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EXPONENTIAL STABILITY AND NUMERICAL TREATMENT FOR PIEZOELECTRIC BEAMS WITH MAGNETIC EFFECT ☆

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Abstract. In this paper, we consider a one-dimensional dissipative system of piezoelectric beams with magnetic effect, inspired by the model studied by Morris and Özer (*Proc. of 52nd IEEE Conference on Decision & Control* (2013) 3014–3019). Our main interest is to analyze the issues relating to exponential stability of the total energy of the continuous problem and reproduce a numerical counterpart in a totally discrete domain, which preserves the important decay property of the numerical energy.

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1. Introduction

Piezoelectric materials have the ability to convert mechanical energy into electric and *vice versa* [16]. Because of this special ability, these materials have been widely used as sensors and actuators in the area of structures and intelligent systems [3, 4]. As actuators, these materials are electrically controllable positions elements in which guarantee offsets with a resolution of approximately 1 nm [31]. These characteristics, along with the fact they have fast response, high rigidity, without clearance and without friction, make the piezoelectric actuators an essential component in ultra precision machines, active vibration control and also are used in microelectromechanical systems [3, 31]. However, actuators, in addition to transform mechanical energy into electric one, it also turns a small portion of which into magnetic energy [12]. For being a relatively small effect, models that often describe, for example, piezoelectric beams, ignore such magnetic effects because the magnetic energy has a relatively small effect on the overall dynamics. However, this magnetic contribution may limit the system performance. For example, it can cause oscillations in the output, which in turn, can lead to system instability in a closed loop [16, 31]. In this way, models based on electrostatic or near-static theories, completely ignore the magnetic energy stored/produced in the process. These models are known to be exactly observable and have exponential stability in energy space. The Figure 1 illustrates a piezoelectric beam model.

In the last three years, researches on piezoelectric beams systems have received much attention in the specialized literature, especially considering the presence of the magnetic effect. Some of the main results for the

☆ *Dedicated to Prof. Midori Makino on the occasion of his 68th Birthday.*

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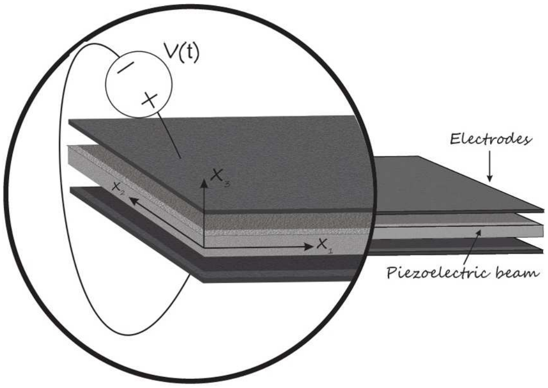

FIGURE 1. Piezoelectric beam model.

one-dimensional evolution equation model were introduced by Morris and Özer in [12], where authors used a variational approach to construct a coupled model of piezoelectric beams with magnetic effect. The Lagrangian due to the model is

$$\mathcal{L} = \int_0^T \Big[K - (P+E) + B + W\Big]\mathrm{d}t, \tag{1.1}$$

where K, P, E and B denote kinetic, potential, magnetic and electric energies, respectively and W represents the work done by external forces. In addition, to a beam of length L and thickness h, we assume that

$$P + E = \frac{h}{2}\int_0^L \Big(\alpha|v_x|^2 - 2\gamma\beta v_x p_x + \beta|p_x|^2\Big)\mathrm{d}x, \quad B = \frac{\mu h}{2}\int_0^L |p_t|^2\mathrm{d}x, \tag{1.2}$$

$$K = \frac{\rho h}{2}\int_0^L |v_t|^2\mathrm{d}x \quad \text{and} \quad W = -\int_0^L p_x V(t)\mathrm{d}x. \tag{1.3}$$

A simple application of the Hamilton principle for admissible displacement variations $\{v, p\}$ of $\mathcal{L}$ the zero shows that

$$\delta\mathcal{L} = -h\int_0^L \Big(\rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx}\Big)\delta v\,\mathrm{d}x - h\int_0^L \Big(\mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx}\Big)\delta p\,\mathrm{d}x$$
$$-h\Big(\alpha v_x - \gamma\beta p_x\Big)\delta v\Big|_0^L + h\left(\gamma\beta v_x - \beta p_x - \frac{V(t)}{h}\right)\delta p\Big|_0^L. \tag{1.4}$$

Therefore, the piezoelectric beams system with magnetic effect is given by

$$\rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} = 0, \tag{1.5}$$
$$\mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, \tag{1.6}$$

with the boundary conditions

$$v(0,t) = p(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, \tag{1.7}$$
$$\beta p_x(L,t) - \gamma\beta v_x(L,t) = -\frac{V(t)}{h}, \tag{1.8}$$

and initial conditions

$$v(x,0)=v_0(x),\ v_t(x,0)=v_1(x),\ p(x,0)=p_0(x),\ p_t(x,0)=p_1(x), \tag{1.9}$$

where $\rho,\alpha,\gamma,\mu,\beta$ and V denote the mass density per unit volume, elastic rigidity, piezoelectric coefficient, magnetic permeability, water resistance coefficient of the beam and the prescribed voltage on electrodes of beam respectively, and in addition, the relationship is considered

$$\alpha=\alpha_1+\gamma^2\beta. \tag{1.10}$$

Here the functions v and p are used to denote the transverse displacement of the beam and the total load of the electric displacement along the transverse direction at each point x respectively.

In [13], Morris and Özer proved that the magnetic effect, despite being relatively small, has a strong interference in stabilizing and controlling the equation system below,

$$\rho v_{tt}-\alpha v_{xx}+\gamma\beta p_{xx}=0,\quad (x,t)\in(0,L)\times\mathbb{R}^+, \tag{1.11}$$

$$\mu p_{tt}-\beta p_{xx}+\gamma\beta v_{xx}=0,\quad (x,t)\in(0,L)\times\mathbb{R}^+, \tag{1.12}$$

$$v(0,t)=p(0,t)=\alpha v_x(L,t)-\gamma\beta p_x(L,t)=0,\quad t\in\mathbb{R}^+, \tag{1.13}$$

$$\beta p_x(L,t)-\gamma\beta v_x(L,t)+\frac{p_t(L,t)}{h}=0,\quad t\in\mathbb{R}^+, \tag{1.14}$$

despite being strongly stable, it is not exponentially stable for almost all system parameters, unlike the classical model, consisting of a single wave equation studied in [9, 26], where the magnetic effect is despised and so the decay is exponential. In [14], the same author also proves an exact observability inequality, in an energy space less regular, and in addition, he proves an estimate of a polynomial decay. In paper [15], the author also shows a three-layer smart piezoelectric laminate totally dynamic, adopting the Rao-Nakra thin compliant layer assumptions [21] to model a sandwich-like structure, keeping all magnetic effects of piezoelectric layers. Other problems related to piezoelectric systems can be found in the following references [5, 10, 19, 22, 24, 27, 28, 30].

In [29], Wang and Guo studied a one-dimensional porous-elastic system. They proved that the system is exponentially stable using the spectral method, and furthermore, they showed that the system's generalized self-functions form a Riesz basis for energy space. Despite the similarity to the problem (1.11)–(1.14), we emphasize that the physical phenomenon and the parameters involved in the model are totally different.

The main objective of this work is to investigate the exponential decay of total energy and some numerical aspects associated with the dissipative system of piezoelectric beams with magnetic effect given by

$$\rho v_{tt}-\alpha v_{xx}+\gamma\beta p_{xx}+\delta v_t=0,\quad \text{in}\quad (0,L)\times(0,T) \tag{1.15}$$

$$\mu p_{tt}-\beta p_{xx}+\gamma\beta v_{xx}=0,\quad \text{in}\quad (0,L)\times(0,T) \tag{1.16}$$

$$v(0,t)=\alpha v_x(L,t)-\gamma\beta p_x(L,t)=0,\quad 0\le t\le T \tag{1.17}$$

$$p(0,t)=p_x(L,t)-\gamma v_x(L,t)=0,\quad 0\le t\le T \tag{1.18}$$

$$v(x,0)=v_0(x),\quad v_t(x,0)=v_1(x),\quad p(x,0)=p_0(x),\quad p_t(x,0)=p_1(x),\quad 0\le x\le L, \tag{1.19}$$

where $\rho,\alpha,\gamma,\mu,\beta,\delta$ are positive constants with $\alpha=\alpha_1+\gamma^2\beta$. The main difference between the system (1.15)–(1.19) and the system (1.11)–(1.14) is in the insertion of the dissipative term δv_t in the mechanical equation and in the use of the boundary condition $\gamma v_x(L,t)-p_x(L,t)=0$, for all $t\ge 0$.

A approach used to prove the exponential decay of the system, consists in the use of multiplicative techniques combined with the energy method.

The total energy of the system of equations (1.15)–(1.19) is given by

$$E(t) := \frac{\rho}{2}\int_0^L |v_t|^2 \mathrm{d}x + \frac{\mu}{2}\int_0^L |p_t|^2 \mathrm{d}x + \frac{\alpha_1}{2}\int_0^L |v_x|^2 \mathrm{d}x + \frac{\beta}{2}\int_0^L |\gamma v_x - p_x|^2 \mathrm{d}x, \quad \forall\, t \geq 0, \tag{1.20}$$

and it satisfies the following of dissipation law

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\delta \int_0^L |v_t|^2 \mathrm{d}x, \quad \forall\, t \geq 0. \tag{1.21}$$

This shows that

$$E(t) \leq E(0), \quad \forall\, t \geq 0. \tag{1.22}$$

In fact, the procedure used to obtain this result is quite simple and will be described in detail in the following section. Note that it only needs to multiply the equations (1.15) and (1.16) by v_t and p_t respectively, we promote an integration by parts using the boundary conditions (1.17)–(1.18) and add the two resulting equations to establish the proof.

On the other hand, in the case of numerical methods, more precisely, the Finite Differences Method (FDM) applied to piezoelectric systems, a work like this is very rare (see [11]). Up to now, we have not had any knowledge about any work in the literature that analyzes the system by the FDM (1.11)–(1.14) and much less the system (1.15)–(1.19). Therefore, our contribution to the numerical study of the piezoelectric beams system with magnetic effect is new.

Therefore, our main goals in this paper are:

- (i) Prove that the dissipation produced by damping δv_t is strong enough to stabilize exponentially the system solution (1.15)–(1.19) for whatever the physical parameters of the model;
- (ii) Discretize the system of piezoelectric beams (1.15)–(1.19) and make a detailed analysis of the behavior of the numerical energy to ensure its property numeric decay;
- (iii) Demonstrate by means of numerical simulations the results set out in items (i) and (ii).

1.1. Outline of the paper

To achieve our goals, this paper takes the following route: in Section 2, we prove the exponential decay using energy method. In Section 3, we introduced the finite difference numerical method, where we built the numerical energy and prove its decay. In Section 4, we prove our results through numerical simulations. We finish in Section 5, making a brief comment about our future investigations.

2. The exponential stability

In this section, we have established and proved the result of exponential stability for energy (1.15)–(1.19) by using a quite elementary method known as Energy Method, introduced in [18]. To achieve the purpose of this section, we need some auxiliary lemmas, but let us first look at the following proposition which gives us the rate of change of the total energy of the system.

Proposition 2.1. *Let (v,p) the system solution (1.15)–(1.19), this way, the total energy given by*

$$E(t) := \frac{\rho}{2}\int_0^L |v_t|^2 \mathrm{d}x + \frac{\mu}{2}\int_0^L |p_t|^2 \mathrm{d}x + \frac{\alpha_1}{2}\int_0^L |v_x|^2 \mathrm{d}x + \frac{\beta}{2}\int_0^L |\gamma v_x - p_x|^2 \mathrm{d}x, \quad \forall\, t \geq 0, \tag{2.1}$$

satisfies the following rate of change

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\delta\int_0^L |v_t|^2\mathrm{d}x, \quad \forall\, t \geq 0. \tag{2.2}$$

Proof. Multiplying the equation (1.15) by v_t, integrating by parts in the interval $(0, L)$ and using the boundary conditions (1.17) and (1.18) we obtained

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\rho}{2}\int_0^L |v_t|^2\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\alpha_1}{2}\int_0^L |v_x|^2\,\mathrm{d}x + \gamma\beta\int_0^L (\gamma v_x - p_x)v_{xt}\,\mathrm{d}x + \delta\int_0^L |v_t|^2\mathrm{d}x = 0. \tag{2.3}$$

In an analogous way, multiplying the equation (1.16) by p_t we assume that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mu}{2}\int_0^L |p_t|^2\,\mathrm{d}x - \beta\int_0^L (\gamma v_x - p_x)p_{xt}\,\mathrm{d}x = 0. \tag{2.4}$$

Adding the two equations above, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\rho}{2}\int_0^L |v_t|^2\mathrm{d}x + \frac{\mu}{2}\int_0^L |p_t|^2\mathrm{d}x + \frac{\alpha_1}{2}\int_0^L |v_x|^2\mathrm{d}x + \frac{\beta}{2}\int_0^L |\gamma v_x - p_x|^2\mathrm{d}x\right) = -\delta\int_0^L |v_t|^2\mathrm{d}x$$

for all $t \geq 0$, and therefore, the proof is complete. □

Next, we enunciate and prove the auxiliary lemmas necessary to prove the exponential decay of the total energy.

Lemma 2.2. *Let (v, p) the system solution (1.15)–(1.19). So, the functional*

$$\mathcal{F}(t) := \rho\int_0^L v_t v\,\mathrm{d}x + \gamma\mu\int_0^L p_t v\,\mathrm{d}x + \frac{\delta}{2}\int_0^L |v|^2\,\mathrm{d}x \tag{2.5}$$

satisfies the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t) \leq C_\varepsilon\int_0^L |v_t|^2\,\mathrm{d}x + \frac{\gamma\mu}{4\varepsilon}\int_0^L |p_t|^2\,\mathrm{d}x - \alpha_1\int_0^L |v_x|^2\,\mathrm{d}x, \quad \forall\, t \geq 0, \tag{2.6}$$

with ε and $C_\varepsilon = \rho + \varepsilon\gamma\mu$ positive constants.

Proof. Multiplying the equation (1.15) by v, integrating by parts into $(0, L)$ and using the boundary conditions (1.17)–(1.18) we have

$$\rho\int_0^L v_{tt}v\,\mathrm{d}x + \alpha_1\int_0^L |v_x|^2\,\mathrm{d}x - \gamma\beta\int_0^L (\gamma v - p)_{xx}v\,\mathrm{d}x + \delta\int_0^L v_t v\,\mathrm{d}x = 0. \tag{2.7}$$

Using the identity $v_{tt}v = \frac{\mathrm{d}}{\mathrm{d}t}v_t v - |v_t|^2$ and the equation (1.16) from where do we have $\beta(\gamma v - p)_{xx} = -\mu p_{tt}$, it turns out that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\int_0^L v_t v\,\mathrm{d}x\right) - \rho\int_0^L |v_t|^2\,\mathrm{d}x + \alpha_1\int_0^L |v_x|^2\,\mathrm{d}x + \gamma\mu\int_0^L p_{tt}v\,\mathrm{d}x + \frac{\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L |v|^2\,\mathrm{d}x = 0. \tag{2.8}$$

Next, we use the identity $p_{tt}v = \frac{\mathrm{d}}{\mathrm{d}t}p_t v - p_t v_t$ to write

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\int_0^L v_t v\,\mathrm{d}x + \gamma\mu\int_0^L p_t v\,\mathrm{d}x + \frac{\delta}{2}\int_0^L |v|^2\,\mathrm{d}x\right) - \rho\int_0^L |v_t|^2\,\mathrm{d}x - \gamma\mu\int_0^L p_t v_t\,\mathrm{d}x + \alpha_1\int_0^L |v_x|^2\,\mathrm{d}x = 0. \quad (2.9)$$

Using the inequality of Young $xy \leq x^2/4\varepsilon + \varepsilon y^2$ with $x, y \in \mathbb{R}$ and $\varepsilon > 0$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\int_0^L v_t v\,\mathrm{d}x + \gamma\mu\int_0^L p_t v\,\mathrm{d}x + \frac{\delta}{2}\int_0^L |v|^2\,\mathrm{d}x\right) \leq C_\varepsilon\int_0^L |v_t|^2\,\mathrm{d}x + \frac{\gamma\mu}{4\varepsilon}\int_0^L |p_t|^2\,\mathrm{d}x - \alpha_1\int_0^L |v_x|^2\,\mathrm{d}x, \quad (2.10)$$

with $C_\varepsilon = \rho + \varepsilon\gamma\mu$. This concludes the proof.

□

Lemma 2.3. *Let (v, p) the system solution (1.15)–(1.19). So, the functional*

$$\mathcal{G}(t) := \rho\int_0^L v_t(\gamma v - p)\,\mathrm{d}x + \gamma\mu\int_0^L p_t(\gamma v - p)\,\mathrm{d}x + \frac{\gamma\delta}{2}\int_0^L |v|^2\mathrm{d}x - \delta\int_0^L vp\,\mathrm{d}x \quad (2.11)$$

satisfies the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}(t) \leq C_1\int_0^L |v_t|^2\mathrm{d}x - \frac{\gamma\mu}{2}\int_0^L |p_t|^2\mathrm{d}x + C_{\eta_4}\int_0^L |v_x|^2\mathrm{d}x + \alpha_1\eta_4\int_0^L |\gamma v_x - p_x|^2\mathrm{d}x, \quad \forall\, t \geq 0, \quad (2.12)$$

with η_4, $C_1 = \rho\gamma + 3\rho^2/2\gamma\mu + 3\gamma^3\mu/2$ and $C_{\eta_4} = \alpha_1/4\eta_4 + 3\delta^2 c_p/2\gamma\mu$ positive constants.

Proof. Multiplying the equation (1.15) by $(\gamma v - p)$, integrating by parts into $(0, L)$ and using the boundary conditions (1.17)–(1.18) we have

$$\rho\int_0^L v_{tt}(\gamma v - p)\,\mathrm{d}x + \alpha_1\int_0^L v_x(\gamma v - p)_x\,\mathrm{d}x - \gamma\beta\int_0^L (\gamma v - p)_{xx}(\gamma v - p)\,\mathrm{d}x + \delta\int_0^L v_t(\gamma v - p)\,\mathrm{d}x = 0. \quad (2.13)$$

Using the equation (1.16), we assume that $\beta(\gamma v - p)_{xx} = -\mu p_{tt}$. Therefore,

$$\rho\int_0^L v_{tt}(\gamma v - p)\,\mathrm{d}x + \alpha_1\int_0^L v_x(\gamma v - p)_x\,\mathrm{d}x + \gamma\mu\int_0^L p_{tt}(\gamma v - p)\,\mathrm{d}x + \delta\int_0^L v_t(\gamma v - p)\,\mathrm{d}x = 0. \quad (2.14)$$

Next, we use the identities $v_{tt}(\gamma v - p) = \frac{\mathrm{d}}{\mathrm{d}t}v_t(\gamma v - p) - v_t(\gamma v - p)_t$ and $p_{tt}(\gamma v - p) = \frac{\mathrm{d}}{\mathrm{d}t}p_t(\gamma v - p) - p_t(\gamma v - p)_t$ to write

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\int_0^L v_t(\gamma v - p)\,\mathrm{d}x + \gamma\mu\int_0^L p_t(\gamma v - p)\,\mathrm{d}x + \frac{\gamma\delta}{2}\int_0^L |v|^2\mathrm{d}x\right) + \alpha_1\int_0^L v_x(\gamma v - p)_x\,\mathrm{d}x$$
$$-\rho\gamma\int_0^L |v_t|^2\,\mathrm{d}x + \rho\int_0^L v_t p_t\,\mathrm{d}x - \gamma^2\mu\int_0^L v_t p_t\,\mathrm{d}x + \gamma\mu\int_0^L |p_t|^2\mathrm{d}x - \delta\int_0^L v_t p\,\mathrm{d}x = 0. \quad (2.15)$$

In view of the identity $v_tp = \frac{\mathrm{d}}{\mathrm{d}t}vp - vp_t$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\int_0^L v_t(\gamma v-p)\,\mathrm{d}x + \gamma\mu\int_0^L p_t(\gamma v-p)\,\mathrm{d}x + \frac{\gamma\delta}{2}\int_0^L |v|^2\mathrm{d}x - \delta\int_0^L vp\,\mathrm{d}x\right) + \alpha_1\int_0^L v_x(\gamma v-p)_x\,\mathrm{d}x$$
$$-\rho\gamma\int_0^L |v_t|^2\,\mathrm{d}x + \gamma\mu\int_0^L |p_t|^2\mathrm{d}x + \rho\int_0^L v_tp_t\,\mathrm{d}x - \gamma^2\mu\int_0^L v_tp_t\,\mathrm{d}x + \delta\int_0^L vp_t\,\mathrm{d}x = 0. \tag{2.16}$$

Using the inequalities of Poincaré and Young $xy \le x^2/4\eta_i + \eta_i y^2$ with $x, y \in \mathbb{R}$ and $\eta_i > 0$ to $i = 1, 2, 3, 4$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\int_0^L v_t(\gamma v-p)\,\mathrm{d}x + \gamma\mu\int_0^L p_t(\gamma v-p)\,\mathrm{d}x + \frac{\gamma\delta}{2}\int_0^L |v|^2\mathrm{d}x - \delta\int_0^L vp\,\mathrm{d}x\right) =$$
$$\left(\rho\gamma + \frac{\rho}{4\eta_1} + \frac{\gamma^2\mu}{4\eta_2}\right)\int_0^L |v_t|^2\mathrm{d}x - \left(\gamma\mu - \rho\eta_1 - \gamma^2\mu\eta_2 - \delta\eta_3\right)\int_0^L |p_t|^2\mathrm{d}x \tag{2.17}$$
$$+\left(\frac{\alpha_1}{4\eta_4} + \frac{\delta c_p}{4\eta_3}\right)\int_0^L |v_x|^2\mathrm{d}x + \alpha_1\eta_4\int_0^L |\gamma v_x - p_x|^2\mathrm{d}x, \tag{2.18}$$

where c_p is the Poincaré constant. Choosing the constants appropriately η_1, η_2 and η_3 we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\int_0^L v_t(\gamma v-p)\,\mathrm{d}x + \gamma\mu\int_0^L p_t(\gamma v-p)\,\mathrm{d}x + \frac{\gamma\delta}{2}\int_0^L |v|^2\mathrm{d}x - \delta\int_0^L vp\,\mathrm{d}x\right) \le$$
$$C_1\int_0^L |v_t|^2\mathrm{d}x - \frac{\gamma\mu}{2}\int_0^L |p_t|^2\mathrm{d}x + C_{\eta_4}\int_0^L |v_x|^2\mathrm{d}x + \alpha_1\eta_4\int_0^L |\gamma v_x - p_x|^2\mathrm{d}x, \tag{2.19}$$

with η_4, $C_1 = \rho\gamma + \frac{3\rho^2}{2\gamma\mu} + \frac{3\gamma^3\mu}{2}$ and $C_{\eta_4} = \frac{\alpha_1}{4\eta_4} + \frac{3\delta^2 c_p}{2\gamma\mu}$ positive constants. So we concluded the proof. □

Lemma 2.4. *Let (v, p) the system solution (1.15)–(1.19). So, the functional*

$$\mathcal{H}(t) := \rho\int_0^L v_tv\,\mathrm{d}x - \frac{\delta}{2}\int_0^L |v|^2\,\mathrm{d}x + \mu\int_0^L p_tp\,\mathrm{d}x \tag{2.20}$$

satisfies the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t) = \rho\int_0^L |v_t|^2\mathrm{d}x + \mu\int_0^L |p_t|^2\mathrm{d}x - \alpha_1\int_0^L |v_x|^2\mathrm{d}x - \beta\int_0^L |\gamma v_x - p_x|^2\,\mathrm{d}x, \quad \forall\, t \ge 0. \tag{2.21}$$

Proof. Multiplying the equation (1.15) by v, integrating by parts into $(0, L)$ and using the boundary conditions (1.17)–(1.18) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\int_0^L v_tv\,\mathrm{d}x - \frac{\delta}{2}\int_0^L |v|^2\,\mathrm{d}x\right) = \rho\int_0^L |v_t|^2\mathrm{d}x - \alpha_1\int_0^L |v_x|^2\mathrm{d}x - \gamma\beta\int_0^L (\gamma v_x - p_x)v_x\,\mathrm{d}x. \tag{2.22}$$

Similarly, we get to the equation (1.16)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu\int_0^L p_tp\,\mathrm{d}x\right) = \mu\int_0^L |p_t|^2\mathrm{d}x + \beta\int_0^L (\gamma v_x - p_x)p_x\,\mathrm{d}x. \tag{2.23}$$

Adding the equations (2.22) and (2.23) we get the expected result,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\int_0^L v_t v\,\mathrm{d}x - \frac{\delta}{2}\int_0^L |v|^2\,\mathrm{d}x + \mu\int_0^L p_t p\,\mathrm{d}x\right) = \rho\int_0^L |v_t|^2\mathrm{d}x + \mu\int_0^L |p_t|^2\mathrm{d}x - \alpha_1\int_0^L |v_x|^2\mathrm{d}x - \beta\int_0^L |\gamma v_x - p_x|^2\,\mathrm{d}x. \tag{2.24}$$

□

Now, we are in a position to prove the main result of this section that is the exponential decay of the system (1.15)–(1.19) and for that we define the functional Lyapunov (see [7, 20])

$$\mathcal{L}(t) := N_1 E(t) + N_2\mathcal{F}(t) + N_3\mathcal{G}(t) + N_4\mathcal{H}(t), \tag{2.25}$$

where N_i for $i = 1, 2, 3, 4$ are positive constants defined later, E is the total energy of the system and the functional $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are given in Lemmas 2.2–2.4, respectively.

Theorem 2.5. *For all solution (v, p) of the system (1.15)–(1.19), there are constants $M > 0$ and $\omega > 0$ independent of the initial conditions, such that*

$$E(t) \le ME(0)e^{-\omega t}, \quad \forall\, t \ge 0. \tag{2.26}$$

Proof. Followed by Lemmas 2.2–2.4 and the law of dissipation (2.2) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \le{}& -N_1\delta\int_0^L |v_t|^2\mathrm{d}x + N_2C_\varepsilon\int_0^L |v_t|^2\,\mathrm{d}x + N_2\frac{\gamma\mu}{4\varepsilon}\int_0^L |p_t|^2\,\mathrm{d}x - N_2\alpha_1\int_0^L |v_x|^2\,\mathrm{d}x\\ &+N_3C_1\int_0^L |v_t|^2\mathrm{d}x - N_3\frac{\gamma\mu}{2}\int_0^L |p_t|^2\mathrm{d}x + N_3C_{\eta_4}\int_0^L |v_x|^2\mathrm{d}x + N_3\eta_4\alpha_1\int_0^L |\gamma v_x - p_x|^2\mathrm{d}x\\ &+N_4\rho\int_0^L |v_t|^2\mathrm{d}x + N_4\mu\int_0^L |p_t|^2\mathrm{d}x - N_4\alpha_1\int_0^L |v_x|^2\mathrm{d}x - N_4\beta\int_0^L |\gamma v_x - p_x|^2\,\mathrm{d}x,\end{aligned}$$

or equivalently,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \le{}& -\Big(N_1\delta - N_2C_\varepsilon - N_3C_1 - N_4\rho\Big)\int_0^L |v_t|^2\mathrm{d}x - \left(N_3\frac{\gamma\mu}{2} - N_2\frac{\gamma\mu}{4\varepsilon} - N_4\mu\right)\int_0^L |p_t|^2\mathrm{d}x\\ &-\Big(N_2\alpha_1 + N_4\alpha_1 - N_3C_{\eta_4}\Big)\int_0^L |v_x|^2\mathrm{d}x - \Big(N_4\beta - N_3\eta_4\alpha_1\Big)\int_0^L |\gamma v_x - p_x|^2\mathrm{d}x,\end{aligned}$$

with $\eta_4 > 0$, $C_\varepsilon = \rho + \varepsilon\gamma\mu$, $C_1 = \rho\gamma + \frac{3\rho^2}{2\gamma\mu} + \frac{3\gamma^3\mu}{2}$ and $C_{\eta_4} = \frac{\alpha_1}{4\eta_4} + \frac{3\delta^2 c_p}{2\gamma\mu}$. Choosing the constants appropriately

$$\varepsilon = \frac{N_2}{2} \quad \text{and} \quad \eta_4 = \frac{\beta}{\alpha_1 N_3}, \tag{2.27}$$

we get $C_{N_2} = \rho + N_2\gamma\mu \big/ 2$ and $C_{N_3} = \alpha_1^2 N_3 \big/ 4\beta + 3\delta^2 c_p \big/ 2\gamma\mu$ and consequently

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \leq -\left(\frac{N_1\delta - N_2 C_{N_2} - N_3 C_1}{\rho} - N_4\right)\rho\int_0^L |v_t|^2 \mathrm{d}x - \left(\frac{(N_3-1)\gamma}{2} - N_4\right)\mu\int_0^L |p_t|^2\mathrm{d}x$$
$$-\left(N_2 + N_4 - \frac{N_3 C_{N_3}}{\alpha_1}\right)\alpha_1\int_0^L |v_x|^2\mathrm{d}x - \Big(N_4 - 1\Big)\beta\int_0^L |\gamma v_x - p_x|^2\mathrm{d}x. \tag{2.28}$$

Here, we choose $N_4 > 1$, followed by $N_3 > 1 + 2N_4 \big/ \gamma$ and $N_2 > N_3 C_{N_3} \big/ \alpha_1$. Once N_2, N_3 and N_4 are fixed, we choose N_1 sufficiently large, *i.e.*, $N_1 > \Big(N_2 C_{N_2} + N_3 C_1 + N_4\rho\Big) \Big/ \delta$. This way, we ensured that

$$\xi_1 := \frac{N_1\delta - N_2 C_{N_2} - N_3 C_1}{\rho} - N_4 > 0, \quad \xi_2 := \frac{(N_3-1)\gamma}{2} - N_4 > 0$$

and

$$\xi_3 := N_2 + N_4 - \frac{N_3 C_{N_3}}{\alpha_1} > 0, \quad \xi_4 := N_4 - 1 > 0.$$

Thus, we can conclude that there is a constant $N_0 := 2\min\limits_{1\leq i\leq 4}\{\xi_i\} > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}(t) \leq -N_0 E(t), \quad \forall\, t \geq 0. \tag{2.29}$$

As $\mathcal{L}(t)$ is equivalent to the energy (check out Appendix A), there are constants $M > 0$ and $\omega > 0$ such as

$$E(t) \leq M E(0) e^{-\omega t}, \quad \forall\, t \geq 0. \tag{2.30}$$

□

3. Numerical approach

In this section, we apply the explicit integration method to the dissipative system of piezoelectric beams with magnetic effects (1.15)–(1.19), where we can prove the result of the decay of the numerical energy in a way analogous to the continuous.

3.1. Fully-discrete scheme in finite differences and properties

For our purposes, considering $J, N \in \mathbb{N}$, we define $\Delta x = \dfrac{L}{J+1}$, $\Delta t = \dfrac{T}{N+1}$ and we introduced the point-network

$$0 = x_0 < x_1 = \Delta x < \cdots < x_j = j\Delta x < \cdots < x_J < x_{J+1} = (J+1)\Delta x = L, \tag{3.1}$$
$$0 = t_0 < t_1 = \Delta t < \cdots < t_n = n\Delta t < \cdots < t_N < t_{N+1} = (N+1)\Delta t = T, \tag{3.2}$$

for all $j = 0, 1, 2, \ldots, J+1$ and $n = 0, 1, 2, \ldots, N+1$.

Keeping in mind the system of equations (1.15)–(1.19), the problem is to obtain (v_j^n, p_j^n) such that

$$\rho\overline{\partial}_t\partial_t v_j^n - \alpha\overline{\partial}_x\partial_x v_j^n + \gamma\beta\overline{\partial}_x\partial_x p_j^n + \delta\left(\frac{\overline{\partial}_t + \partial_t}{2}\right)v_j^n = 0, \tag{3.3}$$

$$\mu\overline{\partial}_t\partial_t p_j^n - \beta\overline{\partial}_x\partial_x p_j^n + \gamma\beta\overline{\partial}_x\partial_x v_j^n = 0, \tag{3.4}$$

for all $j = 1, 2, \ldots, J$ and $n = 1, 2, \ldots, N$, where we assume the following numerical operators in finite differences

$$\overline{\partial}_t\partial_t v_j^n := \frac{v_j^{n+1} - 2v_j^n + v_j^{n-1}}{\Delta t^2}, \quad \overline{\partial}_x\partial_x v_j^n := \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{\Delta x^2}, \quad \left(\frac{\overline{\partial}_t + \partial_t}{2}\right)v_j^n := \frac{v_j^{n+1} - v_j^{n-1}}{2\Delta t},$$

with similar expressions for $\overline{\partial}_t\partial_t p_j^n$ and $\overline{\partial}_x\partial_x p_j^n$.

Here, we denote by v_j^n and p_j^n the numerical solution for the solutions v and p in points (x_j, t_n) of the network. The initial conditions are

$$v_j^0 = v_{0j}, \quad \frac{v_j^1 - v_j^{-1}}{2\Delta t} = v_{1j}, \quad p_j^0 = p_{0j}, \quad \frac{p_j^1 - p_j^{-1}}{2\Delta t} = p_{1j}, \quad j = 0, 1, \ldots, J+1, \tag{3.5}$$

followed by the boundary conditions

$$v_0^n = \alpha\left(v_{J+1}^n - v_J^n\right) - \gamma\beta\left(p_{J+1}^n - p_J^n\right) = 0, \quad \forall n = 0, 1, \ldots, N+1, \tag{3.6}$$

$$p_0^n = \left(p_{J+1}^n - p_J^n\right) - \gamma\left(v_{J+1}^n - v_J^n\right) = 0, \quad \forall n = 0, 1, \ldots, N+1. \tag{3.7}$$

The numerical energy associated with the numerical scheme (3.3)–(3.7) is given by

$$\begin{aligned} E^n := {}& \rho\frac{\Delta x}{2}\sum_{j=0}^{J}\left(\frac{v_j^{n+1} - v_j^n}{\Delta t}\right)^2 + \mu\frac{\Delta x}{2}\sum_{j=0}^{J}\left(\frac{p_j^{n+1} - p_j^n}{\Delta t}\right)^2 + \alpha_1\frac{\Delta x}{2}\sum_{j=0}^{J}\left(\frac{v_{j+1}^{n+1} - v_j^{n+1}}{\Delta x}\frac{v_{j+1}^n - v_j^n}{\Delta x}\right) \\ & + \beta\frac{\Delta x}{2}\sum_{j=0}^{J}\left(\gamma\frac{v_{j+1}^{n+1} - v_j^{n+1}}{\Delta x} - \frac{p_{j+1}^{n+1} - p_j^{n+1}}{\Delta x}\right)\left(\gamma\frac{v_{j+1}^n - v_j^n}{\Delta x} - \frac{p_{j+1}^n - p_j^n}{\Delta x}\right), \quad \forall\, n = 0, 1, \ldots, N. \end{aligned} \tag{3.8}$$

Remark 3.1. The construction of numerical energy (3.8) is based on the work of Strauss and Vazquez (see [25], Sect. 2) it should be noted that the main differences between the functional (3.8) and (1.20) are the numerical approximations

$$\alpha_1\frac{\Delta x}{2}\sum_{j=0}^{J}\left(\frac{v_{j+1}^{n+1} - v_j^{n+1}}{\Delta x}\frac{v_{j+1}^n - v_j^n}{\Delta x}\right) \approx \frac{\alpha_1}{2}\int_0^L |v_x|^2 \mathrm{d}x,$$

$$\beta\frac{\Delta x}{2}\sum_{j=0}^{J}\left(\gamma\frac{v_{j+1}^{n+1} - v_j^{n+1}}{\Delta x} - \frac{p_{j+1}^{n+1} - p_j^{n+1}}{\Delta x}\right)\left(\gamma\frac{v_{j+1}^n - v_j^n}{\Delta x} - \frac{p_{j+1}^n - p_j^n}{\Delta x}\right) \approx \frac{\beta}{2}\int_0^L |\gamma v_x - p_x|^2 \mathrm{d}x,$$

that are not positive definite quantities and, therefore, the numerical energy is not a positive functional defined. Let's check out the following theorem, which deals with the construction of energy and its rate of change.

Theorem 3.2. *Let* (v_j^n, p_j^n) *the solution of the finite difference numerical method (3.3)–(3.7). So, for all* Δt, $\Delta x \in (0,1)$, *the rate of change of the numerical energy*

$$E^n := \rho\frac{\Delta x}{2}\sum_{j=0}^{J}\left(\frac{v_j^{n+1}-v_j^n}{\Delta t}\right)^2 + \mu\frac{\Delta x}{2}\sum_{j=0}^{J}\left(\frac{p_j^{n+1}-p_j^n}{\Delta t}\right)^2 + \alpha_1\frac{\Delta x}{2}\sum_{j=0}^{J}\left(\frac{v_{j+1}^{n+1}-v_j^{n+1}}{\Delta x}\frac{v_{j+1}^n-v_j^n}{\Delta x}\right)$$
$$+\beta\frac{\Delta x}{2}\sum_{j=0}^{J}\left(\gamma\frac{v_{j+1}^{n+1}-v_j^{n+1}}{\Delta x}-\frac{p_{j+1}^{n+1}-p_j^{n+1}}{\Delta x}\right)\left(\gamma\frac{v_{j+1}^n-v_j^n}{\Delta x}-\frac{p_{j+1}^n-p_j^n}{\Delta x}\right), \tag{3.9}$$

at the instant of time t_n *is given by*

$$\frac{E^n-E^{n-1}}{\Delta t} = -\delta\Delta x\sum_{j=0}^{J}\left|\left(\frac{\overline{\partial}_t+\partial_t}{2}\right)v_j^n\right|^2, \quad \forall\, n=1,2,\ldots,N \tag{3.10}$$

and, therefore, satisfies

$$E^n \le E^0, \quad \forall\, n=1,2,\ldots,N. \tag{3.11}$$

Proof. Multiply the equation (3.3) by $\Delta x\left(\frac{\overline{\partial}_t+\partial_t}{2}\right)v_j^n$ and added to $j\in\{1,2,\ldots,J\}$. So we get

$$\rho\Delta x\sum_{j=1}^{J}(\overline{\partial}_t\partial_t v_j^n)\left(\frac{\overline{\partial}_t+\partial_t}{2}\right)v_j^n - \alpha\Delta x\sum_{j=1}^{J}(\overline{\partial}_x\partial_x v_j^n)\left(\frac{\overline{\partial}_t+\partial_t}{2}\right)v_j^n + \gamma\beta\Delta x\sum_{j=1}^{J}(\overline{\partial}_x\partial_x p_j^n)\left(\frac{\overline{\partial}_t+\partial_t}{2}\right)v_j^n$$
$$+\delta\Delta x\sum_{j=1}^{J}\left|\left(\frac{\overline{\partial}_t+\partial_t}{2}\right)v_j^n\right|^2 = 0. \tag{3.12}$$

Some simplifications in the sum of equation (3.12), in view of the boundary condition $v_0^n = 0$ for all $n = 0,1,\ldots,N+1$, show us that

$$\begin{aligned}\rho\Delta x\sum_{j=1}^{J}(\overline{\partial}_t\partial_t v_j^n)\left(\frac{\partial_t+\overline{\partial}_t}{2}v_j^n\right) &= \frac{\rho\Delta x}{2\Delta t^3}\sum_{j=1}^{J}\Big[(v_j^{n+1}-v_j^n)(v_j^{n+1}-v_j^n+v_j^n-v_j^{n-1})\\ &\quad -(v_j^n-v_j^{n-1})(v_j^{n+1}-v_j^n+v_j^n-v_j^{n-1})\Big]\\ &= \frac{\rho\Delta x}{2\Delta t^3}\sum_{j=0}^{J}\Big[(v_j^{n+1}-v_j^n)^2-(v_j^n-v_j^{n-1})^2\\ &\quad +\underbrace{\rho(v_j^{n+1}-v_j^n)(v_j^n-v_j^{n-1})-\rho(v_j^n-v_j^{n-1})(v_j^{n+1}-v_j^n)}_{=0}\Big].\end{aligned}$$

So, we assume that

$$\rho\Delta x\sum_{j=1}^{J}(\overline{\partial}_t\partial_t v_j^n)\left(\frac{\partial_t+\overline{\partial}_t}{2}v_j^n\right) = \frac{\rho\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{v_j^{n+1}-v_j^n}{\Delta t}\right)^2 - \frac{\rho\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{v_j^n-v_j^{n-1}}{\Delta t}\right)^2. \tag{3.13}$$

After several simplifications, we conclude that

$$-\alpha\Delta x\sum_{j=1}^{J}(\overline{\partial}_x\partial_x v_j^n)\left(\frac{\partial_t+\overline{\partial}_t}{2}v_j^n\right)=\frac{\alpha\Delta x}{\Delta x}\sum_{j=0}^{J}\left(\frac{v_{j+1}^n-v_j^n}{\Delta x}\frac{v_{j+1}^{n+1}-v_{j+1}^{n-1}}{2\Delta t}-\frac{v_{j+1}^n-v_j^n}{\Delta x}\frac{v_j^{n+1}-v_j^{n-1}}{2\Delta t}\right), \\ +(\mathcal{B}_1)^n, \tag{3.14}$$

where the boundary terms $(\mathcal{B}_1)^n$ are given by

$$(\mathcal{B}_1)^n=\frac{\alpha\Delta x}{\Delta x}\left(\frac{v_1^n-v_0^n}{\Delta x}\frac{v_0^{n+1}-v_0^{n-1}}{2\Delta t}-\frac{v_{J+1}^n-v_J^n}{\Delta x}\frac{v_{J+1}^{n+1}-v_{J+1}^{n-1}}{2\Delta t}\right).$$

Analogously, we also write

$$\gamma\beta\Delta x\sum_{j=1}^{J}(\overline{\partial}_x\partial_x p_j^n)\left(\frac{\partial_t+\overline{\partial}_t}{2}v_j^n\right)=\frac{\gamma\beta\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{p_{j+1}^n-p_j^n}{\Delta x}\frac{v_{j+1}^{n-1}-v_j^{n-1}}{\Delta x}-\frac{p_{j+1}^n-p_j^n}{\Delta x}\frac{v_{j+1}^{n+1}-v_j^{n+1}}{\Delta x}\right), \\ +(\mathcal{B}_2)^n, \tag{3.15}$$

where the boundary terms $(\mathcal{B}_2)^n$ are given by

$$(\mathcal{B}_2)^n=\frac{\gamma\beta\Delta x}{\Delta x}\left(\frac{p_{J+1}^n-p_J^n}{\Delta x}\frac{v_{J+1}^{n+1}-v_{J+1}^{n-1}}{2\Delta t}-\frac{p_1^n-p_0^n}{\Delta x}\frac{v_0^{n+1}-v_0^{n-1}}{2\Delta t}\right).$$

On the other hand, using the boundary condition $v_0^n=0$ for all $n=0,1,\dots,N+1$, it's easy to see that

$$\delta\Delta x\sum_{j=1}^{J}\left|\left(\frac{\overline{\partial}_t+\partial_t}{2}\right)v_j^n\right|^2=\delta\Delta x\sum_{j=0}^{J}\left|\left(\frac{\overline{\partial}_t+\partial_t}{2}\right)v_j^n\right|^2. \tag{3.16}$$

Combining the equations (3.12), (3.13), (3.14), (3.15), (3.16) and keeping in mind that $\alpha=\alpha_1+\gamma^2\beta$ we get

$$\begin{aligned}
&\frac{\rho\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{v_j^{n+1}-v_j^n}{\Delta t}\right)^2+\frac{\alpha_1\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{v_{j+1}^{n+1}-v_j^{n+1}}{\Delta x}\frac{v_{j+1}^n-v_j^n}{\Delta x}\right)\\
&+\frac{\beta\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\gamma\frac{v_{j+1}^{n+1}-v_j^{n+1}}{\Delta x}\right)\left(\gamma\frac{v_{j+1}^n-v_j^n}{\Delta x}-\frac{p_{j+1}^n-p_j^n}{\Delta x}\right)\\
&-\frac{\rho\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{v_j^n-v_j^{n-1}}{\Delta t}\right)^2-\frac{\alpha_1\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{v_{j+1}^n-v_j^n}{\Delta x}\frac{v_{j+1}^{n-1}-v_j^{n-1}}{\Delta x}\right)\\
&-\frac{\beta\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\gamma\frac{v_{j+1}^{n-1}-v_j^{n-1}}{\Delta x}\right)\left(\gamma\frac{v_{j+1}^n-v_j^n}{\Delta x}-\frac{p_{j+1}^n-p_j^n}{\Delta x}\right)+\delta\Delta x\sum_{j=0}^{J}\left|\left(\frac{\overline{\partial}_t+\partial_t}{2}\right)v_j^n\right|^2\\
&+\frac{\alpha\Delta x}{\Delta x}\left(\frac{v_0^{n+1}-v_0^{n-1}}{2\Delta t}\frac{v_1^n-v_0^n}{\Delta x}\right)+\frac{v_{J+1}^{n+1}-v_{J+1}^{n-1}}{2\Delta t\Delta x}\Delta x\left(\gamma\beta\frac{p_{J+1}^n-p_J^n}{\Delta x}-\alpha\frac{v_{J+1}^n-v_J^n}{\Delta x}\right)\\
&-\frac{\gamma\beta\Delta x}{\Delta x}\left(\frac{v_0^{n+1}-v_0^{n-1}}{2\Delta t}\frac{p_1^n-p_0^n}{\Delta x}\right)=0.
\end{aligned} \tag{3.17}$$

Analogously, we have for the equation (3.4)

$$\frac{\mu\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{p_j^{n+1}-p_j^n}{\Delta t}\right)^2+\frac{\beta\Delta x}{2\Delta t}\sum_{j=0}^{J}\frac{p_{j+1}^{n+1}-p_j^{n+1}}{\Delta x}\left(\frac{p_{j+1}^n-p_j^n}{\Delta x}-\gamma\frac{v_{j+1}^n-v_j^n}{\Delta x}\right)$$
$$-\frac{\mu\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{p_j^{n}-p_j^{n-1}}{\Delta t}\right)^2-\frac{\beta\Delta x}{2\Delta t}\sum_{j=0}^{J}\frac{p_{j+1}^{n-1}-p_j^{n-1}}{\Delta x}\left(\frac{p_{j+1}^n-p_j^n}{\Delta x}-\gamma\frac{v_{j+1}^n-v_j^n}{\Delta x}\right)$$
$$+\frac{\beta\Delta x}{\Delta x}\frac{p_0^{n+1}-p_0^{n-1}}{2\Delta t}\left(\frac{p_1^n-p_0^n}{\Delta x}-\gamma\frac{v_1^n-v_0^n}{\Delta x}\right)$$
$$-\frac{\beta\Delta x}{\Delta x}\frac{p_{J+1}^{n+1}-p_{J+1}^{n-1}}{2\Delta t}\left(\frac{p_{J+1}^n-p_J^n}{\Delta x}-\gamma\frac{v_{J+1}^n-v_J^n}{\Delta x}\right)=0. \tag{3.18}$$

Adding the equations (3.17) and (3.18) we have

$$\frac{\rho\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{v_j^{n+1}-v_j^n}{\Delta t}\right)^2+\frac{\mu\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{p_j^{n+1}-p_j^n}{\Delta t}\right)^2+\frac{\alpha_1\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{v_{j+1}^{n+1}-v_j^{n+1}}{\Delta x}\frac{v_{j+1}^n-v_j^n}{\Delta x}\right)$$
$$+\frac{\beta\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\gamma\frac{v_{j+1}^{n+1}-v_j^{n+1}}{\Delta x}-\frac{p_{j+1}^{n+1}-p_j^{n+1}}{\Delta x}\right)\left(\gamma\frac{v_{j+1}^n-v_j^n}{\Delta x}-\frac{p_{j+1}^n-p_j^n}{\Delta x}\right)$$
$$-\frac{\rho\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{v_j^{n}-v_j^{n-1}}{\Delta t}\right)^2-\frac{\mu\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{p_j^{n}-p_j^{n-1}}{\Delta t}\right)^2-\frac{\alpha_1\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\frac{v_{j+1}^{n}-v_j^{n}}{\Delta x}\frac{v_{j+1}^{n-1}-v_j^{n-1}}{\Delta x}\right)$$
$$-\frac{\beta\Delta x}{2\Delta t}\sum_{j=0}^{J}\left(\gamma\frac{v_{j+1}^{n}-v_j^{n}}{\Delta x}-\frac{p_{j+1}^{n}-p_j^{n}}{\Delta x}\right)\left(\gamma\frac{v_{j+1}^{n-1}-v_j^{n-1}}{\Delta x}-\frac{p_{j+1}^{n-1}-p_j^{n-1}}{\Delta x}\right)$$
$$+\delta\Delta x\sum_{j=0}^{J}\left|\left(\frac{\overline{\partial}_t+\partial_t}{2}\right)v_j^n\right|^2+\frac{v_0^{n+1}-v_0^{n-1}}{2\Delta t\Delta x}\Delta x\left(\alpha\frac{v_1^n-v_0^n}{\Delta x}-\gamma\beta\frac{p_1^n-p_0^n}{\Delta x}\right)$$
$$+\frac{v_{J+1}^{n+1}-v_{J+1}^{n-1}}{2\Delta t\Delta x}\Delta x\left(\gamma\beta\frac{p_{J+1}^n-p_J^n}{\Delta x}-\alpha\frac{v_{J+1}^n-v_J^n}{\Delta x}\right)+\frac{p_0^{n+1}-p_0^{n-1}}{2\Delta t\Delta x}\Delta x\left(\beta\frac{p_1^n-p_0^n}{\Delta x}-\gamma\beta\frac{v_1^n-v_0^n}{\Delta x}\right)$$
$$-\frac{p_{J+1}^{n+1}-p_{J+1}^{n-1}}{2\Delta t\Delta x}\Delta x\left(\beta\frac{p_{J+1}^n-p_J^n}{\Delta x}-\gamma\beta\frac{v_{J+1}^n-v_J^n}{\Delta x}\right)=0. \tag{3.19}$$

Once the energy E^n is defined in (3.8), use the boundary conditions (3.6)–(3.7) in order to achieve

$$\frac{E^n-E^{n-1}}{\Delta t}=-\delta\Delta x\sum_{j=0}^{J}\left|\left(\frac{\overline{\partial}_t+\partial_t}{2}\right)v_j^n\right|^2,\quad\forall\, n=1,2,\ldots,N \tag{3.20}$$

and, consequently,

$$E^n\le E^0,\quad\forall\, n=1,2,\ldots,N.$$

□

Corollary 3.3 (Conservation of energy)**.** *Let (v_j^n,p_j^n) the solution of the finite difference numerical method (3.3)–(3.7) with $\delta=0$. So, for all $\Delta t,\Delta x\in(0,1)$, worth*

$$E^n=E^0,\quad\forall\, n=1,2,\ldots,N. \tag{3.21}$$

Remark 3.4. Numerical explicit time integration schemes such as those adopted in (3.3)–(3.7) are conditionally stable and, therefore, depend on a relationship between the mesh parameters Δt and Δx, known as a condition of Courant-Friedrichs-Lewy (CFL) [17, 23], given by

$$\Delta t \leq \frac{\Delta x}{\eta c}, \tag{3.22}$$

where $c := \max\left\{\sqrt{\alpha/\rho},\ \sqrt{\beta/\mu}\ \right\}$ is the maximum speed of the domain and η is a positive empirically chosen constant. Thus, this relationship works as a sufficient condition for guaranteeing the stability of the system (3.3)–(3.7). To be more precise about the stability criterion and to find a necessary and sufficient condition, we must make a more in-depth analysis based on Wright's work [8].

4. Numerical simulations

In this section, we present the results of numerical simulations of the finite difference methods (3.3)–(3.7) and of numerical energy E^n given in (3.8), as regards the solution and exponential decay. It is important to note that at this point, we are not interested in analyzing the issues of convergence of the numerical solution for the exact solution, because our intention here is just to illustrate with examples the numerical analytical results proven in the previous sections.

The main issue in the case of a finite-dimensional model is to determine the appropriate parameters to ensure the properties mentioned earlier and that the relationship (3.22) is fundamental.

For the numerical example, we consider $L = 3$, $\rho = 7.6 \cdot 10^3$, $\mu = 1.2 \cdot 10^{-6}$, $\gamma = 3 \cdot 10^{-4}$, $\beta = 1.9 \cdot 10^{-5}$ and $\delta = 1.8 \cdot 10^4$. In the initial conditions adopted

$$v_{0j} = \sin\left(\frac{(2n+1)\pi x_j}{2L}\right), \quad p_{0j} = 3\sin\left(\frac{(2n+1)\pi x_j}{2L}\right), \quad v_{1j} = p_{1j} = 0, \quad \text{with} \quad n \in \mathbb{Z}, \quad j = 0, 1, \ldots, J+1. \tag{4.1}$$

The graphics are given in Figure 2.

Using Matlab, we apply numerical methods (3.3)–(3.7) and (3.8) to plot the solution and the approximate numerical energy of the (1.15)–(1.19) in two distinct cases which will be described below. For this purpose, we choose $J = 300$, $T = 3$ and $N = 2390$. Below are the results.

4.1. Case without damping and internal damping in the mechanical equation

Here, we consider the numerical scheme (3.3)–(3.7) in two very different physical situations. In the first case, we consider a situation without damping, attributing the value $\delta = 0$ in internal dissipation δv_t in (3.3) and in the second case, we consider the model with the internal dissipation introduced by the term δv_t with $\delta > 0$.

4.1.1. Without internal damping

The following results show the conservative behavior of the numerical solution (v_j^n, p_j^n) over time t_n and the constant behavior of the numerical energy E^n in relation to the conservation law (see Cor. 3.3).

Comments I: In Figure 3, we have an overview of the evolution of the numerical solution over time. On the other hand, Figure 4 shows the numerical energy graph, which ensures a measure of precision of the explicit method to reproduce the conservative character of the solutions of the dynamic numerical model of piezoelectric beams with magnetic effect.

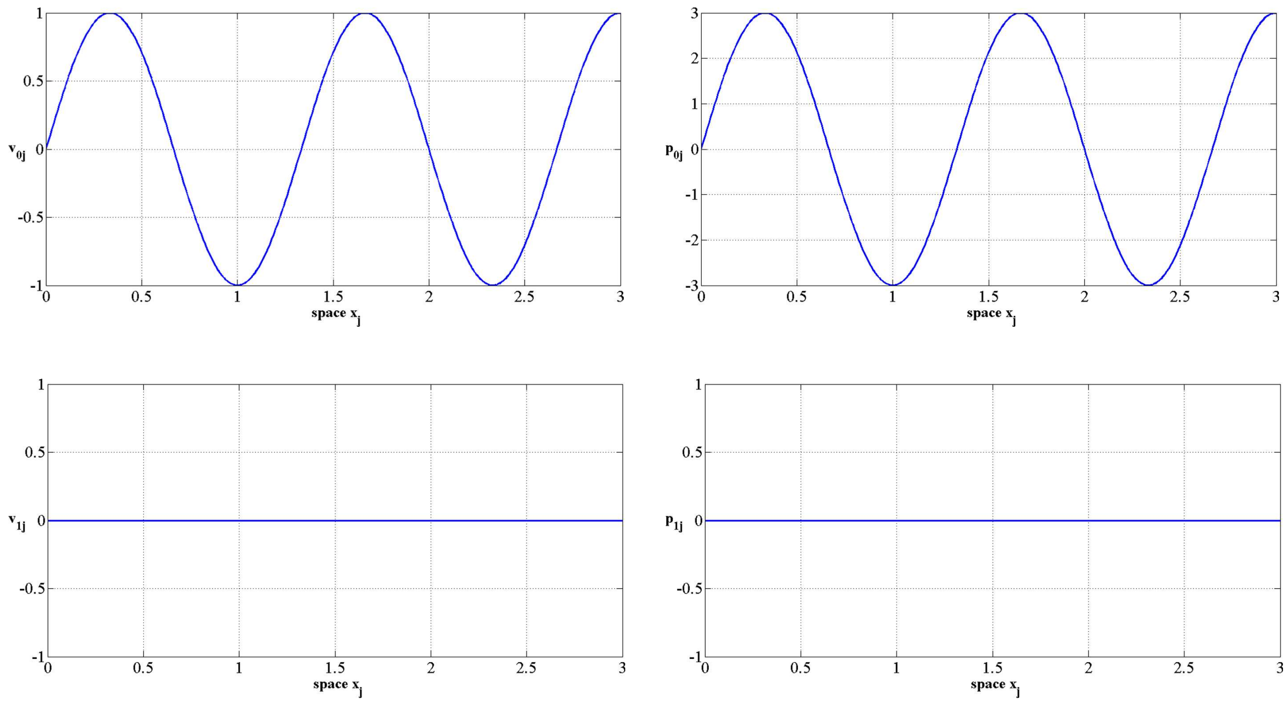

FIGURE 2. Graphics of initial conditions $v_{0j},\ p_{0j},\ v_{1j},\ p_{1j}$ respectively.

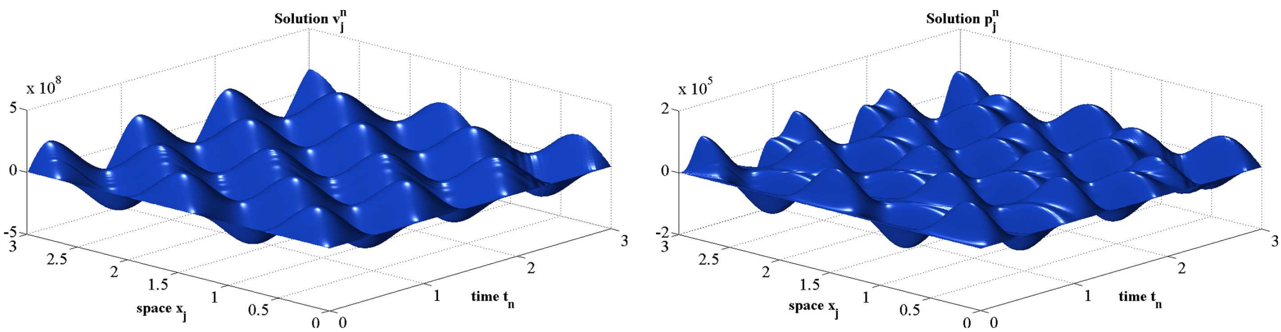

FIGURE 3. Longitudinal displacement v of the beam axis (*left side*) and the total electric displacement load p (*right side*).

4.1.2. With internal damping

The following results show the dissipative behavior of the numerical solution (v_j^n, p_j^n) over time t_n and the monotonous behavior decreasing to zero, of the numerical energy E^n in relation to the dissipation law (see Thm. 3.2).

Comments II: The graph of Figure 5, shows the dissipative behavior of the numerical solution in the presence of an internal damping introduced in the mechanical equation. In a similar way to the conservative case, we perceive Figure 6, that the numerical energy is capable of reproducing the dissipative configuration of the solution of the numerical model.

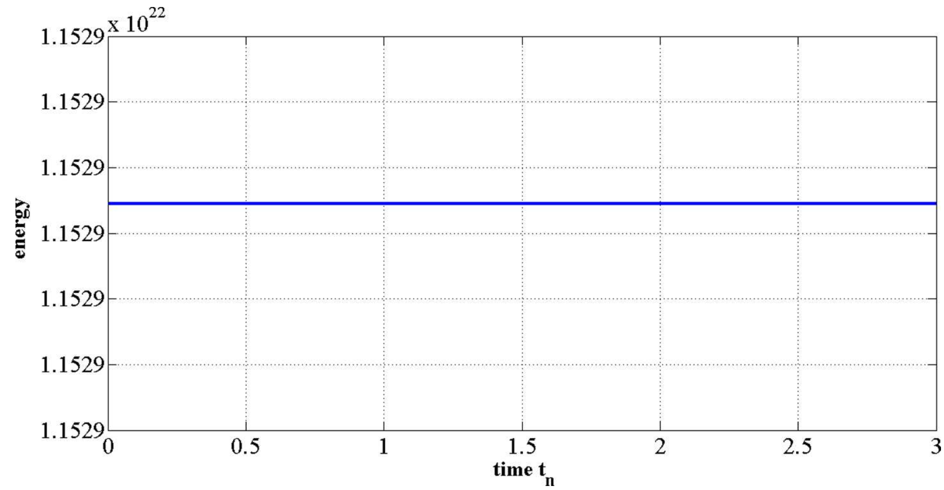

FIGURE 4. Numerical energy of the conservative system: case $\delta = 0$.

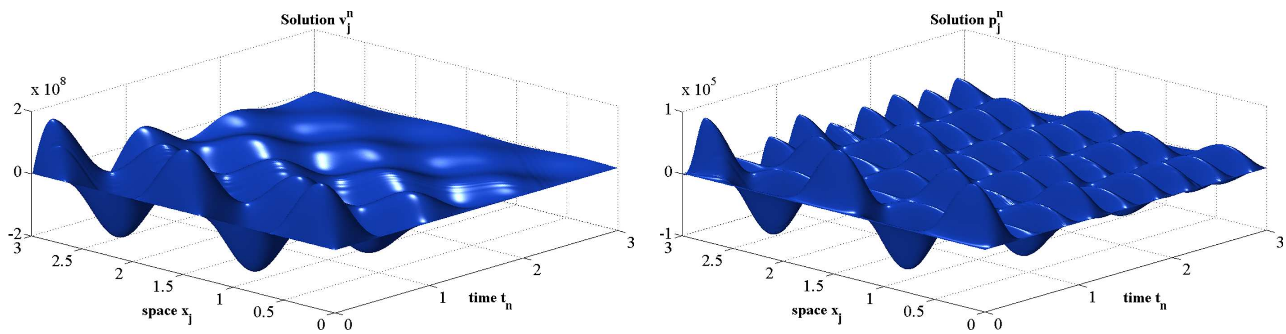

FIGURE 5. Longitudinal displacement v of the beam axis (*left side*) and the total electric displacement load p (*right side*).

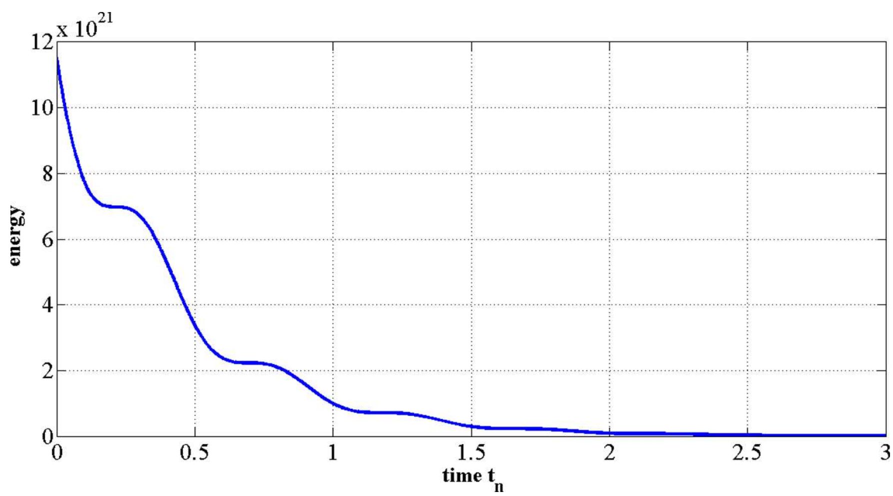

FIGURE 6. Numerical energy of the dissipative system: case $\delta > 0$.

The main objective of these simulations about the numerical solution and the decay of the numerical energy, is to empirically prove that the explicit numerical scheme of finite differences is robust enough to reproduce the properties present in the continuous, provided that due restriction is observed (3.22) between mesh parameters Δt and Δx.

5. Final comment and future research

Here, we show a dissipative system of piezoelectric beams with magnetic effect and we prove the exponential decay of the total energy at the level of the continuous. Thus, we discretized the equations of the system and analyzed the behavior of the numerical energy, where we perceive that the functional energy is not positive definite, different from what happens in the continuous; however, we show that this does not prevent it from being decreasing with $n \to \infty$.

Other issues will be discussed in the future. We highlight some of them.

- **Lack of numerical observability.** In a recent paper, Almeida Jr. *et al.* [1], proved the lack of numerical observability for a semi-discretized finite difference system applied to coupled-wave equations using multiplicative techniques adapted for the semi-discrete domain, combined with Fourier series developments. An open problem would be to analyze the lack of numerical observability for the conservative system of piezoelectric beams semi-discretized by finite differences, following the steps of these authors.
- **Polynomial stabilization.** Analyzing Theorem 3.2, more precisely the equation (3.19) we can consider some numerical boundary conditions. Among them, we highlight those that frequently occur in practice.

Boundary conditions: Type I

$$v_0^n = p_0^n = 0, \quad \forall\, n = 0, 1, \ldots, N+1, \tag{5.1}$$

$$\alpha\left(\frac{v_{J+1}^n - v_J^n}{\Delta x}\right) - \gamma\beta\left(\frac{p_{J+1}^n - p_J^n}{\Delta x}\right) = 0, \quad \forall\, n = 0, 1, \ldots, N+1, \tag{5.2}$$

$$\left(\frac{p_{J+1}^n - p_J^n}{\Delta x}\right) - \gamma\left(\frac{v_{J+1}^n - v_J^n}{\Delta x}\right) = 0, \quad \forall\, n = 0, 1, \ldots, N+1. \tag{5.3}$$

Boundary conditions: Type II

$$v_0^n = p_0^n = 0, \quad \forall\, n = 1, 2, \ldots, N, \tag{5.4}$$

$$\alpha\left(\frac{v_{J+1}^n - v_J^n}{\Delta x}\right) - \gamma\beta\left(\frac{p_{J+1}^n - p_J^n}{\Delta x}\right) = 0, \quad \forall\, n = 1, 2, \ldots, N, \tag{5.5}$$

$$\beta\left(\frac{p_{J+1}^n - p_J^n}{\Delta x}\right) - \gamma\beta\left(\frac{v_{J+1}^n - v_J^n}{\Delta x}\right) + \frac{1}{h}\left(\frac{p_{J+1}^{n+1} - p_{J+1}^{n-1}}{2\Delta t}\right) = 0, \quad \forall\, n = 1, 2, \ldots, N. \tag{5.6}$$

Note that the numerical boundary conditions (5.4)–(5.6) correspond to the analogous continuous (1.11)–(1.14) studied in [13]. Thus, the discretization of the system (1.11)–(1.14) can be obtained from our numerical scheme (3.3)–(3.4) assuming $\delta = 0$, together with the boundary conditions (5.4)–(5.6).

Another problem that we want to study in the future is the polynomial semi-discrete energy decay

$$\mathcal{E}_{\Delta x}(t) := \frac{\Delta x}{2}\sum_{j=0}^{J}\left[\frac{\rho}{3}|v_j'(t)|^2 + \frac{\mu}{3}|p_j'(t)|^2 + \frac{\rho}{6}\left|v_{j+1}'(t)+v_j'(t)\right|^2 + \frac{\mu}{6}\left|p_{j+1}'(t)+p_j'(t)\right|^2 \right.$$
$$\left. +\alpha_1\left|\frac{v_{j+1}(t)-v_j(t)}{\Delta x}\right|^2 + \beta\left|\gamma\frac{v_{j+1}(t)-v_j(t)}{\Delta x} - \frac{p_{j+1}(t)-p_j(t)}{\Delta x}\right|^2\right], \quad \forall t\geq 0 \tag{5.7}$$

associated to the semi-discretized problem by the finite element method

$$\rho\Theta\Big(v_j'(t)\Big) - \alpha\frac{v_{j+1}(t)-2v_j(t)+v_{j-1}(t)}{\Delta x^2} + \gamma\beta\frac{p_{j+1}(t)-2p_j(t)+p_{j-1}(t)}{\Delta x^2} = 0, \tag{5.8}$$

$$\mu\Theta\Big(p_j'(t)\Big) - \beta\frac{p_{j+1}(t)-2p_j(t)+p_{j-1}(t)}{\Delta x^2} + \gamma\beta\frac{v_{j+1}(t)-2v_j(t)+v_{j-1}(t)}{\Delta x^2} = 0, \tag{5.9}$$

$$v_0(t) = \alpha\left(\frac{v_{J+1}(t)-v_J(t)}{\Delta x}\right) - \gamma\beta\left(\frac{p_{J+1}(t)-p_J(t)}{\Delta x}\right) = 0, \tag{5.10}$$

$$p_0(t) = \beta\left(\frac{p_{J+1}(t)-p_J(t)}{\Delta x}\right) - \gamma\beta\left(\frac{v_{J+1}(t)-v_J(t)}{\Delta x}\right) + \frac{p_{J+1}'(t)}{h} = 0, \tag{5.11}$$

for all $j = 1,2,\ldots,J$ and $t\geq 0$, where $\Theta(\cdot)$ is an operator given by

$$\Theta\Big(v_j'(t)\Big) := \frac{2}{3}v_j'(t) + \frac{1}{6}\Big(v_{j+1}'(t)+v_{j-1}'(t)\Big), \quad \forall \quad j=1,2,\ldots,J \quad \text{and} \quad t\geq 0, \tag{5.12}$$

with an analogous expression for $\Theta\big(p_j'(t)\big)$. Here, the main question is whether the polynomial decay of the system is uniform with respect to the mesh parameter Δx for sufficiently regular initial data. We remind you that the (5.8)–(5.11) is obtained by an approximation of Galerkin of the continuous system (1.11)–(1.14), constructed using finite elements. See, for example, the paper by Infante and Zuazua (see [6], Sect. 3.1), where the authors studied an analogous semi-discretization in finite elements for the problem of lack of observability of the one-dimensional wave equation. Still speaking of polynomial stabilization, we find in the work of Banks *et al.* [2], some important notes about the polynomial decay for different numerical methods, among them, the authors highlight that the finite element method is one of the most suitable for studying the polynomial decay rate.

To conclude, we emphasize that the explicit numerical scheme, applied to piezoelectric beam systems with magnetic effect we analyzed, had never been studied before, therefore, this work constitutes the first contribution to the numerical study of this system.

Appendix A. Equivalence between the total energy and the Lyanupov functional

In this section, we have established the equivalence between the functional result of Lyapunov and the total energy of the system of piezoelectric beams (1.15)–(1.19). But we precisely prove the following result.

Lemma A.1. *For $N_1 > 0$ big enough, there are positive constants k_1 and k_2 such that of the functional*

$$\mathcal{L}(t) := N_1E(t) + N_2\mathcal{F}(t) + N_3\mathcal{G}(t) + N_4\mathcal{H}(t), \tag{A.1}$$

satisfies

$$k_1E(t) \leq \mathcal{L}(t) \leq k_2E(t), \quad \forall\ t\geq 0. \tag{A.2}$$

Proof. In fact, considering the functional $\mathcal{L}(t)$ defined above, it follows that

$$
\begin{aligned}
|\mathcal{L}(t) - N_1 E(t)| \leq N_2\rho \int_0^L |v_t v|\mathrm{d}x + N_2\gamma\mu \int_0^L |p_t v|\mathrm{d}x + \frac{N_2\delta}{2}\int_0^L |v|^2\mathrm{d}x \\
+N_3\rho \int_0^L |v_t(\gamma v - p)|\mathrm{d}x + N_3\gamma\mu \int_0^L |p_t(\gamma v - p)|\mathrm{d}x \\
+N_3\frac{\gamma\delta}{2}\int_0^L |v|^2\mathrm{d}x + N_3\delta \int_0^L |vp|\mathrm{d}x + N_4\rho\int_0^L |v_t v|\mathrm{d}x \\
+\frac{N_4\delta}{2}\int_0^L |v|^2\mathrm{d}x + N_4\mu\int_0^L |p_t p|\mathrm{d}x. \qquad \text{(A.3)}
\end{aligned}
$$

By applying Young and Poincaré's inequalities, we conclude that there is a constant $c > 0$ such that

$$|\mathcal{L}(t) - N_1 E(t)| \leq cE(t), \quad \forall\, t \geq 0. \qquad \text{(A.4)}$$

Consequently,

$$(N_1 - c)E(t) \leq \mathcal{L}(t) \leq (N_1 + c)E(t), \quad \forall\, t \geq 0. \qquad \text{(A.5)}$$

Choosing N_1 sufficiently large, that is, $N_1 := \max\Big\{c,\ \Big(N_2 C_{N_2} + N_3 C_1 + N_4\rho\Big)/\delta\Big\}$, (see Eq. (2.28)) we get the proof.

$\square$

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