\ell}|+1} \left(\sum_{|\boldsymbol{n}| \leq (|\boldsymbol{\ell}|-1)_+} \frac{1}{\boldsymbol{n}!} \left\| \partial^{n_R}_{R} (\mathcal{G}^0_\beta \tilde{\boldsymbol{\varphi}}) \right\|_{L^2(\gamma^{\pm}_1)} + \|\tilde{\boldsymbol{\varphi}}\|_{L^2(\gamma^{\pm}_1)} \right) \end{aligned}$$

which also reads

$$\begin{aligned} \frac{1}{\boldsymbol{\ell}!} \left\| \partial^{\boldsymbol{\ell}}_{R,3} (\mathrm{e}^{\beta R}\boldsymbol{\varphi}) \right\|_{L^2(\Sigma_2)} \leq C^{|\boldsymbol{\ell}|+1} & \left(\sum_{|\boldsymbol{n}| \leq (|\boldsymbol{\ell}|-2)_+} \frac{1}{\boldsymbol{n}!} \left\| \partial^{\boldsymbol{n}}_{R,3} (\mathrm{e}^{\beta R} \mathcal{B}^0 \boldsymbol{\varphi}) \right\|_{L^2(\Sigma_1)} + \left\| \mathrm{e}^{\beta R}\boldsymbol{\varphi} \right\|_{L^2(\Sigma_1)} \right) \\ & + C^{|\boldsymbol{\ell}|+1} \left(\sum_{|\boldsymbol{n}| \leq (|\boldsymbol{\ell}|-1)_+} \frac{1}{\boldsymbol{n}!} \left\| \partial^{n_R}_{R} (\mathrm{e}^{\beta R} \mathcal{G}^0 \boldsymbol{\varphi}) \right\|_{L^2(\gamma^{\pm}_1)} + \left\| \mathrm{e}^{\beta R}\boldsymbol{\varphi} \right\|_{L^2(\gamma^{\pm}_1)} \right). \end{aligned}$$

Noting that $\partial^{\boldsymbol{\ell}}_{R,3}(\mathrm{e}^{\beta R}\boldsymbol{\varphi}) = \mathrm{e}^{\beta R}(\partial_R+\beta)^{n_R}\partial_3^{n_3}\boldsymbol{\varphi}$, we can deduce from the last estimate that

$$\frac{1}{\boldsymbol{\ell}!}\left\|\mathrm{e}^{\beta R}\partial^{\boldsymbol{\ell}}_{R,3}\boldsymbol{\varphi}\right\|_{L^2(\Sigma_2)} \leq C^{|\boldsymbol{\ell}|+1}\left(\sum_{|\boldsymbol{n}|\leq(|\boldsymbol{\ell}|-2)_+}\frac{1}{\boldsymbol{n}!}\left\|\mathrm{e}^{\beta R}\partial^{\boldsymbol{n}}_{R,3}(\mathcal{B}^0\boldsymbol{\varphi})\right\|_{L^2(\Sigma_1)} + \left\|\mathrm{e}^{\beta R}\boldsymbol{\varphi}\right\|_{L^2(\Sigma_1)}\right)$$
$$+C^{|\boldsymbol{\ell}|+1}\left(\sum_{|\boldsymbol{n}|\leq(|\boldsymbol{\ell}|-1)_+}\frac{1}{\boldsymbol{n}!}\left\|\mathrm{e}^{\beta R}\partial^{n_R}_{R}(\mathcal{G}^0\boldsymbol{\varphi})\right\|_{L^2(\gamma_1^\pm)} + \left\|\mathrm{e}^{\beta R}\boldsymbol{\varphi}\right\|_{L^2(\gamma_1^\pm)}\right).$$

Whence $\boldsymbol{\varphi}\in\mathfrak{H}^{\mathcal{A}}_{\beta,\delta}(\Sigma_2)$ if $(\mathcal{B}^0\boldsymbol{\varphi},\mathcal{G}^0\boldsymbol{\varphi})$ belongs to $\mathfrak{K}^{\mathcal{A}}_{\beta,\delta}(\Sigma_1)$ and if $\mathrm{e}^{\beta R}\boldsymbol{\varphi}$ is in $L^2(\Sigma_1)$.

Step (ii): Analytic estimates in $\Sigma_{0,2}$.

Since the differential operators $\mathcal{B}^0,\mathcal{G}^0$ in (6.3) have constant coefficients and are in divergence form, and since $\mathbf{P}$ is analytic on $R=0$, $|X_3|\leq 1$, the regularity theory of Babuška and Guo [13,14] (see also Th. IV.1 in [3]) implies that $\boldsymbol{\varphi}\in\mathfrak{H}^{\mathcal{A}}_{\beta,\delta}(\Sigma_{0,2})$. □

Then, combining Lemma 6.5 and Lemma 6.6 we prove by induction on $k\geq 1$:

Lemma 6.7. *Notations are as in Lemma* 6.3 *and* β *and* δ *as in Lemma* 6.2. *Then for any integer* $k\geq 1$ *all the boundary layer profiles* $\boldsymbol{\varphi}^{k,\ell}$ *and* $\boldsymbol{\theta}^{k,j}$ *belong to* $\mathfrak{H}^{\mathcal{A}}_{\beta,\delta}(\Sigma^+)$ *and estimates* (6.6) *hold with a constant* $C>0$ *depending on* k.

7. *hp*-Approximation of the boundary layer profiles

Unscaling expansion (2.10) we obtain that the terms $\widetilde{\boldsymbol{u}}^k$ (2.11) of the expansion of $\widetilde{\boldsymbol{u}}$ are the sum of the outer terms $\widetilde{\boldsymbol{v}}^k$ and of the inner terms $\chi\boldsymbol{\varphi}^k$. In this section, we investigate the approximation of the inner expansion terms $\chi\boldsymbol{\varphi}^k$ by mapped piecewise polynomials.

7.1. *hp*-Approximation of layer profiles on the half-strip Σ^+

For the approximation of the profiles $\boldsymbol{\varphi}^k$, we subdivide Σ^+ into three regions

$$\Sigma^+_{\mathrm{I}} := (0,2)\times(-1,1),\quad \Sigma^+_{\mathrm{II}} := (2,\widehat{R})\times(-1,1),\quad \Sigma^+_{\mathrm{III}} := (\widehat{R},\infty)\times(-1,1) \tag{7.1}$$

where $\widehat{R}\geq 3$ is an integer at our disposal which will be selected below. In each subregion Σ^+_ν, $\nu\in\{\mathrm{I},\mathrm{II},\mathrm{III}\}$, we introduce a FE-mesh $\mathcal{M}_\nu$ as follows.

In Σ^+_{I}, we need a parameter n which is an integer ≥ 1: $\mathcal{M}^n_{\mathrm{I}}$ consists of axiparallel quadrilaterals with hanging nodes which are geometrically refined toward the "corners" of Σ^+ with n layers and a grading ratio $\sigma\in(0,1)$, *cf.* (7.19) below (see Chap. 4 in [19], for more details on geometric meshes with hanging nodes).

In Σ^+_{II}, we define

$$\mathcal{M}_{\mathrm{II}} := \left\{(i,i+1)\times(-1,1) : i=2,\ldots,\widehat{R}-1\right\} \tag{7.2}$$

and finally, $\mathcal{M}_{\mathrm{III}} = \{(\widehat{R},\infty)\times(-1,1)\}$. The mesh $\mathcal{M}^n$ in Σ^+ is the union of the meshes in the subregions:

$$\mathcal{M}^n = \mathcal{M}^n_{\mathrm{I}}\cup\mathcal{M}_{\mathrm{II}}\cup\mathcal{M}_{\mathrm{III}}\,.$$

We next define the *hp*-FE space in Σ^+ which we will use to approximate the profiles. Let p be an integer ≥ 1. We denote by $\mathcal{Q}_p$ the usual spaces of polynomials of degree p in each variable and we define

$$S^p(\Sigma^+,\mathcal{M}^n) := \left\{\varphi\in H^1(\Sigma^+) : \varphi|_K\in\mathcal{Q}_p(K)\ \ \forall K\in\mathcal{M}^n, \varphi(R,\cdot)=0 \text{ for } R>\widehat{R}\right\}\cdot \tag{7.3}$$

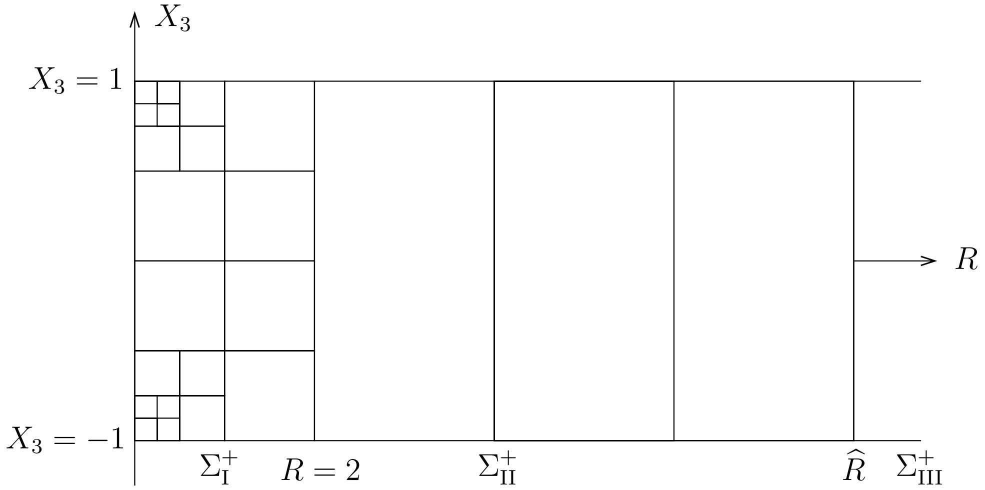

FIGURE 7.1. The regions Σ^+_ν, $\nu \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$, of Σ^+ and the meshes $\mathcal{M}^n_{\mathrm{I}}$ with $n = 3$, $\sigma = 0.5$, $\mathcal{M}_{\mathrm{II}}$ and $\mathcal{M}_{\mathrm{III}}$.

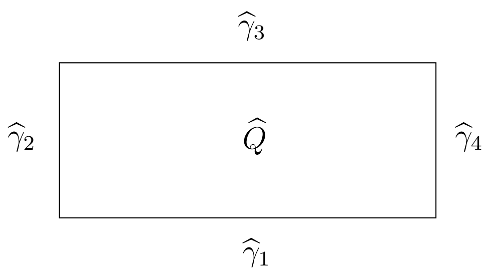

FIGURE 7.2. $\widehat{Q}$ and notation.

The next theorem addresses the approximation of the boundary layer profile space $\mathfrak{H}^{\mathcal{A}}_{\beta,\delta}(\Sigma^+)$ from $S^p(\Sigma^+, \mathcal{M}^n)$ and is the main result of this subsection.

Theorem 7.1. *Let $\varphi \in \mathfrak{H}^{\mathcal{A}}_{\beta,\delta}(\Sigma^+)$ for $\beta \in (0,1)$ and $\delta > 0$, be a boundary layer profile. Then there exist $C > 0$ and $b > 0$ such that, for any $p \geq 1$*

$$\exists \boldsymbol{\varphi}_p \in S^p(\Sigma^+, \mathcal{M}^p)^3 \quad \textit{such that} \quad \big\| \boldsymbol{\varphi} - \boldsymbol{\varphi}_p \big\|_{H^1(\Sigma^+)} \leq C\big(\mathrm{e}^{-bp} + \mathrm{e}^{-\delta \widehat{R}}\big). \tag{7.4}$$

Remark. Note that the number n of layers in the geometric mesh $\mathcal{M}^n_{\mathrm{I}}$ is taken equal to p.

Proof. To prove Theorem 7.1 we construct $\boldsymbol{\varphi}_p$ separately in each subdomain Σ^+_ν, for ν in $\{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$. The following Lemma of approximation on the model square $\widehat{Q} = (-1,1) \times (-1,1)$ will be used throughout.

(i) ESTIMATES IN THE MODEL SQUARE

Lemma 7.2. *Notation as in Figure 7.2. Let $\pi^0_p : L^2(-1,1) \to \mathcal{Q}_p(-1,1)$ denote the L^2-projection and define $\pi^1_p u$ for $u \in H^1(-1,1)$ by*

$$(\pi^1_p u)(x) := u(-1) + \int_{-1}^{x} (\pi^0_{p-1} u')(\xi)\, \mathrm{d}\xi\,,$$

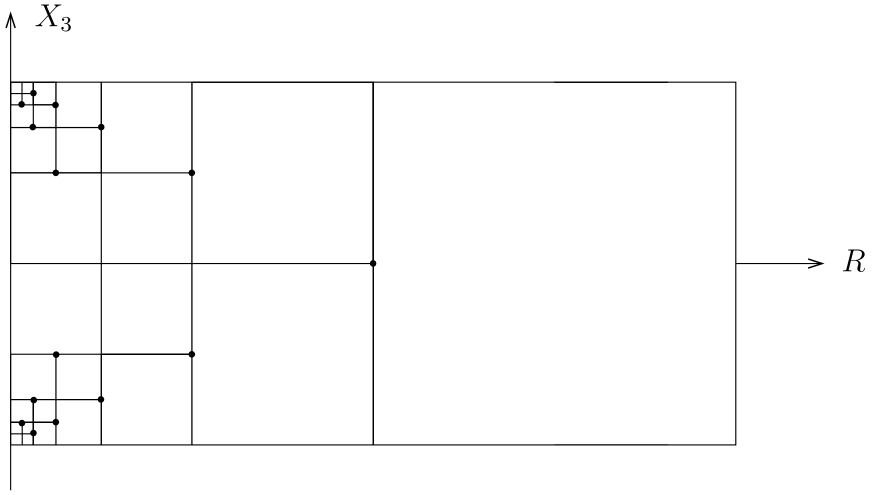

FIGURE 7.3. Geometric mesh $\mathcal{M}_{\mathrm{I}}^{n}$ in Σ_{I}^{+} with hanging nodes.

and denote by $\widehat{\Pi}_p := \pi_p^1\,\pi_p^2$ *the tensor product interpolant on* $\widehat{Q}$ *(here* π_p^2 *is the analogue of* π_p^1 *in the vertical direction). Then, for any* $u \in H^{1+k}(\widehat{Q})$, $k>0$, *holds*

$$\widehat{\Pi}_p u = u \quad \textit{at the vertices of } \widehat{Q}\,, \tag{7.5}$$

$$\widehat{\Pi}_p u\big|_{\widehat{\gamma}_i} = \pi_p^1(u|_{\widehat{\gamma}_i}),\ \ i=1,3 \quad \textit{and} \quad \widehat{\Pi}_p u\big|_{\widehat{\gamma}_i} = \pi_p^2(u|_{\widehat{\gamma}_i}),\ \ i=2,4 \tag{7.6}$$

and, for any $p \geq 1$ *and* $0 \leq s \leq \min(p,k)$ *the estimates*

$$\left\|\nabla(u-\widehat{\Pi}_p u)\right\|_{L^2(\widehat{Q})}^2 \leq C\,\Phi(p,s)\,\left\|D^{s+1}u\right\|_{L^2(\widehat{Q})}^2\,, \tag{7.7}$$

$$\left\|u-\widehat{\Pi}_p u\right\|_{L^2(\widehat{Q})}^2 \leq C\,\frac{\Phi(p,s)}{p(p+1)}\,\left\|D^{s+1}u\right\|_{L^2(\widehat{Q})}^2\,. \tag{7.8}$$

Here $\left\|D^k u\right\|_{L^2(\widehat{Q})}^2 = \sum_{|\alpha|=k}\|D^\alpha u\|_{L^2(\widehat{Q})}^2$. *The constant* $C>0$ *is independent of* s *and* p, *and* Φ *is given by*

$$\Phi(p,s) := \frac{(p-s)!}{(p+s)!} + \frac{1}{p(p+1)}\,\frac{(p-s+1)!}{(p+s-1)!}\,, \qquad 0 \leq s \leq p\,.$$

(ii) ESTIMATES IN Σ_{I}^{+}

Next, we address the interpolation on geometric meshes.

Lemma 7.3. *In* $\Sigma_{\mathrm{I}}^{+} = (0,2)\times(-1,1)$, *consider the geometric mesh* $\mathcal{M}_{\mathrm{I}}^{n}$ *shown in Figure* 7.3. *Then, for* $u \in H^{k+1}(\Sigma_{\mathrm{I}}^{+})$ *and* $p \geq 1$, *exists* $\widetilde{\Pi} u \in S^p(\Sigma_{\mathrm{I}}^{+},\mathcal{M}_{\mathrm{I}}^{n})$ *such that* $\widetilde{\Pi} u$ *is continuous in* Σ_{I}^{+} *and that there hold the*

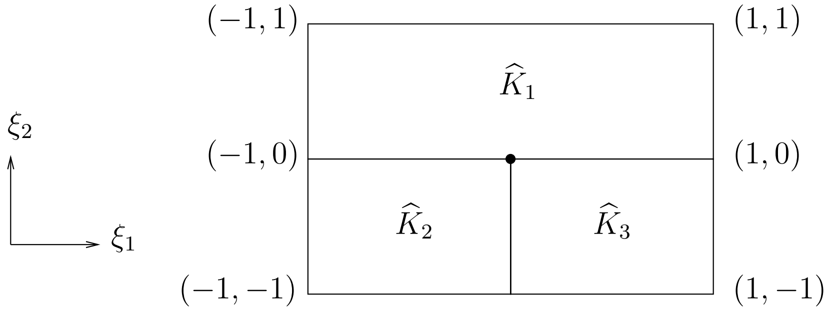

FIGURE 7.4. Mesh patch with hanging node.

following estimates for any $0 \le s \le \min(p,k)$

$$\left\|u - \widetilde{\Pi}u\right\|^2_{L^2(\Sigma_{\mathrm{I}}^+)} \le C \sum_{K \in \mathcal{M}_{\mathrm{I}}^n} \left(\frac{h_K}{2}\right)^{2s+2} \frac{\Phi(p,s)}{p+1} \left\|D^{s+1}u\right\|^2_{L^2(K)} , \tag{7.9}$$

$$\left\|\nabla(u - \widetilde{\Pi}u)\right\|^2_{L^2(\Sigma_{\mathrm{I}}^+)} \le C \sum_{K \in \mathcal{M}_{\mathrm{I}}^n} \left(\frac{h_K}{2}\right)^{2s} \Phi(p,s) \left\|D^{s+1}u\right\|^2_{L^2(K)} . \tag{7.10}$$

Proof. For $K \in \mathcal{M}_{\mathrm{I}}^n$, let $F_K : \widehat{Q} \to K$ be the affine element map. Define

$$(\Pi u)|_K := \left(\widehat{\Pi}_p\,(u \circ F_K)\right) \circ F_K^{-1} .$$

Then applying Lemma 7.2 elementwise and a scaling argument imply (7.9, 7.10). By (7.5, 7.6), Πu is continuous across edges which do not contain hanging nodes. It remains therefore to remove jumps of the interpolant on edges with hanging nodes.

Assume now that we are on an edge with hanging node as shown in Figure 7.4.
Denote $\gamma_{ij} := \overline{\widehat{K}}_i \cap \overline{\widehat{K}}_j$ and by $[u - \Pi u]_{ij} = -[\pi u]_{ij}$ the jump of $u - \Pi u$ across γ_{ij}.
By (7.6), $[u - \Pi u]_{23} \equiv 0$ and $[\Pi u]_{ij} \in \mathcal{Q}_p(\gamma_{ij})$. We remove the discontinuity by lifting $[\Pi u]$. Put

$$V(\xi) := -(\xi_2 + 1) \begin{cases} [\Pi u]_{12}(\xi_1) \text{ on } \widehat{K}_2 , \\ [\Pi u]_{13}(\xi_1) \text{ on } \widehat{K}_3 . \end{cases}$$

Now $[\Pi u]_{23} = 0$ implies that $V \in C^0(\overline{\widehat{K}}_2 \cup \overline{\widehat{K}}_3)$ and

$$\|\nabla V\|_{L^2(\widehat{K}_2 \cup \widehat{K}_3)} \le C \left\| [\pi u] \right\|_{H^{1/2}(\gamma_{12} \cup \gamma_{13})} , \tag{7.11}$$

with C independent of p. The Trace Theorem in $\widehat{K}$ implies

$$\begin{aligned} \|[\pi u]\|_{H^{1/2}(\gamma_{12}\cup\gamma_{13})} &= \|[u - \pi u]\|_{H^{1/2}(\gamma_{12}\cup\gamma_{13})} \\ &\le \|(u - \Pi u)^+\|_{H^{1/2}(\gamma_{12}\cup\gamma_{13})} + \|(u - \Pi u)^-\|_{H^{1/2}(\gamma_{12}\cup\gamma_{13})} \\ &\le C \textstyle\sum_{i=1}^3 \|u - \Pi u\|_{H^1(\widehat{K}_i)} . \end{aligned} \tag{7.12}$$

Put

$$\widetilde{\Pi}u := \begin{cases} \Pi u & \text{in } \widehat{K}_1 \\ V + \Pi u & \text{in } \widehat{K}_2 \cup \widehat{K}_3 \,. \end{cases} \tag{7.13}$$

Then $\widetilde{\Pi}u \in C^0(\widehat{K})$ by construction and from (7.11)–(7.12) we obtain

$$\left\| \nabla(u - \widetilde{\Pi}u) \right\|_{L^2(\widehat{K})} \leq C \sum_{i=1}^{3} \|\nabla(u - \Pi u)\|_{L^2(\widehat{K}_i)} \,.$$

Concerning the L^2 estimate we have

$$\begin{aligned} \|V\|_{L^2(\widehat{K}_2 \cup \widehat{K}_3)} &\leq C \, \|[u - \Pi u]\|_{L^2(\gamma_{12} \cup \gamma_{13})} \\ &\leq C \left\{ \|(u - \Pi u)^+\|_{L^2(\gamma_{12} \cup \gamma_{13})} + \|(u - \Pi u)^-\|_{L^2(\gamma_{12} \cup \gamma_{13})} \right\} \\ &\leq C \, \textstyle\sum_{i=1}^3 \left(\|u - \Pi u\|_{L^2(\widehat{K}_i)} + \|u - \Pi u\|_{L^2(\widehat{K}_i)}^{1/2} \|\nabla(u - \Pi u)\|_{L^2(\widehat{K}_i)}^{1/2} \right) \end{aligned}$$

and we arrive at

$$\left\| u - \widetilde{\Pi}u \right\|_{L^2(\widehat{K})} \leq C \, \textstyle\sum_{i=1}^3 \left(\|u - \Pi u\|_{L^2(\widehat{K}_i)} + \|u - \Pi u\|_{L^2(\widehat{K}_i)}^{1/2} \|\nabla(u - \Pi u)\|_{L^2(\widehat{K}_i)}^{1/2} \right) .$$

Now assume that K_i are of size h. To obtain error estimates, we first use (7.7, 7.8) and then we scale $\widehat{K}$, $\widehat{K}_i$ to this size. Summing over all patches in Figure 6.3 gives (7.9, 7.10) since in the geometric mesh $\mathcal{M}_{\mathrm{I}}^n$ the modification V in (7.11) is applied at most twice per element. $\square$

Later on, we have the problem that if $K \in \mathcal{M}_{\mathrm{I}}^n$ abuts at the vertices $(0, \pm 1)$, then the layer profile $\boldsymbol{\varphi}$ does not belong to $H^2(K)^3$, in general. Let us denote by K_{11}^+ the element $K \in \mathcal{M}_{\mathrm{I}}^n$ such that $(0,1) \in \overline{K}$, and likewise for K_{11}^- with $(0,-1)$. Put

$$\widetilde{\Sigma}_{\mathrm{I}}^+ := \Sigma_{\mathrm{I}}^+ \setminus \left(\overline{K_{11}^+} \cup \overline{K_{11}^-} \right), \quad \widetilde{\mathcal{M}}_{\mathrm{I}}^n := \left\{ K \in \mathcal{M}_{\mathrm{I}}^n : \ K \neq K_{11}^{\pm} \right\}.$$

Then if $u \in H^{k+1}(\widetilde{\Sigma}_{\mathrm{I}}^+)$ we obtain like for Lemma 7.3 that the interpolation estimates (7.9, 7.10) hold with $\mathcal{M}_{\mathrm{I}}^n$ replaced by $\widetilde{\mathcal{M}}_{\mathrm{I}}^n$ and Σ_{I}^+ by $\widetilde{\Sigma}_{\mathrm{I}}^+$.

To deal with the corner singularities we rely on, see *e.g.* [19]:

Lemma 7.4 (Hardy-type estimate)**.** *Let $Q = (0,h)^2$ and assume that $u \in H^1(Q)$ satisfies for a $\gamma \in (0,1)$:*

$$|u|_{H_\gamma^2(Q)}^2 := \int_Q r^{2\gamma} \, |D^2 u|^2 \, \mathrm{d}x < \infty \,. \tag{7.14}$$

Then $u \in C^0(\overline{Q})$ and the bilinear interpolant $J_Q u$ satisfies the estimate

$$\|u - J_Q u\|_{H^1(Q)} \leq C h^{1-\gamma} \, |u|_{H_\gamma^2(Q)} \,. \tag{7.15}$$

We proceed to *hp*-approximation. Let $\boldsymbol{\varphi}$ belong to $\mathfrak{H}_{\beta,\delta}^{\mathcal{A}}(\Sigma^+)$ for $\beta \in (0,1)$ and $\delta > 0$. We note that $\boldsymbol{\varphi}$ belongs to $H_\gamma^2(Q)^3$ for $\gamma = 1 - \beta$.

Without loss of generality, we consider only $\widetilde{\mathcal{M}}_{\mathrm{I},+}^n$, the upper half $X_3 > 0$ of $\widetilde{\mathcal{M}}_{\mathrm{I}}^n$. We number the elements in this mesh by K_{ij}, $1 \leq i \leq n$ and $j = 1$ if $i = 1$ (vertex element) and $1 \leq j \leq 3$ otherwise, where $i = 2$ in the layer surrounding the vertex element, $i = n$ in the largest element layer. For any K_{ij}, $i \geq 2$, denote by

$$h_{ij} = \operatorname{diam}(K_{ij}), \quad r_{ij} = \operatorname{dist}\big(K_{ij}, (0,1)\big) \,. \tag{7.16}$$

Then there exists $\lambda \in (0,1)$, independent of n, s.t.

$$h_{ij} \le \lambda\, r_{ij} \qquad \text{and} \qquad \forall x \in K_{ij}, \quad r_{ij} \le \rho(x) \le r_{ij} + 2h_{ij} \le (2+\lambda)\, r_{ij}\,. \tag{7.17}$$

Now consider a layer profile $\boldsymbol{\varphi} \in \mathfrak{H}^{\mathcal{A}}_{\beta,\delta}(\Sigma^+)$. Then a typical term in the error bounds (7.10) can be estimated as follows:

$$\begin{aligned}\left(\frac{h}{2}\right)^{2s} \Phi(p,s) \int_{K_{ij}} |D^{s+1}\boldsymbol{\varphi}|^2 \,\mathrm{d}x &\le \left(\frac{\lambda r}{2}\right)^{2s} \Phi(p,s)\, r^{-2(s-1-\beta)} \int_{K_{ij}} \rho^{2(s-1-\beta)} |D^{s+1}\boldsymbol{\varphi}|^2 \,\mathrm{d}x \\ &\le \left(\frac{\lambda}{2}\right)^{2s} \Phi(p,s)\, r^{2(1-\beta)}\, \left\|\rho^{s-1-\beta} D^{s+1}\boldsymbol{\varphi}\right\|^2_{L^2(K_{ij})}\,.\end{aligned} \tag{7.18}$$

Now, since the mesh is geometric with grading ratio $0 < \sigma < 1$, for all $2 \le i \le n$ and $1 \le j \le 3$, we also have

$$\sigma^{n-i+1} \le r_{ij} \le \sqrt{2}\, \sigma^{n-i+1}. \tag{7.19}$$

Summing the error over all K_{ij} gives with (7.18) and (7.19) in (7.10) with the regularity (6.6) that

$$\begin{aligned}\left\|\nabla(\boldsymbol{\varphi} - \widetilde{\Pi}\boldsymbol{\varphi})\right\|^2_{L^2(\widetilde{\Sigma}^+_{\mathrm{I}})} &\le C \sum_{i=2}^{n} \left(\frac{\lambda}{2}\right)^{2s} \Phi(p,s)\, \sigma^{2(1-\beta)(n-i+1)}\, C^{2s+2} (s+1)!^2 \\ &\le C\, \Phi(p,s) \, \left(\frac{\lambda C}{2}\right)^{2s+2} (s+1)!^2\, \sigma^{2(n+1)(1-\beta)} \, \textstyle\sum_{i=2}^{n} \sigma^{-2(1-\beta)i}\,. \\ &\le C\, \Phi(p,s) \, \left(\frac{\lambda C}{2}\right)^{2s+2} (s+1)!^2\,.\end{aligned}$$

If we take $s = \alpha p$ for an $\alpha \in (0,1)$, Stirling's formula implies that

$$\Phi(p,s)(s+1)!^2 \left(\frac{\lambda C}{2}\right)^{2s} \le \frac{(p-s)!}{(p+s)!}\,(s+1)!^2 \left(\frac{\lambda C}{2}\right)^{2s+2} \overset{s\,=\,\alpha p}{\le} C\, p^3 \left(F(\alpha, \lambda C/2)\right)^p$$

where

$$F(\alpha, d) := \frac{(1-\alpha)^{1-\alpha}}{(1+\alpha)^{1+\alpha}}\, (\alpha d)^{2\alpha}\,.$$

Since for $d > 1$

$$\min_{0<\alpha<1} F(\alpha,d) = F(\alpha_{\min}, d) = F_{\min} < 1,\ \alpha_{\min} = \frac{1}{\sqrt{1+d^2}} < 1\,,$$

Sin we get

$$\Phi(p, \alpha p)(s+1)!^2 \left(\frac{\lambda d}{2}\right)^{2\alpha p} \le C\, p^3\, F^p_{\min}\,.$$

Then

$$\left\|\nabla(\boldsymbol{\varphi} - \widetilde{\Pi}\boldsymbol{\varphi})\right\|^2_{L^2(\widetilde{\Sigma}^+_{\mathrm{I}})} \le C\, p^3\, F^p_{\min} \le C\, \mathrm{e}^{-2bp}$$

since $F_{\min} < 1$, for some C, $b > 0$ independent of p. Analogous bounds hold for the L^2 norm of $\boldsymbol{\varphi} - \widetilde{\Pi}\boldsymbol{\varphi}$. Summarizing, we obtain that there exist C and $b > 0$, such that for any $\boldsymbol{\varphi} \in \mathfrak{H}^{\mathcal{A}}_{\beta,\delta}(\Sigma^+)$ and any n and p

$$\left\|\boldsymbol{\varphi} - \widetilde{\Pi}\boldsymbol{\varphi}\right\|^2_{L^2(\widetilde{\Sigma}^+_{\mathrm{I}})} + \left\|\nabla(\boldsymbol{\varphi} - \widetilde{\Pi}\boldsymbol{\varphi})\right\|^2_{L^2(\widetilde{\Sigma}^+_{\mathrm{I}})} \le C\,\mathrm{e}^{-2bp}\,. \tag{7.20}$$

It remains to estimate the error on K_{11}, the vertex element: recalling that any $\boldsymbol{\varphi}$ in $\mathfrak{H}^{\mathcal{A}}_{\beta,\delta}(\Sigma^+)$ belongs to $H^2_\gamma(K_{11})$ for $\gamma = 1 - \beta$, we deduce from Lemma 7.4 that

$$\|\boldsymbol{\varphi} - J_{K_{11}}\boldsymbol{\varphi}\|_{H^1(K_{11})} \le C\,\sigma^{n(1-\beta)}\,. \tag{7.21}$$

Now a continuous interpolant in Σ^+_{I} is obtained by joining the bilinear interpolant $J_{K_{11}}\boldsymbol{\varphi}$ and $\widetilde{\Pi}\boldsymbol{\varphi}$ continuously on $\overline{K}_{11} \cap \overline{\widetilde{\Sigma}}{}^+_{\mathrm{I}}$, by liftings in $K_{21} \cup K_{23}$. Finally, estimates (7.20) and (7.21) yield the hp type approximation estimate if we choose

$$n = p\,.$$

(iii) ESTIMATES IN Σ^+_{II}

Consider now the approximation in $\Sigma^+_{\mathrm{II}} = (2, \widehat{R}) \times (-1,1)$. We write

$$\left\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_p\right\|^2_{H^1(\Sigma^+_{\mathrm{II}})} = \sum_{i=2}^{\widehat{R}} \left\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_p\right\|^2_{H^1(K_i)}$$

where $K_i = (i, i+1) \times (-1,1) \in \mathcal{M}_{\mathrm{II}}$. Applying again Lemma 7.2, we construct $\boldsymbol{\varphi}_p$ elementwise. By (7.6), $\boldsymbol{\varphi}_p$ is continuous in Σ^+_{II} and

$$\left\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_p\right\|^2_{H^1(K_i)} \le C\,\Phi(p,s)\,\left\|D^{s+1}\varphi\right\|^2_{L^2(K_i)}\,.$$

The analytic regularity $\mathfrak{H}^{\mathcal{A}}_{\beta,\delta}$ of $\boldsymbol{\varphi}$ in Σ^+_{II}, *cf.* (6.6), then gives

$$\left\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_p\right\|^2_{H^1(K_i)} \le C\,\Phi(p,s)\,C^{2(s+1)}(s+1)!^2\,\mathrm{e}^{-2\delta i}\,. \tag{7.22}$$

Choosing again $s = \alpha_{\min}\, p$ as in Σ^+_{I}, we find

$$\left\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_p\right\|^2_{H^1(K_i)} \le C\,\mathrm{e}^{-2(bp+\delta i)},\ \ i = 2, \ldots, \widehat{R}\,. \tag{7.23}$$

Summing (7.23) over all i, we get for constants C, $b > 0$ independent of $\widehat{R}$ and ε

$$\left\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_p\right\|^2_{H^1(\Sigma^+_{\mathrm{II}})} \le C\,\mathrm{e}^{-2bp}. \tag{7.24}$$

Remark. The bound (7.23) indicates that the polynomial degree p necessary for the boundary layer approximation in Σ^+_{II} may actually decrease with i: we only need $bp_i + \delta i \ge p_{\max}$, whence $p_i \ge b^{-1}(p_{\max} - \delta i)$ for $i = 2, \ldots, \widehat{R}$ is sufficient to ensure (7.24).

(iv) ESTIMATES IN Σ^+_{III}

Finally, we discuss the region $\Sigma^+_{\mathrm{III}} = (\widehat{R}+1, \infty) \times (-1,1)$. Here we choose $\boldsymbol{\varphi}_p \equiv \mathbf{0}$ and get from (6.6) with $|\boldsymbol{\ell}| = 1$

$$\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_p\|^2_{H^1(\Sigma^+_{\mathrm{III}})} = \|\boldsymbol{\varphi}\|^2_{H^1(\Sigma^+_{\mathrm{III}})} \le C\,\mathrm{e}^{-2\delta\widehat{R}}\,. \tag{7.25}$$

The choice $\boldsymbol{\varphi}_p \equiv \mathbf{0}$ in Σ^+_{III} introduces a jump

$$\mathbf{0} \neq [\boldsymbol{\varphi} - \boldsymbol{\varphi}_p] = -[\boldsymbol{\varphi}_p] \in \mathcal{Q}_p(-1,1) \;\text{ on }\; \{R = \widehat{R}+1\} \times (-1,1)\,.$$

We lift this jump into the last element $K_{\widehat{R}} = (\widehat{R}, \widehat{R}+1) \times (-1,1) \in \Sigma^+_{\mathrm{II}}$ by

$$V(R,X_3) = (R - \widehat{R})[\varphi_p](X_3), \quad (R,X_3) \in K_{\widehat{R}}\,. \tag{7.26}$$

Then, $(\boldsymbol{\varphi}_p - V)(\widehat{R}+1, X_3) = 0$, and there is $C > 0$ independent of $\widehat{R}$ and of p, such that

$$\|V\|_{H^1(K_{\widehat{R}})} \leq C \left\|[\boldsymbol{\varphi}_p]\right\|_{L^2(\{\widehat{R}+1\}\times(-1,1))}\,.$$

Since $\boldsymbol{\varphi}_p|_{\Sigma^+_{\mathrm{III}}} \equiv \mathbf{0}$, we have by the trace theorem in $K_{\widehat{R}}$

$$\begin{aligned} \left\|[\boldsymbol{\varphi}_p]\right\|_{L^2(\{\widehat{R}+1\}\times(-1,1))} &= \left\|[\boldsymbol{\varphi} - \boldsymbol{\varphi}_p]\right\|_{L^2(\{\widehat{R}+1\}\times(-1,1))} \\ &\leq C \left\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_p\right\|_{H^1(K_{\widehat{R}})} \overset{(7.23)}{\leq} C\,\mathrm{e}^{-(bp+\delta\widehat{R})}\,. \end{aligned} \tag{7.27}$$

(v) Conclusion

This yields a continuous approximation $\boldsymbol{\varphi}_p \in S^p(\Sigma^+, \mathcal{M}^p)$ which satisfies (7.4), if we combine all 3 approximations in Σ^+_ν. Theorem 7.1 is proved. □

Corollary 7.5. *Let $\boldsymbol{\varphi} \in \mathfrak{H}^{\mathcal{A}}_{\beta,\delta}(\Sigma^+)$ for $\beta \in (0,1)$ and $\delta > 0$, be a boundary layer profile. On the scaled strip $\varepsilon\Sigma^+$, let $\boldsymbol{\varphi}^\varepsilon$ be defined as $\boldsymbol{\varphi}^\varepsilon(r,x_3) := \boldsymbol{\varphi}(\varepsilon^{-1}r, \varepsilon^{-1}x_3)$. Then, if we take $\widehat{R} = p$, we have*

$$\exists \boldsymbol{\varphi}^\varepsilon_p \in S^p(\varepsilon\Sigma^+, \varepsilon\mathcal{M}^p)^3 \quad \textit{such that} \quad \left\|\boldsymbol{\varphi}^\varepsilon - \boldsymbol{\varphi}^\varepsilon_p\right\|_{H^1(\varepsilon\Sigma^+)} \leq C\,e^{-bp}\,.$$

Proof. Let $\boldsymbol{\varphi}_p$ be the approximant of $\boldsymbol{\varphi}$ given by Theorem 7.1, and let $\boldsymbol{\varphi}^\varepsilon_p$ be defined as $\boldsymbol{\varphi}^\varepsilon_p(r,x_3) := \boldsymbol{\varphi}_p(\varepsilon^{-1}r, \varepsilon^{-1}x_3)$. Scaling $r = \varepsilon\,R$, $x_3 = \varepsilon\,X_3$ implies $\mathrm{d}R\,\mathrm{d}X_3 = \varepsilon^{-2}\,\mathrm{d}r\,\mathrm{d}x_3$ and there holds

$$\left\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_p\right\|_{L^2(\Sigma^+)} = \varepsilon^{-1}\left\|\boldsymbol{\varphi}^\varepsilon - \boldsymbol{\varphi}^\varepsilon_p\right\|_{L^2(\varepsilon\Sigma^+)} \quad \text{and} \quad \left|\boldsymbol{\varphi} - \boldsymbol{\varphi}_p\right|_{H^1(\Sigma^+)} = \left|\boldsymbol{\varphi}^\varepsilon - \boldsymbol{\varphi}^\varepsilon_p\right|_{H^1(\varepsilon\Sigma^+)}\,.$$

Theorem 7.1 implies the assertion. □

7.2. *hp*-Finite Element space in $\boldsymbol{\Omega}^\varepsilon$

To prove approximation results in the three dimensional domain Ω^ε for layer profiles $\boldsymbol{\psi}$ of the tensor form $\boldsymbol{\psi} = \gamma(s)\boldsymbol{\varphi}(R,X_3)$ with $\gamma \in \mathcal{A}(\partial\omega)$ and $\boldsymbol{\varphi} \in \mathfrak{H}^{\mathcal{A}}_{\beta,\delta}(\Sigma^+)$, we define first the Finite Element space.

Our approximation shall be based on a regular partition τ_ω of ω like that used in Section 5.3, with the new request that τ_ω has one layer of quadrilateral elements along its boundary as we are going to describe. Let us define the tubular layer

$$\omega_{\mathrm{b}} = \left\{x_* \in \omega : \;\mathrm{dist}(x_*, \partial\omega) < \rho_0\right\} \tag{7.28}$$

where ρ_0 is chosen less than one half of the minimal radius of curvature of $\partial\omega$.

Let L be the length of the curve $\partial\omega$ and let $s \mapsto \big(x_1(s), x_2(s)\big)$ be an analytic, L-periodic parametric representation of $\partial\omega$. The mapping $m(s,r)$ given by

$$m(s,r) := \big(x_1(s) - r\,x_2'(s),\; x_2(s) + r x_1'(s)\big) \tag{7.29}$$

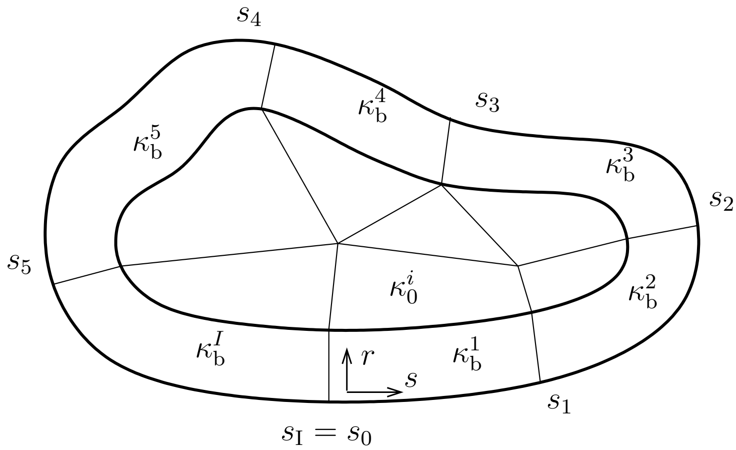

FIGURE 7.5. Boundary fitted mesh τ_ω in the midsurface ω.

is an analytic map of $(0, L) \times (0, \rho_0)$ onto ω_b.

In ω, a fixed, regular partition τ_ω is introduced as follows, see Figure 7.5: partition the interval $(0, L)$ in a fixed number of subintervals $\tau_i := (s_{i-1}, s_i)$, $i = 1, \ldots, I$, $0 = s_0 < s_1 < \cdots < s_I = L$, and set $\kappa_b^i := m(\tau_i \times (0, \rho_0))$, $i = 1, \ldots, I$. The remaining interior $\omega_0 := \omega \backslash \overline{\omega}_b$ is then covered by a fixed curvilinear partition τ_ω of triangular or quadrilateral elements κ_0^i which are images of a reference element $\widehat{\kappa}$ under analytic element maps m_0^i: $\widehat{\kappa} \to \kappa_0^i \in \tau_\omega$.

For each integer $n \geq 1$ we define now a three-dimensional mesh $\mathcal{T}_\varepsilon^n$ corresponding to the mesh $\mathcal{M}^n$ in the half-strip Σ^+ constructed in the previous subsection to resolve the layer profiles, *cf.* Figure 7.1:

(a) In $\Omega_0^\varepsilon := \omega_0 \times (-\varepsilon, \varepsilon)$, we pick tensorized elements $K_0^i := \kappa_0^i \times (-\varepsilon, \varepsilon)$, $\kappa_0^i \in \tau_\omega$, which are fixed, *i.e.* independent of n (their number is also independent of ε).

(b) In the three-dimensional boundary layer region $\Omega_b^\varepsilon \equiv \omega_b \times (-\varepsilon, \varepsilon)$, we select $\mathcal{T}_\varepsilon^n$ to be the tensor product of $\varepsilon\mathcal{M}^n$ in the (r, x_3)-plane times the intervals τ_i along $\partial\omega$:

$$\mathcal{T}_\varepsilon^n|_{\Omega_b^\varepsilon} := \underline{m}\big(\tau_{\partial\omega} \otimes \varepsilon\mathcal{M}^n\big) \cap \overline{\Omega}_b^\varepsilon\,, \tag{7.30}$$

where $\tau_{\partial\omega} = \{\tau_i : 1 = 1, \ldots, I\}$ and $\underline{m}(s, r, x_3) = \big(m(s, r), x_3\big)$. In order for this mesh to be well defined, we assume

$$\varepsilon\widehat{R} < \rho_0\,, \tag{7.31}$$

which ensures that the internal boundary $\overline{\Omega}_b^\varepsilon \cap \overline{\Omega}_0^\varepsilon$ is covered by $m(\varepsilon\mathcal{M}_{\mathrm{III}} \otimes \tau_{\partial\omega})$.

Each element $K \in \mathcal{T}_\varepsilon^n$ is then the image of a hexahedral or prismatic reference element under an analytic element map

$$x = M_K\big(\widehat{x}_*, X_3\big) = \big(m_K(\widehat{x}_*),\ \varepsilon a_K(X_3)\big), \tag{7.32}$$

where m_K is analytic and $a_K(\cdot)$ is affine.

The *hp*-FE spaces $S^p(\Omega^\varepsilon, \mathcal{T}_\varepsilon^n)$ are then defined by

$$S^p(\Omega^\varepsilon, \mathcal{T}_\varepsilon^n) = \big\{u \in H^1(\Omega^\varepsilon) :\ u \circ M_K \in \mathcal{Q}^p(\widehat{K}),\ \ K \in \mathcal{T}_\varepsilon^n\big\} \tag{7.33}$$

where $\widehat{K}$ denotes a hexahedral or prismatic reference element of unit size.

7.3. *hp*-Boundary layer approximation in Ω^ε

Let us define now the approximations of the profiles $\boldsymbol{\psi} = \gamma(s)\boldsymbol{\varphi}(r/\varepsilon, x_3/\varepsilon)$. Here $\gamma(s)$ is an analytic, L-periodic function independent of ε, therefore can be approximated by polynomials at an exponential rate:

Lemma 7.6. *Let γ be analytic and L-periodic in s. Let $S^p_{\rm per}(\partial\omega, \tau_{\partial\omega})$ denote the space of continuous, L-periodic piecewise polynomial functions of degree p in $(0, L)$. Then for any integer $p \geq 1$ there exist interpolants $j_p\gamma \in S^p_{\rm per}(\partial\omega, \tau_{\partial\omega})$ such that*

$$\|\gamma - j_p\gamma\|_{H^1(0,L)} \leq C\mathrm{e}^{-bp}\,. \tag{7.34}$$

Here $b > 0$ depends only on the domain of analyticity of γ.

We can now construct the *hp*-approximation of a generic boundary layer profile $\boldsymbol{\psi}$.

Proposition 7.7. *For ρ_0 as in (7.28), assume that the integer p satisfies*

$$\varepsilon p < \rho_0\,, \tag{7.35}$$

and that the mesh $\mathcal{M}^p$ in Σ^+ is such that $\widehat{R} = p$. Let $\boldsymbol{\psi} = \gamma(s)\boldsymbol{\varphi}(R, X_3)$ with $\gamma \in \mathcal{A}(\partial\omega)$ and a layer profile $\boldsymbol{\varphi} \in \mathfrak{H}^{\mathcal{A}}_{\beta,\delta}(\Sigma^+)$ with $\beta \in (0, 1)$ and $\delta > 0$. Let $\boldsymbol{\psi}^\varepsilon$ be defined as $\boldsymbol{\psi}^\varepsilon(s, r, x_3) := \boldsymbol{\psi}(s, \varepsilon^{-1}r, \varepsilon^{-1}x_3)$. Then there exists an interpolant

$$\mathcal{J}_p\boldsymbol{\psi}^\varepsilon \in S^p(\Omega^\varepsilon, \mathcal{T}^p_\varepsilon)^3, \quad \textit{with support in } \Omega^\varepsilon_{\rm b}$$

such that there holds the error bound, with constants C, $b > 0$ independent of ε and p

$$\|\boldsymbol{\psi}^\varepsilon - \mathcal{J}_p\boldsymbol{\psi}^\varepsilon\|_{H^1(\Omega^\varepsilon)} \leq C\,\mathrm{e}^{-bp}\,. \tag{7.36}$$

Proof. Define in $\Omega^\varepsilon_{\rm b}$

$$(\mathcal{J}_p\boldsymbol{\psi}^\varepsilon)(s, r, x_3) := (j_p\gamma)(s)\,\boldsymbol{\varphi}^\varepsilon_p(r, x_3) \tag{7.37}$$

with j_p defined in Lemma 7.6 and $\boldsymbol{\varphi}^\varepsilon_p$ in Corollary 7.5. Evidently, condition (7.35) implies that the support of $\mathcal{J}_p\boldsymbol{\psi}^\varepsilon$ is contained in $\Omega^\varepsilon_{\rm b}$, and we have the estimate

$$\|\boldsymbol{\psi}^\varepsilon - \mathcal{J}_p\boldsymbol{\psi}^\varepsilon\|_{H^1(\Omega^\varepsilon_0)} = \|\boldsymbol{\psi}^\varepsilon\|_{H^1(\Omega^\varepsilon_0)} \leq Ce^{-\delta\rho_0/\varepsilon} \leq Ce^{-\delta p}$$

where we used (7.35) and the exponential decay of the profile $\boldsymbol{\varphi}(R, X_3)$ with respect to R.

In the boundary layer $\Omega^\varepsilon_{\rm b}$, we may go back to the stretched tubular coordinates (s, R, X_3). The application $x \mapsto (s, R, X_3)$ maps $\Omega^\varepsilon_{\rm b}$ onto the product $(0, L) \times (0, \rho_0/\varepsilon) \times (-1, 1)$. With

$$(\mathcal{J}_p\boldsymbol{\psi})(s, R, X_3) := (j_p\gamma)(s)\,\boldsymbol{\varphi}_p(R, X_3)$$

there holds $(\mathcal{J}_p\boldsymbol{\psi}^\varepsilon)(s, r, x_3) = (\mathcal{J}_p\boldsymbol{\psi})(s, \varepsilon^{-1}r, \varepsilon^{-1}x_3)$, hence

$$\begin{aligned}\|\boldsymbol{\psi}^\varepsilon - \mathcal{J}_p\boldsymbol{\psi}^\varepsilon\|_{H^1(\Omega^\varepsilon_{\rm b})} &\simeq \|\boldsymbol{\psi} - \mathcal{J}_p\boldsymbol{\psi}\|_{H^1((0,L)\times(0,\rho_0/\varepsilon)\times(-1,1))}\\ &\leq \|\boldsymbol{\psi} - \mathcal{J}_p\boldsymbol{\psi}\|_{H^1((0,L)\times\Sigma^+)}\,.\end{aligned}$$

Therefore, it is sufficient to bound the right hand side. There holds

$$\begin{aligned}\|\boldsymbol{\psi} - \mathcal{J}_p\boldsymbol{\psi}\|_{H^1((0,L)\times\Sigma^+)} &= \left\|\gamma\boldsymbol{\varphi} - (j_p\gamma)\,\boldsymbol{\varphi}_p\right\|_{H^1((0,L)\times\Sigma^+)}\\ &\le \|(\gamma - j_p\gamma)\boldsymbol{\varphi}\|_{H^1((0,L)\times\Sigma^+)} + \left\|\gamma(\boldsymbol{\varphi} - \boldsymbol{\varphi}_p)\right\|_{H^1((0,L)\times\Sigma^+)}\\ &\le \|\gamma - j_p\gamma\|_{H^1(0,L)}\,\|\boldsymbol{\varphi}\|_{H^1(\Sigma^+)} + \|\gamma\|_{H^1(0,L)}\left\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_p\right\|_{H^1(\Sigma^+)}\end{aligned}$$

Lemma 7.6 and Theorem 7.1 with $\widehat{R} = p$ yield finally the exponential bound $C\mathrm{e}^{-bp}$. □

For $K \ge 0$, let us denote by $\widetilde{\boldsymbol{w}}^{[K]}$ the truncated series of the inner expansion

$$\widetilde{\boldsymbol{w}}^{[K]} = \sum_{k=0}^{K} \varepsilon^k \chi(r)\boldsymbol{\varphi}^k(s, \varepsilon^{-1}r, \varepsilon^{-1}x_3), \tag{7.38}$$

where the profiles $\boldsymbol{\varphi}^k$ are the coefficients of the series $\boldsymbol{\varphi}[\varepsilon] = \boldsymbol{\Phi}[\varepsilon]\boldsymbol{\zeta}[\varepsilon] + \boldsymbol{\Theta}[\varepsilon]\boldsymbol{f}[\varepsilon]$, see (6.1). Note that, although the series $\boldsymbol{\zeta}[\varepsilon]$ and $\boldsymbol{f}[\varepsilon]$ start with $k = -1$, the series $\boldsymbol{\varphi}[\varepsilon]$ starts with $k = 0$ because the operators $\boldsymbol{\Phi}^k$ and $\boldsymbol{\Theta}^k$ are zero for any $k \le 0$.

The final result on the *hp* approximation of the inner expansion now reads

Theorem 7.8. *For the definition of the discrete space $S^p(\Omega^\varepsilon, \mathcal{T}^p_\varepsilon)$ we assume that $\varepsilon p < \rho_0$ and $\widehat{R} = p$. Let $K \ge 0$. Let the volume load $\overline{\boldsymbol{f}}$ be such that $\overline{\boldsymbol{f}}(x_*, 0) \not\equiv 0$. Then we have, with constants b, $C > 0$ independent of ε and p but depending on K:*

$$\exists \widetilde{\boldsymbol{w}}^{[K]}_N \in S^p(\Omega^\varepsilon, \mathcal{T}^p_\varepsilon) \quad \textit{such that} \quad \left\|\widetilde{\boldsymbol{w}}^{[K]} - \widetilde{\boldsymbol{w}}^{[K]}_N\right\|_{E(\Omega^\varepsilon)} \le C\varepsilon^{1/2}\mathrm{e}^{-bp}\,\|\widetilde{\boldsymbol{u}}\|_{E(\Omega^\varepsilon)}\,. \tag{7.39}$$

Proof. (i) Let us take $k \le K$. First recall that according to (6.1), $\boldsymbol{\varphi}^k = \sum_{\ell=1}^{k} \boldsymbol{\Phi}^\ell\boldsymbol{\zeta}^{k-\ell} + \boldsymbol{\Theta}^\ell\boldsymbol{f}^{k-\ell}$ and that according Lemmas 6.3 and 6.7 each term of the above sum is itself a linear combination of terms of the form $\boldsymbol{\psi} = \gamma(s)\boldsymbol{\varphi}(R, X_3)$ with $\gamma \in \mathcal{A}(\partial\omega)$ and $\boldsymbol{\varphi} \in \mathfrak{H}^{\mathcal{A}}_{\beta,\delta}(\Sigma^+)$ for $\beta \in (0,1)$ and some $\delta > 0$. Therefore Proposition 7.7 applies. We note that χ can be chosen such that $\chi(r) \equiv 1$ for $0 \le r \le \rho_0$, $\chi(r) \equiv 0$ for $r \ge 2\rho_0$. Therefore, as in $\Omega^\varepsilon_{\mathrm{b}}$, $\chi(r)\,\boldsymbol{\varphi}^k \equiv \boldsymbol{\varphi}^k$, there holds

$$\left\|\chi(r)\,\boldsymbol{\varphi}^k - \mathcal{J}_p\boldsymbol{\varphi}^k\right\|_{E(\Omega^\varepsilon_{\mathrm{b}})} = \left\|\boldsymbol{\varphi}^k - \mathcal{J}_p\boldsymbol{\varphi}^k\right\|_{E(\Omega^\varepsilon_{\mathrm{b}})} \le C\,\left\|\boldsymbol{\varphi}^k - \mathcal{J}_p\boldsymbol{\varphi}^k\right\|_{H^1(\Omega^\varepsilon_{\mathrm{b}})} \le C\,\mathrm{e}^{-bp}.$$

In Ω^ε_0, $\mathcal{J}_p\boldsymbol{\varphi}^k \equiv 0$ by construction. Therefore

$$\left\|\chi(r)\,\boldsymbol{\varphi}^k - \mathcal{J}_p\boldsymbol{\varphi}^k\right\|_{E(\Omega^\varepsilon_0)} \le \left\|\chi(r)\,\boldsymbol{\varphi}^k\right\|_{H^1(\Omega^\varepsilon_0)} \le C\mathrm{e}^{-\delta\widehat{R}}.$$

Whence the upper bound $C\,\mathrm{e}^{-bp}$ on Ω^ε_0 since $\widehat{R} = p$.

(ii) By superposition we find that for any $K_0 \le K$ the partial sum satisfies:

$$\left\|\sum_{k=K_0}^{K} \varepsilon^k\chi(r)\boldsymbol{\varphi}^k - \sum_{k=K_0}^{K} \varepsilon^k\mathcal{J}_p\boldsymbol{\varphi}^k\right\|_{E(\Omega^\varepsilon)} \le C\varepsilon^{K_0}\mathrm{e}^{-bp}$$

(iii) If $\overline{f}_3(x_*, 0) \not\equiv 0$, then the energy norm of $\widetilde{\boldsymbol{u}}$ is equivalent to $\varepsilon^{-1/2}$ and the inner expansion starts with $K_0 = 0$, whence (7.39).

(iv) If $\overline{f}_3(x_*, 0) \equiv 0$, then $\overline{\boldsymbol{f}}_*(x_*, 0) \not\equiv 0$ and then the energy norm of $\widetilde{\boldsymbol{u}}$ is equivalent to $\varepsilon^{1/2}$. Moreover the inner expansion starts with $K_0 = 1$, whence (7.39). □

8. *hp*-Approximation of 3-d plates

To obtain the *hp*-approximation of the full problem (2.2–2.5), it suffices to combine the results of Theorems 5.8 and 7.8.

For this, we only have to note that for any $n \geq 1$ the geometric boundary layer meshes $\mathcal{T}_\varepsilon^n$ defined in Section 7.2 are refinements of the regular mesh $\mathcal{T}_\varepsilon^0$ defined in Section 5.3, provided $\mathcal{T}_\varepsilon^0$ is based on the same boundary fitted mesh τ_ω on ω. Moreover we have the inclusion

$$V^{p,q}(\Omega^\varepsilon, \mathcal{T}_\varepsilon^0) \subset S^p(\Omega^\varepsilon, \mathcal{T}_\varepsilon^n) \tag{8.1}$$

for all $q \leq p$, $n \geq 1$, and $\varepsilon > 0$.

As a corollary of Theorems 5.8 and 7.8, we obtain our main results, namely *a-priori* estimates for *hp*-approximations of the three-dimensional plate problem.

Theorem 8.1. *Let* $\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)$ *be a plate of thickness* 2ε *and midsurface* ω *with analytic boundary, and let* $\widetilde{\boldsymbol{f}} = \overline{\boldsymbol{f}}\big|_{\Omega^\varepsilon}$ *be a volume loading where* $\overline{\boldsymbol{f}}$ *is analytic in* $\overline{\omega} \times (-\varepsilon_0, \varepsilon_0)$ *for some* $\varepsilon_0 > \varepsilon > 0$ *and such that* $\overline{\boldsymbol{f}}\big|_{\omega\times\{0\}}$ *is not identically* 0. *We consider the hp-approximation* $\widetilde{\boldsymbol{u}}_N$ *of the three dimensional solution* $\widetilde{\boldsymbol{u}}$ *of the hard clamped plate problem* (2.2)–(2.5) *based on the subspace of dimension* $N = \mathcal{O}(p^4)$

$$\boldsymbol{H}_N = \left\{ \widetilde{\boldsymbol{v}}_N \in S^p(\Omega^\varepsilon, \mathcal{T}_\varepsilon^p)^3, \quad \widetilde{\boldsymbol{v}}_N\big|_{\partial\omega\times(-\varepsilon,\varepsilon)} = 0 \right\} \cdot \tag{8.2}$$

(i) *There holds for every* $K \geq 0$ *the error bound*

$$\|\widetilde{\boldsymbol{u}} - \widetilde{\boldsymbol{u}}_N\|_{E(\Omega^\varepsilon)} \leq C_K(\varepsilon^K + \varepsilon^{-1}\,\mathrm{e}^{-bp})\,\|\widetilde{\boldsymbol{u}}\|_{E(\Omega^\varepsilon)} \tag{8.3}$$

for some $b, C_K > 0$ *independent of* ε, p *as* $\varepsilon \to 0$, $p \to \infty$.

(ii) *If the condition of Theorem* 5.8 (A.II) *is satisfied, or if* $\overline{f}_3\big|_{\omega\times\{0\}} \equiv 0$, *then we have the robust estimate of the error bound*

$$\|\widetilde{\boldsymbol{u}} - \widetilde{\boldsymbol{u}}_N\|_{E(\Omega^\varepsilon)} \leq C_K(\varepsilon^K + \mathrm{e}^{-bp})\,\|\widetilde{\boldsymbol{u}}\|_{E(\Omega^\varepsilon)}\,. \tag{8.4}$$

Proof. Using Theorem 2.1 and the triangle inequality (2.15), we only have to estimate the *hp*-approximation of the asymptotic expansion $\sum_{k=-2}^{K} \varepsilon^k \widetilde{\boldsymbol{u}}^k$ of the three-dimensional solution $\widetilde{\boldsymbol{u}}$. Since, according to notations (5.15) and (7.38), we have

$$\sum_{k=-2}^{K} \varepsilon^k\, \widetilde{\boldsymbol{u}}^k = \widetilde{\boldsymbol{v}}^{[K]} + \widetilde{\boldsymbol{w}}^{[K]}$$

the sum $\widetilde{\boldsymbol{v}}_N^{[K]} + \widetilde{\boldsymbol{w}}_N^{[K]}$ of the interpolants constructed in Theorems 5.8 and 7.8 yields an interpolant $\widetilde{\boldsymbol{u}}_N^{[K]}$ in the space $S^p(\Omega^\varepsilon, \mathcal{T}_\varepsilon^p)^3$. It remains the problem of the trace of $\widetilde{\boldsymbol{u}}_N^{[K]}$ on Γ_0^ε which could be non-zero.

By construction, for any $k \geq -2$, the trace of $\widetilde{\boldsymbol{v}}^k + \widetilde{\boldsymbol{w}}^k$ on Γ_0^ε is zero. In particular, the traces of $\widetilde{\boldsymbol{v}}^{-2}$ and $\widetilde{\boldsymbol{v}}^{-1}$ are zero, and thanks to the property of the interpolation operator i_p in ω, *cf.* Proposition 5.6, the interpolants $\widetilde{\boldsymbol{v}}_N^k$ of $\widetilde{\boldsymbol{v}}^k$ for $k = -2, -1$ can be chosen with zero traces on Γ_0^ε with the same error bound. We have the same situation for $k = 0$ in the case when $\overline{f}_3(x_*, 0) \equiv 0$. Therefore we have to consider the trace:

$$\sum_{k=K_0}^{K} \varepsilon^k (\widetilde{\boldsymbol{v}}^k + \boldsymbol{\varphi}^k)\big|_{\Gamma_0^\varepsilon}$$

with $K_0 = 0$ if $\overline{f}_3(x_*, 0) \not\equiv 0$ and $K_0 = 1$ otherwise. Let us fix $k \geq K_0$. Inspecting the constructions and proofs in Section 5.3 and Sections 7.1, 7.3, and taking advantage that in the layer $\Omega^\varepsilon_{\rm b}$ the finite elements are tensorial in the three directions, we find that there also hold error bounds in the norm $H^1(\Gamma^\varepsilon_0)$. When scaled to $\partial\omega \times (-1,1)$, these estimates are uniform with respect to ε, and scaled back to $\Gamma^\varepsilon_0 = \partial\omega \times (-\varepsilon, \varepsilon)$ their behavior in ε is $\mathcal{O}(\varepsilon^{-1/2})$. In p, we still have the exponential rate, *cf.* (5.14) and (7.36), which means that the interpolants $\widetilde{\boldsymbol{v}}^k_N$ and $\widetilde{\boldsymbol{w}}^k_N$ satisfy

$$\left\| (\widetilde{\boldsymbol{v}}^k - \widetilde{\boldsymbol{v}}^k_N)\big|_{\Gamma^\varepsilon_0} \right\|_{H^1(\Gamma^\varepsilon_0)} \leq C\,\varepsilon^{-1/2}\mathrm{e}^{-bp} \quad \text{and} \quad \left\| (\boldsymbol{\varphi}^k - \widetilde{\boldsymbol{w}}^k_N)\big|_{\Gamma^\varepsilon_0} \right\|_{H^1(\Gamma^\varepsilon_0)} \leq C\,\varepsilon^{-1/2}\mathrm{e}^{-bp},$$

whence, as $(\widetilde{\boldsymbol{v}}^k + \boldsymbol{\varphi}^k)\big|_{\Gamma^\varepsilon_0} = 0$:

$$\left\| (\widetilde{\boldsymbol{v}}^k_N + \widetilde{\boldsymbol{w}}^k_N)\big|_{\Gamma^\varepsilon_0} \right\|_{H^1(\Gamma^\varepsilon_0)} \leq C\,\varepsilon^{-1/2}\mathrm{e}^{-bp}.$$

Let us consider the lifting

$$\ell^k_N(s, r, x_3) := (\widetilde{\boldsymbol{v}}^k_N + \widetilde{\boldsymbol{w}}^k_N)(s, 0, x_3)\,(1 - r/\rho_0) \quad \text{in} \quad \Omega^\varepsilon_{\rm b} \quad \text{and} \quad 0 \quad \text{in} \quad \Omega^\varepsilon_0.$$

This defines an element of $S^p(\Omega^\varepsilon, \mathcal{T}^p_\varepsilon)^3$ which also satifies the estimate

$$\left\| \ell^k_N \right\|_{H^1(\Omega^\varepsilon)} \leq C\,\varepsilon^{-1/2}\mathrm{e}^{-bp}.$$

Then the element of $S^p(\Omega^\varepsilon, \mathcal{T}^p_\varepsilon)^3$ defined as $\widetilde{\boldsymbol{v}}^k_N + \widetilde{\boldsymbol{w}}^k_N - \ell^k_N$ is an interpolant of $\widetilde{\boldsymbol{u}}^k$ in $\boldsymbol{H}_N$. The extra contribution to the error is

$$\left\| \sum_{k=K_0}^{K} \varepsilon^k \ell^k_N \right\|_{E(\Omega^\varepsilon)} \leq C\,\varepsilon^{K_0 - 1/2}\mathrm{e}^{-bp}.$$

Combining with the behavior of the energy of $\widetilde{\boldsymbol{u}}$ in Ω^ε as $\varepsilon \to 0$, we finally obtain (8.3) and (8.4). □

Corollary 8.2. *For every $K \geq 0$ there is $C_* > 0$ such that*

$$\|\widetilde{\boldsymbol{u}} - \widetilde{\boldsymbol{u}}_N\|_{E(\Omega^\varepsilon)} \leq C\,\varepsilon^K\,\|\widetilde{\boldsymbol{u}}\|_{E(\Omega^\varepsilon)} \quad \textit{as } \varepsilon \to 0\,, \tag{8.5}$$

provided that $p \geq C_*\,|\log\varepsilon|$ in the general case (i) *of Theorem* 8.1 *and provided $p \geq C_*$ in the case* (ii) *of the same theorem.*

The preceding results assumed that the transverse degree of $S^p(\Omega^\varepsilon, \mathcal{T}^n_\varepsilon)$ is increased uniformly throughout the domain. If, however, the volume load $\boldsymbol{f}[\varepsilon]$ vanishes or is constant in subdomains, substantial simplifications are possible, if the transverse polynomial degree is taken variable.

Remark 8.3.

(i) If the plate deforms due to a constant bending volume force $(0, 0, 1)$, (8.3)–(8.5) hold even if $\deg_3(\boldsymbol{H}_N) = \binom{3}{4}$ in all $\kappa^i_0 \subset \omega_0$.

(ii) For a constant membrane volume force $(a, b, 0)$ throughout Ω^ε, (8.3)–(8.5) hold if $\deg_3(\boldsymbol{H}_N) = \binom{2}{1}$ in ω_0.

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