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SCALAR PROBLEMS IN JUNCTIONS OF RODS AND A PLATE
II. SELF-ADJOINT EXTENSIONS AND SIMULATION MODELS

Renata Bunoiu[1], Giuseppe Cardone[2,*] and Sergey A. Nazarov[3]

Abstract. In this work we deal with a scalar spectral mixed boundary value problem in a spacial junction of thin rods and a plate. Constructing asymptotics of the eigenvalues, we employ two equipollent asymptotic models posed on the skeleton of the junction, that is, a hybrid domain. We, first, use the technique of self-adjoint extensions and, second, we impose algebraic conditions at the junction points in order to compile a problem in a function space with detached asymptotics. The latter problem is involved into a symmetric generalized Green formula and, therefore, admits the variational formulation. In comparison with a primordial asymptotic procedure, these two models provide much better proximity of the spectra of the problems in the spacial junction and in its skeleton. However, they exhibit the negative spectrum of finite multiplicity and for these "parasitic" eigenvalues we derive asymptotic formulas to demonstrate that they do not belong to the service area of the developed asymptotic models.

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1. Introduction

1.1. Motivations

As it was observed in [4], an asymptotic expansion of a solution of the Poisson scalar mixed boundary-value problem in a junction of thin rods and a thin plate in a certain range of physical parameters gains the rational dependence on the big parameter $|\ln h|$ where $h > 0$ is a small parameter characterizing the diameters of the rods and the thickness of the plate. Similar asymptotic forms had been discovered for other elliptic problems stated in domains with singular perturbations of the boundaries, see the books [9,11,16]. The aim of this paper is to study the scalar spectral problem associated to the Poisson problem studied in [4].

Keywords and phrases. Junction of thin rods and plate, scalar spectral problem, asymptotics, dimension reduction, self-adjoint extensions of differential operators, function space with detached asymptotics.

[1] University of Lorraine, IECL, CNRS UMR 7502, 3 rue Augustin Fresnel, 57073, Metz, France.

[2] Università del Sannio, Department of Engineering, Corso Garibaldi, 107, 82100 Benevento, Italy.
[*] Corresponding author: giuseppe.cardone@unisannio.it

[3] Mathematics and Mechanics Faculty, St. Petersburg State University, 198504, Universitetsky pr., 28, Stary Peterhof, Russia; Peter the Great St. Petersburg State Polytechnical University, Polytechnicheskaya ul., 29, St. Petersburg, 195251, Russia; Institute of Problems of Mechanical Engineering RAS, V.O., Bolshoj pr., 61, St. Petersburg, 199178, Russia.

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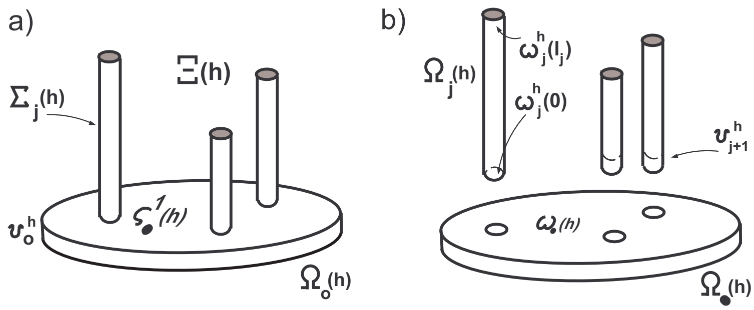

FIGURE 1. The junction (a) and its elements (b).

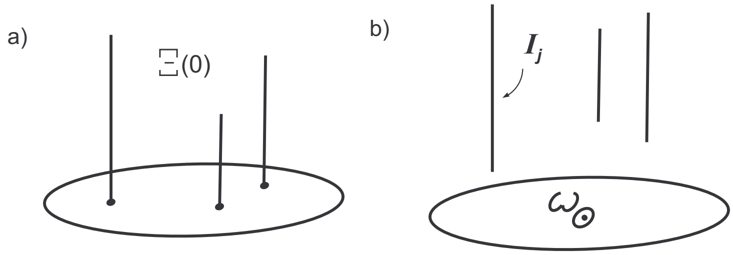

FIGURE 2. The skeleton of the junction, the coupled (a) and disjoint (b) hybrid domain.

Based on the asymptotic procedure in ([16], Sects. 2.2.4, 5.5.2, 9.1.3) and [4,17], it is quite predictable that the asymptotic expansions of the eigenpairs "eigenvalue/eigenfunction" become much more complicated than the one obtained for the solution of the Poisson problem and purchase the holomorphic dependence on the parameter $|\ln h|^{-1}$. Such sophistication of the asymptotic expansions and the lack of algorithms allowing to clarify the appearing holomorphic functions make them almost useless in applications, especially for combined numerical and asymptotic methods.

In [14] Lions announced as an open question an application of the technique of self-adjoint extensions of differential operators for modeling boundary-value problems with singular perturbations. This technique has been employed in [18–20] and others to deal with particular types of perturbations but at our knowledge, was never used before for the study of spectral problems for junctions of thin domains with different limit dimension like 1d and 2d for the rods and plate in our 3d junction.

The use of such techniques in the asymptotic analysis here is the main novelty of our work. We observe that to the spacial junction $\varXi(h)$ represented in Figure 1a, corresponds the hybrid domain $\varXi^0$ depicted in Figure 2a, which consists of several line segments joined to some interior points of a planar domain. Supplying these elements of $\varXi^0$ with differential structures, we describe a family of all self-adjoint operators which is parametrized by a finite set of free parameters and choose an appropriate set by examining the boundary layer phenomenon in the vicinity of the junction zones, namely where the rods are inserted into small sockets in the plate, Figure 1b. As a result, we obtain a model which provides the satisfactory proximity $O(h^{1/2}(1+|\ln h|)^3)$ instead of the uncomfortable one $O((1+|\ln h|)^{-2})$ within a simplified but comprehensible version of the conventional asymptotic procedure. It should be mentioned that our model involves only one scalar integral characteristic of each junction zone.

1.2. Formulation of the spectral problem

Given a small parameter $h \in (0, h_0]$, we introduce the thin plate and rods

$$\Omega_0(h) = \left\{x = (y, z) \in \mathbb{R}^3 : y = (y_1, y_2) \in \omega_0,\ \zeta := h^{-1}z \in (0,1)\right\}, \tag{1.1}$$

$$\Omega_j(h) = \left\{x : \eta^j = h^{-1}\left(y - P^j\right) \in \omega_j,\quad z \in I_j := (0, l_j)\right\},\ j = 1, \ldots, J. \tag{1.2}$$

Here, ω_p, $p = 0, 1, \ldots, J$, are domains in the plane $\mathbb{R}^2$ with smooth (for simplicity) boundaries $\partial\omega_p$ and the compact closures $\overline{\omega}_p = \omega_p \cup \partial\omega_p$; $l_1, \ldots, l_J$ are positive numbers independent of h, and $P^j \in \omega_0$, $P^j \neq P^k$ for $j \neq k$. Reducing a characteristic size of ω_0 to 1, we make Cartesian coordinates and all geometric parameters dimensionless. We fix some $h_0 \in (0, \min\{l_1, \ldots, l_J\})$ such that, for $h \in (0, h_0]$, $\overline{\omega}^h_j \subset \omega_0$ and $\overline{\omega}^h_j \cap \overline{\omega}^h_k = \emptyset$, $j \neq k$, where $\omega^h_j = \left\{y : \eta^j \in \omega_j\right\}$ is the cross section of the rod $\Omega_j(h)$ while $\omega^h_j(0)$ and $\omega^h_j(l_j)$ are its lower and upper ends. In the sequel the bound h_0 can be diminished but always remains strictly positive.

The rods (1.2) are plugged into sockets, *i.e.*, holes in the plate, Figure 1b and 1a,

$$\Omega_\bullet(h) = \omega_\bullet(h) \times (0, h), \quad \omega_\bullet(h) = \omega_0 \setminus \left(\overline{\omega}^h_1 \cup \ldots \cup \overline{\omega}^h_J\right), \tag{1.3}$$

and compose the junction

$$\Xi(h) = \Omega_\bullet(h) \cup \Omega_1(h) \cup \ldots \cup \Omega_J(h). \tag{1.4}$$

The lateral side of the plate (1.1) and (1.3) and their bases are denoted by $\upsilon_0(h) = \partial\omega_0 \times (0, h)$ and

$$\varsigma^i_0(h) = \{x : y \in \omega_0,\ z = ih\}, \quad \varsigma^i_\bullet(h) = \{x : y \in \omega_\bullet,\ z = ih\}, \quad i = 0, 1. \tag{1.5}$$

The lateral side of the rod $\Omega_j(h)$ is divided into two parts

$$\Sigma_j(h) = \partial\omega^h_j \times (h, l_j), \quad \upsilon^h_j = \partial\omega^h_j \times (0, h),$$

where the latter is the junctional boundary of the socket.

In the junction (1.4) we consider a spectral problem consisting of the differential equations with the Laplace operator

$$-\Delta_x u_0(h, x) = \lambda(h) u_0(h, x), \quad x \in \Omega_\bullet(h), \tag{1.6}$$

$$-\gamma_j(h)\Delta_x u_j(h, x) = \lambda(h)\rho_j(h) u_j(h, x), \quad x \in \Omega_j(h), \tag{1.7}$$

the Neumann and Dirichlet boundary conditions

$$\partial_\nu u_0(h, x) = 0, \quad x \in \Sigma_\bullet(h) = \upsilon_0(h) \cup \varsigma^0_\bullet(h) \cup \varsigma^1_\bullet(h), \tag{1.8}$$

$$\gamma_j(h)\partial_\nu u_j(h, x) = 0, \quad x \in \Sigma_j(h) \cup \omega^h_j(0), \tag{1.9}$$

$$u_j(h, x) = 0, \quad x \in \omega^h_j(l_j), \tag{1.10}$$

and the transmission conditions

$$u_0(h, x) = u_j(h, x), \quad x \in \upsilon^h_j, \tag{1.11}$$

$$\partial_\nu u_0(h, x) = \gamma_j(h)\partial_\nu u_j(h, x), \quad x \in \upsilon^h_j. \tag{1.12}$$

Here, $\lambda(h)$ is a spectral parameter, $j = 1, \ldots, J$, u_0 and u_j are restrictions of the function u on $\Omega_\bullet(h)$ and $\Omega_j(h)$, respectively, and ∂_ν is the outward normal derivative at the surface of $\overline{\Xi(h)}$ in (1.8) and (1.9) while in the second transmission condition ∂_ν is outward with respect to the rods. In what follows we mainly deal with the coefficients

$$\gamma_j(h) = \gamma_j h^{-\alpha},\ \gamma_j > 0, \qquad \rho_j(h) = \rho_j h^{-\alpha},\ \rho_j > 0, \tag{1.13}$$

in the most informative but complicated case

$$\alpha = 1. \tag{1.14}$$

We emphasize that in the case (1.14) the limit passage $h \to +0$ in the problem (1.6)−(1.12) leads to the eigenvalue problem for a self-adjoint operator in the skeleton of (1.4) in Figure 2a, a hybrid domain, while, for $\alpha > 1$ and $\alpha < 1$, this limit operator decouples, *cf.* Figure 2b. We discuss the latter cases in Section 5 and pay attention to the homogeneous junction with

$$\alpha = 0, \quad \gamma_j = \rho_j = 1, \quad j = 1, \dots, J. \tag{1.15}$$

Other cases will be discussed in Section 5. The factors γ_j and ρ_j in (1.13) are real positive numbers.

The variational formulation of the problem (1.6)−(1.12) reads:

$$a\left(u, w; \varXi\left(h\right)\right) = \lambda(h) b\left(u, w; \varXi\left(h\right)\right) \quad \forall w \in H_0^1\left(\varXi\left(h\right); \varGamma\left(h\right)\right) \tag{1.16}$$

where $H_0^1\left(\varXi\left(h\right); \varGamma\left(h\right)\right)$ is the Sobolev space of functions satisfying the Dirichlet conditions (1.10) on $\varGamma\left(h\right) = \omega_1^h\left(l_1\right) \cup \ldots \cup \omega_J^h\left(l_J\right)$,

$$a\left(u, v; \varXi\left(h\right)\right) = \left(\nabla_x u_0, \nabla_x w_0\right)_{\varOmega_\bullet(h)} + \sum\nolimits_j \gamma_j\left(h\right)\left(\nabla_x u_j, \nabla_x w_j\right)_{\varOmega_j(h)}, \tag{1.17}$$
$$b\left(u, v; \varXi\left(h\right)\right) = \left(u_0, w_0\right)_{\varOmega_\bullet(h)} + \sum\nolimits_j \rho_j\left(h\right)\left(u_j, w_j\right)_{\varOmega_j(h)},$$

$(\ )_{\varXi(h)}$ is the natural scalar product in the Lebesgue space $L^2\left(\varXi\left(h\right)\right)$ and $\sum_j$ everywhere stands for summation over $j = 1, \dots, J$.

In view of the compact embedding $H^1\left(\varXi\left(h\right)\right) \subset L^2\left(\varXi\left(h\right)\right)$ the spectral problem (1.16) possesses, for every fixed h, the following positive monotone unbounded sequence of eigenvalues

$$0 < \lambda^1(h) < \lambda^2(h) \le \ldots \le \lambda^n(h) \le \ldots \to +\infty \tag{1.18}$$

listed according to their multiplicity. The corresponding eigenfunctions $u^1(h, \cdot),\ u^2(h, \cdot), \dots,\ u^n(h, \cdot), \dots \in H_0^1\left(\varXi\left(h\right); \varGamma\left(h\right)\right)$ can be subject to the normalization and orthogonality conditions

$$b\left(u^n, v^m; \varXi\left(h\right)\right) = \delta_{n,m}, \quad n, m \in \mathbb{N} = \left\{1, 2, 3, \ldots\right\}, \tag{1.19}$$

where $\delta_{n,m}$ is the Kronecker symbol.

1.3. The hybrid domain

The asymptotic analysis which has been presented at length, for example, in [4–6] and, in particular, includes the dimension reduction procedure, converts the differential equations (1.6) and (1.7) with the Neumann boundary conditions (1.8) and (1.9), respectively, into the limit equations

$$-\varDelta_y v_0(y) = \mu v_0(y), \quad y \in \omega_\odot, \tag{1.20}$$
$$-\gamma_j |\omega_j| \partial_z^2 v_j(z) = \mu \rho_j |\omega_j| v_j(z), \quad z \in (0, l_j), \tag{1.21}$$

where $\omega_\odot$ is the punctured domain $\omega_0 \setminus \mathcal{P}$, $\mathcal{P} = \left\{P^1, \dots, P^J\right\}$ and $|\omega_j|$ is the area of the domain ω_j. Moreover, a primary examination of the boundary layer phenomenon near the lateral side $\upsilon_0\left(h\right)$ of the plate and the end $\omega_j^h(l_j)$ of the rod $\varOmega_j\left(h\right)$, respectively, gives the following boundary conditions

$$\partial_\nu v_0(y) = 0, \quad y \in \partial\omega_0, \tag{1.22}$$
$$v_j(l_j) = 0. \tag{1.23}$$

However, the one-dimensional and two-dimensional problems are not completed yet due to the lack of boundary conditions at the endpoints $z = 0$ of the intervals $(0, l_j)$ and because the differential equation (1.20) is fulfilled for sure only outside the points $P^1, \ldots, P^J$, since near the sockets $\omega_j^h \times (0, h)$ the geometrical structure of the junction (1.4) changes and becomes crucially spacial so that the dimension reduction does not work. As was shown in [4], this observation requires to consider solutions of the problem (1.20), (1.22) with logarithmic singularities at the points in the set $\mathcal{P}$. We also will take in the sequel such singular solutions into account, however further considerations in this paper diverge from the asymptotic analysis used in [4]. Indeed, we will provide two abstract but applicable formulations of a spectral problem in the hybrid domain in Figure 2a, which give an approximation of the spectrum (1.18) with relatively high precision. First, we detect a self-adjoint operator as an extension of the differential operator of the problem (1.20)−(1.23) supplied (*cf.* (2.3) and (2.1)) with the restrictive conditions

$$v_0(P^j) = 0, \quad v_j(0) = \partial_z v_j(0) = 0, \quad j = 1, \ldots, J. \tag{1.24}$$

Second, we construct certain point conditions at $P^1, \ldots, P^J$ which tie the independent problems (1.20), (1.22) and (1.21), (1.23) into a formally self-adjoint problem in the hybrid domain Ξ^0. These two formulations happen to be equivalent and both are realized as operators with the discrete spectrum which, in the low-frequency range, approximate the spectrum of the problem (1.6)−(1.12) (or, equivalently, (1.16)) with admissible precision[4] $O(h^{1/2}|\ln h|^3)$. Unfortunately, serving for a particular range of the spectrum, both the operators lose the positivity property and gain so called *parasite* eigenvalues which are negative and big, of order h^{-2}; therefore, we prove that they lay outside the scope of the asymptotic models and have no relation to the original problem. In order to furnish, for example, an application of the minimum principle, (*cf.* [2], Thm. 10.2.1), we construct detailed asymptotics of these *parasite* eigenvalues and of the corresponding eigenfunctions, which are located in the very vicinity of the points $P^1, \ldots, P^J$ and decay exponentially at a distance from them.

Parameters of the self-adjoint extension and ingredients of the point conditions are found out with the help of the method of matched asymptotic expansions on the basis of special solutions described in the first part [4] of our work. Both linearly depend on $|\ln h|$ and this makes the eigenvalues and eigenvectors to be real analytic functions in $|\ln h|^{-1}$. However, a possible numerical realization of the models with a small but fixed parameter h does not require to take into account such complication of asymptotic expansions. We, of course, compute explicitly couple of initial terms of the convergent series in $|\ln h|^{-1}$.

1.4. Outline of the paper

In Section 2 we describe all self-adjoint extensions of the operator of the problem (1.20)−(1.24) as well as the point conditions which involve the problems (1.20), (1.22) and (1.21), (1.23) into a symmetric generalized Green formula. This material is known and is presented in a condensed form, mainly in order to introduce the notation and explain some technicalities used throughout the paper. We refer to the review papers [3,21,29] for a detailed information. If the skeleton $\Xi(0)$ decouples in the limit, see Figure 2b, then the extended operator of the Neumann problem (1.20), (1.22) may require for "potentials of zero radii" [1,29] or "pseudo-Laplacian" in the terminology [7], but in this case the ordinary differential equations (1.21), (1.23) are supplied with either Neumann, or Dirichlet condition at the endpoints $z = 0$ of the interval (*cf.*, Rem. 2.5, 2.6 and see the paper [3] which provides the complete description of the techniques of self-adjoint extensions).

The most interesting situation occurs under the restriction (1.14) when the skeleton does not decouple in the limit, see Figure 2a. The asymptotic procedure in [4] allow us to determine in Section 3 appropriate parameters of a particular self-adjoint extension serving for the original problem (1.6)−(1.12) as well as all ingredients of the point condition in the corresponding hybrid model.

The most cumbersome part of our analysis is concentrated in Section 4, where we perform the justification of our asymptotic models with the help of weighted estimates obtained in [4]. We state here the main result of the

[4]The error estimates are derived in the paper with quite simple tools. Advanced estimation may detect the accuracy $O(h|\ln h|)$ and extend the proximity property of the models to a part of the mid-frequency range, *cf.* [20]. The latter, however, enlarges enormously massif of calculations.

paper, Theorem 4.4. Section 5 contains some simple asymptotic formulas and the example of the homogeneous junction, see (1.15).

2. General statement of problems on the hybrid domain

2.1. Unbounded operators and their adjoints

Let A_j be an unbounded operator in the Lebesgue space $L^2(I_j)$ with the differential expression $-\gamma_j|\omega_j|\partial_z^2$ and the domain

$$\mathcal{D}(A_j) = \left\{w_j \in H^2(I_j) : v_j(l_j) = 0,\ v_j(0) = \partial_z v_j(0) = 0\right\}. \tag{2.1}$$

The operator is symmetric and closed. By a direct calculation, it follows that the adjoint operator A_j^* gets the same differential expression but its domain is bigger, namely

$$\mathcal{D}(A_j^*) = \left\{w_j \in H^2(I_j) : v_j(l_j) = 0\right\}. \tag{2.2}$$

Hence, $\dim(\mathcal{D}(A_j^*)/\mathcal{D}(A_j)) = 2$ and the defect index of A_j is $1:1$.

Analogously, we introduce the unbounded operator A_0 in $L^2(\omega_0)$ with the differential expression $-\Delta_y = -\partial^2/\partial y_1^2 - \partial^2/\partial y_2^2$ and the domain

$$\mathcal{D}(A_0) = \left\{w_0 \in H^2(\omega_0) : \partial_\nu w_0 = 0 \text{ on } \partial\omega_0,\ w_j(P^k) = 0,\ k = 1, \dots, J\right\}. \tag{2.3}$$

By virtue of the Sobolev embedding theorem $H^2(\omega_0) \subset C(\omega_0)$ and the classical Green formula

$$-(\Delta_y w_0, v_0)_{\omega_0} + (\partial_\nu w_0, v_0)_{\partial\omega_0} = -(w_0, \Delta_y v_0)_{\omega_0} + (w_0, \partial_\nu v_0)_{\partial\omega_0}, \tag{2.4}$$

the operator A_0 is closed and symmetric. The following lemma, in particular, shows that the defect index of A_0 is $J \times J$.

Lemma 2.1. *The adjoint operator A_0^* for A_0 has the differential expression $-\Delta_y$ and the domain*

$$\mathcal{D}(A_0^*) = \Big\{V_0 \in L^2(\omega_0) : V_0(y) = \widehat{V}_0(y) - \frac{1}{2\pi}\textstyle\sum_j b_j \chi_j(y) \ln|y - P^j|, \tag{2.5}$$
$$b_j \in \mathbb{C},\ \widehat{V}_0(y) \in H^2(\omega_0),\ \partial_\nu \widehat{V}_0 = 0 \quad \text{on} \quad \partial\omega_0\Big\},$$

where $\chi_1, \dots, \chi_J \in C_c^\infty(\omega_0)$ are cut-off functions such that

$$\chi_j(P^j) = 1, \quad \chi_j(y)\chi_k(y) = 0 \text{ for } j \neq k, \quad \operatorname{supp}\chi_j \subset \omega_0.$$

Proof. By definition, a function $V_0 \in L^2(\omega_0)$ belongs to $\mathcal{D}(A_0^*)$ if and only if the following integral identity holds:

$$-(V_0, \Delta_y v_0)_{\omega_0} = (F_0, v_0)_{\omega_0} \quad \forall v_0 \in \mathcal{D}(A_0). \tag{2.6}$$

At the first step we take $v_0 \in C_c^\infty(\overline{\omega}_0 \setminus \mathcal{P}) \cap \mathcal{D}(A_0)$. Based on the Green formula (2.4), we recall the Neumann boundary condition in (2.3) and apply classical results ([15], Sects. 3−6, Chap. 2) on lifting smoothness of solutions to elliptic problems. In this way we conclude that $V_0 \in H^2_{loc}(\overline{\omega}_0 \setminus \mathcal{P})$ and

$$-\Delta_y V_0(y) = F_0(y),\ y \in \omega_\odot = \omega_0 \setminus \mathcal{P}, \quad \partial_\nu V_0(y) = 0,\ y \in \partial\omega_0. \tag{2.7}$$

The next step requires for results [8] of the theory of elliptic problem in domains with conical points. Indeed, regarding P^j as the top of the "complete cone" $\mathbb{R}^2 \setminus P^j$, that is, the punctured plane, we introduce the Kondratiev space $V^l_\beta(\omega_0)$ with $l \in \{0,1,2,\dots\}$ and $\beta \in \mathbb{R}$ as the completion of $C^\infty_c(\overline{\omega}_0 \setminus \mathcal{P})$ with respect to the weighted norm

$$||v_0; V^l_\beta(\omega_0)|| = \left(\sum_{k=0}^{l} || \min\{r_1,\dots,r_J\}^{\beta-l-k} \nabla^k_y v_0; L^2(\omega_0)^2||\right)^{1/2} \tag{2.8}$$

where $r_j = |y-P^j|$ and $\nabla^k_y v_0$ stands for a collection of all order k derivatives of v_0. Clearly, $L^2(\omega_0) \subset V^0_\delta(\omega_0)$ and $V^2_\delta(\omega_0) \subset H^1(\omega_0)$ for $\delta \in [0,1]$.

Since $V_0 \in L^2(\omega_0) \subset V^0_0(\omega_0)$, the theorem on asymptotics [8], (see, *e.g.*, [25], Sects. 3.5, 4.2, 6.4) and also the introductory chapters in the books [10, 25], gives the representation

$$V_0(y) = \widetilde{V}_0(y) + \textstyle\sum_j \chi_j(y)\left(a_j - \dfrac{b_j}{2\pi}\ln r_j\right) \tag{2.9}$$

as well as the inclusion $\widetilde{V}_0 \in V^2_\delta(\omega_0)$ with any $\delta > 0$ and the estimate

$$||\widetilde{V}_0; V^2_\delta(\omega_0)|| + \textstyle\sum_j(|a_j|+|b_j|) \le c_\delta(\|F_0; L^2(\omega_0)\| + \|V_0; L^2(\omega_0)\|). \tag{2.10}$$

Hence, the sum

$$\widehat{V}_0(y) = V_0(y) + \frac{1}{2\pi}\sum\nolimits_j b_j \ln r_j \tag{2.11}$$

belongs to $H^1(\omega_0)$ and still solves the problem (2.7) with a new right-hand side $\widehat{F}_0 \in L^2(\omega_0)$ in the Poisson equation. Thus, referring to ([15], Sect. 9, Chap. 2) we have $\widehat{V}_0 \in H^2(\omega_0)$ and, therefore, V_0 falls into the linear set (2.5). $\square$

Remark 2.2. The representation (2.9) can be derived by means of the Fourier method but the Kondratiev theory [8] helps to avoid any calculation.

2.2. The generalized Green formula

The norm, *cf.* the left-hand side of (2.10),

$$\|V_0; \mathfrak{H}_0\| = (||\widehat{V}_0; H^2(\omega_0)||^2 + \textstyle\sum_j |b_j|^2)^{1/2} \tag{2.12}$$

brings Hilbert structure to the linear space (2.5) denoted by $\mathfrak{H}_0$. By $\mathfrak{H}$, we understand the direct product

$$\mathfrak{H} = \mathfrak{H}_0 \times \mathfrak{H}_1 \times \ldots \times \mathfrak{H}_J \tag{2.13}$$

where $\mathfrak{H}_j$ is the linear space (2.2) with the Sobolev H^2-norm. Moreover, taking into account the right-hand sides of the equations (1.20) and (1.21) we supply the vector Lebesgue space $\mathfrak{L}$ with the special norm

$$||v; \mathfrak{L}|| = (||v_0; L^2(\omega_0)||^2 + \textstyle\sum_j \rho_j |\omega_j| \; ||v_j; L^2(I_j)||^2)^{1/2}, \tag{2.14}$$

with $v = (v_0, v_1, \dots, v_J) \in \mathfrak{L} := L^2(\omega_0) \times L^2(I_1) \times \ldots \times L^2(I_J)$.

We also introduce two continuous projections $\wp_\pm : \mathfrak{H} \to \mathbb{R}^{2J}$ by the formulas

$$\begin{aligned} \wp_+ v &= (\wp'_+ v, \wp''_+ v) = (\widehat{v}_0(P^1), \dots, \widehat{v}_0(P^J), v_1(0), \dots, v_J(0)), \\ \wp_- v &= (\wp'_- v, \wp''_- v) = (b_1, \dots, b_J, -\gamma_1|\omega_1|\partial_z v_1(0), \dots, -\gamma_J|\omega_J|\partial_z v_J(0)), \end{aligned} \tag{2.15}$$

where $v = (v_0, v_1, \dots, v_J) \in \mathfrak{H}$ and $b_1, \dots, b_J$, $\widehat{v}_0$ are attributes of the decomposition (2.9) of $v_0 \in \mathfrak{H}_0$.

The next assertion is but a concretization of a general result in [24, 26, 27], see also ([25], Sect. 6.2), however we give a condensed and much simplified proof for reader's convenience.

Proposition 2.3. *For v and $w = (w_0, w_1, \dots, w_J)$ in $\mathfrak{H}$, the generalized Green formula*

$$q(v,w) := -(\Delta_y v_0, w_0)_{\omega_0} + (v_0, \Delta_y w_0)_{\omega_0} - \sum_j \gamma_j |\omega_j| ((\partial_z^2 v_j, w_j)_{I_j} - (v_j, \partial_z^2 w_j)_{I_j}) \\ = \langle \wp_+ v, \wp_- w\rangle - \langle \wp_- v, \wp_+ w\rangle \tag{2.16}$$

is valid, where $\langle\ \rangle$ stands for the natural scalar product in $\mathbb{R}^{2J}$.

Proof. First of all, we write the evident identity

$$-(\partial_z^2 v_j, w_j)_{I_j} + (v_j, \partial_z^2 w_j)_{I_j} = w_j(0)\partial_z v_j(0) - v_j(0)\partial_z w_j(0) \tag{2.17}$$

and multiply it with $\gamma_j|\omega_j|$. Then we take $v_0, w_0 \in \mathfrak{H}_0$ and write the standard Green formula

$$-(\Delta_y v_0, w_0)_{\omega_0} + (v_0, \Delta_y w_0)_{\omega_0} = \lim_{\delta\to+0}(-(\Delta_y v_0, w_0)_{\omega_\delta} + (v_0, \Delta_y w_0)_{\omega_\delta}) \\ = -\lim_{\delta\to+0}\sum_j \delta \int_0^{2\pi} \left(\left(\widehat{w}(P^j) - \frac{b_j^w}{2\pi}\ln r_j\right)\frac{\partial}{\partial r_j}\frac{b_j^v}{2\pi}\ln r_j - \left(\widehat{v}(P^j) - \frac{b_j^v}{2\pi}\ln r_j\right)\frac{\partial}{\partial r_j}\frac{b_j^w}{2\pi}\ln r_j\right)\mathrm{d}\varphi_j \\ = -\sum_j (\widehat{w}(P^j) b_j^v - \widehat{v}(P^j) b_j^w) \tag{2.18}$$

where $\omega_\delta = \omega_0 \setminus \mathbb{B}_\delta(P^j)$, $\mathbb{B}_\delta(P^j) = \{y : r_j < \delta\}$ is a disk and $(r_j, \varphi_j) \in \mathbb{R}_+ \times [0, 2\pi)$ is the polar coordinate system centered at P^j. Now (2.16) follows from (2.17), (2.18) and (2.15). $\square$

2.3. Self-adjoint extensions

Calculations in Section 2.2 detect the defect index $2J : 2J$ of the operator $A = (A_0, A_1, \dots, A_J)$ with the differential expression $(-\Delta_y v_0, -\gamma_1\rho_1^{-1}\partial_z^2, \dots, -\gamma_J\rho_J^{-1}\partial_z^2)$ in the Hilbert space $\mathfrak{L}$, see (2.14). Hence, this operator admits a self-adjoint extension $\mathcal{A}$, that is, $A \subset \mathcal{A} \subset A^*$ and $\mathcal{A} = \mathcal{A}^*$.

Since $\mathcal{A}$ is a restriction of $A^* = (A_0^*, A_1^*, \dots, A_J^*)$, we conclude that

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}^*) = \{v \in \mathcal{D}(\mathcal{A}) : (\wp_+ v, \wp_- v) \in \mathcal{R}\} \tag{2.19}$$

where a linear subspace $\mathcal{R} \subset \mathbb{R}^{4J}$ of dimension $2J$ must be chosen such that in accord with Proposition 2.3

$$q(v,w) = 0 \quad \forall v, w \in \mathcal{D}(\mathcal{A}). \tag{2.20}$$

The symplectic form (2.16) is actually defined on the factor space $\mathcal{D}(\mathcal{A}^*)/\mathcal{D}(\mathcal{A}) \approx \mathbb{R}^{4J}$ because the generalized Green formula demonstrates that

$$0 = q(v,w) = -\overline{q(w,v)} \ \text{ for } v \in \mathcal{D}(\mathcal{A}),\ w \in \mathcal{D}(\mathcal{A}^*). \tag{2.21}$$

Description of null spaces of a symplectic form in Euclidean space is a primary algebraic question, *cf.* [13], but it gives a direct identification of all self-adjoint extensions of our operator A, see [1,7,31] and, *e.g.*, [3,22,29].

Proposition 2.4. *Let $\mathcal{R}^+ \oplus \mathcal{R}^0 \oplus \mathcal{R}^-$ be an orthogonal decomposition of $\mathbb{R}^{2J}$ and let $\mathcal{S}$ be a symmetric invertible operator in $\mathcal{R}^0$. The restriction $\mathcal{A}$ of the operator A^* onto the domain*

$$\mathcal{D}(\mathcal{A}) = \left\{v \in \mathcal{D}(A^*) : \wp_+ v = t^+ + \mathcal{T} t^0,\ \wp_- v = t^- + t^0,\ t^\alpha \in \mathcal{R}^\alpha,\ \alpha = 0, \pm\right\} \tag{2.22}$$

is a self-adjoint extension of the operator A in $\mathfrak{L}$. Any self-adjoint extension of A can be obtained in this way.

Remark 2.5. If we put

$$\mathcal{R}^- = \mathcal{R}^0 = \{0\}^{2J}, \quad \mathcal{R}^+ = \mathbb{R}^{2J}, \tag{2.23}$$

then the self-adjoint extension $\mathcal{A}^0$ given in Proposition 2.4 is nothing but the set of the two-dimensional Neumann problem

$$-\Delta_y v_0(y) = f_0(y), \quad y \in \omega_0, \qquad \partial_\nu v_0(y) = 0, \quad y \in \partial\omega_0, \tag{2.24}$$

and the one-dimensional mixed boundary-value problems

$$-\gamma_j|\omega_j|\partial_z^2 v_j(z) = f_j(z), \ z \in (0, l_j), \quad v_j(l_j) = \gamma_j|\omega_j|\partial_z v_j(0) = 0. \tag{2.25}$$

These problems are independent and are posed on the spaces $H^2(\omega_0)$ and $H^2(I_j)$, respectively. The corresponding operator has the kernel spanned over the constant vectors $(c_0, 0, \dots, 0)$.

Remark 2.6. In the case $\mathcal{R}^- = \{0\}^J \times \mathbb{R}^J$, $\mathcal{R}^0 = \{0\}^{2J}$, $\mathcal{R}^+ = \mathbb{R}^J \times \{0\}$, we obtain a self-adjoint extension which gives rise to the Neumann problem (2.24) and a set of the Dirichlet problems

$$-\gamma_j|\omega_j|\partial_z^2 v_j(z) = f_j(z), \ z \in (0, l_j), \quad v_j(l_j) = v_j(0) = 0.$$

In Section 3 we come across a self-adjoint extension $\mathcal{A} = \mathcal{A}^h$ (the superscript will appear in Sect. 3.2) with the following attributes in (2.22):

$$\mathcal{R}^- = \{0\}^{2J}, \ \mathcal{R}^0 = \left\{ \begin{pmatrix} c \\ -c \end{pmatrix} : c \in \mathbb{R}^J \right\}, \ \mathcal{R}^+ = \left\{ \begin{pmatrix} c \\ c \end{pmatrix} : c \in \mathbb{R}^J \right\}, \ \mathcal{T} = \begin{pmatrix} \mathbb{O} & \frac{1}{2}S \\ \frac{1}{2}S & \mathbb{O} \end{pmatrix} \tag{2.26}$$

where $S = S^h$ is a symmetric non-degenerate matrix of size $J \times J$. In the next section we will give a different formulation of the abstract equation

$$\mathcal{A}^h v = f \in \mathcal{L} \tag{2.27}$$

which will help us to study the spectrum of $\mathcal{A}^h$.

2.4. The differential problem with point conditions

Let $\wp'_\pm$ and $\wp''_\pm$ be projections $: \mathfrak{H} \to \mathbb{R}^J$ defined in (2.15). Following [22,23,28], we rewrite relations imposed on $\wp'_\pm v$ and $\wp''_\pm v$ in (2.22) according to (2.26), as the *point conditions*

$$\wp''_+ v - \wp'_+ v - S\wp'_- v = 0 \in \mathbb{R}^J, \tag{2.28}$$
$$\wp'_- v + \wp''_- v = 0 \in \mathbb{R}^J. \tag{2.29}$$

We also will deal with the inhomogeneous equation

$$\wp''_+ v - \wp'_+ v - S\wp'_- v = k \in \mathbb{R}^J. \tag{2.30}$$

The problems

$$-\Delta_y v_0(y) = f_0(y), \quad y \in \omega_\odot, \qquad \partial_\nu v_0(y) = 0, \quad y \in \partial\omega \tag{2.31}$$
$$-\gamma_j|\omega_j|\partial_z^2 v_j(z) = f_j(z), \ z \in (0, l_j), \quad v_j(l_j) = 0 \tag{2.32}$$

with the point conditions (2.29), (2.30) give rise to the continuous mapping

$$\mathfrak{A} : \mathfrak{H}_- = \{v \in \mathfrak{H} : \wp'_- v + \wp''_- v = 0\} \to \mathfrak{R} := \mathfrak{L} \times \mathbb{R}^J. \tag{2.33}$$

Remark 2.7. Simple algebraic transformations demonstrate that, under circumstances (2.22) and (2.26), a solution of the equation (2.27) is a solution of the problem (2.31), (2.32), (2.28), (2.29) and *vice versa*.

Proposition 2.8. *The operator* $\mathfrak{A}$ *in* (2.33) *is Fredholm of index zero.*

Proof. The point conditions $\wp'_- v = 0$, $\wp''_+ v = 0$ in Remark 2.6 generate the Fredholm operator of index zero

$$H^2(\omega_0) \times \textstyle\prod_{j=1}^{J} (H^2(I_j) \cap H^1_0(I_j)) \to \mathfrak{L}. \tag{2.34}$$

The operator (2.33) is a finite dimensional, *i.e.* compact, perturbation of (2.34) and thus keeps the Fredholm property. Since $S = S^T$, the generalized Green formula (2.16) can be written in the symmetric form reflecting the particular point conditions (2.28), (2.29)

$$\begin{aligned} q(v, w) = &\left\langle \wp'_+ v - \wp''_+ v + S\wp'_- v, \wp'_- w \right\rangle - \left\langle \wp'_- v, \wp'_+ w - \wp''_+ w + S\wp'_- w \right\rangle \\ &+ \left\langle \wp''_+ v, \wp'_- w + \wp''_- w \right\rangle - \left\langle \wp'_- v + \wp''_- v, \wp''_+ w \right\rangle \end{aligned} \tag{2.35}$$

and hence an argument in ([15], Sects. 2.2.5, 2.5.3, *cf.* [25], Sect. 6.2), shows that

$$\operatorname{Ind} \mathfrak{A} = \dim\ker\mathfrak{A} - \dim\operatorname{coker}\,\mathfrak{A} = 0, \quad \operatorname{coker}\,\mathfrak{A} = \{(v, \wp'_- v) \in \mathfrak{R} : v \in \ker\mathfrak{A}\}. \tag{2.36}$$

□

Let G be the generalized Green function of the Neumann problem (2.24), see, *e.g.*, [32], namely a distributional solution of

$$\begin{gathered} -\Delta_y G(y, \mathbf{y}) = \delta(y - \mathbf{y}) - |\omega_0|^{-1},\ y \in \omega_0, \quad \partial_{\nu(y)} G(y, \mathbf{y}) = 0,\ y \in \partial\omega_0, \\ \int_{\omega_0} G(y, \mathbf{y}) \mathrm{d}y = 0,\ \mathbf{y} \in \omega_0, \end{gathered} \tag{2.37}$$

where δ is the Dirac mass. We put $G^j(y) = G(y, P^j)$ and write

$$G^j(y) = -\chi_j(y)(2\pi)^{-1} \ln r_j + \widehat{G}^j(y), \quad \widehat{G}^j \in H^2(\omega_0). \tag{2.38}$$

The $J \times J$-matrix $\mathcal{G}$ with entries $\mathcal{G}^j_k = \widehat{G}^j(P^k)$ is symmetric, (see [4], Sect. 2.2). We compose the vectors

$$\mathbf{G}^j = (G^j, \delta_{j1}\gamma_1^{-1}|\omega_1|^{-1}(z - l_1), \dots, \delta_{jJ}\gamma_J^{-1}|\omega_J|^{-1}(z - l_J)) \in \mathfrak{H}, \tag{2.39}$$

which fall into the subspace $\mathfrak{H}_-$, see (2.33), because

$$\begin{aligned} &\wp'_- \mathbf{G}^j = -\wp''_- \mathbf{G}^j = \mathbf{e}_{(j)}, \\ &\wp'_+ \mathbf{G}^j = \mathcal{G}^j = (\mathcal{G}^j_1, \dots, \mathcal{G}^j_J), \quad \wp''_+ \mathbf{G}^j = -\gamma_j^{-1}|\omega_j|^{-1} l_j \mathbf{e}_{(j)}. \end{aligned} \tag{2.40}$$

Here, $\mathbf{e}_{(j)} = (\delta_{j1}, \dots, \delta_{jJ})$, $j = 1, \dots, J$, is the natural basis in $\mathbb{R}^J$.

Let $\mathcal{E}$ be a subspace spanned over the vector $\varepsilon = (1, \dots, 1) \in \mathbb{R}^J$, $|\varepsilon| = \sqrt{J}$, and $\mathbb{R}^J = \mathcal{E} \oplus \mathcal{E}^\perp$ with the orthogonal projector $\mathcal{P}^\perp$ onto $\mathcal{E}^\perp$, $\dim\mathcal{E}^\perp = J - 1$. We also introduce the diagonal matrix

$$\mathcal{Q} = \operatorname{diag}\left\{\gamma_1|\omega_1|l_1^{-1}, \dots, \gamma_J|\omega_J|l_J^{-1}\right\}. \tag{2.41}$$

Theorem 2.9. *If the operator*

$$\mathcal{P}^\perp(S + \mathcal{G} + \mathcal{Q}^{-1})\mathcal{P}^\perp : \mathcal{E}^\perp \to \mathcal{E}^\perp \tag{2.42}$$

is invertible, then the problem (2.29)–(2.32) *on the hybrid domain* $\varXi^0 = \omega_\odot \cup \bigcup_{j=1}^J (P^j \cup I_j)$, *Figure* 2a, *has a unique solution* $v \in \mathfrak{H}_-$ *for any* $\{f, k\} \in \mathfrak{R}$. *In other words, the operator* (2.33) *is an isomorphism.*

Proof. We search for a solution of the problem in the form

$$v = \mathbf{v} + \alpha_1 \mathbf{G}^1 + \ldots + \alpha_J \mathbf{G}^J \tag{2.43}$$

where $\alpha = (\alpha_1, \ldots, \alpha_J) \in \mathbb{R}^J$, $\mathbf{G}^j$ is given in (2.39) and $\mathbf{v} = (\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_J) \in \mathfrak{H}_-$ with

$$\mathbf{v}_0(y) = \alpha_0^0 + \mathbf{v}_0^0(y),\ \mathbf{v}_0^0 \in H^2(\omega_0),\ \int_{\omega_0} \mathbf{v}_0^0(y)\mathrm{d}y = 0,\ \mathbf{v}_j \in H^2(I_j). \tag{2.44}$$

In view of (2.37)−(2.39) these functions must satisfy the problems

$$\begin{gathered} -\Delta_y \mathbf{v}_0(y) = f_0(y) - |\omega_0|^{-1}(\alpha_1 + \ldots + \alpha_J),\quad y \in \omega_0, \qquad \partial_\nu \mathbf{v}_0(y) = 0,\quad y \in \omega_0, \\ -\gamma_j|\omega_j|\partial_z^2 \mathbf{v}_j(z) = f_j(z),\ z \in (0, l_j), \qquad \mathbf{v}_j(l_j) = 0, \quad -\gamma_j|\omega_j|\partial_z \mathbf{v}_j(0) = 0. \end{gathered} \tag{2.45}$$

Under the condition

$$\textstyle\sum_j \alpha_j = |\omega_0| \displaystyle\int_{\omega_0} f_0(y)\mathrm{d}y, \tag{2.46}$$

the problems (2.45) have unique solutions (2.44) but with arbitrary constant α_0^0. The vector function (2.43) fulfils the point condition (2.29) while (2.30) turns into

$$-\mathcal{G}\alpha - \alpha_0^0 \varepsilon - \mathcal{Q}^{-1}\alpha - S\alpha = k + \wp'_+(\mathbf{v}_0^0, 0, \ldots, 0) - \wp''_+(0, \mathbf{v}_1, \ldots, \mathbf{v}_J) \in \mathbb{R}^J. \tag{2.47}$$

Applying the projector $\mathcal{P}^\perp$, we annul the term $\alpha_0^0\varepsilon$ in (2.47) and determine $\mathcal{P}^\perp\alpha$, thanks to our assumption on the mapping (2.42). Then the equation (2.46) gives the remaining part of the coefficient vector in (2.43). Recalling (2.47) yields a value of α_0^0.

Since we have found a solution (2.43), the operator (2.33) is an epimorphism and becomes isomorphism by virtue of Proposition 2.8. □

2.5. The variational formulation of the problem with point conditions

Similarly to [22, 23, 28] we associate the problem (2.29)−(2.32) with the quadratic form

$$\begin{aligned} \mathfrak{E}(v; f, k) = &- \frac{1}{2}(\Delta_y v_0, v_0)_{\omega_0} - (f_0, v_0)_{\omega_0} - \textstyle\sum_j (\gamma_j|\omega_j|(\partial_z^2 v_j, v_j)_{I_j} + (f_j, v_j)_{I_j} \\ &+ \frac{1}{2}\left\langle \wp''_+ v - \wp'_+ v - S\wp'_- v, \wp'_- v \right\rangle - \left\langle k, \wp'_- v \right\rangle \end{aligned} \tag{2.48}$$

defined properly in the subspace $\mathfrak{H}_-$ of the Hilbert space (2.13). We call (2.48) an *energy functional* for the problem with point conditions.

Remark 2.10. In the case $k = 0$ the form $\mathfrak{E}(v; f, 0)$ restricted onto $\mathcal{D}(\mathcal{A}) \times \mathcal{L} \subset \mathfrak{H}_- \times \mathcal{L}$ coincides with the energy functional

$$\frac{1}{2}(\mathcal{A}v, v)_{\mathcal{L}} - (f, v)_{\mathcal{L}}$$

generated by the self-adjoint extension $\mathcal{A}$ with the parameters (2.26) in Proposition 2.4. This follows from the fact that two scalar products in $\mathbb{R}^J$ on the right-hand side of (2.48) vanish.

Theorem 2.11. *A vector function $v \in \mathfrak{H}$ is a solution of the problem (2.29)−(2.32) if and only if v is a stationary point of the energy functional (2.48).*

Proof. Calculating the variation of the functional (2.48), we obtain

$$\begin{aligned}\delta\mathfrak{E}(v,w;f,k) = &-\frac{1}{2}(\varDelta_y v_0, w_0)_{\omega_0} - \frac{1}{2}(\varDelta_y w_0, v_0)_{\omega_0} - (f_0, v_0)_{\omega_0} \\ &- \sum\nolimits_j \left(\frac{1}{2}\gamma_j|\omega_j|(\partial_z^2 v_j, w_j)_{I_j} + \frac{1}{2}\gamma_j|\omega_j|(\partial_z^2 w_j, v_j)_{I_j} + (f_j, w_j)_{I_j}\right) \\ &+ \frac{1}{2}\left\langle \wp_+'' v - \wp_+' v - S\wp_-' v, \wp_-' w\right\rangle + \frac{1}{2}\left\langle \wp_+'' w - \wp_+' w - S\wp_-' w, \wp_-' v\right\rangle - \left\langle k, \wp_-' w\right\rangle.\end{aligned}$$

We make use of the generalized Green formula (2.16) while interchanging positions of v and w. Recalling the relation $S = S^\top$ and the point condition (2.29) for $v, w \in \mathfrak{H}_-$, we have

$$\begin{aligned}\delta\mathfrak{E}(v,w;f,k) = &-\frac{1}{2}(\varDelta_y v_0, w_0)_{\omega_0} - \frac{1}{2}(w_0, \varDelta_y v_0)_{\omega_0} - (f_0, v_0)_{\omega_0} \\ &- \sum\nolimits_j \left(\frac{1}{2}\gamma_j|\omega_j|(\partial_z^2 v_j, w_j)_{I_j} + \frac{1}{2}\gamma_j|\omega_j|(w_j, \partial_z^2 v_j)_{I_j} + (f_j, w_j)_{I_j}\right) \\ &+ \frac{1}{2}\left\langle \wp_+ w, \wp_- v\right\rangle + \frac{1}{2}\left\langle \wp_- w, \wp_+ v\right\rangle + \frac{1}{2}\left\langle \wp_+'' v - \wp_+' v, \wp_-' w\right\rangle \\ &+ \frac{1}{2}\left\langle \wp_+'' w - \wp_+' w, \wp_-' v\right\rangle + \frac{1}{2}\left\langle S\wp_-' v, \wp_-' w\right\rangle - \frac{1}{2}\left\langle \wp_-' w, S\wp_-' v\right\rangle - \left\langle k, \wp_-' w\right\rangle \\ = &(-\varDelta_y v_0 - f_0, w_0)_{\omega_0} + \sum\nolimits_j (-\gamma_j|\omega_j|\partial_z^2 v_j - f_j, w_j)_{I_j} + \left\langle \wp_+'' v - \wp_+' v - S\wp_-' v, \wp_-' w\right\rangle.\end{aligned} \tag{2.49}$$

Here we, in particular, used the relation (2.29). It also should be mentioned that all functions are real as well as the matrix S and the vector k.

We see that a solution $v \in \mathfrak{H}$ of the problem (2.29)−(2.32), annuls the variation (2.49) of the functional (2.48). On the other hand, for any test vector $w \in \mathfrak{H}_-$, the expression with the stationary point $v \in \mathfrak{H}_-$ of $\mathfrak{E}$ vanishes; in particular, taking $w \in C_c^\infty(\omega_0) \times C_c^\infty(I_1) \times \ldots \times C_c^\infty(I_J)$ brings the differential equations in (2.31) and (2.32). Thus, the last scalar product in (2.49) is null and (2.30) is fulfilled because $\wp_-' \mathfrak{H}_- = \mathbb{R}^J$. It remains to mention that the boundary conditions (1.22), (1.23) and the point condition (2.29) are kept in the space $\mathfrak{H}_-$. □

3. Determination of parameters of an appropriate hybrid model

3.1. The boundary layer phenomenon

An asymptotic analysis performed in [4] gave a detailed description of the behavior of solutions to the stationary problem in $\varXi(h)$ near the junction zones. The internal constitution of the boundary layers which appear in the vicinity of the sockets $\theta_j^h = \omega_j^h \times (0,h)$ and are written in the rapid variables

$$\xi^j = (\eta^j, \zeta^j), \quad \eta^j = h^{-1}(y - P^j), \quad \zeta^j = h^{-1} z, \tag{3.1}$$

depends crucially on the exponent α in (1.13). In the case $\alpha = 1$, see (1.14), the transmission conditions (1.11), (1.12) decouple and the coordinate dilation leads to two independent limit problems in the semi-infinite cylinder $Q_j = \omega_j \times \mathbb{R}^+$ and the perforated layer $\varLambda_j = (\mathbb{R}^2 \setminus \overline{\omega}_j) \times (0,1)$. In ([4], Sect. 2.4), we have examined these problems, namely the Neumann problem

$$\begin{aligned}&-\gamma_j \varDelta_\xi W_j(\xi) = 0, \ \xi \in Q_j, \quad \gamma_j \partial_\nu W_j(\xi) = g_j(\xi), \quad \xi \in \partial\omega_j \times \mathbb{R}_+, \\ &-\gamma_j \partial_\zeta W_j(\eta, 0) = 0, \quad \eta \in \omega_j,\end{aligned} \tag{3.2}$$

where ∂_ν is the outward normal derivative, and the mixed boundary-value problem

$$\begin{aligned}&-\varDelta_\xi W_0(\xi) = 0, \ \xi \in \varLambda_j, \quad -\partial_\zeta W_0(\eta, 0) = \partial_\zeta W_0(\eta, 1) = 0, \quad \eta \in \mathbb{R}^2 \setminus \overline{\omega}_j, \\ &W_0(\xi) = g_0(\xi), \quad \xi \in \partial\omega_j \times (0,1).\end{aligned} \tag{3.3}$$

We now point out several special solutions of these problems, that we need in the sequel. First of all, the homogeneous ($g_j = 0$) problem (3.2) has a constant solution, say $\mathbf{w}_j(\xi) = 1$, and the problem (3.3) with $g_0(\xi) = 1$ is also satisfied by $\mathbf{w}_j^0(\xi) = 1$. The homogeneous ($g_0 = 0$) problem (3.3) admits a solution with the logarithmic growth at infinity

$$\mathbf{W}_j^0(\eta) = (2\pi)^{-1}\left(\ln|\eta| + \ln c_{\log}(\omega_j)\right) + \widetilde{\mathbf{W}}_j^0(\eta), \quad \widetilde{\mathbf{W}}_j^0(\eta) = O\left(|\eta|^{-1}\right), \quad |\xi| \to +\infty, \tag{3.4}$$

where $c_{\log}(\omega_j)$ is the logarithmic capacity of the set $\overline{\omega}_j \subset \mathbb{R}^2$. Note that the function $\mathbf{W}_j^0$ in (3.4) is independent of ζ and is called the logarithmic capacity potential, see [12,30]. Finally, according to ([4], Lem. 6),

$$-1 = \int_{\partial\omega_j} g_j(\eta)\, ds_\eta, \quad g_j(\eta) = \partial_\nu \mathbf{W}_j^0(\eta) \tag{3.5}$$

and a solution of the Neumann problem (3.2) with the datum in (3.5) can be found in the form

$$\mathbf{W}_j(\eta,\zeta) = \gamma_j^{-1}|\omega_j|^{-1}\zeta + O(\mathrm{e}^{-\delta\zeta}), \quad \delta > 0. \tag{3.6}$$

3.2. Individual choice of the self-adjoint extension

Applying the method of matched asymptotic expansions, see for example ([11,16,33], Chap. 2), we take some functions $v_0 \in \mathfrak{H}_0$, $v_j \in \mathfrak{H}_j$ and write the *outer* expansions in the plate $\Omega_0(h)$ and the rod $\Omega_j(h)$ near but outside the socket θ_j^h

$$v_0(y) = \frac{b_j}{2\pi}\ln\frac{1}{r_j} + \widehat{v}_0(P^j) + \ldots = \frac{b_j}{2\pi}\ln\frac{1}{|\eta_j|} - \frac{b_j}{2\pi}\ln h + \widehat{v}_0(P^j) + \ldots, \tag{3.7}$$

$$v_j(z) = v_j(0) + z\partial_z v_j(0) + \ldots = v_j(0) + h\zeta^j v_j(0) + \ldots \tag{3.8}$$

Here, ellipses stand for higher order terms of no importance in our asymptotic procedure. The *inner* expansions in the immediate vicinity of the sockets are composed from the above described solutions of the problems (3.3) and (3.2)

$$c_j^0\mathbf{w}_j^0(\xi^j) + c_j^1\mathbf{W}_j^0(\xi^j) = c_j^1\frac{1}{2\pi}\ln\frac{1}{|\eta_j|} + c_j^1\frac{1}{2\pi}\ln c_{\log}(\omega_j) + c_j^0 + \ldots, \ (P^j + h\eta^j, h\zeta^j) \in \Omega_\bullet(h), \tag{3.9}$$

$$c_j^0\mathbf{w}_j(\xi^j) + hc_j^1\mathbf{W}_j(\xi^j) = hc_j^1\gamma_j^{-1}|\omega_j|^{-1}\zeta^j + c_j^0 + \ldots, \quad (P^j + h\eta^j, h\zeta^j) \in \Omega_j(h). \tag{3.10}$$

We emphasize that, by definition of those solutions, the transmission condition (1.12) is wholly satisfied by (3.9) and (3.10) but the condition (1.12) with the reasonable precision $O(h)$.

Comparing (3.7) with (3.9) and (3.8) with (3.10) yields the equations

$$\begin{aligned} b_j = c_j^1, \quad \widehat{v}_0(P^j) - b_j(2\pi)^{-1}\ln h = c_j^0 + c_j^1(2\pi)^{-1}\ln c_{\log}(\omega_j), \\ \partial_z v_j(0) = c_j^1\gamma_j^{-1}|\omega_j|^{-1}, \quad v_j(0) = c_j^0. \end{aligned} \tag{3.11}$$

Excluding the coefficients c_j^1 and c_j^0, we derive the relations

$$\begin{aligned} b_j - \gamma_j|\omega_j|\partial_z v_j(0) = 0, \\ \widehat{v}_0(P^j) - v_j(0) = b_j(2\pi)^{-1}(\ln h + \ln c_{\log}(\omega_j)). \end{aligned} \tag{3.12}$$

We also introduce the diagonal $J \times J$-matrix

$$S^h = -(2\pi)^{-1}\mathrm{diag}\{\ln h + \ln c_{\log}(\omega_1), \ldots, \ln h + \ln c_{\log}(\omega_J)\} = (2\pi)^{-1}(|\ln h|\mathbb{I} + C_{\log}) \tag{3.13}$$

which is positive definite for $h \in (0, h_0)$ with a small $h_0 \in (0,1)$ and further substitutes for S in (2.26). Here, $\mathbb{I}$ is the unit $J \times J$-matrix and $C_{\log} = \mathrm{diag}\{\ln c_{\log}(\omega_1), \ldots, \ln c_{\log}(\omega_J)\}$.

The main conclusion from the above consideration is that the relations (3.12) between the functions $v_0 \in \mathfrak{H}_0$ and $v_j \in \mathfrak{H}_j$, $J = 1, \ldots, J$ are equivalent to conditions in (2.22), (2.26) defining a self-adjoint extension $\mathcal{A}^h$ of the operator $A = (A_0, A_1, \ldots, A_J)$ in the space $\mathfrak{L}$, see (2.14), examined in Section 2. In what follows we deal with this operator $\mathcal{A}^h$.

3.3. The spectral problem

Based on results in Section 2, we give two models of the spectral problem (1.6)−(1.12). First, we use the introduced self-adjoint operator $\mathcal{A}^h$ and write the equation

$$\mathcal{A}^h v^h = \mu^h v^h \ \text{ in } \mathfrak{L}. \tag{3.14}$$

Second, we supply the problems (1.20), (1.22) and (1.21), (1.23) with the point conditions (2.28), (2.29) and formulate the abstract equation

$$\mathfrak{A}^h v^h = \mu^h (v^h, 0) \ \text{ in } \mathfrak{L}\times\mathbb{R}^J \tag{3.15}$$

where the operator $\mathfrak{A}^h : \mathfrak{H}_- \to \mathfrak{L}\times\mathbb{R}^J$ involves the point condition (2.28) with the operator $S = S^h$ in (3.13) while zero at the last position in (3.15) indicates that this condition is homogeneous, namely $k = 0$ in (2.30).

According to Remark 2.7, the spectral equations (3.14) and (3.15) are equivalent with each other. Unfortunately, the operator $\mathcal{A}^h$ intended to model the problem (1.6)−(1.12) with the positive spectrum (1.18), is not positive definite in view of Remark 2.10 and formula (2.48) involving the negative definite matrix $-S = -S^h$. However, in Remark 2.5 we mentioned the positive operator $\mathcal{A}^0$ with the attributes (2.23) in (2.22) which differs from the operator $\mathcal{A}^h$ only in subspace of dimension J. Thus, the max-min principle, (*cf.* [2], Thm. 10.2.2) demonstrates that the total multiplicity of the negative part of the spectrum of $\mathcal{A}^h$ cannot exceed J. In the next section we will construct asymptotics of negative eigenvalues which we call *parasitic*.

3.4. Asymptotics of parasitic eigenvalues

We will need the fundamental solution Φ of the operator $-\Delta_y + 1$ in the plane $\mathbb{R}^2$. Its expression is well known, in particular

$$\Phi(y) = O(\mathrm{e}^{-\psi|y|}) \text{ as } |y| \to +\infty, \quad \Phi(y) = \frac{1}{2\pi}\ln\frac{1}{|y|} + \Psi + O(|y|) \text{ as } |y| \to +0 \tag{3.16}$$

but exact values of $\psi > 0$ and Ψ are of no importance here. Let

$$\mu^h = -h^{-2}(\mathrm{e}^{2m} + O(h)). \tag{3.17}$$

We set

$$v_0^h(y) = \textstyle\sum_j \alpha_j \chi_j(y) \Phi(h^{-1}\mathrm{e}^m(y - P^j)), \tag{3.18}$$

$$v_j^h(y) = \alpha_j X_j(z) \gamma_j^{-1} |\omega_j|^{-1} h \mathrm{e}^{-m} \mathrm{e}^{-h^{-1}\mathrm{e}^m z}. \tag{3.19}$$

where the column $\alpha = (\alpha_1, \dots, \alpha_J) \in \mathbb{R}^J$ and the number $m \in \mathbb{R}$ are to be determined and $X_j \in C^\infty(I_j)$, $X_j(z) = 1$ for $[0, l_j/3]$ and $X_j(z) = 0$ for $[2l_j/3, l_j]$. Clearly, the functions (3.18), (3.19) satisfy the boundary conditions (1.22), (1.23) and leave small discrepancies $O(\mathrm{e}^{-\delta/h})$, $\delta > 0$, in the differential equations (1.20), (1.21) with the spectral parameter $-h^{-2}\mathrm{e}^{2m}$. It should be mentioned that the exponential decay of the boundary layer terms in (3.18) and (3.19) is due to the negative value of μ^h in (3.17).

The vector $v^h = (v_0^h, v_1^h, \dots, v_J^h)$ has the following projections:

$$\begin{aligned} &\wp'_+ v^h = (2\pi)^{-1}(m - \ln h) + \phi_0, \quad \wp'_- v^h = \alpha, \\ &\wp''_+ v^h = Qh\mathrm{e}^{-m}, \quad \wp''_- v^h = -\alpha, \end{aligned} \tag{3.20}$$

where $Q = \mathrm{diag}\{\gamma_1^{-1}|\omega_1|^{-1}, \dots, \gamma_J^{-1}|\omega_J|^{-1}\}$. Hence, the point condition (2.29) is fulfilled while, in view of (3.13) and (3.20), the condition (2.28) converts into

$$Qh\mathrm{e}^m\alpha - (2\pi)^{-1}(m - \ln h)\alpha + (2\pi)^{-1}(\ln h\mathbb{I} + C_{\log})\alpha = 0$$

or, what is the same,

$$C_{\log}\alpha = m\alpha - 2\pi h e^{-m} Q\alpha. \tag{3.21}$$

Since both matrices are diagonal, the system (3.21) splits into J independent transcendental equations. The small factor h of the exponent e^{-m} allows us to apply the implicit function theorem and to obtain the solutions

$$m_j^h = \log c_{\log}(\omega_j) + O(h), \quad \alpha_{(j)}^h = \mathbf{e}_{(j)} + O(h), \quad j = 1, \dots, J. \tag{3.22}$$

We have derived "good approximations" for J negative eigenvalues of the spectral equations (3.14) and (3.15). Recalling that their number cannot exceed J, we come in position to formulate an assertion on the whole negative part of the spectrum.

Proposition 3.1. *There exist positive numbers h_- and c_- such that, for $h \in (0, h_-]$, the equation (3.14) or (3.15) possesses exactly J eigenvalues on the semi-axis $\mathbb{R}_- = (-\infty, 0)$. These eigenvalues obey the asymptotic form*

$$|\mu_{-j}^h + h^{-2}(c_{\log}(\omega_j))^2| \le c_- h^{-1} \tag{3.23}$$

where $c_{\log}(\omega_j) > 0$ is the logarithmic capacity of the set $\overline{\omega}_j \subset \mathbb{R}^2$, see [12, 30].

We will outline the proof in Remark 4.5. Notice that the negative eigenvalues $\mu_{-j}^h = O(h^{-2})$, $j = 1, \dots, J$, are situated very far away from the positive part of the spectrum, that is, outside the scope of the asymptotic models under consideration.

4. Justification of the asymptotic models

4.1. The first convergence theorem

Let μ^h be an eigenvalue of the self-adjoint operator $\mathcal{A}^h$ in (3.14). The corresponding eigenvector $v^h = (v_0^h, v_1^h, \dots, v_J^h) \in \mathcal{D}(\mathcal{A}^h) \subset \mathfrak{H}_-$ can be normed as follows:

$$||v_0^h; L^2(\omega_0)||^2 + \textstyle\sum_j \rho_j |\omega_j| \; ||v_j^h; L^2(I_j)||^2 \tag{4.1}$$

Assuming that

$$|\mu^h| \le c, \tag{4.2}$$

in particular, rejecting negative eigenvalues in Proposition 3.1, we recall the Kondratiev theory used in Section 2.1. Then, a solution $v_0^h \in H^2_{loc}(\overline{\omega}_0 \setminus \mathcal{P}) \cap L^2(\omega_0)$ of the problem (1.20), (1.22) admits the decomposition (2.9) with the ingredients $a_j^h = \widehat{v}^h(P^j)$, $b_j^h \in \mathbb{R}$ and $\widehat{v}_0^h \in H^2(\omega_0) \cap H_0^1(\omega_0)$ while

$$||\widehat{v}_0^h; H^2(\omega_0)||^2 + \textstyle\sum_j |b_j^h| \le c(1 + |\mu^h|)||v_0^h; L^2(\omega_0)||^2 \le C, \tag{4.3}$$

where $\widehat{v}_0^h$ is given in (2.11).

Furthermore, a solution $v_j^h \in L^2(I_j)$ of the ordinary differential equation (1.21) with the Dirichlet condition (1.23) falls into the Sobolev space $H^2(I_j)$ and fulfills the estimate

$$|v_j^h(0)| + |\partial_z v_j^h(0)| \le c||v_j^h; H^2(I_j)|| \le c(1 + |\mu^h|)||v_0^h; L^2(I_j)||^2 \le C. \tag{4.4}$$

The inequalities (4.2)−(4.4) help to conclude the following convergence along an infinitesimal positive sequence $\{h_k\}_{k\in\mathbb{N}}$:

$$\begin{aligned} &\mu^h \to \mu^0 \in \mathbb{R}, \quad b^h = (b_1^h, \dots, b_J^h) \to b^0 \in \mathbb{R}^J, \\ &\widehat{v}_0^h \rightharpoonup \widehat{v}_0^0 \quad \text{weakly in} \quad H^2(\omega_0), \;\; v_j^h \rightharpoonup v_j^0 \quad \text{weakly in} \quad H^2(I_j). \end{aligned} \tag{4.5}$$

This and the embeddings $H^2(\omega_0) \subset C(\omega_0)$, $H^2(I_j) \subset C^1(I_j)$ imply the convergence of the projections (2.15)

$$\wp_{\pm} v^h \to \wp_{\pm} v^0 \in \mathbb{R}^J. \tag{4.6}$$

We emphasize that formulas in (4.5) guarantee the strong convergences $v_0^h \to v_0^0$ in $L^2(\omega_0)$ and $v_j^h \to v_j^0$ in $L^2(I_j)$ so that the normalization condition (4.1) is kept by the limit $v^0 = (v_0^0, v_1^0, \dots, v_J^0)$. Moreover, the differential equations (1.20), (1.21) with $\mu = \mu^h$ and the boundary conditions (1.22), (1.23) for v_0^h, v_j^h are passed to the limits

$$\mu^0, \quad v_0^0 = \widehat{v}_0^0 - (2\pi)^{-1} \textstyle\sum_j \chi_j b_j \ln r_j, \quad v_j^h, \ j = 1, \dots, J. \tag{4.7}$$

In order to formulate the next assertion it suffices to mention that the point conditions (2.28), (2.29) with the matrix (3.13) containing the big component $-(2\pi)^{-1} \ln h \mathbb{I}$ turn in the limit into the relations

$$\wp'_- v^0 = 0, \quad \wp'_- v^0 + \wp''_- v^0 = 0 \quad \Rightarrow \quad \wp'_- v^0 = \wp''_- v^0 = 0 \in \mathbb{R}^J. \tag{4.8}$$

These provide the self-adjoint extension $\mathcal{A}^0$ with the attributes (2.23) in (2.22) that corresponds to the Neumann and mixed boundary-value problems (2.24) and (2.25). The spectra $\{\varkappa_n^0\}_{n\in\mathbb{N}}$ and $\{\varkappa_n^j = \frac{\pi^2}{l_j^2} \frac{\gamma_j}{\rho_j} (n + \frac{1}{2})^2\}_{n\in\mathbb{N}}$, $j = 1, \dots, J$, of the above mentioned problems are united into the common monotone sequence

$$\{\mu_n^0\}_{n\in\mathbb{N}}, \quad \mu_1^0 = 0 < \mu_{2.}^0 \tag{4.9}$$

Here, eigenvalues are listed while counting their multiplicity in.

Theorem 4.1. *If an eigenvalue μ^h of the operator $\mathcal{A}^h$, cf.* (3.14) *and* (3.15)*, and the corresponding eigenvector v^h fulfil the requirements* (4.2) *and* (4.1)*, then the limits μ^0 and v^0 in* (4.5) *along an infinitesimal sequence $\{h_n\}_{n\in\mathbb{N}}$ are an eigenvalue of the operator $\mathcal{A}^0$ described in Remark* 2.5 *and the corresponding eigenvector normed in the space $\mathcal{L}$, see* (2.14)*.*

4.2. The second convergence theorem

In the next section we will verify that entries of the eigenvalue sequence (1.18) of the original problem (1.6)–(1.12) in the junction $\varXi(h) \subset \mathbb{R}^3$ satisfy the inequalities

$$0 < \lambda^n(h) \le c_n \text{ for } h \in (0, h_n] \tag{4.10}$$

with some positive h_n and c_n which depend on the eigenvalue number n but are independent of h. The corresponding eigenfunction $u^n(h,\cdot) \in H_0^1(\varXi(h); \varGamma(h))$ is subject to the normalization condition (1.19). We introduce the functions

$$v_0^n(h,y) = \frac{1}{\sqrt{h}} \int_0^h u^n(h,y,z) \mathrm{d}z, \quad y \in \omega_0, \tag{4.11}$$

$$v_j^n(h,z) = \frac{1}{h^{3/2}|\omega_j|} \int_{\omega_j^h} u_j^n(h,y,z) \mathrm{d}y, \quad y \in (0, l_j), \ j = 1, \dots, J, \tag{4.12}$$

and write

$$\begin{aligned} \int_{\omega_0} |v_0^n(h,y)|^2 \mathrm{d}y &= \frac{1}{h} \int_{\omega_0} \left| \int_0^h u^n(h,y,z) \mathrm{d}z \right|^2 \mathrm{d}y \le \int_{\omega_0} \int_0^h |u^n(h,x)|^2 \, \mathrm{d}x \\ &\le b(u^n, u^n; \varOmega_0(h)), \\ \rho_j |\omega_j| \int_0^{l_j} |v_j^n(h,z)|^2 \mathrm{d}z &= \rho_j |\omega_j|^{-1} \frac{1}{h^3} \int_0^{l_j} \left| \int_{\omega_j^h} u_j^n(h,y,z) \mathrm{d}y \right|^2 \mathrm{d}z \\ &\le \frac{\rho_j}{h} \frac{|\omega_j^h|}{h^2 |\omega_j|} \int_{\varOmega_j^h(h)} |u^n(h,x)|^2 \mathrm{d}x \le b(u^n, u^n; \varOmega_j(h)). \end{aligned} \tag{4.13}$$

Here, we used formulas (1.17) and (1.13), (1.14) while taking the relation $1 \le h^{-1}\rho_j$ on $\Omega_0(h) \cap \Omega_j(h)$ into account. Hence, the vector function $v^n = (v_0^n, v_1^n, \dots, v_J^n)$ satisfies the estimate

$$||v^n(h,\cdot); \mathfrak{L}|| \le 1. \tag{4.14}$$

A similar calculation gives us the formula

$$\begin{aligned} ||\nabla v_0^n; L^2(\omega_0)||^2 + \textstyle\sum_j ||\partial_z v_j^n; L^2(I_j)||^2 &\le c(||\nabla_y u^n; L^2(\Omega_0(h))||^2 + \textstyle\sum_j ||\partial_z u^n; L^2(\Omega_j(h))||^2) \\ &\le ca(u^n, u^n; \Xi(h)) = c\lambda(h) b(u^n, u^n; \Xi(h)) \le C_n. \end{aligned}$$

Moreover, the Poincaré inequalities in $(0,h)$ and ω_j^h show that the functions

$$\begin{aligned} u_0^{n\perp}(h,x) &= u^n(h,x) - h^{-1/2} v_0^n(h,y) \qquad \text{in} \qquad \Omega_0(h), \\ u_j^{n\perp}(h,x) &= u_j^n(h,x) - h^{-1/2} v_j^n(h,y) \qquad \text{in} \qquad \Omega_j(h), \end{aligned}$$

which are of mean zero in $z \in (0,h)$ and $y \in \omega_j^h$, respectively, enjoy the relations

$$\begin{aligned} ||u_0^{n\perp}; L^2(\Omega_0(h))||^2 &\le c_0 h^2 ||\partial_z(u^n - h^{-1/2} v_0^n); L^2(\Omega_0(h))||^2 \\ &= c_0 h^2 ||\partial_z u^n; L^2(\Omega_0(h))||^2 \le c_0 a(u^n, u^n; \Omega_0(h)), \\ ||u_j^{n\perp}; L^2(\Omega_j(h))||^2 &\le c_j h^2 ||\nabla_y(u_j^n - h^{-1/2} v_j^n); L^2(\Omega_j(h))||^2 \\ &= c_j h^2 ||\nabla_y u^n; L^2(\Omega_j(h))||^2 \le C_j h^3 a(u^n, u^n; \Omega_j(h)). \end{aligned} \tag{4.15}$$

Hence, we obtain

$$\begin{aligned} 1 &= ||h^{-1/2} v_0^n - u_0^{n\perp}; L^2(\Omega_\bullet(h))||^2 + h^{-1} \textstyle\sum_j \rho_j ||h^{-1/2} v_j^n - u_j^{n\perp}; L^2(\Omega_j(h))||^2 \\ &\le (1+h)(||v_0^n; L^2(\omega_\bullet(h))||^2 + \textstyle\sum_j \rho_j |\omega_j| \; ||v_j^n; L^2(I_j)||^2) \\ &\quad + (1+h^{-1})(||u_0^{n\perp}; L^2(\Omega_\bullet(h))||^2 + h^{-1} \textstyle\sum_j \rho_j ||u_j^{n\perp}; L^2(\Omega_j(h))||^2). \end{aligned} \tag{4.16}$$

To estimate the norm $||v_0^n; L^2(\omega_\bullet(h))||^2$, we apply the weighted inequality in ([4], Thm. 9)

$$\begin{aligned} &(1+|\ln h|)^{-2} ||r^{-1}(1+|\ln r|)^{-1} u; L^2(\Omega_0(h))||^2 + h^{-1} \textstyle\sum_j ||(l_j - z)^{-1} u_j; L^2(\Omega_j(h))||^2 \\ &\le c_\Xi a(u, u; \Xi(h)) \end{aligned} \tag{4.17}$$

where $r = \min\{r_1, \dots, r_J\}$ and c_Ξ is independent of $h \in (0, h_0]$ and $u \in H_0^1(\Xi(h); \Gamma(h))$. Since $r_j(1+|\ln r_j|) \le c_j h(1+|\ln h|)$ in the socket $\theta_j^h = \omega_j^h \times (0,h)$, we recall (1.16), (4.10) and conclude that

$$\begin{aligned} ||v_0{}^n; L^2(\omega_j^h)||^2 &\le ||u^n; L^2(\theta_j^h)||^2 \le ch^2(1+|\ln h|)^4 ||r_j^{-1}(1+|\ln r_j|)^{-1} u^n; L^2(\theta_j^h)||^2 \\ &\le ch^2(1+|\ln h|)^4 a(u^n, u^n; \Xi(h)) = ch^2(1+|\ln h|)^4 \lambda^n(h) b(u^n, u^n; \Xi(h)) \le C_n h^2 (1+|\ln h|)^4. \end{aligned} \tag{4.18}$$

The estimates (4.10) and (4.13), (4.14) provide the following convergence along an infinitesimal sequence $\{h_k\}_{k\in\mathbb{N}}$:

$$\begin{aligned} &\lambda^n(h) \to \lambda^0, \\ &v_0^n(h,\cdot) \rightharpoonup v_0^{n0} \text{ weakly in } H^1(\omega_0) \text{ and strongly in } L^2(\omega_0), \\ &v_j^n(h,\cdot) \rightharpoonup v_j^{n0} \text{ weakly in } H^1(I_j) \text{ and strongly in } L^2(I_j). \end{aligned} \tag{4.19}$$

We compose a test function w from components $w_0 \in C_c^\infty(\overline{\omega}_0 \setminus \mathcal{P})$ and $w_j \in C_c^\infty(I_j)$, $j = 1, \dots, J$. Then according to (1.17) and (4.11), (4.12), we transform the integral identity into the formula

$$\begin{aligned} &\sqrt{h}(\nabla_y v_0^n, \nabla_y w_0)_{\omega_0} + \sqrt{h} \textstyle\sum_j \gamma_j |\omega_j| (\partial_z v_j^n, \partial_z w_j)_{I_j} = (\nabla_y u^n, \nabla_y w_0)_{\Omega_\bullet(h)} + h^{-1} \textstyle\sum_j \gamma_j (\partial_z u_j^n, \partial_z w_j)_{\Omega_j(h)} \\ &\quad = \lambda(h) \left((u^n, w_0)_{\Omega_\bullet(h)} + h^{-1} \textstyle\sum_j \rho_j (u_j^n, w_j)_{\Omega_j(h)} \right) = \sqrt{h} \lambda(h) \left((v_0^n, w_0)_{\omega_0} + \textstyle\sum_j \rho_j |\omega_j| (v_j^n, w_j)_{I_j} \right). \end{aligned}$$

We multiply this with $h^{-1/2}$ and perform the limit passage along the sequence $\{h_k\}_{k\in\mathbb{N}}$ to obtain

$$(\nabla_y v_0^{n0}, \nabla_y w_0)_{\omega_0} + \sum_j \gamma_j |\omega_j| (\partial_z v_j^{n0}, \partial_z w_j)_{I_j} = \lambda^0 \Big((v_0^{n0}, w_0)_{\omega_0} + \sum_j \rho_j |\omega_j| (v_j^{n0}, w_j)_{I_j} \Big). \tag{4.20}$$

Thanks to ([15], Sect. 9 Chap. 2), the weak solution $v_0^{n0} \in H^1(\omega_0)$ of the variational problem (4.20) where $w_j = 0$, $j = 1, \ldots, J$, falls into $H^2(\omega_0)$ and satisfies the Neumann problem (1.20), (1.22) with $\mu = \lambda^0$. We emphasize that $C_c^\infty(\overline{\omega}_0 \setminus \mathcal{P})$ is dense in $H^1(\omega_0)$ so that any test function $w \in H^1(\omega_0)$ is available. At the same time, $w_j \in C_c^\infty(I_j)$ vanishes near the points $z = l_j$ and $z = 0$. Hence, we may conclude that $v_j^{n0} \in H^2(I_j)$ and the differential equation (1.21) with $\mu = \lambda^0$. However, the boundary conditions

$$v_j^{n0}(l_j) = 0, \ -\gamma_j |\omega_j| \partial_z v_j^{n0}(0) = 0 \tag{4.21}$$

still must be derived. The Dirichlet condition in (4.21) is inherited from the conditions $v_j^n(h, l_j) = 0$ and $u_j^n(h, y, l_j) = 0$, $y \in \omega_j^h$. To conclude with the Neumann condition, we observe that the inequality (4.18) allows us to repeat the above transformations with the "very special" test vector function

$$w_0^j(y) = \chi_j(y), \quad w_k^j(z) = X_k(z)\delta_{k,j}, \quad j, k = 1, \ldots, J, \tag{4.22}$$

where χ_j and X_k are taken from (2.5) and (3.19). As a result, the obtained information on v_0^{n0} and v_j^{n0} reduces the integral identity (4.21) with (4.22) to the formula

$$0 = -((\Delta_y + \lambda^0) v_0^{n0}, w_0^j)_{\omega_0} - \gamma_j |\omega_j| (((\partial_z^2 + \lambda^0) v_j^{n0}, \partial_z w_j^j)_{I_j} + \partial_z v_j^{n0}(0) w_j^j(0)) = -\gamma_j |\omega_j| \partial_z v_j^{n0}(0).$$

We are in position to formulate the convergence theorem.

Theorem 4.2. *The limits λ^0 and $v^{n0} = (v_0^{n0}, v_1^{n0}, \ldots, v_J^{n0})$ in (4.19) are an eigenvalue and the corresponding eigenvector normed by (4.1) of the problems (1.20), (1.22) and (1.21), (4.21).*

4.3. An abstract formulation of the original problem

In the Hilbert space $H^h = H_0^1(\Xi(h); \Gamma(h))$ we introduce the scalar product

$$(u^h, v^h)_h = a(u^h, v^h; \Xi(h)) + b(u^h, v^h; \Xi(h)) \tag{4.23}$$

and positive, symmetric and continuous, therefore, self-adjoint operator T^h,

$$(T^h u^h, v^h)_h = b(u^h, v^h; \Xi(h)) \quad \forall u^h, v^h \in H^h. \tag{4.24}$$

Bilinear form on the right-hand side of (4.23) are defined in (1.17). Comparing (1.16) with (4.23), (4.24), we see that the variational formulation of the problem (1.6)–(1.12) in $\Xi(h)$ is equivalent to the abstract equation

$$T^h u^h = \tau(h) u^h \ \text{ in } H^h \tag{4.25}$$

with the new spectral parameter

$$\tau(h) = (1 + \lambda(h))^{-1}. \tag{4.26}$$

The operator T^h is compact and, hence, the essential spectrum of T^h consists of the only point $\tau = 0$, (see, *e.g.*, [2], Thm. 10.1.5), while the discrete spectrum composes the positive monotone infinitesimal sequence

$$1 > \tau^1(h) > \tau^2(h) \geq \ldots \geq \tau^n(h) \geq \ldots . \to +0 \tag{4.27}$$

obtained from (1.18) according to formula (4.26).

The following assertion is known as the lemma on "near eigenvalues and eigenvectors" [34] following directly from the spectral decomposition of resolvent, (see, *e.g.*, [2], Chap. 6).

Lemma 4.3. *Let* $\mathcal{U}^h \in H^h$ *and* $t^h \in \mathbb{R}_+$ *satisfy*

$$||\mathcal{U}^h; H^h|| = 1, \ ||T^h \mathcal{U}^h - t^h \mathcal{U}^h; H^h|| := \delta^h \in (0, t^h). \tag{4.28}$$

Then there exists an eigenvalue τ_n^h *of the operator* T^h *such that*

$$|t^h - \tau_n^h| \leq \delta^h. \tag{4.29}$$

Moreover, for any $\delta_*^h \in (\delta^h, t^h)$, *one finds coefficients* c_k^h *to fulfil the relations*

$$\left\| \mathcal{U}^h - \sum\nolimits_{k=N^h}^{N^h+X^h-1} c_k^h u_{(k)}^h; H^h \right\| \leq 2\frac{\delta^h}{\delta_*^h}, \qquad \sum\nolimits_{k=N^h}^{N^h+X^h-1} |c_k^h|^2 = 1 \tag{4.30}$$

where $\tau_{N^h}^h, \ldots, \tau_{N^h+X^h-1}^h$ *are all eigenvalues in the segment* $[t^h - \delta_*^h, t^h + \delta_*^h]$ *and* $u_{(N^h)}^h, \ldots, u_{(N^h+X^h-1)}^h$ *are the corresponding eigenvectors subject to the normalization and orthogonality conditions*

$$(u_{(p)}^h, u_{(q)}^h)_h = \delta_{p,q}. \tag{4.31}$$

4.4. Detecting eigenvalues with prescribed asymptotic form

Let $\mu_p^h \in \mathbb{R}_+$ and $v_{(p)}^h = (v_{(p)0}^h, v_{(p)1}^h, \ldots, v_{(p)J}^h) \in \mathfrak{H}_-$ be an eigenvalue of the equation (3.15) and the corresponding eigenvector enjoing the normalization and orthogonality conditions

$$(v_{(p)}^h, v_{(q)}^h)_{\mathfrak{L}} = \delta_{p,q} \tag{4.32}$$

where $(\ , \ )_{\mathfrak{L}}$ denotes the scalar product in the Lebesgue space $\mathfrak{L}$ induced by the norm (2.14). In Lemma 4.3 we set

$$t_p^h = (1 + \mu_p^h)^{-1} \tag{4.33}$$

and build an asymptotic approximation $\mathcal{U}_{(p)}^h$ of an eigenfunction of the problem (1.6)$-$(1.12) in the junction (1.4). To mimic the method of matched asymptotic expansions, we use asymptotic structures with "overlapping" cut-off functions (see [16], Chap. 2), [18, 22] *etc.*, namely in addition to the functions χ_j in (2.5) and X_j in (3.19) we introduce

$$\begin{aligned} &\mathcal{X}^h(y) = 0 \text{ for } r_j \leq Rh, \ j = 1, \ldots, J, \qquad \mathcal{X}^h(y) = 1 \text{ for } \min\{r_1, \ldots, r_J\} \geq 2Rh, \\ &X^h(z) = 0 \text{ for } z \leq 2h, \qquad X^h(z) = 1 \text{ for } z \geq 3h. \end{aligned} \tag{4.34}$$

Radius R is chosen such that $\mathcal{X}^h(y) = 0$ on ω_j^h.

The functions $\mathcal{U}_{(p)}^h$ and $\mathcal{V}_{(p)}^h$ are determined by formulas

$$\mathcal{U}_{(p)}^h = ||\mathcal{V}_{(p)}^h; H^h||^{-1} \mathcal{V}_{(p)}^h, \tag{4.35}$$

$$\begin{aligned} \mathcal{V}_{(p)0}^h(x) = {} & \mathcal{X}^h(y) v_{(p)0}^h(y) + \sum\nolimits_j \chi_j(y) (c_j^0 + c_j^1 (\mathbf{W}_j^0(\xi^j) + h \widehat{\mathbf{w}}_j(\xi^j)) \\ & - \sum\nolimits_j \mathcal{X}^h(y) \chi_j(y) \left(c_j^0 + c_j^1 \frac{1}{2\pi} (\ln \frac{1}{|\eta^j|} + \ln c_{\log}(\omega_j) \right), \end{aligned} \tag{4.36}$$

$$\mathcal{V}_{(p)j}^h(x) = X^h(z) v_{(p)j}^h(z) + X_j(z)(c_j^0 + hc_j^1 \mathbf{W}_j(\xi^j)) - X^h(z) X_j(z) (c_j^0 + hc_j^1 \gamma_j^{-1} |\omega_j|^{-1} \zeta^j) \tag{4.37}$$

where ingredients are taken from (3.9), (3.10) and $\widehat{\mathbf{w}}_j \in H^1(\Lambda_j)$ is a function with compact support such that

$$\widehat{\mathbf{w}}_j(\xi^j) = \mathbf{W}_j(\xi^j), \quad \xi^j \in \partial\omega_j \times (0, 1). \tag{4.38}$$

The latter condition and the cut-off functions $\mathcal{X}^h, X^h$ in (4.36), (4.37) assure that $\mathcal{V}_{(p)0}^h$ and $\mathcal{V}_{(p)j}^h$ coincide with each other on $\upsilon_j(h)$, *cf.* (1.11), and the composite function $\mathcal{V}_{(p)}^h$ falls into $H_0^1(\Xi(h); \Gamma(h))$.

First of all, we compute the scalar products $(\mathcal{V}^h_{(p)}, \mathcal{V}^h_{(q)})_h$. To this end, we observe that, according to (3.12), we have

$$\textstyle\sum_j(|b^h_{(p)j}| + |\partial_z v^h_{(p)j}(0)|) \le c|\ln h|^{-1}\sum_j(|\widehat{v}^h_{(p)0}(P^j)| + |v^h_{(p)j}(0)|). \tag{4.39}$$

The estimate (1.9) applied in the problem (1.20), (1.21) shows that

$$||\widehat{v}^h_{(p)0}; H^1(\omega_0)|| + \textstyle\sum_j|\widehat{v}^h_{(p)0}(P^j)| \le c\mu_p||v^h_{(p)0}; L^2(\omega_0)|| \le C_p. \tag{4.40}$$

Moreover, a solution of (1.21), (1.23) satisfies

$$||v_{(p)j}; H^2(I_j)|| \le c(\mu_p||v_{(p)j}; L^2(I_j)|| + |\partial_z v_{(p)j}(0)|) \tag{4.41}$$

while the last bound is due to the estimate (4.40) and the small factor $|\ln h|^{-1}$ on the right of (4.39).

Recalling that $c^0_j = \widehat{v}^h_{(p)0}(P^j)$ and $c^1_j = \gamma_j^{-1}|\omega_j|\partial_z v_{(p)j}(0)$, see (3.11), we rewrite (4.36) as follows:

$$\mathcal{V}^h_{(p)0} = \widehat{v}^h_{(p)0} + (\mathcal{X}^h - 1)\textstyle\sum_j\chi_j(\widehat{v}^h_{(p)0} - \widehat{v}^h_{(p)0}(P^j)) - (2\pi)^{-1}\mathcal{X}^h\sum_j\chi_j b^h_{(p)j}\ln r_j + \sum_j\chi_j c^1_j(h\widehat{\mathbf{w}}_j + \widetilde{\mathbf{W}}^0_j).$$

We list the estimates

$$\begin{aligned}
&||(1-\mathcal{X}^h)\chi_j(\widehat{v}^h_{(p)0} - \widehat{v}^h_{(p)0}(P^j)); H^1(\varOmega_\bullet(h))|| \le ch^{3/2-\delta}||\widetilde{v}^h_{(p)0}; V^2_\delta(\omega_0)|| \le ch, \\
&||b^h_{(p)j}\chi_j\ln r_j; H^1(\varOmega_\bullet(h))|| \le ch^{1/2}|\ln h|^{-1}\left(\int_h^R (r_j^{-2}+1)r_j\mathrm{d}r_j\right)^{1/2} \le ch^{1/2}|\ln h|^{-1/2}, \\
&h||\chi_j c^1_j h\widehat{\mathbf{w}}_j; H^1(\varOmega_\bullet(h))|| \le ch^{3/2}|\ln h|^{-1}||\widehat{\mathbf{w}}_j; H^1(\varLambda_j)||, \\
&||c^1_j\chi_j\widetilde{\mathbf{W}}^0_j; H^1(\varOmega_\bullet(h))|| \le ch^{1/2}|\ln h|^{-1}\left(\int_h^R\left(\frac{h^2}{r_j^4} + \frac{1}{r_j^2}\right)r_j\mathrm{d}r_j\right)^{1/2} \le ch^{1/2}|\ln h|^{-1/2}.
\end{aligned} \tag{4.42}$$

These must be commented. Notice that the factor $h^{1/2}$ comes due to integration in $z \in (0,h)$. In the first estimate we took into account that $1 - \mathcal{X}^h = 0$ outside the disk $\mathbb{B}_{2Rh}(P^j)$ where $r_j \le 2Rh$ and applied the weighted inequality (1.9) for $\widetilde{v}^h_{(p)0}(y) = \widehat{v}^h_{(p)0}(y) - \widehat{v}^h_{(p)0}(P^j)$ with $\delta \in (0, 1/2)$. The second and fourth estimates were derived by a direct calculation of norms and using the decomposition (3.4) of $\mathbf{W}^0_j$ and the bound $c|\ln h|^{-1}$ for $|c^1_j|$, *cf.* (4.39)–(4.41). The third estimate is obtained by the coordinate change $x \mapsto \xi^j$, see (3.1).

The above listed estimates support the following relation:

$$\begin{aligned}
|(\nabla_x\mathcal{V}^h_{(p)0}, \nabla_x\mathcal{V}^h_{(q)0})_{\varOmega_\bullet(h)} + (\mathcal{V}^h_{(p)0}, \mathcal{V}^h_{(q)0})_{\varOmega_\bullet(h)} - h(\nabla_y\widehat{v}^h_{(p)0}, \nabla_y\widehat{v}^h_{(q)0})_{\omega_\bullet(h)} \\
-h(\widehat{v}^h_{(p)0}, \widehat{v}^h_{(q)0})_{\omega_\bullet(h)}| \le c|\ln h|^{-1/2}.
\end{aligned} \tag{4.43}$$

Moreover,

$$|(\widehat{v}^h_{(p)0}, \widehat{v}^h_{(q)0})_{\omega_\bullet(h)} - (v^h_{(p)0}, v^h_{(q)0})_{\omega_0}| \le c|\ln h|^{-1/2}, \tag{4.44}$$

$$|(\nabla_y\widehat{v}^h_{(p)0}, \nabla_y\widehat{v}^h_{(q)0})_{\omega_\bullet(h)} - \mu^h_p(v^h_{(p)0}, v^h_{(q)0})_{\omega_0}| \le c|\ln h|^{-1/2}. \tag{4.45}$$

Indeed, in (4.44) we got rid of $\ln r_j$ by using the second estimate (4.42) and evaluate $||\widehat{v}^h_{(p)0}; L^2(\omega^h_j)||$ by means of the weighted inequality (1.9) again. To conclude (4.45), we observed additionally that

$$|(\nabla_y\widehat{v}^h_{(p)0}, \nabla_y\widehat{v}^h_{(q)0})_{\omega_0} - \mu^h_p(\widehat{v}^h_{(p)0}, \widehat{v}^h_{(q)0})_{\omega_0}| \le c|\ln h|^{-1/2}$$

because $\widehat{v}^h_{(p)0}$ is a solution of the problem

$$-\Delta\widehat{v}^h_{(p)0} - \mu^h_p\widehat{v}^h_{(p)0} = \widehat{f}^h_{(p)0} \quad \text{in} \quad \omega_0, \quad \partial_\nu\widehat{v}^h_{(p)0} = 0 \quad \text{on} \quad \partial\omega_0$$

with a right-hand side which is caused by abolition of $-b^h_{(p)j}\chi_j(2\pi)^{-1}\ln r_j$ and therefore has the $L^2(\omega_0)$-norm of order $|\ln h|^{-1}$.

From (4.43)–(4.45) we derive that

$$|(\nabla_x\mathcal{V}^h_{(p)0},\nabla_x\mathcal{V}^h_{(q)0})_{\Omega_\bullet(h)}+(\mathcal{V}^h_{(p)0},\mathcal{V}^h_{(q)0})_{\Omega_\bullet(h)}-h(1+\mu^h_p)(v^h_{(p)0},v^h_{(q)0})_{\omega_0}|\le c_{pq}|\ln h|^{-1/2}. \tag{4.46}$$

In a similar way but with much simpler calculations (recall that v^h_j is a smooth function on $[0,l_j]$) we derive the inequalities

$$\Big|h^{-1}\gamma_j(\nabla_x\mathcal{V}^h_{(p)j},\nabla_x\mathcal{V}^h_{(q)j})_{\Omega_j(h)}+h^{-1}\rho_j(\mathcal{V}^h_{(p)j},\mathcal{V}^h_{(q)j})_{\Omega_j(h)}-h(1+\mu^h_p)\rho|\omega_j|(v^h_{(p)j},v^h_{(q)j})_{I_j}\Big|\le c|\ln h|^{-1/2}. \tag{4.47}$$

Notice that the factor h^{-1} is compensated due to the relation $|\omega^h_j|=h^2|\omega_j|$. According to (1.17), (4.23) and (4.32), (2.14), the inequalities (4.46) and (4.47) lead us to

$$|(\mathcal{V}^h_{(p)},\mathcal{V}^h_{(q)})_h-h(1+\mu^h_p)\delta_{p,q}|\le c|\ln h|^{-1/2}. \tag{4.48}$$

Now we evaluate the norm in (4.28), namely

$$\delta^h_p=||T^h\mathcal{U}^h_{(p)}-t^h_p\mathcal{U}^h_{(p)};H^h||=(1+\mu^h_p)||\mathcal{V}^h_{(p)};H^h||^{-1}||\mathcal{V}^h_{(p)}-(1+\mu^h_p)T^h\mathcal{V}^h_{(p)};H^h||. \tag{4.49}$$

The first factor on the right is bounded, see Theorem 4.2, and the second one does not exceed c_ph^{-1} owing to (4.48) with $p=q$. Using a definition of a Hilbert norm together with formulas (4.23) and (4.24), we obtain that the last factor is equal to

$$\sup\Big|(\mathcal{V}^h_{(p)}-(1+\mu^h_p)T^h\mathcal{V}^h_{(p)},W^h)_h\Big|=\sup\Big|a(\mathcal{V}^h_{(p)},W^h;\varXi(h))-\mu^h_pb(\mathcal{V}^h_{(p)},W^h;\varXi(h))\Big| \tag{4.50}$$

where supremum is computed over the unit ball in H^h. Inequality (4.17), see ([4], Thm. 9), indicates bounds for weighted Lebesgue norms of $\mathcal{W}^h$.

We insert representations (4.36) and (4.37) into the last expressions $\mathcal{J}^h$ between the modulo sign in (4.50) and detach the elementary term

$$\mathcal{J}^h_{\mathbf{w}}=h\textstyle\sum_j c^1_j(\nabla_x\widehat{\mathbf{w}}_j,\nabla_xW^h)_{\Omega_\bullet(h)}-\mu^h_p(\widehat{\mathbf{w}}_j,W^h)_{\Omega_\bullet(h)},\quad|\mathcal{J}^h_{\mathbf{w}}|\le ch^{3/2}|\ln h|^{-1}. \tag{4.51}$$

We derived the estimate in the same way as above, that is, the information on $\widehat{\mathbf{w}}_j$, c^1_j and the coordinate change $x\mapsto\xi^j$.

Other ingredients are sufficiently smooth while integration by parts and commuting the Laplace operator with cut-off functions yield

$$\mathcal{J}^h_0=(\varDelta_xv^h_{(p)0}+\mu^h_pv^h_{(p)0},\mathcal{X}^hW^h)_{\Omega_\bullet(h)}+([\varDelta_x,\mathcal{X}^h]\widetilde{v}^h_{(p)0},W^h_0)_{\Omega_\bullet(h)}, \tag{4.52}$$

$$\begin{aligned}\mathcal{J}^h_{j0}=&(\varDelta_x\mathbf{W}^0_j,\chi_jW^h_0)_{\Omega_\bullet(h)}+([\varDelta_x,\chi_j]\widetilde{\mathbf{W}}^0_j,W^h_0)_{\Omega_\bullet(h)}\\&+c^1_j\mu^h_p(\mathbf{W}^0_j+\mathcal{X}^h(2\pi)^{-1}(\ln|\eta|-\ln c_{\log}(\omega_j)),\chi_jW^h)_{\Omega_\bullet(h)},\end{aligned} \tag{4.53}$$

$$\mathcal{J}^h_j=h^{-1}(\gamma_j\varDelta_xv^h_{(p)j}+\mu^h_p\rho_jv^h_{(p)j},X^hW^h)_{\Omega_j(h)}+h^{-1}\gamma_j([\varDelta_x,X^h]\widetilde{v}^h_{(p)j},W^h)_{\Omega_j(h)}, \tag{4.54}$$

$$\begin{aligned}\mathcal{J}^h_{jj}=&c^1_j\gamma_j(\varDelta_x\mathbf{W}_j,X_jW^h)+c^1_j\gamma_j([\varDelta_x,X_j]\widetilde{\mathbf{W}}_j,W^h)_{\Omega_j(h)}\\&+c^1_j\mu^h_p\rho_j(\mathbf{W}_j-X^h\gamma_j^{-1}|\omega_j|^{-1}\zeta^j,W^h)_{\Omega_j(h)}.\end{aligned} \tag{4.55}$$

Note that integrals over the surfaces $\upsilon_0(h)$, $\omega^0_j(0)$ and $\upsilon_j(h)$ vanish due to our choice of cut-off functions and boundary conditions for v^h_0, v^h_j and $\mathbf{W}^0_j$, $\mathbf{W}_j$, see Section 1.2 and 3.1, respectively. We will estimate all scalar products in (4.52)–(4.54) and explain how their sum converts into $\mathcal{J}^h-\mathcal{J}^h_{\mathbf{w}}$.

In view of the equation (1.20) the first term in (4.52) is null. The next term $\mathcal{J}_0^{h2}$ in (4.52) is obtained from the first and third terms in (4.36) after commuting the Laplace operator with the cut-off function $\mathcal{X}^h$; notice that

$$\begin{aligned}[][\Delta_x, \mathcal{X}^h\chi_j] = [\Delta_x, \mathcal{X}^h] + [\Delta_x, \chi_j],\\ [\Delta_x, \mathcal{X}^h] = 2\nabla_x\mathcal{X}^h\cdot\nabla_x + \Delta_x\mathcal{X}^h, \quad [\Delta_x, \chi_j] = 2\nabla_x\chi_j\cdot\nabla_x + \Delta_x\chi_j.\end{aligned} \tag{4.56}$$

Since $\widetilde{v}^h_{(p)0}(P^j) = 0$, see (2.9), and supports of coefficients in the differential operator $[\Delta_x, \mathcal{X}^h]$ belong to the disk $\mathbb{B}_{2Rh}(P^j)$, see (4.34), the direct consequence of the one-dimensional Hardy inequality

$$\begin{aligned}||r_j^{-2}(1+|\ln r_j|)^{-1}\widetilde{v}^h_{(p)0}; L^2(\mathbb{B}_{2Rh}(P^j))|| &\le c||r_j^{-2}(1+|\ln r_j|)^{-1}\nabla_y\widetilde{v}^h_{(p)0}; L^2(\mathbb{B}_{2Rh}(P^j))||\\ &\le ||\widetilde{v}^h_{(p)0}; H^2(\mathbb{B}_{2Rh}(P^j))||\end{aligned} \tag{4.57}$$

provides that

$$\begin{aligned}|\mathcal{J}_0^{h1}| \le (h^{-2}h^2(1+|\ln h|) + h^{-1}h(1+|\ln h|))h^{1/2}||\widetilde{v}^h_{(p)0}; H^2(\mathbb{B}_{2Rh}(P^j))||\\ \times h(1+|\ln h|)||r^{-1}(1+|\ln r|)^{-1}W_0^h; L^2(\Omega_\bullet(h))|| \le ch^{3/2}|\ln h|^3.\end{aligned} \tag{4.58}$$

Here, we took into account that $r = r_j < 2Rh$ for $y \in \mathbb{B}_{2Rh}(P^j)$. The first (long) multiplier in the middle of (4.58) is written according to the relation $|\nabla_y^k\mathcal{X}^h(y)| \le C_k h^{-k}$, formula for $[\Delta_x, \mathcal{X}^h]$ in (4.56) and weights in (4.57). The factor $h^{1/2}$ is due to integration in $z \in (0,h)$ and finally the weights $r^{-1}(1+|\ln r|)^{-1}$ and $(1+|\ln h|)^{-1}$ from (4.17) were considered. It should be also mentioned that $h_0 < 1$ and, therefore, $|\ln h| \ge c > 0$ for $h \in (0, h_0]$.

Dealing with (4.53) we observe that the Laplace equation for $\mathbf{W}_j^0$ in (3.3) annuls the first term $\mathcal{J}_{j0}^{h1}$. Supports of coefficients of $[\Delta_x, \chi_j]$, see (4.56), are located in the annulus $\{x \in \overline{\Omega_0(h)} : R_j^- \le r_j \le R_j^+\}$. Hence, using the decay rate in $|\eta^j| = \rho^j = h^{-1}r_j$ of the remainder $\widetilde{\mathbf{W}}_j^0$, see (3.4), we derive the following estimates for the second and third terms in (4.53):

$$\begin{aligned}|\mathcal{J}_{0j}^{h2}| &\le c|c_j^1|h^{1/2}\left(\int_{R_j^-}^{R_j^+}\left(\frac{1}{h^2}\frac{1}{(1+\rho_j)^4} + \frac{1}{(1+\rho_j)^2}\right)r_j\mathrm{d}r_j\right)^{1/2}||W^h; L^2(\Omega_\bullet(h))|| \le ch^{3/2},\\ |\mathcal{J}_{0j}^{h3}| &\le c|c_j^1|h^{1/2}\left(\int_{Rh}^{R_j^+}\frac{r_j\mathrm{d}r_j}{(1+\rho_j)^2}\right)^{1/2}||W^h; L^2(\Omega_\bullet(h))|| \le ch^{3/2}|\ln h|.\end{aligned} \tag{4.59}$$

Here, we applied (4.17) again and recalled that $|c_j^1| \le c|\ln h|^{-1}$. It should be mentioned that $\mathcal{J}_{0j}^{h2}$ and $\mathcal{J}_{0j}^{h3}$, respectively, involve the commutator $[\Delta_x, \chi_j]$ and the multiplication operator μ_p^h acting on $c_j^1\mathbf{W}_j^0$ and the last term in (4.36). In this way the sum $\mathcal{J}_0^h + \mathcal{J}_{01}^h + \ldots + \mathcal{J}_{0J}^h + \mathcal{J}_{\mathbf{w}}^h$ exhibits the whole part of $\mathcal{J}^h$ generated by (4.36).

Referring to (4.54), we see that $\mathcal{J}_j^{h1} = 0$ in view of (1.21). The second term $\mathcal{J}_j^{h2}$ in (4.54) meets the estimate

$$|\mathcal{J}_j^{h2}| \le ch^{-1}(h^{-2}h^2 + h^{-1}h)h^{3/2}||W_j^h; L^2(\Omega_j^h)|| \le ch^{3/2}.$$

Here, h^{-1} came from (4.54), the relations $|\nabla_x^k X^h(z)| \le c_k h^{-k}$ and $\widetilde{v}^h_{(p)j}(z) = v^h_{(p)j}(z) - v^h_{(p)j}(0) - z\partial_z v^h_{(p)j}(0) = O(z^2)$ were used, the factor $h^{3/2}$ is due to integration over $\Omega_j^h = \{x \in \overline{\Omega_j(h)} : z \le 3h\} \supset\mathrm{supp}\partial_z X^h$ and finally the direct consequence of the Newton-Leibnitz formula

$$h^{-2}||W_j^h; L^2(\Omega_j^h)||^2 \le ch^{-1}(||\partial_z W_j^h; L^2(\Omega_j^h)||^2 + ||W_j^h; L^2(\Omega_j^h)||^2) \le c||W_j^h; H^h||^2 = c \tag{4.60}$$

together with the inequality (4.17) were applied.

Similarly to the above considerations, the first term $\mathcal{J}_{jj}^{h1}$ in (4.55) vanishes, *cf.* (3.2), and the other couple of terms can be estimated as follows:

$$\begin{aligned}|\mathcal{J}_{jj}^{h2}| + |\mathcal{J}_{jj}^{h3}| &\leq c|c_j^1| \left(\int_{\Omega_j^h} \mathrm{d}x ||W_j^h; L^2(\Omega_j^h)||^2 + \int_{\Omega_j(h)} \mathrm{e}^{-2\delta z/h} \mathrm{d}x ||W_j^h; L^2(\Omega_j(h))||^2 \right)^{1/2} \\ &\leq c|\ln h|^{-1}(h^3h^2 + h^3h)^{1/2} \leq c|\ln h|^{-1}h^2.\end{aligned}$$

Here, we took into account the exponential decay in (3.6) together with formulas (4.60) and (4.17).

In the same way as above we detect a proper redistribution of commutators of Δ_x with X^h, X_j and conclude that $\mathcal{J}_j^h + \mathcal{J}_{jj}^h$ equals a part of $\mathcal{J}^h$ generated by (4.37).

We summarize our calculations and find that the worst bound in estimates derived for components of (4.50), occurs in (4.58). Hence, according to (4.48) with $q = p$, we obtain the following inequality for the quantity (4.49):

$$\delta_p^h \leq c_p h^{1/2} |\ln h|^3.$$

Lemma 4.3 gives us eigenvalues $\tau^n(h)$ and $\lambda^n(h) = \tau^n(h)^{-1} - 1$, see (4.26), of the problems (4.25) and(1.6)$-$(1.12), respectively, such that, by virtue of (4.33),

$$\begin{aligned}&\tau^n(h) \in [t_p^h - c_p h^{1/2}|\ln h|^3, t_p^h + c_p h^{1/2}|\ln h|^3] \\ &\qquad \Rightarrow \lambda^n(h) \in \left[\mu_p^h - C_p h^{1/2}|\ln h|^3, \mu_p^h + C_p h^{1/2}|\ln h|^3\right] \qquad \forall h \in (0, h_p)\end{aligned} \tag{4.61}$$

where c_p, C_p and h_p are some positive numbers and the index n may depend on h.

4.5. Theorem on asymptotics

We are in position to conclude with the main assertion of the paper.

Theorem 4.4. *For any $N \in \mathbb{N}$, there exist positive $h(N)$ and $c(N)$ such that the entries $\lambda^1(h), \ldots, \lambda^N(h)$ of the eigenvalue sequence* (1.18) *of the original problem* (1.6)$-$(1.14) (*or* (1.16) *in the variational formulation*) *and the first N positive eigenvalues*

$$0 < \mu_1^h \leq \mu_2^h \leq \ldots \leq \mu_N^h \tag{4.62}$$

of the equation (3.14) *or* (3.15) *are in the relationship*

$$|\lambda^n(h) - \mu_n^h| \leq c(N)h^{1/2}|\ln h|^3, \quad n = 1, \ldots, N, \ h \in (0, h(N)). \tag{4.63}$$

Proof. The result directly stems from (4.61) and all the previous considerations. We still need to utter several important remarks, in order to complete the proof. First, if μ_p^h is eigenvalue of multiplicity $\varkappa_p^h \geq 1$, then Lemma 4.3 provides us with $\varkappa_p^h$ different eigenvalues $\lambda^n(h), \ldots, \lambda^{n+\varkappa_p^h-1}(h)$ satisfying (4.61) with a bigger constant C_p. Indeed, owing to (4.48), the constructed approximate eigenfunctions $\mathcal{U}_{(p)}^h, \ldots, \mathcal{U}_{(p+\varkappa_p^h-1)}^h$ are almost orthonormalized, that is

$$|(\mathcal{U}_{(k)}^h, \mathcal{U}_{(q)}^h)_h - \delta_{p,q}| \leq c_p |\ln h|^{-1/2}, \quad k, q = p, \ldots, p + \varkappa_p^h - 1.$$

Moreover, setting $\delta_*^h = R\max\{\delta_p^h, \ldots, \delta_{p+\varkappa_p^h-1}^h\}$ in (4.30), we see that their projection onto the linear hull $\mathcal{L}(u_{N^h}^h, \ldots, u_{N^h+X^h-1}^h)$ are linear independent for a small h and a big R that is possible in the case $X^h \geq \varkappa_p^h$ only. Thus, changing $C_p \mapsto RC_p$ in (4.61) gives us at least $\varkappa_p^h$ desired eigenvalues.

Second, since each eigenvalue μ_p^h in (4.62) has an eigenvalue $\lambda^{n^h(p)}(h)$ in its small neighborhood and $n(p) \neq n(q)$ for $p \neq q$, we obtain $\lambda^n(h) \leq \mu_n^h + c_n h^{1/2}|\ln h|^3$ and confirm the Assumption (4.10) so that Theorem 4.2 becomes true.

Finally, we assume without loss of generality that N is fixed such that the eigenvalues μ_N^0 and μ_{N+1}^0 of the operator $\mathcal{A}^0$ obey the relation $\mu_N^0 < \mu_{N+1}^0$. If it happens that the index $n^h(N)$ of the eigenvalue $\lambda^{n^h(N)}$ in the vicinity of μ_N^h is strictly bigger that N, then, for an infinitesimal positive sequence $\{h_k\}_{k\in\mathbb{N}}$, we have $\lambda^{N+1}(h_k) \le \mu_{N+1}^0 - \varepsilon$ with some $\varepsilon > 0$. We can apply Theorem 4.2 and conclude that $\lambda^{N+1}(0) = \lim \lambda^{N+1}(h_k)$ as $k \to +0$ is an eigenvalue in the interval $(0, \mu_{N+1}^0 - \varepsilon]$, while the limits of (4.11), (4.12) constructed from the corresponding eigenfunctions $u^{N+1}(h_k, \cdot)$ are orthogonal in $\mathcal{L}$ to the vector eigenfunctions $(v_0^p, v_1^p, \dots, v_J^p)$, $p = 1, \dots, N$, of the operator $\mathcal{A}^0$. Since $\lambda^{N+1}(0)$ belongs to the spectrum of $\mathcal{A}^0$ but $\lambda^{N+1}(0) \le \mu_N^0$, the latter is absurd. $\square$

Remark 4.5. A proof of Proposition 3.1 may follow the same scheme and meets crucial simplifications because, first, the total multiplicity of the negative spectrum is known *a priori* and, second, the corresponding approximate eigenfunctions (3.18), (3.19) decay exponentially at a distance from the points $P^1, \dots, P^J$.

5. Final remarks

5.1. Simplified and rough asymptotics

Since the point condition (2.28) contains the big parameter $\ln h$ in the matrix (3.13), it is straightforward to write an asymptotics in $|\ln h|$ for eigenvalues (1.18). In view of the precision estimate (4.63) in Theorem 4.4 it suffices to find a decomposition of eigenvalues in the model problem (1.20)−(1.23), (2.28), (2.29), in particular, of the projections (2.15) of the corresponding eigenvectors.

Let us demonstrate the simplest ansatz for the first eigenvalue

$$\lambda^1(h) \sim \mu^1(h) \sim |\ln h|^{-1}\mu_1^1 + |\ln h|^{-2}\mu_2^1 + \dots \tag{5.1}$$

The corresponding eigenvector of the model problem is searched in the asymptotic form

$$v^1(h,\cdot) \sim v_{(0)}^1 + |\ln h|^{-1}v_{(1)}^1 + |\ln h|^{-2}v_{(2)}^1 + \dots \tag{5.2}$$

while the absence in (5.1) of the term $|\ln h|^0\mu_0^1$ clearly requires that

$$v_{(0)}^1 = (1, 0, \dots, 0) \tag{5.3}$$

and therefore

$$\wp'_+ v_{(0)}^1 = \varepsilon = (1, \dots, 1) \in \mathbb{R}^J, \quad \wp''_+ v_{(0)}^1 = \wp'_- v_{(0)}^1 = \wp''_- v_{(0)}^1 = 0 \in \mathbb{R}^J. \tag{5.4}$$

Hence, a problem to determine $v_{(1)}^1 = (v_{(1)0}^1, v_{(1)1}^1, \dots, v_{(1)J}^1)$ involves the Neumann problem

$$-\Delta_y v_{(1)0}^1(y) = \mu_1^1 v_{(1)0}^1(y) = \mu_1^1, \quad y \in \omega_\odot, \qquad \partial_\nu v_{(1)0}^1(y) = 0, \quad y \in \partial\omega_0, \tag{5.5}$$

which is derived by inserting (5.1)−(5.3) into the equations (1.20)−(1.22) and extracting terms of order $|\ln h|^{-1}$. Furthermore, we obtain from the point condition (2.28) with the matrix $S^h = (2\pi)^{-1}|\ln h|\mathbb{I} + \dots$ from (3.13) that

$$\wp'_+ v_{(0)}^1 + (2\pi)^{-1}\wp'_- v_{(1)}^1 = 0 \Rightarrow \wp'_- v_{(1)}^1 = -2\pi\varepsilon. \tag{5.6}$$

The generalized Green formula (2.16) delivers the compatibility condition in this problem, namely

$$\begin{aligned}\mu_1^1|\omega_0| &= -(\Delta_y v_{(1)0}^1, v_{(0)0}^1)_{\omega_0} + (v_{(1)0}^1, \Delta_y v_{(0)0}^1)_{\omega_0} = \left\langle \wp_+ v_{(1)}^1, \wp_- v_{(0)}^1 \right\rangle - \left\langle \wp_- v_{(1)}^1, \wp_+ v_{(0)}^1 \right\rangle \\ &= -\left\langle \wp'_- v_{(1)}^1, \wp'_+ v_{(0)}^1 \right\rangle = 2\pi\left\langle \varepsilon, \varepsilon \right\rangle = 2\pi J.\end{aligned}$$

Note that we obtained in the ansatz (5.1) the first term

$$\mu_1^1 = 2\pi J|\omega_0|^{-1} \tag{5.7}$$

which does not contain much information about the rod elements $\Omega_1(h), \ldots, \Omega_J(h)$ of the junction $\Xi(h)$, namely only their number J.

In order to construct the second term μ_2^1, we, first of all, observe that a solution of the problem (5.5) can be subject to the orthogonality condition

$$\int_{\omega_0} v^1_{(1)0}(y)\mathrm{d}y = 0 \tag{5.8}$$

and, thus, formulas (5.5), (5.6) and (2.37), (2.39) lead to the representation

$$v^1_{(1)}(y) = J^{-1}|\omega_0|\mu_1^1(\mathbf{G}^1(y) + \ldots + \mathbf{G}^J(y)) = 2\pi(\mathbf{G}^1(y) + \ldots + \mathbf{G}^J(y))$$

so that

$$\wp'_+ v^1_{(1)} = 2\pi\mathcal{G}\varepsilon, \quad \wp''_+ v^1_{(1)} = -2\pi\mathcal{Q}\varepsilon. \tag{5.9}$$

Owing to (5.8), the compatibility condition in the problem

$$-\Delta_y v^1_{(2)0}(y) = \mu_2^1 v^1_{(0)0}(y) + \mu_1^1 v^1_{(1)0}(y), \quad y \in \omega_\odot, \qquad \partial_z v^1_{(2)0}(y) = 0, \quad y \in \partial\omega_0,$$

reads

$$\mu_2^1|\omega_0| = -(\Delta_y v^1_{(2)0}, v^1_{(0)0})_{\omega_0} = -\left\langle \wp'_- v^1_{(2)}, \wp'_+ v^1_{(0)} \right\rangle$$

and implies

$$\mu_2^1 = 2\pi|\omega_0|^{-1}\left\langle (C_{\log} + 2\pi\mathcal{G} + 2\pi\mathcal{Q})\varepsilon, \varepsilon \right\rangle. \tag{5.10}$$

The diagonal matrices $\mathcal{Q}$ and $C_{\log}$ depend on length l_j and the reduced cross-section ω_j of the rod $\Omega_j(h)$, $j = 1, \ldots, J$, while the matrix $\mathcal{G}$ reflects disposition of the junction points $P^1, \ldots, P^J$ in the base of the plate $\Omega_0(h)$.

The terms (5.7) and (5.10) detached in (5.1) have been computed. An estimate of the asymptotic remainder $\widetilde{\mu}^1(h)$ is a simple algebraic task and Theorem 4.4 converts the estimate into

$$|\lambda^1(h) - |\ln h|^{-1}\mu_1^1 - |\ln h|^{-2}\mu_2^1| \le c_1|\ln h|^{-3}. \tag{5.11}$$

5.2. The homogeneous junction

By setting in (1.13) the restrictions (1.15), the problem (1.6)$-$(1.12) reduces to the mixed boundary-value problem

$$\begin{aligned} -\Delta_x u(h,x) = \lambda(h)u(h,x), \quad & x \in \Xi_\sqcup(h), \\ u(h,x) = 0,\ x \in \Gamma(h), \quad & \partial_\nu u(h,x) = 0,\ x \in \Xi_\sqcup(h) \setminus \Gamma(h), \end{aligned} \tag{5.12}$$

stated in the intact domain $\Xi_\sqcup(h) = \Omega_0(h) \cup \Omega_1(h) \cup \ldots \cup \Omega_J(h)$, *cf.* (1.4) and (1.1), (1.3). An asymptotic analysis of the stationary problem (5.12) with a source term $f(h,\cdot) \in L^2(\Xi_\sqcup(h))$ instead of $\lambda(h)u(h,\cdot)$ has been developed in ([4], Sect. 2). Let us outline certain peculiarities of asymptotic models in the case (1.15), which are to be derived along the same scheme as in Sections 3 and 4.

As was mentioned in Section 1.2, a specific feature of the homogeneous junction $\Xi_\sqcup(h)$ is that the limit operator decouples, see (5.17), *i.e.*, instead of the connected skeleton in Figure 2a, we obtain in the limit the disjoint domain $\omega \subset \mathbb{R}^2$ and intervals $I_1, \ldots, I_J \subset \mathbb{R}$ as drawn in Figure 2b.

Both the asymptotic models can be applied for the stationary problem of type (1.6)$-$(1.12) with the source term f_p instead of λu_p on the right-hand side of the Poisson equations. However, a result in [4] displays an explicit rational dependence on $|\ln h|$ of the corresponding solution that furnishes its complete asymptotic form in finite steps. Moreover, it reduces an evident importance of the models for the spectral problem (1.6)$-$(1.12) where an argument based on the big parameter $|\ln h|$ in (3.13) and (2.28) detects the holomorphic dependence on $|\ln h|^{-1}$ which clearly cannot be described in finite steps.

Trying to formulate point conditions connecting the limit problems (1.20), (1.22) and (1.21), (1.23) in the skeleton Ξ^0 of the junction $\Xi(h)$, see Figures 1 and 2, we need a special solution of the homogeneous Neumann problem for the Laplace equation in the union $\Upsilon_j = \Lambda \cup Q_j$ of a layer and a semi-infinite cylinder. As was shown in [4], this solution gets the asymptotic behavior

$$W(\xi) = \frac{1}{2\pi}\left(\ln\frac{1}{|\eta|} + \ln c_{\log}(\Lambda \cup Q_j)\right) + \sum_{i=1,2} K_i(\Lambda \cup Q_j)\frac{\eta_j}{|\eta|^2} + O(|\eta|^{-2}),\ \xi \in \Lambda,\ |\eta| \to +\infty, \tag{5.13}$$

$$W(\xi) = |\omega_j|^{-1}\zeta + O(\mathrm{e}^{-\delta\zeta}),\ \delta > 0,\ \xi \in Q_j,\ \zeta \to +\infty. \tag{5.14}$$

It should be noticed that a proper choice of the coordinate origin eliminates the constants $K_i = K_i(\Lambda \cup Q_j)$ because the change $\eta \mapsto \eta' = \eta + K$ with $K \in \mathbb{R}^2$ provides the formulas

$$\ln|\eta| = \ln|\eta' - K| = \ln|\eta'| - |\eta'|^{-2}(K\cdot\eta') + O(|\eta'|^{-3}),$$
$$|\eta|^{-2}\eta_i = |\eta'|^{-2}\eta'_i + O(|\eta'|^{-3}).$$

In what follows we assume that the points P^j and, therefore, the coordinates $y^j = h\eta^j$ are fixed such that $K_i(\Lambda \cup Q_j) = 0$, $i = 1, 2$, in (5.13).

Repeating the matching procedure performed in Section 3.2, we keep the expansions (3.7), (3.8) but, according to (5.13), (5.14), replace (3.9), (3.10) with the following ones:

$$c_j^0 + c_j^1 W_j(\xi) = c_j^1\frac{1}{2\pi}\ln\frac{1}{|\eta^j|} + c_j^1\frac{1}{2\pi}\ln c_{\log}(\Lambda \cup Q_j)) + c_j^0 + \dots \quad \text{in } \Omega_\bullet(h),$$
$$c_j^0 + c_j^1 W_j(\xi) = c_j^0 + c_j^1|\omega_j|^{-1}\zeta + \dots \qquad \text{in} \qquad \Omega_j(h).$$

Thus, we obtain the relations

$$b_j^h - h|\omega_j|^{-1}\partial_z v_j^h(0) = 0, \quad \widehat{v}_0^h(P^j) - v_j^h(0) = b_j^h(2\pi)^{-1}(\ln h + \ln c_{\log}(\Lambda \cup Q_j)) \tag{5.15}$$

which look quite similar to (3.12) but we have the small factor h on the derivatives $\partial_z v_j^h(0)$, that is, on the projection $\wp''_- v$. To perform the correct limit passage $h \to +0$, we recall our previous asymptotic analysis in ([4], Sect. 2) and make the substitution

$$v^h = (v_0^h, v_1^h, \dots, v_J^h) \Rightarrow \mathbf{v}^h = (\mathbf{v}_0^h, \mathbf{v}_1^h, \dots, \mathbf{v}_J^h) = (v_0^h, h^{-1/2}v_1^h, \dots, h^{-1/2}v_J^h). \tag{5.16}$$

As a result, we conclude the point conditions

$$b_j = 0,\ v_j^0(0) = 0,\ j = 1, \dots, J \quad \Leftrightarrow \quad \wp'_-\mathbf{v}^0 = 0 \in \mathbb{R}^J,\ \wp''_+\mathbf{v}^0 = 0 \in \mathbb{R}^J \tag{5.17}$$

corresponding to the self-adjoint extension described in Remark 2.6. In other words, the limit spectrum (4.9) is composed from the spectrum $\{\kappa_h^0\}_{n\in\mathbb{N}}$ of the Neumann problem in ω_0 and the spectra $\{\kappa_h^j = \pi^2 l_j^{-2} n^2\}_{n\in\mathbb{N}}$ of the Dirichlet problems in I_j, $j = 1, \dots, J$. It is worth to mention that, as was observed in [4], the substitution (5.16) has a clear physical reason, namely the energy functional for the problem (5.12) gets an appropriate approximation by the sum of the energy functionals for the above mentioned limit problems multiplied with the common factor h.

Let us construct the first correction term in the asymptotics

$$\lambda^1(h) = 0 + h\mu_1^1 + \widetilde{\lambda}^1(h) \tag{5.18}$$

of the first eigenvalue of the problem (5.12). Recalling an asymptotic procedure in ([4], Sect. 2), we search for the corresponding eigenfunction in the form

$$u^h(h,x) = 1 + hv_{(1)0}^1(x) + \dots \qquad \text{in} \qquad \Omega_\bullet(h), \tag{5.19}$$
$$u^h(h,x) = 1 - l_j^{-1}z + \dots \qquad \text{in} \qquad \Omega_j(h).$$

Regarding (5.19) as outer expansions, we write the inner expansions in the vicinity of the sockets θ_j^h as follows:

$$u^h(h,x) = 1 - hl_j^{-1}|\omega_j|W_j(h^{-1}(y-P^j), h^{-1}z) + \dots$$

Finally, we take the representations (5.13), (5.14) and apply the matching procedure, *cf.* Section 3.2, to close the problem (5.5) with the asymptotic conditions near the points $P^1, \dots, P^J$

$$v^1_{(1)0}(x) = -\frac{1}{2\pi}\sum\nolimits_j \chi_j(y)\frac{|\omega_j|}{l_j}\ln\frac{1}{r_j} + \widehat{v}^1_{(1)0}(x), \quad \widehat{v}^1_{(1)0} \in H^2(\omega_0). \tag{5.20}$$

Similarly to Section 5.1 the compatibility condition in the problem (5.5), (5.20) converts into the formula

$$\mu_1^1 = |\omega_0|^{-1}\left(|\omega_1|l_1^{-1} + \dots + |\omega_J|l_J^{-1}\right).$$

Combining approaches in Section 4 and ([4], Sect. 2), the asymptotic formula (5.18) can be justified by means of the estimate $|\widetilde{\lambda}^1(h)| \le c_1 h^2(1+|\ln h|)$ for the remainder.

Let us consider the problems (1.20), (1.22) and (1.21), (1.23) connected through the point conditions

$$\wp'_-\mathbf{v}^h + h^{1/2}\wp''_-\mathbf{v}^h = 0, \quad h^{-1/2}\wp''_+\mathbf{v}^h - \wp'_+\mathbf{v}^h - S^h\wp'_-\mathbf{v}^h = 0 \in \mathbb{R}^J, \tag{5.21}$$

where $\gamma_j = \rho_j = 1$ because the junction is homogeneous and the $J \times J$-matrix S^h is given by formula (3.13) with $c_{\log}(\varLambda \cup Q_j)$ instead of $c_{\log}(\omega_j)$, see (5.13) and (3.4).

The conditions (5.21) follow immediately from (5.15) after substitution (5.16). They are involved into the symmetric generalized Green formula of type (2.35)

$$\begin{aligned} q(v^h,w^h) &= \left\langle \wp'_+v^h - h^{-1/2}\wp''_+v^h + S^h\wp'_-v^h, \wp'_-w^h\right\rangle - \left\langle \wp'_-v^h, \wp'_+w^h - h^{-1/2}\wp''_+w^h + S^h\wp'_-w^h\right\rangle \\ &+ h^{-1/2}\left\langle \wp''_+v^h, \wp'_-w^h + h^{1/2}\wp''_-w^h\right\rangle - h^{-1/2}\left\langle \wp'_-v^h + h^{1/2}\wp''_-v^h, \wp''_+w^h\right\rangle, \end{aligned}$$

and therefore, all requirements in Sections 2 and 3 with slight modifications apply to the model in the case (1.15), too. Moreover, a simple analysis requiring only for algebraic operations as in Section 5.1, provides the representation

$$\mu^1(h) = h\mu_1^1 + O(h^2(1+|\ln h|)) \tag{5.22}$$

of an eigenvalue in the model (1.20)−(1.23), (5.21) which is supplied with an operator of type (2.33) on the function space (2.13) with detached asymptotics.

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