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UNIFIED FORMULATION AND ANALYSIS OF MIXED AND PRIMAL DISCONTINUOUS SKELETAL METHODS ON POLYTOPAL MESHES*,**

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Abstract. We propose in this work a unified formulation of mixed and primal discretization methods on polyhedral meshes hinging on globally coupled degrees of freedom that are discontinuous polynomials on the mesh skeleton. To emphasize this feature, these methods are referred to here as discontinuous skeletal. As a starting point, we define two families of discretizations corresponding, respectively, to mixed and primal formulations of discontinuous skeletal methods. Each family is uniquely identified by prescribing three polynomial degrees defining the degrees of freedom, and a stabilization bilinear form which has to satisfy two properties of simple verification: stability and polynomial consistency. Several examples of methods available in the recent literature are shown to belong to either one of those families. We then prove new equivalence results that build a bridge between the two families of methods. Precisely, we show that for any mixed method there exists a corresponding equivalent primal method, and the converse is true provided that the gradients are approximated in suitable spaces. A unified convergence analysis is carried out delivering optimal error estimates in both energy- and L^2-norms.

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1. Introduction

Over the last few years, discretization methods that support general polytopal meshes have received a great amount of attention. Such methods are often formulated in terms of two sets of degrees of freedom (DOFs) located inside mesh elements and on the mesh skeleton, respectively. The former can often be eliminated (possibly after hybridization) by static condensation, whereas the latter are responsible for the transmission of information among elements, and are therefore globally coupled. To emphasize the role of the second set of DOFs, these methods are referred to here as "skeletal". Skeletal methods can be classified according to the continuity property of skeletal DOFs on the mesh skeleton. We focus here on "discontinuous skeletal" methods, where skeletal DOFs are single-valued polynomials over faces fully discontinuous at the face boundaries. Since this terminology is not classical in the sense of standard finite elements, we explicitly point out that here

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single-valued means that interface values match from one element to the adjacent one. Discontinuous, on the other hand, refers to the fact that skeletal DOFs are discontinuous at vertices in 2d and edges in 3d.

Let $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$, denote an open, bounded, connected polytopal set, and let $f \in L^2(\Omega)$. To avoid unnecessary complications, we consider the following pure diffusion model problem: Find $u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} -\triangle u &= f && \text{in} && \Omega, \\ u &= 0 && \text{on} && \partial\Omega. \end{aligned} \tag{1.1}$$

We introduce a unified formulation of discontinuous skeletal discretizations of problem (1.1) which encompasses a large number of schemes from the literature. As a starting point, we define two families of discretizations corresponding, respectively, to mixed and primal discontinuous skeletal methods. Each family is uniquely identified by prescribing three polynomial degrees defining element-based and skeletal DOFs, and a stabilization bilinear form which has to satisfy two properties of simple verification: stability expressed in terms of a uniform norm equivalence, and polynomial consistency. Several examples of methods available in the recent literature are shown to belong to either one of those families. We then prove new equivalence results, collected in Theorems 6.4, 6.5, and 7.2 below, which build a bridge between the two families of methods. Precisely, we show that for any mixed method there exists a corresponding equivalent primal method, and the converse is true provided that the gradients are approximated in suitable spaces. A unified convergence analysis is also carried out delivering optimal error estimates in both energy- and L^2-norms; *cf.* Theorems 8.2 and 8.4 below.

A fundamental and motivating example is presented in Section 3: it refers to the well-known equivalence between the lowest-order Raviart–Thomas element [52] and the nonconforming Crouzeix–Raviart element [30] on triangular meshes. In some sense, the framework presented in this paper extends, with suitable modifications, this equivalence to recent methods supporting general polytopal meshes.

Polytopal methods were first investigated in the context of lowest-order discretizations starting from different points of view. In the context of finite volume schemes, several families of polyhedral methods have been developed as an effort to weaken the conditions on the mesh required for the consistency of classical five-point schemes. The resulting methods are expressed in terms of local balances, and an explicit expression for the numerical fluxes is usually available. Discontinuous skeletal methods in this context include the Mixed and Hybrid Finite Volume schemes of [42, 47]. Continuous skeletal methods have also been considered, *e.g.*, in [48].

Relevant features of the continuous problem different from local conservation have inspired other approaches. Mimetic Finite Difference methods are derived by using discrete integration by parts formulas to define the counterparts of differential operators and L^2-products; *cf.* [15] for an introduction. Discontinuous skeletal methods in this context include, in particular, the mixed Mimetic Finite Difference scheme of [21]. An example of continuous skeletal method is provided, on the other hand, by the nodal scheme of [18]. In the Discrete Geometric Approach [29], the formal links with the continuous operators are expressed in terms of Tonti diagrams [53]. We also cite in this context the Compatible Discrete Operator framework of [17]. To different extents, all of the previous methods can be linked to the seminal ideas of Whitney on geometric integration. Other methods that deserve to be cited here are the cell centered Galerkin methods of [31, 32], which can be regarded as discontinuous Galerkin methods with only one unknown per element where consistency is achieved by the use of cleverly-tailored reconstructions.

The close relation among the Mixed [42] and Hybrid [47] Finite Volume schemes and mixed Mimetic Finite Difference methods [21] has been investigated in [43], where equivalence at the algebraic level is demonstrated for generalized versions of such schemes; *cf.* also ([54], Sect. 7) for further insight into the link with submesh-based polyhedral implementations of classical mixed finite elements. The results of [43] are recovered here as a special case. A unifying point of view for the convergence analysis has been recently proposed in [44] under the name of Gradient Schemes. Finally, the methods discussed above can often be regarded as lowest-order versions of more recent polytopal technologies such as, *e.g.*, Virtual Elements and Hybrid High-Order methods.

A natural development of polytopal methods was to increase the approximation order. It has been known for quite some time that high-order polyhedral discretizations can be obtained by fully nonconforming approaches such as the discontinuous Galerkin method. An exposition of the basic analysis tools in this framework can be found in [36]; *cf.* also [33,34] for polynomial approximation results on polyhedral elements based on the Dupont-Scott theory [2, 10, 22, 46] for further developments. Particularly interesting among discontinuous Galerkin methods is the hybridizable version introduced in [23, 28], which constitutes a first example of high-order discontinuous skeletal method.

Very recent works have shown other possible approaches to the design of high-order polytopal discretizations combining element-based and skeletal unknowns. A first example of arbitrary-order discontinuous skeletal methods are primal [35, 39] and mixed [38] Hybrid High-Order methods. Hybrid High-Order methods were originally introduced in [37] in the context of linear elasticity and later extended to more general linear and nonlinear problems (see [41] for an up-to-date introduction including a list of references). The main idea consists in reconstructing high-order differential operators based on suitably selected DOFs and discrete integration by parts formulas. These reconstructions are then used to formulate the local contributions to the discrete problem including a cleverly tailored stabilization that penalizes high-order face-based residuals. A study of the relations among primal Hybrid High-Order methods, Hybridizable Discontinuous Galerkin (HDG) methods, and High-Order Mimetic Finite Differences [50] can be found in [25], where the corresponding numerical fluxes in the spirit of HDG methods are identified. The hybridization of the original mixed Hybrid High-Order method was studied in [1] (these results are recovered as a special case in this work). We also cite here [40], where the above ideas are illustrated for variable diffusion problems with more general boundary conditions.

Another framework including both continuous and discontinuous skeletal methods is provided by Virtual Elements [11,12]. Virtual Elements can be described as finite elements where the expressions of the basis functions are not available at each point, but suitable projections thereof can be computed using the selected DOFs. Such computable projections are then used to approximate bilinear forms, which also include a stabilization term that penalizes differences between the DOFs and the computable projection. We are particularly interested here in mixed [13,14,20] and nonconforming [8] Virtual Elements, both of which are discontinuous skeletal methods.

Very recently, other discontinuous skeletal discretizations supporting various polytopal shapes have been introduced, whose relation with the present framework will deserve further investigation in the future. We mention here, in particular, the M-decompositions studied in two dimensions in [26] and in three dimensions in [27]. M-decompositions provide a means to recover within HDG methods the superconvergence properties of classical mixed methods including, *e.g.*, the Raviart–Thomas and Brezzi–Douglas–Marini [19] methods on simplicial meshes; see also [5,6] concerning quadrilateral meshes.

The rest of this paper is organized as follows. In Section 2 we formulate the assumptions on the mesh and introduce the corresponding notation. In Section 3 we recall the classical equivalence of lowest-order Raviart–Thomas and nonconforming finite element methods. In Sections 4 and 5 we introduce the families of mixed and primal discontinuous skeletal methods under study, and provide several examples of lowest-order and high-order methods that fall in each category. In Section 6 we show how to obtain, starting from a discontinuous skeletal method in mixed formulation, an equivalent primal method. Conversely, in Section 7, we show how to derive an equivalent mixed formulation starting from a discontinuous skeletal method in primal formulation. Section 8 contains a unified convergence analysis yielding optimal error estimates in the energy- and L^2-norms.

2. Mesh and notation

Let $\mathcal{H} \subset \mathbb{R}_*^+$ denote a countable set of meshsizes having 0 as its unique accumulation point. We consider refined mesh sequences $(\mathcal{T}_h)_{h\in\mathcal{H}}$ where, for all $h \in \mathcal{H}$, $\mathcal{T}_h = \{T\}$ is a finite collection of nonempty disjoint open polytopal elements such that $\overline{\Omega} = \bigcup_{T\in\mathcal{T}_h} \overline{T}$ and $h = \max_{T\in\mathcal{T}_h} h_T$ (h_T stands for the diameter of T). For $X \subset \mathbb{R}^d$, we denote by $|X|_N$ the N-dimensional Hausdorff measure of X and, for all $T \in \mathcal{T}_h$, we let $\overline{\boldsymbol{x}}_T := |T|_d^{-1} \int_T \boldsymbol{x}$ denote the barycenter of T.

A hyperplanar closed connected subset F of $\overline{\Omega}$ is called a face if $|F|_{d-1} > 0$ and (i) either there exist distinct $T_1, T_2 \in \mathcal{T}_h$ such that $F = \partial T_1 \cap \partial T_2$ (and F is an interface) or (ii) there exists $T \in \mathcal{T}_h$ such that $F = \partial T \cap \partial\Omega$ (and F is a boundary face). The set of interfaces is denoted by $\mathcal{F}^{\mathrm{i}}$, the set of boundary faces by $\mathcal{F}^{\mathrm{b}}$, and we let $\mathcal{F}_h := \mathcal{F}^{\rm i} \cup \mathcal{F}^{\rm b}$. The set of faces partitions the mesh skeleton in the sense that distinct faces have disjoint interiors and that $\bigcup_{T\in\mathcal{T}_h} \partial T = \bigcup_{F\in\mathcal{F}_h} F$. For all $F \in \mathcal{F}_h$, we denote by $\overline{\boldsymbol{x}}_F := |F|_{d-1}^{-1} \int_F \boldsymbol{x}$ the barycenter of F.

For all $T \in \mathcal{T}_h$, the sets $\mathcal{F}_T := \{F \in \mathcal{F}_h \mid F \subset \partial T\}$ and $\mathcal{F}_T^{\rm i} := \mathcal{F}_T \cap \mathcal{F}^{\rm i}$ collect, respectively, the faces and interfaces lying on the boundary of T and, for all $F \in \mathcal{F}_T$, we denote by $\boldsymbol{n}_{TF}$ the normal to F pointing out of T. Symmetrically, for all $F \in \mathcal{F}_h$, $\mathcal{T}_F := \{T \in \mathcal{T}_h \mid F \subset \partial T\}$ is the set containing the one or two elements sharing F.

We assume that $(\mathcal{T}_h)_{h\in\mathcal{H}}$ is regular in the sense of ([36], Chapt. 1), *i.e.*, for all $h \in \mathcal{H}$, $\mathcal{T}_h$ admits a matching simplicial submesh $\mathfrak{T}_h$ and there exists a real number $\varrho > 0$ (the mesh regularity parameter) independent of h such that the following conditions hold: (i) For all $h \in \mathcal{H}$ and any simplex $S \in \mathfrak{T}_h$ of diameter h_S and inradius r_S, $\varrho h_S \leqslant r_S$; (ii) for all $h \in \mathcal{H}$, all $T \in \mathcal{T}_h$, and all $S \in \mathfrak{T}_h$ such that $S \subset T$, $\varrho h_T \leqslant h_S$. We refer to ([36], Chap. 1 and [33,34]) for a set of geometric and functional analytic results valid on regular meshes.

Let X be a mesh element or face. For an integer $l \geqslant 0$, we denote by $\mathbb{P}^l(X)$ the space spanned by the restriction to X of d-variate polynomials of total degree at most l. We also conventionally set $\mathbb{P}^{-1}(X) := \{0 \in \mathbb{R}\}$. We denote by $(\cdot,\cdot)_X$ and $\|\cdot\|_X$ the usual inner product and norm of $L^2(X)$. The index is dropped when $X = \Omega$. The L^2-projector $\pi_X^l : L^1(X) \to \mathbb{P}^l(X)$ is defined such that, for all $v \in L^1(X)$,

$$(\pi_X^l v - v, w)_X = 0 \qquad \forall w \in \mathbb{P}^l(X). \tag{2.1}$$

Let now a mesh element $T \in \mathcal{T}_h$ be fixed. For any integer $l \geqslant -1$ we set

$$\mathbb{G}_T^l := \nabla \mathbb{P}^{l+1}(T), \qquad \overline{\mathbb{G}}_T^l := \left\{ \boldsymbol{\tau} \in \mathbb{P}^l(T)^d \mid (\boldsymbol{\tau}, \nabla w)_T = 0 \quad \forall w \in \mathbb{P}^{l+1}(T) \right\}, \tag{2.2}$$

and denote by $\boldsymbol{\pi}_{\mathbb{G},T}^l : L^1(T)^d \to \mathbb{G}_T^l$ and $\boldsymbol{\pi}_{\overline{\mathbb{G}},T}^l : L^1(T)^d \to \overline{\mathbb{G}}_T^l$ the L^2-orthogonal projectors on $\mathbb{G}_T^l$ and $\overline{\mathbb{G}}_T^l$, respectively. Notice that (2.2) with $l = -1$ gives $\mathbb{G}_T^{-1} = \nabla \mathbb{P}^0(T) = \{\boldsymbol{0} \in \mathbb{R}^d\}$. Clearly, we have the direct decomposition

$$\mathbb{P}^l(T)^d = \mathbb{G}_T^l \oplus \overline{\mathbb{G}}_T^l. \tag{2.3}$$

For further use, at the global level we also define the space of broken polynomials

$$\mathbb{P}^l(\mathcal{T}_h) := \left\{ v_h \in L^2(\Omega) \mid v_T := v_{h|T} \in \mathbb{P}^l(T) \quad \forall T \in \mathcal{T}_h \right\}.$$

Throughout the paper, to avoid naming constants, we abridge as $a \lesssim b$ the inequality $a \leqslant Cb$ with real number $C > 0$ independent of h. We will also write $a \approx b$ to mean $a \lesssim b \lesssim a$.

3. A motivating example

In order to put the following discussion into perspective, we start by recalling an important motivating example, *i.e.*, the well-known equivalence between lowest-order Raviart–Thomas element and nonconforming Crouzeix–Raviart element on triangular meshes.

The Raviart–Thomas element [52] is widely used for the approximation of problems involving $\boldsymbol{H}(\mathrm{div};\Omega)$ when $\mathcal{T}_h$ is a matching triangular mesh. A popular implementation of the Raviart–Thomas scheme makes use of a hybridization procedure, introducing a Lagrange multiplier in order to enforce the continuity of the normal component of vectors from one element to the other. As a starting point, problem (1.1) is written in mixed form as follows: Find the flux $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div};\Omega)$ and the potential $u \in L^2(\Omega)$ such that

$$\begin{aligned} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathrm{div}\,\boldsymbol{\tau}, u) &= 0 && \forall \boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div};\Omega), \\ -(\mathrm{div}\,\boldsymbol{\sigma}, v) &= (f, v) && \forall v \in L^2(\Omega). \end{aligned}$$

Taking the Raviart–Thomas finite element space [52]

$$\mathbb{RT}^0(\mathcal{T}_h) := \left\{ \boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div};\Omega) \mid \boldsymbol{\tau}_{|T} \in \mathbb{RT}^0(T) := \mathbb{P}^0(T)^d + \boldsymbol{x}\mathbb{P}^0(T) \quad \forall T \in \mathcal{T}_h \right\} \tag{3.1}$$

for the flux and the space of piecewise constants $\mathbb{P}^0(\mathcal{T}_h) \subset L^2(\Omega)$ for the potential, its discretization reads: Find $\boldsymbol{\sigma}_h \in \mathbb{RT}^0(\mathcal{T}_h)$ and $u_h \in \mathbb{P}^0(\mathcal{T}_h)$ such that

$$\begin{aligned} (\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (\operatorname{div} \boldsymbol{\tau}_h, u_h) &= 0 \qquad && \forall \boldsymbol{\tau}_h \in \mathbb{RT}^0(\mathcal{T}_h), \\ -(\operatorname{div} \boldsymbol{\sigma}_h, v_h) &= (f, v_h) \qquad && \forall v_h \in \mathbb{P}^0(\mathcal{T}_h). \end{aligned} \tag{3.2}$$

The hybridized version of (3.2) consists in introducing the space Λ_h of piecewise constants on the internal portion of the mesh skeleton, and in solving the following problem which involves the *discontinuous* Raviart–Thomas space $\mathbb{RT}^{0,\mathrm{d}}(\mathcal{T}_h)$: Find $\boldsymbol{\sigma}_h \in \mathbb{RT}^{0,\mathrm{d}}(\mathcal{T}_h)$, $u_h \in \mathbb{P}^0(\mathcal{T}_h)$, and $\lambda_h \in \Lambda_h$ such that

$$\begin{aligned} (\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + (\operatorname{div} \boldsymbol{\tau}_h, u_h) + \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T^{\mathrm{i}}} (\boldsymbol{\tau}_h \cdot \boldsymbol{n}_{TF}, \lambda_h)_F &= 0 \qquad && \forall \boldsymbol{\tau}_h \in \mathbb{RT}^{0,\mathrm{d}}(\mathcal{T}_h), \\ -(\operatorname{div} \boldsymbol{\sigma}_h, v_h) &= (f, v_h) \qquad && \forall v_h \in \mathbb{P}^0(\mathcal{T}_h), \\ \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T^{\mathrm{i}}} (\boldsymbol{\sigma}_h \cdot \boldsymbol{n}_{TF}, \mu_h)_F &= 0 \qquad && \forall \mu_h \in \Lambda_h. \end{aligned} \tag{3.3}$$

The usual way of solving problem (3.3) is to invert the (block-diagonal) mass matrix corresponding to the variables in $\mathbb{RT}^{0,\mathrm{d}}(\mathcal{T}_h)$ and to consider a statically condensed linear system of the form

$$\mathbf{A}\mathsf{\Lambda} = \mathsf{F}$$

where $\mathbf{A}$ is symmetric and positive definite.

Let now $\mathbb{NC}(\mathcal{T}_h)$ be the nonconforming Crouzeix–Raviart space of [30] on the same mesh $\mathcal{T}_h$; *i.e.*, the space of piecewise affine functions which are continuous on the midnodes of the interelement edges. Denoting by $\mathbb{NC}_0(\mathcal{T}_h)$ the subspace of $\mathbb{NC}(\mathcal{T}_h)$ with DOFs lying on $\partial\Omega$ set to zero, the approximation of problem (1.1) reads: Find $u_h \in \mathbb{NC}_0(\mathcal{T}_h)$ such that

$$(\nabla_h u_h, \nabla_h v_h) = (f, v_h) \qquad \forall v_h \in \mathbb{NC}_0(\mathcal{T}_h), \tag{3.4}$$

where ∇_h denotes the broken gradient operator on $\mathcal{T}_h$. The matrix form of (3.4) is

$$\mathbf{B}\mathsf{U} = \mathsf{G}$$

with $\mathbf{B}$ symmetric and positive definite. It is now well understood that the matrices $\mathbf{A}$ and $\mathbf{B}$ are identical, as well as the corresponding right hand sides F and G. This important equivalence is a consequence of the results of [7, 51], [4, 24], and has been reported in this form in [54].

A natural question is whether results of this type can be obtained for higher order schemes on general polytopal meshes. The results that we are going to present aim at describing a unified setting where the equivalence of primal, mixed, and hybrid formulation can be proved. For a discussion of lowest-order Raviart–Thomas and Crouzeix–Raviart elements in the framework introduced in the following sections, we refer to Examples 4.6 and 5.7, respectively.

4. A family of mixed discontinuous skeletal methods

In this section we introduce a family of mixed discontinuous skeletal methods and provide a few examples of members of this family.

4.1. Local spaces

For a given integer $k \geqslant 0$ corresponding to the skeletal polynomial degree, we let l and m be two integers such that

$$\max(0, k-1) \leqslant l \leqslant k+1, \qquad m \in \{0, k\}. \tag{4.1}$$

Let a mesh element $T \in \mathcal{T}_h$ be given. We define the following space of flux degrees of freedom (DOFs):

$$\underline{\boldsymbol{\Sigma}}_T^{k,l,m} := (\mathbb{G}_T^{l-1} \oplus \overline{\mathbb{G}}_T^m) \times \left(\bigtimes_{F \in \mathcal{F}_T} \mathbb{P}^k(F) \right). \tag{4.2}$$

For a generic element $\underline{\boldsymbol{\tau}}_T$ of $\underline{\boldsymbol{\Sigma}}_T^{k,l,m}$ we use the notation $\underline{\boldsymbol{\tau}}_T = (\boldsymbol{\tau}_T, (\tau_{TF})_{F \in \mathcal{F}_T})$ with $\boldsymbol{\tau}_T = \boldsymbol{\tau}_{\mathbb{G},T} + \boldsymbol{\tau}_{\overline{\mathbb{G}},T}$. For a fixed Lebesgue index $s > 2$, we let $\boldsymbol{\Sigma}^+(T) := \{\boldsymbol{\tau} \in L^s(T)^d \mid \operatorname{div} \boldsymbol{\tau} \in L^2(T)\}$ and define the local flux reduction map $\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,m} : \boldsymbol{\Sigma}^+(T) \to \underline{\boldsymbol{\Sigma}}_T^{k,l,m}$ such that, for all $\boldsymbol{\tau} \in \boldsymbol{\Sigma}^+(T)$,

$$\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,m} \boldsymbol{\tau} := \left(\boldsymbol{\pi}_{\mathbb{G},T}^{l-1} \boldsymbol{\tau} + \boldsymbol{\pi}_{\overline{\mathbb{G}},T}^m \boldsymbol{\tau}, \left(\pi_F^k (\boldsymbol{\tau} \cdot \boldsymbol{n}_{TF}) \right)_{F \in \mathcal{F}_T} \right). \tag{4.3}$$

The additional regularity in $\boldsymbol{\Sigma}^+(T)$ is classically needed for the face reductions to be well-defined (see, *e.g.*, [16], Sect. 2.5.1) for a detailed discussion of this point. The space $\underline{\boldsymbol{\Sigma}}_T^{k,l,m}$ is equipped with the $L^2(T)^d$-like norm $\|\cdot\|_{\boldsymbol{\Sigma},T}$ such that, for all $\underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{\Sigma}}_T^{k,l,m}$,

$$\begin{aligned} \|\underline{\boldsymbol{\tau}}_T\|_{\boldsymbol{\Sigma},T}^2 &:= \|\boldsymbol{\tau}_T\|_T^2 + \sum_{F \in \mathcal{F}_T} h_F \|\tau_{TF}\|_F^2 \\ &= \|\boldsymbol{\tau}_{\mathbb{G},T}\|_T^2 + \|\boldsymbol{\tau}_{\overline{\mathbb{G}},T}\|_T^2 + \sum_{F \in \mathcal{F}_T} h_F \|\tau_{TF}\|_F^2, \end{aligned} \tag{4.4}$$

where to pass to the second line we have used the orthogonal decomposition (2.3). Finally, we define the following space of local potential DOFs:

$$U_T^l := \mathbb{P}^l(T). \tag{4.5}$$

4.2. Local reconstruction operators

The family of mixed discretizations of problem (1.1) relies on operator reconstructions defined at the element level. Let $T \in \mathcal{T}_h$. The discrete divergence $\mathrm{D}_T^l : \underline{\boldsymbol{\Sigma}}_T^{k,l,m} \to U_T^l$ is such that, for all $\underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{\Sigma}}_T^{k,l,m}$,

$$(\mathrm{D}_T^l \underline{\boldsymbol{\tau}}_T, q)_T = -(\boldsymbol{\tau}_T, \nabla q)_T + \sum_{F \in \mathcal{F}_T} (\tau_{TF}, q)_F \qquad \forall q \in U_T^l. \tag{4.6}$$

The right-hand side of (4.6) resembles an integration by parts formula where the role of the vector function represented by $\underline{\boldsymbol{\tau}}_T$ in volumetric and boundary integrals is played by the element-based and face-based DOFs, respectively.

The local reconstruction $\mathbf{P}_T^k : \underline{\boldsymbol{\Sigma}}_T^{k,l,m} \to \mathbb{G}_T^k$ of the irrotational component of the flux is such that, for all $\underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{\Sigma}}_T^{k,l,m}$,

$$(\mathbf{P}_T^k \underline{\boldsymbol{\tau}}_T, \nabla w)_T = -(\mathrm{D}_T^l \underline{\boldsymbol{\tau}}_T, w)_T + \sum_{F \in \mathcal{F}_T} (\tau_{TF}, w)_F \qquad \forall w \in \mathbb{P}^{k+1}(T), \tag{4.7}$$

where again the right-hand side is designed to resemble an integration by parts formula where the continuous divergence operator is replaced by D_T^l, while the role of the normal trace of the vector function represented by $\underline{\boldsymbol{\tau}}_T$ is played by boundary DOFs.

Remark 4.1. The definitions of D_T^l and $\mathbf{P}_T^k$ are independent of the flux DOFs $\boldsymbol{\tau}_{\overline{\mathbb{G}},T} \in \overline{\mathbb{G}}_T^m$.

Finally, we define the full vector field reconstruction $\mathbf{S}_T^k : \underline{\boldsymbol{\Sigma}}_T^{k,l,m} \to \mathbb{P}^k(T)^d$ such that, for all $\underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{\Sigma}}_T^{k,l,m}$,

$$\mathbf{S}_T^k \underline{\boldsymbol{\tau}}_T := \mathbf{P}_T^k \underline{\boldsymbol{\tau}}_T + \boldsymbol{\tau}_{\overline{\mathbb{G}},T}. \tag{4.8}$$

The following properties hold:

$$\mathrm{D}_T^l \underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,m} \boldsymbol{\tau} = \pi_T^l(\operatorname{div} \boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}^+(T), \tag{4.9}$$

$$\mathbf{P}_T^k \underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,m} \boldsymbol{\tau} = \boldsymbol{\tau} \qquad \forall \boldsymbol{\tau} \in \mathbb{G}_T^k. \tag{4.10}$$

Defining the space

$$\mathbb{S}_T^{k,m} := \begin{cases} \mathbb{G}_T^k & \text{if } m = 0, \\ \mathbb{P}^k(T)^d & \text{if } m = k, \end{cases} \tag{4.11}$$

it follows from (4.10) together with the orthogonal decomposition (2.3) and the definitions (4.3) of the reduction map $\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,m}$ and (4.8) of $\mathbf{S}_T^k$ that

$$\mathbf{S}_T^k \underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,m} \boldsymbol{\tau} = \boldsymbol{\tau} \qquad \forall \boldsymbol{\tau} \in \mathbb{S}_T^{k,m}, \tag{4.12}$$

which expresses the polynomial consistency of $\mathbf{S}_T^k$.

4.3. Local bilinear form

Let $T \in \mathcal{T}_h$. We approximate the $L^2(T)^d$-product of fluxes by means of the bilinear form $\mathrm{m}_T : \underline{\boldsymbol{\Sigma}}_T^{k,l,m} \times \underline{\boldsymbol{\Sigma}}_T^{k,l,m} \to \mathbb{R}$ such that

$$\mathrm{m}_T(\underline{\boldsymbol{\sigma}}_T, \underline{\boldsymbol{\tau}}_T) := \left(\mathbf{S}_T^k \underline{\boldsymbol{\sigma}}_T, \mathbf{S}_T^k \underline{\boldsymbol{\tau}}_T\right)_T + \mathrm{s}_{\boldsymbol{\Sigma},T}(\underline{\boldsymbol{\sigma}}_T, \underline{\boldsymbol{\tau}}_T) \tag{4.13a}$$

$$= \left(\mathbf{P}_T^k \underline{\boldsymbol{\sigma}}_T, \mathbf{P}_T^k \underline{\boldsymbol{\tau}}_T\right)_T + \left(\boldsymbol{\sigma}_{\overline{\mathbb{G}},T}, \boldsymbol{\tau}_{\overline{\mathbb{G}},T}\right)_T + \mathrm{s}_{\boldsymbol{\Sigma},T}(\underline{\boldsymbol{\sigma}}_T, \underline{\boldsymbol{\tau}}_T), \tag{4.13b}$$

where the right-hand side of (4.13a) is composed of a consistency and a stabilization term.

Assumption 4.2 (Bilinear form $\mathrm{s}_{\boldsymbol{\Sigma},T}$)**.** The symmetric, positive semi-definite bilinear form $\mathrm{s}_{\boldsymbol{\Sigma},T} : \underline{\boldsymbol{\Sigma}}_T^{k,l,m} \times \underline{\boldsymbol{\Sigma}}_T^{k,l,m} \to \mathbb{R}$ satisfies the following properties:

(S1) *Stability.* It holds, for all $\underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{\Sigma}}_T^{k,l,m}$, with norm $\|\cdot\|_{\boldsymbol{\Sigma},T}$ defined by (4.4),

$$\|\underline{\boldsymbol{\tau}}_T\|_{\mathrm{m},T}^2 := \mathrm{m}_T\left(\underline{\boldsymbol{\tau}}_T, \underline{\boldsymbol{\tau}}_T\right) \approx \|\underline{\boldsymbol{\tau}}_T\|_{\boldsymbol{\Sigma},T}^2;$$

(S2) *Polynomial consistency.* For all $\boldsymbol{\chi} \in \mathbb{S}_T^{k,m}$, with local flux reduction map $\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,m}$ defined by (4.3),

$$\mathrm{s}_{\boldsymbol{\Sigma},T}\left(\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,m} \boldsymbol{\chi}, \underline{\boldsymbol{\tau}}_T\right) = 0 \qquad \forall \underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{\Sigma}}_T^{k,l,m}.$$

4.4. Global spaces and mixed problem

We define the following global discrete spaces for the flux:

$$\underline{\check{\boldsymbol{\Sigma}}}_h^{k,l,m} := \bigtimes_{T \in \mathcal{T}_h} \underline{\boldsymbol{\Sigma}}_T^{k,l,m}, \qquad \underline{\boldsymbol{\Sigma}}_h^{k,l,m} := \left\{ \underline{\boldsymbol{\tau}}_h \in \underline{\check{\boldsymbol{\Sigma}}}_h^{k,l,m} \;\middle|\; \sum_{T \in \mathcal{T}_F} \tau_{TF} = 0 \quad \forall F \in \mathcal{F}^{\mathrm{i}} \right\}. \tag{4.14}$$

The restriction of a DOF vector $\underline{\boldsymbol{\tau}}_h \in \underline{\check{\boldsymbol{\Sigma}}}_h^{k,l,m}$ to a mesh element $T \in \mathcal{T}_h$ is denoted by $\underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{\Sigma}}_T^{k,l,m}$, and we equip $\underline{\check{\boldsymbol{\Sigma}}}_h^{k,l,m}$ (hence also $\underline{\boldsymbol{\Sigma}}_h^{k,l,m}$) with the $L^2(\Omega)^d$-like norm (*cf.* (4.4) for the definition of $\|\cdot\|_{\boldsymbol{\Sigma},T}$)

$$\|\underline{\boldsymbol{\tau}}_h\|_{\boldsymbol{\Sigma},h}^2 := \sum_{T \in \mathcal{T}_h} \|\underline{\boldsymbol{\tau}}_T\|_{\boldsymbol{\Sigma},T}^2. \tag{4.15}$$

TABLE 1. Examples of methods originally introduced in mixed formulation.

Reference	Name	k	l	m	$\mathrm{s}_{\boldsymbol{\Sigma},T}$
[52]	$\mathbb{RT}^0$ Finite Element	0	0	0	Equation (4.23)
[21]	Mimetic Finite Difference	0	0	0	Equation (4.20)
[43]	Mixed Finite Volume				
[29]	Discrete Geometric Approach	0	0	0	Equation (4.27)
[38]	Mixed High-Order	$\geqslant 0$	k	0	Equation (4.28)
[20]	Mixed Virtual Element	$\geqslant 1$	$k-1$	0	Equation (4.29)
[14]	Mixed Virtual Element	$\geqslant 0$	k	k	Equation (4.30)

The global space for the potential is spanned by broken polynomials of total degree l:

$$U_h^l := \mathbb{P}^l(\mathcal{T}_h). \tag{4.16}$$

The global $L^2(\Omega)^d$-like product on $\underline{\check{\boldsymbol{\Sigma}}}_h^{k,l,m}$ is defined by element-by-element assembly by setting, for all $\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h \in \underline{\check{\boldsymbol{\Sigma}}}_h^{k,l,m}$,

$$\mathrm{m}_h(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h) := \sum_{T \in \mathcal{T}_h} \mathrm{m}_T(\underline{\boldsymbol{\sigma}}_T, \underline{\boldsymbol{\tau}}_T). \tag{4.17}$$

We also need the global divergence operator $\mathrm{D}_h^l : \underline{\check{\boldsymbol{\Sigma}}}_h^{k,l,m} \to U_h^l$ such that, for all $\underline{\boldsymbol{\tau}}_h \in \underline{\check{\boldsymbol{\Sigma}}}_h^{k,l,m}$,

$$(\mathrm{D}_h^l \underline{\boldsymbol{\tau}}_h)_{|T} = \mathrm{D}_T^l \underline{\boldsymbol{\tau}}_T \qquad \forall T \in \mathcal{T}_h.$$

We consider the following

Problem 4.3 (Mixed problem)**.** Find $(\underline{\boldsymbol{\sigma}}_h, u_h) \in \underline{\boldsymbol{\Sigma}}_h^{k,l,m} \times U_h^l$ such that,

$$\begin{aligned} \mathrm{m}_h(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\tau}}_h) + (u_h, \mathrm{D}_h^l \underline{\boldsymbol{\tau}}_h) &= 0 \qquad && \forall \underline{\boldsymbol{\tau}}_h \in \underline{\boldsymbol{\Sigma}}_h^{k,l,m}, && (4.18a) \\ -(\mathrm{D}_h^l \underline{\boldsymbol{\sigma}}_h, v_h) &= (f, v_h) \qquad && \forall v_h \in U_h^l. && (4.18b) \end{aligned}$$

Using standard arguments relying on the coercivity of m_h (a consequence of (S1)) and the existence of a Fortin interpolator (*cf.* (4.9)), one can prove that problem (4.18) is well-posed; *cf.*, *e.g.*, [16].

Remark 4.4 (Hybridization and static condensation)**.** Various possibilities are available to make the actual implementation of the method (4.18) more efficient. A first option consists in implementing the equivalent primal reformulation (6.14) described in detail below; *cf.* also Remark 5.4. Another option, in the spirit of [3], consists in locally eliminating element-based flux DOFs and element-based potential DOFs of degree $\geqslant 1$ by locally solving small mixed problems. The resulting global problem is expressed in terms of the skeletal flux DOFs plus one potential DOF per element.

4.5. Examples

We provide in this section a few examples of discontinuous skeletal methods originally introduced in a mixed formulation which can be traced back to (4.18). Each method is uniquely defined by prescribing the three polynomial degrees k, l, and m (in accordance with (4.1)) and the expression of the local stabilization bilinear form $s_{\boldsymbol{\Sigma},T}$ for a generic mesh element $T \in \mathcal{T}_h$. A synopsis is provided in Table 1.

Example 4.5 (The Mimetic Finite Difference method of [21] and the Mixed Finite Volume method of [43])**.** The Mimetic Finite Difference method of [21] and the Mixed Finite Volume method of ([43], Sect. 2.3) (which is a variation of the one originally introduced in [42]) correspond to the choice $k = l = m = 0$. We present them together since an equivalence result was already proved in [43]. The equivalence therein also includes the Hybrid

Finite Volume method of [47] (see also Exp. 5.5 below), suggesting the acronym HMM (Hybrid–Mixed–Mimetic) used in subsequent papers from the same authors. In the lowest-order case, explicit expressions can be found for both D_T^0 and $\mathbf{S}_T^0 = \mathbf{P}_T^0$: For all $\underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{\Sigma}}_T^{0,0,0}$,

$$\mathrm{D}_T^0 \underline{\boldsymbol{\tau}}_T = \frac{1}{|T|_d} \sum_{F \in \mathcal{F}_T} |F|_{d-1} \tau_{TF}, \qquad \mathbf{S}_T^0 \underline{\boldsymbol{\tau}}_T = \mathbf{P}_T^0 \underline{\boldsymbol{\tau}}_T = \frac{1}{|T|_d} \sum_{F \in \mathcal{F}_T} |F|_{d-1} \tau_{TF} (\overline{\boldsymbol{x}}_F - \overline{\boldsymbol{x}}_T). \tag{4.19}$$

Following [42,43], $\overline{\boldsymbol{x}}_T$ can be replaced by a more general point $\boldsymbol{x}_T$ in the above formula: this requires to modify the definition (4.3) of the local reduction map $\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,m}$ to preserve polynomial consistency for $\mathbf{S}_T^0$, as recently explained in [45]. The stabilization is parametrized by a symmetric, positive definite matrix $\boldsymbol{B}^T = (B_{FF'}^T)_{F,F' \in \mathcal{F}_T}$:

$$\mathrm{s}_{\boldsymbol{\Sigma},T}(\underline{\boldsymbol{\sigma}}_T, \underline{\boldsymbol{\tau}}_T) = \sum_{F \in \mathcal{F}_T} \sum_{F' \in \mathcal{F}_T} (\mathbf{S}_T^0 \underline{\boldsymbol{\sigma}}_T \cdot \boldsymbol{n}_{TF} - \sigma_{TF}) B_{FF'}^T (\mathbf{S}_T^0 \underline{\boldsymbol{\tau}}_T \cdot \boldsymbol{n}_{TF'} - \tau_{TF'}). \tag{4.20}$$

In order to have the uniform norm equivalence (S1), the matrix $\boldsymbol{B}^T$ should have an appropriate scaling as detailed in ([43], Eqs. (4.2)–(4.4)) (the latter conditions are essentially equivalent to (S1)). Straightforward choices are, *e.g.*, $\boldsymbol{B}^T = |T|_d \boldsymbol{I}_d$ or $\boldsymbol{B}^T = \mathrm{diag}\,(h_F |F|_{d-1})_{F \in \mathcal{F}_T}$. It is worth noting that the original Mixed Finite Volume method of [42] does not enter the present framework as the corresponding stabilization bilinear form $\mathrm{s}_{\boldsymbol{\Sigma},T}(\underline{\boldsymbol{\sigma}}_T, \underline{\boldsymbol{\tau}}_T) = \sum_{F \in \mathcal{F}_T} h_T |F|_{d-1} \sigma_{TF} \tau_{TF}$ violates (S2) (it is, however, weakly consistent).

Example 4.6 (The lowest-order Raviart–Thomas element)**.** We assume that T is an element from a matching simplicial mesh $\mathcal{T}_h$, and we consider the local lowest order Raviart–Thomas space $\mathbb{RT}^0(T)$ defined by (3.1). Clearly, the vector space $\underline{\boldsymbol{\Sigma}}_T^{0,0,0}$ contains the standard DOFs for $\mathbb{RT}^0(T)$ defined by the flux reduction map $\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{0,0,0}$ as the average values of the normal components on each face. It can be checked that $\mathbb{RT}^0(T) = \mathrm{span}\left(\boldsymbol{\varphi}_F^T\right)_{F \in \mathcal{F}_T}$ with

$$\boldsymbol{\varphi}_F^T(\boldsymbol{x}) := \frac{|F|_{d-1}}{d|T|_d} (\boldsymbol{x} - \boldsymbol{P}_F) \qquad \forall \boldsymbol{x} \in T, \tag{4.21}$$

where $\boldsymbol{P}_F$ denotes the vertex opposite to the face F (this formula generalizes [9], Eq. (4.3) to any $d \geqslant 1$). For all $F \in \mathcal{F}_T$, the basis function $\boldsymbol{\varphi}_F^T$ satisfies $(\boldsymbol{\varphi}_F^T \cdot \boldsymbol{n}_{TF})_{|F} \equiv 1$ and $(\boldsymbol{\varphi}_F^T \cdot \boldsymbol{n}_{TF'})_{|F'} \equiv 0$ for all $F' \in \mathcal{F}_T \backslash \{F\}$. Moreover, inserting $\pm \overline{\boldsymbol{x}}_T$ inside the parentheses in (4.21), and using the fact that $(\overline{\boldsymbol{x}}_T - \boldsymbol{P}_F) = d(\overline{\boldsymbol{x}}_F - \overline{\boldsymbol{x}}_T)$, we arrive at the following equivalent expression for $\boldsymbol{\varphi}_F^T$:

$$\boldsymbol{\varphi}_F^T(\boldsymbol{x}) = \frac{|F|_{d-1}}{|T|_d} (\overline{\boldsymbol{x}}_F - \overline{\boldsymbol{x}}_T) + \frac{|F|_{d-1}}{d|T|_d} (\boldsymbol{x} - \overline{\boldsymbol{x}}_T) \qquad \forall \boldsymbol{x} \in T.$$

Let now $\mathbf{t}_T \in \mathbb{RT}^0(T)$ and $\underline{\boldsymbol{\tau}}_T = (\tau_{TF})_{F \in \mathcal{F}_T} := \underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{0,0,0} \mathbf{t}_T$, so that $\mathbf{t}_T = \sum_{F \in \mathcal{F}_T} \boldsymbol{\varphi}_F^T \tau_{TF}$. Straightforward computations show that

$$\mathrm{div}\, \mathbf{t}_T = \mathrm{D}_T^0 \underline{\boldsymbol{\tau}}_T, \qquad \boldsymbol{\pi}_T^0 \mathbf{t}_T = \mathbf{S}_T^0 \underline{\boldsymbol{\tau}}_T = \mathbf{P}_T^0 \underline{\boldsymbol{\tau}}_T,$$

with explicit expressions for D_T^0 and $\mathbf{S}_T^0 = \mathbf{P}_T^0$ given by (4.19). Hence, we can rewrite the L^2-product of two functions $\mathfrak{s}_T, \mathbf{t}_T \in \mathbb{RT}^0(T)$ with DOFs $\underline{\boldsymbol{\sigma}}_T := \underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{0,0,0} \mathfrak{s}_T$ and $\underline{\boldsymbol{\tau}}_T := \underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{0,0,0} \mathbf{t}_T$ as

$$(\mathfrak{s}_T, \mathbf{t}_T)_T = (\pi_T^0 \mathfrak{s}_T, \pi_T^0 \mathbf{t}_T)_T + (\mathfrak{s}_T - \pi_T^0 \mathfrak{s}_T, \mathbf{t}_T - \pi_T^0 \mathbf{t}_T)_T = (\mathbf{S}_T^0 \underline{\boldsymbol{\sigma}}_T, \mathbf{S}_T^0 \underline{\boldsymbol{\tau}}_T)_T + \mathrm{s}_{\boldsymbol{\Sigma},T}(\underline{\boldsymbol{\sigma}}_T, \underline{\boldsymbol{\tau}}_T), \tag{4.22}$$

where, observing that $(\boldsymbol{\varphi}_F^T - \pi_T^0 \boldsymbol{\varphi}_F^T)(\boldsymbol{x}) = \frac{|F|_{d-1}}{d|T|_d} (\boldsymbol{x} - \overline{\boldsymbol{x}}_T)$,

$$\mathrm{s}_{\boldsymbol{\Sigma},T}(\underline{\boldsymbol{\sigma}}_T, \underline{\boldsymbol{\tau}}_T) := \sum_{F \in \mathcal{F}_T} \sum_{F' \in \mathcal{F}_T} \sigma_{TF} B_{FF'}^T \tau_{TF'}, \quad B_{FF'}^T := \frac{|F|_{d-1} |F'|_{d-1}}{d^2 |T|_d^2} \int_T \|\boldsymbol{x} - \overline{\boldsymbol{x}}_T\|_2^2. \tag{4.23}$$

From (4.22) it is clear that $\mathrm{s}_{\boldsymbol{\Sigma},T}$ satisfies both (S1) and (S2).

Example 4.7 (The Discrete Geometric Approach of [29])**.** Assume T star-shaped with respect to $\overline{\boldsymbol{x}}_T$. The Discrete Geometric Approach of [29] is a lowest-order method corresponding to $k = l = m = 0$ based on a stable piecewise constant flux reconstruction obtained by setting, for all $\underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{\Sigma}}_T^{0,0,0}$,

$$\mathbf{S}_T^{\mathrm{dga}}\underline{\boldsymbol{\tau}}_T := \sum_{G\in\mathcal{F}_T} |G|_{d-1}\tau_{TG}\boldsymbol{\varphi}_G^T. \tag{4.24}$$

In (4.24), for all $G \in \mathcal{F}_T$, the restriction of the basis function $\boldsymbol{\varphi}_G^T$ to any pyramid $\mathcal{P}_{TF}$ of apex $\overline{\boldsymbol{x}}_T$ and base $F \in \mathcal{F}_T$ satisfies, denoting by $\mathfrak{h}_{TF}$ the altitude of $\mathcal{P}_{TF}$ (*i.e.*, the distance of $\overline{\boldsymbol{x}}_T$ from the hyperplane containing F),

$$(\boldsymbol{\varphi}_G^T)_{|\mathcal{P}_{TF}} \equiv \frac{(\overline{\boldsymbol{x}}_G - \overline{\boldsymbol{x}}_T)}{|T|_d} + \left(\frac{(\overline{\boldsymbol{x}}_F - \overline{\boldsymbol{x}}_T)\otimes\boldsymbol{n}_{TF}}{|T|_d\mathfrak{h}_{TF}} - \frac{\delta_{FG}}{|G|_{d-1}\mathfrak{h}_{TG}}\boldsymbol{I}_d\right)(\overline{\boldsymbol{x}}_T - \overline{\boldsymbol{x}}_G), \tag{4.25}$$

where $\delta_{FG} = 1$ if $F = G$, 0 otherwise. In the previous definition, $\overline{\boldsymbol{x}}_T$ can be replaced by a point $\boldsymbol{x}_T$ in T with respect to which T is star-shaped (as in Exp. 4.5, this requires to modify the definition (4.3) of the local reduction map $\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,m}$ to preserve polynomial consistency). We stress that the function $\boldsymbol{\varphi}_G^T$ defined by (4.25) is piecewise constant on the pyramidal partition $\{\mathcal{P}_{TF}\}_{F\in\mathcal{F}_T}$ of the element T. The local bilinear form m_T is then defined by setting, for all $\underline{\boldsymbol{\sigma}}_T, \underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{\Sigma}}_T^{0,0,0}$,

$$\mathrm{m}_T(\underline{\boldsymbol{\sigma}}_T, \underline{\boldsymbol{\tau}}_T) := \left(\mathbf{S}_T^{\mathrm{dga}}\underline{\boldsymbol{\sigma}}_T, \mathbf{S}_T^{\mathrm{dga}}\underline{\boldsymbol{\tau}}_T\right)_T. \tag{4.26}$$

Plugging (4.25) into (4.24), and using the second formula in (4.19), we can identify in the expression of $\mathbf{S}_T^{\mathrm{dga}}$ two $L^2(T)^d$-orthogonal contributions observing that, for all $\underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{\Sigma}}_T^{0,0,0}$ and all $F \in \mathcal{F}_T$, it holds

$$(\mathbf{S}_T^{\mathrm{dga}}\underline{\boldsymbol{\tau}}_T)_{|\mathcal{P}_{TF}} \equiv (\mathbf{S}_T^0\underline{\boldsymbol{\tau}}_T)_{|\mathcal{P}_{TF}} + \mathfrak{h}_{TF}^{-1}(\mathbf{S}_T^0\underline{\boldsymbol{\tau}}_T\cdot\boldsymbol{n}_{TF} - \tau_{TF})(\overline{\boldsymbol{x}}_T - \overline{\boldsymbol{x}}_F),$$

where the first term in the right-hand side represents the consistent part of the flux, while the second acts as a stabilization. Hence, a straightforward computation shows that the bilinear form m_T defined by (4.26) can be recast in the form (4.13a) with stabilization bilinear form

$$\mathrm{s}_{\boldsymbol{\Sigma},T}(\underline{\boldsymbol{\sigma}}_T, \underline{\boldsymbol{\tau}}_T) = \sum_{F\in\mathcal{F}_T} \frac{\|\overline{\boldsymbol{x}}_T - \overline{\boldsymbol{x}}_F\|_2^2}{d\mathfrak{h}_{TF}}(\mathbf{S}_T^0\underline{\boldsymbol{\sigma}}_T\cdot\boldsymbol{n}_{TF} - \sigma_{TF}, \mathbf{S}_T^0\underline{\boldsymbol{\tau}}_T\cdot\boldsymbol{n}_{TF} - \tau_{TF})_F. \tag{4.27}$$

Note that this expression can be recovered from (4.20) taking $\boldsymbol{B}^T = \mathrm{diag}(\frac{\|\overline{\boldsymbol{x}}_T - \overline{\boldsymbol{x}}_F\|_2^2|F|_{d-1}}{d\mathfrak{h}_{TF}})_{F\in\mathcal{F}_T}$.

Example 4.8 (The Mixed High-Order method of [38])**.** The Mixed High-Order method of [38] corresponds to the choice $l = k$ and $m = 0$, for which $\mathbf{S}_T^k = \mathbf{P}_T^k$ holds. The stabilization term is defined by penalizing face-based residuals in a least-square fashion:

$$\mathrm{s}_{\boldsymbol{\Sigma},T}(\underline{\boldsymbol{\sigma}}_T, \underline{\boldsymbol{\tau}}_T) = \sum_{F\in\mathcal{F}_T} h_F(\mathbf{S}_T^k\underline{\boldsymbol{\sigma}}_T\cdot\boldsymbol{n}_{TF} - \sigma_{TF}, \mathbf{S}_T^k\underline{\boldsymbol{\tau}}_T\cdot\boldsymbol{n}_{TF} - \tau_{TF})_F. \tag{4.28}$$

When $k = 0$, this stabilization bilinear form coincides with (4.20) with $\boldsymbol{B}^T = \mathrm{diag}(h_F|F|_{d-1})_{F\in\mathcal{F}_T}$.

Example 4.9 (The Virtual Element method of [20])**.** Let $d = 2$. We consider the Mixed Virtual Element method of [20] when the diffusion tensor (denoted by $\mathbb{K}$ in the reference) is the 2×2 identity matrix $\boldsymbol{I}_2$. In this case, while the DOFs for the flux in ([20], Eq. (3.8)) do not coincide with the ones in (4.2), the resulting method ([20], Eq. (6.1)) can be recast in the form (4.18). For a given integer $k \geqslant 1$, the underlying finite-dimensional local virtual space is

$$\begin{aligned}\mathfrak{S}^{\mathrm{vem},1}(T) := \{\mathbf{t}_T \in \boldsymbol{H}(\mathrm{div};T) \cap \boldsymbol{H}(\mathrm{rot};T) \mid \\ \mathrm{div}\,\mathbf{t}_T \in \mathbb{P}^{k-1}(T),\ \mathrm{rot}\,\mathbf{t}_T \in \mathbb{P}^{k-1}(T),\ \text{and}\ \mathbf{t}_{T|F}\cdot\boldsymbol{n}_{TF} \in \mathbb{P}^k(F) \quad \text{for all} \quad F \in \mathcal{F}_T\},\end{aligned}$$

where $\operatorname{rot} \mathbf{t}_T := \partial_1 \mathbf{t}_{T,2} - \partial_2 \mathbf{t}_{T,1}$. Let $\mathbf{t}_T \in \mathfrak{S}^{\mathrm{vem},1}(T)$, and observe that $\operatorname{rot} \mathbf{t}_T$ does not contribute to defining $\operatorname{div} \mathbf{t}_T$ (see [20], Eq. (3.15)) nor the projection on $\mathbb{G}_T^k$ given by ([20], Eq. (5.5)). As a result, due to the presence of the stabilization term in ([20], Eq. (5.6)), the first line in ([20], Eq. (6.1)) actually enforces a zero-rot condition on the discrete solution. Hence, we can equivalently reformulate the method in terms of the zero-rot subspace

$$\mathfrak{S}^{\mathrm{vem},1}(\mathrm{rot}_0; T) := \left\{ \mathbf{t}_T \in \mathfrak{S}^{\mathrm{vem},1}(T) \mid \operatorname{rot} \mathbf{t}_T = 0 \right\}.$$

This was essentially already observed at the end of ([20], Rem. 6.3). This equivalent reformulation corresponds to the mixed form (4.18) with polynomial degrees $l = k-1$, and $m = 0$, and stabilization bilinear form $\mathrm{s}_{\boldsymbol{\Sigma},T}$ defined as described hereafter. We preliminarily observe that the reduction map $\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,k-1,0}$ (*cf.* (4.3)) defines an isomorphism from $\mathfrak{S}^{\mathrm{vem},1}(\mathrm{rot}_0; T)$ to $\underline{\boldsymbol{\Sigma}}_T^{k,k-1,0}$. Assume that the scaled monomial basis ([20], Eqs. (3.6)–(3.7)) has been fixed for $\underline{\boldsymbol{\Sigma}}_T^{k,k-1,0}$, and denote by $\mathrm{S}_{\boldsymbol{\Sigma},T}^{\mathrm{vem},1}$ the bilinear form on $\mathfrak{S}^{\mathrm{vem},1}(\mathrm{rot}_0; T) \times \mathfrak{S}^{\mathrm{vem},1}(\mathrm{rot}_0; T)$ represented by the identity matrix in this basis. The stabilization bilinear form is then given by

$$\mathrm{s}_{\boldsymbol{\Sigma},T}(\underline{\boldsymbol{\sigma}}_T, \underline{\boldsymbol{\tau}}_T) := \mathrm{S}_{\boldsymbol{\Sigma},T}^{\mathrm{vem},1} \left(\mathbf{P}_T^k \underline{\boldsymbol{\sigma}}_T - \mathfrak{s}_T, \mathbf{P}_T^k \underline{\boldsymbol{\tau}}_T - \mathbf{t}_T \right)_T, \tag{4.29}$$

where $\mathfrak{s}_T$ and $\mathbf{t}_T$ are the unique functions of $\mathfrak{S}^{\mathrm{vem},1}(\mathrm{rot}_0; T)$ such that $\underline{\boldsymbol{\sigma}}_T = \underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,k-1,0} \mathfrak{s}_T$ and $\underline{\boldsymbol{\tau}}_T = \underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,k-1,0} \mathbf{t}_T$. This stabilization amounts to penalising in a least-square sense the high-order differences $\boldsymbol{\pi}_{\mathbb{G},T}^{k-2}(\mathbf{P}_T^k \underline{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_{\mathbb{G},T})$ and $(\mathbf{P}_T^k \underline{\boldsymbol{\tau}}_T \cdot \boldsymbol{n}_{TF} - \tau_{TF})$, $F \in \mathcal{F}_T$.

Example 4.10 (The Virtual Element method of [14])**.**
A different Virtual Element method in dimension $d = 2$ was presented in [14] in the context of more general elliptic problems featuring variable diffusion as well as advective and reactive terms. In the pure diffusion case considered here (which, in the original notation from the reference, corresponds to $\kappa = \boldsymbol{I}_2$, $\boldsymbol{b} = \boldsymbol{0}$, and $\gamma = 0$), the method is obtained by choosing $l = m = k$ with $k \geqslant 0$. The underlying virtual space is, this time,

$$\begin{aligned}\mathfrak{S}^{\mathrm{vem},2}(T) := \{ \mathbf{t}_T \in \boldsymbol{H}(\mathrm{div}; T) \cap \boldsymbol{H}(\mathrm{rot}; T) \mid \\ \operatorname{div} \mathbf{t}_T \in \mathbb{P}^k(T),\ \operatorname{rot} \mathbf{t}_T \in \mathbb{P}^{k-1}(T), \text{ and } (\mathbf{t}_T \cdot \boldsymbol{n}_{TF})_{|F} \in \mathbb{P}^k(F) \text{ for all } F \in \mathcal{F}_T \}.\end{aligned}$$

The local flux reduction map $\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,k,k}$ defines an isomorphism from $\mathfrak{S}^{\mathrm{vem},2}$ to $\underline{\boldsymbol{\Sigma}}_T^{k,k,k}$, which contains the DOFs defined by ([14], Eqs. (16)–(18)). The stabilization bilinear form is defined in a similar manner as in the previous example: Given a bilinear form $\mathrm{S}_{\boldsymbol{\Sigma},T}^{\mathrm{vem},2}$ on $\mathfrak{S}^{\mathrm{vem},2}(T) \times \mathfrak{S}^{\mathrm{vem},2}(T)$ with the same scaling as the $L^2(T)^d$-inner product of fluxes, we set

$$\mathrm{s}_{\boldsymbol{\Sigma},T}(\underline{\boldsymbol{\sigma}}_T, \underline{\boldsymbol{\tau}}_T) := \mathrm{S}_{\boldsymbol{\Sigma},T}^{\mathrm{vem},2} \left(\mathbf{S}_T^k \underline{\boldsymbol{\sigma}}_T - \mathfrak{s}_T, \mathbf{S}_T^k \underline{\boldsymbol{\tau}}_T - \mathbf{t}_T \right)_T, \tag{4.30}$$

where $\mathfrak{s}_T$ and $\mathbf{t}_T$ are the unique functions of $\mathfrak{S}^{\mathrm{vem},2}(T)$ such that $\underline{\boldsymbol{\sigma}}_T = \underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,k,k} \mathfrak{s}_T$ and $\underline{\boldsymbol{\tau}}_T = \underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,k,k} \mathbf{t}_T$. This stabilization essentially corresponds to penalising in a least-square sense the high-order differences $\boldsymbol{\pi}_{\mathbb{G},T}^{k-1}(\mathbf{P}_T^k \underline{\boldsymbol{\tau}}_T - \boldsymbol{\tau}_{\mathbb{G},T})$ and $(\mathbf{S}_T^k \underline{\boldsymbol{\tau}}_T \cdot \boldsymbol{n}_{TF} - \tau_{TF})$, $F \in \mathcal{F}_T$. For further developments on $\boldsymbol{H}(\mathrm{div}; \Omega)$- and $\boldsymbol{H}(\mathrm{curl}; \Omega)$-conforming Virtual Elements we refer to [13].

5. A family of primal discontinuous skeletal methods

We introduce in this section a family of primal discontinuous skeletal methods and provide a few examples of members of this family.

5.1. Local space

Let a mesh element $T \in \mathcal{T}_h$ and three polynomial degrees k, l, and m as in (4.1) be fixed. We define the following local space for the potential:

$$\underline{U}_T^{k,l} := U_T^l \times \left(\bigtimes_{F \in \mathcal{F}_T} \mathbb{P}^k(F) \right),$$

where, recalling (4.5), $U_T^l = \mathbb{P}^l(T)$. The local potential reduction map $\underline{I}_{U,T}^{k,l} : H^1(T) \to \underline{U}_T^{k,l}$ is such that, for all $v \in H^1(T)$,

$$\underline{I}_{U,T}^{k,l} v := \left(\pi_T^l v, (\pi_F^k v)_{F \in \mathcal{F}_T} \right). \tag{5.1}$$

We define on $\underline{U}_T^{k,l}$ the $H^1(T)$-like seminorm $\|\cdot\|_{U,T}$ such that, for all $\underline{v}_T \in \underline{U}_T^{k,l}$,

$$\|\underline{v}_T\|_{U,T}^2 := \|\nabla v_T\|_T^2 + \sum_{F \in \mathcal{F}_T} h_F^{-1} \|v_F - v_T\|_F^2, \tag{5.2}$$

and observe that, by virtue of a local Poincaré inequality, the map $\|\cdot\|_{U,T}$ defines a norm on the quotient space

$$\underline{U}_{T,*}^{k,l} := \underline{U}_T^{k,l} / \underline{I}_{U,T}^{k,l} \mathbb{P}^0(T), \tag{5.3}$$

where two elements of $\underline{U}_T^{k,l}$ belong to the same equivalence class if their difference is the interpolate of a constant function over T. Clearly, $\dim(\underline{U}_{T,*}^{k,l}) = \dim(\underline{U}_T^{k,l}) - 1$.

5.2. Local gradient reconstruction

Let $T \in \mathcal{T}_h$. The family of primal methods hinges on the local gradient reconstruction operator $\mathbf{G}_T^k : \underline{U}_T^{k,l} \to \mathbb{S}_T^{k,m}$ (*cf.* (4.11)) defined such that, for all $\underline{v}_T \in \underline{U}_T^{k,l}$,

$$\left(\mathbf{G}_T^k \underline{v}_T, \boldsymbol{\tau} \right)_T = -(v_T, \operatorname{div} \boldsymbol{\tau})_T + \sum_{F \in \mathcal{F}_T} (v_F, \boldsymbol{\tau} \cdot \boldsymbol{n}_{TF})_F \qquad \forall \boldsymbol{\tau} \in \mathbb{S}_T^{k,m}, \tag{5.4}$$

where the right-hand side is devised so as to resemble an integration by parts formula where the role of the function represented by $\underline{v}_T$ inside volumetric and boundary terms is played by element- and face-based DOFs, respectively.

Remark 5.1 (Polynomial degree m)**.** The polynomial degree m does not appear in the definition (5.1) of the local space of potential DOFs. Its role is to determine the codomain of the discrete gradient operator $\mathbf{G}_T^k$ which, recalling (4.11), is either $\mathbb{G}_T^k$ (if $m = 0$) or $\mathbb{P}^k(T)^d$ (if $m = k$).

Adapting the arguments of ([39], Lem. 3) (*cf.*, in particular, Eq. (17) therein), it can be checked that the following commuting property holds: For all $v \in H^1(T)$,

$$\mathbf{G}_T^k \underline{I}_{U,T}^{k,l} v = \boldsymbol{\pi}_{\mathbb{S},T}^{k,m} \nabla v, \tag{5.5}$$

where $\boldsymbol{\pi}_{\mathbb{S},T}^{k,m}$ denotes the L^2-orthogonal projector on $\mathbb{S}_T^{k,m}$ and the potential reduction map $\underline{I}_{U,T}^{k,l}$ is defined by (5.1).

5.3. Local bilinear form

We define, for all $T \in \mathcal{T}_h$, the local bilinear form $\mathrm{a}_T : \underline{U}_T^{k,l} \times \underline{U}_T^{k,l} \to \mathbb{R}$ as follows:

$$\mathrm{a}_T(\underline{u}_T, \underline{v}_T) := \left(\mathbf{G}_T^k \underline{u}_T, \mathbf{G}_T^k \underline{v}_T\right)_T + \mathrm{s}_{U,T}(\underline{u}_T, \underline{v}_T), \tag{5.6}$$

where, as for the bilinear form m_T defined by (4.13a), the right-hand side is composed of a consistency and a stabilization term.

Assumption 5.2 (Bilinear form $\mathrm{s}_{U,T}$)**.** The symmetric, positive semi-definite bilinear form $\mathrm{s}_{U,T} : \underline{U}_T^{k,l} \times \underline{U}_T^{k,l} \to \mathbb{R}$ satisfies the following properties:

(S1′) *Stability.* It holds, for all $\underline{v}_T \in \underline{U}_T^{k,l}$, with seminorm $\|\cdot\|_{U,T}$ defined by (5.2),

$$\|\underline{v}_T\|_{\mathrm{a},T}^2 := \mathrm{a}_T(\underline{v}_T, \underline{v}_T) \approx \|\underline{v}_T\|_{U,T}^2.$$

(S2′) *Polynomial consistency.* For all $w \in \mathbb{P}^{k+1}(T)$, with local potential reduction map $\underline{I}_{U,T}^{k,l}$ defined by (5.1),

$$\mathrm{s}_{U,T}(\underline{I}_{U,T}^{k,l} w, \underline{v}_T) = 0 \qquad \forall \underline{v}_T \in \underline{U}_T^{k,l}.$$

5.4. Global space and primal problem

We define the following global spaces of potential DOFs with single-valued interface unknowns:

$$\underline{U}_h^{k,l} := U_h^l \times \left(\bigtimes_{F \in \mathcal{F}_h} \mathbb{P}^k(F)\right), \qquad \underline{U}_{h,0}^{k,l} := \left\{\underline{v}_h \in \underline{U}_h^{k,l} \mid v_F = 0 \quad \forall F \in \mathcal{F}^{\mathrm{b}}\right\}, \tag{5.7}$$

where the subspace $\underline{U}_{h,0}^{k,l}$ embeds the homogeneous Dirichlet boundary condition. For a generic DOF vector $\underline{v}_h \in \underline{U}_h^{k,l}$ we use the notation $\underline{v}_h = ((v_T)_{T \in \mathcal{T}_h}, (v_F)_{F \in \mathcal{F}_h})$, and we denote by $\underline{v}_T \in \underline{U}_T^{k,l}$ its restriction to T. We also denote by $v_h \in U_h^l$ the piecewise polynomial function obtained from element-based DOFs such that $v_{h|T} = v_T$ for all $T \in \mathcal{T}_h$. On $\underline{U}_h^{k,l}$, we define the global $H^1(\Omega)$-like seminorm $\|\cdot\|_{U,h}$ such that, for all $\underline{v}_h \in \underline{U}_h^{k,l}$,

$$\|\underline{v}_h\|_{U,h}^2 := \sum_{T \in \mathcal{T}_h} \|\underline{v}_T\|_{U,T}^2, \tag{5.8}$$

with $\|\cdot\|_{U,T}$ given by (5.2). Following a reasoning analogous to that of ([37], Prop. 5), it can be easily checked that the map $\|\cdot\|_{U,h}$ defines a norm on $\underline{U}_{h,0}^{k,l}$. We will also need the global potential reduction map $\underline{I}_{U,h}^{k,l} : H^1(\Omega) \to \underline{U}_h^{k,l}$ such that, for all $v \in H^1(\Omega)$,

$$\underline{I}_{U,h}^{k,l} v = \left((\pi_T^l v)_{T \in \mathcal{T}_h}, (\pi_F^k v)_{F \in \mathcal{F}_h}\right).$$

Clearly, the restriction of $\underline{I}_{U,h}^{k,l}$ to a mesh element $T \in \mathcal{T}_h$ coincides with the local potential reduction map defined by (5.1). Also, $\underline{I}_{U,h}^{k,l}$ maps elements of $H_0^1(\Omega)$ to elements of $\underline{U}_{h,0}^{k,l}$. Finally, we define the global bilinear form $\mathrm{a}_h : \underline{U}_h^{k,l} \times \underline{U}_h^{k,l} \to \mathbb{R}$ by element-by-element assembly by setting

$$\mathrm{a}_h(\underline{u}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} \mathrm{a}_T(\underline{u}_T, \underline{v}_T).$$

We consider the following

Problem 5.3 (Primal problem)**.** Find $\underline{u}_h \in \underline{U}_{h,0}^{k,l}$ such that

$$\mathrm{a}_h(\underline{u}_h, \underline{v}_h) = (f, v_h) \qquad \forall \underline{v}_h \in \underline{U}_{h,0}^{k,l}. \tag{5.9}$$

Remark 5.4 (Static condensation)**.** In the actual implementation of the method (5.9), element-based DOFs can be locally eliminated by static condensation. The procedure is essentially analogous to the one described, *e.g.*, in ([25], Sect. 2.4), to which we refer for further details.

TABLE 2. Examples of methods originally introduced in primal formulation. * The High-Order Mimetic method enters the present framework only for $k \geqslant 1$.

Reference	Name	k	l	m	$\mathrm{s}_{U,T}$
[47]	Hybrid Finite Volume	0	0	0	Equation (5.10)
[43]	Hybrid Finite Volume	0	0	0	Equation (5.12)
[49]	Hybridizable Discontinuous Galerkin	$\geqslant 0$	$k+1$	k	Equation (5.17)
[25]	Hybridizable Discontinuous Galerkin	$\geqslant 0$	Equation (4.1)	k	Equation (5.14)
[39]	Hybrid High-Order	$\geqslant 0$	k	0	Equation (5.14)
[25]	Hybrid High-Order	$\geqslant 0$	Equation (4.1)	0	Equation (5.14)
[8, 50]	High-Order Mimetic	$\geqslant 0^*$	$k-1$	0	Equation (5.18)

5.5. Examples

We collect in this section a few examples of discontinuous skeletal methods originally introduced in a primal formulation which can be traced back to (5.9). Each method is uniquely defined by prescribing the three polynomial degrees k, l, and m (in accordance with (4.1)) and the expression of the local stabilization bilinear form $s_{U,T}$ for a generic mesh element $T \in \mathcal{T}_h$. A synopsis is provided in Table 2.

Example 5.5 (The Hybrid Finite Volume method of [47] and its generalization of [43])**.** The Hybrid Finite Volume method of ([47], Sect. 2.1) corresponds to $k = l = m = 0$. In this case, an explicit expression for the gradient operator $\mathbf{G}_T^0$ defined by (5.4) is available: For all $\underline{v}_T \in \underline{U}_T^{0,0}$,

$$\mathbf{G}_T^0 \underline{v}_T = \frac{1}{|T|_d} \sum_{F \in \mathcal{F}_T} |F|_{d-1} v_F \boldsymbol{n}_{TF}.$$

For every element $T \in \mathcal{T}_h$, the stabilization bilinear form is such that

$$\mathrm{s}_{U,T}(\underline{u}_T, \underline{v}_T) = \sum_{F \in \mathcal{F}_T} |F|_{d-1} \frac{\eta}{\mathfrak{h}_{TF}} \delta_{TF}^0 \underline{u}_T \delta_{TF}^0 \underline{v}_T, \tag{5.10}$$

where $\eta > 0$ is a user-dependent stabilization parameter, $\mathfrak{h}_{TF}$ is as in Example 4.7, and the face-based residual operator $\delta_{TF}^0 : \underline{U}_T^{0,0} \to \mathbb{P}^0(F)$ is such that

$$\delta_{TF}^0 \underline{v}_T := v_T + \mathbf{G}_T^0 \underline{v}_T \cdot (\overline{\boldsymbol{x}}_F - \overline{\boldsymbol{x}}_T) - v_F. \tag{5.11}$$

In the previous definition, $\overline{\boldsymbol{x}}_T$ can be replaced by a point $\boldsymbol{x}_T$ which may or may not belong to T (more general choices have indeed been considered in [42, 43]) In this case, the definition (5.1) of the local reduction map has to be modified in order to ensure that condition (S2′) is verified. In ([43], Sect. 2.2), the following generalization of (5.10) is proposed: For a given positive definite matrix $\boldsymbol{B}^T = (B_{FF'}^T)_{F,F' \in \mathcal{F}_T}$,

$$\mathrm{s}_{U,T}(\underline{u}_T, \underline{v}_T) = \sum_{F \in \mathcal{F}_T} \sum_{F' \in \mathcal{F}_T} \delta_{TF}^0 \underline{u}_T B_{FF'}^T \delta_{TF'}^0 \underline{v}_T. \tag{5.12}$$

In order to have the uniform norm equivalence (S1′), the matrix $\boldsymbol{B}^T$ should have an appropriate scaling as detailed in ([43], Eqs. (4.2)–(4.4)). Straightforward choices are, *e.g.*, $\boldsymbol{B}^T = h_T^{d-2} \boldsymbol{I}_d$ or $\boldsymbol{B}^T = \mathrm{diag}\left(h_F^{-1} |F|_{d-1}\right)_{F \in \mathcal{F}_T}$ (see also the following example concerning the latter choice).

Example 5.6 (The Hybrid High-Order method of [39] and the variants of [25])**.** The original Hybrid High-Order method of [39] corresponds to the choice $l = k$ and $m = 0$. In [25], variants corresponding to $l = k - 1$ (when $k \geqslant 1$) and $l = k + 1$ have also been proposed. Let an element $T \in \mathcal{T}_h$ be fixed, and define the potential reconstruction operator $\mathrm{p}_T^{k+1} : \underline{U}_T^{k,l} \to \mathbb{P}^{k+1}(T)$ such that, for all $\underline{v}_T \in \underline{U}_T^{k,l}$,

$$\nabla \mathrm{p}_T^{k+1} \underline{v}_T = \mathbf{G}_T^k \underline{v}_T \quad \text{and} \quad (\mathrm{p}_T^{k+1} \underline{v}_T - v_T, 1)_T = 0. \tag{5.13}$$

Notice that the first condition makes sense since, having supposed $m = 0$, $\mathbf{G}_T^k \underline{v}_T \in \mathbb{G}_T^k$. The stabilization bilinear form is defined as follows:

$$\mathrm{s}_{U,T}(\underline{u}_T, \underline{v}_T) = \sum_{F \in \mathcal{F}_T} h_F^{-1} \left(\delta_{TF}^k \underline{u}_T, \delta_{TF}^k \underline{v}_T \right)_F, \tag{5.14}$$

where, for all $F \in \mathcal{F}_T$, the face-based residual operator $\delta_{TF}^k : \underline{U}_T^{k,l} \to \mathbb{P}^k(F)$ is such that, for all $\underline{v}_T \in \underline{U}_T^{k,l}$,

$$\delta_{TF}^k \underline{v}_T = \pi_F^k \left(\mathrm{p}_T^{k+1} \underline{v}_T - v_F - \pi_T^l \left(\mathrm{p}_T^{k+1} \underline{v}_T - v_T \right) \right). \tag{5.15}$$

As already observed in ([39], Sect. 2.5), in the lowest-order case $k = 0$ the face-based residuals defined by (5.11) and (5.15) coincide, and the stabilization (5.14) can be recovered from (5.12) selecting $\boldsymbol{B}^T = \mathrm{diag}(h_F^{-1}|F|_{d-1})_{F \in \mathcal{F}_T}$ (the only difference with respect to (5.10) is the change of local scaling $\mathfrak{h}_{TF} \leftarrow h_F$).

Example 5.7 (The Crouzeix–Raviart finite element)**.** We study the solution of problem (5.9) using the Hybrid High-Order method of Example 5.6 with $k = l = m = 0$ but with right-hand side modified as follows:

$$(f, v_h) \leftarrow (f, \mathrm{p}_h^1 \underline{v}_h), \tag{5.16}$$

where $(\mathrm{p}_h^1 \underline{v}_h)_{|T} = \mathrm{p}_T^1 \underline{v}_T$ for all $T \in \mathcal{T}_h$, and the local potential reconstruction $\mathrm{p}_T^1 : \underline{U}_T^{0,0} \to \mathbb{P}^1(T)$ is defined according to (5.13) but with average value on T set to $\frac{1}{d+1} \sum_{F \in \mathcal{F}_T} v_F$. With this choice, for all $\underline{v}_h \in \underline{U}_h^{0,0}$ it holds that $\pi_F^0 \mathrm{p}_T^1 \underline{v}_T = \mathrm{p}_T^1 \underline{v}_T(\overline{\boldsymbol{x}}_F) = v_F$ for all $T \in \mathcal{T}_h$ and all $F \in \mathcal{F}_T$. As a result, $\mathrm{p}_h^1 \underline{v}_h \in \mathbb{NC}_0(\mathcal{T}_h)$, the Crouzeix–Raviart space defined in Section 3. Moreover, it can easily be checked that the DOFs collected in $\underline{U}_{h,0}^{0,0}$ coincide with the standard ones for $\mathbb{NC}_0(\mathcal{T}_h)$, so that $\mathbb{NC}_0(\mathcal{T}_h) = \mathrm{span}\{\mathrm{p}_h^1 \underline{v}_h \mid \underline{v}_h \in \underline{U}_{h,0}^{0,0}\}$. Using these facts, for all $T \in \mathcal{T}_h$ and all $F \in \mathcal{F}_T$ the face-based residual operator (5.15) with $k = l = 0$ becomes

$$\delta_{TF}^0 \underline{v}_T = -\pi_T^0 (\mathrm{p}_T^1 \underline{v}_T - v_T) = v_T - \mathrm{p}_T^1 \underline{v}_T (\overline{\boldsymbol{x}}_T).$$

Denote now by $\underline{u}_h \in \underline{U}_{h,0}^{0,0}$ the solution of problem (5.9) with right-hand side modified as in (5.16). Observing that element-based DOFs do not contribute to the consistency term in (5.6) nor to the right-hand side (5.16), we infer that the stabilization term is actually enforcing the condition $u_T = \mathrm{p}_T^1 \underline{u}_T(\overline{\boldsymbol{x}}_T)$ for all $T \in \mathcal{T}_h$. As a result, $\mathrm{p}_h^1 \underline{u}_h$ coincides with the Crouzeix–Raviart solution (3.4).

Example 5.8 (The Hybridizable Discontinuous Galerkin method of [49] and the variants of [25])**.** The Hybridizable Discontinuous Galerkin originally proposed in ([49], Rem. 1.2.4) corresponds to the case $l = k + 1$, $m = k$, and stabilization

$$\mathrm{s}_{U,T}(\underline{u}_T, \underline{v}_T) = \sum_{F \in \mathcal{F}_T} h_F^{-1} \left(\pi_F^k (u_T - u_F), \pi_F^k (v_T - v_F) \right)_F. \tag{5.17}$$

As pointed out in ([25], Rem. 2), this stabilization coincides with (5.14) when $l = k + 1$. Motivated by this observation, variants corresponding to the choices $l = k - 1$ (when $k \geqslant 1$) and $l = k$ and $m = k$ are proposed therein. It is worth noticing here that the original Hybridizable Discontinuous Galerkin method of [23, 28] does not fit in the present framework since the corresponding stabilization bilinear form is only polynomially consistent up to degree k, *i.e.*, it does not satisfy (S2′). Correspondingly, the orders of convergence are reduced (*cf.* [25], Tab. 1 for further details).

Example 5.9 (The High-Order Mimetic method of [8, 50])**.** The High-Order Mimetic method of [50] (subsequently referred to as Nonconforming Virtual Element method in [8]) provides a high-order generalization of the concepts underlying Mimetic Difference Methods (*cf.*, *e.g.*, [15]). Its lowest-order version, corresponding to the case $k = 0$ and $l = -1$, violates (4.1), and therefore does not enter our unified framework. For $k \geqslant 1$, on the other hand, it corresponds to the choices $l = k - 1$ and $m = 0$. To write the corresponding bilinear form, define the finite-dimensional local virtual space

$$\mathfrak{U}^k(T) := \left\{ \mathfrak{v}_T \in H^1(T) \mid \triangle \mathfrak{v}_T \in \mathbb{P}^{k-1}(T) \text{ and } (\nabla \mathfrak{v}_T)_{|F} \cdot \boldsymbol{n}_{TF} \in \mathbb{P}^k(F) \text{ for all } F \in \mathcal{F}_T \right\}.$$

Clearly, $\mathbb{P}^{k+1}(T) \subset \mathfrak{U}^k(T)$, and it can be proved that $\underline{I}_{U,T}^{k,k-1}$ defines an isomorphism from $\mathfrak{U}^k(T)$ to $\underline{U}_T^{k,k-1}$ (this is essentially a consequence of ([8], Lem. 3.1) after observing that the DOFs in $\underline{U}_T^{k,k-1}$ are equivalent to those defined by Eqs. (3.5)–(3.6) therein). Denote by $\mathrm{S}_T^{\mathrm{hom}} : \mathfrak{U}^k(T) \times \mathfrak{U}^k(T) \to \mathbb{R}$ a bilinear form whose representation in the canonical basis of $\mathfrak{U}^k(T)$ is spectrally equivalent to the unit matrix. The stabilization bilinear form is obtained by setting, for all $\underline{u}_T, \underline{v}_T \in \underline{U}_T^{k,l}$,

$$\mathrm{s}_{U,T}(\underline{u}_T, \underline{v}_T) := h_T^{d-2} \mathrm{S}_T^{\mathrm{hom}}(\mathrm{p}_T^{k+1}\underline{u}_T - \mathfrak{u}_T, \mathrm{p}_T^{k+1}\underline{v}_T - \mathfrak{v}_T), \tag{5.18}$$

where $\mathfrak{u}_T$ and $\mathfrak{v}_T$ are the unique functions in $\mathfrak{U}^k(T)$ such that $\underline{u}_T = \underline{I}_{U,T}^{k,k-1}\mathfrak{u}_T$ and $\underline{v}_T = \underline{I}_{U,T}^{k,k-1}\mathfrak{v}_T$, while the operator p_T^{k+1} is defined by (5.13). The stabilization (5.18) essentially corresponds to penalizing in a least-square sense the high-order differences $\pi_T^l(\mathrm{p}_T^{k+1}\underline{v}_T - v_T)$ and $\pi_F^k(\mathrm{p}_T^{k+1}\underline{v}_T - v_F)$, $F \in \mathcal{F}_T$, with scaling factor chosen so that the uniform equivalence in (S1′) holds.

6. From mixed to primal methods

In this section we obtain from (4.18) an equivalent primal problem by hybridization. The primal hybrid problem is then shown to belong to the family (5.9) of primal discontinuous skeletal methods.

6.1. Mixed hybrid formulation of mixed methods

We define the bilinear form $\mathrm{b}_h : \underline{\check{\boldsymbol{\Sigma}}}_h^{k,l,m} \times \underline{U}_h^{k,l} \to \mathbb{R}$ (with spaces $\underline{\check{\boldsymbol{\Sigma}}}_h^{k,l,m}$ and $\underline{U}_h^{k,l}$ defined by (4.14) and (5.7), respectively) such that, for all $(\underline{\boldsymbol{\tau}}_h, \underline{v}_h) \in \underline{\check{\boldsymbol{\Sigma}}}_h^{k,l,m} \times \underline{U}_h^{k,l}$,

$$\mathrm{b}_h(\underline{\boldsymbol{\tau}}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} \mathrm{b}_T(\underline{\boldsymbol{\tau}}_T, \underline{v}_T), \qquad \mathrm{b}_T(\underline{\boldsymbol{\tau}}_T, \underline{v}_T) := (\mathrm{D}_T^l \underline{\boldsymbol{\tau}}_T, v_T)_T - \sum_{F \in \mathcal{F}_T} (\tau_{TF}, v_F)_F. \tag{6.1}$$

For further use, we note that it holds for all $T \in \mathcal{T}_h$, all $\underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{\Sigma}}_T^{k,l,m}$, and all $\underline{v}_T \in \underline{U}_T^{k,l}$,

$$\mathrm{b}_T(\underline{\boldsymbol{\tau}}_T, \underline{v}_T) = -(\boldsymbol{\tau}_{\mathbb{G},T}, \nabla v_T)_T + \sum_{F \in \mathcal{F}_T} (\tau_{TF}, v_T - v_F)_F, \tag{6.2}$$

as can be easily checked replacing D_T^l by its definition (4.6) and accounting for Remark 4.1. Hence, using the Cauchy–Schwarz inequality and recalling the definitions (4.4) and (5.2) of $\|\cdot\|_{\boldsymbol{\Sigma},T}$ and $\|\cdot\|_{U,T}$, we infer the following boundedness result for b_T:

$$|\mathrm{b}_T(\underline{\boldsymbol{\tau}}_T, \underline{v}_T)| \leqslant \|\underline{\boldsymbol{\tau}}_T\|_{\boldsymbol{\Sigma},T} \|\underline{v}_T\|_{U,T}. \tag{6.3}$$

Consider the following

Problem 6.1 (Mixed hybrid problem)**.** Find $(\underline{\boldsymbol{\sigma}}_h, \underline{u}_h) \in \underline{\check{\boldsymbol{\Sigma}}}_h^{k,l,m} \times \underline{U}_{h,0}^{k,l}$ such that,

$$\forall T \in \mathcal{T}_h, \qquad \mathrm{m}_T(\underline{\boldsymbol{\sigma}}_T, \underline{\boldsymbol{\tau}}_T) + \mathrm{b}_T(\underline{\boldsymbol{\tau}}_T, \underline{u}_T) = 0 \qquad \forall \underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{\Sigma}}_T^{k,l,m}, \tag{6.4a}$$

$$-\mathrm{b}_h(\underline{\boldsymbol{\sigma}}_h, \underline{v}_h) = (f, v_h) \qquad \forall \underline{v}_h \in \underline{U}_{h,0}^{k,l}, \tag{6.4b}$$

where we remind the reader that the local bilinear form m_T that approximates the $L^2(T)^d$-product of fluxes is defined by (4.13a).

Compared to the mixed problem (4.18), the single-valuedness of interface flux unknowns is enforced here by Lagrange multipliers (corresponding to the skeletal DOFs in $\underline{U}_{h,0}^{k,l}$) instead of being embedded in the discrete space. Equation (6.4a) defines a set of local constitutive relations connecting flux to potential DOFs inside each mesh element. Equation (6.4b), on the other hand, expresses local balances and a global transmission condition. In what follows, we will eliminate flux unknowns by locally inverting (6.4a), ending up with a problem in the hybrid potential unknowns only.

6.2. Mixed-to-primal potential-to-flux operator

For all $T \in \mathcal{T}_h$, we define the local mixed-to-primal potential-to-flux operator $\underline{\varsigma}_T^{k,l,m} : \underline{U}_T^{k,l} \to \underline{\boldsymbol{\Sigma}}_T^{k,l,m}$ such that, for all $\underline{v}_T \in \underline{U}_T^{k,l}$,

$$\mathrm{m}_T(\underline{\varsigma}_T^{k,l,m}\underline{v}_T, \underline{\boldsymbol{\tau}}_T) = -\mathrm{b}_T(\underline{\boldsymbol{\tau}}_T, \underline{v}_T) \qquad \forall \underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{\Sigma}}_T^{k,l,m}. \tag{6.5}$$

Recalling the reformulation (6.2) of b_T, (6.5) equivalently rewrites

$$\mathrm{m}_T(\underline{\varsigma}_T^{k,l,m}\underline{v}_T, \underline{\boldsymbol{\tau}}_T) = (\nabla v_T, \boldsymbol{\tau}_{\mathbb{G},T})_T + \sum_{F\in\mathcal{F}_T} (v_F - v_T, \tau_{TF})_F \qquad \forall \underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{\Sigma}}_T^{k,l,m}. \tag{6.6}$$

We next state some useful properties for the potential-to-flux operator.

Lemma 6.2 (Properties of the mixed-to-primal potential-to-flux operator)**.** *Let a mesh element $T \in \mathcal{T}_h$ be given and let $\mathrm{s}_{\boldsymbol{\Sigma},T}$ be a bilinear form satisfying Assumption 4.2. Then, the corresponding potential-to-flux operator $\underline{\varsigma}_T^{k,l,m}$ given by (6.5) is well defined and has the following properties:*

(1) Stability and continuity. *For all $\underline{v}_T \in \underline{U}_T^{k,l}$, it holds*

$$\|\underline{\varsigma}_T^{k,l,m}\underline{v}_T\|_{\boldsymbol{\Sigma},T} \approx \|\underline{v}_T\|_{U,T}, \tag{6.7}$$

with norms $\|\cdot\|_{\boldsymbol{\Sigma},T}$ and $\|\cdot\|_{U,T}$ defined by (4.4) and (5.2), respectively.

(2) Commuting property. *For all $w \in \mathbb{P}^{k+1}(T)$, we have*

$$\underline{\varsigma}_T^{k,l,m}\underline{I}_{U,T}^{k,l}w = \underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,m}\nabla w. \tag{6.8}$$

(3) Link with the discrete gradient operator. *It holds, with operators $\mathbf{G}_T^k$ and $\mathbf{S}_T^k$ defined by (5.4) and (4.8), respectively, that*

$$\mathbf{G}_T^k := \mathbf{S}_T^k \circ \underline{\varsigma}_T^{k,l,m}. \tag{6.9}$$

Proof. Problem (6.5) is well-posed owing to assumption (S1) expressing the coercivity of m_T. As a result, $\underline{\varsigma}_T^{k,l,m}$ is well defined.

(1) *Stability and continuity.* Using (S1) followed by the definition (6.5) of $\underline{\varsigma}_T^{k,l,m}$ and the boundedness (6.3) of b_T, we infer, for all $\underline{v}_T \in \underline{U}_T^{k,l}$,

$$\|\underline{\varsigma}_T^{k,l,m}\underline{v}_T\|_{\boldsymbol{\Sigma},T}^2 \lesssim \|\underline{\varsigma}_T^{k,l,m}\underline{v}_T\|_{\mathrm{m},T}^2 = -\mathrm{b}_T(\underline{\varsigma}_T^{k,l,m}\underline{v}_T, \underline{v}_T) \leqslant \|\underline{\varsigma}_T^{k,l,m}\underline{v}_T\|_{\boldsymbol{\Sigma},T}\|\underline{v}_T\|_{U,T}.$$

To prove the converse inequality, let $\underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{\Sigma}}_T^{k,l,m}$ in (6.6) be such that $\boldsymbol{\tau}_T = \nabla v_T$ and $\tau_{TF} = h_F^{-1}(v_F - v_T)$ for all $F \in \mathcal{F}_T$, and observe that

$$\|\underline{v}_T\|_{U,T}^2 = \mathrm{m}_T\left(\underline{\varsigma}_T^{k,l,m}\underline{v}_T, \underline{\boldsymbol{\tau}}_T\right) \lesssim \|\underline{\varsigma}_T^{k,l,m}\underline{v}_T\|_{\boldsymbol{\Sigma},T}\|\underline{\boldsymbol{\tau}}_T\|_{\boldsymbol{\Sigma},T} = \|\underline{\varsigma}_T^{k,l,m}\underline{v}_T\|_{\boldsymbol{\Sigma},T}\|\underline{v}_T\|_{U,T},$$

where we have used the Cauchy–Schwarz inequality together with (S1) to bound m_T, and the definitions (4.4) of $\|\cdot\|_{\boldsymbol{\Sigma},T}$ and (5.8) of $\|\cdot\|_{U,T}$ to infer $\|\underline{\boldsymbol{\tau}}_T\|_{\boldsymbol{\Sigma},T} = \|\underline{v}_T\|_{U,T}$ and conclude.

(2) *Commuting property.* Let $w \in \mathbb{P}^{k+1}(T)$. Using the definition (6.5) of $\underline{\varsigma}_T^{k,l,m}$ with $\underline{v}_T = \underline{I}_{U,T}^{k,l}w$ and recalling (6.1), we infer, for all $\underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{\Sigma}}_T^{k,l,m}$,

$$\begin{aligned}\mathrm{m}_T(\underline{\varsigma}_T^{k,l,m}\underline{I}_{U,T}^{k,l}w, \underline{\boldsymbol{\tau}}_T) &= -(\pi_T^l w, \mathrm{D}_T^l\underline{\boldsymbol{\tau}}_T)_T + \sum_{F\in\mathcal{F}_T}(\pi_F^k w, \tau_{TF})_F \\ &= -(w, \mathrm{D}_T^l\underline{\boldsymbol{\tau}}_T)_T + \sum_{F\in\mathcal{F}_T}(w, \tau_{TF})_F = (\nabla w, \mathbf{P}_T^k\underline{\boldsymbol{\tau}}_T)_T,\end{aligned} \tag{6.10}$$

where we have used the definitions (2.1) of π_T^l and π_F^k to pass to the second line and the definition (4.7) of $\mathbf{P}_T^k$ to conclude. On the other hand, using the definition (4.13a) of m_T followed by the polynomial consistency (4.12) of $\mathbf{S}_T^k$ together with (S2), for all $\underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{\Sigma}}_T^{k,l,m}$ we have that

$$\begin{aligned}\mathrm{m}_T(\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,m}\nabla w, \underline{\boldsymbol{\tau}}_T) &= (\mathbf{S}_T^k\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,m}\nabla w, \mathbf{S}_T^k\underline{\boldsymbol{\tau}}_T)_T + \mathrm{s}_{\boldsymbol{\Sigma},T}(\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,m}\nabla w, \underline{\boldsymbol{\tau}}_T) \\ &= (\nabla w, \mathbf{S}_T^k\underline{\boldsymbol{\tau}}_T)_T = (\nabla w, \mathbf{P}_T^k\underline{\boldsymbol{\tau}}_T)_T,\end{aligned} \tag{6.11}$$

where the last equality follows from the definition (4.8) of $\mathbf{S}_T^k$ together with the orthogonal decomposition (2.3). Subtracting (6.11) from (6.10), we infer, for all $\underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{\Sigma}}_T^{k,l,m}$,

$$\mathrm{m}_T(\underline{\boldsymbol{\varsigma}}_T^{k,l,m}\underline{I}_{U,T}^{k,l}w - \underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,m}\nabla w, \underline{\boldsymbol{\tau}}_T) = 0,$$

from which (6.8) follows since m_T is coercive on $\underline{\boldsymbol{\Sigma}}_T^{k,l,m}$ owing to (S1).

(3) *Link with the discrete gradient operator.* Let $\underline{v}_T \in \underline{U}_T^{k,l}$, $\boldsymbol{\tau} \in \mathbb{S}_T^{k,m}$, and set $\underline{\boldsymbol{\tau}}_T := \underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,m}\boldsymbol{\tau}$. Recalling the definition (4.13a) of m_T, and using the polynomial consistency (4.12) of $\mathbf{S}_T^k$ together with (S2), it is readily inferred that

$$\mathrm{m}_T(\underline{\boldsymbol{\varsigma}}_T^{k,l,m}\underline{v}_T, \underline{\boldsymbol{\tau}}_T) = \Big((\mathbf{S}_T^k \circ \underline{\boldsymbol{\varsigma}}_T^{k,l,m})\underline{v}_T, \boldsymbol{\tau}\Big)_T. \tag{6.12}$$

On the other hand, recalling the definitions (4.3) of $\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,m}$ and (6.1) of b_T, we get

$$\begin{aligned}\mathrm{b}_T(\underline{\boldsymbol{\tau}}_T, \underline{v}_T) &= (v_T, \mathrm{D}_T^l\underline{\boldsymbol{\tau}}_T)_T - \sum_{F\in\mathcal{F}_T}(v_F, \tau_{TF})_F \\ &= (v_T, \pi_T^l(\operatorname{div}\boldsymbol{\tau}))_T - \sum_{F\in\mathcal{F}_T}(v_F, \pi_F^k(\boldsymbol{\tau}\cdot\boldsymbol{n}_{TF}))_F \\ &= (v_T, \operatorname{div}\boldsymbol{\tau})_T - \sum_{F\in\mathcal{F}_T}(v_F, \boldsymbol{\tau}\cdot\boldsymbol{n}_{TF})_F = -(\mathbf{G}_T^k\underline{v}_h, \boldsymbol{\tau})_T,\end{aligned} \tag{6.13}$$

where we have used the commuting property (4.9) of D_T^l in the second line and the definition (2.1) of π_T^l and π_F^k and (5.4) of $\mathbf{G}_T^k$ in the third. To conclude, plug (6.12) and (6.13) into the definition (6.5) of $\underline{\boldsymbol{\varsigma}}_T^{k,l,m}$. □

6.3. Equivalent primal formulations of mixed methods

We start by showing a link among problems (4.18), (6.4), and the following

Problem 6.3 (Primal hybrid problem)**.** Find $(\underline{\boldsymbol{\sigma}}_h, \underline{u}_h) \in \check{\underline{\boldsymbol{\Sigma}}}_h^{k,l,m} \times \underline{U}_{h,0}^{k,l}$ such that

$$\underline{\boldsymbol{\sigma}}_T = \underline{\boldsymbol{\varsigma}}_T^{k,l,m}\underline{u}_T \qquad \forall T \in \mathcal{T}_h, \tag{6.14a}$$

with potential-to-flux operator $\underline{\boldsymbol{\varsigma}}_T^{k,l,m}$ defined by (6.5) and $\underline{u}_h$ solution of

$$\mathrm{a}_h(\underline{u}_h, \underline{v}_h) = (f, v_h) \qquad \forall \underline{v}_h \in \underline{U}_{h,0}^{k,l}, \tag{6.14b}$$

where the bilinear form a_h on $\underline{U}_h^{k,l} \times \underline{U}_h^{k,l}$ is such that

$$\mathrm{a}_h(\underline{u}_h, \underline{v}_h) := \sum_{T\in\mathcal{T}_h}\mathrm{a}_T(\underline{u}_T, \underline{v}_T), \qquad \mathrm{a}_T(\underline{u}_T, \underline{v}_T) := \mathrm{m}_T\left(\underline{\boldsymbol{\varsigma}}_T^{k,l,m}\underline{u}_T, \underline{\boldsymbol{\varsigma}}_T^{k,l,m}\underline{v}_T\right). \tag{6.15}$$

The well-posedness of (6.14b) is an immediate consequence of point (1) in Theorem 6.5 below.

Theorem 6.4 (Link among the mixed, mixed hybrid and primal hybrid problems)**.** *For all $T \in \mathcal{T}_h$, let $\mathrm{s}_{\boldsymbol{\Sigma},T}$ satisfy Assumption 4.2. Let $(\underline{\boldsymbol{\sigma}}_h, \underline{u}_h) \in \underline{\check{\boldsymbol{\Sigma}}}_h^{k,l,m} \times \underline{U}_{h,0}^{k,l}$, and let $u_h \in U_h^l$ be such that $u_{h|T} = u_T$ for all $T \in \mathcal{T}_h$. Then, the following statements are equivalent:*

(i) $(\underline{\boldsymbol{\sigma}}_h, \underline{u}_h)$ *solves the mixed hybrid problem (6.4);*
(ii) $\underline{\boldsymbol{\sigma}}_h \in \underline{\boldsymbol{\Sigma}}_h^{k,l,m}$ *and $(\underline{\boldsymbol{\sigma}}_h, u_h)$ solves the mixed problem (4.18);*
(iii) $(\underline{\boldsymbol{\sigma}}_h, \underline{u}_h)$ *solves the primal hybrid problem (6.14).*

Proof. The equivalence (i) $\iff$ (ii) classically follows from the theory of Lagrange multipliers. Let us prove the equivalence (i) $\iff$ (iii). We show that if $(\underline{\boldsymbol{\sigma}}_h, \underline{u}_h)$ solves the mixed hybrid problem (6.4), then it solves the primal hybrid problem (6.14). Equation (6.14a) immediately follows from (6.4a) recalling the definition (6.5) of the potential-to-flux operator. As a consequence, it holds for all $T \in \mathcal{T}_h$ and all $\underline{v}_T \in \underline{U}_T^{k,l}$,

$$-\mathrm{b}_T\left(\underline{\boldsymbol{\sigma}}_T, \underline{v}_T\right) = -\mathrm{b}_T\left(\underline{\boldsymbol{\varsigma}}_T^{k,l,m}\underline{u}_T, \underline{v}_T\right) = \mathrm{m}_T\left(\underline{\boldsymbol{\varsigma}}_T^{k,l,m}\underline{u}_T, \underline{\boldsymbol{\varsigma}}_T^{k,l,m}\underline{v}_T\right) = \mathrm{a}_T\left(\underline{u}_T, \underline{v}_T\right),$$

where we have used the definition (6.5) of the potential-to-flux operator together with the symmetry of m_T in the second equality and the definition (6.15) of the bilinear form a_T to conclude. This implies that (6.4b) is equivalent to (6.14b). By similar arguments, we can prove that if $(\underline{\boldsymbol{\sigma}}_h, \underline{u}_h)$ solves the primal hybrid problem (6.14), then it solves the mixed hybrid problem (6.4), thus concluding the proof. □

We close this section with our main result, *i.e.*, the existence of a primal method belonging to the family (5.9) whose solution coincides with that of the mixed method (4.18) for given stabilization bilinear forms satisfying Assumption 4.2. In the light of Theorem 6.4, it suffices to state the equivalence with respect to the corresponding mixed hybrid formulation (6.4).

Theorem 6.5 (Link with the family of primal discontinuous skeletal methods)**.** *For all $T \in \mathcal{T}_h$, let $\mathrm{s}_{\boldsymbol{\Sigma},T}$ satisfy Assumption 4.2 and set with $\underline{\boldsymbol{\varsigma}}_T^{k,l,m}$ defined by (6.5):*

$$\mathrm{s}_{U,T}\left(\underline{u}_T, \underline{v}_T\right) := \mathrm{s}_{\boldsymbol{\Sigma},T}\left(\underline{\boldsymbol{\varsigma}}_T^{k,l,m}\underline{u}_T, \underline{\boldsymbol{\varsigma}}_T^{k,l,m}\underline{v}_T\right). \tag{6.16}$$

Then,

(1) Properties of $\mathrm{s}_{U,T}$. *The stabilization bilinear forms $\mathrm{s}_{U,T}$, $T \in \mathcal{T}_h$, satisfy Assumption 5.2*;
(2) Link with primal methods. *$\underline{u}_h \in \underline{U}_{h,0}^{k,l}$ solves the primal problem (5.9) with stabilization as in (6.16) if and only if $(\underline{\boldsymbol{\sigma}}_h, \underline{u}_h) \in \underline{\check{\boldsymbol{\Sigma}}}_h^{k,l,m} \times \underline{U}_{h,0}^{k,l}$ with $\underline{\boldsymbol{\sigma}}_h$ such that $\underline{\boldsymbol{\sigma}}_T = \underline{\boldsymbol{\varsigma}}_T^{k,l,m}\underline{u}_T$ for all $T \in \mathcal{T}_h$ solves the mixed hybrid problem (6.4).*

Proof.

(1) *Properties of* $\mathrm{s}_{U,T}$. Let $T \in \mathcal{T}_h$. The bilinear form $\mathrm{s}_{U,T}$ is clearly symmetric and positive semi-definite. It then suffices to prove conditions (S1′) and (S2′). To prove (S1′) observe that, for all $\underline{v}_T \in \underline{U}_T^{k,l}$, we have

$$\|\underline{v}_T\|_{\mathrm{a},T} = \|\underline{\boldsymbol{\varsigma}}_T^{k,l,m}\underline{v}_T\|_{\mathrm{m},T} \approx \|\underline{\boldsymbol{\varsigma}}_T^{k,l,m}\underline{v}_T\|_{\boldsymbol{\Sigma},T} \approx \|\underline{v}_T\|_{U,T},$$

where we have used the definition (6.15) of a_T, (S1), and the stability and continuity (6.7) of $\underline{\boldsymbol{\varsigma}}_T^{k,l,m}$. Let us prove (S2′). Letting $w \in \mathbb{P}^{k+1}(T)$, for all $\underline{v}_T \in \underline{U}_T^{k,l}$ we have

$$\mathrm{s}_{U,T}\left(\underline{I}_{U,T}^{k,l}w, \underline{v}_T\right) = \mathrm{s}_{\boldsymbol{\Sigma},T}\left(\underline{\boldsymbol{\varsigma}}_T^{k,l,m}\underline{I}_{U,T}^{k,l}w, \underline{\boldsymbol{\varsigma}}_T^{k,l,m}\underline{v}_T\right) = \mathrm{s}_{\boldsymbol{\Sigma},T}\left(\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,m}\nabla w, \underline{\boldsymbol{\varsigma}}_T^{k,l,m}\underline{v}_T\right) = 0,$$

where we have used the definition (6.16) of $\mathrm{s}_{U,T}$, the commuting property (6.8), and (S2).
(2) *Link with primal methods.* Compare the primal hybrid formulation (6.14) with the primal formulation (5.9) and recall the equivalence with the mixed hybrid formulation (6.4) stated in Theorem 6.4. □

7. From primal to mixed methods

In this section we show that the primal discontinuous skeletal methods of Section 5 with $m = 0$ can be recast into the mixed formulation introduced in Section 4. This closes the circle and shows a precise equivalence relation between the family (4.18) of mixed discontinuous skeletal methods and the family (5.9) of primal discontinuous skeletal methods.

7.1. Primal-to-mixed potential-to-flux operator

We assume from this point on that, for a given integer $k \geqslant 0$, l is as in (4.1), and

$$m = 0.$$

The crucial ingredient is the primal-to-mixed potential-to-flux operator $\underline{\varsigma}_T^{k,l} : \underline{U}_T^{k,l} \to \underline{\boldsymbol{\Sigma}}_T^{k,l,0}$ such that, for all $\underline{w}_T \in \underline{U}_T^{k,l}$, $\underline{\varsigma}_T^{k,l}\underline{w}_T$ solves

$$-\mathrm{b}_T(\underline{\varsigma}_T^{k,l}\underline{w}_T, \underline{v}_T) = \mathrm{a}_T(\underline{w}_T, \underline{v}_T) \qquad \forall \underline{v}_T \in \underline{U}_T^{k,l}. \tag{7.1}$$

The use of a similar notation as for the mixed-to-primal potential-to-flux operator defined by (6.5) is motivated by the fact that these two operators share the same properties (compare Lemmas 6.2 and 7.1) and play very much the same role.

Lemma 7.1 (Properties of the primal-to-mixed potential-to-flux operator)**.** *Let a mesh element $T \in \mathcal{T}_h$ be given and let $\mathrm{s}_{U,T}$ be a bilinear form satisfying Assumption 5.2. Then, the corresponding potential-to-flux operator $\underline{\varsigma}_T^{k,l}$ given by (7.1) is well defined and has the following properties:*

(1) Stability and continuity. *For all $\underline{v}_T \in \underline{U}_T^{k,l}$, it holds with norms $\|\cdot\|_{\boldsymbol{\Sigma},T}$ and $\|\cdot\|_{U,T}$ defined by (4.4) and (5.2), respectively,*

$$\|\underline{\varsigma}_T^{k,l}\underline{v}_T\|_{\boldsymbol{\Sigma},T} \approx \|\underline{v}_T\|_{U,T}. \tag{7.2}$$

(2) Commuting property. *For all $w \in \mathbb{P}^{k+1}(T)$, we have*

$$\underline{\varsigma}_T^{k,l}\underline{I}_{U,T}^{k,l}w = \underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,0}\nabla w. \tag{7.3}$$

(3) Link with the discrete gradient operator. *It holds, with operators $\mathbf{G}_T^k$, $\mathbf{P}_T^k$, and $\mathbf{S}_T^k$ defined by (5.4), (4.7), and (4.8), respectively, that*

$$\mathbf{G}_T^k = \mathbf{P}_T^k \circ \underline{\varsigma}_T^{k,l} = \mathbf{S}_T^k \circ \underline{\varsigma}_T^{k,l}. \tag{7.4}$$

Additionally, $\underline{\varsigma}_T^{k,l}$ defines an isomorphism from $\underline{U}_{T,*}^{k,l}$ (cf. (5.3)) to $\underline{\boldsymbol{\Sigma}}_T^{k,l,0}$.

Proof. Let $T \in \mathcal{T}_h$. To show that $\underline{\varsigma}_T^{k,l}$ is well defined we prove the following inf-sup condition: For all $\underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{\Sigma}}_T^{k,l,0}$,

$$\|\underline{\boldsymbol{\tau}}_T\|_{\boldsymbol{\Sigma},T} \leqslant \mathsf{S} := \sup_{\underline{v}_T \in \underline{U}_{T,*}^{k,l}\setminus\{\underline{0}_{U,T}\}} \frac{\mathrm{b}_T(\underline{\boldsymbol{\tau}}_T, \underline{v}_T)}{\|\underline{v}_T\|_{U,T}}. \tag{7.5}$$

Let $\underline{v}_{\boldsymbol{\tau},T} \in \underline{U}_T^{k,l}$ be such that $\nabla v_{\boldsymbol{\tau},T} = \boldsymbol{\tau}_T$ and $v_{\boldsymbol{\tau},F} - v_{\boldsymbol{\tau},T} = h_F \tau_{TF}$ ($\underline{v}_{\boldsymbol{\tau},T}$ is defined up to an element of $\underline{I}_{U,T}^{k,l}\mathbb{P}^0(T)$, coeherently with the fact that we write $\underline{U}_{T,*}^{k,l}$ in the supremum). It can be checked that $\|\underline{v}_{\boldsymbol{\tau},T}\|_{U,T} = \|\underline{\boldsymbol{\tau}}_T\|_{\boldsymbol{\Sigma},T}$ and it holds, recalling the reformulation (6.2) of the bilinear form b_T,

$$\|\underline{\boldsymbol{\tau}}_T\|_{\boldsymbol{\Sigma},T}^2 = -\mathrm{b}_T(\underline{\boldsymbol{\tau}}_T, \underline{v}_{\boldsymbol{\tau},T}) \leqslant \mathsf{S}\|\underline{v}_{\boldsymbol{\tau},T}\|_{U,T} = \mathsf{S}\|\underline{\boldsymbol{\tau}}_T\|_{\boldsymbol{\Sigma},T},$$

which proves (7.5). To prove the well-posedness of problem (7.1) it only remains to observe that, for all $\underline{v}_T \in \underline{I}_{U,T}^{k,l}\mathbb{P}^0(T)$, equation (7.1) becomes the trivial identity $0 = 0$, which can be intepreted as a compatibility condition. Finally, the fact that $\underline{\varsigma}_T^{k,l}$ defines an isomorphism from $\underline{U}_{T,*}^{k,l}$ to $\underline{\boldsymbol{\Sigma}}_T^{k,l,0}$ follows observing that $\underline{\varsigma}_T^{k,l}$ is injective as a result of (7.5) and $\dim(\underline{U}_{T,*}^{k,l}) = \dim(\underline{\boldsymbol{\Sigma}}_T^{k,l,0})$.

(1) *Stability and continuity.* Combining the inf-sup condition (7.5) with the definition (7.1) of $\underline{\varsigma}_T^{k,l}$, and using the Cauchy–Schwarz inequality followed by (S1'), we get for all $\underline{v}_T \in \underline{U}_T^{k,l}$ that

$$\|\underline{\varsigma}_T^{k,l}\underline{v}_T\|_{\boldsymbol{\Sigma},T} \leqslant \sup_{\underline{w}_T \in \underline{U}_{T,*}^{k,l}\setminus\{\underline{0}_{U,T}\}} \frac{\mathrm{b}_T(\underline{\varsigma}_T^{k,l}\underline{v}_T, \underline{w}_T)}{\|\underline{w}_T\|_{U,T}} = \sup_{\underline{w}_T \in \underline{U}_{T,*}^{k,l}\setminus\{\underline{0}_{U,T}\}} \frac{\mathrm{a}_T(\underline{v}_T, \underline{w}_T)}{\|\underline{w}_T\|_{U,T}} \lesssim \|\underline{v}_T\|_{U,T}.$$

On the other hand, (S1') followed by the definition (7.1) of $\underline{\varsigma}_T^{k,l}$ and the boundedness (6.3) of the bilinear form b_T yields

$$\|\underline{v}_T\|_{U,T}^2 \lesssim \mathrm{a}_T(\underline{v}_T, \underline{v}_T) = -\mathrm{b}_T(\underline{\varsigma}_T^{k,l}\underline{v}_T, \underline{v}_T) \leqslant \|\underline{\varsigma}_T^{k,l}\underline{v}_T\|_{\boldsymbol{\Sigma},T}\|\underline{v}_T\|_{U,T},$$

which concludes the proof of (7.2).

(2) *Commuting property.* Let $w \in \mathbb{P}^{k+1}(T)$. For all $\underline{v}_T \in \underline{U}_T^{k,l}$ it holds

$$-\mathrm{b}_T\left(\underline{\varsigma}_T^{k,l}\underline{I}_{U,T}^{k,l}w, \underline{v}_T\right) = \mathrm{a}_T\left(\underline{I}_{U,T}^{k,l}w, \underline{v}_T\right) = (\nabla w, \mathbf{G}_T^k\underline{v}_T)_T = -\mathrm{b}_T\left(\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,0}\nabla w, \underline{v}_T\right),$$

where we have used the definition (7.1) of $\underline{\varsigma}_T^{k,l}$ in the first equality, the definition (5.6) of a_T together with (S2′) in the second equality, and concluded recalling the definitions (5.4) of $\mathbf{G}_T^k$, (4.3) of $\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,0}$, and (6.1) of b_T. As a consequence,

$$\mathrm{b}_T\left(\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,0}\nabla w - \underline{\varsigma}_T^{k,l}\underline{I}_{U,T}^{k,l}w, \underline{v}_T\right) = 0 \qquad \forall \underline{v}_T \in \underline{U}_T^{k,l},$$

which, accounting for the inf-sup condition (7.5), implies (7.3).

(3) *Link with the discrete gradient operator.* Let $\underline{v}_T \in \underline{U}_T^{k,l}$ and $w \in \mathbb{P}^{k+1}(T)$. Recalling the definitions (6.1) of b_T and (5.1) of $\underline{I}_{U,T}^{k,l}$, we infer that

$$\begin{aligned}
-\mathrm{b}_T(\underline{\varsigma}_T^{k,l}\underline{v}_T, \underline{I}_{U,T}^{k,l}w) &= -(\mathrm{D}_T^l\underline{\varsigma}_T^{k,l}\underline{v}_T, \pi_T^l w)_T + \sum_{F\in\mathcal{F}_T}((\underline{\varsigma}_T^{k,l}\underline{v}_T)_{TF}, \pi_F^k w)_F \\
&= -(\mathrm{D}_T^l\underline{\varsigma}_T^{k,l}\underline{v}_T, w)_T + \sum_{F\in\mathcal{F}_T}((\underline{\varsigma}_T^{k,l}\underline{v}_T)_{TF}, w)_F = ((\mathbf{P}_T^k \circ \underline{\varsigma}_T^{k,l})\underline{v}_T, \nabla w)_T,
\end{aligned}$$

where we have used the definition (2.1) of π_T^l and π_F^k to pass to the second line and the definition (4.7) of $\mathbf{P}_T^k$ to conclude. On the other hand, by the definition (5.6) of a_T together with the polynomial consistency of $\mathbf{G}_T^k$ (a consequence of (5.5)) and (S2′), we have

$$\mathrm{a}_T\left(\underline{v}_T, \underline{I}_{U,T}^{k,l}w\right) = (\mathbf{G}_T^k\underline{v}_T, \nabla w)_T.$$

Substituting the above relations into the definition (7.1) of $\underline{\varsigma}_T^{k,l}$ we infer that $\mathbf{G}_T^k\underline{v}_T = \mathbf{P}_T^k \circ \underline{\varsigma}_T^{k,l}$. Additionally, since we have supposed $m = 0$, we also have $\mathbf{S}_T^k = \mathbf{P}_T^k$, thus concluding the proof. □

7.2. Equivalent mixed formulation of primal methods

We close this section by showing the existence of a mixed method belonging to the family (4.18) whose solution coincides with that of the primal problem (5.9). In the light of Theorem 6.4, we state the equivalence result in terms of the corresponding mixed hybrid formulation (6.4).

Theorem 7.2 (Link with the family of mixed discontinuous skeletal methods)**.** *For all $T \in \mathcal{T}_h$, let $\mathrm{s}_{U,T}$ satisfy Assumption 5.2 and set, for all $\underline{\boldsymbol{\sigma}}_T, \underline{\boldsymbol{\tau}}_T \in \underline{\boldsymbol{\Sigma}}_T^{k,l,0}$,*

$$\mathrm{s}_{\boldsymbol{\Sigma},T}(\underline{\boldsymbol{\sigma}}_T, \underline{\boldsymbol{\tau}}_T) := \mathrm{s}_{U,T}\left((\underline{\varsigma}_T^{k,l})^{-1}\underline{\boldsymbol{\sigma}}_T, (\underline{\varsigma}_T^{k,l})^{-1}\underline{\boldsymbol{\tau}}_T\right), \tag{7.6}$$

where it is understood that $(\underline{\varsigma}_T^{k,l})^{-1}\underline{\boldsymbol{\tau}}_T$ and $(\underline{\varsigma}_T^{k,l})^{-1}\underline{\boldsymbol{\sigma}}_T$ are defined up to an element of $\underline{I}_{U,T}^{k,l}\mathbb{P}^0(T)$. Then,

(1) Properties of $\mathrm{s}_{\boldsymbol{\Sigma},T}$. *The stabilization bilinear forms* $\mathrm{s}_{\boldsymbol{\Sigma},T}$, $T\in\mathcal{T}_h$ *satisfy Assumption 4.2*;

(2) Link with mixed methods. $(\underline{\boldsymbol{\sigma}}_h,\underline{u}_h)\in\check{\underline{\boldsymbol{\Sigma}}}_h^{k,l,0}\times\underline{U}_{h,0}^{k,l}$ *solves the mixed hybrid problem (6.4) with stabilization as in (7.6) if and only if* $\underline{u}_h$ *solves the primal problem (5.9) and, for all* $T\in\mathcal{T}_h$, $\underline{\boldsymbol{\sigma}}_T=\underline{\boldsymbol{\varsigma}}_T^{k,l}\underline{u}_T$ *with* $\underline{\boldsymbol{\varsigma}}_T^{k,l}$ *defined by (7.1).*

Proof.

(1) *Properties of* $\mathrm{s}_{\boldsymbol{\Sigma},T}$. Let $T\in\mathcal{T}_h$. The bilinear form $\mathrm{s}_{\boldsymbol{\Sigma},T}$ is clearly symmetric and positive semi-definite. It then suffices to prove conditions (S1) and (S2). Let us start by (S1). Recalling the definition (4.13a) of the bilinear form m_T, property (7.4) for the potential-to-flux operator $\underline{\boldsymbol{\varsigma}}_T^{k,l}$ defined by (7.1), and (7.6), we infer for all $\underline{w}_T,\underline{v}_T\in\underline{U}_T^{k,l}$ that

$$\mathrm{m}_T(\underline{\boldsymbol{\varsigma}}_T^{k,l}\underline{w}_T,\underline{\boldsymbol{\varsigma}}_T^{k,l}\underline{v}_T)=\mathrm{a}_T(\underline{w}_T,\underline{v}_T). \tag{7.7}$$

Let now $\underline{\boldsymbol{\tau}}_T\in\underline{\boldsymbol{\Sigma}}_T^{k,l,0}$ be such that $\underline{\boldsymbol{\tau}}_T=\underline{\boldsymbol{\varsigma}}_T^{k,l}\underline{v}_T$ with $\underline{v}_T\in\underline{U}_T^{k,l}$ (the existence of such $\underline{v}_T$, defined up to an element of $\underline{I}_{U,T}^{k,l}\mathbb{P}^0(T)$, follows from Lemma 7.1). We have that

$$\|\underline{\boldsymbol{\tau}}_T\|_{\boldsymbol{\Sigma},T}\approx\|\underline{v}_T\|_{U,T}\approx\|\underline{v}_T\|_{\mathrm{a},T}=\|\underline{\boldsymbol{\tau}}_T\|_{\mathrm{m},T},$$

where the first norm equivalence follows from (7.2), the second from (S2′), and the last one from (7.7). Property (S1) follows.
Let us now prove (S2). Let $\boldsymbol{\chi}\in\mathbb{G}_T^k$ be such that $\boldsymbol{\chi}=\nabla w$ with $w\in\mathbb{P}^{k+1}(T)$. For all $\underline{v}_T\in\underline{U}_T^{k,l}$ it holds,

$$\mathrm{s}_{\boldsymbol{\Sigma},T}(\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,0}\boldsymbol{\chi},\underline{\boldsymbol{\varsigma}}_T^{k,l}\underline{v}_T)=\mathrm{s}_{U,T}((\underline{\boldsymbol{\varsigma}}_T^{k,l})^{-1}\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,0}\boldsymbol{\chi},\underline{v}_T)=\mathrm{s}_{U,T}(\underline{I}_{U,T}^{k,l}w,\underline{v}_T)=0,$$

where we have used the definition (7.6) of $\mathrm{s}_{\boldsymbol{\Sigma},T}$, the commuting property (7.3), and concluded using (S2′).

(2) *Link with mixed methods.* We let $(\underline{\boldsymbol{\sigma}}_h,\underline{u}_h)\in\check{\underline{\boldsymbol{\Sigma}}}_h^{k,l,0}\times\underline{U}_{h,0}^{k,l}$ solve the mixed hybrid problem (6.4) with $s_{\boldsymbol{\Sigma},T}$ given by (7.6), and we show that $\underline{u}_h$ solves (5.9) and $\underline{\boldsymbol{\sigma}}_T=\underline{\boldsymbol{\varsigma}}_T^{k,l}\underline{u}_T$ for all $T\in\mathcal{T}_h$. Making $\underline{\boldsymbol{\tau}}_T=\underline{\boldsymbol{\varsigma}}_T^{k,l}\underline{v}_T$ with $\underline{v}_T\in\underline{U}_T^{k,l}$ in (6.4a), it is inferred

$$0=\mathrm{m}_T(\underline{\boldsymbol{\sigma}}_T,\underline{\boldsymbol{\varsigma}}_T^{k,l}\underline{v}_T)+\mathrm{b}_T(\underline{\boldsymbol{\varsigma}}_T^{k,l}\underline{v}_T,\underline{u}_T)=\mathrm{m}_T(\underline{\boldsymbol{\sigma}}_T-\underline{\boldsymbol{\varsigma}}_T^{k,l}\underline{u}_T,\underline{\boldsymbol{\varsigma}}_T^{k,l}\underline{v}_T).$$

Since $\underline{\boldsymbol{\Sigma}}_T^{k,l,0}=\underline{\boldsymbol{\varsigma}}_T^{k,l}\underline{U}_T^{k,l}$ as a result of Lemma 7.1 and $\underline{v}_T$ is arbitrary in $\underline{U}_T^{k,l}$, this means that

$$\underline{\boldsymbol{\sigma}}_T=\underline{\boldsymbol{\varsigma}}_T^{k,l}\underline{u}_T\qquad\forall T\in\mathcal{T}_h.$$

Plugging this relation into (6.4b), and recalling the definition (7.1) of $\underline{\boldsymbol{\varsigma}}_T^{k,l}$, we infer that it holds for all $\underline{v}_h\in\underline{U}_{h,0}^{k,l}$,

$$(f,v_h)=-\sum_{T\in\mathcal{T}_h}\mathrm{b}_T(\underline{\boldsymbol{\sigma}}_T,\underline{v}_T)=-\sum_{T\in\mathcal{T}_h}\mathrm{b}_T(\underline{\boldsymbol{\varsigma}}_T^{k,l}\underline{u}_T,\underline{v}_T)=\mathrm{a}_h(\underline{u}_h,\underline{v}_h),$$

which shows that $\underline{u}_h$ solves the primal problem (5.9). Following a similar reasoning one can prove that, if $\underline{u}_h$ solves (5.9), then $(\underline{\boldsymbol{\sigma}}_h,\underline{u}_h)$ with $\underline{\boldsymbol{\sigma}}_T=\underline{\boldsymbol{\varsigma}}_T^{k,l}\underline{u}_T$ for all $T\in\mathcal{T}_h$ solves (6.4). ☐

8. Analysis

In this section we carry out a unified convergence analysis encompassing both mixed and primal discontinuous skeletal methods. Recalling Theorems 6.4, 6.5, and 7.2, we focus on the mixed hybrid problem (6.4). Let three integers $k\geqslant 0$ and l,m as in (4.1) be fixed, set $\underline{\boldsymbol{X}}_h^{k,l,m}:=\check{\underline{\boldsymbol{\Sigma}}}_h^{k,l,m}\times\underline{U}_{h,0}^{k,l}$, and define the bilinear form $\mathcal{A}_h:\underline{\boldsymbol{X}}_h^{k,l,m}\times\underline{\boldsymbol{X}}_h^{k,l,m}\to\mathbb{R}$ such that

$$\mathcal{A}_h((\underline{\boldsymbol{\sigma}}_h,\underline{u}_h),(\underline{\boldsymbol{\tau}}_h,\underline{v}_h)):=\mathrm{m}_h(\underline{\boldsymbol{\sigma}}_h,\underline{\boldsymbol{\tau}}_h)+\mathrm{b}_h(\underline{\boldsymbol{\tau}}_h,\underline{u}_h)-\mathrm{b}_h(\underline{\boldsymbol{\sigma}}_h,\underline{v}_h).$$

Problem (6.4) admits the following equivalent reformulation: Find $(\underline{\boldsymbol{\sigma}}_h, \underline{u}_h) \in \underline{\breve{\boldsymbol{\Sigma}}}_h^{k,l,m} \times \underline{U}_{h,0}^{k,l}$ such that

$$\mathcal{A}_h((\underline{\boldsymbol{\sigma}}_h, \underline{u}_h), (\underline{\boldsymbol{\tau}}_h, \underline{v}_h)) = (f, v_h) \qquad \forall (\underline{\boldsymbol{\tau}}_h, \underline{v}_h) \in \underline{\breve{\boldsymbol{\Sigma}}}_h^{k,l,m} \times \underline{U}_{h,0}^{k,l}. \tag{8.1}$$

8.1. Stability and well-posedness

We equip the space $\underline{\boldsymbol{X}}_h^{k,l,m}$ with the norm $\|\cdot\|_{\boldsymbol{X},h}$ such that, for all $(\underline{\boldsymbol{\tau}}_h, \underline{v}_h) \in \underline{\boldsymbol{X}}_h^{k,l,m}$,

$$\|(\underline{\boldsymbol{\tau}}_h, \underline{v}_h)\|_{\boldsymbol{X},h}^2 := \|\underline{\boldsymbol{\tau}}_h\|_{\boldsymbol{\Sigma},h}^2 + \|\underline{v}_h\|_{U,h}^2,$$

with norms $\|\cdot\|_{\boldsymbol{\Sigma},h}$ on $\underline{\breve{\boldsymbol{\Sigma}}}_h^{k,l,m}$ and $\|\cdot\|_{U,h}$ on $\underline{U}_h^{k,l}$ defined by (4.15) and (5.8), respectively.

Lemma 8.1 (Well-posedness)**.** *For all $(\underline{\boldsymbol{\chi}}_h, \underline{w}_h) \in \underline{\boldsymbol{X}}_h^{k,l,m}$ it holds*

$$\|(\underline{\boldsymbol{\chi}}_h, \underline{w}_h)\|_{\boldsymbol{X},h} \lesssim \sup_{(\underline{\boldsymbol{\tau}}_h, \underline{v}_h) \in \underline{\boldsymbol{X}}_h^{k,l,m} \setminus \{\underline{\boldsymbol{0}}_{\boldsymbol{X},h}\}} \frac{\mathcal{A}_h((\underline{\boldsymbol{\chi}}_h, \underline{w}_h), (\underline{\boldsymbol{\tau}}_h, \underline{v}_h))}{\|(\underline{\boldsymbol{\tau}}_h, \underline{v}_h)\|_{\boldsymbol{X},h}}. \tag{8.2}$$

Consequently, problem (8.1) is well-posed.

Proof. We start by proving the following inf-sup condition for b_h: For all $\underline{v}_h \in \underline{U}_{h,0}^{k,l}$,

$$\|\underline{v}_h\|_{U,h} \lesssim \sup_{\underline{\boldsymbol{\tau}}_h \in \underline{\breve{\boldsymbol{\Sigma}}}_h^{k,l,m} \setminus \{\underline{\boldsymbol{0}}_{\boldsymbol{\Sigma},h}\}} \frac{\mathrm{b}_h(\underline{\boldsymbol{\tau}}_h, \underline{v}_h)}{\|\underline{\boldsymbol{\tau}}_h\|_{\boldsymbol{\Sigma},h}}. \tag{8.3}$$

Fix an element $\underline{v}_h \in \underline{U}_{h,0}^{k,l}$, and let $\underline{\boldsymbol{\tau}}_{v,h} \in \underline{\breve{\boldsymbol{\Sigma}}}_h^{k,l,m}$ be such that, for all $T \in \mathcal{T}_h$, $\boldsymbol{\tau}_{v,T} = \nabla v_T$ and $\tau_{v,TF} = h_F^{-1}(v_F - v_T)$ for all $F \in \mathcal{F}_T$. Denoting by S the supremum in (8.3), from (6.2) it is inferred that

$$\|\underline{v}_h\|_{U,h}^2 = \mathrm{b}_h(\underline{\boldsymbol{\tau}}_{v,h}, \underline{v}_h) \leqslant \mathsf{S}\|\underline{\boldsymbol{\tau}}_{v,h}\|_{\boldsymbol{\Sigma},h},$$

and (8.3) readily follows observing that, by the definitions (4.4) and (5.2) of the local norms, $\|\underline{\boldsymbol{\tau}}_{v,T}\|_{\boldsymbol{\Sigma},T} = \|\underline{v}_T\|_{U,T}$. The inf-sup condition (8.2) on $\mathcal{A}_h$ and the well-posedness of problem (6.4) are then classical consequences of the $\|\cdot\|_{\boldsymbol{\Sigma},h}$-coercivity of m_h (itself a consequence of (S1)) and the inf-sup condition (8.3) on b_h; *cf.*, *e.g.*, [16]. □

8.2. Energy error estimate

We estimate the error defined as the difference between the solution of the mixed hybrid problem (6.4) and the projection $(\underline{\widehat{\boldsymbol{\sigma}}}_h, \underline{\widehat{u}}_h) \in \underline{\breve{\boldsymbol{\Sigma}}}_h^{k,l,m} \times \underline{U}_{h,0}^{k,l}$ of the exact solution defined as follows:

$$\underline{\widehat{\boldsymbol{\sigma}}}_h := \underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},h}^{k,l,m} \nabla_h \breve{u}_h \quad \forall T \in \mathcal{T}_h, \qquad \underline{\widehat{u}}_h := \underline{I}_{U,h}^{k,l} u,$$

where $\breve{u}_h \in \mathbb{P}^{k+1}(\mathcal{T}_h)$ is such that, for all $T \in \mathcal{T}_h$, $\breve{u}_T := \breve{u}_{h|T}$ is the local elliptic projection of u satisfying

$$\nabla \breve{u}_T = \boldsymbol{\pi}_{\mathbb{G},T}^k \nabla u \quad \text{and} \quad (\breve{u}_T - u, 1)_T = 0, \tag{8.4}$$

while $\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},h}^{k,l,m}$ is the global flux reduction map on $\underline{\breve{\boldsymbol{\Sigma}}}_h^{k,l,m}$ whose restriction to every mesh elements $T \in \mathcal{T}_h$ coincides with $\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,m}$ defined by (4.3). Optimal approximation properties for $\breve{u}_h$ on regular mesh sequences are proved in ([39], Lem. 3) and, in a more general framework, in [34].

Theorem 8.2 (Energy error estimate)**.** *Let $u \in H_0^1(\Omega)$ be the weak solution of problem (1.1), and assume the additional regularity $u \in H^{k+2}(\Omega)$. Then, it holds*

$$\|(\underline{\boldsymbol{\sigma}}_h - \underline{\widehat{\boldsymbol{\sigma}}}_h, \underline{u}_h - \underline{\widehat{u}}_h)\|_{\boldsymbol{X},h} \lesssim h^{k+1}\|u\|_{H^{k+2}(\Omega)}. \tag{8.5}$$

Proof. The following error equation descends from (8.1): For all $(\underline{\boldsymbol{\tau}}_h, \underline{v}_h) \in \underline{\check{\boldsymbol{\Sigma}}}_h^{k,l,m} \times \underline{U}_{h,0}^{k,l}$,

$$\mathcal{A}_h((\underline{\boldsymbol{\sigma}}_h - \underline{\widehat{\boldsymbol{\sigma}}}_h, \underline{u}_h - \underline{\widehat{u}}_h), (\underline{\boldsymbol{\tau}}_h, \underline{v}_h)) = \mathcal{E}_h(\underline{\boldsymbol{\tau}}_h, \underline{v}_h),$$

with conformity error

$$\mathcal{E}_h(\underline{\boldsymbol{\tau}}_h, \underline{v}_h) := (f, v_h) + \mathrm{b}_h(\underline{\widehat{\boldsymbol{\sigma}}}_h, \underline{v}_h) - \mathrm{m}_h(\underline{\widehat{\boldsymbol{\sigma}}}_h, \underline{\boldsymbol{\tau}}_h) - \mathrm{b}_h(\underline{\boldsymbol{\tau}}_h, \underline{\widehat{u}}_h). \tag{8.6}$$

Recalling the inf-sup condition (8.2), we then have that

$$\|(\underline{\boldsymbol{\sigma}}_h - \underline{\widehat{\boldsymbol{\sigma}}}_h, \underline{u}_h - \underline{\widehat{u}}_h)\|_{\boldsymbol{X},h} \lesssim \sup_{(\underline{\boldsymbol{\tau}}_h, \underline{v}_h) \in \underline{\boldsymbol{X}}_h^{k,l,m} \setminus \{\underline{\mathbf{0}}_{\boldsymbol{X},h}\}} \frac{\mathcal{E}_h(\underline{\boldsymbol{\tau}}_h, \underline{v}_h)}{\|(\underline{\boldsymbol{\tau}}_h, \underline{v}_h)\|_{\boldsymbol{X},h}}. \tag{8.7}$$

To conclude, it suffices to bound $\mathcal{E}_h(\underline{\boldsymbol{\tau}}_h, \underline{v}_h)$. Denote by $\mathfrak{T}_1, \ldots, \mathfrak{T}_4$ the addends in the right-hand side of (8.6). Recalling that $f = -\triangle u$ a.e. in Ω, integrating by parts element-by-element, and using the fact that the normal component of ∇u is continuous across all interfaces $F \in \mathcal{F}^{\mathrm{i}}$ and that v_F vanishes on boundary faces $F \in \mathcal{F}^{\mathrm{b}}$, we have that

$$\mathfrak{T}_1 = \sum_{T \in \mathcal{T}_h} \left((\nabla u, \nabla v_T) + \sum_{F \in \mathcal{F}_T} (\nabla u \cdot \boldsymbol{n}_{TF}, v_F - v_T)_F \right).$$

Using the commuting property (4.9) of D_T^l to infer $\mathrm{D}_T^l \underline{\widehat{\boldsymbol{\sigma}}}_T = \triangle \check{u}_T$, and integrating by parts element-by-element, we have that

$$\mathfrak{T}_2 = -\sum_{T \in \mathcal{T}_h} \left((\nabla u, \nabla v_T)_T + \sum_{F \in \mathcal{F}_T} (\nabla \check{u}_T \cdot \boldsymbol{n}_{TF}, v_F - v_T)_F \right),$$

where we have used the definition (8.4) of $\check{u}_T$ to write ∇u instead of $\nabla \check{u}_T$ in the first term. Summing the above equations, passing to the absolute value, and using the Cauchy–Schwarz inequality yields

$$|\mathfrak{T}_1 + \mathfrak{T}_2| \leqslant \left(\sum_{F \in \mathcal{F}_T} h_F \|\nabla(u - \check{u}_T)\|_F^2 \right)^{\frac{1}{2}} \times \left(\sum_{F \in \mathcal{F}_T} h_F^{-1} \|v_F - v_T\|_F^2 \right)^{\frac{1}{2}} \lesssim h^{k+1}\|u\|_{H^{k+2}(\Omega)} \|\underline{v}_h\|_{U,h}, \tag{8.8}$$

where we have used the optimal approximation properties of $\check{u}_T$ to conclude.

Recalling the definition (4.13b) of m_T, using the polynomial consistency (4.10) of $\mathbf{P}_T^k$ together with (S2), and expanding $\mathbf{P}_T^k \underline{\boldsymbol{\tau}}_T$ according to its definition (4.7) (with $w = \check{u}_T$), it is inferred that

$$\mathfrak{T}_3 = -\sum_{T \in \mathcal{T}_h} (\nabla \check{u}_T, \mathbf{P}_T^k \underline{\boldsymbol{\tau}}_T)_T = \sum_{T \in \mathcal{T}_h} \left((\check{u}_T, \mathrm{D}_T^l \underline{\boldsymbol{\tau}}_T)_T - \sum_{F \in \mathcal{F}_T} (\check{u}_T, \tau_{TF})_F \right).$$

Recalling (6.1) together with the definitions (5.1) of $\underline{I}_{U,T}^{k,l}$ and (2.1) of π_T^l and π_F^k, we get that

$$\mathfrak{T}_4 = \sum_{T \in \mathcal{T}_h} \left(-(u, \mathrm{D}_T^l \underline{\boldsymbol{\tau}}_T)_T + \sum_{F \in \mathcal{F}_T} (u, \tau_{TF})_F \right).$$

Summing the above equations, passing to the absolute value, and using the Cauchy–Schwarz inequality, we then obtain

$$\begin{aligned}|\mathfrak{T}_3+\mathfrak{T}_4| &\leqslant \left[\sum_{T\in\mathcal{T}_h}\left(h_T^{-2}\|u-\breve{u}_T\|_T^2+\sum_{F\in\mathcal{F}_T}h_F^{-1}\|u-\breve{u}_T\|_F^2\right)\right]^{\frac{1}{2}}\times\left[\sum_{T\in\mathcal{T}_h}\left(h_T^2\|\mathrm{D}_T^l\underline{\boldsymbol{\tau}}_T\|_T^2+\sum_{F\in\mathcal{F}_T}h_F\|\tau_{TF}\|_F^2\right)\right]^{\frac{1}{2}}\\ &\lesssim h^{k+1}\|u\|_{H^{k+2}(\Omega)}\|\underline{\boldsymbol{\tau}}_T\|_{\boldsymbol{\Sigma},T},\end{aligned} \tag{8.9}$$

where we have used the optimal approximation properties of $\breve{u}_T$ and the inverse inequality $\|\mathrm{D}_T^l\underline{\boldsymbol{\tau}}_T\|_T \lesssim h_T^{-1}\|\underline{\boldsymbol{\tau}}_T\|_{\boldsymbol{\Sigma},T}$ to pass to the second line. Combining (8.8) with (8.9), we infer the bound

$$|\mathcal{E}_h(\underline{\boldsymbol{\tau}}_h,\underline{v}_h)| \lesssim h^{k+1}\|u\|_{H^{k+2}(\Omega)}\|(\underline{\boldsymbol{\tau}}_h,\underline{v}_h)\|_{\boldsymbol{X},h},$$

which, plugged into (8.7), yields the desired result. □

8.3. L^2-error estimate

In this section we prove a sharp L^2-error estimate on the potential under the following usual elliptic regularity assumption: For all $g\in L^2(\Omega)$, the unique solution $z\in H_0^1(\Omega)$ of the problem

$$(\nabla z,\nabla v)=(g,v)\qquad\forall v\in H_0^1(\Omega), \tag{8.10}$$

satisfies

$$\|z\|_{H^2(\Omega)}\leqslant C_\Omega\|g\|_{L^2(\Omega)}, \tag{8.11}$$

with real number $C_\Omega>0$ only depending on Ω. In the proof we will need the following consistency property for the bilinear form b_h.

Proposition 8.3 (Consistency of b_h)**.** *For all $\boldsymbol{\chi}\in\boldsymbol{H}(\mathrm{div};\Omega)$ such that $\boldsymbol{\chi}_{|T}\in\boldsymbol{\Sigma}^+(T)$ for all $T\in\mathcal{T}_h$, it holds*

$$\mathrm{b}_h\left(\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},h}^{k,l,m}\boldsymbol{\chi},\underline{v}_h\right)=(\mathrm{div}\,\boldsymbol{\chi},v_h)\qquad\forall\underline{v}_h\in\underline{U}_{h,0}^{k,l}. \tag{8.12}$$

Proof. Recall the expression (6.1) of b_h and use commuting property (4.9) for D_T^l together with the fact that $\boldsymbol{\chi}$ has continuous normal components across interfaces $F\in\mathcal{F}^{\mathrm{i}}$ and $v_F=0$ on all $F\in\mathcal{F}^{\mathrm{b}}$. □

Theorem 8.4 (L^2-error estimate)**.** *Let the assumptions of Theorem 8.2 hold true, and further assume $(k,l)\neq(1,0)$, elliptic regularity, $f\in H^{k+\delta}(\Omega)$ with $\delta=1$ if $k=l=0$, $\delta=0$ otherwise. Then, it holds*

$$\|\widehat{u}_h-u_h\|\lesssim h^{k+2}\|u\|_{H^{k+2}(\Omega)}+h^{k+2}\|f\|_{H^{k+\delta}(\Omega)}.$$

Remark 8.5 (The case $(k,l)=(1,0)$)**.** If $(k,l)=(1,0)$, the term $\mathfrak{T}_1$ in the proof below limits the convergence order to h^2. We refer to ([50], Sect. 2.7) for a modification of the right-hand side that aims at avoiding this shortcoming.

Proof. Let z solve (8.10) with $g=u_h-\widehat{u}_h$ and set, for the sake of brevity,

$$\underline{\widehat{\boldsymbol{\chi}}}_h:=\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},h}^{k,l,m}\nabla z,\qquad \underline{\widehat{z}}_h:=\underline{I}_{U,h}^{k,l}z.$$

Then, we have

$$\|\widehat{u}_h-u_h\|^2=(u-u_h,\triangle z)=-(f,z)-\mathrm{b}_h(\underline{\widehat{\boldsymbol{\chi}}}_h,\underline{u}_h), \tag{8.13}$$

where for the first addend we have integrated by parts and used the fact that $(\nabla u,\nabla z)=(f,z)$, while for the second addend we have used the consistency property (8.12) of b_h with $\boldsymbol{\chi}=\nabla z$ and $\underline{v}_h=\underline{u}_h$. Using (6.4a) we get, denoting by $\underline{\boldsymbol{\varsigma}}_h^{k,l,m}$ the global mixed-to-primal potential-to-flux operator whose restriction to every mesh element $T \in \mathcal{T}_h$ coincides with $\underline{\boldsymbol{\varsigma}}_T^{k,l,m}$ defined by (6.5),

$$\begin{aligned}
-\mathrm{b}_h(\underline{\widehat{\boldsymbol{\chi}}}_h, \underline{u}_h) &= \mathrm{m}_h\left(\underline{\widehat{\boldsymbol{\chi}}}_h, \underline{\boldsymbol{\sigma}}_h\right) \\
&= \mathrm{m}_h\left(\underline{\widehat{\boldsymbol{\chi}}}_h - \underline{\boldsymbol{\varsigma}}_h^{k,l,m}\underline{\widehat{z}}_h, \underline{\boldsymbol{\sigma}}_h\right) + \mathrm{a}_h(\underline{\widehat{z}}_h, \underline{u}_h) \\
&= \mathrm{m}_h\left(\underline{\widehat{\boldsymbol{\chi}}}_h - \underline{\boldsymbol{\varsigma}}_h^{k,l,m}\underline{\widehat{z}}_h, \underline{\boldsymbol{\sigma}}_h - \underline{\widehat{\boldsymbol{\sigma}}}_h\right) + \mathrm{m}_h\left(\underline{\widehat{\boldsymbol{\chi}}}_h - \underline{\boldsymbol{\varsigma}}_h^{k,l,m}\underline{\widehat{z}}_h, \underline{\widehat{\boldsymbol{\sigma}}}_h\right) + (f, \widehat{z}_h),
\end{aligned} \tag{8.14}$$

where we have inserted $\pm\underline{\boldsymbol{\varsigma}}_h^{k,l,m}\underline{\widehat{z}}_h$ and used the fact that $\underline{\boldsymbol{\sigma}}_h = \underline{\boldsymbol{\varsigma}}_h^{k,l,m}\underline{u}_h$ together with the definition (6.15) of the primal hybrid bilinear form a_h to pass to the second line, and we have inserted $\pm\underline{\widehat{\boldsymbol{\sigma}}}_h$ and used (6.14b) (with $\underline{v}_h = \underline{\widehat{z}}_h$) to conclude. Plugging (8.14) into (8.13), and observing that $(f, \widehat{z}_h) = (\pi_h^l f, z)$ with π_h^l denoting the L^2-orthogonal projector on U_h^l (*cf.* (4.16)), we arrive at

$$\|\widehat{u}_h - u_h\|^2 = (\pi_h^l f - f, z - \pi_h^l z) + \mathrm{m}_h\left(\underline{\widehat{\boldsymbol{\chi}}}_h - \underline{\boldsymbol{\varsigma}}_h^{k,l,m}\underline{\widehat{z}}_h, \underline{\boldsymbol{\sigma}}_h - \underline{\widehat{\boldsymbol{\sigma}}}_h\right) + \mathrm{m}_h\left(\underline{\widehat{\boldsymbol{\chi}}}_h - \underline{\boldsymbol{\varsigma}}_h^{k,l,m}\underline{\widehat{z}}_h, \underline{\widehat{\boldsymbol{\sigma}}}_h\right). \tag{8.15}$$

Denote by $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3$ the terms in the right-hand side of (8.15). For $\mathfrak{T}_1$, if $k = l = 0$, we have

$$|\mathfrak{T}_1| \leqslant \|\pi_h^l f - f\|\|z - \pi_h^l z\| \lesssim h^2\|f\|_{H^1(\Omega)}\|z\|_{H^1(\Omega)}, \tag{8.16}$$

while, in all the other cases,

$$|\mathfrak{T}_1| \leqslant \|\pi_h^l f - f\|\|z - \pi_h^l z\| \lesssim h^{k+2}\|f\|_{H^k(\Omega)}\|z\|_{H^2(\Omega)}. \tag{8.17}$$

For $\mathfrak{T}_2$, the Cauchy–Schwarz inequality followed by (S1) and the energy error estimate (8.5) yields

$$|\mathfrak{T}_2| \lesssim \|\underline{\widehat{\boldsymbol{\chi}}}_h - \underline{\boldsymbol{\varsigma}}_h^{k,l,m}\underline{\widehat{z}}_h\|_{\boldsymbol{\Sigma},h}\|\underline{\boldsymbol{\sigma}}_h - \underline{\widehat{\boldsymbol{\sigma}}}_h\|_{\boldsymbol{\Sigma},h} \lesssim h^{k+2}\|z\|_{H^2(\Omega)}\|u\|_{H^{k+2}(\Omega)}. \tag{8.18}$$

To estimate the quantity $\|\underline{\widehat{\boldsymbol{\chi}}}_h - \underline{\boldsymbol{\varsigma}}_h^{k,l,m}\underline{\widehat{z}}_h\|_{\boldsymbol{\Sigma},h}$ in (8.18), let $\breve{z}_h \in \mathbb{P}^{k+1}(\mathcal{T}_h)$ be the broken elliptic projection such that $\breve{z}_T := \breve{z}_{h|T}$ is defined as in (8.4) with u replaced by z, observe that $\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},h}^{k,l,m}\nabla_h\breve{z}_h = \underline{\boldsymbol{\varsigma}}_h^{k,l,m}\underline{I}_{U,h}^{k,l}\breve{z}_h$ by (6.8), and use (6.7) to infer

$$\begin{aligned}
\|\underline{\widehat{\boldsymbol{\chi}}}_h - \underline{\boldsymbol{\varsigma}}_h^{k,l,m}\underline{\widehat{z}}_h\|_{\boldsymbol{\Sigma},h} &\leqslant \|\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},h}^{k,l,m}(\nabla z - \nabla_h\breve{z}_h)\|_{\boldsymbol{\Sigma},h} + \|\underline{\boldsymbol{\varsigma}}_h^{k,l,m}\underline{I}_{U,h}^{k,l}(z - \breve{z}_h)\|_{\boldsymbol{\Sigma},h} \\
&\lesssim \|\underline{\boldsymbol{I}}_{\boldsymbol{\Sigma},T}^{k,l,m}(\nabla z - \nabla_h\breve{z}_h)\|_{\boldsymbol{\Sigma},h} + \|\underline{I}_{U,h}^{k,l}(z - \breve{z}_h)\|_{U,h} \lesssim h\|z\|_{H^2(\Omega)},
\end{aligned}$$

where the conclusion follows from the stability of the L^2-projector and the optimal approximation properties of $\breve{z}_h$.

For $\mathfrak{T}_3$, recalling the definitions (4.17) of m_h, (4.13a) of m_T, and (S2), we have

$$\begin{aligned}
\mathfrak{T}_3 &= \sum_{T\in\mathcal{T}_h}\left(\mathbf{S}_T^k(\underline{\widehat{\boldsymbol{\chi}}}_T - \underline{\boldsymbol{\varsigma}}_T^{k,l,m}\underline{\widehat{z}}_T), \mathbf{S}_T^k\underline{\widehat{\boldsymbol{\sigma}}}_T\right)_T \\
&= \sum_{T\in\mathcal{T}_h}\left(\mathbf{P}_T^k\underline{\widehat{\boldsymbol{\chi}}}_T - \nabla\breve{z}_T, \nabla\breve{u}_T\right)_T \\
&= \sum_{T\in\mathcal{T}_h}\left((\nabla(z - \breve{z}_T), \nabla\breve{u}_T)_T + \sum_{F\in\mathcal{F}_T}(\pi_F^k(\nabla z{\cdot}\boldsymbol{n}_{TF}) - \nabla z{\cdot}\boldsymbol{n}_{TF}, \breve{u}_T)_F\right) \\
&= \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_T}(\pi_F^k(\nabla z{\cdot}\boldsymbol{n}_{TF}) - \nabla z{\cdot}\boldsymbol{n}_{TF}, \breve{u}_T - u)_F,
\end{aligned}$$

where we have used the definition (4.8) of $\mathbf{S}_T^k$ together with the orthogonal decomposition (2.3) and the fact that $(\mathbf{S}_T^k \circ \underline{\boldsymbol{\varsigma}}_T^{k,l,m})\underline{\widehat{z}}_T = \mathbf{G}_T^k\underline{\widehat{z}}_T = \nabla\breve{z}_T$ (*cf.* (6.9) and (5.5)) to pass to the second line, the definition (4.7) of $\mathbf{P}_T^k$

(with $\underline{\boldsymbol{\tau}}_T = \underline{\widehat{\boldsymbol{\chi}}}_T$ and $w = \breve{u}_T$) together with the fact that $\mathrm{D}_T^l \underline{\widehat{\boldsymbol{\chi}}}_T = \triangle z$ and an integration by parts to pass to the third line, and concluded in the fourth line using the fact that $\breve{z}_T$ is a local elliptic projection to cancel the first term together with the fact that the quantity $(\pi_F^k(\nabla z \cdot \boldsymbol{n}_{TF}) - \nabla z \cdot \boldsymbol{n}_{TF})$ is single-valued on every interface $F \in \mathcal{F}^{\mathrm{i}}$ and $u = 0$ on all $F \in \mathcal{F}^{\mathrm{b}}$ to insert u into the second term. Using the Cauchy–Schwarz inequality and the optimal approximation properties of π_F^k and $\breve{u}_T$, we conclude

$$|\mathfrak{T}_3| \lesssim h^{k+2} \|u\|_{H^{k+2}(\Omega)} \|z\|_{H^2(\Omega)}. \tag{8.19}$$

Using (8.16)–(8.19) to estimate the right-hand side of (8.15) followed by the elliptic regularity (8.11) to bound $\|z\|_{H^2(\Omega)} \lesssim \|\widehat{u}_h - u_h\|$, the desired result follows. □

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