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VALUE FUNCTION AND OPTIMAL TRAJECTORIES FOR A MAXIMUM RUNNING COST CONTROL PROBLEM WITH STATE CONSTRAINTS. APPLICATION TO AN ABORT LANDING PROBLEM[☆]

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Abstract. The aim of this article is to study the Hamilton Jacobi Bellman (HJB) approach for state-constrained control problems with maximum cost. In particular, we are interested in the characterization of the value functions of such problems and the analysis of the associated optimal trajectories, without assuming any controllability assumption. The rigorous theoretical results lead to several trajectory reconstruction procedures for which convergence results are also investigated. An application to a five-state aircraft abort landing problem is then considered, for which several numerical simulations are performed to analyse the relevance of the theoretical approach.

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1. Introduction

Let $T > 0$ be a finite time horizon and consider the following dynamical system:

$$\dot{\mathbf{y}}(s) = f(s, \mathbf{y}(s), \mathbf{u}(s)), \text{ a.e. } s \in (t, T), \tag{1.1a}$$

$$\mathbf{y}(t) = y, \tag{1.1b}$$

where $f : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}^d$ is a Lipschitz continuous function, U is a compact set, and $\mathbf{u} : [0, T] \to U$ is a measurable function. Denote $\mathbf{y}^{\mathbf{u}}_{t,y}$ the absolutely continuous solution of (1.1) associated to the control function $\mathbf{u}$ and with the initial position y at initial time $t \in [0, T]$. Let $\mathcal{K} \subset \mathbb{R}^d$ be a given non-empty closed set and

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consider the following control problem and its associated value function:

$$\vartheta(t,y) := \min_{\mathbf{u}\in L^\infty((t,T),U)} \left\{ \max_{s\in[t,T]} \Phi(s,\mathbf{y}^{\mathbf{u}}_{t,y}(s)) \bigvee \varphi(\mathbf{y}^{\mathbf{u}}_{t,y}(T)) \;\middle|\; \mathbf{y}^{\mathbf{u}}_{t,y}(s)\in\mathcal{K} \quad \forall s\in[t,T] \right\},$$

with the convention that $\inf\emptyset = +\infty$ and where the notation $a\bigvee b$ stands for $\max(a,b)$. The cost functions $\Phi : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ and $\varphi : \mathbb{R}^d \to \mathbb{R}$ are given Lipschitz continuous functions.

In the case when $\mathcal{K} = \mathbb{R}^d$, control problems with maximum cost have been already studied in the literature, for instance in [6, 7] where the control problem with maximum cost is approximated by a sequence of control problems with L^p-cost. Then the value function is characterized as the unique solution of a Hamilton-Jacobi-Bellman (HJB) equation. In [24], the case of lower semi-continuous cost function has been considered and the epigraph of the value function is characterized as a viability kernel for a specific dynamics.

In the general case where the set of state constraints $\mathcal{K}$ is a non-empty closed subset of $\mathbb{R}^d$ ($\mathcal{K} \subsetneq \mathbb{R}^d$), the value function is merely l.s.c. and its characterization as unique solution of a HJB equation requires some assumptions that involve an interplay between the dynamics f and the set of constraints $\mathcal{K}$. A most popular assumption, called inward pointing condition, has been introduced in [26] and requires, at each point of the boundary of $\mathcal{K}$, the existence of a control variable that lets the dynamics points in the interior of the set $\mathcal{K}$. This assumption, when it is satisfied, provides a nice framework for analysing the value function and also the optimal trajectories. However, in many applications the inward pointing condition is not satisfied and then the characterization of the value function as solution of a HJB equation becomes much more delicate, see for instance [1, 19].

Here we shall follow an idea introduced in [1] (in the context of Bolza problems) and that consists in characterizing the epigraph of ϑ by means of a Lipschitz continuous value function solution of an adequate unconstrained control problem. Actually, it is known that when the value function is only l.s.c., the characterization of its epigraph becomes much more relevant than the characterization of the graph (see for instance [4, 13, 14]). In the present work we consider a running cost problem, and we show that the epigraph of ϑ can be described by using a Lipschitz continuous value function of an auxiliary control problem free of state constraints. Furthermore, the auxiliary value function can be characterized as the unique Lipschitz continuous viscosity solution of a HJB equation. This HJB equation is posed on a neighborhood $\mathcal{K}_\eta$ of $\mathcal{K}$ with precise and rigorous Dirichlet boundary conditions. This new result turns out to be very useful for numerical purposes.

Another new contribution of the paper focuses on the analysis of the optimal trajectories associated with the state constrained optimal control problem with maximum cost. Several procedures for reconstruction of optimal trajectories are discussed. The auxiliary value function w being a Lipschitz continuous function, we show that the approximation procedures based on w provide a convergent sequence of sub-optimal trajectories for the original control problem. More precisely, we extend the result of [25] to the optimal control problem with maximum criterion and with state constraints (without imposing any controllability assumption on the set of constraints $\mathcal{K}$).

The theoretical study of this paper is then applied to an aircraft landing problem in presence of windshear. This meteorological phenomenon is defined as a difference in wind speed and/or direction over a relatively short distance in the atmosphere. This change of the wind affects the aircraft motion relative to the ground and it has more significant effects during the landing case. When landing, windshear is a hazard as it affects the aircraft motion relative to the ground, particularly when the winds are strong [11]. In a high altitude, the abort landing is probably the best strategy to avoid the failed landing. This procedure consists in steering the aircraft to the maximum altitude that can be reached in order to prevent a crash on the ground. In the references [21, 22] the authors propose a Chebyshev-type optimal control for which an approximate solution for the problem is derived along with the associated feedback control. This solution was improved in [11, 12] by considering the switching structure of the problem that has bang-bang subarcs and singular arcs.

Here, we consider the same problem formulation as in [11, 12, 21, 22]. The Hamilton Jacobi approach is used in order to characterize the value function and compute its numerical approximations. Next, we will reconstruct the associated optimal trajectories and feedback control using different algorithms of reconstruction. Let us mention some recent works [2, 10] where numerical analysis of the abort landing problem has been also investigated with a simplified model involving four-dimensional controlled systems. Here we consider the full five-dimensional control problem as in [11, 12]. Many simulations will be included in this paper involving data of a Boeing 727 aircraft model, see [11].

Notations. Throughout this paper, $|\cdot|$ is the Euclidean norm and $\langle\cdot,\cdot\rangle$ is the Euclidean inner product on $\mathbb{R}^N$ (for any $N \geq 1$). Let E be a Banach space, we denote by $\mathbb{B}_E$ the unit closed ball $\{x \in E : \|x\|_E \leq 1\}$ of E.

For any set $K \subseteq \mathbb{R}^d$, $\overline{K}$, ${}^cK$ and $\overset{\circ}{K}$ denote its closure, complement in $\mathbb{R}^d$ and interior, respectively. The distance function to K is $\mathrm{dist}(x, K) = \inf\{|x-y| : \ y \in K\}$. We will also use the notation d_K for the signed distance to K (*i.e.*, $\mathrm{d}_K(x) = -\mathrm{dist}(x, {}^cK)$ if $x \in K$ otherwise $\mathrm{d}_K(x) = \mathrm{dist}(x, K)$).

For any $a, b \in \mathbb{R}$, the notation $a \bigvee b$ stands for the $\max(a, b)$ while $a \bigwedge b$ stands for $\min(a, b)$. Finally, the notation $W^{1,1}([a, b])$ is used for the usual Sobolev space of functions $\{f \in L^1(a, b), f' \in L^1(a, b)\}$.

2. Setting and formulation of the problem

Let $T > 0$ be a fixed time horizon and consider the differential system obeying

$$\begin{cases} \dot{\mathbf{y}}(s) := f(s, \mathbf{y}(s), \mathbf{u}(s)), & \text{a.e} \ \ s \in (t, T), \\ \mathbf{y}(t) := y, \end{cases} \tag{2.1}$$

where $\mathbf{u}(\cdot)$ is a measurable function and the dynamics f satisfies:

$(\mathbf{H_1})$ $f : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}^d$ is continuous. For any $R > 0$, $\exists L_R \geq 0$ such that for every $u \in U$ and $s \in [0, T]$:

$$|f(s, y_1, u) - f(s, y_2, u)| \leq L_R(|y_1 - y_2|) \qquad \forall y_1, y_2 \in \mathbb{R}^d \text{ with } |y_1| \leq R, |y_2| \leq R.$$

A measurable function $\mathbf{u} : [0, T] \to \mathbb{R}^m$ is said admissible if $\mathbf{u}(s) \in U$, where U is a given compact subset of $\mathbb{R}^m$. The set of all admissible controls will be denoted by $\mathcal{U}$:

$$\mathcal{U} := \Big\{\mathbf{u} : (0, T) \to \mathbb{R}^m \text{ measurable}, \ \mathbf{u}(s) \in U \text{ a.e.}\Big\}.$$

Under assumption $(\mathbf{H_1})$, for any control $\mathbf{u} \in \mathcal{U}$, the differential equation (2.1) admits a unique absolutely continuous solution in $W^{1,1}([t, T])$. The set of all absolutely continuous solutions of (2.1) on $[t, T]$, starting from the position y at initial time t and associated to control functions in $\mathcal{U}$, is defined by:

$$S_{[t,T]}(y) := \{\mathbf{y}^{\mathbf{u}}_{t,y} \in W^{1,1}([t, T]), \ \ \mathbf{y}^{\mathbf{u}}_{t,y} \text{ solution of (2.1) associated to } \mathbf{u} \in \mathcal{U}\}.$$

Let $\mathcal{K} \subset \mathbb{R}^d$ be a closed subset of $\mathbb{R}^d$. For any $y \in \mathbb{R}^d$ and $t \in [0, T]$, a trajectory $\mathbf{y} \in S_{[t,T]}(y)$ will be said admissible on $[t, T]$ if and only if:

$$\forall \ s \in [t, T], \ \mathbf{y}(s) \in \mathcal{K}.$$

The set of all admissible trajectories on $[t, T]$ starting from the position y will be denoted by $S^{\mathcal{K}}_{[t,T]}(y)$:

$$S^{\mathcal{K}}_{[t,T]}(y) := \{\mathbf{y} \in S_{[t,T]}(y), \ \ \text{s.t.} \ \forall \ s \in [t, T], \ \mathbf{y}(s) \in \mathcal{K}\}.$$

This set may be empty if no trajectory can remain in the set $\mathcal{K}$ during the time interval $[t, T]$. Let us recall (see [3]) that under assumption $(\mathbf{H_1})$, the set-valued map $y \rightsquigarrow S_{[t,T]}(y)$ is locally Lipschitz continuous in the sense that for any $R > 0$, there exists some $L > 0$, $S_{[t,T]}(y_1) \subset S_{[t,T]}(y_2) + L|y_1 - y_2|\mathbb{B}_{W^{1,1}([0,T])}$ for all $y_1, y_2 \in \mathbb{R}^d$ with $|y_1| \leq R$ and $|y_2| \leq R$. This is no longer the case for the set-valued map $y \rightsquigarrow S^{\mathcal{K}}_{[t,T]}(y)$ even for simple sets $\mathcal{K}$ and linear dynamics f. Moreover, if we assume that:

$(\mathbf{H_2})$ for every $s \in [0,T]$ and $y \in \mathbb{R}^d$, the set $f(s,y,U) = \{f(s,y,u),\ u \in U\}$ is a convex set

then, by Filippov's Theorem, for every $y \in \mathbb{R}^d$, the set of trajectories $S_{[t,T]}(y)$ is a compact subset of $W^{1,1}([t,T])$ endowed with the C^0-topology.

Now, consider cost functions $\Phi : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ and $\varphi : R^d \to \mathbb{R}$ satisfying:

$(\mathbf{H_3})$ Φ is a Lipschitz continuous function on $[0,T] \times \mathbb{R}^d$ and φ is Lipschitz continuous on $\mathbb{R}^d$:

$$\exists L_\Phi \geq 0, \quad |\Phi(s,y) - \Phi(s',y')| \leq L_\Phi(|s-s'| + |y-y'|) \quad \forall s,s' \in [0,T],\ \forall y,y' \in \mathbb{R}^d;$$
$$\exists L_\varphi \geq 0, \quad |\varphi(y) - \varphi(y')| \leq L_\varphi |y-y'| \quad \forall y,y' \in \mathbb{R}^d.$$

In this paper, we are interested in the following control problem with supremum cost:

$$\vartheta(t,y) := \inf\left\{ \max_{s\in[t,T]} \Phi(s, \mathbf{y}^{\mathbf{u}}_{t,y}(s)) \bigvee \varphi(\mathbf{y}^{\mathbf{u}}_{t,y}(T)) \ \Big| \ \mathbf{y}^{\mathbf{u}}_{t,y} \in S^{\mathcal{K}}_{[t,T]}(y) \right\}, \tag{2.2}$$

where $\vartheta : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is the value function, and with the classical convention that $\inf\{\emptyset\} := +\infty$. The aim of this paper is to use Hamilton-Jacobi-Bellman (HJB) approach in order to describe the value function ϑ and to analyze some algorithms for reconstruction of optimal trajectories. Note that, in general (when $\mathcal{K} \neq \mathbb{R}^d$), the value function ϑ is discontinuous and its characterization as unique solution of a HJB equation may not be possible without further controllability assumptions, see [9, 18–20, 24–26]. In the present work, we shall follow an idea introduced in [1] to describe the epigraph of ϑ by using an auxiliary optimal control problem free of constraints whose value function is continuous.

3. Main results: characterization of the value function ϑ

This section is devoted to the characterization of the value function ϑ. For this, following [1], we shall first consider (in Sect. 3.1) an auxiliary control problem free of state constraints and whose value function can provide the value of ϑ. In this new control problem the state space is increased by one more component. We shall then characterize the auxiliary value function as the unique solution of a HJB equation posed on $[0,T] \times \mathbb{R}^d \times \mathbb{R}$. This characterization is similar to a result already proved in [1].

Furthermore, in the next Section 3.2, we shall prove that the auxiliary control problem can be adequately defined to get a characterization of the auxiliary value function by a HJB equation on $[0,T] \times \mathcal{K}_\eta \times \mathbb{R}$, where $\mathcal{K}_\eta$ is a neighbourhood of $\mathcal{K}$. This new HJB equation admits a Dirichlet condition on the boundary of $\mathcal{K}_\eta$. It is worth-noting that the Dirichlet condition is very useful for numerical approximation purposes.

In Section 3.3, we will also investigate the link between the auxiliary control problem and the "*reachability time*" problem that consists of maximizing the length of time interval during which a trajectory remains in the set of constraints $\mathcal{K}$ before leaving it. The *reachability time* function will be precisely defined in Section 3.3 and the link with the auxiliary value function will be investigated in the case of autonomous control problems.

Later, in Section 4, we shall study some new results for trajectory reconstruction using the auxiliary value function.

3.1. Auxiliary control problem free of state-constraints

First, consider the following augmented dynamics $\widehat{f}$ for $s \in [0,T]$, $u \in U$ and $\widehat{y} := (y,z) \in \mathbb{R}^d \times \mathbb{R}$:

$$\widehat{f}(s,\widehat{y},u) = \begin{pmatrix} f(s,y,u) \\ 0 \end{pmatrix}.$$

Let $\widehat{\mathbf{y}}(\cdot) := (\mathbf{y}^{\mathbf{u}}_{t,y}(\cdot), \mathbf{z}^{\mathbf{u}}_{t,y,z}(\cdot))$ (where $\mathbf{z}^{\mathbf{u}}_{t,y,z}(\cdot) \equiv z$) be the associated augmented solution of:

$$\dot{\widehat{\mathbf{y}}}(s) = \widehat{f}(s, \widehat{\mathbf{y}}(s), \mathbf{u}(s)), \quad s \in (t, T), \tag{3.1a}$$

$$\widehat{\mathbf{y}}(t) = (y, z)^T. \tag{3.1b}$$

Define the corresponding set of feasible trajectories, for $\widehat{y} = (y, z) \in \mathbb{R}^d \times \mathbb{R}$, by:

$$\widehat{S}_{[t,T]}(\widehat{y}) := \{\widehat{\mathbf{y}} = (\mathbf{y}^{\mathbf{u}}_{t,y}, \mathbf{z}^{\mathbf{u}}_{t,y,z}), \quad \widehat{\mathbf{y}} \text{ satisfies (3.1) for some } \mathbf{u} \in \mathcal{U}\}. \tag{3.2}$$

Remark 3.1. Under the assumptions $(\mathbf{H_1})$ and $(\mathbf{H_2})$, for every $\widehat{y} \in \mathbb{R}^d \times \mathbb{R}$, the set $\widehat{S}_{[0,T]}(\widehat{y})$ is a compact subset of $W^{1,1}([0,T])$ for the topology of $C([0,T]; \mathbb{R}^{d+1})$ (see [3]).

By adapting an idea introduced in [1], we define an auxiliary optimal control problem without state constraints whose value function can help to compute ϑ in an efficient manner. For this, we consider $g : \mathbb{R}^d \to \mathbb{R}$ a Lipschitz continuous function characterizing the constraints set $\mathcal{K}$ as follows:

$$\forall y \in \mathbb{R}^d, \; g(y) \le 0 \Leftrightarrow y \in \mathcal{K}. \tag{3.3}$$

In the sequel, we denote by $L_g > 0$ the Lipschitz constant of g. Note that a Lipschitz function g satisfying (3.3) always exists since $\mathcal{K}$ is a closed set (for instance the signed distance $\mathrm{d}_{\mathcal{K}}(\cdot)$ to $\mathcal{K}$ is a Lipschitz function that satisfies the condition (3.3)). Therefore, for $\mathbf{u} \in \mathcal{U}$, the following equivalence holds:

$$\mathbf{y}^{\mathbf{u}}_{t,y}(s) \in \mathcal{K}, \forall s \in [t,T] \;\Leftrightarrow\; \max_{s \in [t,T]} g(\mathbf{y}^{\mathbf{u}}_{t,y}(s)) \le 0. \tag{3.4}$$

Now, consider the auxiliary control problem and its value function w:

$$w(t,y,z) := \inf \left\{ \max_{s \in [t,T]} \varPsi(s, \mathbf{y}(s), \mathbf{z}(s)) \bigvee \Big(\varphi(\mathbf{y}(T)) - \mathbf{z}(T)\Big) \;\Big|\; \widehat{\mathbf{y}} = (\mathbf{y}, \mathbf{z}) \in \widehat{S}_{[t,T]}((y,z)) \right\}, \tag{3.5}$$

where for $(y, z) \in \mathbb{R}^d \times \mathbb{R}$, we define the function $\varPsi$ as:

$$\varPsi(s, y, z) := (\varPhi(s, y) - z) \bigvee g(y). \tag{3.6}$$

By definition, the function $\varPsi$ is Lipschitz continuous under assumption $(\mathbf{H_3})$. In the sequel, we shall denote by $L_\varPsi$ a bound of the Lipschitz constant for $\varPsi$. The following proposition shows that the level sets of this new value function w characterize the epigraph of ϑ.

Proposition 3.2. *Assume* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $(\mathbf{H_3})$. *The value function w is related to ϑ by the following relations, for every* $(t, y, z) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}$*:*

$$(i) \quad \vartheta(t,y) - z \le 0 \Leftrightarrow w(t,y,z) \le 0,$$

$$(ii) \quad \vartheta(t,y) = \min \Big\{ z \in \mathbb{R} \;,\; w(t,y,z) \le 0 \Big\}.$$

Proof. (i) Assume $\vartheta(t,y) \le z$. This implies first that $S^{\mathcal{K}}_{[t,T]}(y)$ is not empty and, by $(\mathbf{H_1})$–$(\mathbf{H_2})$, it is a compact subset of $W^{1,1}(0,T)$ (endowed with the C^0-topology). Thus there exists an admissible trajectory $\overline{\mathbf{y}} \in S^{\mathcal{K}}_{[0,T]}(y)$ such that

$$\max_{t \le s \le T} \Big(\varPhi(s, \overline{\mathbf{y}}(s)) - z\Big) \bigvee \Big(\varphi(\overline{\mathbf{y}}(T)) - z\Big) = \vartheta(t,y) - z \le 0.$$

By using (3.4), we obtain:

$$w(t,y,z) \le \max_{t\le s\le T}\Big(\Phi(s,\overline{\mathbf{y}}(s)) - z\Big) \bigvee \max_{t\le s\le T} g(\overline{\mathbf{y}}(s)) \bigvee \Big(\varphi(\overline{\mathbf{y}}(T)) - z\Big) \le 0.$$

Conversely, assume $w(t,y,z)\le 0$. By Remark 3.1, there exists a trajectory $\widehat{\mathbf{y}} = (\mathbf{y},\mathbf{z}) \in \widehat{S}_{[t,T]}(y,z)$ starting from $\widehat{y} = (y,z)$ such that

$$0 \ge w(t,y,z) = \max_{t\le s\le T} \Psi(s,\mathbf{y}(s),z) \bigvee \left(\varphi(\mathbf{y}(T))\right),$$

which gives:

$$\left(\max_{t\le s\le T}\Phi(s,\mathbf{y}(s)) \bigvee \varphi(\mathbf{y}(T))\right) \le z, \quad \text{ and } \quad \max_{t\le s\le T} g(\mathbf{y}(s)) \le 0.$$

It follows that $\mathbf{y}$ is admissible on $[t,T]$ and $\vartheta(t,y)\le z$. This ends the proof of (i). Assertion (ii) follows directly from (i). □

Remark 3.3. Note that the value function $\vartheta(t,.)$ is l.s.c. and then its epigraph is a closed set. Moreover, from Proposition 3.2, for every $t\in[0,T]$:

$$Epi\big(\vartheta(t,.)\big) = \Big\{(y,z)\in\mathcal{K}\times\mathbb{R} \mid w(t,y,z)\le 0\Big\}.$$

The value function w enjoys more regularity properties. It is then more interesting to characterize first w and then to recover the values of ϑ from those of w.

Proposition 3.4. *Assume* $(\mathbf{H_1})$ *and* $(\mathbf{H_3})$ *hold.*
(i) The value function w *is locally Lipschitz continuous on* $[0,T]\times\mathbb{R}^d\times\mathbb{R}$.
(ii) For any $t\in[0,T]$, $h\ge 0$, *such that* $t+h\le T$,

$$w(t,y,z) = \inf_{\widehat{\mathbf{y}}:=(\mathbf{y},\mathbf{z})\in\widehat{S}_{[t,t+h]}(y,z)} \Big\{ w(t+h,\mathbf{y}(t+h),z) \bigvee \max_{s\in[t,t+h]} \Psi(s,\mathbf{y}(s),z)\Big\}.$$

(iii) Furthermore, the function w *is the unique continuous viscosity solution of the following HJ equation:*

$$\min\Big(-\partial_t w(t,y,z) + H(t,y,\nabla_y w)\ ,\ w(t,y,z) - \Psi(t,y,z)\Big) = 0 \qquad in\ \ [0,T[\times\mathbb{R}^d\times\mathbb{R}, \tag{3.7a}$$

$$w(T,y,z) = \Psi(T,y,z)\bigvee\left(\varphi(y) - z\right) \qquad in\ \mathbb{R}^d\times\mathbb{R}, \tag{3.7b}$$

where the Hamiltonian H *is defined, for* $y,p\in\mathbb{R}^d$ *and* $t\in[0,T]$ *by:*

$$H(t,y,p) := \sup_{u\in U}\big(-f(t,y,u)\cdot p\big), \tag{3.8}$$

and the notations $\partial_t w$ *and* $\nabla_y w$ *stand for the partial derivatives of* w *with respect to the variable* t *and* y*, respectively.*

Proof. The proof of the local Lipschitz continuity can be obtained as in Proposition 3.3 from [1].

The dynamic programming principle (stated in (ii)) is a classical result and its proof can be found in [1, 7] where a HJB equation is also derived for the value function associated to a control problem with maximum cost. Besides, the uniqueness result is shown in Appendix A from [1].

Finally, note that the characterization of the function w does not require assumption $(\mathbf{H_2})$ to be satisfied. If $(\mathbf{H_2})$ happens to be fulfilled, then $\widehat{S}_{[t,t+h]}(y,z)$ is a compact subset of $W^{1,1}([t,t+h])$ and therefore the minimum in the dynamic Programming principle, stated in Proposition 3.4 (ii), is achieved. □

3.2. A particular choice of g

The main feature of the auxiliary control problem consists on that it is free of state constraints. However, the new control problem involves one more state component, and the HJB equation that characterizes w is defined on $\mathbb{R}^d \times \mathbb{R}$. To restrict the domain of interest for w to a neighbourhood of $\mathcal{K} \times \mathbb{R}$, it is possible to define w with a more specific function g so that the auxiliary value function w keeps a constant value outside a neighbourhood of $\mathcal{K} \times \mathbb{R}$.

Indeed, in all the sequel, let $\eta > 0$ be a fixed parameter and define a neighbourhood $\mathcal{K}_\eta$ of $\mathcal{K}$ by:

$$K_\eta := K + \eta \mathbb{B}_{\mathbb{R}^d}. \tag{3.9}$$

Consider a Lipschitz continuous function g_η satisfying, for $y \in \mathbb{R}^d$:

$$g_\eta(y) \le 0 \Leftrightarrow y \in \mathcal{K}, \quad g_\eta(y) \le \eta \;\; \forall y \in \mathbb{R}^d \quad \text{and} \quad g_\eta(y) = \eta \;\; \forall y \notin \mathcal{K}_\eta. \tag{3.10}$$

Such a Lipschitz function always exists since $\mathcal{K}$ is a closed set. For instance, g_η can be defined as:

$$g_\eta(y) := \mathrm{d}_{\mathcal{K}}(y) \bigwedge \eta, \quad \text{for any } y \in \mathbb{R}^d.$$

Now, we consider also a truncation of Ψ given by

$$\Psi_\eta(s,y,z); = \Big((\Phi(s,y) - z) \bigwedge \eta \Big) \bigvee g_\eta(y). \tag{3.11}$$

Clearly, by definition of Ψ_η and with (3.6), we have: $\Psi_\eta(s,y,z) = \Psi(s,y,z) \bigwedge \eta$.

Furthermore, we introduce a truncated final cost φ_η by:

$$\varphi_\eta(y,z) := \Big(\varphi(y) - z \Big) \bigwedge \eta. \tag{3.12}$$

Finally, we define the value function w_η, for $\widehat{y} = (y,z) \in \mathbb{R}^d \times \mathbb{R}$, as:

$$w_\eta(t,y,z) := \inf_{(\mathbf{y},\mathbf{z}) \in \widehat{S}_{[t,T]}(\widehat{y})} \left[\max_{s \in [t,T]} \Psi_\eta(s, \mathbf{y}(s), \mathbf{z}(s)) \bigvee \varphi_\eta(\mathbf{y}(T), \mathbf{z}(T)) \right]. \tag{3.13}$$

Note that with the above definitions and if $g_\eta(y) := g(y) \bigwedge \eta$ where g is defined as in (3.3), then the new value function w_η satisfies:

$$w_\eta(t,y,z) = w(t,y,z) \bigwedge \eta, \quad \forall (t,y,z) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}.$$

The epigraph of ϑ can be also characterized by the function w_η, and under assumptions $(\mathbf{H_1})$, $(\mathbf{H_2})$ and $(\mathbf{H_3})$, all statements of Proposition 3.2 are still valid with w_η defined as in (3.13).

Now, let us emphasize that the function w_η has been defined in such a way it takes a constant value outside $\mathcal{K}_\eta$. This information can be used as a Dirichlet boundary condition in the HJB equation satisfied by w_η.

Theorem 3.5. *Assume* $(\mathbf{H_1})$, $(\mathbf{H_3})$ *hold. Let* g_η, Ψ_η, φ_η *and* w_η *defined as in* (3.10), (3.11), (3.12) *and* (3.13) *respectively.*

The value function w_η *is the unique continuous viscosity solution of the following Hamilton Jacobi equation:*

$$\min\Big(-\partial_t w_\eta(t,y,z) + H(t,y,\nabla_y w_\eta)\ ,\ w_\eta(t,y,z) - \Psi_\eta(t,y,z)\Big) = 0, \qquad \textit{in}\ \ [0,T[\times \overset{\circ}{\mathcal{K}_\eta} \times\mathbb{R}, \tag{3.14a}$$

$$w_\eta(T,y,z) = \Psi_\eta(T,y,z) \bigvee \varphi_\eta(y,z), \qquad \textit{in}\ \ \overset{\circ}{\mathcal{K}_\eta} \times\mathbb{R}, \tag{3.14b}$$

$$w_\eta(t,y,z) = \eta, \qquad \textit{for all}\ t\in[0,T],\ y \notin \overset{\circ}{\mathcal{K}_\eta}\ \ \textit{and}\ \ \ z\in\mathbb{R}. \tag{3.14c}$$

Proof. Equations (3.14a) and (3.14b) are obtained as in Proposition 3.4 (iii). Let us prove assertion (3.14c). First, notice that:

$$\eta \geq w_\eta(t,y,z) \geq \Psi_\eta(t,y,z) \geq g_\eta(y) \quad \forall t\in[0,T],\ y\in\mathbb{R}^d,\ z\in\mathbb{R}.$$

Moreover, by definition of g_η, for any $y \notin \overset{\circ}{\mathcal{K}_\eta}$, we have $g_\eta(y) = \eta$. It follows that

$$w_\eta(t,y,z) = \eta \quad \forall y \notin \overset{\circ}{\mathcal{K}_\eta}\ .$$

This concludes the proof. □

Remark 3.6. If the cost function Φ is bounded and satisfies:

$$\Phi(y) \in [\underline{m}, \overline{M}], \quad \forall y\in\mathcal{K}_\eta,$$

then, it suffices to consider the variable z in the interval $[\underline{m},\overline{M}]$. Indeed, in this case, we still have the relation:

$$\vartheta(t,y) = \inf\Big\{ z\in[\underline{m},\overline{M}]\ |\ w_\eta(t,y,z)\leq 0\Big\}.$$

In addition, the function w_η is the unique continuous viscosity solution of the following HJ equation:

$$\min\Big(-\partial_t w_\eta(t,y,z) + H(t,y,\nabla_y w_\eta), w_\eta(t,y,z) - \Psi_\eta(t,y,z)\Big) = 0 \qquad \text{in}\ \ [0,T[\times \overset{\circ}{\mathcal{K}_\eta} \times[\underline{m},\overline{M}], \tag{3.15a}$$

$$w_\eta(T,y,z) = \Psi_\eta(T,y,z) \bigvee \varphi_\eta(y,z) \qquad \text{in}\ \ \overset{\circ}{\mathcal{K}_\eta} \times[\underline{m},\overline{M}], \tag{3.15b}$$

$$w_\eta(t,y,z) = \eta \qquad \text{for all}\ t\in[0,T],\ y\notin \overset{\circ}{\mathcal{K}_\eta}\ \ \text{and}\ \ \ z\in[\underline{m},\overline{M}]. \tag{3.15c}$$

Let us point on that there is no need for any boundary condition on the z-axis because the dynamics is zero $\dot{\mathbf{z}}(t) = 0$.

Remark 3.7. The function w_η depends on the choice of the parameter η. However, the region of interest ($\{w_\eta \leq 0\}$) is always the same, for any $\eta > 0$, and the characterization of the original value $\vartheta(t,x) = \min\{z,\ w_\eta(t,x,z) \leq 0\}$ holds for any $\eta > 0$. In the sequel, we will denote by w any auxiliary value function corresponding to an adequate function g (or g_η).

3.3. Case of autonomous control problems: link with the reachability time function

The aim of this subsection is to make a link between the control problem discussed in the previous subsection and an optimal *reachability time* that we will define correctly in the following. This link can be established in a general case, however it turns out to be of a particular interest when the control problem is autonomous. This interest will be clarified throughout this section. Here, we consider that all the functions involved in the control problem (2.2) are time independent (*i.e.*, $f(t,x,u) = f(x,u)$ and $\Phi(t,x) = \Phi(x)$), and introduce the sets:

$$\begin{aligned}
&\mathcal{D} := \left\{ \widehat{y} = (y,z) \in \mathbb{R}^{d+1} \;\middle|\; y \in \mathcal{K} \quad \text{and} \quad \widehat{y} \in \mathcal{E}pi(\Phi) \right\} = \mathcal{E}pi(\Phi) \cap (\mathcal{K} \times \mathbb{R}),\\
&\mathcal{C} := \mathcal{E}pi(\varphi).
\end{aligned}$$

Let us define also the *reachability time* function $\mathcal{T} : \mathbb{R}^{d+1} \to [0,T]$, which associates to each initial position $\widehat{y} = (y,z) \in \mathbb{R}^{d+1}$, the first time $t \in [0,T]$ such that there exists an admissible trajectory $\widehat{\mathbf{y}} \in \widehat{\mathcal{S}}_{[t,T]}(\widehat{y})$ remaining in $\mathcal{E}pi(\Phi) \bigcap \mathcal{K}$ and that reaches $\mathcal{E}pi(\varphi)$ at time T:

$$\mathcal{T}(y,z) := \inf \left\{ t \in [0,T] \;\middle|\; \exists \widehat{\mathbf{y}} \in \widehat{\mathcal{S}}_{[t,T]}(\widehat{y}) \quad s.t. \quad \widehat{\mathbf{y}}(s) \in \mathcal{D},\ \forall s \in [t,T], \quad \text{and} \quad \widehat{\mathbf{y}}(T) \in \mathcal{C} \right\}. \tag{3.16}$$

Remark 3.8. Let us point out that, from the definition of $\mathcal{D}$, one can easily check that the two following assertions are equivalent.

(a) there exists $\widehat{\mathbf{y}} = (\mathbf{y},\mathbf{z}) \in \widehat{S}_{[t,T]}(\widehat{y})$ such that: $\widehat{\mathbf{y}}(s) \in \mathcal{D}$ for every $s \in [t,T]$ and $\widehat{\mathbf{y}}(T) \in \mathcal{C}$;
(b) there exists $\widehat{\mathbf{y}} = (\mathbf{y},\mathbf{z}) \in \widehat{S}_{[t,T]}(\widehat{y})$ such that: $\max\limits_{s\in[t,T]} \Phi(s,\mathbf{y}(s)) \bigvee \varphi(\mathbf{y}(T)) \le z$ and $\mathbf{y}(s) \in \mathcal{K}\ \forall s \in [t,T]$.

The following theorem gives a link between the value functions w, ϑ and the *reachability time* function $\mathcal{T}$.

Theorem 3.9. *Assume* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $(\mathbf{H_3})$ *hold. Then we have:*

$$\begin{aligned}
&(i) \quad \mathcal{T}(y,z) = \inf \left\{ t \in [0,T] \;\middle|\; w(t,y,z) \le 0 \right\},\\
&(ii) \quad \mathcal{T}(y,z) = t \Rightarrow\ w(t,y,z) = 0,\\
&(iii) \quad \vartheta(t,y) = \inf \left\{ z \;\middle|\; \mathcal{T}(y,z) \le t \right\}.
\end{aligned}$$

Proof. Let $\widehat{y} = (y,z)$ be in $\mathbb{R}^d \times \mathbb{R}$. Let $t \in [0,T]$ such that $w(t,y,z) \le 0$. Then there exists $\widehat{\mathbf{y}} = (\mathbf{y},\mathbf{z}) \in \widehat{S}_{[t,T]}(\widehat{y})$ such that: $\max\limits_{s\in[t,T]} \Phi(\mathbf{y}(s)) \le z$ and $\mathbf{y}(s) \in \mathcal{K}$ for every $s \in [t,T]$. This implies that there exists $\widehat{\mathbf{y}} = (\mathbf{y},\mathbf{z}) \in \widehat{S}_{[t,T]}(\widehat{y})$ such that:

$$\widehat{\mathbf{y}}(s) \in \mathcal{D}, \quad \forall s \in [t,T],$$

which proves that $\mathcal{T}(y,z) \le t$. Therefore, $\mathcal{T}(y,z) \le \inf \left\{ t \in [0,T] \;\middle|\; w(t,y,z) \le 0 \right\}$.

Now, let $t := \mathcal{T}(y,z)$ and assume that $t < \infty$. By definition of $\mathcal{T}$ and Remark 3.8, there exists an admissible trajectory $\widehat{\mathbf{y}}_{\widehat{y}} = (\mathbf{y},\mathbf{z}) \in \widehat{S}_{[t,T]}(\widehat{y})$ such that

$$\max_{s\in[t,T]} \Phi(\mathbf{y}(s)) - z \le 0, \qquad \varphi(\mathbf{y}(T)) - z \le 0, \qquad \text{and} \qquad \max_{s\in[t,T]} g(\mathbf{y}(s)) \le 0.$$

This implies that $w(t,y,z) \le 0$, and then the proof of (i) is completed. Furthermore, for any $\tau < t$ we have $w(\tau,y,z) > 0$ (since otherwise we would have $t = \mathcal{T}(x,z) \le \tau$). Then by continuity of w, we conclude that $w(t,y,z) = 0$.

It remains to prove claim (iii). For this, note that in the autonomous case the value function $w(\cdot, y, z)$ is decreasing in time. This property along with assertion (i) yield to:

$$\mathcal{T}(y,z) \leq t \Longleftrightarrow w(t,y,z) \leq 0. \tag{3.17}$$

Thus, statement (iii) follows from the fact that:

$$\vartheta(t,y) = \inf\left\{z \;\middle|\; w(t,y,z) \leq 0\right\}.$$

□

Remark 3.10. Statement (iii) of Theorem 3.9 is no more valid if the problem is non-autonomous. Actually, in this case the equivalence (3.17) would not be true and only the implication:

$$w(t,y,z) \leq 0 \Longrightarrow \mathcal{T}(y,z) \leq t \tag{3.18}$$

is fulfilled. Indeed, the reverse implication of (3.18) may fail to be true in the non-autonomous case since the function $w(\cdot, y, z)$ can change signs several times over the time interval (while in the autonomous case the function $w(\cdot, y, z)$ can only change sign from positive to negative once during the time interval $[0, T]$).

Remark 3.11. For numerical purposes, the function $\mathcal{T}$ presents a major feature as it allows to recover the values of the original function ϑ. There is no need to store the function w on a grid of $d+2$ dimensions (d for the state components y + variable z and the time variable). Indeed, it is sufficient to store the values of the reachability time function on a grid of $d+1$ dimensions.

In the following, a link is established between an optimal trajectory associated with the original state-constrained control problem, an optimal trajectory for the auxiliary control problem, and an optimal trajectory of the optimal reachability time function.

Proposition 3.12. *Assume* ($\mathbf{H_1}$), ($\mathbf{H_2}$) *and* ($\mathbf{H_3}$) *hold. Let* $y \in \mathcal{K}$ *and* $t \in [0,T]$ *such that* $\vartheta(t,y) < \infty$. *Define* $z := \vartheta(t,y)$.

(i) Let $\widehat{\mathbf{y}}^* = (\mathbf{y}^*, \mathbf{z}^*)$ *be an optimal trajectory for the auxiliary control problem* (3.5) *on* $[t,T]$ *associated with the initial point* (y,z). *Then,* $\mathbf{z}^*(s) \equiv z$ *on* $[t,T]$, *and the trajectory* $\mathbf{y}^*$ *is optimal for the control problem* (2.2) *on* $[t,T]$ *associated to the initial position* y.

(ii) Assume that $t = \mathcal{T}(y,z)$. *Let* $\widehat{\mathbf{y}}^* = (\mathbf{y}^*, \mathbf{z}^*)$ *be an optimal trajectory for the reachability problem* (3.16) *associated with the initial point* $(y,z) \in \mathcal{K} \times \mathbb{R}$. *Then,* $\widehat{\mathbf{y}}^*$ *is also optimal for the auxiliary control problem* (3.5).

Proof. Let $(y,z) \in \mathcal{K} \times \mathbb{R}$ such that $\vartheta(t,y) = z$.

(i) Let $\widehat{\mathbf{y}}^* = (\mathbf{y}^*, \mathbf{z}^*)$ be an optimal trajectory for the auxiliary control problem (3.5) associated with the initial point $(y,z) \in \mathcal{K} \times \mathbb{R}$. Using Proposition 3.2, we have that

$$\vartheta(t,y) = z \Rightarrow w(t,y,z) \leq 0.$$

It follows that

$$w(t,y,z) = \max_{s \in [t,T]} \Psi(\mathbf{y}^*(s), z) \bigvee (\varphi(\mathbf{y}^*(T)) - z) \leq 0.$$

Using the definition of Ψ, we get,

$$\max_{s \in [t,T]} \Phi(\mathbf{y}^*(s)) \leq z, \quad \varphi(\mathbf{y}^*(T)) \leq z \;\text{ and }\; \max_{s \in [t,T]} g(\mathbf{y}^*(s)) \leq 0.$$

Since $\vartheta(t,y)=z$, it follows that:

$$\max_{s\in[t,T]} \Phi(\mathbf{y}^*(s)) \bigvee \varphi(\mathbf{y}^*(T)) \leq \vartheta(t,x) \quad \text{and} \quad \mathbf{y}^*(s)\in\mathcal{K},\ \forall s\in[t,T].$$

By definition of ϑ one can conclude that

$$\vartheta(t,x) = \max_{s\in[t,T]} \Phi(\mathbf{y}^*(s)) \bigvee \varphi(\mathbf{y}^*(T)) \quad \text{and} \quad \mathbf{y}^*(s)\in\mathcal{K},\ \forall s\in[t,T].$$

Therefore, $\mathbf{y}^*$ is an optimal trajectory for (2.2) with the initial position y and the proof of assertion (i) is achieved.

(ii) Assume that $t=\mathcal{T}(y,z)$ and let $\widehat{\mathbf{y}}^*=(\mathbf{y}^*,\mathbf{z}^*)$ be an optimal trajectory for problem (3.16) associated with the initial point (y,z). It follows from the definition of $\mathcal{T}$ that,

$$\widehat{\mathbf{y}}^*(s) := (\mathbf{y}^*(s),\mathbf{z}^*(s))\in\mathcal{D},\ \forall s\in[t,T], \quad \text{and } \mathbf{y}^*(T)\in\mathcal{C}.$$

Then, we have,

$$\max_{s\in[t,T]} \Phi(\mathbf{y}^*(s)) \bigvee \varphi(\mathbf{y}^*(T)) \leq z, \qquad \text{and} \quad \max_{s\in[t,T]} g(\mathbf{y}^*(s)) \leq 0.$$

Since $\vartheta(t,y)=z$ and by definition of g, we obtain that

$$\max_{s\in[t,T]} \Phi(\mathbf{y}^*(s)) \bigvee \varphi(\mathbf{y}^*(T)) \leq \vartheta(t,x) \quad \text{and} \quad \mathbf{y}^*(s)\in\mathcal{K},\ \forall s\in[t,T].$$

We conclude that $\mathbf{y}^*$ is an optimal trajectory for (2.2) on the time interval $[t,T]$ with the initial position y. $\square$

Remark 3.13. For the sake of clarity, we have chosen to state Proposition 3.12 under the assumption that the value of $\vartheta(t,y)$ is known. By Proposition 3.2, Remark (3.6) and Theorem 3.9, we know that this value can be obtained from the auxiliary function w or by the reachability time function $\mathcal{T}$.

Assertion (i) of Proposition 3.12 states that each optimal trajectory for the auxiliary control problem corresponds to an optimal solution of the original problem. The converse is also true. More precisely, let $\mathbf{y}^*$ be an optimal trajectory for the control problem (2.2) on $[t,T]$ associated to the initial position y. Then by setting $\mathbf{z}^*(s)\equiv z$ on $[t,T]$, the augmented trajectory $\widehat{\mathbf{y}}^*=(\mathbf{y}^*,\mathbf{z}^*)$ is an optimal trajectory for the auxiliary control problem (3.5), on the time interval $[t,T]$, with the initial point (y,z).

4. Reconstruction procedure based on the value function

In the case of Bolza or Mayer optimal control problems, reconstruction algorithms were proposed for instance in Appendix A from [5] or in [25]. In our setting, the control problem involves a maximum cost function, we shall discuss a reconstruction procedure based on the knowledge of the auxiliary value function w or an approximation of it. For simplicity, we consider the trajectory reconstruction on the time interval $[0,T]$. However, all the results remain valid for a reconstruction on any subinterval $[t,T]$.

Consider a numerical approximation f_h of the dynamics f such that, for every $R>0$, we have:

$$|f_h(t,x,u)-f(t,x,u)| \leq C_R h, \qquad \forall t\in[0,T],\ |x|\leq R, u\in U, \tag{4.1}$$

where the constant C_R is independent of $h \in [0,1]$. Hence, an approximation scheme for the differential equation $\dot{\mathbf{y}}(t) = f(t, \mathbf{y}(t), u)$ (for a constant control u, discrete times s_k and time step $h_k = s_{k+1} - s_k$) can be written

$$y_{k+1} = y_k + h_k f_{h_k}(s_k, y_k, u), \quad k \geq 0. \tag{4.2}$$

The case of the Euler forward scheme corresponds to the choice

$$f_h := f.$$

Higher order Runge Kutta schemes can also be written as (4.2) and with a function f_h satisfying (4.1). For instance, the Heun scheme corresponds to the choice

$$f_h(t, y, u) := \frac{1}{2}(f(t, y, u) + f(t + h, y + hf(t, y, u), u)).$$

Now, consider also, for each $h > 0$, a function w^h being an approximation of the value function w, and define E_h as a uniform bound on the error:

$$|w^h(t, y, z) - w(t, y, z)| \leq E_h, \quad \forall t \in [0, T], |y| \leq R, |z| \leq R,$$

with $R > 0$ large enough. The function w^h could be a numerical approximation obtained by solving a discretized form of the HJB equation.

Algorithm 4.1. Fix $y \in \mathbb{R}^d$ and $z \in \mathbb{R}$. For $h > 0$ we consider an integer $n_h \in \mathbb{N}$, a partition $s_0 = 0 < s_1 < \cdots < s_{n_h} = T$ of $[0, T]$, denote $h_k := s_{k+1} - s_k$ and assume that

$$\max_k h_k \leq h.$$

We define the positions $(y^h_k)_{k=0,\ldots,n_h}$, and control values $(u^h_k)_{k=0,\ldots,n_h-1}$, by recursion as follows. First we set $y^h_0 := y$. Then for $k = 0, \ldots, n_h - 1$, knowing the state y^h_k we define

(i) an optimal control value $u^h_k \in U$ such that

$$u^h_k \in \operatorname*{argmin}_{u \in U} w^h\big(s_k, y^h_k + h_k\, f_h(s_k, y^h_k, u), z\big) \bigvee \Psi(s_k, y^h_k, z) \tag{4.3}$$

(ii) a new state position y^h_{k+1}

$$y^h_{k+1} := y^h_k + h_k f_h(s_k, y^h_k, u^h_k). \tag{4.4}$$

Note that in (4.3) the value of u^h_k can also be defined as a minimizer of $u \to w^h\big(s_k, y^h_k + h_k\, f_h(s_k, y^h_k, u), z\big)$, since this will imply in turn to be a minimizer of (4.3).

We also associate to this sequence of controls a piecewise constant control $\mathbf{u}^h(s) := u^h_k$ on $s \in [s_k, s_{k+1}[$, and an approximate trajectory $\mathbf{y}^h$ such that

$$\dot{\mathbf{y}}^h(s) = f_h(s_k, y^h_k, u^h_k) \quad \text{a.e} \;\; s \in (s_k, s_{k+1}), \tag{4.5a}$$
$$\mathbf{y}^h(s_k) = y^h_k. \tag{4.5b}$$

In particular the value of $\mathbf{y}^h(s_{k+1})$ obtained by (4.5a) and (4.5b) does correspond to y^h_{k+1} as defined in (4.4) (notice that in general $\mathbf{y}^{\boldsymbol{u}^h} \neq \mathbf{y}^h$).

We shall show that any cluster point $\bar{\mathbf{y}}$ of $(\mathbf{y}^h)_{h>0}$ is an optimal trajectory that realizes a minimum in the definition of $w(0,x,z)$. This claim is based on some arguments introduced in [25]. The precise statement and proof are given in Theorem 4.2 below.

Theorem 4.2. *Assume* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $(\mathbf{H_3})$ *hold true. Assume also that the approximation* (4.1) *is valid and the error estimate* $E_h := \|w - w^h\|$ *satisfies:*

$$E_h/h \to 0 \quad \textit{as } h \to 0. \tag{4.6}$$

Let (y,z) *be in* $\mathbb{R}^d \times \mathbb{R}$ *and let* (y_k^h) *be the sequence generated by Algorithm 4.1.*
(i) The approximate trajectories $(y_k^h)_{k=0,\ldots,n_h}$ *constitute a minimizing sequence in the following sense:*

$$w(0,y,z) = \lim_{h\to 0} \left(\max_{0\le k\le n_h} \Psi(s_k, y_k^h, z) \right) \bigvee \varphi(y_{n_h}^h, z). \tag{4.7}$$

(ii) Moreover, the family $(\mathbf{y}^h)_{h>0}$ *admits cluster points, for the* L^∞ *norm, when* $h \to 0$. *For any such cluster point* $\bar{\mathbf{y}}$, *we have* $\bar{\mathbf{y}} \in \mathcal{S}_{[0,T]}(y)$ *and* $(\bar{\mathbf{y}}, z)$ *is an optimal trajectory for* $w(0,y,z)$.

Proof. First, by using similar arguments as in [25], one can prove that assertion (ii) is a consequence of (i). So, we shall focus on assertion (i) whose proof will be splitted in several steps.

For simplicity of the presentation, we shall consider only the case of uniform time partition

$$s_0 = 0 \le s_1 \le \cdots \le s_{n_h} = T,$$

with step size $h = \frac{T}{n_h}$. Throughout the proof, we shall use the simple notation n instead of n_h.

Let (y,z) be in $\mathbb{R}^d \times \mathbb{R}$ and consider, for every $h > 0$, the discrete trajectory $y^h = (y_0^h, \ldots, y_n^h)$ and discrete control input $u^h := (u_0^h, \ldots, u_{n-1}^h)$ given by Algorithm 4.1. In the sequel of the proof, and for simplicity of notations, we shall denote y_k (resp. u_k) instead of y_k^h (resp. u_k^h).

Assumption (H1) and (4.1) imply that there exists $R > 0$ such that for any $h > 0$ and any $k \le n$, we have $|y_k^h| \le R$. This constant R can be chosen large enough such that every trajectory on a time interval $I \subset [0,T]$, starting from an initial position y_k^h would still remain in a ball of $\mathbb{R}^d$ centred at 0 and with radius R. We set $M_R > 0$ a constant such that

$$|f(s,y,u)| \le M_R \quad \text{for every } s \in [0,T],\ y \in \mathbb{B}_R \text{ and } u \in U.$$

Step 1. Let us first establish that there exists $\varepsilon_h > 0$ such that $\lim\limits_{h\to 0} \varepsilon_h = 0$, and

$$w(s_0, y_0, z) \ge w(s_1, y_0 + h f_h(s_0, y_0, u_0), z) \bigvee \Psi(s_0, y_0, z) + h\varepsilon_h - 2E_h. \tag{4.8}$$

The dynamic programming principle for w gives (recall that $s_0 = 0$ and $y_0 = y$):

$$\begin{aligned} w(s_0, y_0, z) &= \min_{\boldsymbol{u}\in\mathcal{U}} w(s_1, \mathbf{y}^{\mathbf{u}}_{s_0,y_0}(s_1), z) \bigvee \max_{\theta\in(s_0,s_1)} \Psi(\theta, \mathbf{y}^{\mathbf{u}}_{s_0,y_0}(\theta), z) \\ &\ge \min_{\boldsymbol{u}\in\mathcal{U}} w(s_1, \mathbf{y}^{\mathbf{u}}_{s_0,y_0}(s_1), z) \bigvee \Psi(s_0, y_0, z). \end{aligned} \tag{4.9}$$

Consider $\boldsymbol{u}_0^* \in \mathcal{U}$ a minimizer of the term (4.9). By using the convexity of the set $f(s_0, y_0, U)$ (assumption (H2)), there exists $u_0^* \in U$ such that $\int_{s_0}^{s_1} f(s_0, y_0, \boldsymbol{u}_0^*(s))ds = hf(s_0, y_0, u_0^*)$, and therefore

$$y_0 + \int_{s_0}^{s_1} f(s_0, y_0, \boldsymbol{u}_0^*(s))ds = y_0 + hf(s_0, y_0, u_0^*).$$

Consider the trajectory $\mathbf{y}_{s_0,y_0}^{\boldsymbol{u}_0^*}$ solution of (2.1) corresponding to the control $\boldsymbol{u}_0^*$ and starting at time s_0 from y_0. Hence, $|\mathbf{y}_{s_0,y_0}^{\boldsymbol{u}_0^*} - y_0| \leq M_R h$, for $s \in [s_0, s_1]$, and

$$|\mathbf{y}_{s_0,y_0}^{\boldsymbol{u}_0^*}(s_1) - y_0 + hf(s_0, y_0, u_0^*)| \leq \int_{s_0}^{s_1} |f(s, \mathbf{y}_{s_0,y_0}^{\boldsymbol{u}_0^*}(s), \boldsymbol{u}_0^*(s)) - f(s_0, y_0, \boldsymbol{u}_0^*(s))|ds.$$

On the other hand, by (H1), there exists $\delta(h) > 0$ the modulus of continuity of f defined as:

$$\delta(h) := \max\{|f(s, \xi, u) - f(s', \xi, u)|, \quad \text{for } \xi \in \mathbb{B}_R, u \in U \text{ and } s, s' \in [0, T] \text{ with } |s - s'| \leq h\}.$$

We get (L_f being the Lipschitz constant of f as in (H1)):

$$|\mathbf{y}_{s_0,y_0}^{\boldsymbol{u}_0^*}(s_1) - y_0 + hf(s_0, y_0, u_0^*)| \leq \int_{s_0}^{s_1} h\delta(h) + L_f|\mathbf{y}_{s_0,y_0}^{\boldsymbol{u}_0^*} - y_0|ds \leq h\delta(h) + L_f M_R h^2.$$

By using Assumption (4.1), it also holds:

$$|\mathbf{y}_{s_0,y_0}^{\boldsymbol{u}_0^*}(s_1) - y_0 + hf_h(s_0, y_0, u_0^*)| \leq \int_{s_0}^{s_1} h\delta(h) + L_f|\mathbf{y}_{s_0,y_0}^{\boldsymbol{u}_0^*} - y_0|ds \leq h\delta(h) + (L_f M_R + C_R)h^2.$$

This estimate along with (4.9), and by using the Lipschitz continuity of w, yield:

$$\begin{aligned} w(0, y, z) &\geq w(s_1, \mathbf{y}_{s_0,y_0}^{\boldsymbol{u}_0^*}(s_1), z) \bigvee \Psi(s_0, y_0, z) \\ &\geq w(s_1, y_0 + hf_h(s_0, y_0, u_0^*), z) \bigvee \Psi(s_0, y_0, z) - hL_w\big(\delta(h) + (L_f M_R + C_R)h\big). \end{aligned} \tag{4.10}$$

Then, by the definition of the minimizer u_0 we finally obtain

$$w(s_0, y_0, z) \geq w(s_1, y_0 + hf_h(s_0, y_0, u_0), z) \bigvee \Psi(s_0, y_0, z) - h\varepsilon_h, \tag{4.11}$$

where $\varepsilon_h := L_w\big(\delta(h) + (L_f M_R + C_R)h\big)$. Knowing that $y_1 = y_0 + hf_h(s_0, y_0, u_0)$ and that $\|w - w^h\| \leq E_h$, we finally get the desired result:

$$w^h(s_0, y_0, z) \geq \Big(w^h(s_1, y_1, z) \bigvee \Psi(s_0, y_0, z)\Big) - h\varepsilon_h - 2E_h.$$

With exactly the same arguments, for all $k = 0, \ldots, n-1$, we obtain:

$$w^h(s_k, y_k, z) \geq \Big(w(s_{k+1}, y_{k+1}, z) \bigvee \Psi(s_k, y_k, z)\Big) - h\varepsilon_h - 2E_h. \tag{4.12}$$

Step 2. From (4.12), we get:

$$w^h(0,y,z) = w^h(s_0,y_0,z) \geq \Big(w^h(s_1,y_1,z)\bigvee \Psi(s_0,y_0,z)\Big) - h\varepsilon_h - 2E_h$$
$$\geq \bigg(\Big(\Big(w^h(s_2,y_2,z)\bigvee \Psi(s_1,y_1,z)\Big) - h\varepsilon_h - 2E_h\Big)\bigvee \Psi(s_0,y_0,z)\bigg) - h\varepsilon_h - 2E_h.$$

Now, notice that for any $a, b \in \mathbb{R}$ and $c \geq 0$, we have: $(a-c)\vee b \geq a \vee b - c$. Therefore:

$$w^h(0,y,z) \geq \Big(w^h(s_2,y_2,z)\bigvee \Psi(s_0,y_0,z)\bigvee \Psi(s_1,y_1,z)\Big) - 2h\varepsilon_h - 4E_h.$$

By induction, we finally get:

$$w^h(0,y,z) \geq \Big(w^h(s_n,y_n,z)\bigvee \Psi(s_0,y_0,z)\bigvee \cdots \bigvee \Psi(s_{n-1},y_{n-1},z)\Big) - nh\varepsilon_h - 2nE_h. \tag{4.13}$$

Step 3. Since $s_n = T$ and $w(T,y_n,z) = \Psi(T,y_n,z)\bigvee(\varphi(y_n)-z)$, we deduce from (4.13) that:

$$w^h(0,y,z) \geq (w(s_n,y_n,z) - E_h)\bigvee \left(\bigvee_{k=0}^{n-1}\Psi(y_k,z)\right) - nh\varepsilon_h - 2nE_h$$
$$\geq \left(\bigvee_{k=0}^{n}\Psi(s_k y_k,z)\right)\bigvee \varphi(y_n,z) - T\varepsilon_h - (\frac{2T}{h}+1)E_h.$$

By passing to the limit when $h \to 0$, and using (4.6) it follows that:

$$w(0,y,z) \geq \limsup_{h\to 0}\left(\bigvee_{k=0}^{n}\Psi(s_k,y_k,z)\right)\bigvee \varphi(y_n,z). \tag{4.14}$$

Step 4. Let $\mathbf{y}^{\boldsymbol{u}^h}(s)$ denote the trajectory obtained with piecewise constant controls $u_0,\dots,u_{n-1}$ (*i.e.*, $\boldsymbol{u}^h(s) := u_k$ for all $s \in [s_k, s_{k+1}[$) and solution of $\mathbf{y}^{\boldsymbol{u}^h}(0) = x$ and $\dot{\mathbf{y}}^{\boldsymbol{u}^h}(s) = f(s,\mathbf{y}(s),\boldsymbol{u}^h(s))$ a.e. $s \geq 0$. Consider also $\mathbf{y}^h(.)$ as the approximate trajectory, satisfying $\mathbf{y}(0) = x$ and $\dot{\mathbf{y}}(s) = f_h(s_k,y_k,u_k)$ for all $s \in [s_k, s_{k+1}[$. By using the same arguments as in Step 1, we obtain the following estimate:

$$\max_{\theta\in[0,T]}|\mathbf{y}^{\boldsymbol{u}^h}(\theta) - \mathbf{y}^h(\theta)| \leq \delta(h) + (L_f M_R + C_R)h. \tag{4.15}$$

Step 5. Now, we claim that the following bound holds:

$$\left|\bigvee_{k=0}^{n-1}\Psi(s_k,y_k,z) - \max_{\theta\in[0,T]}\Psi(s_k,\mathbf{y}^{\mathbf{u}^h}(\theta),z)\right| \leq O(\max(\delta(h),h)). \tag{4.16}$$

In order to prove this claim, let us first remark that, by using the Lipschitz regularity of $t \to \mathbf{y}^{\boldsymbol{u}^h}(t)$, there exists $M_R > 0$ such that:

$$\max_{\theta\in[s_k,s_{k+1}]}|\mathbf{y}^{\boldsymbol{u}^h}(\theta) - \mathbf{y}^{\boldsymbol{u}^h}(s_k)| \leq M_R h. \tag{4.17}$$

Then, by straightforward calculations, we obtain:

$$
\begin{aligned}
&\Big|\bigvee_{k=0}^{n-1}\Psi(s_k,y_k,z)-\max_{\theta\in[0,T]}\Psi(\theta,\mathbf{y}^{\boldsymbol{u}^h}(\theta),z)\Big| = \Big|\bigvee_{k=0}^{n-1}\Psi(s_k,y_k,z)-\bigvee_{k=0}^{n-1}\max_{\theta\in[s_k,s_{k+1}]}\Psi(\theta,\mathbf{y}^{\boldsymbol{u}}(\theta),z)\Big|\\
&\le \Big|\bigvee_{k=0}^{n-1}\Psi(s_k,y_k,z)-\bigvee_{k=0}^{n-1}\Psi(s_k,\mathbf{y}^{\boldsymbol{u}^h}(s_k),z)\Big| + \Big|\bigvee_{k=0}^{n-1}\Psi(s_k,\mathbf{y}^{\boldsymbol{u}^h}(s_k),z)-\bigvee_{k=0}^{n-1}\max_{\theta\in[s_k,s_{k+1}]}\Psi(\theta,\mathbf{y}^{\boldsymbol{u}^h}(\theta),z)\Big|\\
&\le \max_{k=0,..,n-1}|\Psi(s_k,y_k,z)-\Psi(s_k,\mathbf{y}^{\boldsymbol{u}^h}(s_k),z)| + \max_{k=0,..,n-1}|\Psi(s_k,\mathbf{y}^{\boldsymbol{u}^h}(s_k),z)-\max_{\theta\in[s_k,s_{k+1}]}\Psi(\theta,\mathbf{y}^{\boldsymbol{u}^h}(\theta),z)|\\
&\le \max_{k=0,..,n-1}L_\Psi|y_k-\mathbf{y}^{\boldsymbol{u}^h}(s_k)| + \max_{k=0,..,n-1}L_\Psi\max_{\theta\in[s_k,s_{k+1}]}|\mathbf{y}^{\boldsymbol{u}^h}(s_k)-\mathbf{y}^{\boldsymbol{u}^h}(\theta)|\\
&\le L_\Psi\delta(h)+L_\Psi(L_fM_R+C_R+M_R)h,
\end{aligned}
$$

which proves (4.16). In the same way, we have also:

$$
\Big|\Big[\bigvee_{k=0}^{n}\Psi(s_k,y_k,z)\bigvee(\varphi(y_n)-z)\Big]-\Big[\max_{\theta\in[0,T]}\Psi(\theta,\mathbf{y}^{\boldsymbol{u}}(\theta),z)\bigvee(\varphi(\mathbf{y}^{\boldsymbol{u}^h}(T))-z)\Big]\Big|\le O(\max(\delta(h),h)). \tag{4.18}
$$

Step 6. Combining the previous estimates (4.14) and (4.18), we obtain

$$
w(0,y,z)\ge\limsup_{h\to0}\max_{\theta\in[0,T]}\Psi(\theta,\mathbf{y}^{\boldsymbol{u}^h}(\theta),z)\bigvee(\varphi(\mathbf{y}^{\boldsymbol{u}^h}(T))-z). \tag{4.19}
$$

On the other hand, by definition of w the following reverse inequality holds :

$$
w(0,y,z)\le\liminf_{h\to0}\max_{\theta\in[0,T]}\Psi(\theta,\mathbf{y}^{\boldsymbol{u}^h}(\theta),z)\bigvee(\varphi(\mathbf{y}^{\boldsymbol{u}^h}(T))-z). \tag{4.20}
$$

Hence the right-hand side term has a limit and

$$
w(0,y,z)=\lim_{h\to0}\max_{\theta\in[0,T]}\Psi(\theta,\mathbf{y}^{\boldsymbol{u}^h}(\theta),z)\bigvee(\varphi(\mathbf{y}^{\boldsymbol{u}^h}(T))-z). \tag{4.21}
$$

Also the discrete constructed trajectory reaches the same value:

$$
w(0,y,z)=\lim_{h\to0}\bigvee_{k=0}^{n}\Psi(s_k,y_k,z)\bigvee(\varphi(y_n)-z)). \tag{4.22}
$$

This concludes to the desired result. □

In a second algorithm we consider a trajectory reconstruction procedure with a perturbation term in the definition of the optimal control value. This perturbation takes the form of a penalization term on the variation of the control with respect to the previously computed control values. To this end, for every $k\ge1$, we introduce a function $q_k:\mathbb{R}\times\mathbb{R}^k\to\mathbb{R}^+$ that represents a penalization term for the control value. For instance, if $\mathbb{U}_k:=(u_0,...,u_{k-1})$ is a vector in $\mathbb{R}^k$, we may choose

$$
q_k(u,\mathbb{U}_k):=\|u-u_{k-1}\|,\quad\text{or}\quad q_k(u,\mathbb{U}_k):=\|u-\frac{1}{p}\sum_{i=1}^{p}u_{k-i}\|\ \text{ for some } p\ge1. \tag{4.23}
$$

Let $(\lambda_h)_{h>0}$ be a family of positive constants.

Algorithm 4.3. Let $y \in \mathbb{R}^d$ and $z \in \mathbb{R}$. For $h > 0$ we consider an integer $n_h \in \mathbb{N}$, a partition $s_0 = 0 < s_1 < \cdots < s_{n_h} = T$ of $[0,T]$, denote $h_k := s_{k+1} - s_k$ and assume that

$$\max_k h_k \leq h.$$

We define positions $(y_k^h)_{k=0,\ldots,n_h}$ and controls $(u_k^h)_{k=0,\ldots,n_h-1}$ by recursion as follows. First we set $y_0^h := y$. For $k = 0$, we compute u_0^h and y_0^h as in Algorithm 4.1. Then, for $k \geq 1$ we define $\mathbb{U}_k := (u_0^h, \cdots, u_{k-1}^h)$ and compute:

(i) an optimal control value $u_k^h \in U$ such that

$$u_k^h \in \operatorname*{argmin}_{u \in U} \left[\left(w^h\big(s_k, y_k^h + h_k\, f_h(s_k, y_k^h, u), z^h\big) \bigvee \Psi(s_k, y_k^h, z^h) \right) \;+\; \lambda_h q_k(u, \mathbb{U}_k) \right]; \tag{4.24}$$

(ii) a new state position y_{k+1}^h as follows

$$y_{k+1}^h := y_k^h + h_k f_h(s_k, y_k^h, u_k^h).$$

We shall prove that this second algorithm provides also a minimizing sequence $(\mathbf{y}^h, \mathbf{u}^h, z^h)_{h>0}$ as soon as λ_h decreases sufficiently fast as $h \to 0$.

In the reconstruction process (Algorithm 4.1), the formula (4.3) suggests that the control input is a value that minimizes the function $u \longmapsto \left(w^h\big(s_k, y_k^h + h_k\, f_h(s_k, y_k^h, u), z^h\big) \bigvee \Psi(s_k, y_k^h, z^h) \right)$. Such a function may admit several minimizers and the reconstruction procedure does not give any further information on which minimizer to choose. Adding the term $\lambda_h q_k(u, \mathbb{U}_k)$ can be seen as a penalization term. For example, by choosing $q_k(u, \mathbb{U}_k) := u_{k-1}$, we force the value u_k to stay as close as possible to u_{k-1}. Here we address the convergence result of Algorithm 4.3 with a penalization term q_k. However, the choice of a relevant function q_k is not a trivial task and depends on the control problem under study.

Theorem 4.4. *Assume (*$\mathbf{H_1}$*), (*$\mathbf{H_2}$*) and (*$\mathbf{H_3}$*) hold true, and* (4.1) *and and* (4.6) *are fulfilled. Let* (y_k^h) *be the family generated by Algorithm 4.3. Assume furthermore that the penalization term is bounded: there exists* $M_q > 0$ *such that* $|q_k(u, \mathbb{U})| \leq M_q$ *for every* $u \in U$ *and every* $\mathbb{U} \in U^k$*, and*

$$\lambda_h / h \to 0.$$

(i) The approximate trajectories $(y_k^h)_{k=0,\ldots,n}$ *are minimizing sequences in the following sense:*

$$w(0, y, z) = \lim_{h \to 0} \left(\max_{0 \leq k \leq n_h} \Psi(s_k, y_k^h, z) \right) \bigvee \varphi(y_{n_h}^h, z). \tag{4.25}$$

(ii) There exist cluster points for the sequence $(\mathbf{y}^h)_{h>0}$ *as* $h \to 0$*, for the* L^∞ *norm. Moreover, any such cluster point* $\bar{\mathbf{y}}$ *is an admissible trajectory belonging to* $S_{[0,T]}(y)$ *and* $\bar{\mathbf{y}}$ *is an optimal trajectory for* $w(0, y, z)$*.*

Proof. The arguments of the proof are similar to the ones used in the proof of Theorem 4.2. The only change is in the estimate derived in Step 1, where instead of (4.12), we get now:

$$w^h(s_k, y_k, z) \geq \Big(w(s_{k+1}, y_{k+1}, z) \bigvee \Psi(s_k, y_k, z) \Big) - h\varepsilon_h - 2E_h - M_q \lambda_h. \tag{4.26}$$

The rest of the proof remains unchanged. $\square$

Before we end this section, we introduce a third algorithm that will be tested in the numerical section. This algorithm uses the reachability time function as defined in Section 3.3. We assume that the control problem is autonomous (all the involved functions in the control problem do not depend in the time variable). We assume that an approximation $\mathcal{T}^h$ of the reachability time function is computed. The reconstruction algorithm reads as follows.

Algorithm 4.5. Fix $y \in \mathbb{R}^d$ and $z \in \mathbb{R}$. For $h > 0$ we consider an integer $n_h \in \mathbb{N}$, a partition $s_0 = 0 < s_1 < \cdots < s_{n_h} = T$ of $[0, T]$, denote $h_k := s_{k+1} - s_k$ and assume that

$$\max_k h_k \le h.$$

We define the positions $(y_k^h)_{k=0,\ldots,n_h}$, and control values $(u_k^h)_{k=0,\ldots,n_h-1}$, by recursion as follows. First we set $y_0^h := y$. Then for $k = 0, \ldots, n_h - 1$, knowing the state y_k^h we define

(i) an optimal control value $u_k^h \in U$ such that

$$u_k^h \in \operatorname*{argmin}_{u \in U} \mathcal{T}^h\big(y_k^h + h_k\, f_h(y_k^h, u), z\big) \tag{4.27}$$

(ii) a new state position y_{k+1}^h

$$y_{k+1}^h := y_k^h + h_k f_h(y_k^h, u_k^h). \tag{4.28}$$

Without further assumption on the control problem, the reachability time function may be discontinuous and we do not have any convergence proof for Algorithm 4.5. However, there is an obvious numerical advantage of using Algorithm 4.5 rather than Algorithm 4.1 or 4.3. Indeed, while the two first algorithms require the auxiliary function w^h to be stored on a grid of dimension $d + 1$, at every time s_k, the third algorithm requires $\mathcal{T}^h$ to be stored only once on a grid of dimension $d + 1$.

5. The aircraft landing abort problem: model

5.1. The flight aerodynamics

Consider the flight of an aircraft in a vertical plane over a flat earth where the thrust force, the aerodynamic force and the weight force act on the center of gravity $\mathbf{G}$ of the aircraft and lie in the same plane of symmetry. Let $\mathbf{V}$ be the velocity vector of the aircraft relative to the atmosphere. In order to obtain the equations of motion, the following system of coordinates is considered:

(i) the ground axes system $Ex_ey_ez_e$, fixed to the surface of earth at mean sea level.

(ii) the wind axes system denoted by $Ox_wy_wz_w$ moving with the aircraft and the x_w axis coincides with the velocity vector.

The path angle γ defines the wing axes orientation with respect to the ground horizon axes. Let $\mathbf{G}$ be the center of the gravity.

We write Newton's law as $\mathbf{F} = m\frac{d\mathbf{V_G}}{dt}$, where where $\mathbf{V_G} = \mathbf{V} + \mathbf{w}$ is the resultant velocity of the aircraft relative to the ground axis system, and $\mathbf{w}$ denotes the velocity of the atmosphere relative to the ground axis system. The different forces are the following:

- the thrust force $\mathbf{F_T}$ is directed along the aircraft. The modulus of the thrust force is of the form $F_T(t, v) := \beta(t)F_T(v)$ where $v = |\mathbf{V}|$ is the modulus of the velocity and $\beta(t) \in [0, 1]$ is the power setting of the engine. In the present study

$$F_T(v) := A_0 + A_1\, v + A_2\, v^2.$$

- the lift and drag forces $\mathbf{F_L}$, $\mathbf{F_D}$. The norm of these forces are supposed to satisfy the following relations:

$$F_L(v,\alpha)=\frac{1}{2}\rho S v^2 c_\ell(\alpha), \quad F_D(v,\alpha)=\frac{1}{2}\rho S v^2 c_d(\alpha), \tag{5.1}$$

  where ρ is the air density on altitude, S is the wing area. The coefficients $c_d(\alpha)$ and $c_\ell(\alpha)$ depend on the angle of attack α and the nature of the aircraft. As in [11, 12] we consider here:

$$\begin{aligned} c_d(\alpha) &= B_0 + B_1\alpha + B_2\alpha^2 \\ c_\ell(\alpha) &= \begin{cases} C_0 + C_1\alpha & \alpha \le \alpha_*, \\ C_0 + C_1\alpha + C_2(\alpha-\alpha_*)^2 & \alpha_* \le \alpha, \end{cases} \end{aligned} \tag{5.2}$$

  (The coefficient c_ℓ depends linearly on the angle α until a swiching point α^* where the dependency becomes polynomial.)
- the weight force $\mathbf{F_P}$: its modulus satisfies $|\mathbf{F_P}| = mg$ where m is the aircraft mass and g the gravitational constant.

The constants ρ, S, α^*, $(A_i)_{i=0,1,2}$, $(B_i)_{i=0,1,2}$, $(C_i)_{i=0,1,2}$, m, g are given in Table A.1 of Appendix A. We assume that the motion of the aircraft can be controlled by the angular velocity $u(t) := \dot\alpha(t)$.

By using Newton's law, the equation of motion are [11]:

$$\dot x = v\cos\gamma + w_x \tag{5.3a}$$

$$\dot h = v\sin\gamma + w_h \tag{5.3b}$$

$$\dot v = \frac{\beta F_T(v)}{m}\cos(\alpha+\delta) - \frac{F_D}{m} - g\sin\gamma - \dot w_x\cos\gamma - \dot w_h\sin\gamma \tag{5.3c}$$

$$\dot\gamma = \frac{1}{v}\left(\frac{\beta F_T(v)}{m}\sin(\alpha+\delta) + \frac{F_L}{m} - g\cos\gamma + \dot w_x\sin\gamma - \dot w_h\cos\gamma\right) \tag{5.3d}$$

$$\dot\alpha = u$$

where $\delta > 0$ is also a parameter of the model, given in Table A.1, and where w_x and w_h are respectively the horizontal and the vertical components of the wind velocity vector $\mathbf{w}$, and

$$\begin{aligned} \dot w_x &:= \frac{\partial w_x}{\partial x}(v\cos\gamma + w_x) + \frac{\partial w_x}{\partial h}(v\sin\gamma + w_h). \\ \dot w_h &:= \frac{\partial w_h}{\partial x}(v\cos\gamma + w_x) + \frac{\partial w_h}{\partial h}(v\sin\gamma + w_h). \end{aligned}$$

The precise model is of the form $w_x \equiv w_x(x)$, $w_h \equiv w_h(x,h)$, as shown in Figure 1, and is provided in Appendix A.

In the more general setting, one can consider the variables u and β as controls of the dynamical system associated with the motion equations (5.3). In this work different scenarios of the dynamical system are be considered, in accordance with the role that plays the power factor β. They are described in detail in the following section.

In the sequel, the state variables are represented by a vector of $\mathbb{R}^5$:

$$\mathbf{y}(\cdot) = (x(\cdot), h(\cdot), v(\cdot), \gamma(\cdot), \alpha(\cdot))^T.$$

Therefore the differential system (5.3) will be also denoted as follows:

$$\dot{\mathbf{y}}(t) = f(\mathbf{y}(t), \mathbf{u}(t)) \tag{5.4}$$

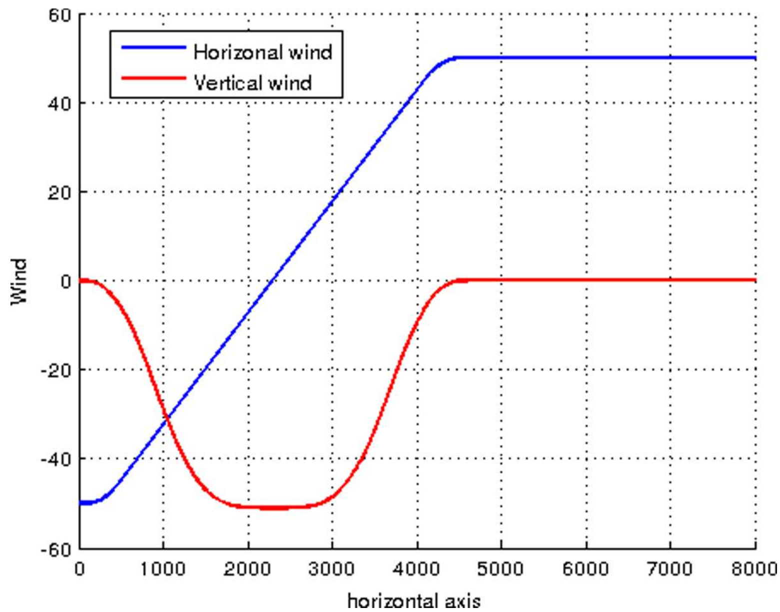

FIGURE 1. Wind components $w_x(x)$ and $w_h(x,h)$ as functions of x ($h = 1000\,\mathrm{ft}$). (Color online.)

where the dynamics f stands for the right-hand-side of (5.3), and the control is $\mathbf{u} = (u, \beta)$.

The state is subject to the constraints $y(t) \in \mathcal{K}$ with

$$\mathcal{K} := \big[x_{\min}, x_{\max}\big] \times \big[h_{\min}, h_{\max}\big] \times \big[v_{\min}, v_{\max}\big] \times \big[\gamma_{\min}, \gamma_{\max}\big] \times \big[\alpha_{\min}, \alpha_{\max}\big], \tag{5.5}$$

where $h_{\min}$ defines the lower altitude below which the abort landing is very difficult, $h_{\max}$ is a reference altitude (the cruise altitude for instance), $v_{\max}$ is given by the aircraft constructor, $v_{\min} > 0$ is the desired minimum velocity value, $[\gamma_{\min}, \gamma_{\max}] \subset [-\frac{\pi}{2}, \frac{\pi}{2}]$. All the numerical values for the boundary constraints are given in Table 1 in Section 6.2.

The constraints on the control u is of the type $u_{min} \leq u \leq u_{\max}$ with constants $u_{min}, u_{\max}$ as in Table 1. Moreover, the control $\beta(t) \in [0,1]$ will be also subject to different type of restrictions as made precise in each test of Section 6.3.

5.2. Optimality criterion

In the case of windshear, the Airport Traffic Control Tower has to choose between two options. The first one is to penetrate inside the windshear area and try to make a successful landing. If the altitude is high enough, it is safer to choose another option : the abort landing, in order to avoid any unexpected instability of the aircraft. In this article we focus on this second option.

Starting from an initial point $y \in \mathbb{R}^d$, the optimal control problem is to maximize the lower altitude over a given time interval, that is,

$$\text{maximize} \left(\min_{\theta \in [0,T]} h(\theta) \right)$$

where $h(\theta)$ is the altitude at time θ corresponding to the second component of the vector $\mathbf{y}_y^{\mathbf{u}}(\theta)$ solution of (5.4) at time θ and such that $\mathbf{y}_y^{\mathbf{u}}(0) = y$. For commodity, the problem is recasted into a minimization problem as follows. Let $H_r > 0$ be a given reference altitude, and set

$$\Phi(y) := H_r - h, \tag{5.6}$$

where h is the second component of the vector y.

The state constrained control problem with a maximum running cost associated to Φ, denoted $(\mathcal{P}_\infty)$, is the following:

$$\inf_{\mathbf{y}\in S^{\mathcal{K}}_{[0,T]}} \max_{\theta\in[0,T]} \Phi(\mathbf{y}^{\mathbf{u}}_y(\theta)). \tag{$\mathcal{P}_\infty$}$$

Note that in the above formulation of the abort landing problem, we do not impose a minimum altitude constraint because the value of the threshold to impose for this minimum altitude is not supposed to be known. Actually, problem $(\mathcal{P}_\infty)$ aims at computing an optimal trajectory that realizes a safe abort landing and with the *highest* minimal altitude, see [11, 12, 21, 22].

6. The abort landing problem: numerical results

6.1. The finite difference scheme

It is well known, since the work of Crandall and Lions [15], that the Hamilton Jacobi equation (3.7) can be approximated by using finite difference (FD) schemes. In our case we consider a slightly more precise scheme; namely an Essentially Non Oscillatory (ENO) scheme of second order, see [23]. Such a scheme has been numerically observed to be efficient. Notice that we could have also considered other discretization methods such as Semi-Lagrangian methods (see [16, 17]). For the present application we prefer to use the ENO scheme because there is no need of a control discretization in the definition of the numerical Hamiltonian function (see $\mathcal{H}$ below).

For given non-negative mesh steps h, $\Delta y = (\mathrm{d}y_i)_{1\le i\le d}$, and Δz, for a given multi-index $i = (i_1, \ldots, i_d)$, let $y_i := y_{\min} + i\Delta x \equiv (y_{k,min} + i_k \Delta y_k)_{1\le k\le d}$, $z_j := z_{\min} + j\Delta z$ and $t_n = nh$. Let us define the following grid of $\mathcal{K}_\eta \times [z_{\min}, z_{\max}]$:

$$\mathcal{G} := \big\{(y_i, z_j),\ i \in \mathbb{Z}^d,\ j \in \mathbb{Z},\ (y_i, z_j) \in \mathcal{K}_\eta \times [z_{\min}, z_{\max}]\big\}.$$

Let us furthermore denote $\psi_{i,j} := \Psi(y_i, z_j)$. In the following, $w^n_{i,j}$ will denote an approximation of the solution $w(t_n, y_i, z_j)$.

Given a numerical Hamiltonian $\mathcal{H}$: $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ (see Rems. 6.1–6.2 below) the following "explicit" scheme is considered, as in [8]. First, the numerical approximation is initialized with

$$w^N_{i,j} = \psi_{i,j},\ (y_i, z_j) \in \mathcal{G}. \tag{6.1a}$$

Then, for $n \in \{N-1, N-2, \ldots, 1, 0\}$ we compute recursively

$$w^n_{i,j} \ = \ \max\left(w^{n+1}_{i,j} - \Delta t\, \mathcal{H}(y_i,\, D^- w^{n+1}_{i,j},\, D^+ w^{n+1}_{i,j}),\ \psi_{i,j}\right), \qquad (y_i, z_j) \in \mathcal{G} \tag{6.1b}$$

(where $\mathcal{H}$ is made precise later on).

A monotone (first order) finite difference approximation is obtained using $D^\pm w^n_{i,j} = (D^\pm_k w^n_{i,j})_{1\le k\le d}$ with

$$D^\pm_k w^n_{i,j} := \pm \frac{w^n_{i\pm e_k,j} - w^n_{i,j}}{\Delta y_k},$$

TABLE 1. Constants for the domain $\mathcal{K}$ and control constraints.

State variable	x (ft)	h (ft)	v (ft s^{-1})	γ (°)	α (°)	Control variable	u (°)
min	−100	450	160	−7.0	4.0	$u_{\min}$	−3.0
max	9900	1000	260	15.0	17.2	$u_{\max}$	3.0

and where $\{e_k\}_{k=1,\dots,d}$ is the canonical basis of $\mathbb{R}^d$ ($(e_k)_k = 1$ and $(e_k)_j = 0$ if $j \neq k$). In this paper, we use a second order ENO scheme in order to estimate more precisely the terms $D_k^{\pm} w_{i_k,j}^n$, see [23]:

$$D_k^{\pm} w_{i,j}^n := \pm \frac{w_{i\pm e_k,j}^n - w_{i,j}^n}{\Delta y_k} \mp \frac{1}{2}\Delta y_k \; m\big(D_{k,0}^2 w_{i,j}^n,\; D_{k,\pm 1}^2 w_{i,j}^n\big)$$

with $D_{k,\varepsilon}^2 w_{i,j}^n := (-w_{i+(-1+\varepsilon)e_k,j}^n + 2w_{i+\varepsilon e_k,j}^n - w_{i+(1+\varepsilon)e_k,j}^n)/(\Delta y_k)^2$, and where $m(a,b) := a$ if $ab > 0$ and $|a| \leq |b|$, $m(a,b) = b$ if $ab > 0$ and $|a| > |b|$, and $m(a,b) = 0$ if $ab \leq 0$.

Remark 6.1. If the numerical Hamiltonian $\mathcal{H}$ is Lipschitz continuous on all its arguments, consistent with H ($\mathcal{H}(y,p,p) = H(y,p)$) and monotone (*i.e.* $\frac{\partial \mathcal{H}}{\partial p_k^-}(y,p^-,p^+) \geq 0$, $\frac{\partial \mathcal{H}}{\partial p_k^+}(y,p^-,p^+) \leq 0$) together with the following Courant-Friedrich-Levy (CFL) condition

$$\Delta t \sum_{k=1}^{d} \frac{1}{\Delta y_k} \left\{ \left| \frac{\partial \mathcal{H}}{\partial p_k^-}(y,p^-,p^+) \right| + \left| \frac{\partial \mathcal{H}}{\partial p_k^+}(y,p^-,p^+) \right| \right\} \leq 1, \tag{6.2}$$

then the numerical solution $(w_{i,j}^n)$ converges to the desired solution (see [8] for more details).

Remark 6.2. Since the control variable u enters linearly and only in the 5th component of the dynamics f in (5.4), the Hamiltonian $H(y,p)$ takes the following simple analytic form (here in the case of $u_{\max} = -u_{\min} \geq 0$):

$$H(y,p) := \sum_{i=1}^{4} -f_i(y)p_i + u_{\max}|p_5|$$

where $(p_i)_{1\leq i\leq d}$ are the components of p and $(f_1(y),\dots,f_4(y))$ are the first four components of f.

For this particular situation we will use the following numerical Hamiltonian:

$$\mathcal{H}(y,p^-,p^+) = \sum_{i=1}^{4} \Big(\max(-f_i(y),0)\, p_i^- + \min(-f_i(y),0)\, p_i^+ \Big) + u_{\max} \max(p_5^-,\, -p_5^+,\, 0).$$

It satisfies all the required conditions of Remark 6.1 for a sufficiently small time step h satisfying the CFL condition (6.2), which can be written here :

$$\Delta t \Bigg(\sum_{1\leq i\leq 4} \frac{|f_i(y)|}{\Delta y_i} + \frac{u_{\max}}{\Delta y_5} \Bigg) \leq 1.$$

6.2. Computational domain, control constraints

To solve the control problem $(\mathcal{P}_\infty)$, we will use the HJB approach as introduced in Section 3. Let us mention other recent works [2, 10] where an approximated control problem of $(\mathcal{P}_\infty)$, involving a 4-dimensional model, is also considered by using HJB approach.

In all our computations, the boundary of the domain $\mathcal{K}$ is defined as in Table 1.

The computational domain is slightly extended in all directions $\mathcal{K}_\eta := \mathcal{K} + \eta B_\infty$, where $B_\infty := [-1,1]^d$ is the unit ball centered in the origin for the ℓ^∞ norm. The parameter η is fixed to a stricly positive, small value ($\eta = 0.05$ in our computations).

6.3. Numerical experiments and analysis

In this section, we will perform different tests to investigate numerical aspects for computing an approximation of the optimal value $\vartheta(0, y_0)$ of $(\mathcal{P}_\infty)$ through the use of the auxiliary value function w. The computations are performed as follows[1]:

- We first consider a grid $\mathcal{G}_h$ on $\mathcal{K}_\eta$ (the grid's size will be made precise for each test). Then solve numerically the HJB equation (3.15) and get an approximation w^h of the auxiliary value function corresponding to $(\mathcal{P}_\infty)$.
- Let $y_0 \in \mathcal{K}$ be a given initial state of the dynamical system (5.3). Following Proposition 3.2, an approximation of the minimum value of $(\mathcal{P}_\infty)$ can be then defined as:

$$z_h^* := \min\{z \in [0, 550ft] \mid w^h(0, y_0, z) \leq 0\}.$$

- By using Algorithm 4.1 or 4.3 on a time partition $0 = s_0 \leq s_1 \leq \cdots s_{n_h} = T$, we get a suboptimal trajectory for $w^h(0, y_0, z_h^*)$ that we shall denote as $y^{h,w}$.
- In the sequel, we shall also use Algorithm 4.5 to reconstruct a trajectory $y^{h,\mathcal{T}}$ corresponding to the initial condition (y_0, z_h^*), by using the reachability time function (here we don't have any theoretical basis to guarantee that $y^{h,\mathcal{T}}$ is a approximation of an optimal trajectory, but numerical experiments will show that $y^{h,\mathcal{T}}$ is as good an approximation as $y^{h,w}$).
- Then, we will define

$$J_{h,w} := \max_{0 \leq k \leq n_h} \Phi(y^{h,w}(s_k)), \tag{6.3}$$

$$J_{h,\mathcal{T}} := \max_{0 \leq k \leq n_h} \Phi(y^{h,\mathcal{T}}(s_k)). \tag{6.4}$$

By combining Proposition 3.12 and Theorem 4.2, we know that a subsequence of $(y^{h,w})_{h>0}$ converges to an optimal trajectory of $(\mathcal{P}_\infty)$, when h goes to 0. Moreover,

$$\lim_{h \to 0} J_{h,w} = \vartheta(0, y_0).$$

6.3.1. Test 1: Running cost problem. Comparison of different methods for computing the optimal value

In this first test, we assume that the power thrust is maximal, *i.e.*, $\beta(t) \equiv 1$ which implies that the corresponding dynamical system is autonomous, controlled only by the function $u(.)$. The initial state used is chosen as in [11] and [12]:

$$y_0 := (0.0, 600.0, 239.7, -2.249°, 7.373°). \tag{6.5}$$

First, we choose a uniform grid on $\mathcal{K}_\eta$ (for the variable y) with $40 \times 20 \times 16 \times 8 \times 24$ nodes. The auxiliary variable interval (for variable z) is fixed to $[0, H_r - h_{\min}] = [0, 550\,\mathrm{ft}]$. The aim is to have a good approximation of the optimal value z_h^* when the computational grid of the variable z is refined. Recall that the dynamics of the z variable is zero. In order to keep a reasonable number of grid points, we will rather fix the number of grid points to 5 in the z variable, and refine the z interval by a dichotomy approach. Therefore, the whole computation grid contains $40 \times 20 \times 16 \times 8 \times 24 \times 5$ nodes.

[1] All the computations are performed by using our software ROC-HJ available at https://uma.ensta-paristech.fr/soft/ROC-HJ.

TABLE 2. (Test 1) Dichotomy on the interval of z variable.

z interval	z_h^*	$J_{h,w}$	$J_{h,\mathcal{T}}$
$[0, 550]$	542.30	506.16	500.03
$[275, 550]$	525.65	487.60	482.41
$[412.5, 550]$	519.72	482.22	476.14
$[481.25, 550]$	518.98	481.95	473.18

TABLE 3. (Test 1) Convergence with space grid refinements for the y variable only.

Grid	z^*	$J_{h,w}$	$J_{h,\mathcal{T}}$
$40 \times 20 \times 16 \times 8 \times 24 \times 5$	518.98	481.95	473.18
$60 \times 30 \times 24 \times 12 \times 36 \times 5$	487.72	482.94	480.45
$80 \times 40 \times 32 \times 16 \times 48 \times 5$	485.30	487.77	490.13
$100 \times 50 \times 40 \times 20 \times 60 \times 5$	486.10	487.91	489.36

Remark 6.3. The computation grid is defined in such a way the mesh size in each direction give similar CFL ratios, *i.e.*, when the values of $\mu_i := \frac{\Delta t}{\Delta y_i}\|f_i\|_\infty$ are approximately equal for $i = 1, \ldots, 5$.

The numerical results are shown in Table 2, using 4 successive reductions of the z interval, giving in particular the estimated optimal z_h^* in the second column. The values of $J_{h,w}$ and $J_{h,\mathcal{T}}$ are reported respectively in the third and fourth columns. Recall that here we do not refine the computation grid with respect to the state variable y. The aim of these (first) calculations is not to observe a convergence result but rather to locate a narrow set where we can find the optimal value z_h^*.

In a next step, we fix the last interval for the auxiliary z variable (*i.e.*, $z \in [481.25, 550]$) and refine the space grid in the y variable. Table 3 shows the numerical results obtained when the number of grid points is increased by a factor of 1.5^5, 2^5 and then 2.5^5. By this calculation, we notice that the values of z_h^*, $J_{h,w}$ and $J_{h,\mathcal{T}}$ become closer and closer as the grid size is refined.

It should be noted that the computation performed on the grid $100 \times 50 \times 40 \times 20 \times 60 \times 5$ requires a huge calculation effort. It would be difficult to consider a much finer grid without resorting to parallel computing. We prefer in the sequel not to address this issue and rather focus on the optimal trajectories reconstruction on a reasonable grid $80 \times 40 \times 32 \times 16 \times 48 \times 5$.

In Figure 2, we compare the reconstructed trajectories $y^{h,w}$ and $y^{h,\mathcal{T}}$. We notice that these trajectories are very similar and their performances ($J_{h,w}$ and $J_{h,\mathcal{T}}$) are close enough.

Several remarks should be made here. First, the reconstruction by the reachability time function is less CPU time consuming because it requires to store the function $\mathcal{T}$ only on a six-dimensional grid, whereas the reconstruction by using the auxiliary value function requires to store w on a six-dimensional grid for each time step. Secondly, the trajectories in Figure 2 are very similar on the time interval $[0, 30]$, and then they differ on the time interval $[30, 40]$. This can be explained by the fact that the minimum running cost is reached at a time less than $t = 30$. The rest of the trajectory after that time is not relevant anymore for the running cost.

Remark 6.4. We observe in Figure 2 that the constraint on the angle α is saturated during some period of time, which justify the careful treatment of the state constraints for this example.

It is worth to mention that once the value function w^h is computed, it is possible to obtain more information on the original control problem than simply the reconstruction of an optimal trajectory corresponding to a single initial position. Indeed, from the function w^h one can obtain an approximation of the whole feasibility set, *i.e.*, the set of initial conditions of the system for which there exists at least one trajectory satisfying all state constraints until the given time horizon T. For the landing abort problem that means to know all initial flight configurations for which it is possible to abort the landing without danger, when the local dominant wind

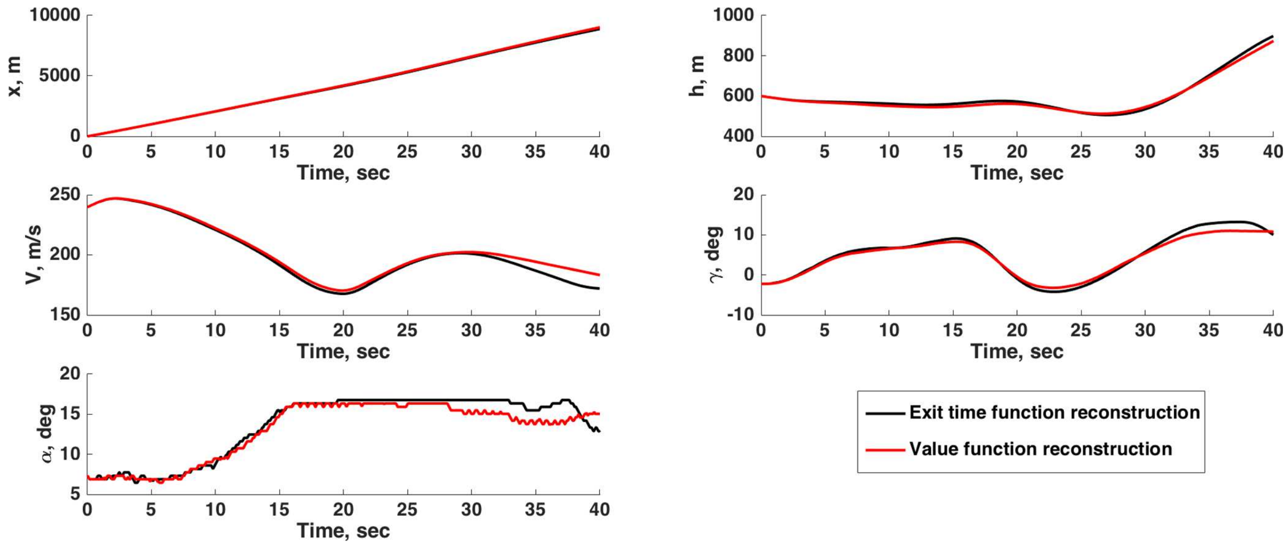

FIGURE 2. (Test 1) Optimal trajectories obtained using value function (red line) and the exit time (black line). (Color online.)

profile is known. Indeed, from the definition of the value function w (see also [8]), the feasibility set is given by:

$$\Omega := \{y \in \mathbb{R}^5,\ \exists z \in [0, 550],\ w(0, y, z) \leq 0\}.$$

Therefore, an approximation of the feasibility set is given by:

$$\Omega^h = \{y \in \mathbb{R}^5,\ \exists z \in [0, 550],\ w^h(0, y, z) \leq 0\}.$$

As an illustration, Figure 3 shows two slices of the feasibility set. The left figure shows the feasible slice obtained in the (v, h) plane, with fixed value $x = 0\,\mathrm{ft}$, $\alpha = 7.373°$ and $\gamma = -2.249°$; the right figure shows the feasible slice obtained in the (v, γ) plane with fixed value $x = 0\,\mathrm{ft}$, $\alpha = 7.373°$ and $h = 600\,\mathrm{ft}$. Both slices where extracted from the value function w^h computed with the grid $80 \times 40 \times 32 \times 16 \times 48 \times 5$.

Figure 4 shows, for illustration purposes, different optimal trajectories corresponding to different initial positions:

- $y_0 = (0.0, 600.0, 239.7, -2.249°, 7.373°)$ (in black);
- $y_1 = (0.0, 550.0, 250.0, -2.249°, 7.373°)$ (in red);
- $y_2 = (0.0, 600.0, 230.0, -1.500°, 7.373°)$ (in blue);
- $y_3 = (0.0, 650.0, 239.7, -3.400°, 7.373°)$ (in green).

The optimal trajectories starting from different initial positions (as shown in Fig. 4) are obtained by using the same approximation of the value function w^h and without requiring any additional numerical efforts.

6.3.2. Test 2: Comparison of different strategies for control β for the maximum running cost problem

In test 1, we have fixed the power factor $\beta \equiv 1$ during the whole time interval $[0, T]$. A more realistic model for β, introduced in [11, 12], considers that at the initial time, when the aircraft begins its landing maneuver, the power factor is equal to a value $\beta_0 < 1$, then the pilot may increase the power until its maximum value, with a constant variation rate, β_1, and then keep it at the maximum level until the end of the maneuver.

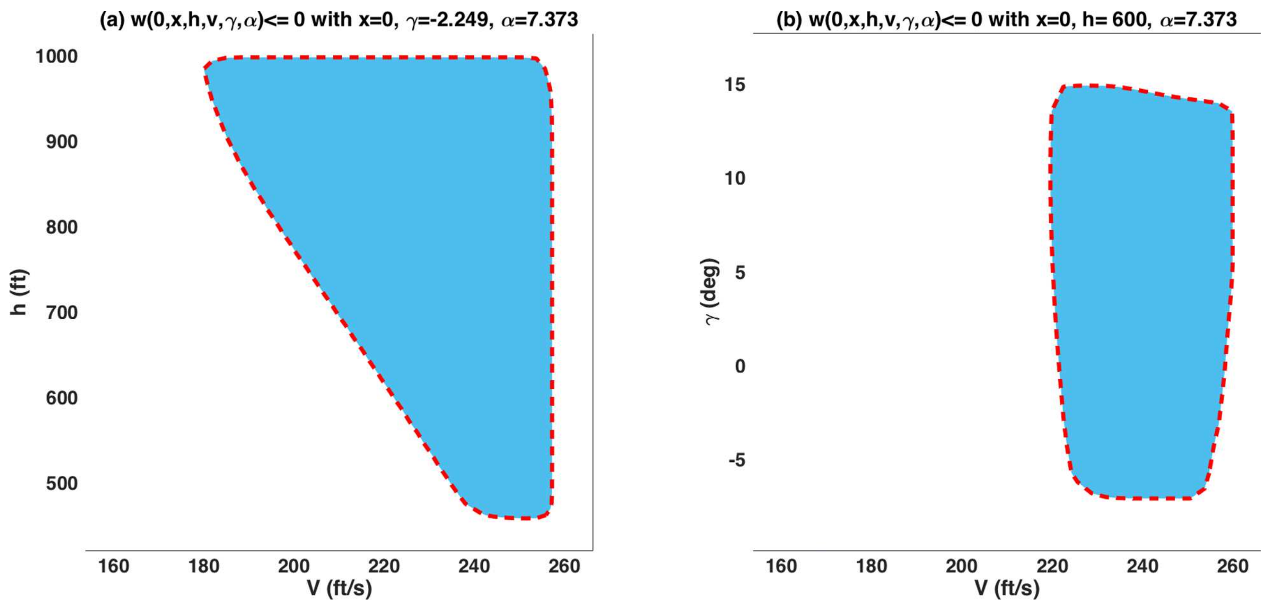

FIGURE 3. (Test 1) Two slices of the negative level set of the value function w. (Color online.)

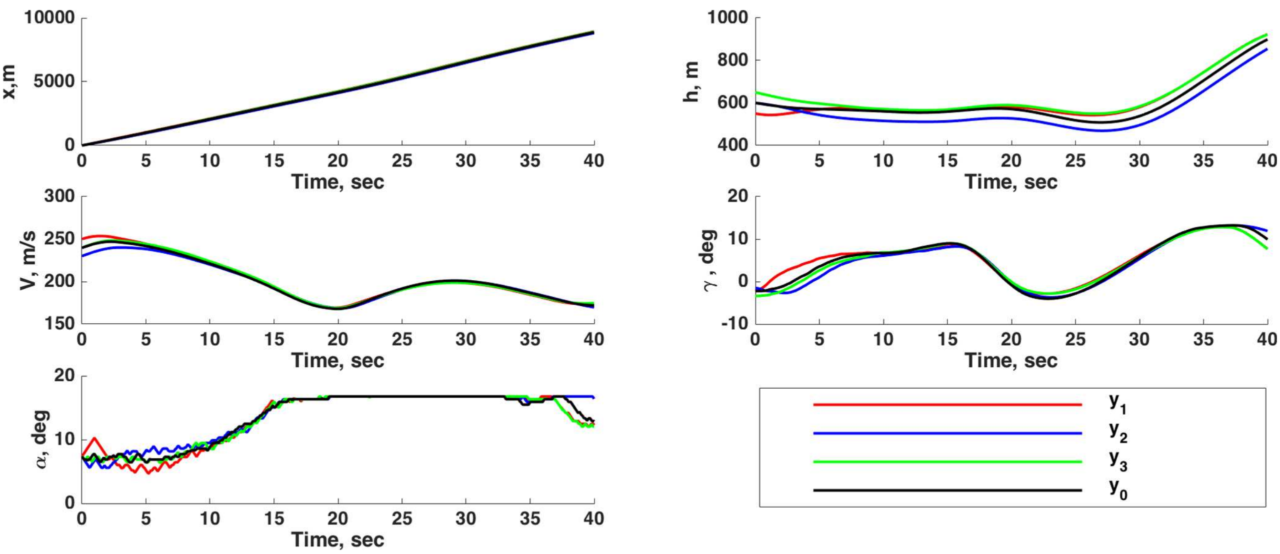

FIGURE 4. (Test 1) Optimal trajectories for different initial conditions. (Color online.)

In this section the following cases are studied and compared (the second case will correspond to the one of [11, 12]):

- **Case 1.** The factor β is fixed to the maximum level : $\beta(t) = 1$. In this case the system is controlled by u, the angular velocity of the trust force orientation angle α:

$$\mathbf{u}(t) := u(t) \in U \equiv [u_{\min}, u_{\max}] \subset \mathbb{R}. \tag{6.6}$$

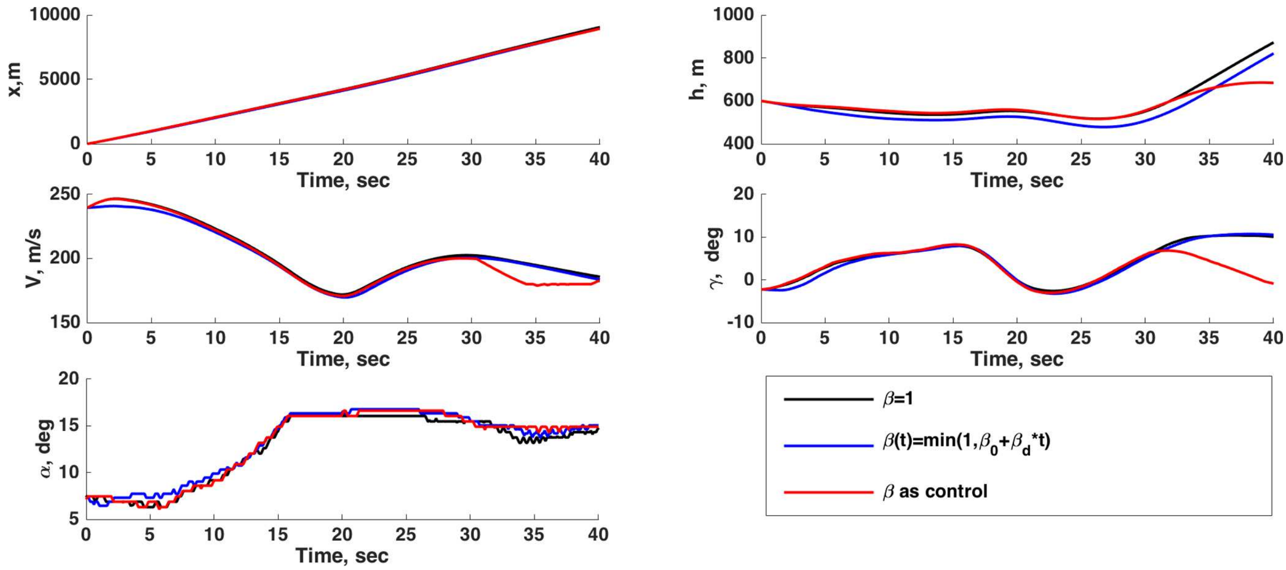

FIGURE 5. (Test 2) Optimal trajectories for different control strategies. (Color online.)

TABLE 4. (Test 2) Higher minimal value h for different control strategies $\beta(\cdot)$.

β strategy		$\min h(.)$
$\beta := 1$	maximum power (case 1)	512.23
$\beta := \min(1, \beta_0 + \beta_d t)$	a prescribed profile (case 2)	478.46
$\beta \in [0, 1]$	controlled power (case 3)	515.90

- **Case 2.** This is the same setting as in Pesch *et al.* [11, 12], where the factor $\beta(t)$ is a known function of time:

$$\beta(t) := \begin{cases} \beta_0 + \beta_d t \text{ if } t \in [0, t_0] \\ 1 \qquad\qquad \text{otherwise} \end{cases} \tag{6.7}$$

where $\beta_0 = 0.3825$, $\beta_d = 0.2$ and $t_0 = (1 - \beta_0)/\beta_d$. In this case the system is again controlled by $\mathbf{u}(\cdot) = u(\cdot)$ as in (6.6).

- **Case 3.** The factor power $\beta(t)$ is considered as a control input. In this case, we have:

$$\mathbf{u}(t) := (u(t), \beta(t)) \in U \equiv [u_{\min}, u_{\max}] \times [\beta_{\min}, \beta_{\max}] \subset \mathbb{R}^2. \tag{6.8}$$

(with $\beta_{\min} = 0$ and $\beta_{\max} = 1$, $u_{\min}$ and $u_{\max}$ defined in Tab. 1).

Let us point out that in cases 1 and 3, the dynamical system (5.3) is autonomous. However, in case 2 where the dynamics depends on a given time-dependent function β (which is not considered as a control input anymore), the control problem becomes non-autonomous. In this case, the link between the reachability time function and the value function does not hold and the reconstruction of optimal trajectories can be performed only by using Algorithm 4.1 or 4.3.

Figure 5 shows the optimal trajectories obtained for the three different cases. From this test, it appears that the strategy of case 2 is not the optimal choice. Indeed, we report in Table 4 the higher minimal values of the altitude $h(\cdot)$ computed by the three different cases presented in this subsection. The computations are done on

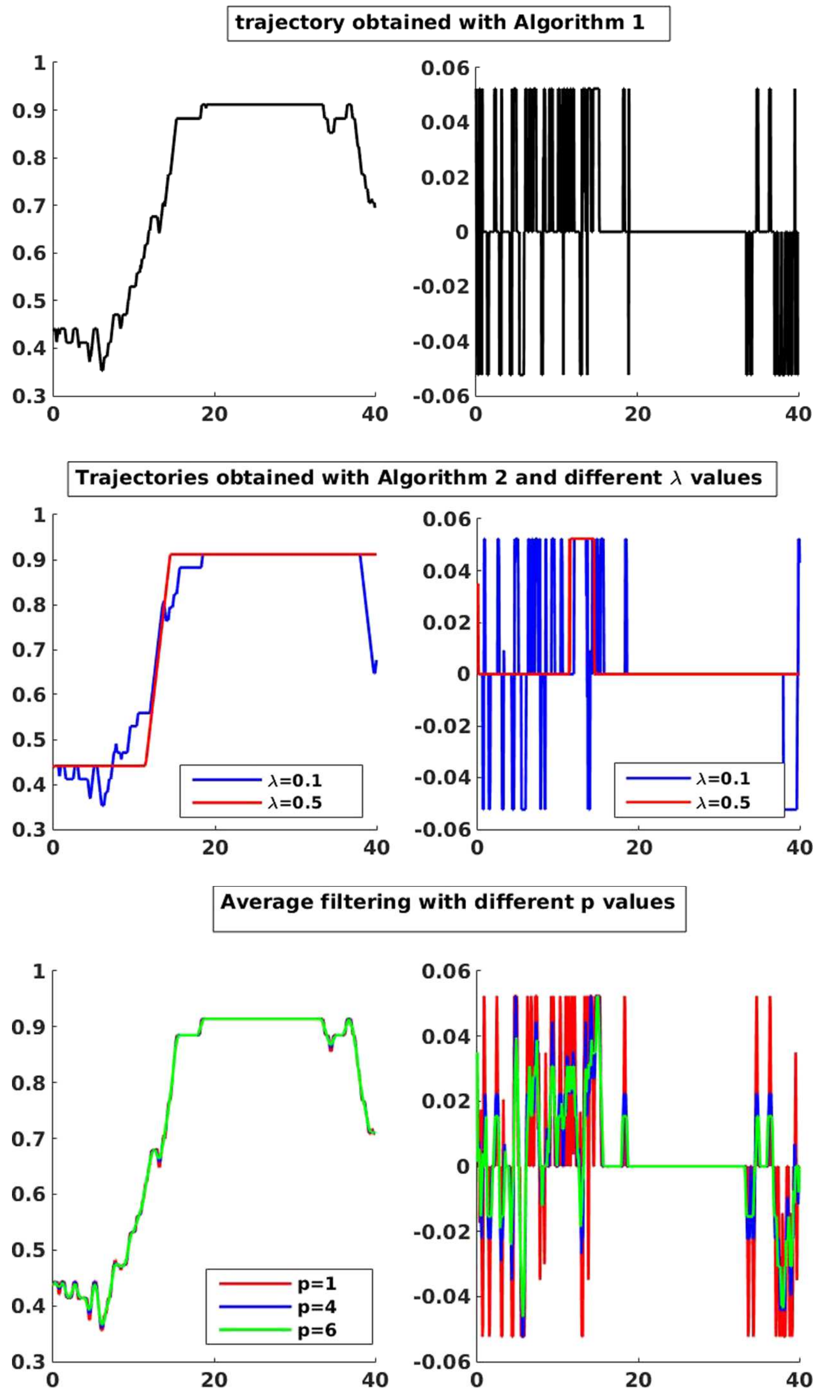

FIGURE 6. (Test 3) Optimal trajectories (*left*) and corresponding control u (*right*), obtained using different reconstruction procedures. (Color online.)

a grid of $80 \times 40 \times 32 \times 16 \times 48 \times 5$ nodes. The results in Table 4 and Figure 5 confirm that optimizing the control β as in case 3 leads to a higher minimal value of $h(.)$.

6.3.3. Test 3: penalisation and post-processing procedures for optimal trajectory reconstruction

In the previous tests, we focused on the reconstruction of optimal trajectories and the corresponding optimal values. However, for an optimal control problem, it is also important to determine a control law associated with the optimal trajectory. An approximation of this law can be obtained by Algorithm 4.1, 4.3 or 4.5. In these algorithms the control value (at each time step) is obtained by a minimization of a simple scalar criterion. It may happen that this criterion admits several minimizers. However, the reconstruction algorithm just requires to take one of these minimizers and can therefore lead to a control function with large oscillations. In order to limit this oscillation phenomenon, it is possible to add a penalisation term in the optimization problems that give the control values (as in the Algorithm 4.3). Another possibility is to perform a reconstruction algorithm to get a (first) control function and then use a posterior filtering process to regularize this function.

We compare different reconstruction procedures here using the exit time function. The aim is to reduce the shattering of the control law. The computational grid used is $60 \times 30 \times 24 \times 12 \times 36 \times 5$ and the same initial point y_0 as in (6.5).

In Figure 6 we show the results obtained with different reconstruction procedures: the fifth state component that is the angle of attack $\alpha(\cdot)$ (*on the left*) and the control $u(.)$ that represents the angular velocity (*on the right*).

The figures on the top line correspond to Algorithm 4.5 (no penalisation term). Here the control function presents large oscillations.

In the he second line of Figure 6, we represent the results obtained by Algorithm 4.5 (but with a penalization term as in Algorithm 4.3). Here, we test two values for λ and we observe that with a quite strong penalization ($\lambda = 0.5$) it is possible to regularize the control function without deteriorating the quality of the trajectory and without significantly modifying the optimum value.

Finally, in the bottom of Figure 6 we have tested a filtering process: we replace the optimal numerical control found by Algorithm 4.5, $(u_k^h)_k$, by an average over a small symmetric window in time

$$\bar{u}_k := \frac{1}{2p+1} \sum_{j=-p,\dots,p} u_{k+j}.$$

We numerically observe a smoothing effect on the control while the trajectory is almost unchanged with respect to the unfiltered solution ($p = 0$).

Appendix A. Numerical data

The data corresponding to a Boeing B 727 aircraft is considered. The wind velocity components relative to the winshear model are satisfying the following relations:

$$w_x(x) = kA(x), \quad w_h(x,h) = k\frac{h}{h_*}B(x), \tag{A.1}$$

where $A(x)$ and $B(x)$ are functions depending only on the x axis given by,

$$A(x) = \begin{cases} -50 + ax^3 + bx^4, & 0 \le x \le 500, \\ \frac{1}{40}(x - 2300), & 500 \le x \le 4100, \\ 50 - a(4600 - x)^3 - b(4600 - x)^4, & 4100 \le x \le 4600, \\ 50, & 4600 \le x, \end{cases}$$

TABLE A.1. Boeing 727 aircraft model and wind data.

	Value	Unit
ρ	2.203×10^{-3}	$\mathrm{Ib\,s^2 ft^{-4}}$
S	1.56×10^3	ft^2
g	32.172	$\mathrm{ft\,s^{-2}}$
mg	1.5×10^5	Ib
δ	3.49×10^{-2}	rad
A_0	4.456×10^4	Ib
A_1	-23.98	$\mathrm{Ib\,s\,ft^{-1}}$
A_2	1.42×10^{-2}	$\mathrm{Ib\,s^2 ft^{-2}}$
B_0	0.1552	
B_1	0.1237	rad^{-1}
B_2	2.4203	rad^{-2}
C_0	0.7125	
C_1	6.0877	rad^{-1}
C_2	$-\,9.0277$	rad^{-2}
α_*	0.2094	rad
k	$\in [0,1]$	
h_*	1000	ft
a	6×10^{-8}	$\mathrm{s^{-1}\,ft^{-2}}$
b	-4×10^{-11}	$\mathrm{s^{-1}\,ft^{-3}}$
c	$-\ln(\frac{25}{30.6}) \times 10^{-12}$	ft^{-4}
d	-8.02881×10^{-8}	$\mathrm{sec^{-1}\,ft^{-2}}$
e	6.28083×10^{-11}	$\mathrm{sec^{-1}\,ft^{-3}}$

$$B(x) = \begin{cases} dx^3 + ex^4, & 0 \le x \le 500, \\ -51\exp(-c(x-2300),^4), & 500 \le x \le 4100, \\ d(4600-x)^3 + e(4600-x)^4, & 4100 \le x \le 4600, \\ 0, & 4600 \le x. \end{cases} \tag{A.2}$$

The constants appearing in the above relations and for the forces and the wind are given in Table A.1.

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