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CORIOLIS EFFECT ON WATER WAVES

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Abstract. This paper is devoted to the study of water waves under the influence of the gravity and the Coriolis force. It is quite common in the physical literature that the rotating shallow water equations are used to study such water waves. We prove a local wellposedness theorem for the water waves equations with vorticity and Coriolis force, taking into account the dependence on various physical parameters and we justify rigorously the shallow water model. We also consider a possible non constant pressure at the surface that can be used to describe meteorological disturbances such as storms or pressure jumps for instance.

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1. Introduction

1.1. Presentation of the problem

There has been a lot of interest on the Cauchy problem for the irrotational water waves problem since the work of Wu ([32,33]). More relevant for our present work is the Eulerian approach developed by Lannes ([17]) in the presence of a bottom. Another program initiated by Craig ([10]) consists in justifying the use of the many asymptotic models that exist in the physical literature to describe the motion of water waves. This requires a local wellposedness result that is uniform with respect to the small parameters involved (typically, the shallow water parameter). This was achieved by Alvarez−Samaniego and Lannes ([4]) for many regimes; other references in this direction are ([16,27,28]). The irrotational framework is however not always the relevant one when dealing with wave-current interactions or, at larger scales, if one wants to take into account the Coriolis force. The latter configuration motivates the present study. Several authors considered the local wellposedness theory for the water waves equations in the presence of vorticity ([9] or [20] for instance). Recently, Castro and Lannes proposed a generalization of the Zakharov−Craig−Sulem formulation (see [1, 11, 12, 34] for an explanation of this formulation), and gave a system of three equations that allow for the presence of vorticity. Then, they used it to derive new shallow water models that describe wave current interactions and more generally the coupling between waves and vorticity effects ([7, 8]). In this paper, we base our study on their formulation.

This paper is organized in three parts: firstly we derive a generalization of the Castro−Lannes formulation that takes into account the Coriolis forcing as well as non flat bottoms and a non constant pressure at the surface; secondly, we prove a local wellposedness result taking account the dependence of small parameters;

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Finally, we justify that the rotational shallow water model is a good asymptotic model of the rotational water waves equations under a Coriolis forcing.

We model the sea by an incompressible ideal fluid bounded from below by the seabed and from above by a free surface. We suppose that the seabed and the surface are graphs above the still water level. The pressure at the surface is of the form $P + P_{\text{ref}}$ where $P(t, \cdot)$ models a meteorological disturbance and P_{ref} is a constant which represents the pressure far from the meteorological disturbance. We denote by d the horizontal dimension, which is equal to 1 or 2. The horizontal variable is $X \in \mathbb{R}^d$ and $z \in \mathbb{R}$ is the vertical variable. H is the typical water depth. The water occupies the domain $\Omega_t := \{(X,z) \in \mathbb{R}^{d+1}, \ -H + b(X) < z < \zeta(t,X)\}$. The water is homogeneous (constant density ρ), inviscid with no surface tension. We denote by $\mathbf{U}$ the velocity of the fluid, $\mathbf{V}$ is the horizontal component of the velocity and $\mathbf{w}$ its vertical component. The water is under the influence of the gravity $\boldsymbol{g} = -g\boldsymbol{e_z}$ and the rotation of the Earth with a rotation vector $\mathbf{f} = \frac{f}{2}\boldsymbol{e_z}$. We assume that we are under the f-plane approximation which means that f is set to a constant value. Finally, we define the pressure in the fluid domain by $\mathcal{P}$. The equations governing the motion of the surface of an ideal fluid under the influence of gravity and Coriolis force are the free surface Euler Coriolis equations [2]

$$\begin{cases} \partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla_{X,z})\, \mathbf{U} + 2\mathbf{f} \times \mathbf{U} = -\frac{1}{\rho} \nabla_{X,z} \mathcal{P} - g\boldsymbol{e_z} \quad \text{in} \quad \Omega_t, \\ \operatorname{div} \mathbf{U} = 0 \quad \text{in} \quad \Omega_t, \end{cases} \tag{1.1}$$

with the boundary conditions

$$\begin{cases} \partial_t \zeta - \underline{\mathrm{U}} \cdot N = 0, \\ \mathrm{U}_b \cdot N_b = 0, \end{cases} \tag{1.2}$$

where $N = \begin{pmatrix} -\nabla\zeta \\ 1 \end{pmatrix}$, $N_b = \begin{pmatrix} -\nabla b \\ 1 \end{pmatrix}$, $\underline{\mathrm{U}} = \begin{pmatrix} \underline{\mathrm{V}} \\ \underline{\mathrm{w}} \end{pmatrix} = \mathbf{U}_{|z=\zeta}$ and $\mathrm{U}_b = \begin{pmatrix} \mathrm{V}_b \\ \mathrm{w}_b \end{pmatrix} = \mathbf{U}_{|z=-H+b}$.

The pressure $\mathcal{P}$ can be decomposed as the surface contribution and the internal pressure

$$\mathcal{P}(t,X,z) = P(t,X) + P_{\text{ref}} + \widetilde{\mathcal{P}}(t,X,z),$$

with $\widetilde{\mathcal{P}}_{|z=\zeta} = 0$.

Remark 1.1. In this paper, we identify functions on $\mathbb{R}^2$ as function on $\mathbb{R}^3$. Then, the gradient, the curl and the divergence operators become in the two dimensional case

$$\nabla_{X,z} f = \begin{pmatrix} \partial_x f \\ 0 \\ \partial_z f \end{pmatrix}, \ \operatorname{curl} \mathbf{A} = \begin{pmatrix} -\partial_z \mathbf{A}_2 \\ \partial_z \mathbf{A}_1 - \partial_x \mathbf{A}_3 \\ -\partial_x \mathbf{A}_2 \end{pmatrix}, \ \operatorname{div} \mathbf{A} = \partial_x \mathbf{A}_1 + \partial_z \mathbf{A}_3.$$

In order to obtain some asymptotic models we nondimensionalize the previous equations. There are five important physical parameters: the typical amplitude of the surface a, the typical amplitude of the bathymetry a_{bott}, the typical horizontal scale L, the characteristic water depth H and the typical Coriolis frequency f. Then we can introduce four dimensionless parameters

$$\varepsilon = \frac{a}{H}, \ \beta = \frac{a_{\text{bott}}}{H}, \ \mu = \frac{H^2}{L^2} \qquad \text{and} \qquad \mathrm{Ro} = \frac{a}{fL}\sqrt{\frac{g}{H}}, \tag{1.3}$$

[2] We consider that the centrifugal potential is constant and included in the pressure term.

where ε is called the nonlinearity parameter, β the bathymetric parameter, μ the shallowness parameter and Ro the Rossby number. We also nondimensionalize the variables and the unknowns. We introduce (see [19, 23] for instance for an explanation of this nondimensionalization)

$$\begin{cases} X' = \dfrac{X}{L},\ z' = \dfrac{z}{H},\ \zeta' = \dfrac{\zeta}{a},\ b' = \dfrac{b}{a_{\text{bott}}},\ t' = \dfrac{\sqrt{gH}}{L}t, \\ \mathbf{V}' = \sqrt{\dfrac{H}{g}}\dfrac{\mathbf{V}}{a},\ \mathbf{w}' = H\sqrt{\dfrac{H}{g}}\dfrac{\mathbf{w}}{aL},\ P' = \dfrac{P}{\rho g a} \quad \text{and} \quad \widetilde{\mathcal{P}}' = \dfrac{\widetilde{\mathcal{P}}}{\rho g H}. \end{cases} \tag{1.4}$$

In this paper, we use the following notations

$$\nabla^{\mu}_{X',z'} = \begin{pmatrix} \sqrt{\mu}\nabla_{X'} \\ \partial_{z'} \end{pmatrix},\ \text{curl}^{\mu} = \nabla^{\mu}_{X',z'} \times,\ \text{div}^{\mu} = \nabla^{\mu}_{X',z'} \cdot. \tag{1.5}$$

We also define

$$\mathbf{U}^{\mu} = \begin{pmatrix} \sqrt{\mu}\mathbf{V}' \\ \mathbf{w}' \end{pmatrix},\ \boldsymbol{\omega}' = \frac{1}{\mu}\text{curl}^{\mu}\mathbf{U}^{\mu},\ \underline{\text{U}}^{\mu} = \mathbf{U}^{\mu}_{|z'=\varepsilon\zeta'},\ \text{U}^{\mu}_{b} = \mathbf{U}^{\mu}_{|z'=-1+\beta b'}, \tag{1.6}$$

and

$$N^{\mu} = \begin{pmatrix} -\varepsilon\sqrt{\mu}\nabla\zeta' \\ 1 \end{pmatrix},\ N^{\mu}_{b} = \begin{pmatrix} -\beta\sqrt{\mu}\nabla b' \\ 1 \end{pmatrix}. \tag{1.7}$$

Notice that our nondimensionalization of the vorticity allows us to consider only weakly sheared flows (see [7, 26, 30]). The nondimensionalized fluid domain is

$$\Omega'_{t'} := \{(X', z') \in \mathbb{R}^{d+1},\ -1 + \beta b'(X') < z' < \varepsilon\zeta'(t', X')\}. \tag{1.8}$$

Finally, if $\mathbf{V} = \begin{pmatrix} \text{V}_1 \\ \text{V}_2 \end{pmatrix} \in \mathbb{R}^2$, we define $\mathbf{V}$ by $\mathbf{V}^{\perp} = \begin{pmatrix} -\text{V}_2 \\ \text{V}_1 \end{pmatrix}$. Then, the Euler Coriolis equations (1.1) become

$$\begin{cases} \partial_{t'}\mathbf{U}^{\mu} + \dfrac{\varepsilon}{\mu}\left(\mathbf{U}^{\mu} \cdot \nabla^{\mu}_{X',z'}\right)\mathbf{U}^{\mu} + \dfrac{\varepsilon\sqrt{\mu}}{\text{Ro}}\begin{pmatrix} \mathbf{V}'^{\perp} \\ 0 \end{pmatrix} = -\sqrt{\mu}\begin{pmatrix} \nabla P' \\ 0 \end{pmatrix} - \dfrac{1}{\varepsilon}\nabla^{\mu}_{X',z'}\widetilde{\mathcal{P}}' - \dfrac{1}{\varepsilon}\boldsymbol{e_z} \quad \text{in} \quad \Omega'_t, \\ \text{div}^{\mu}_{X',z'}\,\mathbf{U}^{\mu} = 0 \quad \text{in} \quad \Omega'_t, \end{cases} \tag{1.9}$$

with the boundary conditions

$$\begin{cases} \partial_{t'}\zeta' - \dfrac{1}{\mu}\underline{\text{U}}^{\mu} \cdot N^{\mu} = 0, \\ \text{U}^{\mu}_{b} \cdot N^{\mu}_{b} = 0. \end{cases} \tag{1.10}$$

In the following we omit the primes. In [8], Castro and Lannes derived a formulation of the water waves equations with vorticity. We outline the main ideas of this formulation and extend it to take into account the Coriolis force; even in absence of Coriolis forcing, our results extend the result of [8] by allowing non flat bottoms. First, applying the curl^{μ} operator to the first equation of (1.9) we obtain an equation on $\boldsymbol{\omega}$

$$\partial_t\boldsymbol{\omega} + \frac{\varepsilon}{\mu}\left(\mathbf{U}^{\mu} \cdot \nabla^{\mu}_{X,z}\right)\boldsymbol{\omega} = \frac{\varepsilon}{\mu}\boldsymbol{\omega} \cdot {\nabla^{\mu}}_{X,z}\mathbf{U}^{\mu} + \frac{\varepsilon}{\mu\text{Ro}}\partial_z\mathbf{U}^{\mu}. \tag{1.11}$$

Furthermore, taking the trace at the surface of the first equation of (1.9) we get

$$\partial_t\underline{\text{U}}^{\mu} + \varepsilon\left(\underline{\text{V}} \cdot \nabla_X\right)\underline{\text{U}}^{\mu} + \frac{\varepsilon\sqrt{\mu}}{\text{Ro}}\begin{pmatrix} \underline{\text{V}}^{\perp} \\ 0 \end{pmatrix} = -\sqrt{\mu}\begin{pmatrix} \nabla P \\ 0 \end{pmatrix} - \frac{1}{\varepsilon}\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{\varepsilon}\left(\partial_z\widetilde{\mathcal{P}}\right)_{|z=\varepsilon\zeta}N^{\mu}. \tag{1.12}$$

Then, in order to eliminate the term $(\partial_z \mathcal{P})_{|z=\varepsilon\zeta}\, N^\mu$, we have to introduce the following quantity. If $\mathbf{A}$ is a vector field on Ω_t, we define $\mathrm{A}_{/\!/}$ as

$$\mathrm{A}_{/\!/} = \frac{1}{\sqrt{\mu}}\underline{\mathrm{A}}_h + \varepsilon \underline{\mathrm{A}}_v \nabla\zeta,$$

where A_h is the horizontal component of A, A_v its vertical component, $\underline{\mathrm{A}} = \mathbf{A}_{|z=\varepsilon\zeta}$ and $\mathrm{A}_b = \mathbf{A}_{|z=-1+\beta b}$. Notice that,

$$\underline{\mathrm{A}} \times N^\mu = \sqrt{\mu}\begin{pmatrix} -\mathrm{A}_{/\!/}^{\perp} \\ -\varepsilon\sqrt{\mu}\mathrm{A}_{/\!/}^{\perp}\cdot\nabla\zeta \end{pmatrix}. \tag{1.13}$$

Therefore, taking the orthogonal of the horizontal component of the vectorial product of (1.12) with N^μ we obtain

$$\partial_t \mathrm{U}_{/\!/}^\mu + \nabla\zeta + \frac{\varepsilon}{2}\nabla\left|\mathrm{U}_{/\!/}^\mu\right|^2 - \frac{\varepsilon}{2\mu}\nabla\left[\left(1+\varepsilon^2\mu\,|\nabla\zeta|^2\right)\underline{\mathrm{w}}^2\right] + \varepsilon\left(\underline{\omega}\cdot N^\mu + \frac{1}{\mathrm{Ro}}\right)\underline{\mathrm{V}}^\perp = -\nabla P. \tag{1.14}$$

Since $\mathrm{U}_{/\!/}^\mu$ is a vector field on $\mathbb{R}^2$, we have the classical Hodge−Weyl decomposition

$$\mathrm{U}_{/\!/}^\mu = \nabla\frac{\nabla}{\Delta}\cdot\mathrm{U}_{/\!/}^\mu + \nabla^\perp\frac{\nabla^\perp}{\Delta}\cdot\mathrm{U}_{/\!/}^\mu. \tag{1.15}$$

In the following we denote by $\psi := \frac{\nabla}{\Delta}\cdot\mathrm{U}_{/\!/}^\mu$ and $\widetilde{\psi} := \frac{\nabla^\perp}{\Delta}\cdot\mathrm{U}_{/\!/}^\mu$ [3]. Applying the operator $\frac{\nabla}{\Delta}\cdot$ to (1.14), we obtain

$$\partial_t\psi + \zeta + \frac{\varepsilon}{2}\left|\mathrm{U}_{/\!/}^\mu\right|^2 - \frac{\varepsilon}{2\mu}\left(1+\varepsilon^2\mu\,|\nabla\zeta|^2\right)\underline{\mathrm{w}}^2 + \varepsilon\frac{\nabla}{\Delta}\cdot\left[\left(\underline{\omega}\cdot N^\mu + \frac{1}{\mathrm{Ro}}\right)\underline{\mathrm{V}}^\perp\right] = -P. \tag{1.16}$$

Moreover, using the following vectorial identity

$$\left(\nabla_{X,z}^\mu \times \mathbf{U}^\mu\right)_{|z=\varepsilon\zeta}\cdot N^\mu = \mu\nabla^\perp\cdot\mathrm{U}_{/\!/}^\mu, \tag{1.17}$$

we have

$$\Delta\widetilde{\psi} = \left(\underline{\omega}\cdot N^\mu\right). \tag{1.18}$$

We can now give the nondimensionalized Castro−Lannes formulation of the water waves equations with vorticity in the presence of Coriolis forcing. It is a system of three equations for the unknowns $(\zeta, \psi, \boldsymbol{\omega})$

$$\begin{cases} \partial_t\zeta - \dfrac{1}{\mu}\underline{\mathrm{U}}^\mu\cdot N^\mu = 0, \\ \partial_t\psi + \zeta + \dfrac{\varepsilon}{2}\left|\mathrm{U}_{/\!/}^\mu\right|^2 - \dfrac{\varepsilon}{2\mu}\left(1+\varepsilon^2\mu\,|\nabla\zeta|^2\right)\underline{\mathrm{w}}^2 + \varepsilon\dfrac{\nabla}{\Delta}\cdot\left[\left(\underline{\omega}\cdot N^\mu + \dfrac{1}{\mathrm{Ro}}\right)\underline{\mathrm{V}}^\perp\right] = -P, \\ \partial_t\boldsymbol{\omega} + \dfrac{\varepsilon}{\mu}\left(\mathbf{U}^\mu\cdot\nabla_{X,z}^\mu\right)\boldsymbol{\omega} = \dfrac{\varepsilon}{\mu}\left(\boldsymbol{\omega}\cdot\nabla_{X,z}^\mu\right)\mathbf{U}^\mu + \dfrac{\varepsilon}{\mu\mathrm{Ro}}\partial_z\mathbf{U}^\mu, \end{cases} \tag{1.19}$$

where $\mathbf{U}^\mu := \mathbf{U}^\mu[\varepsilon\zeta, \beta b](\psi, \boldsymbol{\omega})$ is the unique solution in $H^1(\Omega_t)$ of

$$\begin{cases} \mathrm{curl}^\mu\,\mathbf{U}^\mu = \mu\boldsymbol{\omega} \quad \text{in} \quad \Omega_t, \\ \mathrm{div}^\mu\,\mathbf{U}^\mu = 0 \quad \text{in} \quad \Omega_t, \\ \mathrm{U}_{/\!/}^\mu = \nabla\psi + \dfrac{\nabla^\perp}{\Delta}\left(\underline{\omega}\cdot N^\mu\right), \\ \mathrm{U}_b^\mu\cdot N_b^\mu = 0. \end{cases} \tag{1.20}$$

[3]We define rigorously these operators in the next section.

We add a technical assumption. We assume that the water depth is bounded from below by a positive constant

$$\exists\, h_{\min} > 0,\ \varepsilon\zeta + 1 - \beta b \geq h_{\min}. \tag{1.21}$$

We also suppose that the dimensionless parameters satisfy

$$\exists \mu_{\max},\ 0 < \mu \leq \mu_{\max},\ 0 < \varepsilon \leq 1,\ 0 < \beta \leq 1 \text{ and } \frac{\varepsilon}{\mathrm{Ro}} \leq 1. \tag{1.22}$$

Remark 1.2. The assumption $\varepsilon \leq \mathrm{Ro}$ is equivalent to $fL \leq \sqrt{gH}$. This means that the typical rotation speed due to the Coriolis force is less than the typical water wave celerity. For water waves, this assumption is common (see for instance [25]). Typically for offshore long water waves at mid-latitudes, we have $L = 100$km and $H = 1$km and $f = 10^{-4}$Hz. Then, $\frac{\varepsilon}{\mathrm{Ro}} = 10^{-1}$.

1.2. Notations

- If $\mathbf{A} \in \mathbb{R}^3$, we denote by $\mathbf{A}_h$ its horizontal component and by $\mathbf{A}_v$ its vertical component.
- If $\mathbf{V} = \begin{pmatrix} \mathrm{V}_1 \\ \mathrm{V}_2 \end{pmatrix} \in \mathbb{R}^2$, we define the orthogonal of $\mathbf{V}$ by $\mathbf{V}^{\perp} = \begin{pmatrix} -\mathrm{V}_2 \\ \mathrm{V}_1 \end{pmatrix}$.
- In this paper, $C(\cdot)$ is a nondecreasing and positive function whose exact value has no importance.
- Consider a vector field $\mathbf{A}$ or a function $\mathbf{w}$ defined on Ω. Then, we denote $\mathrm{A} = \mathbf{A} \circ \Sigma$ and $\mathrm{w} = \mathbf{w} \circ \Sigma$, where Σ is defined in (2.16). Furthermore, we denote $\underline{\mathrm{A}} = \mathbf{A}_{|z=\varepsilon\zeta} = \mathrm{A}_{|z=0}$, $\underline{\mathrm{w}} = \mathbf{w}_{|z=\varepsilon\zeta} = \mathrm{w}_{|z=0}$ and $\mathrm{A}_b = \mathbf{A}_{|z=-1+\beta b} = \mathrm{A}_{|z=-1}$, $\mathrm{w}_b = \mathbf{w}_{|z=-1+b} = \mathrm{w}_{|z=-1}$.
  item[−]If $s \in \mathbb{R}$ and f is a function on $\mathbb{R}^d$, $|f|_{H^s}$ is its H^s-norm and $|f|_2$ is its L^2-norm. The quantity $|f|_{W^{k,\infty}}$ is $W^{k,\infty}(\mathbb{R}^d)$-norm of f, where $k \in \mathbb{N}^*$, and $|f|_{L^\infty}$ its $L^\infty(\mathbb{R}^d)$-norm.
- The operator $(\ ,\ )$ is the L^2-scalar product in $\mathbb{R}^d$.
- If $N \in \mathbb{N}^*$, $\mathbf{A}$ is defined on Ω and $\mathrm{A} = \mathbf{A} \circ \Sigma$, $||\mathrm{A}||_{H^N}$ and $||\mathbf{A}||_{H^N}$ are respectively the $H^N(\mathcal{S})$-norm of A and the $H^N(\Omega)$-norm of $\mathbf{A}$. The L^p-norm are denoted $||\cdot||_p$.
- -The norm $||\cdot||_{H^{s,k}}$ is defined in Definition 2.10.
- The space $H^s_*(\mathbb{R}^d)$, $\dot{H}^s(\mathbb{R}^d)$ and $H_b(\mathrm{div}^\mu_0, \Omega)$ are defined in Section 2.1.
- If f is a function defined on $\mathbb{R}^d$, we denote ∇f the gradient of f.
- If $\mathbf{w}$ is a function defined on Ω, $\nabla_{X,z}\mathbf{w}$ is the gradient of $\mathbf{w}$ and $\nabla_X \mathbf{w}$ its horizontal component. We have the same definition for functions defined on $\mathcal{S}$.
- The operators $|D|$, $\mathfrak{P}$ and Λ are Fourier multipliers in $\mathcal{S}'\left(\mathbb{R}^d\right)$ defined by

$$\widehat{|D|\, u}(\xi) = |\xi|\, \widehat{u}(\xi)\ ,\ \mathfrak{P} = \frac{|D|}{\sqrt{1+\sqrt{\mu}\,|D|}} \text{ and } \Lambda = \sqrt{1+|D|^2}.$$

- In the following M_N is a constant of the form

$$M_N = C\left(\mu_{\max}, \frac{1}{h_{\min}}, \varepsilon\,|\zeta|_{H^N}, \beta\,|\nabla b|_{H^N}, \beta\,|b|_{L^\infty}\right). \tag{1.23}$$

1.3. Existence result for the water waves equations

The main result of this article is Theorem 3.6. It is a wellposedness result for the system (3.2) which is a straightened system of the Castro−Lannes formulation (1.19). This result extends Theorems 4.7 and 5.1 in [8] by adding a non flat bottom and a Coriolis forcing. From Theorem 3.6, we get the following wellposedness result for System (1.19).

Theorem 1.3. *Assume that N is an integer large enough, that the initial data, b and P are smooth enough and that the initial vorticity is divergence free. Assume also that conditions* (1.21) *and* (3.10) *are satisfied initially. Then, there exists $T > 0$, and a unique solution to the water waves equations* (1.19) *on* $[0, T]$. *Moreover,*

$$T = \min\left(\frac{T_0}{\max(\varepsilon, \beta, \frac{\varepsilon}{Ro})}, \frac{T_0}{|\nabla P|_{L^\infty_t H^N_X}}\right) \text{ with } \frac{1}{T_0} = c^1,\ \sup_{t\in[0,T]} |\zeta(t)|_{H^N} + |\mathfrak{P}\psi(t)|_{H^{N-\frac{1}{2}}} + ||\omega(t)||_{H^{N-1}(\Omega_t)} = c^2,$$

where c^j is are constants which depend on the initial conditions, P and b.

This theorem allows us to investigate the justification of asymptotic models in the presence of a Coriolis forcing. In the case of a constant pressure at the surface and without a Coriolis forcing, our existence time is similar to Theorem 3.16 in [19] (see also [4]); without a Coriolis forcing, it is as Theorem 2.3 in [23]. Notice finally that condition (3.10) corresponds to the Rayleigh−Taylor criterion (see for instance [19]). Ebin [15] showed that if this condition is not satisfied, the water waves equations are illposed.

2. The div-curl problem

In [8], A. Castro and D. Lannes study the system (1.20) in the case of a flat bottom ($b = 0$). The purpose of this part is to extend their results in the case of a non flat bottom.

2.1. Functional analysis framework

In this paper, we use the Beppo−Levi spaces (see [14])

$$\forall s \geq 0,\ \dot{H}^s(\mathbb{R}^d) = \left\{ f \in L^2_{\mathrm{loc}}(\mathbb{R}^d),\ \nabla f \in H^{s-1}(\mathbb{R}^d) \right\} \text{ and } |\cdot|_{\dot{H}^s} = |\nabla \cdot|_{H^{s-1}}.$$

The dual space of $\dot{H}^s(\mathbb{R}^d)/_{\mathbb{R}}$ is the space (see [6])

$$H^{-s}_*(\mathbb{R}^d) = \left\{ u \in H^{-s}(\mathbb{R}^d),\ \exists v \in H^{-s+1}(\mathbb{R}^d),\ u = |D|\, v \right\} \text{ and } |\cdot|_{H^{-s}_*} = \left| \frac{\cdot}{|D|} \right|_{H^{-s+1}}.$$

Notice that $\dot{H}^1(\mathbb{R}^d)/_{\mathbb{R}}$ is a Hilbert space. Then, we can rigorously define the Hodge−Weyl decomposition and the operators $\frac{\nabla}{\Delta}\cdot$ and $\frac{\nabla^\perp}{\Delta}\cdot$. For $f \in L^2(\mathbb{R}^d)^d$, $u = \frac{\nabla}{\Delta} \cdot f$ is defined as the unique solution, up to a constant, in $\dot{H}^1(\mathbb{R}^d)$ of the variational problem

$$\int_{\mathbb{R}^d} \nabla u \cdot \nabla \phi = \int_{\mathbb{R}^d} f \cdot \nabla \phi,\ \forall \phi \in \dot{H}^1(\mathbb{R}^d).$$

The operator $\frac{\nabla^\perp}{\Delta}\cdot$ can be defined similarly. Then, it is easy to check that the operators $\frac{\nabla^\perp}{\Delta}\cdot$ and $\frac{\nabla^\perp}{\Delta}\cdot$ belong to $\mathcal{L}(H^s(\mathbb{R}^d)^d, \dot{H}^{s+1}(\mathbb{R}^d))$, for all $s \geq 0$.

Let Ω be a domain in $\mathbb{R}^{d+1}$ with a Lipschitz boundary. The subspace of $L^2(\Omega)^3$ of functions whose curl is in $L^2(\Omega)^3$ is the space

$$H(\mathrm{curl}^\mu, \Omega) = \left\{ \mathbf{A} \in L^2(\Omega)^3,\ \mathrm{curl}^\mu \mathbf{A} \in L^2(\Omega)^3 \right\}.$$

The subspace of $L^2(\Omega)^3$ of divergence free vector fields is the space

$$H(\mathrm{div}^\mu_0, \Omega) = \left\{ \mathbf{A} \in L^2(\Omega)^3,\ \mathrm{div}^\mu \mathbf{A} = 0 \right\}.$$

Remark 2.1. Notice that $\mathbf{A} \in H(\mathrm{div}^\mu_0, \Omega)$ *implies that* $\left(\mathbf{A}_{|\partial\Omega} \cdot n\right)$ *belongs to* $H^{-\frac{1}{2}}(\partial\Omega)$ *and* $\mathbf{A} \in H(\mathrm{curl}^\mu, \Omega)$ *implies that* $\left(\mathbf{A}_{|\partial\Omega} \times n\right)$ *belongs to* $H^{-\frac{1}{2}}(\partial\Omega)$ *(see* [13]*).*

Finally, we define $H_b(\mathrm{div}^\mu_0, \Omega)$ as

$$H_b(\mathrm{div}^\mu_0, \Omega) = \left\{ \mathbf{A} \in H(\mathrm{div}^\mu_0, \Omega),\ A_b \cdot N^\mu_b \in H^{-\frac{1}{2}}_*(\mathbb{R}^d) \right\}.$$

Remark 2.2. We have a similar equation to (1.17) at the bottom

$$\frac{1}{\mu} \left(\nabla^\mu_{X,z} \times \mathbf{U}^\mu \right)_{|z=-1+\beta b} \cdot N^\mu_b = \nabla^\perp \cdot \left(\mathbf{V}_b + \beta \mathbf{w}_b \nabla b \right),$$

hence, in the following, we suppose that $\boldsymbol{\omega} \in H_b(\mathrm{div}^\mu_0, \Omega)$ *.*

It is important to notice that, if $\boldsymbol{\omega} \in H_b(\mathrm{div}^\mu_0, \Omega)$, the quantity $\frac{1}{\mathfrak{P}}(\omega_b \cdot N^\mu_b)$ makes sense and belongs to $L^2(\mathbb{R}^d)$.

2.2. Existence and uniqueness for the div-curl problem

In this part, we forget the dependence on t. First, notice that we can split the problem into two parts. Let $\Phi \in \dot{H}^2(\Omega)$ the unique solution of the Laplace problem (see [19])

$$\begin{cases} \Delta^{\mu}_{X,z}\Phi = 0 \quad \text{in} \quad \Omega, \\ \Phi_{|z=\varepsilon\zeta} = \psi, \ \left(N^{\mu}_b \cdot \nabla^{\mu}_{X,z}\Phi\right)_{|z=-1+\beta b} = 0. \end{cases} \tag{2.1}$$

Using the vectorial identity

$$\left(\nabla^{\mu}_{X,z}\Phi\right)_{/\!/} = \nabla\psi,$$

it is easy to check that if $\mathbf{U}^{\mu}$ satisfies (1.20), $\widetilde{\mathbf{U}}^{\mu} := \mathbf{U}^{\mu} - \nabla^{\mu}_{X,z}\Phi$ satisfies

$$\begin{cases} \operatorname{curl}^{\mu} \widetilde{\mathbf{U}}^{\mu} = \mu\,\boldsymbol{\omega} \quad \text{in} \quad \Omega_t, \\ \operatorname{div}^{\mu} \widetilde{\mathbf{U}}^{\mu} = 0 \quad \text{in} \quad \Omega_t, \\ \widetilde{\mathrm{U}}^{\mu}_{/\!/} = \dfrac{\nabla^{\perp}}{\Delta}\left(\underline{\boldsymbol{\omega}} \cdot N^{\mu}\right) \quad \text{at the surface}, \\ \widetilde{\mathbf{U}}^{\mu}_b \cdot N^{\mu}_b = 0 \quad \text{at the bottom}. \end{cases} \tag{2.2}$$

In the following we focus on the system (2.2). We give 4 intermediate results in order to get the existence and uniqueness. The first Proposition shows how to control the norm of the gradient of a function with boundary condition as in (2.2).

Proposition 2.3. *Let $\zeta, b \in W^{2,\infty}(\mathbb{R}^d)$, $\mathbf{A} \in H(\operatorname{div}^{\mu}_0, \Omega) \cap H(\operatorname{curl}^{\mu}, \Omega)$. Then, for all $C \in H^1(\mathbb{R}^d)^3$, we have*

$$\int_{\Omega} \nabla^{\mu}_{X,z}\mathbf{A} : \nabla^{\mu}_{X,z}\mathbf{C} = \int_{\Omega} \operatorname{curl}^{\mu}\mathbf{A} : \operatorname{curl}^{\mu}\mathbf{C} + \langle l^{\mu}[\varepsilon\zeta](\underline{\mathrm{A}}), \underline{C}\rangle_{H^{-\frac{1}{2}}-H^{\frac{1}{2}}} - \langle l^{\mu}[\beta b](\mathrm{A}_b), C_b\rangle_{H^{-\frac{1}{2}}-H^{\frac{1}{2}}}, \tag{2.3}$$

where for $\mathrm{B} \in H^{\frac{1}{2}}\left(\mathbb{R}^2\right)^3$ and for $\eta \in W^{2,\infty}(\mathbb{R}^d)$,

$$l^{\mu}[\eta](\mathrm{B}) = \begin{pmatrix} \sqrt{\mu}\nabla\mathrm{B}_v - \mu\left(\nabla^{\perp}\eta \cdot \nabla\right)\mathrm{B}^{\perp}_h \\ -\sqrt{\mu}\nabla \cdot \mathrm{B}_h \end{pmatrix}. \tag{2.4}$$

Furthermore, if $\widetilde{\psi} \in \dot{H}^{\frac{3}{2}}(\mathbb{R}^d)$ and

$$A_b \cdot N^{\mu}_b = 0 \qquad \text{and} \qquad \mathrm{A}_{/\!/} = \nabla^{\perp}\widetilde{\psi},$$

we have the following estimate

$$\begin{aligned} \left|\left|\nabla^{\mu}_{X,z}\boldsymbol{A}\right|\right|_2^2 \leq{}& ||\operatorname{curl}^{\mu}\boldsymbol{A}||_2^2 + \mu C\left(\varepsilon\left|\zeta\right|_{W^{2,\infty}}, \beta\left|b\right|_{W^{2,\infty}}\right)\left(|\underline{A}|_2^2 + |A_{bh}|_2^2\right) \\ &+ \mu C\left(\mu_{\max}, \varepsilon\left|\zeta\right|_{W^{2,\infty}}, \beta\left|b\right|_{W^{2,\infty}}\right)\left|\sqrt{1+\sqrt{\mu}|D|}\nabla\widetilde{\psi}\right|_2\left|\sqrt{1+\sqrt{\mu}\,|D|}\underline{A}_h\right|_2. \end{aligned} \tag{2.5}$$

Proof. Using the Einstein summation convention and denoting $\nabla^{\mu}_{X,z} = (\partial^{\mu}_1, \partial^{\mu}_2, \partial^{\mu}_3)^T$, a simple computation gives (see Lem. 3.2 in [8] or Chap. 9 in [13]),

$$||\nabla^{\mu}\mathbf{A}||_2^2 = ||\operatorname{curl}^{\mu}\mathbf{A}||_2^2 + ||\operatorname{div}^{\mu}\mathbf{A}||_2^2 + \int_{\partial\Omega} n^{\mu}_i \mathbf{A}_j \partial^{\mu}_j \mathbf{A}_i - n^{\mu}_j \mathbf{A}_j \partial^{\mu}_i \mathbf{A}_i. \tag{2.6}$$

In this case, $\partial\Omega$ is the union of two surfaces and $\overrightarrow{n^\mu} = \pm \begin{pmatrix} -\sqrt{\mu}\nabla\eta \\ 1 \end{pmatrix}$, where η is the corresponding surface. Then, one can check that (see also Lem. 3.8 in [8]),

$$\int_{\{z=\eta\}} n_i^\mu \mathbf{A}_j \partial_j^\mu \mathbf{A}_i - n_j^\mu \mathbf{A}_j \partial_i^\mu \mathbf{A}_i = \pm \int_{\mathbb{R}^d} \mathbf{A}_{\eta,h} \cdot \left(2\sqrt{\mu}\nabla \mathbf{A}_{\eta,v} - \mu \left(\nabla\eta^\perp \cdot \nabla \right) \mathbf{A}_{\eta,h}^\perp \right), \tag{2.7}$$

where $\mathbf{A}_\eta := \mathbf{A}_{|z=\eta}$. The first part of the Proposition follows by polarization of equations (2.6) and (2.7) (as quadratic forms). For the second estimate, since $\mathrm{A}_b \cdot N_b^\mu = 0$, we get at the bottom that

$$\begin{aligned}\int_{\{z=-1+\beta b\}} n_i^\mu \mathbf{A}_j \partial_j^\mu \mathbf{A}_i - n_j^\mu \mathbf{A}_j \partial_i^\mu \mathbf{A}_i &= -2 \int_{\mathbb{R}^d} \mu\beta \big(\partial_x b \mathrm{A}_{bx} \partial_y \mathrm{A}_{by} + \partial_y b \mathrm{A}_{by} \partial_x \mathrm{A}_{bx} + \partial_{xy}^2 b \mathrm{A}_{bx}\mathrm{A}_{by}\big) - \mathrm{A}_z \sqrt{\mu} \mathrm{div}_X \mathrm{A}_{bh} \\ &= -\mu\beta \int_{\mathbb{R}^d} \partial_x^2 b\, \mathrm{A}_{bx}^2 + \partial_y^2 b\, \mathrm{A}_{by}^2 + 2\partial_{xy}^2 b\, \mathrm{A}_{bx}\mathrm{A}_{by}.\end{aligned}$$

At the surface, since $\underline{\mathrm{A}}_h = \sqrt{\mu}\nabla^\perp \widetilde{\psi} - \varepsilon\sqrt{\mu}\underline{\mathrm{A}}_v \nabla\zeta$, we have

$$\begin{aligned}\int_{\{z=\varepsilon\zeta\}} n_i^\mu \mathbf{A}_j \partial_j^\mu \mathbf{A}_i - n_j^\mu \mathbf{A}_j \partial_i^\mu \mathbf{A}_i &= -2 \int_{\mathbb{R}^d} \varepsilon\mu \big(\partial_x \zeta \underline{\mathrm{A}}_y \partial_y \underline{\mathrm{A}}_x + \partial_y \zeta \underline{\mathrm{A}}_x \partial_x \underline{\mathrm{A}}_y + \partial_{xy}^2 \zeta \underline{\mathrm{A}}_x \underline{\mathrm{A}}_y\big) + \sqrt{\mu} \left(\underline{\mathrm{A}}_h \cdot \nabla_X \right) \underline{\mathrm{A}}_z \\ &= \varepsilon\mu \int_{\mathbb{R}^d} \underline{\mathrm{A}}_x^2 \partial_y^2 \zeta + \underline{\mathrm{A}}_y^2 \partial_x^2 \zeta - 2\underline{\mathrm{A}}_x \underline{\mathrm{A}}_y \partial_{xy}^2 \zeta + \underline{\mathrm{A}}_z^2 \left[\partial_x^2 \zeta + \partial_y^2 \zeta \right] \\ &\quad - 2\varepsilon\mu^{\frac{3}{2}} \int_{\mathbb{R}^d} \underline{\mathrm{A}}_h \cdot \nabla^\perp \left(\nabla\widetilde{\psi} \cdot \nabla\zeta \right).\end{aligned}$$

Then,

$$\left| 2\varepsilon\mu^{\frac{3}{2}} \int_{\mathbb{R}^d} \underline{\mathrm{A}}_h \cdot \nabla^\perp \Big(\nabla\widetilde{\psi} \cdot \nabla\zeta \Big) \right| \leq \varepsilon\mu \left| \sqrt{1+\sqrt{\mu}|D|}\, \underline{\mathrm{A}}_h \right|_2 \left| \sqrt{\mu}\mathfrak{P} \left(\nabla\widetilde{\psi} \cdot \nabla\zeta \right) \right|_2 .$$

and estimate (2.5) follows easily from Lemma A.1. □

The second Proposition gives a control of the L^2-norm of the trace.

Proposition 2.4. *Let* $\zeta,\, b \in W^{1,\infty}\left(\mathbb{R}^d\right)$, $\mathbf{A} \in H(\mathrm{div}_0^\mu, \Omega) \cap H(\mathrm{curl}^\mu, \Omega)$ *and* $\widetilde{\psi} \in \dot{H}^1\left(\mathbb{R}^d\right)$ *such that*

$$\mathrm{A}_b \cdot N_b^\mu = 0 \text{ and } \mathrm{A}_{/\!/} = \nabla^\perp \widetilde{\psi}.$$

Then,

$$|\underline{\mathrm{A}}|_2^2 + |\mathrm{A}_b|_2^2 \leq \left(\mu \left| \nabla\widetilde{\psi} \right|_2^2 + ||\mathrm{curl}^{\,\mu} \mathbf{A}||_2 \, ||\mathbf{A}||_2 \right) C. \tag{2.8}$$

Proof. Using the fact that $\partial_z \mathbf{A}_h = -\left(\mathrm{curl}^{\,\mu} \mathbf{A}\right)_h^\perp + \sqrt{\mu}\nabla_X \mathbf{A}_v$, we have

$$\begin{aligned}\int_{\mathbb{R}^d} |\underline{\mathrm{A}}_h|^2 &= \int_{\mathbb{R}^d} |\mathrm{A}_{bh}|^2 + 2\int_\Omega \partial_z \mathbf{A}_h \cdot \mathbf{A}_h \\ &= \int_{\mathbb{R}^d} |\mathrm{A}_{bh}|^2 - 2\int_\Omega \left(\mathrm{curl}^{\,\mu} \mathbf{A}\right)_h^\perp \cdot \mathbf{A}_h + 2\sqrt{\mu}\int_\Omega \nabla_X \mathbf{A}_v \cdot \mathbf{A}_h \\ &= \int_{\mathbb{R}^d} |\mathrm{A}_{bh}|^2 - 2\int_\Omega \left(\mathrm{curl}^{\,\mu} \mathbf{A}\right)_h^\perp \cdot \mathbf{A}_h + 2\int_\Omega \partial_z \mathbf{A}_v \mathbf{A}_v + 2\sqrt{\mu} \left(\int_{\mathbb{R}^d} \beta \left(\nabla b \cdot \mathrm{A}_{bh} \right) \mathrm{A}_{bv} - \int_{\mathbb{R}^d} \left(\varepsilon\nabla\zeta \cdot \underline{\mathrm{A}}_h \right) \underline{\mathrm{A}}_v \right),\end{aligned}$$

where the third equality is obtained by integrating by parts the third integral and by using the fact that $\mathrm{div}^{\mu}\mathbf{A} = 0$. Furthermore, thanks to the boundary conditions and equality (1.13), we have

$$\varepsilon\sqrt{\mu}\,(\nabla\zeta\cdot\underline{\mathrm{A}}_h)\,\underline{\mathrm{A}}_v = \sqrt{\mu}\nabla^{\perp}\widetilde{\psi}\cdot\underline{\mathrm{A}}_h - |\underline{\mathrm{A}}_h|^2 \text{ and } \beta\sqrt{\mu}\,(\nabla b\cdot\mathrm{A}_{bh})\,\mathrm{A}_{bv} = \mathrm{A}_{bv}^2.$$

Then, we get

$$|\underline{\mathrm{A}}|_2^2 + |\mathrm{A}_b|_2^2 = 2\sqrt{\mu}\int_{\mathbb{R}^d}\nabla^{\perp}\widetilde{\psi}\cdot\underline{\mathrm{A}}_h + 2\int_{\mathbb{R}^d}(\mathrm{curl}^{\,\mu}\mathbf{A})_h^{\perp}\cdot\mathbf{A}_h, \tag{2.9}$$

and the inequality follows. □

The third Proposition is a Poincaré inequality.

Proposition 2.5. *Let ζ, $b\in W^{1,\infty}(\mathbb{R}^d)$ and $\mathbf{A}\in H(\mathrm{div}_0^{\mu},\Omega)\cap H(\mathrm{curl}^{\mu},\Omega)$ such that*

$$\mathrm{A}_b\times N_b^{\mu} = 0 \text{ and } \underline{\mathrm{A}}\cdot N^{\mu} = 0.$$

Then,

$$||\mathbf{A}||_2 \le C\,|\varepsilon\zeta - \beta b + 1|_{L^{\infty}}\left(||\mathrm{curl}^{\mu}\mathbf{A}||_2 + ||\partial_z\mathbf{A}||_2\right). \tag{2.10}$$

Proof. We have

$$|\mathbf{A}(X,z)|^2 = |\mathrm{A}_b(X)|^2 + 2\int_{s=-1+\beta b(X)}^{z}\partial_z\mathbf{A}(X,s)\cdot\mathbf{A}(X,s)\mathrm{d}X\mathrm{d}s.$$

Then, the result follows from the following lemma, which is a similar computation to the one in Proposition 2.4.

Lemma 2.6. *Let ζ, $b\in W^{1,\infty}(\mathbb{R}^d)$, $\mathbf{A}\in H(\mathrm{div}_0^{\mu},\Omega)\cap H(\mathrm{curl}^{\mu},\Omega)$ such that*

$$\mathrm{A}_b\times N_b^{\mu} = 0 \text{ and } \underline{\mathrm{A}}\cdot N^{\mu} = 0.$$

Then,

$$|\underline{\mathrm{A}}|_2^2 + |\mathrm{A}_b|_2^2 \le C\,||\mathrm{curl}^{\,\mu}\mathbf{A}||_2\,||\mathbf{A}||_2\,. \tag{2.11}$$

□

Finally, the fourth Proposition links the regularity of $\widetilde{\psi}$ to the regularity of $\omega_b\cdot N_b^{\mu}$.

Proposition 2.7. *Let $\zeta, b\in W^{1,\infty}(\mathbb{R}^d)$ be such that condition (1.21) is satisfied and let $\boldsymbol{\omega}\in H_b(\mathrm{div}_0^{\mu},\Omega)$. Then, there exists a unique solution $\widetilde{\psi}\in\dot{H}^{\frac{3}{2}}(\mathbb{R}^d)$ to the equation $\Delta\widetilde{\psi} = \underline{\boldsymbol{\omega}}\cdot N^{\mu}$ and we have*

$$\left|\nabla\widetilde{\psi}\right|_2 \le \sqrt{\mu}C\left(\frac{1}{\mathrm{h_{min}}},\,\varepsilon\,|\zeta|_{W^{1,\infty}}\,,\,\beta\,|b|_{W^{1,\infty}}\right)\left(||\boldsymbol{\omega}||_2 + \frac{1}{\sqrt{\mu}}\left|\frac{1}{\mathfrak{P}}\left(\omega_b\cdot N_b^{\mu}\right)\right|_2\right),$$

and

$$\left|\sqrt{1+\sqrt{\mu}|D|}\nabla\widetilde{\psi}\right|_2 \le \sqrt{\mu}C\left(\frac{1}{\mathrm{h_{min}}},\,\varepsilon\,|\zeta|_{W^{1,\infty}}\,,\,\beta\,|b|_{W^{1,\infty}}\right)\left(||\boldsymbol{\omega}||_2 + \frac{1}{\sqrt{\mu}}\left|\frac{1}{\mathfrak{P}}\left(\omega_b\cdot N_b^{\mu}\right)\right|_2\right).$$

Proof. The proof is a small adaptation of Lemmas 3.7 and 5.5 in [8]. □

We can now prove an existence and uniqueness result for the system (1.20) and (2.2).

Theorem 2.8. *Let* ζ, $b \in W^{2,\infty}\left(\mathbb{R}^d\right)$ *such that condition* (1.21) *is satisfied,* $\psi \in \dot{H}^{\frac{3}{2}}\left(\mathbb{R}^d\right)$ *and* $\boldsymbol{\omega} \in H_b(\mathrm{div}_0^{\mu}, \Omega)$. There exists a unique solution $\mathbf{U}^{\mu} = \mathbf{U}^{\mu}[\varepsilon\zeta, \beta b](\psi, \boldsymbol{\omega}) \in H^1(\Omega)$ to (1.20). Furthermore, $\mathbf{U}^{\mu} = \nabla^{\mu}_{X,z}\Phi + \mathrm{curl}^{\mu}\mathbf{A}$, where Φ *satisfies* (2.1) *and* $\mathbf{A}$ *satisfies*

$$\begin{cases} \mathrm{curl}^{\mu}\mathrm{curl}^{\mu}\mathbf{A} = \mu\,\boldsymbol{\omega} \quad \text{in} \quad \Omega_t, \\ \mathrm{div}^{\mu}\mathbf{A} = 0 \quad \text{in} \quad \Omega_t, \\ N^{\mu}_b \times \mathrm{A}_b = 0, \\ N^{\mu} \cdot \underline{\mathrm{A}} = 0, \\ \left(\mathrm{curl}^{\mu}\mathbf{A}\right)_{/\!/} = \dfrac{\nabla^{\perp}}{\Delta}\left(\underline{\omega} \cdot N^{\mu}\right), \\ N^{\mu}_b \cdot \left(\mathrm{curl}^{\,\mu}\mathbf{A}\right)_{|z=-1+\beta b} = 0. \end{cases} \tag{2.12}$$

Finally, one has

$$\|\mathbf{U}^{\mu}\|_2 \leq \sqrt{\mu}C\left(\mu_{\max}, \frac{1}{\mathrm{h}_{\min}}, \varepsilon\,|\zeta|_{W^{2,\infty}}, \beta\,|b|_{W^{2,\infty}}\right)\left(\sqrt{\mu}\,\|\boldsymbol{\omega}\|_2 + \left|\frac{1}{\mathfrak{P}}\left(\omega_b \cdot N^{\mu}_b\right)\right|_2 + |\mathfrak{P}\psi|_2\right), \tag{2.13}$$

and

$$\left\|\nabla^{\mu}_{X,z}\mathbf{U}^{\mu}\right\|_2 \leq \mu C\left(\mu_{\max}, \frac{1}{\mathrm{h}_{\min}}, \varepsilon\,|\zeta|_{W^{2,\infty}}, \beta\,|b|_{W^{2,\infty}}\right)\left(\|\boldsymbol{\omega}\|_2 + \left|\frac{1}{\mathfrak{P}}\left(\omega_b \cdot N^{\mu}_b\right)\right|_2 + |\mathfrak{P}\psi|_{H^1}\right). \tag{2.14}$$

Proof. The uniqueness follows easily from the last Propositions. The existence of Φ and the control of its norm are proved in Chapter 2 in [19]. We focus on the existence of a solution of (2.12). The main idea is the following variational formulation for the system (2.12) (we refer to Lem. 3.5 and Prop. 5.3 in [8] for the details). We denote by

$$\mathcal{X} = \left\{\mathbf{C} \in H^1(\Omega),\, \mathrm{div}^{\,\mu}\mathbf{C} = 0,\, \underline{\mathrm{A}} \cdot N^{\mu} = 0 \text{ and } \mathrm{A}_b \times N^{\mu}_b = 0\right\},$$

and $\widetilde{\psi}$ the unique solution in $\dot{H}^1(\mathbb{R}^d)$ of $\Delta\widetilde{\psi} = \underline{\boldsymbol{\omega}} \cdot N^{\mu}$. Then, $\mathrm{A} \in \mathcal{X}$ is a variational solution of System (2.12) if

$$\forall \mathbf{C} \in \mathcal{X},\ \int_{\Omega} \mathrm{curl}^{\mu}\,\mathbf{A} \cdot \mathrm{curl}^{\mu}\,\mathbf{C} = \mu\int_{\Omega} \boldsymbol{\omega} \cdot \mathbf{C} + \mu\int_{\mathbb{R}^d} \nabla\widetilde{\psi} \cdot \mathrm{C}_{/\!/}, \tag{2.15}$$

The existence of such a $\mathbf{A}$ follows Lax-Milgram's theorem. In the following we only explain how we get the coercivity. Thanks to a similar computation that we used to prove Estimate (2.5) we get

$$\left\|\nabla^{\mu}_{X,z}\mathbf{A}\right\|_2^2 \leq \|\mathrm{curl}^{\,\mu}\mathbf{A}\|_2^2 + \mu C\left(\varepsilon\,|\nabla\zeta|_{W^{2,\infty}}, \beta\,|\nabla b|_{W^{2,\infty}}\right)\left(|\underline{\mathrm{A}}|_2^2 + |\mathrm{A}_{bh}|_2^2\right).$$

Then, thanks to the similar computation that in Propositions 2.4 and 2.5 we obtain the coercivity

$$\|\mathbf{A}\|_2 + \left\|\nabla^{\mu}_{X,z}\mathbf{A}\right\|_2 \leq C\left(\mu_{\max}, \frac{1}{h_{\min}}, \varepsilon\,|\zeta|_{W^{2,\infty}}, \beta\,|b|_{W^{2,\infty}}\right)\|\mathrm{curl}^{\mu}\mathbf{A}\|_2.$$

Then, we can easily extend this for all $\mathbf{C}$ in $\left\{\mathbf{C} \in H^1(\Omega),\, \underline{\mathrm{C}} \cdot N^{\mu} = 0 \text{ and } \mathrm{C}_b \times N^{\mu}_b = 0\right\}$ and thanks to the variational formulation of $\mathbf{A}$ we get

$$\|\mathrm{curl}^{\mu}\mathbf{A}\|_2 \leq C\left(\mu_{\max}, \frac{1}{h_{\min}}, \varepsilon\,|\zeta|_{W^{2,\infty}}, \beta\,|b|_{W^{2,\infty}}\right)\left(\mu\,\|\boldsymbol{\omega}\|_2 + \sqrt{\mu}\left|\nabla\widetilde{\psi}\right|_2\right).$$

Using Proposition 2.7, we get the first estimate. The second estimate follows from the first estimate, the inequality (2.5), Proposition 2.4, Proposition 2.6 and the following Lemma.

Lemma 2.9. *Let ζ, $b \in W^{1,\infty}(\mathbb{R}^d)$ be such that condition (1.21) is satisfied. Then, for all $u \in H^1(\Omega)$,*

$$\left|\sqrt{1+\sqrt{\mu}|D|}\underline{u}\right|_2 + \left|\sqrt{1+\sqrt{\mu}|D|}u_b\right|_2 \leq C\left(\frac{1}{h_{\min}}, \varepsilon|\zeta|_{W^{1,\infty}}, \beta|b|_{W^{1,\infty}}\right)\left(\left|\left|\nabla^{\mu}_{X,z}u\right|\right|_2 + ||u||_2\right).$$

Proof. The proof is a small adaptation of Lemma 5.4 in [8]. □

2.3. The transformed div-curl problem

In this section, we transform the div-curl problem in the domain Ω into a variable coefficients problem in the flat strip $\mathcal{S} = \mathbb{R}^d \times (-1, 0)$. We introduce the diffeomorphism Σ,

$$\Sigma := \begin{array}{ccc} S & \rightarrow & \Omega \\ (X,z) & \mapsto & (X, z+\sigma(X,z)), \end{array} \tag{2.16}$$

where

$$\sigma(X,z) := z\left(\varepsilon\zeta(X) - \beta b(X)\right) + \varepsilon\zeta(X).$$

In the following, we will focus on the bottom contribution and we refer to [8] for the other terms. We keep the notations of [8]. We define

$$\mathrm{U}^{\sigma,\mu}\left[\varepsilon\zeta,\beta b\right](\psi,\omega) = \mathrm{U}^{\mu} = \begin{pmatrix}\sqrt{\mu}\mathrm{V} \\ \mathrm{w}\end{pmatrix} = \mathbf{U}^{\mu}\circ\Sigma,\ \omega = \boldsymbol{\omega}\circ\Sigma,$$

and

$$\nabla^{\sigma,\mu}_{X,z} = \left(J_{\Sigma}^{-1}\right)^t\nabla^{\mu}_{X,z},\ \text{where}\ \left(J_{\Sigma}^{-1}\right)^t = \begin{pmatrix} Id_{d\times d} & \frac{-\sqrt{\mu}\nabla\sigma}{1+\partial_z\sigma} \\ 0 & \frac{1}{1+\partial_z\sigma}\end{pmatrix}.$$

Furthermore, for $\mathrm{A} = \mathbf{A}\circ\Sigma$,

$$\mathrm{curl}^{\sigma,\mu}\mathrm{A} = (\mathrm{curl}^{\mu}\ \mathbf{A})\circ\Sigma = \nabla^{\sigma,\mu}_{X,z}\times\mathrm{A},\ \mathrm{div}^{\sigma,\mu}\mathrm{A} = (\mathrm{div}^{\mu}\ \mathbf{A})\circ\Sigma = \nabla^{\sigma,\mu}_{X,z}\cdot\mathrm{A}.$$

Finally, if A is a vector field on $\mathcal{S}$,

$$\underline{\mathrm{A}} = \mathrm{A}_{|z=0},\ \mathrm{A}_b = \mathrm{A}_{|z=-1}\ \text{and}\ \mathrm{A}_{/\!/} = \frac{1}{\sqrt{\mu}}\underline{\mathrm{A}}_h + \varepsilon\underline{\mathrm{A}}_v\nabla\zeta.$$

Then, U^{μ} is the unique solution in $H^1(\mathcal{S})$ of

$$\begin{cases} \mathrm{curl}^{\sigma,\mu}\ \mathrm{U}^{\mu} = \mu\,\omega \quad \text{in} \quad \mathcal{S}, \\ \mathrm{div}^{\sigma,\mu}\ \mathrm{U}^{\mu} = 0 \quad \text{in} \quad \mathcal{S}, \\ \mathrm{U}^{\mu}_{/\!/} = \nabla\psi + \dfrac{\nabla^{\perp}}{\Delta}\left(\underline{\omega}\cdot N^{\mu}\right) \quad \text{on} \quad \{z=0\}, \\ \mathrm{U}^{\mu}_b\cdot N^{\mu}_b = 0 \quad \text{on} \quad \{z=-1\}. \end{cases} \tag{2.17}$$

We also keep the notations in [22]. If $\mathrm{A} = \mathbf{A}\circ\Sigma$, we define

$$\partial^{\sigma}_i\mathrm{A} = \partial_i\mathbf{A}\circ\Sigma,\ i\in\{t,x,y,z\},\ \partial^{\sigma}_i = \partial_i - \frac{\partial_i\sigma}{1+\partial_z\sigma}\partial_z,\ i\in\{x,y,t\}\ \text{and}\ \partial^{\sigma}_z = \frac{1}{1+\partial_z\sigma}\partial_z.$$

Then, by a change of variables and Proposition 2.3 we get the following variational formulation for U^{μ}. For all $\mathrm{C}\in H^1(\mathcal{S})$,

$$\int_{\mathcal{S}}\nabla^{\mu}_{X,z}\mathrm{U}^{\mu}\cdot P(\Sigma)\,\nabla^{\mu}_{X,z}\mathrm{C} = \mu\int_{\mathcal{S}}(1+\partial_z\sigma)\,\omega\cdot\mathrm{curl}^{\sigma,\mu}\mathrm{C} + \int_{\mathbb{R}^d}l^{\mu}[\varepsilon\zeta](\underline{\mathrm{U}}^{\mu})\cdot\underline{C} - \int_{\mathbb{R}^d}l^{\mu}[\beta b](\mathrm{U}^{\mu}_b)\cdot C_b, \tag{2.18}$$

where $P(\Sigma) = (1+\partial_z \sigma) J_{\Sigma}^{-1} \left(J_{\Sigma}^{-1}\right)^t$ and

$$l^{\mu}[\eta]\left(\mathrm{U}^{\mu}_{|z=\eta}\right) = \begin{pmatrix} \sqrt{\mu}\nabla \mathrm{w}_{|z=\eta} - \mu^{\frac{3}{2}} \left(\nabla^{\perp}\eta \cdot \nabla\right) \mathrm{V}^{\perp}_{|z=\eta} \\ -\mu \nabla \cdot \mathrm{V}_{|z=\eta} \end{pmatrix}.$$

In order to obtain higher order estimates on U^{μ}, we have to separate the regularity on z and the regularity on X. We use the following spaces.

Definition 2.10. We define the spaces $H^{s,k}$

$$H^{s,k}(\mathcal{S}) = \bigcap_{0 \leq l \leq k} H^l_z\left(-1,0\,; H^{s-l}_X\left(\mathbb{R}^d\right)\right) \text{ and } |u|_{H^{s,k}} = \sum_{0\leq l\leq k} \left|\Lambda^{s-j}\partial_z^j u\right|_2.$$

Furthermore, if $\alpha \in \mathbb{N}^d \backslash \{0\}$, we define the *Alinhac's good unknown*

$$\psi_{(\alpha)} = \partial^{\alpha}\psi - \varepsilon \underline{\mathrm{w}}\partial^{\alpha}\zeta \text{ and } \psi_{(0)} = \psi. \tag{2.19}$$

This quantities play an important role in the wellposedness of the water waves equations (see [2] or [19]). In fact, more generally, if A is vector field on $\mathcal{S}$, we denote by

$$\mathrm{A}_{(\alpha)} = \partial^{\alpha}\mathrm{A} - \partial^{\alpha}\sigma\partial_z^{\sigma}\mathrm{A},\ \mathrm{A}_{(0)} = \mathrm{A},\ \underline{\mathrm{A}}_{(\alpha)} = \partial^{\alpha}\underline{\mathrm{A}} - \varepsilon\partial^{\alpha}\zeta\underline{\partial_z^{\sigma}\mathrm{A}} \text{ and } \underline{\mathrm{A}}_{(0)} = \underline{\mathrm{A}}. \tag{2.20}$$

We can now give high order estimates on U^{μ}. We recall that M_N is defined in (1.23).

Theorem 2.11. *Let $N \in \mathbb{N}$, $N \geq 5$. Then, under the assumptions of Theorem 2.8, for all $0 \leq l \leq 1$ and $0 \leq l \leq k \leq N-1$, the straightened velocity U^{μ}, satisfies*

$$\left|\left|\nabla^{\mu}_{X,z}\mathrm{U}^{\mu}\right|\right|_{H^{k,l}} \leq \mu M_N \left(\left|\mathfrak{P}\psi\right|_{H^1} + \sum_{1<|\alpha|\leq k+1} \left|\mathfrak{P}\psi_{(\alpha)}\right|_2 + ||\omega||_{H^{k,l}} + \left|\frac{\Lambda^k}{\mathfrak{P}}\left(\omega_b \cdot N^{\mu}_b\right)\right|_2\right).$$

Proof. We start with $l=0$. We follow the proof of Proposition 3.12 and Proposition 5.8 in [8]. Let $k \in [1, N-1]$, $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$. We take $C = \partial^{2\alpha}\mathrm{U}^{\mu}$ in (2.18)[4] and we get

$$\int_{\mathcal{S}} \nabla^{\mu}_{X,z}\mathrm{U}^{\mu} \cdot P(\Sigma)\,\nabla^{\mu}_{X,z}\partial^{2\alpha}\mathrm{U}^{\mu} = \mu\int_{\mathcal{S}}(1+\partial_z\sigma)\,\omega\cdot \mathrm{curl}^{\sigma,\mu}\partial^{2\alpha}\mathrm{U}^{\mu} + \int_{\mathbb{R}^d} l^{\mu}[\varepsilon\zeta](\underline{\mathrm{U}}^{\mu})\cdot\partial^{2\alpha}\underline{\mathrm{U}}^{\mu} - \int_{\mathbb{R}^d} l^{\mu}[\beta b](\mathrm{U}^{\mu}_b)\cdot\partial^{2\alpha}\mathrm{U}^{\mu}_b.$$

We focus on the bottom contribution, which is the last term of the previous equation. Using the fact that $\mathrm{w}_b = \mu\beta\nabla b \cdot \mathrm{V}_b$, we have

$$\begin{aligned} (-1)^{|\alpha|}\int_{\mathbb{R}^d} l^{\mu}[\beta b]\left(\mathrm{U}_b\right)\cdot\partial^{2\alpha}\mathrm{U}_b &= \int_{\mathbb{R}^d} 2\mu\partial^{\alpha}\nabla\mathrm{w}_b\cdot\partial^{\alpha}\mathrm{V}_b - \mu^2\beta\partial^{\alpha}\left[\left(\nabla^{\perp}b\cdot\nabla\right)\mathrm{V}_b^{\perp}\right]\cdot\partial^{\alpha}\mathrm{V}_b \\ &= \int_{\mathbb{R}^d} 2\mu^2\beta\partial^{\alpha}\nabla\left(\nabla b\cdot\mathrm{V}_b\right)\cdot\partial^{\alpha}\mathrm{V}_b - \mu^2\beta\partial^{\alpha}\left[\left(\nabla^{\perp}b\cdot\nabla\right)\mathrm{V}_b^{\perp}\right]\cdot\partial^{\alpha}\mathrm{V}_b \\ &= \underbrace{\int_{\mathbb{R}^d} 2\mu^2\beta\left(\nabla b\right)^t\cdot\partial^{\alpha}\nabla\mathrm{V}_b\cdot\partial^{\alpha}\mathrm{V}_b - \beta\mu^2\left[\left(\nabla^{\perp}b\cdot\nabla\right)\partial^{\alpha}\mathrm{V}_b^{\perp}\right]\cdot\partial^{\alpha}\mathrm{V}_b}_{I_1} \\ &\quad + \underbrace{\int_{\mathbb{R}^d} 2\mu^2\beta\left[\partial^{\alpha}\nabla,\nabla b\right]\mathrm{V}_b\cdot\partial^{\alpha}\mathrm{V}_b - \beta\mu^2\left[\partial^{\alpha},\left(\nabla^{\perp}b\cdot\nabla\right)\right]\mathrm{V}_b^{\perp}\cdot\partial^{\alpha}\mathrm{V}_b}_{I_2}. \end{aligned}$$

[4] A. Castro and D. Lannes explain why we can take such a C in the variational formulation.

Then, a careful computation gives

$$\begin{aligned}|I_1| &= \left|\mu^2\beta\int_{\mathbb{R}^d}\partial_x^2 b\,(\partial^\alpha \mathrm{V}_{bx})^2 + \partial_y^2 b\,(\partial^\alpha \mathrm{V}_{by})^2 + 2\mu^2\beta\int_{\mathbb{R}^d}\partial_{xy}^2 b\,\partial^\alpha \mathrm{V}_{bx}\,\partial^\alpha \mathrm{V}_{by}\right|\\ &\leq \mu C\left(\delta,\frac{1}{h_{\min}},\varepsilon\,|\zeta|_{W^{1,\infty}},\beta\,|b|_{W^{2,\infty}}\right)||\partial^\alpha \mathrm{U}^\mu||_2^2 + \delta\left|\left|\nabla^\mu_{X,z}\partial^\alpha \mathrm{U}^\mu\right|\right|_2^2\\ &\leq C\left(\delta,\frac{1}{h_{\min}},\varepsilon\,|\zeta|_{W^{1,\infty}},\beta\,|b|_{W^{2,\infty}}\right)\left|\left|\nabla^\mu_{X,z}\mathrm{U}^\mu\right|\right|_{H^{k-1}}^2 + \delta\left|\left|\nabla^\mu_{X,z}\partial^\alpha \mathrm{U}^\mu\right|\right|_2^2,\end{aligned}$$

where $\delta > 0$ is small enough and where we use the following Lemma.

Lemma 2.12. *Let ζ, $b \in W^{1,\infty}(\mathbb{R}^d)$, such that condition (1.21) is satisfied. Then, for all $u \in H^1(\mathcal{S})$* and $\delta > 0$,

$$|\underline{u}|_2^2 + |u_b|_2^2 \leq C\left(\delta,\frac{1}{h_{\min}},\varepsilon\,|\zeta|_{W^{1,\infty}},\beta\,|b|_{W^{1,\infty}}\right)||u||_2^2 + \delta\,||\partial_z u||_2^2.$$

Furthermore, using Lemma A.3 and the previous Lemma, we get

$$\begin{aligned}|I_2| &\leq C\mu\beta\,|\nabla b|_{H^{k+1}}\,|\mathrm{U}^\mu_b|_{H^k}\,|\partial^\alpha \mathrm{U}^\mu_b|_2\\ &\leq \mu C\left(\delta,\frac{1}{h_{\min}},\varepsilon\,|\zeta|_{W^{1,\infty}},\beta\,|b|_{W^{1,\infty}},\beta\,|\nabla b|_{H^{k+1}}\right)\left|\left|\nabla^\mu_{X,z}\mathrm{U}^\mu\right|\right|_{H^{k-1}}^2 + \delta\left|\left|\nabla^\mu_{X,z}\partial^\alpha \mathrm{U}^\mu\right|\right|_2^2.\end{aligned}$$

For the surface contribution, we can do the same thing as in Proposition 3.12 and Proposition 5.8 in [8], using the previous Lemma to control $\partial^\alpha \mathrm{w}$. Finally, for the other terms, the main idea is the following Lemma (which is a small adaptation of Lems. 3.13 and Lem. 5.6 in [8]).

Lemma 2.13. *Let $\widetilde{\psi}$ the unique solution of $\Delta\widetilde{\psi} = \underline{\omega}\cdot N^\mu$ in $\dot{H}^1(\mathbb{R}^d)$. Under the assumptions of the Theorem 2.8, we have the following estimate*

$$\left|\mathfrak{P}\nabla^\perp\widetilde{\psi}\right|_{H^k} \leq M_N\left(||\omega||_{H^{k,0}} + \left|\frac{\Lambda^k}{\mathfrak{P}}\left(\omega_b\cdot N^\mu_b\right)\right|_2\right).$$

Gathering the previous estimates with the estimate without the bottom contribution in Proposition 5.8 in [8], we get

$$||\partial^\alpha\nabla^\mu \mathrm{U}^\mu||_2 \leq \mu M_N\left(|\mathfrak{P}\psi|_{H^1} + \sum_{1<|\alpha|\leq k+1}\left|\mathfrak{P}\psi_{(\alpha)}\right| + ||\omega||_{H^{k,0}} + \left|\frac{\Lambda^k}{\mathfrak{P}}\left(\omega_b\cdot N^\mu_b\right)\right|_2\right) + M_N\left|\left|\Lambda^{k-1}\nabla^\mu_{X,z}\mathrm{U}^\mu\right|\right|_2,$$

and the inequality follows by a finite induction on k. If $l = 1$, we can adapt the proof of Corollary 3.14 in [8] easily. □

Remark 2.14. Notice that for $k \geq 2$, we have

$$\left|\frac{\Lambda^k}{\mathfrak{P}}\left(\omega_b\cdot N^\mu_b\right)\right|_2 \leq C\left(\frac{1}{h_{\min}},\mu_{\max},\beta\,|\nabla b|_{H^{k+1}}\right)\left(||\omega||_{H^{k,1}} + \left|\frac{1}{\mathfrak{P}}\left(\omega_b\cdot N^\mu_b\right)\right|_2\right),$$

thanks to Lemma A.5, Lemmas 2.9 and A.4.

2.4. Time derivatives and few remarks about the good unknown

This part is devoted to recall and adapt some results in [8]. Unlike the previous Propositions, adding a non flat bottom is not painful. That is why we do not give proofs. We refer to section 3.5 and 3.6 in [8] for the details. Firstly, in order to obtain an energy estimate of the Castro−Lannes water waves formulation, we need to control $\partial_t \mathrm{U}^{\mu}$. This is the purpose of the following result.

Proposition 2.15. *Let* $T > 0$, $\zeta \in \mathcal{C}^1\left([0,T], W^{2,\infty}\left(\mathbb{R}^d\right)\right)$, $b \in W^{2,\infty}\left(\mathbb{R}^d\right)$ *such that* (1.21) *is satisfied for* $0 \leq t \leq T$, $\psi \in \mathcal{C}^1\left([0,T], \dot{H}^{\frac{3}{2}}\left(\mathbb{R}^d\right)\right)$ *and* $\omega \in \mathcal{C}^1\left([0,T], L^2\left(\mathcal{S}\right)^{d+1}\right)$ *such that* $\nabla^{\mu,\sigma}_{X,z} \cdot \omega = 0$ *for* $0 \leq t \leq T$*. Then,*

$$\partial_t\left(U^{\sigma,\mu}[\varepsilon\zeta,\beta b]\left(\psi,\omega\right)\right) = U^{\sigma,\mu}[\varepsilon\zeta,\beta b]\Big(\partial_t\psi - \varepsilon\underline{w}\partial_t\zeta + \varepsilon\sqrt{\mu}\frac{\nabla}{\Delta}\cdot\left(\underline{\omega_h}^{\perp}\partial_t\zeta\right), \partial_t^{\sigma}\omega\Big) + \partial_t\sigma\partial_z^{\sigma}\left(U^{\mu,\sigma}[\varepsilon\zeta,\beta b]\left(\psi,\omega\right)\right).$$

Furthermore, for $N \geq 5$, $\mathrm{U}^{\mu} = \mathrm{U}^{\sigma,\mu}[\varepsilon\zeta,\beta b]$ *satisfies*

$$\begin{aligned}\sqrt{\mu}\left|\left|\partial_t U^{\mu}\right|\right|_2 + \left|\left|\partial_t\nabla^{\mu}_{X,z}U^{\mu}\right|\right|_{H^{N-2,0}} &\leq \mu\max\left(M_N, \varepsilon\left|\partial_t\zeta\right|_{H^{N-1}}\right)\\ &\times\Bigg(\left|\mathfrak{P}\partial_t\psi\right|_{H^1} + \sum_{1<|\alpha|\leq N-1}\left|\mathfrak{P}\partial_t\psi_{(\alpha)}\right|_2 + \left|\left|\partial_t\omega\right|\right|_{H^{N-2,0}} + \left|\frac{\Lambda^{N-2}}{\mathfrak{P}}\left(\partial_t\omega_b\cdot N^{\mu}_b\right)\right|_2\\ &+ \left|\mathfrak{P}\psi\right|_{H^1} + \sum_{1<|\alpha|\leq N}\left|\mathfrak{P}\psi_{(\alpha)}\right|_2 + \left|\left|\omega\right|\right|_{H^{N-1,1}} + \left|\frac{1}{\mathfrak{P}}\left(\omega_b\cdot N^{\mu}_b\right)\right|_2\Bigg).\end{aligned}$$

Secondly, in the context of water waves, the *Alinhac's good unknowns* play a crucial role. Masmoudi and Rousset remarked in [22] that the *Alinhac's good unknown* $\mathrm{U}^{\mu}_{(\alpha)}$ is almost incompressible and A. Castro and D. Lannes showed that the $\mathrm{curl}^{\sigma,\mu}$ of $\mathrm{U}^{\mu}_{(\alpha)}$ is also well controlled. This is the purpose of the following proposition. We recall that $\mathrm{U}^{\mu}_{(\alpha)}$ is defined in (2.20).

Proposition 2.16. *Let* $N \geq 5$, $\zeta \in H^N(\mathbb{R}^d)$, $b \in L^{\infty} \cap \dot{H}^{N+1}(\mathbb{R}^d)$ *such that condition* (1.21) *is satisfied and* $\omega \in H^{N-1}(\mathcal{S})$ *such that* $\nabla^{\sigma,\mu}\cdot\omega = 0$*. Then if we denote by* $\mathrm{U}^{\mu} = \mathrm{U}^{\mu,\sigma}[\varepsilon\zeta,\beta b]$*, we have for* $1 \leq |\alpha| \leq N$*,*

$$\begin{aligned}&\left|\left|\nabla^{\sigma,\mu}_{X,z}\cdot\mathrm{U}^{\mu}_{(\alpha)}\right|\right|_2 + \left|\left|\nabla^{\sigma,\mu}_{X,z}\times\mathrm{U}^{\mu}_{(\alpha)} - \mu\partial^{\alpha}\omega\right|\right|_2\\ &\quad\leq \mu\left|(\varepsilon\zeta,\beta b)\right|_{H^N} M_N\left(\left|\mathfrak{P}\psi\right|_{H^1} + \sum_{1<|\alpha'|\leq|\alpha|}\left|\mathfrak{P}\psi_{(\alpha')}\right|_2 + \left|\left|\omega\right|\right|_{H^{\max(|\alpha|-1,1)}} + \left|\frac{1}{\mathfrak{P}}\left(\omega_b\cdot N^{\mu}_b\right)\right|_2\right),\end{aligned}$$

and

$$\left|\mathfrak{P}\psi_{(\alpha)}\right|_2 \leq M_N\left(\left|\mathfrak{P}\psi\right|_{H^3} + \frac{1}{\sqrt{\mu}}\sum_{1<|\alpha'|\leq|\alpha|-1}\left|\left|\nabla_X\mathrm{U}^{\mu}_{(\alpha')}\right|\right|_2 + \left|\left|\omega\right|\right|_{H^{N-1}} + \left|\frac{1}{\mathfrak{P}}\left(\omega_b\cdot N^{\mu}_b\right)\right|_2\right).$$

Furthermore, we can control $\left|\mathfrak{P}\psi\right|_{H^3}$ by U^{μ} and ω.

Proposition 2.17. *Let* $N \geq 5$, $\zeta \in H^N(\mathbb{R}^d)$, $b \in L^{\infty} \cap \dot{H}^{N+1}(\mathbb{R}^d)$ *such that condition* (1.21) *is satisfied and* $\omega \in H^{2,1}(\mathcal{S})$ *such that* $\nabla^{\sigma,\mu}\cdot\omega = 0$*. Then,*

$$\left|\mathfrak{P}\psi\right|_{H^3} \leq M_N\left(\frac{1}{\sqrt{\mu}}\left|\left|\Lambda^3\mathrm{U}^{\sigma,\mu}[\varepsilon\zeta,\beta b]\left(\psi,\omega\right)\right|\right|_2 + \left|\left|\omega\right|\right|_{H^{2,1}} + \left|\frac{1}{\mathfrak{P}}\left(\omega_b\cdot N^{\mu}_b\right)\right|_2\right).$$

Proof. The proof is a small adaptation of Lemma 3.23 in [8]. □

Finally, we give a result that is useful for the energy estimate. Since the proof is a little different to Corollary 3.21 in [8], we give it. Notice that the main difference with Corollary 3.21 in [8] is the fact that we do not have a flat bottom.

Proposition 2.18. *Let $N \geq 5$, $\zeta \in H^N(\mathbb{R}^d)$, $b \in L^\infty \cap \dot{H}^{N+1}(\mathbb{R}^d)$ and $\omega \in H^{N-1}(\mathcal{S})$ such that $\nabla^{\sigma,\mu} \cdot \omega = 0$. Then, for $k = x, y$, $|\gamma| \leq N-1$, α such that $\partial^\alpha = \partial_k \partial^\gamma$* and $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^d)$, we have

$$\left(\varphi, \frac{1}{\mu}\partial_k \underline{\mathrm{U}}^\mu_{(\gamma)} \cdot N^\mu\right) \leq M_N \left(|\mathfrak{P}\psi|_{H^1} + \sum_{1<|\alpha'|\leq|\alpha|} \left|\mathfrak{P}\psi_{(\alpha')}\right|_2 + ||\omega||_{H^{|\alpha|-1}} + \left|\frac{1}{\mathfrak{P}}\left(\omega_b \cdot N^\mu\right)\right|_2 \right)$$
$$\times \left[|\mathfrak{P}\varphi|_2 + \left|\frac{1}{\sqrt{1+\sqrt{\mu}|D|}}\varphi\right|_2 \right],$$

where we denote by $\mathrm{U}^\mu = \mathrm{U}^{\sigma,\mu}[\varepsilon\zeta, \beta b]$.

Proof. Notice that when $\gamma \neq 0$,

$$\partial_k \mathrm{U}^\mu_{(\gamma)} = \mathrm{U}^\mu_{(\alpha)} - \partial^\gamma \sigma \partial_k \partial_z^\sigma \mathrm{U}^\mu.$$

Then, using Lemma 2.9, it is easy to check that

$$\left(\varphi, \underline{\partial^\gamma \sigma \partial_k \partial_z^\sigma \mathrm{U}^\mu} \cdot N^\mu\right) \leq M_N \left|\frac{1}{\sqrt{1+\sqrt{\mu}|D|}}\varphi\right|_2 \left|\left|\nabla^\mu_{X,z}\mathrm{U}^\mu\right|\right|_{H^2}.$$

Furthermore, using the Green identity we get

$$\left(\varphi, \mathrm{U}^\mu_{(\alpha)} \cdot N^\mu\right) = \int_{\mathcal{S}} (1+\partial_z\sigma)\,\varphi^\dagger \nabla^{\sigma,\mu}_{X,z} \cdot \mathrm{U}^\mu_{(\alpha)} + \int_{\mathcal{S}} (1+\partial_z\sigma)\,\mathrm{U}^\mu_{(\alpha)} \cdot \nabla^{\sigma,\mu}_{X,z}\varphi^\dagger + \left(\varphi^\dagger_b, \left(\mathrm{U}^\mu_{(\alpha)}\right)_b \cdot N^\mu_b\right),$$

where $\varphi^\dagger = \chi\left(z\sqrt{\mu}|D|\right)\varphi$ and χ is an even positive compactly supported function equal to 1 near 0. Then, using the fact that $\mathrm{U}^\mu_b \cdot N^\mu_b = 0$ and the trace Lemma, we get

$$\begin{aligned}\left(\varphi^\dagger_b, \left(\mathrm{U}^\mu_{(\alpha)}\right)_b \cdot N^\mu_b\right) &= (\chi(\sqrt{\mu}|D|)\varphi, \partial^\alpha \mathrm{U}^\mu_b \cdot N^\mu_b - \beta\partial^\alpha b\,(\partial_z^\sigma \mathrm{U}^\mu)_b \cdot N^\mu_b) \\ &= (\chi(\sqrt{\mu}|D|)\varphi, \mu\beta\,[\nabla b, \partial^\alpha] \cdot \mathrm{V}_b - \beta\partial^\alpha b\,(\partial_z^\sigma \mathrm{U}^\mu)_b \cdot N^\mu_b) \\ &\leq M_N\,(\sqrt{\mu}\,||\mathrm{U}^\mu||_{H^N} + ||\mathrm{U}^\mu||_{H^{2,2}})\,|\chi(\sqrt{\mu}|D|)\varphi|_2.\end{aligned}$$

Therefore, using Proposition 2.16, Theorem 2.11 and the following Lemma (Lems. 2.20 and 2.34 in [19]) we get the control.

Lemma 2.19. *Let $\varphi \in H^{\frac{1}{2}}(\mathbb{R}^d)$ and χ an even positive compactly supported function equal to 1 near 0. Then,*

$$||\chi\left(z\sqrt{\mu}|D|\right)\varphi||_2 \leq C\left|\frac{1}{\sqrt{1+\sqrt{\mu}|D|}}\varphi\right|_2 \text{ and } \left|\left|\nabla^\mu_{X,z}\left(\chi\left(z\sqrt{\mu}|D|\right)\varphi\right)\right|\right|_2 \leq C\sqrt{\mu}\,|\mathfrak{P}\varphi|_2. \qquad \square$$

3. Well-posedness of the water waves equations

3.1. Framework

In this section, we prove a local well-posedness result of the water waves equations. We improve the result of [8] by adding a non flat bottom, a non constant pressure at the surface and a Coriolis forcing. In order to work on a fixed domain, we seek a system of 3 equations on ζ, ψ and $\omega = \boldsymbol{\omega} \circ \Sigma$. We keep the first and the second equations of the Castro−Lannes formulation (1.19). It is easy to check that ω satisfies

$$\partial_t^\sigma \omega + \frac{\varepsilon}{\mu}\left(\mathrm{U}^\mu \cdot \nabla^{\sigma,\mu}_{X,z}\right)\omega = \frac{\varepsilon}{\mu}\left(\omega \cdot \nabla^{\sigma,\mu}_{X,z}\right)\mathrm{U}^\mu + \frac{\varepsilon}{\mu \mathrm{Ro}}\partial_z^\sigma \mathrm{U}^\mu, \tag{3.1}$$

where $\mathrm{U}^\mu = \mathrm{U}^{\sigma,\mu}[\varepsilon\zeta, \beta b]$. Then, in the following the water waves equations will be the system

$$\begin{cases} \partial_t \zeta - \frac{1}{\mu}\underline{\mathrm{U}}^\mu \cdot N^\mu = 0, \\ \partial_t \psi + \zeta + \frac{\varepsilon}{2}\left|\mathrm{U}^\mu_{/\!/}\right|^2 - \frac{\varepsilon}{2\mu}\left(1 + \varepsilon^2 \mu \left|\nabla \zeta\right|^2\right)\underline{\mathrm{w}}^2 + \varepsilon \frac{\nabla}{\Delta} \cdot \left[\left(\underline{\omega} \cdot N^\mu + \frac{1}{\mathrm{Ro}}\right)\underline{\mathrm{V}}^\perp\right] = -P, \\ \partial_t^\sigma \omega + \frac{\varepsilon}{\mu}\left(\mathrm{U}^\mu \cdot \nabla^{\sigma,\mu}_{X,z}\right)\omega = \frac{\varepsilon}{\mu}\left(\omega \cdot \nabla^{\sigma,\mu}_{X,z}\right)\mathrm{U}^\mu + \frac{\varepsilon}{\mu \mathrm{Ro}}\partial_z^\sigma \mathrm{U}^\mu. \end{cases} \tag{3.2}$$

The following quantity is the energy that we will use to get the local wellposedness

$$\mathcal{E}^N(\zeta,\psi,\omega) = \frac{1}{2}\left|\zeta\right|^2_{H^N} + \frac{1}{2}\left|\mathfrak{P}\psi\right|^2_{H^3} + \frac{1}{2}\sum_{1 \leq |\alpha| \leq N}\left|\mathfrak{P}\psi_{(\alpha)}\right|_2^2 + \frac{1}{2}\left|\left|\omega\right|\right|^2_{H^{N-1}} + \frac{1}{2}\left|\frac{1}{\mathfrak{P}}\left(\omega_b \cdot N_b^\mu\right)\right|_2^2,$$

where we recall that $\psi_{(\alpha)}$ is given by (2.19). For $T \geq 0$, we also introduce the energy space

$$\mathrm{E}_T^N = \left\{(\zeta,\psi,\omega) \in \mathcal{C}\left([0,T], H^2(\mathbb{R}^d) \times \dot{H}^2(\mathbb{R}^d) \times H^2(\mathcal{S})\right), \mathcal{E}^N(\zeta,\psi,\omega) \in L^\infty([0,T])\right\}.$$

We also recall that M_N is defined in (1.23). We keep the organization of the Section 4 in [8]. First, we give an *a priori* estimate for the vorticity. Then, we explain briefly how we can quasilinearize the system and how we obtain *a priori* estimates for the full system. The last part of this section is devoted to the proof of the main result.

3.2. *A priori* estimate for the vorticity

In this part, we give *a priori* estimate for the straightened equation of the vorticity.

Proposition 3.1. *Let $N \geq 5$, $T > 0$, $b \in L^\infty \cap \dot{H}^{N+1}(\mathbb{R}^d)$ and $(\zeta,\psi,\omega) \in \mathrm{E}_T^N$ such that (3.1) and condition (1.21) hold on $[0,T]$. We also assume that on $[0,T]$*

$$\partial_t \zeta - \frac{1}{\mu}\mathrm{U}^{\sigma,\mu}[\varepsilon\zeta,\beta b] \cdot N^\mu = 0.$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left|\left|\omega\right|\right|^2_{H^{N-1}} + \left|\frac{1}{\mathfrak{P}}\left(\omega_b \cdot N_b^\mu\right)\right|_2^2\right) \leq M_N\left(\varepsilon\mathcal{E}^N(\zeta,\psi,\omega)^{\frac{3}{2}} + \max\left(\varepsilon, \frac{\varepsilon}{\mathrm{Ro}}\right)\mathcal{E}^N(\zeta,\psi,\omega)\right).$$

Proof. We denote $\mathrm{U}^{\sigma,\mu}[\varepsilon\zeta,\beta b] = \mathrm{U}^\mu = \begin{pmatrix}\sqrt{\mu}\mathrm{V} \\ \mathrm{w}\end{pmatrix}$. We can reformulate equation (3.1) as

$$\partial_t \omega + \varepsilon\left(\mathrm{V} \cdot \nabla_X\right)\omega + \frac{\varepsilon}{\mu}a\partial_z\omega = \frac{\varepsilon}{\mu}\left(\omega \cdot \nabla^{\sigma,\mu}_{X,z}\right)\mathrm{U}^\mu + \frac{\varepsilon}{\mu \mathrm{Ro}}\partial_z^\sigma \mathrm{U}^\mu,$$

where

$$a = \frac{1}{1+\partial_z\sigma}\left(\mathrm{U}^\mu \cdot \begin{pmatrix}-\sqrt{\mu}\nabla_X\sigma \\ 1\end{pmatrix} - (z+1)\underline{\mathrm{U}}^\mu \cdot N^\mu\right).$$

Notice that $\underline{a} = a_b = 0$. Then, we get

$$\partial_t ||\omega||_2^2 = \varepsilon \int_S \left(\nabla_X \cdot \mathrm{V} + \frac{1}{\mu}\partial_z a\right)\omega^2 + \frac{2}{\mu}\left(\omega \cdot \nabla^{\sigma,\mu}_{X,z}\right)\mathrm{U}^\mu \cdot \omega + \frac{1}{\mathrm{Ro}}\partial_z^\sigma \mathrm{U}^\mu \cdot \omega,$$

and

$$\partial_t ||\omega||_2^2 \leq \frac{\varepsilon}{\mu} C\left(\frac{1}{h_{\min}}, \varepsilon|\zeta|_{W^{1,\infty}}, \beta|b|_{W^{1,\infty}}\right)\left(\left[\left|\left|\nabla^\mu_{X,z}\mathrm{U}^\mu\right|\right|_\infty + \sqrt{\mu}||\mathrm{U}^\mu||_\infty\right]||\omega||_2^2 + \frac{1}{\mathrm{Ro}}\left|\left|\nabla^\mu_{X,z}\mathrm{U}^\mu\right|\right|_\infty ||\omega||_2\right),$$

where we use the fact that

$$|\underline{\mathrm{U}}^\mu \cdot N^\mu|_{L^\infty} \leq C\left(\varepsilon|\zeta|_{W^{1,\infty}}, \beta|b|_{W^{1,\infty}}\right)\left(||\partial_z \mathrm{U}^\mu||_\infty + \sqrt{\mu}||\mathrm{U}^\mu||_\infty\right).$$

The estimate for the L^2-norm of ω follows thanks to Theorem 2.8, Theorem 2.11 and Remark 2.14. For the high order estimates, we differentiate equation (3.1) and we easily obtain the control thanks to Theorem 2.11 and Remark 2.14 (see the proof of Prop. 4.1 in [8]). Finally, taking the trace at the bottom of the vorticity equation in system (1.19), we get the following equation for $\omega_b \cdot N_b^\mu$,

$$\partial_t\left(\omega_b \cdot N_b^\mu\right) + \varepsilon\nabla \cdot \left(\left[\omega_b \cdot N_b^\mu + \frac{1}{\mathrm{Ro}}\right]\mathrm{V}_b\right) = 0, \tag{3.3}$$

and then,

$$\partial_t\left|\frac{1}{\mathfrak{P}}\left(\omega_b \cdot N_b^\mu\right)\right|_2^2 \leq 2\varepsilon\left|\sqrt{1+\sqrt{\mu}|D|}\left(\left[\omega_b \cdot N_b^\mu + \frac{1}{\mathrm{Ro}}\right]\mathrm{V}_b\right)\right|_2\left|\frac{1}{\mathfrak{P}}\left(\omega_b \cdot N_b^\mu\right)\right|_2.$$

The control follows easily thanks to and Lemma 2.9, Theorem 2.8, Theorem 2.11 and Remark 2.14. □

Remark 3.2. Notice that we can also take the trace at the surface of the vorticity equation and we obtain a transport equation for $\underline{\omega} \cdot N^\mu$,

$$\partial_t\left(\underline{\omega} \cdot N^\mu\right) + \varepsilon\nabla \cdot \left(\left[\underline{\omega} \cdot N^\mu + \frac{1}{\mathrm{Ro}}\right]\underline{\mathrm{V}}\right) = 0. \tag{3.4}$$

3.3. Quasilinearization and *a priori* estimates

In this part, we quasilinearize the system (1.19). We introduce the Rayleigh−Taylor coefficient

$$\mathfrak{a} := \mathfrak{a}[\varepsilon\zeta, \beta b](\psi, \omega) = 1 + \varepsilon\left(\partial_t + \varepsilon\underline{\mathrm{V}}[\varepsilon\zeta, \beta b](\psi, \omega) \cdot \nabla\right)\underline{\mathrm{w}}[\varepsilon\zeta, \beta b](\psi, \omega). \tag{3.5}$$

It is well-known that the positivity of this quantity is essential for the wellposedness of the water waves equations (see for instance Rem. 4.17 in [19] or [15]). Thanks to equation (1.12), we can easily adapt Section 4.3.5 in [19] and check that the positivity of $\mathfrak{a}$ is equivalent to the classical Rayleigh−Taylor criterion ([29])

$$\inf_{\mathbb{R}^d}\left(-\partial_z \mathcal{P}_{|z=\varepsilon\zeta}\right) > 0,$$

where we recall that $\mathcal{P}$ is the pressure in the fluid domain. We can now give a quasilinearization of (3.2). We recall that the notation $\underline{\mathrm{U}}^\mu_{(\alpha)}$ is defined in (2.20) and $\psi_{(\alpha)}$ is defined in (2.19).

Proposition 3.3. *Let $N \geq 5$, $T > 0$, $b \in L^\infty \cap \dot{H}^{N+1}(\mathbb{R}^d)$, $P \in L^\infty_t\left(\mathbb{R}^+; \dot{H}^{N+1}_X\left(\mathbb{R}^d\right)\right)$ and $(\zeta, \psi, \omega) \in E^N_T$ solution of the system (3.2) such that (ζ, b) satisfy condition (1.21) on $[0, T]$. Then, for $\alpha, \gamma \in \mathbb{N}^d$ and for $k \in \{x, y\}$ such that $\partial^\alpha = \partial_k \partial^\gamma$* and $|\gamma| \leq N - 1$, *we have the following quasilinearization*

$$\begin{aligned}&\left(\partial_t + \varepsilon\underline{\mathrm{V}} \cdot \nabla\right)\partial^\alpha\zeta - \frac{1}{\mu}\partial_k \underline{\mathrm{U}}^\mu_{(\gamma)} \cdot N^\mu = R^1_\alpha,\\ &\left(\partial_t + \varepsilon\underline{\mathrm{V}} \cdot \nabla\right)\left(\underline{\mathrm{U}}^\mu_{(\gamma)/\!/} \cdot e_{\mathbf{k}}\right) + \mathfrak{a}\partial^\alpha\zeta = -\partial^\alpha P + R^2_\alpha,\end{aligned} \tag{3.6}$$

where

$$\left|R^1_\alpha\right|_2+\left|R^2_\alpha\right|_2+\left|\mathfrak{P}R^2_\alpha\right|_2\leq M_N\left(\max\left(\varepsilon,\frac{\varepsilon}{\mathrm{Ro}}\right)\mathcal{E}^N(\zeta,\psi,\omega)+\frac{\varepsilon}{\mathrm{Ro}}\sqrt{\mathcal{E}^N(\zeta,\psi,\omega)}\right). \tag{3.7}$$

Before proving this result, we introduce the following notation. For $\alpha\in\mathbb{N}^d$ and $f,g\in H^{|\alpha|-1}(\mathbb{R}^d)$, we define the symmetric commutator

$$[\partial^\alpha,f,g]=\partial^\alpha(fg)-g\partial^\alpha f-f\partial^\alpha g.$$

Proof. Firstly, we apply ∂^α to the first equation of (3.2)

$$\partial_t\partial^\alpha\zeta+\varepsilon\underline{\mathrm{V}}\cdot\nabla\partial^\alpha\zeta+\varepsilon\partial^\alpha\underline{\mathrm{V}}\cdot\nabla\zeta-\frac{1}{\mu}\partial^\alpha\underline{\mathrm{w}}+\varepsilon\left[\partial^\alpha,\underline{\mathrm{V}},\nabla\zeta\right]=0.$$

Using Theorem 2.11 and the trace Lemma 2.12, we get the first equality. For the second equality we get, after applying ∂_k to the second equation of (1.19),

$$\begin{aligned}\partial_t\partial_k\psi+\partial_k\zeta+\varepsilon\underline{\mathrm{V}}\cdot\left((\partial_k\nabla\psi-\varepsilon\underline{\mathrm{w}}\nabla\partial_k\zeta)+\partial_k\nabla^\perp\widetilde{\psi}\right)-\frac{\varepsilon}{\mu}\underline{\mathrm{w}}\partial_k\left(\underline{\mathrm{U}}^\mu\cdot N^\mu\right)&\\-\varepsilon\partial_k\frac{\nabla^\perp}{\Delta}\cdot\left(\left(\underline{\omega}\cdot N^\mu+\frac{1}{\mathrm{Ro}}\right)\underline{\mathrm{V}}\right)&=-\partial_kP.\end{aligned}$$

Then, applying ∂^γ and using Lemma 4.3 in [8] (we can easily adapt it thanks to Thm. 2.11 and Lem. 2.13) we get

$$\begin{aligned}\partial_t\partial^\alpha\psi+\partial^\alpha\zeta+\varepsilon\underline{\mathrm{V}}\cdot\left((\partial^\alpha\nabla\psi-\varepsilon\underline{\mathrm{w}}\nabla\partial^\alpha\zeta)+\partial^\alpha\nabla^\perp\widetilde{\psi}\right)&\\-\frac{\varepsilon}{\mu}\underline{\mathrm{w}}\partial^\alpha\left(\underline{\mathrm{U}}^\mu\cdot N^\mu\right)-\varepsilon\partial^\alpha\frac{\nabla^\perp}{\Delta}\cdot\left(\left(\underline{\omega}\cdot N^\mu+\frac{1}{\mathrm{Ro}}\right)\underline{\mathrm{V}}\right)&=-\partial^\alpha P+\widetilde{R^2_\alpha},\end{aligned}$$

where $\widetilde{R^2_\alpha}$ is controlled

$$\left|\widetilde{R^2_\alpha}\right|_2+\left|\mathfrak{P}\widetilde{R^2_\alpha}\right|_2\leq\varepsilon M_N\mathcal{E}^N(\zeta,\psi,\omega). \tag{3.8}$$

Using the first equation of (1.19) and the fact that $\Delta\widetilde{\psi}=\underline{\omega}\cdot N^\mu$, we obtain

$$\begin{aligned}\partial_t\psi_{(\alpha)}+\mathfrak{a}\partial^\alpha\zeta+\varepsilon\underline{\mathrm{V}}\cdot\nabla\psi_{(\alpha)}+\frac{\varepsilon}{\mathrm{Ro}}\partial^\alpha\frac{\nabla^\perp}{\Delta}\cdot\underline{\mathrm{V}}+\partial^\alpha P&=\varepsilon\partial^\alpha\frac{\nabla^\perp}{\Delta}\cdot\left(\underline{\omega}\cdot N^\mu\underline{\mathrm{V}}\right)-\varepsilon\underline{\mathrm{V}}\cdot\nabla^\perp\partial^\alpha\widetilde{\psi}+\widetilde{R^2_\alpha}\\&=\varepsilon\sum_{k\in\{1,2\}}(-1)^{k+1}\left[\partial^\alpha\frac{\partial_k}{\Delta},\underline{\mathrm{V}}_{3-k}\right]\left(\underline{\omega}\cdot N^\mu\right)+\widetilde{R^2_\alpha}\\&:=\widetilde{R^3_\alpha}+\widetilde{R^2_\alpha},\end{aligned}$$

where $\partial_1=\partial_x$ and $\partial_2=\partial_y$. Then, using Theorem 3 in [18], Lemmas A.1 and 2.9 we get

$$\left|\widetilde{R^3_\alpha}\right|_2+\left|\mathfrak{P}\widetilde{R^3_\alpha}\right|_2\leq\varepsilon M_N\left|\left|\mathrm{V}\right|\right|_{H^{N,1}}\left|\left|\omega\right|\right|_{H^{N-1,1}}+\varepsilon\left|\mathfrak{P}\frac{\nabla^\perp}{\Delta}\cdot\left(\underline{\omega}\cdot N^\mu\partial^\alpha\underline{\mathrm{V}}\right)\right|_2.$$

Furthermore,

$$\begin{aligned}\left|\mathfrak{P}\frac{\nabla^{\perp}}{\Delta}\cdot(\underline{\omega}\cdot N^{\mu}\partial^{\alpha}\underline{\mathrm{V}})\right|_2 &\leq \left|\frac{1}{\sqrt{1+\sqrt{\mu}|D|}}(\underline{\omega}\cdot N^{\mu}\partial^{\alpha}\underline{\mathrm{V}})\right|_2,\\ &\leq \left|\frac{1}{\sqrt{1+\sqrt{\mu}|D|}}\left(\partial_k\left(\underline{\omega}\cdot N^{\mu}\right)\partial^{\gamma}\underline{\mathrm{V}}\right)\right|_2 + \left|\mathfrak{P}\left(\underline{\omega}\cdot N^{\mu}\partial^{\gamma}\underline{\mathrm{V}}\right)\right|_2,\\ &\leq C\left(\varepsilon\left|\zeta\right|_{H^N}\right)\left|\underline{\omega}\right|_{H^{N-2}}\left(\left|\underline{\mathrm{V}}\right|_{H^{N-1}}+\left|\mathfrak{P}\partial^{\gamma}\underline{\mathrm{V}}\right|_2\right),\end{aligned}$$

where we use Lemma A.2. The first term is controlled thanks to the trace Lemma 2.12 and Theorem 2.11. For the second term, we have

$$\partial^{\gamma}\underline{\mathrm{V}} = \nabla\partial^{\gamma}\psi - \varepsilon\underline{\mathrm{w}}\nabla\partial^{\gamma}\zeta - \varepsilon\partial^{\gamma}\underline{\mathrm{w}}\nabla\zeta + \nabla^{\perp}\partial^{\gamma}\widetilde{\psi} - \varepsilon\left[\partial^{\gamma},\underline{\mathrm{w}},\nabla\zeta\right],$$

and the control follows from Lemma A.1, Lemma 2.9, Theorem 2.11 and Lemma 2.13. Then, we obtain

$$\partial_t\psi_{(\alpha)} + \mathfrak{a}\partial^{\alpha}\zeta + \varepsilon\underline{\mathrm{V}}\cdot\nabla\psi_{(\alpha)} + \frac{\varepsilon}{\mathrm{Ro}}\partial^{\alpha}\frac{\nabla^{\perp}}{\Delta}\cdot\underline{\mathrm{V}} + \partial^{\alpha}P = \widetilde{\widetilde{R_{\alpha}^2}},$$

where $\widetilde{\widetilde{R_{\alpha}^2}}$ satisfied also the estimate (3.8). Finally, we can adapt Lemma 4.4 in [8] thanks to Remark 3.2, Theorem 2.11 and Proposition 2.15 and we get

$$\partial_t\psi_{(\alpha)} = \partial_t\left(\mathrm{U}^{\mu}_{(\gamma)/\!/}\cdot\mathbf{e}_k\right) + \widetilde{R_{\alpha}},$$

where $\widetilde{R_{\alpha}}$ satisfies the same estimate as R_2 in (3.7). The third equality is a direct consequence of Proposition 3.1. ☐

In order to establish an *a priori* estimate we need to control the Rayleigh−Taylor coefficient $\mathfrak{a}$. The following Proposition is adapted from Proposition 2.10 in [23].

Proposition 3.4. *Let $T>0$, $N\geq 5$, $(\zeta,\psi,\omega)\in E^N_T$ is a solution of the water waves equations (3.2), $P\in L^{\infty}(\mathbb{R}^+;\dot{H}^{N+1}(\mathbb{R}^d))$ and $b\in L^{\infty}\cap\dot{H}^{N+1}(\mathbb{R}^d)$, such that condition (1.21) is satisfied. We assume also that ε,β,Ro,μ satisfy (1.22). Then, for all $0\leq t\leq T$,*

$$\left|\mathfrak{a}-1\right|_{W^{1,\infty}} \leq C\left(M_N,\varepsilon\sqrt{\mathcal{E}^N(\zeta,\psi,\omega)}\right)\varepsilon\sqrt{\mathcal{E}^N(\zeta,\psi,\omega)} + \varepsilon M_N\left|\nabla P\right|_{L^{\infty}_t H^N_X}.$$

Furthermore, if $\partial_t P\in L^{\infty}(\mathbb{R}^+;\dot{H}^N(\mathbb{R}^d))$, then,

$$\left|\partial_t\mathfrak{a}\right|_{L^{\infty}} \leq C\Big(M_N,\left|\nabla P\right|_{L^{\infty}_t H^N_X},\varepsilon\sqrt{\mathcal{E}^N(\zeta,\psi,\omega)}\Big)\varepsilon\sqrt{\mathcal{E}^N(\zeta,\psi,\omega)} + \varepsilon M_N\left|\nabla P\right|_{W^{1,\infty}_t H^N_X}.$$

Proof. Using Proposition 2.15 we get that

$$\begin{aligned}\mathfrak{a}[\varepsilon\zeta,\beta b](\psi,\omega) = {}& 1 + \varepsilon^2\underline{\mathrm{V}}\cdot\nabla\underline{\mathrm{w}} + \varepsilon\partial_t\zeta\underline{\partial_z^{\sigma}\mathrm{w}}\\ &+ \varepsilon\underline{\mathrm{w}}[\varepsilon\zeta,\beta b]\left(\partial_t\psi - \varepsilon\underline{\mathrm{w}}[\varepsilon\zeta,\beta b](\psi,\omega)\partial_t\zeta + \varepsilon\sqrt{\mu}\frac{\nabla}{\Delta}\cdot\left(\underline{\omega_h}^{\perp}\partial_t\zeta\right),\partial_t^{\sigma}\omega\right).\end{aligned}\tag{3.9}$$

Then, using the equations satisfied by (ζ,ψ,ω), Theorems 2.8 and 2.11, Remark 2.14 and standard controls, we easily get the first inequality. The second inequality can be proved similarly. ☐

We can now establish an *a priori* estimate for the Castro−Lannes System with a Coriolis forcing under the positivity on the Rayleigh−Taylor coefficient

$$\exists\, \mathfrak{a}_{\min} > 0,\ \mathfrak{a} \geq \mathfrak{a}_{\min}. \tag{3.10}$$

Theorem 3.5. *Let $N \geq 5$, $T > 0$, $b \in L^{\infty} \cap \dot{H}^{N+2}(\mathbb{R}^d)$, $P \in L^{\infty}_t\left(\mathbb{R}^+; \dot{H}^{N+1}(\mathbb{R}^d)\right)$ and $(\zeta, \psi, \omega) \in E^N_T$ solution of the water waves equations (3.2) such that (ζ, b) satisfy condition (1.21) and $\mathfrak{a}[\varepsilon\zeta, \beta b](\psi, \omega)$ satisfies (3.10) on $[0, T]$. We assume also that $\varepsilon, \beta, Ro, \mu$ satisfy (1.22). Then, for all $t \in [0, T]$,*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^N(\zeta,\psi,\omega) \leq C&\left(\mu_{\max}, \frac{1}{h_{\min}}, \varepsilon\sqrt{\mathcal{E}^N(\zeta,\psi,\omega)}, \beta\left|\nabla b\right|_{H^{N+1}}, \beta\left|b\right|_{L^\infty}, \left|\nabla P\right|_{W^{1,\infty}_t H^N_X}\right)\\ &\times\left(\varepsilon\mathcal{E}^N(\zeta,\psi,\omega)^{\frac{3}{2}} + \max\left(\varepsilon, \beta, \frac{\varepsilon}{Ro}\right)\mathcal{E}^N(\zeta,\psi,\omega) + \left|\nabla P\right|_{L^\infty_t H^N_X}\sqrt{\mathcal{E}^N(\zeta,\psi,\omega)}\right).\end{aligned} \tag{3.11}$$

Proof. Compared to [8], we have here a non flat bottom, a Coriolis forcing and a non constant pressure. We focus on these terms. Inspired by [8] we can symmetrize the Castro−Lannes system. We define a modified energy

$$\begin{aligned}\mathcal{F}^N(\psi,\zeta,\omega) = \frac{1}{2}\Bigg(&\left|\left|\boldsymbol{\omega}\right|\right|^2_{H^{N-1}} + \left|\frac{1}{\mathfrak{P}}\left(\omega_b \cdot N^\mu_b\right)\right|^2_2 + \sum_{|\alpha|\leq 3}\left|\partial^\alpha\zeta\right|^2_2 + \frac{1}{\mu}\int_{\mathcal{S}}(1+\partial_z\sigma)\left|\partial^\alpha \mathrm{U}^\mu\right|^2\\ &+ \sum_{k=x,y,1\leq|\gamma|\leq N-1}\left(\mathfrak{a}\partial_k\partial^\gamma\zeta, \partial_k\partial^\gamma\zeta\right) + \frac{1}{\mu}\int_{\mathcal{S}}(1+\partial_z\sigma)\left|\partial_k \mathrm{U}^\mu_{(\gamma)}\right|^2\Bigg).\end{aligned} \tag{3.12}$$

From Propositions 2.16 and 2.17 we get

$$\mathcal{E}^N(\psi,\zeta,\omega) \leq C\left(\frac{1}{\mathfrak{a}_{\min}}, M_N\right)\mathcal{F}^N(\psi,\zeta,\omega),$$

and from Theorem 2.8, Theorem 2.11, Remark 2.14 and Proposition 3.4 we obtain that

$$\mathcal{F}^N(\psi,\zeta,\omega) \leq C\left(\frac{1}{h_{\min}}, \beta\left|b\right|_{L^\infty}, \beta\left|\nabla b\right|_{H^N}, \left|\nabla P\right|_{L^\infty_t H^N_X}, \varepsilon\sqrt{\mathcal{E}^N(\psi,\zeta,\omega)}\right)\mathcal{E}^N(\psi,\zeta,\omega).$$

Hence, in the following we estimate $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}^N(\psi,\zeta,\omega)$. We already did the work for the vorticity in Proposition 3.1. In the following R will be a remainder whose exact value has no importance and satisfying

$$\left|R\right|_2 \leq C\left(\frac{1}{h_{\min}}, \beta\left|b\right|_{L^\infty}, \beta\left|\nabla b\right|_{H^{N+1}}, \left|\nabla P\right|_{W^{1,\infty}_t H^N_X}, \varepsilon\sqrt{\mathcal{E}^N(\psi,\zeta,\omega)}\right)\mathcal{E}^N(\psi,\zeta,\omega). \tag{3.13}$$

We start by the low order terms. Let $\alpha \in \mathbb{N}^d$, $|\alpha| \leq 3$. We apply ∂^α to the first equation of system (3.2) and we multiply it by ζ. Then, we apply ∂^α to the second equation and we multiply it by $\frac{1}{\mu}\underline{\mathrm{U}}^\mu \cdot N^\mu$. By summing these two equations, we obtain, thanks to Theorem 2.8, Theorem 2.11, Remark 2.14 and the trace Lemma,

$$\frac{1}{2}\partial_t\left(\partial^\alpha\zeta, \partial^\alpha\zeta\right) + \left(\partial_t\partial^\alpha\psi, \frac{1}{\mu}\partial^\alpha\underline{\mathrm{U}}^\mu \cdot N^\mu\right) + \frac{\varepsilon}{\mathrm{Ro}}\left(\frac{\nabla}{\Delta}\cdot\partial^\alpha\underline{\mathrm{V}}^\perp, \frac{1}{\mu}\partial^\alpha\underline{\mathrm{U}}^\mu\cdot N^\mu\right) + \left(\partial^\alpha P, \frac{1}{\mu}\partial^\alpha\underline{\mathrm{U}}^\mu\cdot N^\mu\right) \leq \varepsilon\left|R\right|_2. \tag{3.14}$$

Furthermore, using again the same Propositions as before, we get

$$\frac{\varepsilon}{\mathrm{Ro}}\left(\frac{\nabla}{\Delta}\cdot\partial^\alpha\underline{\mathrm{V}}^\perp, \frac{1}{\mu}\partial^\alpha\underline{\mathrm{U}}^\mu\cdot N^\mu\right) + \left(\partial^\alpha P, \frac{1}{\mu}\partial^\alpha\underline{\mathrm{U}}^\mu\cdot N^\mu\right) \leq \frac{\varepsilon}{\mathrm{Ro}}\left|R\right|_2 + M_N\left|\nabla P\right|_{L^\infty_t H^N_X}\sqrt{\mathcal{E}^N(\psi,\zeta,\omega)}.$$

Then, we have to link $(\partial_t \partial^\alpha \psi, \partial^\alpha \underline{\mathrm{U}}^\mu \cdot N^\mu)$ to $\partial_t \int_{\mathcal{S}} (1+\partial_z \sigma) \left|\partial^\alpha \mathrm{U}^\mu\right|^2$. Remarking that $\psi = \underline{\phi}$, where ϕ satisfies

$$\begin{cases} \nabla^\mu_{X,z} \cdot P(\Sigma) \nabla^\mu_{X,z} \phi = 0 \quad \text{in} \quad \mathcal{S}, \\ \phi_{|z=0} = \psi, \ \mathrm{e}_z \cdot P(\Sigma) \nabla^\mu \phi_{|z=-1} = 0, \end{cases} \tag{3.15}$$

we get thanks to Green's identity

$$\left(\partial_t \partial^\alpha \psi, \frac{1}{\mu} \partial^\alpha \underline{\mathrm{U}}^\mu \cdot N^\mu \right) = \frac{1}{\mu} \int_{\mathcal{S}} (1+\partial_z \sigma) \nabla^{\sigma,\mu}_{X,z} (\partial_t \partial^\alpha \phi) \cdot \partial^\alpha \mathrm{U}^\mu$$
$$+ \frac{1}{\mu} \int_{\mathcal{S}} (1+\partial_z \sigma) \partial^\alpha \partial_t \phi \nabla^{\sigma,\mu}_{X,z} \cdot \partial^\alpha \mathrm{U}^\mu + \left(\partial_t \partial^\alpha \phi_b, \frac{1}{\mu} \partial^\alpha \mathrm{U}^\mu_b \cdot N^\mu_b \right).$$

Then, notice that $\partial_k = \partial^\sigma_k + \partial_k \sigma \partial^\sigma_z$ for $k \in \{t,x,y\}$ and ∂^σ_k and $\nabla^{\sigma,\mu}_{X,z}$ commute. We differentiate equation (3.15) with respect to t and we obtain thanks to Theorems 2.8, 2.11, Proposition 2.15 and Lemma 2.38 in [19] (irrotational theory),

$$\left(\partial_t \partial^\alpha \psi, \frac{1}{\mu} \partial^\alpha \underline{\mathrm{U}}^\mu \cdot N^\mu \right) = \frac{1}{\mu} \int_{\mathcal{S}} (1+\partial_z \sigma) \partial^\sigma_t \partial^{\sigma,\alpha} \nabla^{\sigma,\mu}_{X,z} \phi \cdot \partial^\alpha \mathrm{U}^\mu + \left(\partial_t \partial^\alpha \phi_b, \frac{1}{\mu} \partial^\alpha \mathrm{U}^\mu_b \cdot N^\mu_b \right) + \max(\varepsilon, \beta) R.$$

Using the fact that $\mathrm{w}_b = \mu \beta \nabla b \cdot \mathrm{V}_b$, we get

$$\left(\partial_t \partial^\alpha \phi_b, \frac{1}{\mu} \partial^\alpha \mathrm{U}^\mu_b \cdot N^\mu_b \right) \leq \beta M_N \left|\partial_t \partial^\alpha \phi_b\right| \sqrt{\mathcal{E}^N(\psi, \zeta, \omega)}.$$

Then, by the trace Lemma, we finally obtain

$$\left(\partial_t \partial^\alpha \phi_b, \frac{1}{\mu} \partial^\alpha \mathrm{U}^\mu_b \cdot N^\mu_b \right) \leq \beta \left|R\right|_2.$$

Furthermore, remarking that $\mathrm{U}^\mu = \nabla^{\sigma,\mu}_{X,z} \phi + \mathrm{U}^{\sigma,\mu}[\varepsilon\zeta, \beta b](0,\omega)$, we obtain, thanks to Proposition 2.15, Theorems 2.8 and 2.11,

$$\left(\partial_t \partial^\alpha \psi, \frac{1}{\mu} \partial^\alpha \underline{\mathrm{U}}^\mu \cdot N^\mu \right) = \frac{1}{\mu} \int_{\mathcal{S}} (1+\partial_z \sigma) \partial_t \partial^\alpha \mathrm{U}^\mu \cdot \partial^\alpha \mathrm{U}^\mu + \max\left(\varepsilon, \beta, \frac{\varepsilon}{\mathrm{Ro}}\right) R.$$

Using the following identity

$$\partial_t \int_{\mathcal{S}} (1+\partial_z \sigma) f g = \int_{\mathcal{S}} (1+\partial_z \sigma) \partial^\sigma_t f g + \int_{\mathcal{S}} (1+\partial_z \sigma) f \partial^\sigma_t g + \int_{\mathbb{R}^d} \varepsilon \partial_t \zeta \underline{f} \underline{g}, \tag{3.16}$$

we obtain that

$$\frac{1}{\mu} \partial_t \int_{\mathcal{S}} (1+\partial_z \sigma) |\partial^\alpha \mathrm{U}^\mu|^2 \leq \max\left(\varepsilon, \beta, \frac{\varepsilon}{\mathrm{Ro}}\right) |R|_2 + M_N |\nabla P|_{L^\infty_t H^N_X} \sqrt{\mathcal{E}^N(\psi, \zeta, \omega)}.$$

To control the high order terms of $\mathcal{F}^N(\psi, \zeta, \omega)$ we adapt Step 2 in Proposition 4.5 in [19]. Thanks to Proposition 3.3, we have

$$(\partial_t + \varepsilon \underline{\mathrm{V}} \cdot \nabla) \partial^\alpha \zeta - \frac{1}{\mu} \partial_k \underline{\mathrm{U}}^\mu_{(\gamma)} \cdot N^\mu = R^1_\alpha,$$
$$(\partial_t + \varepsilon \underline{\mathrm{V}} \cdot \nabla) \left(\underline{\mathrm{U}}^\mu_{(\gamma)/\!/} \cdot e_{\mathbf{k}}\right) + \mathfrak{a} \partial^\alpha \zeta = -\partial^\alpha P + R^2_\alpha.$$

Then, we multiply the first equation by $\mathfrak{a}\partial^\alpha\zeta$ and the second by $\frac{1}{\mu}\partial_k \underline{\mathrm{U}}^\mu_{(\gamma)} \cdot N^\mu$ and we integrate over $\mathbb{R}^d$. Then, using Propositions 2.8, 2.18 and 3.4,

$$\frac{1}{2}\partial_t \left(\mathfrak{a}\partial^\alpha\zeta, \partial\zeta\right) + \left(\left(\partial_t + \varepsilon \underline{\mathrm{V}} \cdot \nabla\right) \left(\underline{\mathrm{U}}^\mu_{(\gamma)/\!\!/} \cdot e_{\mathbf{k}}\right), \frac{1}{\mu}\partial_k \underline{\mathrm{U}}^\mu_{(\gamma)} \cdot N^\mu \right) \leq \varepsilon \mathcal{E}^N(\psi,\zeta,\omega)^{\frac{3}{2}} + \max\left(\varepsilon, \frac{\varepsilon}{\mathrm{Ro}}\right) |R|_2 + M_N |\nabla P|_{L^\infty_t H^N_X} \sqrt{\mathcal{E}^N(\psi,\zeta,\omega)}.$$

Then, we remark that

$$\left(\partial_t + \varepsilon \underline{\mathrm{V}} \cdot \nabla\right) \left(\underline{\mathrm{U}}^\mu_{(\gamma)/\!\!/} \cdot e_{\mathbf{k}}\right) = \underline{\left(\partial^\sigma_t + \frac{\varepsilon}{\mu}\mathrm{U}^\mu \cdot \nabla^{\sigma,\mu}\right) \left(\mathrm{U}^{\mathfrak{b},\mu}_{(\gamma)/\!\!/} \cdot e_{\mathbf{k}}\right)},$$

where $\mathrm{U}^{\mathfrak{b},\mu}_{(\gamma)/\!\!/} = \mathrm{V}_{(\gamma)} + \mathrm{w}_{(\gamma)}\nabla\sigma$. Then, we have

$$\begin{aligned}\left(\left(\partial_t + \varepsilon \underline{\mathrm{V}} \cdot \nabla\right) \left(\underline{\mathrm{U}}^\mu_{(\gamma)/\!\!/} \cdot e_{\mathbf{k}}\right), \frac{1}{\mu}\partial_k \underline{\mathrm{U}}^\mu_{(\gamma)} \cdot N^\mu \right) &= \frac{1}{\mu}\int_{\mathcal{S}} (1+\partial_z\sigma) \left(\partial^\sigma_t + \frac{\varepsilon}{\mu}\mathrm{U}^\mu \cdot \nabla^{\sigma,\mu}\right) \left(\mathrm{U}^{\mathfrak{b},\mu}_{(\gamma)/\!\!/} \cdot e_{\mathbf{k}}\right) \nabla^{\sigma,\mu}_{X,z} \cdot \left(\partial_k \mathrm{U}^\mu_{(\gamma)}\right) \\ &+ \frac{1}{\mu}\int_{\mathcal{S}} (1+\partial_z\sigma) \nabla^{\sigma,\mu}_{X,z} \left(\partial^\sigma_t + \frac{\varepsilon}{\mu}\mathrm{U}^\mu \cdot \nabla^{\sigma,\mu}\right) \left(\mathrm{U}^{\mathfrak{b},\mu}_{(\gamma)/\!\!/} \cdot e_{\mathbf{k}}\right) \left(\partial_k \mathrm{U}^\mu_{(\gamma)}\right) \\ &+ \left(\left(\partial_t + \varepsilon \mathrm{V}_b \cdot \nabla\right) \left(\mathrm{U}^{\mathfrak{b},\mu}_{(\gamma)/\!\!/} \cdot e_{\mathbf{k}}\right)_b, \frac{1}{\mu}\partial_k \left(\mathrm{U}^\mu_{(\gamma)}\right)_b \cdot N^\mu_b \right).\end{aligned}$$

We focus on the last term (bottom contribution). The two other terms can be controlled as in Step 2 in Proposition 4.5 in [8]. Using the same computations as in Proposition 2.18, we have

$$\frac{1}{\mu}\partial_k \left(\mathrm{U}^\mu_{(\gamma)}\right)_b \cdot N^\mu_b = -\mu\beta\nabla\partial^\alpha b \cdot \mathrm{V}_b + \text{l.o.t},$$

where l.o.t stands for lower order terms that can be controlled by the energy. Then, since $b \in \dot{H}^{N+2}(\mathbb{R}^d)$, we have by standard controls,

$$\left| \frac{1}{\mu}\partial_k \left(\mathrm{U}^\mu_{(\gamma)}\right)_b \cdot N^\mu_b \right|_{H^{\frac{1}{2}}} \leq \beta \left|\nabla b\right|_{H^{N+1}} \sqrt{\mathcal{E}^N(\psi,\zeta,\omega)}.$$

Furthermore, using Propositions 2.8, 2.11 and 2.15 and standard controls, we have

$$\left| \left(\partial_t + \varepsilon \mathrm{V}_b \cdot \nabla\right) \left(\mathrm{U}^{\mathfrak{b},\mu}_{(\gamma)/\!\!/} \cdot e_{\mathbf{k}}\right)_b \right|_{H^{-\frac{1}{2}}} \leq \varepsilon \left|R\right|_2 + M_N \sqrt{\mathcal{E}^N(\psi,\zeta,\omega)},$$

and the control follows easily. □

3.4. Existence result

We can now establish our existence theorem. Notice that thanks to equation (3.9), we can define the Rayleigh−Taylor coefficient at time $t = 0$.

Theorem 3.6. *Let $A > 0$, $N \geq 5$, $b \in L^\infty \cap \dot{H}^{N+2}(\mathbb{R}^d)$, $P \in W^{1,\infty}(\mathbb{R}^+; \dot{H}^{N+1}(\mathbb{R}^d))$, $(\zeta_0,\psi_0,\omega_0) \in \mathrm{E}^N_0$ such that $\nabla^{\sigma,\mu}_{X,z} \cdot \omega_0 = 0$. We suppose that $(\varepsilon,\beta,\mu,\mathrm{Ro})$ satisfy (1.22). We assume also that*

$$\exists\, h_{\min},\, \mathfrak{a}_{\min} > 0,\ \varepsilon\zeta_0 + 1 - \beta b \geq h_{\min} \text{ and } \mathfrak{a}[\varepsilon\zeta, \beta b]\,(\psi,\omega)_{|t=0} \geq \mathfrak{a}_{\min}$$

and

$$\mathcal{E}^N(\zeta_0,\psi_0,\omega_0) + |\nabla P|_{L^\infty_t H^N_X} \leq A.$$

Then, there exists $T>0$, *and a unique solution* $(\zeta,\psi,\omega)\in \mathrm{E}_T^N$ *to the water waves equations* (3.2) *with initial data* $(\zeta_0,\psi_0,\omega_0)$. *Moreover,*

$$T=\min\left(\frac{T_0}{\max(\varepsilon,\beta,\frac{\varepsilon}{\mathrm{Ro}})},\frac{T_0}{|\nabla P|_{L_t^\infty H_X^N}}\right),\ \frac{1}{T_0}=c^1 \text{ and } \sup_{t\in[0,T]}\mathcal{E}^N\left(\zeta(t),\psi(t),\omega(t)\right)=c^2,$$

with $c^j=C(A,\mu_{\max},\frac{1}{h_{\min}},\frac{1}{\mathfrak{a}_{\min}},|b|_{L^\infty},|\nabla b|_{H^{N+1}},|\nabla P|_{W_t^{1,\infty}H_X^N})$.

Proof. We do not give the proof. It is very similar to Theorem 4.7 in [8]. We can regularize the system (3.2) (see Step 2 of the proof of Thm. 4.7 in [8]) and thanks to the energy estimate of Theorem 3.5 we get the existence. The uniqueness mainly follows from a similar proposition to Corollary 3.19 in [8] which shows that the operator $\mathrm{U}^{\sigma,\mu}$ has a Lipschitz dependence on its coefficients. □

4. The nonlinear shallow water equations

4.1. The context

In this part we justify rigorously the derivation of the nonlinear rotating shallow water equations from the water waves equations. We recall that, in this paper, we do not consider fast Coriolis forcing, *i.e.* $\mathrm{Ro}\le\varepsilon$. The nonlinear shallow water equations (or Saint Venant equations) is a model used by the mathematical and physical communities to study the water waves in shallow waters. Coupled with a Coriolis term, we usually describe shallow waters under the influence of the Coriolis force thanks to it (see for instance [5,21] or [31]). But to the best of our knowledge, there is no mathematical justification of this fact. Without the Coriolis term, many authors mathematically justify the Saint Venant equations; for the irrotational case, there are, for instance the works of Iguchi [16] and Alvarez−Samaniego and Lannes ([4]). It is also done in [19]. More recently, Castro and Lannes proposed a way to justify the Saint-Venant equations without the irrotational condition ([7,8]), we address here the case in which the Coriolis force is present. We denote the depth

$$h(t,X)=1+\varepsilon\zeta(t,X)-\beta b(X), \tag{4.1}$$

and the averaged horizontal velocity

$$\overline{\mathbf{V}}=\overline{\mathbf{V}}[\varepsilon\zeta,\beta b](\psi,\boldsymbol{\omega})(t,X)=\frac{1}{h(t,X)}\int_{z=-1+\beta b(X)}^{\varepsilon\zeta(t,X)}\mathbf{V}[\varepsilon\zeta,\beta b](\psi,\boldsymbol{\omega})(t,X,z)\mathrm{d}z. \tag{4.2}$$

The Saint-Venant equations (in the nondimensionalized form) are

$$\begin{cases}\partial_t\zeta+\nabla\cdot(h\overline{\mathbf{V}})=0,\\ \partial_t\overline{\mathbf{V}}+\varepsilon\left(\overline{\mathbf{V}}\cdot\nabla\right)\overline{\mathbf{V}}+\nabla\zeta+\frac{\varepsilon}{\mathrm{Ro}}\overline{\mathbf{V}}^{\perp}=-\nabla P.\end{cases} \tag{4.3}$$

It is well-known that the shallow water equations are wellposed (see Chap. 6 in [19] or [4] without the pressure term and the Coriolis forcing and [5]) and that we have the following Proposition.

Proposition 4.1. *Let* $t_0>\frac{d}{2}$, $s\ge t_0+1$ *and* $\zeta_0,b\in H^s(\mathbb{R}^d)$, $\overline{V}_0\in H^s(\mathbb{R}^d)^d$. *We assume that condition* (1.21) *is satisfied by* (ζ_0,b). *Assume also that* ε,β *and Ro satisfy condition* (1.22). *Then, there exists* $T>0$ *and a unique solution* $(\zeta,\overline{V})\in\mathcal{C}^0(\left[0,\frac{T}{\max(\varepsilon,\beta)}\right],H^s(\mathbb{R}^d)^{d+1})$ *to the Saint-Venant equations* (4.3) *with initial data* $\left(\zeta_0,\overline{V}_0\right)$. *Furthermore, for all* $t\le\frac{T}{\max(\varepsilon,\beta)}$,

$$\frac{1}{T}=c^1 \text{ and } |\zeta(t,\cdot)|_{H^s}+\left|\overline{V}(t,\cdot)\right|_{H^s}\le c^2,$$

with $c^j=C(\frac{1}{\mathrm{h}_{\min}},|\zeta_0|_{H^s},|b|_{H^s},\left|\overline{V}_0\right|_{H^s})$.

4.2. Asymptotic expansion with respect to μ

In this part, we study the dependence of U^μ with respect to μ. The first Proposition shows that $\overline{\mathbf{V}}$ is linked to $\underline{\mathrm{U}}^\mu \cdot N^\mu$.

Proposition 4.2. *Under the assumptions of Theorem 2.8, we have*

$$\underline{\mathrm{U}}^\mu \cdot N^\mu = -\mu\nabla \cdot \left(h\overline{\mathbf{V}}\right).$$

Proof. This proof is similar to Proposition 3.35 in [19]. Consider φ smooth and compactly supported in $\mathbb{R}^d$. Then, a simple computation gives

$$\begin{aligned}\int_{\mathbb{R}^d} \varphi \mathbf{U}^\mu \cdot N^\mu \mathrm{d}X &= \int_{\Omega} \nabla^\mu_{X,z} \cdot (\varphi \mathbf{U}^\mu)\, \mathrm{d}X\mathrm{d}z,\\ &= \int_{\Omega} \mu\nabla\varphi \cdot \mathbf{V}\mathrm{d}X\mathrm{d}z,\\ &= -\mu \int_{\mathbb{R}^d} \varphi\nabla \cdot \left(\int_{z=-1+\beta b}^{\varepsilon\zeta} \mathbf{V}\right)\mathrm{d}X.\end{aligned}$$

□

Then we need an asymptotic expansion with respect to μ of $\mathbf{U}^\mu$.

Proposition 4.3. *Let $t_0 > \frac{d}{2}$, $0 \leq s \leq t_0$, $\zeta \in H^{t_0+2}(\mathbb{R}^d)$, $b \in L^\infty \cap \dot{H}^{t_0+2}(\mathbb{R}^d)$. Under the assumptions of Theorem 2.8, we have*

$$\mathbf{U}^\mu = \begin{pmatrix} \sqrt{\mu}\overline{\mathbf{V}} + \mu\left(\int_z^{\varepsilon\zeta} \boldsymbol{\omega}_h^\perp - \mathbf{Q}\right) + \mu^{\frac{3}{2}}\widetilde{\mathbf{V}} \\ \mu\widetilde{\mathbf{w}} \end{pmatrix},$$

with

$$\mathbf{Q}(X) = \frac{1}{h(X)} \int_{z'=-1+\beta b(X)}^{\varepsilon\zeta(X)} \int_{s=z'}^{\varepsilon\zeta(X)} \boldsymbol{\omega}_h^\perp(X,s),$$

and

$$\left|\left|\widetilde{\mathbf{V}} \circ \Sigma\right|\right|_{H^{s,1}} + \left|\left|\widetilde{\mathbf{w}} \circ \Sigma\right|\right|_{H^{s,1}} \leq C\left(\frac{1}{\mathrm{h}_{\min}}, \varepsilon\left|\zeta\right|_{H^{t_0+2}}, \beta\left|b\right|_{L^\infty}, \beta\left|\nabla b\right|_{H^{t_0+1}}\right) \left|\left|\mathbf{V}\circ\Sigma\right|\right|_{H^{t_0+2,1}}.$$

Proof. This proof is inspired from the computations of Section 2.2 in [7] and Section 5.7.1 [8]. First, using the Previous Proposition, we get that

$$\underline{\mathrm{w}} = \varepsilon\mu\nabla\zeta \cdot \underline{\mathrm{V}} - \mu\nabla \cdot \left(h\overline{\mathbf{V}}\right).$$

Furthermore, using the fact that $\mathbf{U}^\mu$ is divergence free we have

$$\partial_z \mathbf{w} = -\mu\nabla_X \cdot \mathbf{V}.$$

Then, we obtain

$$\begin{aligned}\mathbf{w} &= \varepsilon\mu\nabla\zeta \cdot \underline{\mathrm{V}} - \mu\nabla \cdot \left(h\overline{\mathbf{V}}\right) + \mu\int_z^{\varepsilon\zeta} \nabla_X \mathbf{V}\\ &= -\mu\nabla_X \cdot \left(\int_{-1+\beta b}^{z} \mathbf{V}\right).\end{aligned}$$

The control of $\widetilde{\mathbf{w}}$ follows easily. Furthermore, using the ansatz

$$\mathbf{V} = \overline{\mathbf{V}} + \sqrt{\mu}\mathbf{V}_1, \tag{4.4}$$

and plugging it into the orthogonal of the horizontal part of $\mathrm{curl}^{\mu}\mathbf{U}^{\mu} = \mu\boldsymbol{\omega}$, we get that

$$\partial_z \mathbf{V}_1 = \sqrt{\mu}\nabla_X \widetilde{\mathbf{w}} - \boldsymbol{\omega}_h^{\perp}.$$

Then, integrating with respect to z the previous equation from z to $\varepsilon\zeta(X)$ we get

$$\mathbf{V}_1(X,z) = \int_{s=z}^{\varepsilon\zeta(X)} \boldsymbol{\omega}_h^{\perp}(X,s)\mathrm{d}s + \underline{\mathbf{V}_1}(X) + \mu^{\frac{1}{2}}\mathbf{R}(X,z), \tag{4.5}$$

where $\mathbf{R}$ is a remainder uniformly bounded with respect to μ and

$$\underline{\mathbf{V}_1} = \frac{\underline{\mathbf{V}} - \overline{\mathbf{V}}}{\sqrt{\mu}}.$$

Integrating equation (4.4) with respect to z from $-1+\beta$ to $\varepsilon\zeta$ we obtain that

$$\int_{z=-1+\beta b(X)}^{\varepsilon\zeta(X)} \mathbf{V}_1(X,z)\mathrm{d}z = 0, \, \forall X \in \mathbb{R}^d.$$

Then, we integrate equation (4.5) with respect to z from $-1+\beta b$ to $\varepsilon\zeta$ and we get

$$h\underline{\mathbf{V}_1} = -\int_{z'=-1+\beta b}^{\varepsilon\zeta} \int_{s=z'}^{\varepsilon\zeta} \boldsymbol{\omega}_h^{\perp} + \mu^{\frac{1}{2}}\widetilde{\mathbf{R}},$$

where $\widetilde{\mathbf{R}}$ is a remainder uniformly bounded with respect to μ. Plugging the previous expression into equation (4.5), we get the result. The control of the remainders is straightforward thanks to Lemma 2.9 (see also the comments about the notations of [8] in Sect. 2.3). □

Remark 4.4. Under the assumptions of the previous Proposition, it is easy to check that

$$\mathbf{w} = -\mu\nabla_X \cdot \left([1+z-\beta b]\,\overline{\mathbf{V}}\right) + \mu^{\frac{3}{2}}\mathbf{w}_1,$$

with

$$||\mathbf{w}_1 \circ \Sigma||_{H^{s,1}} \leq C\left(\frac{1}{\mathrm{h_{min}}}, \varepsilon\left|\zeta\right|_{H^{t_0+2}}, \beta\left|b\right|_{L^\infty}, \beta\left|\nabla b\right|_{H^{t_0+1}}\right)||\mathbf{V}\circ\Sigma||_{H^{t_0+2,1}}. \tag{4.6}$$

Then, we define the quantity

$$\mathbf{Q} = \mathbf{Q}[\varepsilon\zeta,\beta b](\psi,\boldsymbol{\omega})(t,X) = \frac{1}{h}\int_{z'=-1+\beta b}^{\varepsilon\zeta}\int_{s=z'}^{\varepsilon\zeta} \boldsymbol{\omega}_h^{\perp}. \tag{4.7}$$

The following Proposition shows that $\mathbf{Q}$ satisfies the evolution equation

$$\partial_t \mathbf{Q} + \varepsilon\left(\overline{\mathbf{V}}\cdot\nabla\right)\mathbf{Q} + \varepsilon\left(\mathbf{Q}\cdot\nabla\right)\overline{\mathbf{V}} + \frac{\varepsilon}{\mathrm{Ro}}\mathbf{Q}^{\perp} = 0, \tag{4.8}$$

up to some small terms.

Proposition 4.5. *Let $T>0$, $t_0>\frac{d}{2}$, $0\leq s\leq t_0$, $0\leq\mu\leq 1$, $\zeta\in\mathcal{C}^1([0,T];H^{t_0+2}(\mathbb{R}^d))$, $b\in L^\infty\cap\dot{H}^{t_0+2}(\mathbb{R}^d)$. Let $\boldsymbol{\omega},\mathbf{V},\mathbf{w}\in\mathcal{C}^1([0,T];H^{t_0+2}(\mathbb{R}^d))$. Suppose that we are under the assumption of Theorem 2.8, that $\boldsymbol{\omega}$ satisfies the third equation of the Castro–Lannes system (1.19) (the vorticity equation) and that $\partial_t\zeta+\nabla\cdot\left(h\overline{\mathbf{V}}\right)=0$, on $[0,T]$. Then $\mathbf{Q}$ satisfies*

$$\partial_t\mathbf{Q}+\varepsilon\left(\overline{\mathbf{V}}\cdot\nabla\right)\mathbf{Q}+\varepsilon\left(\mathbf{Q}\cdot\nabla\right)\overline{\mathbf{V}}+\frac{\varepsilon}{\mathrm{Ro}}\mathbf{Q}^\perp=\sqrt{\mu}\max\left(\varepsilon,\frac{\varepsilon}{\mathrm{Ro}}\right)\widetilde{\mathbf{R}},$$

and

$$\left|\left|\widetilde{\mathbf{R}}\circ\Sigma\right|\right|_{H^{s,1}}\leq C\left(\frac{1}{\mathrm{h_{min}}},\varepsilon\left|\zeta\right|_{H^{t_0+2}},\beta\left|b\right|_{L^\infty},\beta\left|\nabla b\right|_{H^{t_0+1}}\right)\left|\left|\mathbf{V}\circ\Sigma\right|\right|_{H^{t_0+2,1}}.$$

Proof. This proof is inspired from Section 2.3 in [7]. We know that $\boldsymbol{\omega}_h$ satisfies

$$\partial_t\boldsymbol{\omega}_h+\varepsilon\left(\mathbf{V}\cdot\nabla\right)\boldsymbol{\omega}_h+\frac{\varepsilon}{\mu}\mathbf{w}\partial_z\boldsymbol{\omega}_h=\varepsilon\left(\boldsymbol{\omega}_h\cdot\nabla\right)\mathbf{V}+\frac{\varepsilon}{\sqrt{\mu}}\left(\boldsymbol{\omega}_v+\frac{1}{\mathrm{Ro}}\right)\partial_z\mathbf{V}.$$

Using Proposition 4.2 and Remark 4.4 and the fact that $\boldsymbol{\omega}_v=\nabla^\perp\cdot\mathbf{V}$, we get

$$\partial_t\boldsymbol{\omega}_h+\varepsilon\left(\overline{\mathbf{V}}\cdot\nabla\right)\boldsymbol{\omega}_h-\varepsilon\nabla_X\cdot\left(\left[1+z-\beta b\right]\overline{\mathbf{V}}\right)\partial_z\boldsymbol{\omega}_h=\varepsilon\left(\boldsymbol{\omega}_h\cdot\nabla\right)\overline{\mathbf{V}}-\varepsilon\left(\nabla^\perp\cdot\overline{\mathbf{V}}+\frac{1}{\mathrm{Ro}}\right)\boldsymbol{\omega}_h^\perp+\sqrt{\mu}\max\left(\varepsilon,\frac{\varepsilon}{\mathrm{Ro}}\right)\mathbf{R},$$

where $\mathbf{R}\circ\Sigma$ satisfies the same estimate as $\mathbf{w}_1\circ\Sigma$ in (4.6). If we denote $\mathbf{V}_{sh}=\int_z^{\varepsilon\zeta}\boldsymbol{\omega}_h^\perp$, doing the same computations as in Section 2.3 [7] and using the fact that $\partial_t\zeta+\nabla\cdot\left(h\overline{\mathbf{V}}\right)=0$, we get

$$\partial_t\mathbf{V}_{sh}+\varepsilon\left(\overline{\mathbf{V}}\cdot\nabla\right)\mathbf{V}_{sh}+\varepsilon\left(\mathbf{V}_{sh}\cdot\nabla\right)\overline{\mathbf{V}}-\nabla\cdot\left(\left[1+z-\beta b\right]\overline{\mathbf{V}}\right)+\frac{\varepsilon}{\mathrm{Ro}}\mathbf{V}_{sh}^\perp=\sqrt{\mu}\max\left(\varepsilon,\frac{\varepsilon}{\mathrm{Ro}}\right)\int_z^{\varepsilon\zeta}\mathbf{R}.$$

Then, integrating this expression with respect to z and using again the fact that $\partial_t\zeta+\nabla\cdot\left(h\overline{\mathbf{V}}\right)=0$, we get

$$\partial_t\mathbf{Q}+\varepsilon\left(\overline{\mathbf{V}}\cdot\nabla\right)\mathbf{Q}+\varepsilon\left(\mathbf{Q}\cdot\nabla\right)\overline{\mathbf{V}}+\frac{\varepsilon}{\mathrm{Ro}}\mathbf{Q}^\perp=\sqrt{\mu}\max\left(\varepsilon,\frac{\varepsilon}{\mathrm{Ro}}\right)\int_{-1+\beta b}^{\varepsilon\zeta}\int_z^{\varepsilon\zeta}\mathbf{R},$$

and the result follows easily. □

4.3. Rigorous derivation

The purpose of this part is to prove a rigorous derivation of the water waves equations to the shallow water equations. This part is devoted to the proof of the following Theorem. We recall that Σ is defined in (2.16).

Theorem 4.6. *Let $N\geq 6$, $0\leq\mu\leq 1$, $\varepsilon,\beta,\mathrm{Ro}$ satisfying (1.22). We assume that we are under the assumptions of Theorem 3.6. Then, we can define the following quantity $\boldsymbol{\omega}_0=\omega_0\circ\Sigma^{-1}$, $\boldsymbol{\omega}=\omega\circ\Sigma^{-1}$, $\overline{\mathbf{V}}_0=\overline{\mathbf{V}}[\varepsilon\zeta_0,\beta b](\psi_0,\boldsymbol{\omega}_0)$, $\overline{\mathbf{V}}=\overline{\mathbf{V}}[\varepsilon\zeta,\beta b](\psi,\boldsymbol{\omega})$, $\mathbf{Q}_0=\mathbf{Q}[\varepsilon\zeta_0,\beta b](\psi_0,\boldsymbol{\omega}_0)$ and $\mathbf{Q}=\mathbf{Q}[\varepsilon\zeta,\beta b](\psi,\boldsymbol{\omega})$and there exists a time $T>0$ such that*

(i) *T has the form*

$$T=\min\left(\frac{T_0}{\max(\varepsilon,\beta,\frac{\varepsilon}{Ro})},\frac{T_0}{\left|\nabla P\right|_{L_t^\infty H_X^N}}\right)\qquad and\qquad \frac{1}{T_0}=c^1.$$

(ii) *There exists a unique solution $\left(\zeta_{SW},\overline{\mathbf{V}}_{SW}\right)$ of (4.3) with initial conditions $\left(\zeta_0,\overline{\mathbf{V}}_0\right)$ on $[0,T]$.*
(iii) *There exists a unique solution $\mathbf{Q}_{SW}$ to equation (4.8) on $[0,T]$.*
(iv) *There exists a unique solution (ζ,ψ,ω) of (3.2) with initial conditions $(\zeta_0,\psi_0,\omega_0)$ on $[0,T]$.*
(v) *The following error estimates hold, for $0\leq t\leq T$,*

$$\left|\left(\zeta,\overline{\mathbf{V}},\sqrt{\mu}\mathbf{Q}\right)-\left(\zeta_{SW},\overline{\mathbf{V}}_{SW},\sqrt{\mu}\mathbf{Q}_{SW}\right)\right|_{L^{\infty}([0,t]\times\mathbb{R}^{d})}\leq\mu\, tc^{2},$$

and

$$\left|\underline{\mathbf{V}}-\overline{\mathbf{V}}+\sqrt{\mu}\mathbf{Q}\right|_{L^{\infty}([0,T]\times\mathbb{R}^{d})}\leq\mu\, c^{3},$$

with $c^j = C(A, \mu_{\max}, \frac{1}{h_{\min}}, \frac{1}{\mathfrak{a}_{\min}}, |b|_{L^\infty}, |\nabla b|_{H^{N+1}}, |\nabla P|_{W_t^{1,\infty}H_X^N})$.

Remark 4.7. Hence, in shallow waters the rotating Saint-Venant equations are a good model to approximate the water waves equations under a Coriolis forcing. Furthermore, we notice that if we start initially with a irrotational flow, at the order μ, the flow stays irrotational. It means that a Coriolis forcing (not too fast) does not generate a horizontal vorticity in shallow waters and the assumption of a columnar motion, which is the fact that the velocity is horizontal and independent of the vertical variable z, stays valid. It could be interesting to develop an asymptotic model of the water waves equations at the order μ^2 (Green−Naghdi or Boussinesq models) and study the influence a Coriolis forcing in these models. It will be done in a future work [24].

Proof. The point (ii) follows from Proposition 4.1 and the point(iv) from Theorem 3.6. Since, equation (4.8) is linear, the point (iii) is clear. We only need to show that $\left(\zeta,\overline{\mathbf{V}}\right)$ satisfy the shallow water equations up to a remainder of order μ. Then, a small adaptation of Proposition 6.3 in [19] allows us to prove the point (v). First, we know that

$$\partial_t\psi+\zeta+\frac{\varepsilon}{2}\left|\mathrm{U}_{/\!/}^{\mu}\right|^2-\frac{\varepsilon}{2\mu}\left(1+\varepsilon^2\mu\left|\nabla\zeta\right|^2\right)\underline{\mathbf{w}}^2+\varepsilon\frac{\nabla}{\Delta}\cdot\left[\left(\underline{\boldsymbol{\omega}}\cdot N^{\mu}+\frac{1}{\mathrm{Ro}}\right)\underline{\mathbf{V}}^{\perp}\right]=-P,$$

and

$$\partial_t\left(\underline{\omega}\cdot N^{\mu}\right)+\varepsilon\nabla\cdot\left(\left[\underline{\omega}\cdot N^{\mu}+\frac{1}{\mathrm{Ro}}\right]\underline{\mathrm{V}}\right)=0.$$

Since $\mathrm{U}_{/\!/}^{\mu}=\nabla\psi+\frac{\nabla^{\perp}}{\Delta}\left(\underline{\boldsymbol{\omega}}\cdot N^{\mu}\right)$, we get that

$$\partial_t\mathrm{U}_{/\!/}^{\mu}+\nabla\zeta+\frac{\varepsilon}{2}\nabla\left|\mathrm{U}_{/\!/}^{\mu}\right|^2-\frac{\varepsilon}{2\mu}\nabla\left[\left(1+\varepsilon^2\mu\left|\nabla\zeta\right|^2\right)\underline{\mathbf{w}}^2\right]+\varepsilon\left(\underline{\boldsymbol{\omega}}\cdot N^{\mu}+\frac{1}{\mathrm{Ro}}\right)\underline{\mathbf{V}}^{\perp}=-\nabla P.$$

Then, using Proposition 4.3 and plugging the fact that $\mathrm{U}_{/\!/}^{\mu}=\overline{\mathbf{V}}-\sqrt{\mu}\mathbf{Q}+\mu\mathbf{R}$, we get

$$\partial_t\overline{\mathbf{V}}+\varepsilon\left(\overline{\mathbf{V}}\cdot\nabla\right)\overline{\mathbf{V}}+\nabla\zeta+\frac{\varepsilon}{\mathrm{Ro}}\overline{\mathbf{V}}^{\perp}+\nabla P-\sqrt{\mu}\Big(\partial_t\mathbf{Q}+\varepsilon\left(\overline{\mathbf{V}}\cdot\nabla\right)\mathbf{Q}+\varepsilon\left(\mathbf{Q}\cdot\nabla\right)\overline{\mathbf{V}}+\frac{\varepsilon}{\mathrm{Ro}}\mathbf{Q}^{\perp}\Big)=-\mu\partial_t\mathbf{R}+\widetilde{\mathbf{R}},$$

and using the same idea as Proposition 4.3, it is easy to check that

$$\left|\left|\widetilde{\mathbf{R}}\circ\Sigma\right|\right|_{H^{2,1}}+\left|\left|\partial_t\mathbf{R}\circ\Sigma\right|\right|_{H^{2,1}}\leq C\left(\frac{1}{\mathrm{h}_{\min}},\varepsilon\left|\zeta\right|_{H^4},\varepsilon\left|\partial_t\zeta\right|_{H^4},\beta\left|b\right|_{L^\infty},\beta\left|\nabla b\right|_{H^3}\right)\times\left(\left|\left|\mathbf{V}\circ\Sigma\right|\right|_{H^{4,1}}+\left|\left|\partial_t\mathbf{V}\circ\Sigma\right|\right|_{H^{4,1}}\right).$$

Using Proposition 4.5, Theorem 3.6, Theorems 2.8 and 2.11 and Remark 2.14, we get the result. □

Appendix A. Useful estimates

In this part, we give some classical estimates. See [3, 19] or [18] for the proofs.

Lemma A.1. *Let* $u\in W^{1,\infty}(\mathbb{R}^d)$ *and* $v\in H^{\frac{1}{2}}(\mathbb{R}^d)$ *. Then,*

$$\left|\sqrt{\mu}\mathfrak{P}\left(uv\right)\right|_2\leq C\left(\mu_{\max}\right)\left|u\right|_{W^{1,\infty}(\mathbb{R}^d)}\left|\sqrt{1+\sqrt{\mu}|D|}v\right|_2.$$

Lemma A.2. *Let $t_0 > \frac{d}{2}$, $u \in H^{t_0+1}(\mathbb{R}^d)$ and $v \in H^{\frac{1}{2}}(\mathbb{R}^d)$. Then,*

$$\left|[\mathfrak{P}, u]\, v\right|_2 \leq C\, |u|_{H^{t_0+1}}\, |v|_2\, .$$

Lemma A.3. *Let $s > \frac{d}{2} + 1$. Then, for $f, g \in L^2\left(\mathbb{R}^d\right)$,*

$$\left|[\Lambda^s, f]\, g\right|_2 \leq C\, |f|_{H^s}\, |g|_{H^{s-1}}\, .$$

Lemma A.4. *Let s, s_1, $s_2 \in \mathbb{R}$ such that $s_1+s_2 \geq 0$, $s \leq \min(s_1, s_2)$ and $s < s_1+s_2-\frac{d}{2}$. Then, for $f \in H^{s_1}(\mathbb{R}^d)$ and $g \in H^{s_2}(\mathbb{R}^d)$, we have $fg \in H^s(\mathbb{R}^d)$ and*

$$|fg|_{H^s} \leq C\, |f|_{H^{s_1}}\, |g|_{H^{s_2}}\, .$$

We also give a regularity estimate for functions in $H_*^{-\frac{1}{2}}(\mathbb{R}^d)$.

Lemma A.5. *Let $s \geq 0$ and $u \in H_*^{-\frac{1}{2}}(\mathbb{R}^d) \cap H^{s-\frac{1}{2}}(\mathbb{R}^d)$. Then $u \in H_*^{s-\frac{1}{2}}(\mathbb{R}^d)$ and*

$$\left|\frac{1}{\mathfrak{P}} u\right|_{H^s} \leq \left|\frac{1}{\mathfrak{P}} u\right|_2 + \left|\sqrt{1+\sqrt{\mu}|D|}\, u\right|_{H^{s-1}}.$$

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