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STABLE PERFECTLY MATCHED LAYERS FOR A CLASS OF ANISOTROPIC DISPERSIVE MODELS. PART I: NECESSARY AND SUFFICIENT CONDITIONS OF STABILITY *

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Abstract. In this work we consider the problem of modelling of 2D anisotropic dispersive wave propagation in unbounded domains with the help of perfectly matched layers (PMLs). We study the Maxwell equations in passive media with a frequency-dependent diagonal tensor of dielectric permittivity and magnetic permeability. An application of the traditional PMLs to this kind of problems often results in instabilities. We provide a recipe for the construction of new, stable PMLs. For a particular case of non-dissipative materials, we show that a known necessary stability condition of the perfectly matched layers is also sufficient. We illustrate our statements with theoretical and numerical arguments.

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1. Introduction

The problem of wave propagation in anisotropic dispersive media, *e.g.* in negative index metamaterials or plasmas, is both of theoretical and practical interest. For some applications it is necessary to model the time-domain wave propagation in unbounded (or semi-bounded) domains. One of the ways to bound the computational domain is offered by the use of the perfectly matched layers (PMLs), a method introduced by Bérenger in [4,6]. For an application of this technique to some dispersive materials, for example, plasmas, see [7,8,15,28,29], and for the use of the method in isotropic metamaterials see the papers by Cummer [16] and Bécache *et al.* [9]. However, it is well known that the PMLs often exhibits instabilities in the presence of dispersion and/or anisotropy [9,10]. Multiple attempts were made to overcome this problem [1], but, to our knowledge, no recipe to construct stable PMLs for an arbitrary hyperbolic system exists.

Indeed, there are other ways to tackle the problem of the unboundedness of the domain; a non-exhaustive list of those includes the FEM/BEM coupling [2,3], methods based on pole condition [32,47] or various absorbing boundary conditions [13,21,27,31,37].

Keywords and phrases. Perfectly matched layers, stability, Maxwell equations, passive metamaterials, Laplace transform.

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We concentrate on the construction of stable in the time domain PMLs for a class of anisotropic dispersive models that can be described in the frequency domain as a wave equation with frequency-dependent coefficients

$$\varepsilon_1(\omega)^{-1}\partial_x^2 u + \varepsilon_2(\omega)^{-1}\partial_y^2 u + \omega^2\mu(\omega)u = 0,\ \omega \in \mathbb{R}, \tag{1.1}$$

where ε_1, ε_2, μ have the meaning of dielectric permittivity and magnetic permeability. This class of models generalizes isotropic materials considered in [9], where authors use $\varepsilon_1(\omega) = \varepsilon_2(\omega)$. A notable example of such models includes a uniaxial cold plasma model in two dimensions [8]. In this work we will limit ourselves to so-called passive systems [51], *i.e.* systems for which $\varepsilon_{1,2}(\omega),\ \mu(\omega)$ satisfy $\operatorname{Im}(\omega\varepsilon_i(\omega)) > 0,\ i = 1, 2,\ \operatorname{Im}(\omega\mu(\omega)) \geq 0$ for $\operatorname{Im}\omega > 0$, and are analytic in the upper complex half-plane.

The new (to our knowledge) results obtained in this work include:

(1) a simple recipe of the construction of stable PMLs for passive anisotropic systems (1.1), even in the case when the coefficients in (1.1) correspond to non-local in time operators, or when the model (1.1) is dissipative. Our results extend the ideas of Bécache *et al.* [9] to a more general class of models;
(2) an easy to check constructive stability condition of the PML for a sub-class of problems (1.1) where $\varepsilon_i,\ i = 1, 2$ and μ satisfy additional requirements. More precisely, we show that the necessary condition of the stability of the PML formulated in [9] in terms of the directions of phase and group velocity can be rewritten in an easy-to-check form. Moreover, it is also sufficient for the stability of the PMLs. This extends the existing result [9] for isotropic materials.

An important difference between our work and [9] (as well as many other works on PMLs) is that we perform the analysis by using the Laplace transform in time, rather than working with plane waves which come from the Fourier analysis. This allows to avoid the discussion of the primary cause of the instabilities of the PMLs, namely the presence of so-called backward propagating waves; an interested reader may consult [9] and [10]. To our knowledge, in very few works the PML had been studied in the Laplace domain setting, see *e.g.* articles for the well-posedness of non-dispersive systems by Halpern *et al.* [33] or for convergence of the radial PML by Chen [17]. Such an analysis allows to obtain quite easily sufficient conditions of stability of the PML, while it is easier to derive necessary stability conditions using plane wave techniques.

This article is organized as follows. In Section 2 we discuss in more detail a class of problems of interest, providing some important examples. Next, in Section 3 we connect the passivity requirement to the stability of the problem (1.1) in the time domain. In Section 4 we discuss the construction of stable PMLs for general passive materials. In the second part of the article, we concentrate on a subclass of materials, considered in [9], and characterized in Section 5. Based on the properties of these materials and the necessary stability condition of [9], we provide a criterion of the stability of PMLs for (1.1) in Section 6.

We illustrate our results with the numerical experiments in Sections 4.1, 4.3 and 6.4.

2. Problem setting

We consider a problem of the wave propagation in dispersive, anisotropic media, which is described by the Maxwell equations. In particular, in the time domain (using the scaling $\varepsilon_0 = \mu_0 = c = 1$) in $\mathbb{R}^2$ it reads

$$\begin{aligned}
\partial_t D_x &= \partial_y H_z,\\
\partial_t D_y &= -\partial_x H_z,\\
\partial_t B_z &= -\partial_x E_y + \partial_y E_x.
\end{aligned} \tag{2.1}$$

The relation between the fields $\mathbf{D}$ and $\mathbf{E}$, and H_z and B_z is not necessarily explicitly known in the time domain, but often is given in the Laplace domain. For simplicity, we will consider the above problem with nonvanishing source terms, but zero initial conditions. We will denote by s the Laplace variable, by $\hat{u} = \mathcal{L}u$ the Laplace transform of u in time, and by $\mathbb{C}_+$ the right complex half-plane: $\mathbb{C}_+ = \{s \in \mathbb{C}: \operatorname{Re} s > 0\}$. Additionally, given a function $f(s): \mathbb{C}_+ \cup i\mathbb{R} \to \mathbb{C}$, we will denote

$$\tilde{f}(\omega) = f(-i\omega), \ \operatorname{Im}\omega \geq 0. \tag{2.2}$$

Then the following identities hold true: $\hat{\mathbf{D}} = \underline{\underline{\varepsilon}}(s)\hat{\mathbf{E}}, \ \hat{H}_z = \mu(s)^{-1}\hat{B}_z$. In the time domain, we rewrite them as the following convolutions (where $\varepsilon(\partial_t)$ denotes a time-domain distribution with the symbol $\varepsilon(s)$):

$$\mathbf{D} = \underline{\underline{\varepsilon}}(\partial_t)\mathbf{E}, \qquad B_z = \mu(\partial_t)H_z. \tag{2.3}$$

In particular, we concentrate on the case when the tensor of dielectric permittivity is diagonal. In this case the Maxwell's equations are of the following form:

$$\mathbf{curl}\,\mu(s)^{-1}\operatorname{curl}\hat{\mathbf{E}} + s^2\begin{pmatrix}\varepsilon_1(s) & 0\\ 0 & \varepsilon_2(s)\end{pmatrix}\hat{\mathbf{E}} = 0,$$

or, alternatively, *cf.* (1.1),

$$s^2\mu(s)\hat{H}_z - \varepsilon_2(s)^{-1}\partial_x^2\hat{H}_z - \varepsilon_1(s)^{-1}\partial_y^2\hat{H}_z = 0. \tag{2.4}$$

Let us provide a few examples of such models, and comment on the state of the art of the PMLs for those:

(1) **Isotropic Drude materials**:

$$\varepsilon_1(s) = \varepsilon_2(s) = 1 + \frac{\omega_e^2}{s^2}, \qquad \mu(s) = 1 + \frac{\omega_m^2}{s^2}, \qquad \omega_e, \ \omega_m \geq 0. \tag{2.5}$$

The corresponding system in the time domain reads, *cf.* [9],

$$\begin{aligned}
&\partial_t\mathbf{E} - \mathbf{curl}\,H_z + \omega_e^2\mathbf{J} = 0,\\
&\partial_t H_z + \operatorname{curl}\mathbf{E} + \omega_m^2 K = 0,\\
&\partial_t\mathbf{J} = \mathbf{E},\\
&\partial_t K = H_z.
\end{aligned} \tag{2.6}$$

The construction of the stable PMLs for this class of models in the case $\omega_e = \omega_m$ was suggested in [16]. In [9] the authors have extended these ideas to more general cases.

(2) **Uniaxial cold plasma model**:

$$\varepsilon_1(s) = 1 + \frac{\omega_p^2}{s^2}, \ \omega_p > 0, \ \varepsilon_2(s) = \mu(s) = 1. \tag{2.7}$$

In the time domain the corresponding system reads

$$\begin{aligned}
&\partial_t E_x - \partial_y H_z + j = 0,\\
&\partial_t E_y + \partial_x H_z = 0,\\
&\partial_t H_z + \partial_x E_y - \partial_y E_x = 0,\\
&\partial_t j = \omega_p^2 E_x.
\end{aligned}$$

This model was considered in [7, 8]. The stable PMLs were constructed using an idea similar to [9].

(3) **Generalized Lorentz materials**, which generalize the previous two cases:

$$\begin{aligned}
\varepsilon_1(s) &= 1+\sum_{\ell=0}^{n_x}\frac{\varepsilon_{x\ell}}{s^2+\omega_{x\ell}^2}, \qquad \varepsilon_{x\ell}>0, \qquad \omega_{x\ell}\in\mathbb{R}, \qquad \ell=0,\dots,n_x,\\
\varepsilon_2(s) &= 1+\sum_{\ell=0}^{n_y}\frac{\varepsilon_{y\ell}}{s^2+\omega_{y\ell}^2}, \qquad \varepsilon_{y\ell}>0, \qquad \omega_{y\ell}\in\mathbb{R}, \qquad \ell=0,\dots,n_y,\\
\mu(s) &= 1+\sum_{\ell=0}^{n_\mu}\frac{\mu_{\ell}}{s^2+\omega_{\mu\ell}^2}, \qquad \mu_{\ell}>0, \qquad \omega_{\mu\ell}\in\mathbb{R}, \qquad \ell=0,\dots,n_\mu.
\end{aligned} \tag{2.8}$$

The corresponding system of equations can be constructed similarly to (2.6). For the case $\varepsilon_1(s)=\varepsilon_2(s)$, the stable PMLs for this system have been constructed in [9], by extending and justifying mathematically the idea of [16]. A major part of the present work will be devoted to extending the results of this work to an anisotropic case $\varepsilon_1\neq\varepsilon_2$. Moreover, the results of this article show how to construct the PML in a more general case, for example, in the presence of losses, when $\varepsilon_2(s)$ is defined as

$$\varepsilon_2(s)=1+\sum_{\ell=0}^{n_y}\frac{\varepsilon_{y\ell}}{s^2+2\nu_\ell s+\omega_{y\ell}^2}, \qquad \nu_\ell>0, \qquad \ell=0,\dots,n_y.$$

To our knowledge, stable PMLs for this case did not exist even for isotropic dispersive materials.

For other examples of passive materials we refer an interested reader to [18].

3. Properties of the model. Connection between passivity and stability

In the physics literature, when considering the Maxwell equations (2.1) in so-called passive media, one assumes that ε_1, ε_2 and μ satisfy a certain property [38, 51], which we will call passivity, see the definition below.

Definition 3.1. A function $c:\mathbb{C}_+\to\mathbb{C}$ is passive if it is analytic in $\mathbb{C}_+$ and satisfies $\operatorname{Re}(sc(s))>0$ in $\mathbb{C}_+$.

From now on we will assume that ε_1, ε_2, μ are passive.

Remark 3.2 (Passivity in the Fourier domain)**.** Given a passive function $c(s)$, notice that the function $f_c(z):=zc(-iz)$ with z satisfying $\operatorname{Im} z>0$ is a Herglotz function (*i.e.* it is holomorphic in the upper half-plane $\operatorname{Im} z>0$, and $\operatorname{Im} f(z)\geq 0$ there). This can be seen by setting $z=is$:

$$\operatorname{Im}(zc(-iz))=\operatorname{Re}(-izc(-iz))=\operatorname{Re}(sc(s)).$$

Remark 3.3. Although we concentrate our presentation on the wave propagation problems (2.1), the model (2.4), which we will study in the present work, describes other phenomenas, for example, heat transfer (with a formal choice $\varepsilon_1(s)=\varepsilon_2(s)=1$, $\mu(s)=s^{-1}$).

We proceed as follows. First we recall some (partially known) properties of passive functions, and next show that the passivity is **sufficient** for the well-posedness and stability of the problem (2.4).

3.1. Bounds on passive functions

Here we provide bounds on functions passive in the sense of Definition 3.1.

Lemma 3.4 (Bounds on passive functions)**.**
Let $c(s)$ be passive in the sense of Definition 3.1. Then

(1) *there exists $C_1>0$, s.t. for $s\in\mathbb{C}_+$, the function $c(s)$ satisfies*

$$|c(s)|\leq C_1|s|\left(\operatorname{Re} s\right)^{-1}\max\left(1,\left(\operatorname{Re} s\right)^{-2}\right). \tag{3.1}$$

(2) *there exists $C_2 > 0$, s.t. for all $s \in \mathbb{C}_+$, it holds that*

$$\operatorname{Re}(sc(s)) > C_2|s|^{-2}\min(1,(\operatorname{Re} s)^2)\operatorname{Re} s. \tag{3.2}$$

Proof. See Appendix A. □

From the above we immediately obtain a similar result for a function $b(s)$ which satisfies $\operatorname{Re}(\bar{s}b(s)) > 0$ in $\mathbb{C}_+$. Notice that given an analytic in $\mathbb{C}_+$ function $b(s)$

$$\operatorname{Re}(\bar{s}b(s)) > 0,\ s \in \mathbb{C}_+ \iff \operatorname{Re}\left(\frac{b(s)}{s}\right) > 0,\ s \in \mathbb{C}_+ \iff \operatorname{Re}\left(\frac{s}{b(s)}\right) > 0,\ s \in \mathbb{C}_+. \tag{3.3}$$

The function $b(s)^{-1}$ is well-defined in $\mathbb{C}_+$, since $b(s)$ does not vanish there. With the help of the above, we obtain the following simple result on the behaviour of such a function $b(s)$.

Corollary 3.5. *Let $b(s)$ be analytic in $\mathbb{C}_+$ and $\operatorname{Re}(\bar{s}b(s)) > 0$ there. Then*

(1) *there exists $C_1 > 0$, s.t. for $s \in \mathbb{C}_+$, it holds*

$$|b(s)| \leq C_1|s|^3(\operatorname{Re} s)^{-1}\max\left(1,(\operatorname{Re} s)^{-2}\right). \tag{3.4}$$

(2) *there exists $C_2 > 0$, s.t. for $s \in \mathbb{C}_+$*

$$\operatorname{Re}(\bar{s}b(s)) \geq C_2\min\left(1,(\operatorname{Re} s)^2\right)\operatorname{Re} s. \tag{3.5}$$

Proof. The upper and lower bounds can be obtained simply by noticing that $g_b(s) := b(s)s^{-2}$ satisfies $\operatorname{Re}(sg_b(s)) = \operatorname{Re}\left(\bar{s}|s|^{-2}b(s)\right) > 0$ in $\mathbb{C}_+$. Then both bounds (3.4, 3.5) follow immediately from (3.1, 3.2) applied to $g_b(s)$. □

3.2. Passivity implies stability: Laplace domain analysis

The goal of this section is to investigate a connection between the passivity and the stability of the Maxwell system (2.4). The main result of this section is Theorem 3.15 formulated in the end of the section.

For convenience, let us introduce a sesquilinear form $A(u,v) : H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) \to \mathbb{C}$:

$$A(u,v) = a(s)(\partial_x u, \partial_x v) + b(s)(\partial_y u, \partial_y v) + s^2 c(s)(u,v),\ s \in \mathbb{C}_+,\ u,\ v \in H^1(\mathbb{R}^2). \tag{3.6}$$

Here

$$(u,v) = \int_{\mathbb{R}^2} u(x)\bar{v}(x)\mathrm{d}x, \qquad \text{for } u,v \in L^2\left(\mathbb{R}^2\right).$$

Given $f \in H^{-1}(\mathbb{R}^2),\ u \in H^1(\mathbb{R}^2)$, we will denote by $\langle f, u\rangle$ the duality pairing induced by the above inner product (where $H^{-1}(\mathbb{R}^2)$ is a space of antilinear functionals on $H^1(\mathbb{R}^2)$).

Remark 3.6. The operator corresponding to (3.6) is defined as follows

$$\begin{aligned} \mathcal{A}_s :& \quad H^1(\mathbb{R}^2) \to H^{-1}(\mathbb{R}^2), \\ \mathcal{A}_s u =& -a(s)\partial_x^2 u - b(s)\partial_y^2 u + s^2 c(s)u. \end{aligned} \tag{3.7}$$

Indeed, we recognize the problem (2.4) with

$$a(s) = \varepsilon_2(s)^{-1},\ b(s) = \varepsilon_1(s)^{-1},\ c(s) = \mu(s). \tag{3.8}$$

Remark 3.7. The case $a(s) = b(s)$ corresponds to an isotropic medium, see also [9].

Remark 3.8. All the results of this section are valid for a bounded Lipschitz domain Ω, with the corresponding modification of the spaces for which the sesquilinear form $A(u,v)$ is defined (*e.g.* $H_0^1(\Omega)\times H_0^1(\Omega)$ for the homogeneous Dirichlet problem).

An approach that we are going to adopt here is based on the Laplace transform, see *e.g.* [26]. It is possible to obtain some of the results on the stability of the systems presented in the following sections with the help of the plane-wave analysis ([39], Chap. 2), which is discussed in Section 3.3.

The results used here are based on the theory developed in ([43], Sect. 2.1), which was, under stronger assumptions, made more precise in the following theorem of Dominguez and Sayas [22]. We present here a less refined version of this theorem. Let us remark that by causal we mean distributions ϕ defined on $\mathbb{R}$, s.t. $\langle\phi,\lambda\rangle=0$, for all $\lambda:\operatorname{supp}\lambda\subset(-\infty,0)$.

Theorem 3.9 (Prop. 3.2.2 [48], Prop. 3.2.2 [49])**.** *Let X,Y be Banach spaces, and f be an $\mathcal{L}(X,Y)$-valued causal distribution whose Laplace transform $F(s)$ exists for all $s\in\mathbb{C}_+$ and satisfies*

$$\|F(s)\|_{\mathcal{L}(X,Y)}\le C_F(\operatorname{Re}s)|s|^{\gamma},\ s\in\mathbb{C}_+,\tag{3.9}$$

where $\gamma\ge 0$ and $C_F:\ (0,\ \infty)\to(0,\ \infty)$ is a non-increasing function, s.t. $C_F(\lambda)<C\sigma^{-m}$, $C>0,\ m\ge 0$, for $\sigma\in(0,1]$. Define $k=\lfloor\gamma+2\rfloor$ (where $\lfloor k\rfloor$ denotes the integer part of k). Then for all causal $C^{k-1}(\mathbb{R})$ functions $g:\mathbb{R}\to X$ with integrable kth distributional derivative, the distribution $f*g$ is a causal continuous function $\mathbb{R}\to Y$ which satisfies the following bound with $\alpha_{\gamma,m}>0$ independent of t:

$$\|(f*g)(t)\|_Y\le\alpha_{\gamma,m}\max(t^m,1)\int\limits_0^t\|(1+\partial_\tau)^k g(\tau)\|_X\mathrm{d}\tau,\qquad t\ge 0.\tag{3.10}$$

Proof. See [48], *Proposition* 3.2.2 *for the proof.* Notice that in the derivation of the above bounds we use $t^{\varepsilon}(1+t)^{-\varepsilon}\le$ const for $\varepsilon\ge 0$ (see [48], Prop. 3.2.2 for the notation), and the bound $C_F(1/t)\le C\max(t^m,1)$. ☐

Remark 3.10. First of all, as remarked in ([48], p. 45), the bounds in the above result are non-optimal. However, provided *e.g.* a compactly supported right hand side data, the bound in (3.10) grows not faster than a polynomial in the time domain, which implies the time-domain stability.

Now let us come back to the question of the well-posedness of the variational formulation with the sesquilinear form (3.6). We will need the following assumption on coefficients of the sesquilinear form (3.6).

Definition 3.11. We will call the sesquilinear form (3.6) passive if the functions $a(s)^{-1},\ b(s)^{-1},\ c(s)$ are passive in the sense of Definition 3.1.

Our goal in this section is to show that passive sesquilinear forms define stable systems in the time domain. This motivates the name 'passive', since for the Maxwell system (2.1) the passivity requirement is connected to stability [18]. First of all, a direct application of Theorem 3.9 provides the following result.

Proposition 3.12 (Properties of passive sesquilinear forms)**.** *Let the sesquilinear form $A(u,v)$ given by* (3.6) *be passive in the sense of Definition* 3.11*. Then*:

(1) *For some $C_0,\ C_1>0$ independent of s, and for all $s\in\mathbb{C}_+$, it holds*:

$$|A(u,v)|\le C_0|s|^3(\operatorname{Re}s)^{-1}\max\left(1,(\operatorname{Re}s)^{-2}\right)\|u\|_{H^1}\|v\|_{H^1},\qquad u,v\in H^1(\mathbb{R}^2),\tag{3.11}$$

$$\operatorname{Re}\left(\mathrm{e}^{-i\operatorname{Arg}s}A(u,u)\right)\ge C_1\operatorname{Re}s|s|^{-1}\min\left(1,\ (\operatorname{Re}s)^2\right)\|u\|_{H^1}^2,\qquad u\in H^1(\mathbb{R});\tag{3.12}$$

(2) *For all $G\in H^{-1}(\mathbb{R}^2)$, for all $s\in\mathbb{C}_+$, there exists a unique solution $U(s)\in H^1(\mathbb{R}^2)$ to the variational formulation*

$$A\left(U(s),v\right)=\langle G,v\rangle,\qquad v\in H^1\left(\mathbb{R}^2\right).\tag{3.13}$$

Moreover, there exists $C > 0$, s.t. for $s \in \mathbb{C}_+$,

$$\|U(s)\|_{H^1(\mathbb{R}^2)} \leq C|s|(\operatorname{Re} s)^{-1} \max(1, (\operatorname{Re} s)^{-2})\|G\|_{H^{-1}(\mathbb{R}^2)}; \tag{3.14}$$

(3) *Let $G(s) \in H^{-1}(\mathbb{R}^2)$ be the Laplace transform of a causal C^2-function $g(t) : \mathbb{R} \to H^{-1}(\mathbb{R}^2)$, whose 3rd derivative in the sense of distributions $g^{(3)} \in L^1(\mathbb{R}, H^{-1}(\mathbb{R}^2))$. Then the solution $U(s)$, $s \in \mathbb{C}_+$, of (3.13) with $G = G(s)$ is the Laplace transform of a causal continuous function $u(t) : \mathbb{R} \to H^1(\mathbb{R}^2)$, which can be bounded as follows, with $\alpha > 0$ independent of t:*

$$\|u(t)\|_{H^1} \leq \alpha \max(t^3, 1) \int_0^t \left\|(1 + \partial_\tau)^3 g(\tau)\right\|_{H^{-1}} \mathrm{d}\tau, \qquad t \geq 0.$$

Proof. First of all, let us show that $A(u, v)$ is bounded and coercive. Indeed, given $u, v \in H^1(\mathbb{R}^2)$, using (3.1) and (3.4), we obtain, for some $C_0 > 0$ independent of s and for all $s \in \mathbb{C}_+$:

$$|A(u, v)| \leq \max\left(|a(s)|, |b(s)|, |c(s)||s|^2\right) \|u\|_{H^1}\|v\|_{H^1} \leq C_0|s|^3(\operatorname{Re} s)^{-1} \max\left(1, (\operatorname{Re} s)^{-2}\right) \|u\|_{H^1}\|v\|_{H^1}.$$

To show the coercivity, consider $u \in H^1(\mathbb{R}^2)$, and take $A(u, su) = \bar{s}A(u, u)$, for $s \in \mathbb{C}_+$:

$$\begin{aligned}\operatorname{Re}(\bar{s}A(u, u)) &= \operatorname{Re}(\bar{s}a(s)) \|\partial_x u\|_{L^2}^2 + \operatorname{Re}(\bar{s}b(s)) \|\partial_y u\|_{L^2}^2 + |s|^2 \operatorname{Re}(sc(s)) \|u\|_{L^2}^2 \\ &\geq \min(\operatorname{Re}(\bar{s}a(s)), \operatorname{Re}(\bar{s}b(s)), |s|^2 \operatorname{Re}(sc(s)))\|u\|_{H^1}^2 \geq C_1 \min(1, (\operatorname{Re} s)^2) \operatorname{Re} s\|u\|_{H^1}^2, \ C_1 > 0,\end{aligned}$$

where the last inequality was obtained with the help of (3.2) and (3.5). The above coercivity estimate is obtained for $\bar{s}A(u, u)$, hence, after the division by $|s|$ both sides of the above inequality, we obtain the estimate (3.12).

The second statement is obtained by a direct application of the Lax–Milgram lemma.

To obtain the third statement, we use the invertibility of the operator (3.7). Let us take $s_0 \in \mathbb{C}_+$, and define $T_s = \mathcal{A}_{s_0}^{-1}\mathcal{A}_s$ a bounded invertible operator from $H^1(\mathbb{R}^2)$ into $H^1(\mathbb{R}^2)$. Due to ([25], p. 592, Lem. 13), T_s^{-1} is analytic in $\mathbb{C}_+$ as an operator-valued function, which implies, thanks to ([44], Props. 3.1, 3.2), that T_s^{-1}, and thus $\mathcal{A}_s^{-1}$, is a Laplace transform of a causal distribution. Additionally, the bound $\|\mathcal{A}_s^{-1}\|_{H^{-1}\to H^1} \leq C_1^{-1}|s|(\operatorname{Re} s)^{-1} \max\left(1, (\operatorname{Re} s)^{-2}\right)$ holds for $s \in \mathbb{C}_+$. Then the statement of the proposition follows by the application of Theorem 3.9 to $U(s) = \mathcal{A}_s^{-1}G(s)$:

$$\|u(t)\|_{H^1} = \left\|\left(\mathcal{L}^{-1}U(s)\right)(t)\right\|_{H^1} \leq \alpha \max(t^3, 1) \int_0^t \left\|(1 + \partial_\tau)^3 g(\tau)\right\|_{H^{-1}} \mathrm{d}\tau, \qquad t \geq 0, \qquad \alpha > 0. \qquad \square$$

Let us now apply the above results to the model (2.4) with source terms, more precisely,

$$\begin{aligned}&\partial_t \mathbf{D} - \operatorname{curl} H_z = \mathbf{f_D}, \qquad &&\partial_t B_z + \operatorname{curl} \mathbf{E} = f_z, \\ &\mathbf{D} = \underline{\underline{\varepsilon}}(\partial_t)\mathbf{E}, &&B_z = \mu(\partial_t)H_z.\end{aligned} \tag{3.15}$$

Recall that the notation $\mu(\partial_t)H_z$ corresponds to the convolution

$$\mu(\partial_t)H_z = \int_0^t M(t - \tau)H_z(\tau)\mathrm{d}\tau,$$

where M is the inverse Laplace transform of $\mu(s)$, see [45].

We define the regularity for the right-hand side, for any $k, m \in \mathbb{N}_0$, as follows:

$$\begin{aligned}&\mathbf{f}_D \in C^k\left(\mathbb{R}; \left(L^2(\mathbb{R}^2)\right)^2\right), \qquad &&\mathbf{f}_D^{(k+1)} \in L^1\left(\mathbb{R}; \left(L^2(\mathbb{R}^2)\right)^2\right), \\ &f_z \in C^m\left(\mathbb{R}; L^2(\mathbb{R}^2)\right), &&f_z^{(m+1)} \in L^1\left(\mathbb{R}; L^2(\mathbb{R}^2)\right).\end{aligned} \tag{R_{km}}$$

Definition 3.13. We will call the problem (3.15) well-posed, if there exist $k_*, m_* \in \mathbb{N}_0$, s.t. for any causal right-hand side data satisfying (R_{km}) for some $k \geq k_*$ and $m \geq m_*$, the problem (3.15) with zero initial conditions has a unique solution $(\mathbf{E},\ H_z,\ \mathbf{D},\ B_z) \in C^0(\mathbb{R}; L^2(\mathbb{R}^2))$, which satisfies

$$\|\mathbf{E}(t)\|_{L^2} + \|H_z(t)\|_{L^2} + \|\mathbf{D}(t)\|_{L^2} + \|B_z(t)\|_{L^2} \leq C(t)\mathrm{e}^{at}\int\limits_0^t \left(\sum_{\ell=0}^{k}\left\|\mathbf{f}_D^{(\ell)}(\tau)\right\|_{L^2} + \sum_{\ell=0}^{m}\left\|f_z^{(\ell)}(\tau)\right\|_{L^2}\right)\mathrm{d}\tau, \quad (3.16)$$

where $a \geq 0$, $C(t)$ is polynomial in t (*i.e.* $C(t) \leq c(1+t^n)$, for some $n, c \geq 0$). If, additionally, the above bound holds with $a = 0$, we will call the problem (3.15) stable.

Remark 3.14. In our definition of the well-posedness, the couple $(k_*,\ m_*)$ corresponds to a minimal regularity required on the data in order to be able to obtain estimates of the solution *via* the Laplace domain technique. In general, this requirement is not optimal. The minimal regularity is related to the explicit form of $\underline{\underline{\varepsilon}}(s)$, $\mu(s)$. For the classical Maxwell equations, when $\varepsilon_1(s) = \varepsilon_2(s) = \mu(s) = 1$, one has $k_* = m_* = 2$.

While the regularity required by the Hille−Yosida theorem is lower, *e.g.* $\mathbf{f}_D \in C^1(\mathbb{R}_+; (L^2(\mathbb{R}^2))^2)$, the results of this work allow to obtain the explicit bounds on the solutions in terms of the right hand side data, unlike the result of Hille−Yosida.

Applying Proposition 3.12, we immediately obtain the following result.

Theorem 3.15 (Connection of the passivity and the stability)**.** *The problem* (3.15) *with passive diagonal dielectric permittivity and magnetic permeability is well-posed and stable.*

Remark 3.16. In many practical cases $\varepsilon_1(s)$, $\varepsilon_2(s)$ and $\mu(s)$ are rational fractions, *cf.* Lorentz materials (2.8). Then the convolutions in (3.15) can be computed by introducing auxiliary unknowns and coupling (3.15) with an ODE system for the corresponding unknowns, see *e.g.* the system (5.4a)−(5.4e). In a more general setting, this is explained in [18]. The result of Theorem 3.15 holds true in this case as well.

It would be natural to ask whether the passivity requirement is **necessary** for the stability of (3.15). This is true [18] for a class of isotropic dispersive models.

3.3. Passivity implies stability: Plane-wave analysis

In this section we will briefly discuss a connection between the plane-wave analysis and the Laplace domain analysis. A Fourier-based approach, see (*e.g.* [39], Chap. 2), had been used to study the stability of the PMLs in isotropic dispersive media [9], or anisotropic non-dispersive media [10]. When applied to the system (2.4) without the PML, such analysis consists in looking for the plane-wave solutions $\hat{H}_z\mathrm{e}^{-i(\omega t-\mathbf{k}\cdot\mathbf{x})}$, $\mathbf{k},\ \mathbf{x} \in \mathbb{R}^2$, $\omega \in \mathbb{C}$, of (2.4) rewritten in the time domain. This requires examining the corresponding dispersion relation, which, in general, depends on the time domain formulation of the problem. However, some of its solutions satisfy

$$\tilde{\varepsilon}_2(\omega)^{-1}k_x^2 + \tilde{\varepsilon}_1(\omega)^{-1}k_y^2 - \omega^2\tilde{\mu}(\omega) = 0. \quad (3.17)$$

Definition 3.17. Continuous branches of solutions $\omega(\mathbf{k})$ of the dispersion relation are called modes.

In the time domain the system (2.4) is stable if and only if all the modes $\omega(\mathbf{k})$ satisfy

$$\operatorname{Im}\omega(\mathbf{k}) \leq 0, \qquad \text{for all } \mathbf{k} \in \mathbb{R}^2. \quad (3.18)$$

This holds true for passive sesquilinear forms.

Proposition 3.18 (Modal analysis for passive systems)**.** *Let the sesquilinear form* (3.6) *associated to* a, b, c *given in* (3.8) *be passive. Then all the solutions* $\omega(\mathbf{k})$ *of* (3.17) *satisfy* (3.18)*, and therefore, correspond to stable modes of* (2.4).

Proof. Setting $s = -i\omega$, (3.17) can be rewritten as

$$s^2 c(s) + b(s)k_y^2 + a(s)k_x^2 = 0, \qquad \mathbf{k} \in \mathbb{R}^2.$$

After multiplication by $\bar{s}$ this yields:

$$|s|^2 sc(s) + \bar{s}b(s)k_y^2 + \bar{s}a(s)k_x^2 = 0, \qquad \mathbf{k} \in \mathbb{R}^2.$$

The passivity assumption implies that the real part of the LHS of this equation is strictly positive for $s \in \mathbb{C}_+$. Thus, the above equation has no solutions in $\mathbb{C}_+$, and hence all solutions of (3.17) satisfy (3.18). □

4. Construction of stable PMLs for general passive materials

4.1. A brief introduction into PMLs for dispersive media. Instability of classical PMLs for anisotropic dispersive models

4.1.1. Introduction into PMLs for dispersive media

In this section we will briefly present the technique of PMLs, with a particular application to dispersive media. A detailed introduction into this technique can be found *e.g.* in [30] or [12]. There are at least two ways to apply the perfectly matched layers to the system (2.1). The first one is to use splitting of the time-domain system, as in seminal works by Bérenger [4,6]. The second way consists in the change of variables in the frequency domain, as reinterpreted by Chew *et al.* in [19], see also [53]. Such PMLs are called unsplit PMLs. We will adopt here the latter approach. More precisely, consider the following equation

$$a(s)\partial_x^2 u + b(s)\partial_y^2 u - s^2 c(s)u = 0, \qquad s \in \mathbb{C}_+,\ (x, y) \in \mathbb{R}^2. \tag{4.1}$$

Let the perfectly matched layer be located in the region $x \geq 0$. We assume that the above equation is valid for $x = \tilde{x} \in \mathbb{C}_+$, and introduce an analytic continuation of u that we denote $U(\tilde{x}, y)$. Naturally, U will satisfy the above equation, however, for $(\tilde{x}, y) \in \mathbb{C}_+ \times \mathbb{R}$. Then choosing a parameterization of $\tilde{x}$ suggested in [9]

$$\tilde{x} = \begin{cases} x + s^{-1}\psi(s)\displaystyle\int_0^x \sigma(x')\mathrm{d}x', & x \geq 0, \\ x, & x < 0, \end{cases} \qquad \text{where } \sigma(x) = \begin{cases} \sigma(x) \geq 0, & x \geq 0, \\ 0, & x < 0, \end{cases} \tag{4.2}$$

and $\psi(s)$ is an analytic in $\mathbb{C}_+$ function, we obtain the following system:

$$a(s)\left(1 + \frac{\sigma(x)\psi(s)}{s}\right)^{-1} \partial_x \left(\left(1 + \frac{\sigma(x)\psi(s)}{s}\right)^{-1} \partial_x U\right) + b(s)\partial_y^2 U - s^2 c(s)U = 0, \qquad (x, y) \in \mathbb{R}^2. \tag{4.3}$$

For $x < 0$, the above system coincides with (4.1). The original and the PML systems are coupled *via* transmission conditions. Importantly, $u(x, y) = U(x, y)$ for $x < 0$ by analytic continuation. The resulting system needs to be rewritten in the time domain, see Section 5.2.

Other PML directions can be treated similarly, and in the corners the change of multiple variables should be used. To truncate the perfectly matched layer at some $x = L > 0$, since in practice it cannot be chosen infinitely long, zero Dirichlet or Neumann boundary conditions are used on the exterior boundary of the PML.

Classical PMLs (split or unsplit) correspond to the choice $\psi(s) = 1$. The function $\psi(s)$ was introduced in [9,16] in order to take into account the dispersive character of the equations. Indeed, as demonstrated in [9], for isotropic dispersive models classical PMLs can lead to instabilities in the time domain. We demonstrate that this is also the case for anisotropic dispersive models with the help of the following simple numerical experiment.

Remark 4.1. Notice that in terms of stability (*i.e.* the absence of exponential blow-up), there is no difference between split and unsplit PMLs, as they both correspond to the same change of variables.

4.1.2. Numerical illustration: Instability of classical PMLs for anisotropic dispersive models

Before providing numerical data, let us introduce some auxiliary notation, which we will use in all the numerical experiments in this section, as well as Sections 4.3, 6.4. First of all, we compute solutions to the problem (2.1) inside rectangular domains. Their physical (*i.e.* without the PML) dimensions in the direction x and y are L_x, and L_y. The width of the PML in x-direction is denoted by L_x^σ, and in y-direction by L_y^σ. The absorption parameters in directions x and y are $\sigma_x(x)$ and $\sigma_y(y)$, *cf.* (4.2). On all outer boundaries (physical boundary if no PML in one of the directions is used, or external boundary of the PML), we assume zero Dirichlet boundary conditions for the field H_z.

The time domain PML system has a structure similar to the system of equations (5.6)−(5.7), see Section 5.2.2 and Remark 5.8. We use zero initial conditions and a source $f(t,x,y)$ in the third equation of (5.6). The numerical resolution is done with the help of the Yee scheme [52] for dispersive models, where the dispersive terms are discretized by trapezoid rule, and non-dispersive part is computed with a leapfrog. In the scheme, we discretize the field H_z on integer time and spatial steps. The time step is denoted by Δt, and the space step is $\Delta x = \Delta y$.

In all the experiments where we show a solution at different time steps, or compare the stable and unstable PMLs, we use the same color scale in all relevant figures.

Example 4.2 (Instability of classical PMLs for anisotropic Drude material)**.** We model an **anisotropic** Drude material, which extends the isotropic model (2.5), where

$$a(s) = \varepsilon_2(s)^{-1} = \left(1 + \frac{\omega_a^2}{s^2}\right)^{-1}, \qquad b(s) = \varepsilon_1(s)^{-1} = \left(1 + \frac{\omega_b^2}{s^2}\right)^{-1}, \qquad \mu(s) = 1. \tag{4.4}$$

In particular, we choose $\omega_a = 8$, $\omega_b = 4$. The passivity of the parameters can be checked by a direct calculation.

We use the classical PMLs in the direction x, *i.e.* $\psi(s) = 1$ in (4.2), and zero Dirichlet BCs in the direction y, in order to show that the instability occurs not necessarily in the corner, but in a certain direction. The rest of the parameters can be found in Table 1. We show the field H_z computed with the help of the classical PMLs in Figure 1. One can see the instability developing in this case. In the rightmost plot we demonstrate an exponential blow-up of the L^2-norm of the solution H_z.

TABLE 1. Parameters for the experiment with dielectric permittivity and magnetic permeability (4.4). The notation can be found in the beginning of Section 4.1.2.

L_x	L_y	L_x^σ	L_y^σ	$\sigma_x(x)$	$\sigma_y(y)$	Δx	Δt	$f(t,x,y)$
8	24	2	0	$20x^2$	0	0.05	0.025	$10(t-1.6)\mathrm{e}^{-15(t-1.6)^2}\mathrm{e}^{-300x^2-300y^2}$

4.2. Construction of stable PMLs

4.2.1. The well-posedness and the stability of the PMLs

Let us consider the PML system (4.3) in the Lipschitz domain Ω. We associate with it the sesquilinear form

$$\begin{aligned} A_p(u,v) = \ & a(s)\left(\left(1+\frac{\sigma(x)\psi(s)}{s}\right)^{-1}\partial_x u, \partial_x v\right) + b(s)\left(\left(1+\frac{\sigma(x)\psi(s)}{s}\right)\partial_y u, \partial_y v\right) \\ & + s^2 c(s)\left(\left(1+\frac{\sigma(x)\psi(s)}{s}\right)u, v\right), \qquad u, v \in H_0^1(\Omega). \end{aligned} \tag{4.5}$$

Under mild technical assumptions on the coefficients of the above sesquilinear form (which covers in particular the case when $\psi(s) = 1$), and assuming $\sigma(x) \in L^\infty(\Omega)$, it is possible to show the well-posedness of the corresponding time-domain model, in the sense similar to that of Definition 3.13, see [11]. This is not surprising and extends existing results for the classical PMLs for non-dispersive models, see for example [14, 33–35].

However, this well-posedness result is valid even for the problem with the parameters of Example 4.2 and the classical PML, yet our experiments clearly indicate the instability. Therefore, in this work we concentrate on the problem of the *stability* of the time-domain PML system. This question is somewhat more subtle than the

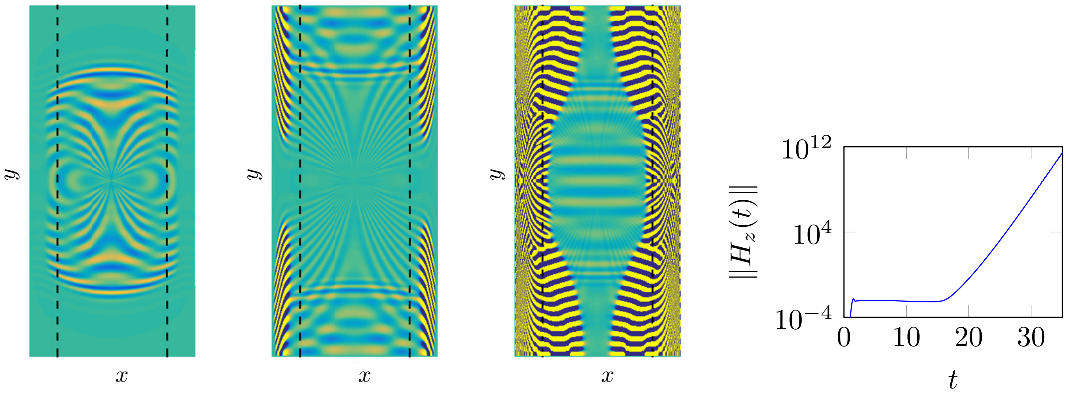


FIGURE 1. From left to right: a solution H_z of Example 4.2 computed with the classical PML at $t = 10,\ 20,\ 35$; the dependence of the L^2-norm of H_z with respect to time. The boundary between the PML and the physical domain is marked in black.

well-posedness. There exist very few works, which provide a full stability analysis of the PMLs for nonconstant absorption parameters even for the case of the classical perfectly matched layers applied to nondispersive anisotropic models. For example, in [23] the authors prove the stability and convergence result of the 2D classical PML for isotropic nondispersive wave equation and extend it to stable PMLs for the advective wave equation. In [17] some convergence estimates are provided for the radial PMLs for 2D acoustic wave equation.

Due to complexity of rigorous analysis, the construction of stable PMLs is often done in two stages:

- assuming that $\sigma(x) = \sigma = \text{const} > 0$ in $\mathbb{R}^2$ in (4.5) and performing the analysis for the resulting problem. This can be viewed as a very particular case of an arbitrary $\sigma(x) \geq 0$. Such a simplification was used, for example, in works [1,9,10,24,36]. It facilitates finding the right PML change of variables (in our case $\psi(s)$), which is potentially stable for variable $\sigma(x)$. And, as many numerical experiments show (including those in the present work), this analysis often results in stable PMLs for $\sigma(x) \neq \text{const}$.
- proving that the PML constructed with the help of the simplified analysis is stable for a more general class of absorption parameters. This part is trickier, as we discussed before, and is an open question for most existing stabilized PMLs.

In this work we will concentrate on the first part of the analysis. For the newly constructed PML the full rigorous proof of the stability is the subject of future research, see, in particular, [40].

4.2.2. Stability analysis

Let us consider a simplified case, when the absorption function is a non-negative constant and is defined in $\mathbb{R}$. This amounts to taking $\sigma(x) = \sigma = \text{const}$ in (4.3) and considering the corresponding sesquilinear form for $(x, y) \in \mathbb{R}^2$. Alternatively, this can be viewed as the following change of variables:

$$x \to x\left(1 + s^{-1}\psi(s)\sigma\right), \qquad \sigma \geq 0, \qquad x \in \mathbb{R}. \tag{4.6}$$

As shown before, the choice $\psi(s) = 1$ may lead to time-domain instabilities. Our goal is to provide stable choices of $\psi(s)$ for an arbitrary passive material. The main results of this section are Theorems 4.8 and 4.11.

Upon the application of (4.6), the sesquilinear form (3.6) is transformed to

$$A_\sigma(u, v) = a(s)\left(1 + \frac{\sigma\psi(s)}{s}\right)^{-2}(\partial_x u, \partial_x v) + b(s)(\partial_y u, \partial_y v) + s^2 c(s)(u, v),\ s \in \mathbb{C}_+, \qquad u,\ v \in H^1(\mathbb{R}^2). \tag{4.7}$$

We suggest that taking in the above $\psi(s) = a(s)$ produces a passive sesquilinear form. This result is generalized in Lemma 4.3 and is explained in detail in Theorem 4.8. Although Lemma 4.3 does not provide an explicit way to construct the function $\psi(s)$, it will play an important role in Section 6.

Lemma 4.3 (Sufficient stability condition for the PML (4.6))**.** *Let the sesquilinear form $A(u,v)$ given by (3.6) be passive in the sense of Definition 3.11. Let an analytic $\psi(s):\mathbb{C}_+\to\mathbb{C}$ satisfy for all $s\in\mathbb{C}_+$:*

(S1) $\operatorname{Re}\left(\bar{s}\psi(s)\right)>0$,
(S2) $\operatorname{Re}\left(\bar{s}\psi(s)a(s)^{-1}b(s)\right)>0$,
(S3) $\operatorname{Re}\left(s\psi(s)a(s)^{-1}c(s)\right)>0$.

Then for all $\sigma\geq 0$, the sesquilinear form

$$\tilde{A}_\sigma(u,v)=\psi(s)a(s)^{-1}A_\sigma(u,v),\qquad u,v\in H^1(\mathbb{R}^2), \tag{4.8}$$

where $A_\sigma(u,v)$ is defined in (4.7), is passive.

Proof. Notice that since $a(s)$ satisfies $\operatorname{Re}\left(\bar{s}a(s)\right)>0$, $s\in\mathbb{C}_+$, it never vanishes in $\mathbb{C}_+$. Let us verify the conditions on the coefficients of the sesquilinear form (4.8) required by passivity. The first coefficient

$$\operatorname{Re}\left(s\psi(s)^{-1}\left(1+\frac{\sigma\psi(s)}{s}\right)^2\right)=\operatorname{Re}\left(s\psi(s)^{-1}\right)+2\sigma+\operatorname{Re}\left(s^{-1}\psi(s)\right)>0,$$

for all $s\in\mathbb{C}_+$, thanks to (3.3) and (S1). It is obviously analytic in $\mathbb{C}_+$. The passivity condition on the rest of the coefficients is included explicitly into the conditions of the lemma. □

Combining the above with Theorem 3.9 and Proposition 3.12, we obtain the following simple result of the time-domain stability of the PML system.

Corollary 4.4 (Stability bounds of the PML of Lemma 4.3)**.** *Let $A_\sigma(u,v)$ be defined by (4.7), and let $\psi(s)$ satisfy the conditions of Lemma 4.3 Let, additionally,*

$$\left|\frac{\psi(s)}{a(s)}\right|\leq |s|^\gamma C_f(\operatorname{Re}s),\qquad s\in\mathbb{C}_+, \tag{4.9}$$

with $\gamma\geq -1$ and $C_f(\lambda):\ (0,\infty)\to(0,\infty)$ being non-increasing and satisfying $C_f(\lambda)<C_0\lambda^{-m}$, $C_0>0$, $m\geq 0$. Then the following holds true for the sesquilinear form (4.7):

(1) *for all $G\in H^{-1}(\mathbb{R}^2)$, for all $s\in\mathbb{C}_+$, there exists a unique solution $U(s)\in H^1(\mathbb{R}^2)$ to the variational formulation*

$$A_\sigma\left(U(s),v\right)=\langle G,v\rangle,\qquad v\in H^1\left(\mathbb{R}^2\right), \tag{4.10}$$

which satisfies, for all $s\in\mathbb{C}_+$,

$$\|U(s)\|_{H^1(\mathbb{R}^2)}\leq C|s|^{\gamma+1}(\operatorname{Re}s)^{-1}\max(1,(\operatorname{Re}s)^{-2})C_f(\operatorname{Re}s)\|G\|_{H^{-1}(\mathbb{R}^2)}, \tag{4.11}$$

where $C>0$ is a constant independent of s;

(2) *let $k=\lfloor\gamma+3\rfloor$, and $\ell=3+m$. Let $G(s)\in H^{-1}(\mathbb{R}^2)$ be the Laplace transform of a causal function $g(t):\mathbb{R}\to H^{-1}(\mathbb{R}^2)$, which belongs to $C^{k-1}(\mathbb{R};H^{-1}(\mathbb{R}^2))$ and has an integrable $k-$th distributional derivative. Then the solution $U(s)$, $s\in\mathbb{C}_+$, of (4.10) with $G=G(s)$ is a Laplace transform of a causal continuous function $u(t):\ \mathbb{R}\to H^1(\mathbb{R}^2)$, which can be bounded as follows, with $\alpha>0$ independent of t:*

$$\|u(t)\|_{H^1}\leq\alpha\max(t^\ell,1)\int_0^t\|\left(1+\partial_\tau)^k g\right)(\tau)\|_{H^{-1}}\mathrm{d}\tau,\qquad t\geq 0. \tag{4.12}$$

Proof. The first statement follows directly from Proposition 3.12, applied to the following variational formulation, which is equivalent to (4.10):

$$\tilde{A}_\sigma(U(s),v)=\psi(s)a(s)^{-1}\langle G,v\rangle,\qquad v\in H^1(\mathbb{R}^2),\ s\in\mathbb{C}_+.$$

Notice that $\tilde{A}_\sigma(u,v)$ is passive, according to Lemma 4.3. The second statement is obtained as in the proof of Proposition 3.12; it is important to notice that $\psi(s)a(s)^{-1}$ is analytic in $\mathbb{C}_+$. The bound in the time domain follows by the application of Theorem 3.9. $\square$

Remark 4.5. Indeed, if (4.9) holds with $\gamma < -1$, we can always bound $|s|^\gamma C_f(\operatorname{Re} s) = |s|^{-1}|s|^{1+\gamma} C_f(\operatorname{Re} s) \leq |s|^{-1} (\operatorname{Re} s)^{\gamma+1} C_f(\operatorname{Re} s)$, which satisfies the conditions of Corollary 4.4.

The time-domain bounds in Corollary 4.4 are non-optimal, since they are based on the extensive use of Proposition 3.12 and non-optimal bounds of Theorem 3.9. Therefore this result cannot be used to compare the bounds on the solution of the PML and non-PML system. Using *a priori* bounds on $\left|a(s)\psi(s)^{-1}\right|$ which can be computed with the help of Corollary 3.5, we obtain the following simple result.

Corollary 4.6 (*A priori* bounds for Corollary 4.4)**.** *Let $a(s)^{-1}$, $\psi(s)^{-1}$ be passive. Then the estimate (4.9) holds true with $\gamma \leq 4$, $m \leq 6$. If additionally $a(s)^{-1}$, $\psi(s)^{-1}$ are generalized Lorentz, i.e. have an expansion as in (2.8), the estimate (4.9) holds true with $\gamma \leq 1$, $m \leq 3$.*

Proof. See Appendix B. $\square$

All the above can be summarized in the following theorem.

Theorem 4.7 (Stability of the time-domain PML of Lemma 4.3)**.** *Let the sesquilinear form $\tilde{A}_\sigma(u,v)$ satisfy the conditions of Lemma 4.3. Then the system (3.15) with $\varepsilon_1(s) = b(s)^{-1}$, $\varepsilon_2(s) = a(s)^{-1}\left(1+\sigma s^{-1}\psi(s)\right)^2$ and $\mu(s) = c(s)$ is well-posed and stable.*

The following direct corollary of Lemma 4.3 provides a simple way to construct a stable PML.

Theorem 4.8 (Construction of stable PMLs in the direction x)**.** *Let a sesquilinear form $A(u,v)$ given by (3.6) be passive in the sense of Definition 3.11. Then for all $\sigma \geq 0$, the form*

$$A_\sigma(u,v) = a(s)\left(1+\frac{\sigma a(s)}{s}\right)^{-2}(\partial_x u, \partial_x v) + b(s)(\partial_y u, \partial_y v) + s^2 c(s)(u,v), \qquad u,v \in H^1(\mathbb{R}^2),$$

obtained from (4.7) with a particular choice $\psi(s) = a(s)$, is passive in the sense of Definition 3.11.

Proof. Setting $\psi(s) = a(s)$ shows that the assumptions of Lemma 4.3 are satisfied. $\square$

Example 4.9 (Uniaxial cold plasma)**.** Consider the plasma model (2.7), with the PML in y-direction chosen as $\psi(s) = \varepsilon_1(s)^{-1}$. The stability of this PML was confirmed in [7,8]. In this case in Corollary 4.4 $m = \mu = 0$.

Theorem 4.8 applied to the isotropic Drude model (2.5) suggests that the choice $\psi(s) = \varepsilon(s)^{-1}$ in the change of variables (4.6) leads to a stable PML. This is confirmed by the analysis and numerical experiments in [9]. In the same work it was demonstrated that a more general change of variables leads to a stable PML, more precisely, $\psi(s) = (1+\frac{\omega_\ell^2}{s^2})^{-1}$, where $\omega_\ell \in [\min(\omega_e,\ \omega_m),\ \max(\omega_e,\ \omega_m)]$. We explain this in the following proposition.

Proposition 4.10 (Other stable PMLs for isotropic models)**.** *Let a sesquilinear form $A(u,v)$ given by (3.6) with $a(s) = b(s)$ be passive in the sense of Definition 3.11. Let $\psi(s) = \left(\alpha a(s)^{-1} + (1-\alpha)c(s)\right)^{-1}$, $0 \leq \alpha \leq 1$.*

Then for all $\sigma \geq 0$, the form $\tilde{A}_\sigma(u,v)$ defined in (4.8) is passive.

Proof. It is sufficient to verify conditions of Lemma 4.3:

(1) thanks to (3.3), it is sufficient to check $\operatorname{Re}\left(s\psi(s)^{-1}\right) = \alpha \operatorname{Re}\left(sa(s)^{-1}\right) + (1-\alpha)\operatorname{Re}\left(sc(s)\right) > 0$.
(2) since $b(s) = a(s)$, $\psi(s)a(s)^{-1}b(s) = \psi(s)$.
(3) to show the required bound for $a(s)^{-1}\psi(s)c(s)$, we use (3.3):

$$\operatorname{Re}\left(\bar{s}\psi(s)^{-1}a(s)c(s)^{-1}\right) = \alpha \operatorname{Re}(\bar{s}c(s)^{-1}) + (1-\alpha)\operatorname{Re}(\bar{s}a(s)) > 0. \qquad \square$$

Finally, to construct a stable PML in corners, we extend the statement of Theorem 4.8. Recall that in a corner we perform changes of several variables:

$$x \to x(1+s^{-1}\psi_x(s)\sigma_x), \qquad y \to y(1+s^{-1}\psi_y(s)\sigma_y). \tag{4.13}$$

Theorem 4.11 (PML stable in a corner)**.** *Let a sesquilinear form $A(u,v)$ given by* (3.6) *be passive in the sense of Definition* 3.11*. Then for all $\sigma_x,\ \sigma_y \geq 0$, the form*

$$A_{\sigma_x,\sigma_y}(u,v) = a(s)\left(1+\frac{\sigma_x a(s)}{s}\right)^{-2}(\partial_x u, \partial_x v) + b(s)\left(1+\frac{\sigma_y b(s)}{s}\right)^{-2}(\partial_y u, \partial_y v) + s^2 c(s)(u,v),\ u,v \in H^1(\mathbb{R}^2),$$

obtained from (3.6) by applying the PML (4.13) *with $\psi_x = a$, $\psi_y = b$, is passive in the sense of Definition* 3.11.

Proof. The proof of this result follows the same arguments as the proof of Lemma 4.3. □

The corresponding time-domain stability result can be formulated as in Theorem 4.7.

Remark 4.12. The stability of the PML as proved in this work implies the uniform stability of the PML as defined in [9], see also Proposition 3.18.

While the changes of variables suggested in this section lead to stable systems, it is not obvious whether the resulting layer is absorbing, *i.e.* that it leads to the energy decay. This question is addressed in [40].

4.3. Numerical verification of the results

Example 4.13 (The system of Example 4.2; verification of Thm. 4.8)**.**

First of all, we apply the obtained results to the system described in Example 4.2. We use exactly the same parameters, see (4.4) and Table 1, however, in this case we construct the PML system with the help of the PML change of variables (4.2) with $\psi(s) = \varepsilon_2(s)^{-1}$. As before, in y-direction we use zero Dirichlet boundary conditions.

The results of this experiment are shown in Figure 2. One clearly sees that the solution is stable. Interestingly, in this case the norm of the solution decays very lightly inside the domain and the PML. This is partially due to the dispersive, anisotropic nature of the problem (the solution remains non-zero inside a bounded domain for a fairly long time), and partially because the PML was used only in one direction (*cf. e.g.* Figure 3, where the PML was used in two directions). Notice that the oscillations in the norm of the solution are likely due to the dispersive behaviour of the problem.

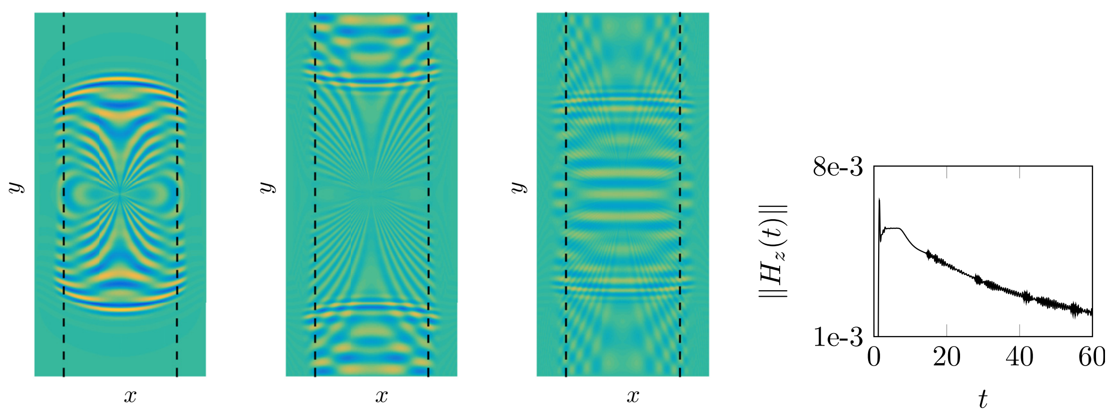


FIGURE 2. From left to right: A solution H_z of Example 4.13 computed with the new PML at $t = 10,\ 20,\ 35$; the dependence of the L^2-norm of H_z with respect to time. The boundary between the domain and the layer is marked in black.

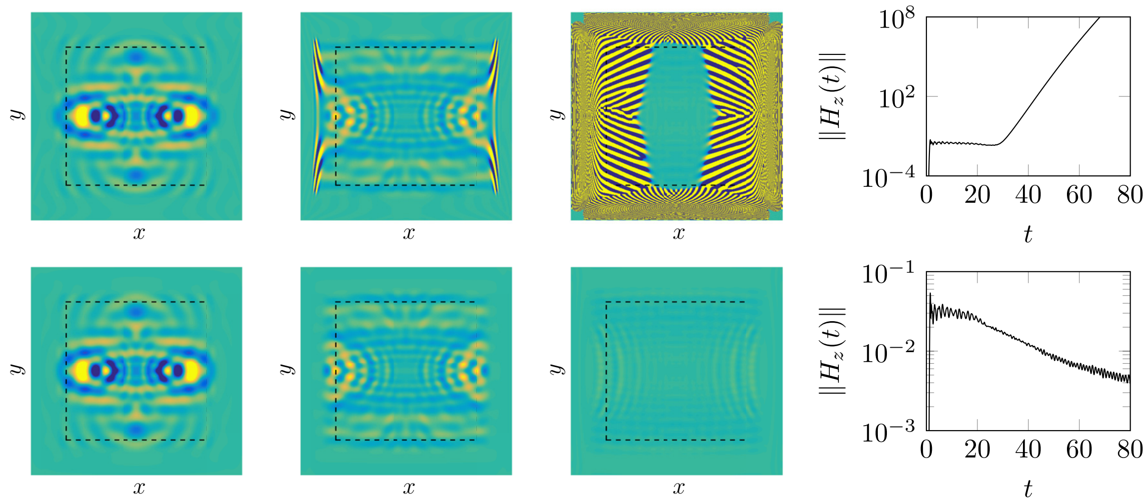


FIGURE 3. Top row: The experiment was done with the help of the classical PML. Bottom row: The experiment is done with the new PML. From left to right: a solution H_z of Example 4.14 at the times $t = 15,\ 30,\ 80$, computed with the help of corresponding PMLs; The dependence on time of the L^2-norm of this solution measured in the whole domain. The boundary between the physical domain and the PML is marked in black.

Example 4.14 (Anisotropic Lorentz material; verification of Thm. 4.11)**.**
We consider an anisotropic Lorentz material (2.8) with the parameters

$$\varepsilon_1(s) = 1 + \frac{1}{12}\left(\frac{325}{s^2+4^2} + \frac{119}{s^2+8^2}\right), \qquad \varepsilon_2(s) = 1 + \frac{16}{s^2+1} + \frac{16}{s^2+5^2}, \qquad \mu(s) = 1 + \frac{3}{s^2+2^2}. \tag{4.14}$$

Notice that the corresponding time-domain system is stable, since $a = \varepsilon_2^{-1}$, $b = \varepsilon_1^{-1}$, $c = \mu$ satisfy conditions of Definition 3.11, *cf.* Theorem 5.4. This time we apply the PML change of variables in both directions (4.13) with $\psi_x = \varepsilon_2^{-1}$ and $\psi_y = \varepsilon_1^{-1}$. The stability of such a PML is proven in Theorem 4.11.

The rest of the parameters for this experiment are provided in Table 2. The results of the experiment are shown in Figure 3. Notice that the classical PMLs are unstable, whereas with the use of the new PMLs the norm of the solution decays. Due to the anisotropy of the model, when using the classical PMLs, the instability in the x-direction is more pronounced and occurs earlier than the instability in the y-direction.

TABLE 2. Parameters for the experiment with dielectric permittivity and magnetic permeability (4.14). The notation can be found in the beginning of Section 4.1.2.

L_x	L_y	L_x^σ	L_y^σ	$\sigma_x(x)$	$\sigma_y(y)$	Δx	Δt	$f(t,x,y)$
8	8	2	2	$20x^2$	$20y^2$	0.025	0.0125	$10(t-1.6)\mathrm{e}^{-15(t-1.6)^2}\mathrm{e}^{-20x^2-20y^2}$

Example 4.15 (Verification of Thm. 4.8 for materials with losses)**.**
Let us consider the problem (2.4) with the parameters:

$$\varepsilon_1(s) = 1, \qquad \varepsilon_2(s) = 1 + \frac{2}{3(s+3)} + \frac{25}{12s} + \frac{3}{4(s+2)}, \qquad \mu(s) = 1. \tag{4.15}$$

TABLE 3. Parameters for the experiment with dielectric permittivity and magnetic permeability (4.15). The notation can be found in the beginning of Section 4.1.2.

L_x	L_y	L_x^σ	L_y^σ	$\sigma_x(x)$	$\sigma_y(y)$	Δx	Δt	$f(t,x,y)$
8	12	2	0	$50x^2$	0	0.05	0.025	$120\mathrm{e}^{-16(t-2)^2}\mathrm{e}^{-300x^2-800y^2}$

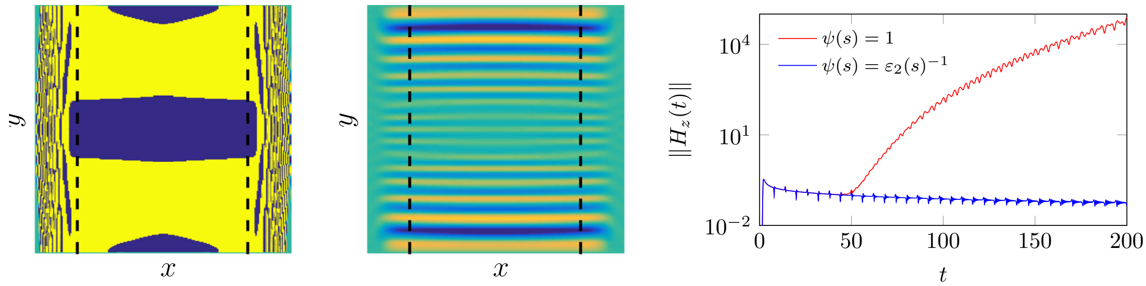


FIGURE 4. From left to right: A solution H_z to the problem of Example 4.15 at $t = 200$ computed with the classical PMLs; The same quantity computed with the new PMLs; The comparison of L^2-norms of the solution H_z, computed with two different PMLs, measured in the whole domain. The boundary between the physical domain and the PML is in black.

By a direct computation one can verify that the corresponding sesquilinear form is passive. Unlike the previous two examples, this problem does not fit the framework of Lorentz materials (2.8). Moreover, the existing necessary PML stability condition [9] does not cover this class of models.

The parameters for this experiment are provided in Table 3. We use the PML only in the direction x. A numerical comparison of the new PMLs (4.2) with $\psi(s) = \varepsilon_2(s)^{-1}$ with the classical perfectly matched layers is shown in Figure 4. Contrary to the classical PMLs, the new change of variables leads to a stable system.

For other examples involving non-local materials we refer an interested reader to [11].

5. Dispersive systems with rational parameters. Their properties. Time-domain formulations

In the previous section we have shown how to construct a stable PML for a model described by a passive sesquilinear form (3.6). In this section we will study a special case when the coefficients of the sesquilinear form are rational even functions (however, most of the results can be generalized to meromorphic even functions).

In the first part of this section we will study the properties of these coefficients. Next, we will formulate the time-domain systems that correspond to passive models with such parameters.

5.1. Properties of even rational functions that satisfy the passivity condition

We consider rational functions $r(z)$ satisfying the following assumptions (they coincide with those of [9]).

Assumption 5.1. A function $r(z)$ can be represented as $r(z) = 1 + \frac{p(z^2)}{q(z^2)}$, where $p(z)$ and $q(z)$ are polynomials with real coefficients that have no common roots, s.t. $\deg p < \deg q$.

Recall that given $r(z)$, $z \in \mathbb{C}_+$, by $\tilde{r}(\omega)$ we denote the restriction $\tilde{r}(\omega) = r(-i\omega)$, $\operatorname{Im}\omega \geq 0$, see (2.2).

Remark 5.2 (Physical relevance)**.** Later on we will assume that $\varepsilon_j(s)$ for $j = 1, 2$, and $\mu(s)$ in (2.4) belong to the above class. These requirements are indeed physically meaningful:

- requiring that ε_1, ε_2 and μ are rational functions is equivalent to the fact that in the time domain the system (2.4) can be written with the help of local (*i.e.* linear combinations of $\partial_t^{(k)}$, for $k \in \mathbb{N}_0$) operators;

- $\varepsilon_1(\omega),\ \varepsilon_2(\omega),\ \mu(\omega) \to 1$ as $\omega \to \infty$ means that for high frequencies there is no dispersion ([42], Sect. 59);
- in a so-called transparency regime ([42], Sect. 64, p. 260) the rational functions $\varepsilon_1(s),\ \varepsilon_2(s),\ \mu(s)$ are real on the imaginary axis. Moreover, in this case $\tilde{\varepsilon}_j,\ \tilde{\mu}$ are even functions of ω (see [42], p. 250). Since they correspond to real in time operators, they should have real coefficients.

We can formulate the following *a priori* property of passive functions. For the proof see [18] or [11].

Lemma 5.3. *Let $r(s)$ be a passive function satisfying Assumption 5.1. Then all of its poles and zeros are real.*

The result below provides a characterization of rational passive functions, (see [46], Cor. 10.1, [5], Thm. 5).

Theorem 5.4 (Characterization of passive functions)**.** *Let $r(s)$ be a rational function satisfying Assumption 5.1. Then this function is passive, i.e.* $\operatorname{Re}(sr(s)) > 0,\ s \in \mathbb{C}_+$, *if and only if*

$$r(s) = 1 + \sum_{\ell=0}^{n} \frac{r_\ell}{s^2 + \omega_\ell^2}, \qquad r_\ell > 0,\ \omega_\ell \in \mathbb{R},\ \ell = 0, \dots, n. \tag{5.1}$$

Moreover, in this case $\operatorname{Re}(sr(s)) > \operatorname{Re} s$.

This result shows that under Assumption 5.1 the passivity requirement can be satisfied **only** by the Lorentz materials (2.8). Another property of passive materials, which is exploited in [9], is a so-called growing property. More precisely, a rational function $r(s)$ satisfies the growing property if for all $\omega \in \mathbb{R}$ except for poles of $\tilde{r}(\omega)$, it holds $(\omega\tilde{r}(\omega))' > 0$. It is interesting that in physical literature ([42], p. 256) and [50] the above property is shown to be crucial for the positivity of the energy density. In what follows we will implicitly rely on Lemma 5.3.

Lemma 5.5 (Properties of passive functions)**.**

Let $r(s)$ satisfy Assumption 5.1 and be passive. Let $(\omega_\ell)_{\ell=1}^n$ be non-negative poles of $\tilde{r}(\omega)$ ordered in ascending order. If $\omega = 0$ is a pole, and its order is $2m$ (the order is necessarily even because of the evenness of $r(s)$), this pole is counted m times (i.e. $0 = \omega_1 = \omega_2 = \ldots = \omega_m$), while the rest of the poles are counted with their multiplicities $0 \leq \omega_1 \leq \omega_2 \leq \ldots \leq \omega_n$. Let $(r_\ell)_{\ell=1}^n$ be non-negative zeros of $\tilde{r}(\omega)$, ordered in ascending order (counting multiplicities). Then

($M1$) $(\omega\tilde{r}(\omega))' > 0$ *in all points but poles of* $\tilde{r}(\omega)$;
($M2$) $0 \leq \omega_1 < r_1 < \omega_2 < r_2 < \ldots < \omega_n < r_n$.

Proof. See [18] or [11]. □

The next result characterizes reciprocals of rational passive functions.

Theorem 5.6. *Let $r(s)$ satisfy Assumption 5.1. Then* $\operatorname{Re}(\bar{s}r(s)) > 0,\ s \in \mathbb{C}_+$, *if and only if*

$$r(s) = 1 - \sum_{\ell=0}^{n} \frac{r_\ell}{s^2 + \omega_\ell^2}, \qquad r_\ell > 0,\ \omega_\ell \in \mathbb{R} \setminus \{0\},\ \ell = 0, \dots, n, \qquad r(0) = 1 - \sum_{\ell=0}^{n} \frac{r_\ell}{\omega_\ell^2} \geq 0. \tag{5.2}$$

Proof. The function $r(s)$ satisfies $\operatorname{Re}(\bar{s}r(s)) > 0$ for $s \in \mathbb{C}_+$ if and only if $r(s)^{-1}$ is passive, see (3.3). Using (M1) of Lemma 5.5, $(\omega\tilde{r}(\omega)^{-1})' > 0$. Therefore all the zeros of $\omega\tilde{r}(\omega)^{-1}$ are simple, and so are the poles of $\omega^{-1}\tilde{r}(\omega)$. Additionally, since $\tilde{r}(\omega)$ is even with real coefficients and $\tilde{r}(\omega) \to 1$ as $\omega \to +\infty$, its partial fraction expansion is

$$\tilde{r}(\omega) = 1 + \sum_{\ell=0}^{n} \frac{r_\ell}{\omega^2 - \omega_\ell^2}, \qquad r_\ell \in \mathbb{R}, \qquad \omega_\ell \in \mathbb{R} \setminus \{0\}.$$

It remains to check the condition on the signs of r_ℓ. To do so, we rewrite the above expansion

$$\tilde{r}(\omega)=1+\sum_{\ell=0}^{n}\frac{r_\ell}{\omega^2-\omega_\ell^2}=1-\sum_{\ell=0}^{n}\frac{r_\ell}{\omega_\ell^2}+\sum_{\ell=0}^{n}\frac{r_\ell}{\omega_\ell^2}+\sum_{\ell=0}^{n}\frac{r_\ell}{\omega^2-\omega_\ell^2}=r(0)+\sum_{\ell=0}^{n}\frac{r_\ell\omega^2}{\omega_\ell^2(\omega^2-\omega_\ell^2)}\cdot \tag{5.3}$$

Then the condition $\operatorname{Re}(\bar{s}r(s))>0$ for $s\in\mathbb{C}_+$ reads:

$$\operatorname{Re}(\bar{s}r(s))=r(0)\operatorname{Re}\bar{s}+\operatorname{Re}\sum_{\ell=0}^{n}\frac{r_\ell}{\omega_\ell^2}\frac{s^2\bar{s}}{s^2+\omega_\ell^2}=r(0)\operatorname{Re}\bar{s}+\operatorname{Re}\sum_{\ell=0}^{n}\frac{r_\ell}{\omega_\ell^2}\frac{|s|^2s(\bar{s}^2+\omega_\ell^2)}{|s^2+\omega_\ell^2|^2}>0,\qquad s\in\mathbb{C}_+.$$

Let us show that $r(0)\geq 0$. Assume by contradiction that $r(0)<0$. In the vicinity of $s=0$, the sign of $\operatorname{Re}(\bar{s}r(s))$ coincides with the sign of $r(0)\operatorname{Re}s$, which is negative for $\operatorname{Re}s>0$. Hence, necessarily, $r(0)\geq 0$. To see that $r_\ell>0$, notice that in the vicinity of $s=\pm i\omega_\ell$, the sign of $\operatorname{Re}(\bar{s}r(s))$ coincides with the sign of the largest term in the partial fraction expansion

$$\operatorname{Re}\frac{r_\ell}{\omega_\ell^2}\frac{|s|^2s(\bar{s}^2+\omega_\ell^2)}{|s^2+\omega_\ell^2|^2}=\frac{r_\ell}{\omega_\ell^2}\frac{|s|^2}{|s^2+\omega_\ell^2|^2}(|s|^2+\omega_\ell^2)\operatorname{Re}s.$$

For the above to be positive when $\operatorname{Re}s>0$, it is necessary that $r_\ell>0$. □

A result analogical to Lemma 5.5 can be formulated for functions $r(s)$, which satisfy $\operatorname{Re}(\bar{s}r(s))>0$ in $\mathbb{C}_+$.

Lemma 5.7 (Properties of reciprocals of passive functions)**.**
Let $r(s)$ satisfy Assumption 5.1, and $\operatorname{Re}(\bar{s}r(s))>0$ for $s\in\mathbb{C}_+$. Let $(\omega_\ell)_{\ell=1}^n$ be non-negative poles of $\tilde{r}(\omega)$ ordered in ascending order, counted with their multiplicities. Let $(r_\ell)_{\ell=1}^n$ be non-negative zeros of $\tilde{r}(\omega)$, ordered in ascending order. If $\omega=0$ is a zero, and its order is $2m$ (the order is necessarily even because of the evenness of $r(s)$), this zero is counted m times (i.e. $0=r_1=r_2=\ldots=r_m$), while the rest of the zeros are counted with their multiplicities: $0\leq r_1\leq r_2\leq\ldots\leq r_n$. Then
($\overline{M}1$) *$(\omega^{-1}\tilde{r}(\omega))'<0$ in all points but poles of $\omega^{-1}\tilde{r}(\omega)$;*
($\overline{M}2$) *$0\leq r_1<\omega_1<r_2<\omega_2<\ldots<r_n<\omega_n$.*

Proof. The property ($\overline{M}1$) follows by a direct computation, with the use of (5.3):

$$(\omega^{-1}\tilde{r}(\omega))'=-\frac{r(0)}{\omega^2}-\sum_{\ell=0}^{n}\frac{r_\ell}{\omega_\ell^2}\frac{\omega^2+\omega_\ell^2}{(\omega^2-\omega_\ell^2)^2}\cdot$$

The above expression is well-defined and strictly negative in all points except for poles of $\tilde{r}(\omega)$ and possibly $\omega=0$. If $\omega^{-1}\tilde{r}(\omega)$ has no pole in $\omega=0$ (*i.e.* $r(0)=0$), the above expression is defined and negative in $\omega=0$.

The property ($\overline{M}2$) follows using the same arguments as in the proof of the growing property in [18]. More precisely, we notice that $\omega^{-1}\tilde{r}(\omega)$ is strictly decreasing, and all its zeros are simple. From this it follows that it cannot have extrema, and thus between two zeros of $\omega^{-1}\tilde{r}(\omega)$ there is always at least one pole.

Since $\tilde{r}(\omega)$ has the same number of poles as zeros, the number n_z of zeros of $\omega^{-1}\tilde{r}(\omega)$ is less by one than the number of its poles. Moreover, all the poles of $\omega^{-1}\tilde{r}(\omega)$ are simple. From these two arguments it follows that the poles and the zeros of $\omega^{-1}\tilde{r}(\omega)$ interlace. Since $\lim\limits_{\omega\to+\infty}\tilde{r}(\omega)=1$, and this function decays (which can be verified by a direct computation, see (5.2)), the largest by absolute value pole of $\omega^{-1}\tilde{r}(\omega)$ is larger than all of the zeros of $\omega^{-1}\tilde{r}(\omega)$. From this we obtain the interlacing property ($\overline{M}2$). □

5.2. PMLs for anisotropic Lorentz materials: Time-Domain formulation

In the previous section we showed that provided that the dielectric permittivity and magnetic permeability satisfy the conditions of passivity and Assumption 5.1, the corresponding materials are Lorentz (2.8). This class of models can be expressed in the time domain as a system of partial differential equations coupled with ordinary differential equations. The goal of this section is to present the corresponding formulation, as well as provide a PML system obtained after the change of variables (4.2), which can be used in practical calculations.

5.2.1. Time-Domain system without the PML

Recall that the Lorentz dielectric permittivity and magnetic permeability are given by (2.8). To rewrite the Maxwell system (2.1) in the time domain, we first notice that one of the ways to express the relation $s\hat{D}_x = s\varepsilon_1(s)\hat{E}_x$ in the time domain is *via* the introduction of auxiliary unknowns:

$$\begin{aligned}
&\partial_t D_x = \partial_t E_x + \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} j_{x\ell},\\
&\partial_t j_{x\ell} + \omega_{x\ell}^2 p_{x\ell} = E_x, \qquad \partial_t p_{x\ell} = j_{x\ell}, \qquad \ell = 0, \dots, n_x.
\end{aligned}$$

One can verify that in the frequency domain it holds $s\hat{D}_x = s\varepsilon_1(s)\hat{E}_x$, provided the initial conditions $D_x|_{t=0} = E_x|_{t=0}$, and zero initial conditions for the unknowns $j_{x\ell}$ and $p_{x\ell}$, $\ell = 0, \dots, n_x$. Similarly we deal with $s\hat{B}_z = s\mu(s)H_z$. Then the Maxwell system (2.1) can be rewritten in the following form:

$$\partial_t E_x + \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} j_{x\ell} = \partial_y H_z, \tag{5.4a}$$

$$\partial_t E_y + \sum_{\ell=0}^{n_y} \varepsilon_{y\ell} j_{y\ell} = -\partial_x H_z, \tag{5.4b}$$

$$\partial_t H_z + \sum_{\ell=0}^{n_\mu} \mu_\ell j_{\mu\ell} = \partial_y E_x - \partial_x E_y, \tag{5.4c}$$

$$\partial_t j_{m\ell} + \omega_{m\ell}^2 p_{m\ell} = E_m, \qquad \partial_t p_{m\ell} = j_{m\ell},\ \ell = 0, \dots, n_m,\ m \in \{x, y\}, \tag{5.4d}$$

$$\partial_t j_{\mu\ell} + \omega_{\mu\ell}^2 p_{\mu\ell} = H_z, \qquad \partial_t p_{\mu\ell} = j_{\mu\ell},\ \ell = 0, \dots, n_\mu. \tag{5.4e}$$

5.2.2. Time-Domain system with the PML

Our goal is to rewrite the system (5.4a)–(5.4e) in the time domain with the PML change of variables (4.2) and non-constant $\sigma(x) \geq 0$. Let us limit our discussion to the functions $\psi(s)$ which satisfy Assumption 5.1. Additionally, let us assume that $\psi(s)^{-1}$ is of the following form (this is clarified in Thm. 6.7):

$$\psi(s)^{-1} = 1 + \sum_{\ell=0}^{n_\psi} \frac{c_{\psi\ell}}{s^2 + r_{\psi\ell}^2}, \qquad r_{\psi\ell} \in \mathbb{R},\ c_{\psi\ell} \in \mathbb{R},\ \ell = 0, \dots, n_\psi.$$

Recall that (4.2) amounts to substituting all ∂_x by $(1 + \frac{\sigma\psi}{s})^{-1}\partial_x$ in the frequency-domain formulation of (5.4a)–(5.4e). For example, consider the equation for E_y of (5.4a)–(5.4e) in the Laplace domain, with the PML (4.2):

$$s\hat{E}_y + \sum_{\ell=0}^{n_y} \varepsilon_{y\ell} \hat{j}_{y\ell} = -\left(1 + \frac{\sigma\psi(s)}{s}\right)^{-1} \partial_x \hat{H}_z = -\frac{s}{s + \sigma\psi}\partial_x \hat{H}_z = -\left(1 - \frac{\sigma}{s\psi^{-1} + \sigma}\right)\partial_x \hat{H}_z. \tag{5.5}$$

Thus, let us define $\hat{E}_y^* = -(s\psi^{-1} + \sigma)^{-1}\partial_x \hat{H}_z$:

$$\begin{aligned}
&s\hat{E}_y^* + \sum_{\ell=0}^{n_\psi} c_{\psi\ell} \hat{j}_{E_y,\ell}^* + \sigma(x)\hat{E}_y^* = -\partial_x \hat{H}_z,\\
&s\hat{j}_{E_y,\ell}^* + r_{\psi\ell}^2 \hat{p}_{E_y,\ell}^* = \hat{E}_y^*, \qquad s\hat{p}_{E_y,\ell}^* = \hat{j}_{E_y,\ell}^*, \qquad \ell = 0, \dots, n_\psi.
\end{aligned}$$

Using the above strategy to derive all the remaining equations with the PML, the system (5.4a)−(5.4e) with the change of variables (4.2) becomes in the time domain:

$$
\begin{aligned}
&\partial_t E_x + \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} j_{x\ell} = \partial_y H_z, \\
&\partial_t E_y + \sum_{\ell=0}^{n_y} \varepsilon_{y\ell} j_{y\ell} = -\partial_x H_z - \sigma(x) E_y^*, \\
&\partial_t H_z + \sum_{\ell=0}^{n_\mu} \mu_\ell j_{\mu\ell} = \partial_y E_x - \partial_x E_y - \sigma(x) H_z^*, \\
&\partial_t j_{m\ell} + \omega_{m\ell}^2 p_{m\ell} = E_m, \qquad \partial_t p_{m\ell} = j_{m\ell},\ \ell = 0, \ldots, n_m,\ m \in \{x, y\}, \\
&\partial_t j_{\mu\ell} + \omega_{\mu\ell}^2 p_{\mu\ell} = H_z, \qquad \partial_t p_{\mu\ell} = j_{\mu\ell},\ \ \ell = 0, \ldots, n_\mu,
\end{aligned}
\tag{5.6}
$$

coupled with the system for the auxiliary unknowns

$$
\begin{aligned}
&\partial_t E_y^* + \sum_{\ell=0}^{n_\psi} c_{\psi\ell} j_{E_y,\ell}^* + \sigma(x) E_y^* = -\partial_x H_z, \\
&\partial_t j_{E_y,\ell}^* + r_{\psi\ell}^2 p_{E_y,\ell}^* = E_y^*, \qquad \partial_t p_{E_y,\ell}^* = j_{E_y,\ell}^*,\ \ell = 0, \ldots, n_\psi, \\
&\partial_t H_z^* + \sum_{\ell=0}^{n_\psi} c_{\psi\ell} j_{H_z,\ell}^* + \sigma(x) H_z^* = -\partial_x E_y, \\
&\partial_t j_{H_z,\ell}^* + r_{\psi\ell}^2 p_{H_z,\ell}^* = H_z^*, \qquad \partial_t p_{H_z,\ell}^* = j_{H_z,\ell}^*,\ \ell = 0, \ldots, n_\psi.
\end{aligned}
\tag{5.7}
$$

Remark 5.8. There is no unique way to write the time-domain formulation depending on the choice of the family of auxiliary unknowns. However, all these choices are equivalent in terms of stability, see *e.g.* [14].

6. Necessary and sufficient conditions of stability of PMLs for models with Lorentz parameters

While Theorem 4.8 provides an explicit way to construct stable perfectly matched layers for anisotropic systems of type (3.6) by choosing $\psi(s) = a(s)$, such a choice of the function $\psi(s)$ may appear to be non-optimal, in a sense that it does not necessarily lead to the smallest number of unknowns in the resulting system in the time domain. On the other hand, while Lemma 4.3 provides a sufficient condition which should be satisfied by $\psi(s)$, it does not provide a constructive way of choosing such a function $\psi(s)$.

In the work [9], which is based on examining the behaviour of modes of the dispersion relation, a necessary condition of the stability of perfectly matched layers is formulated. Such a condition is easier to analyze compared to the conditions of Lemma 4.3, since instead of dealing with the right-half complex plane (as in Lem. 4.3), one studies the behaviour of functions restricted to the imaginary axis. It is well-known [10] that such conditions are not necessarily sufficient for the stability. For example, when applied to non-dispersive cases, they account only for the behaviour in the high-frequency regimes. Nevertheless, in [9] the authors have demonstrated that for a class of isotropic dispersive models (described by the sesquilinear form (3.6) with $a(s) = b(s)$) the necessary condition becomes sufficient.

The goal of this section is to demonstrate that the necessary condition derived in [9] is sufficient for the stability of the perfectly matched layers for a more general class of models of the form (3.6), which, unlike the models considered in [9], are anisotropic. However, we restrict our considerations to passive materials only.

The results of this section will enable us to find a family of $\psi(s)$ that would result in stable PMLs, among which there is an optimal choice in terms of number of auxiliary unknowns in the resulting time-domain PML system. Finding such an optimal $\psi(s)$ is however not always trivial.

First of all, we recall the necessary stability condition derived in [9], and then formulate the main result (Thm. 6.11) in Section 6.2. The proof of this result can be found in Section 6.3. An easy to use method to construct the function $\psi(s)$ can be deduced from [9], see also [11]. Finally, in Section 6.4 we show some numerical experiments with the new PML confirming the main results of Theorem 6.11.

6.1. The necessary stability condition

In this section we limit ourselves to passive sesquilinear forms (3.6) whose coefficients satisfy Assumption 5.1. This is summarized in the following assumption, see also (3.3).

Assumption 6.1. The coefficients of the sesquilinear form (3.6) $a(s)^{-1},\ b(s)^{-1},\ c(s)$ are passive in the sense of Definition 3.1 and satisfy Assumption 5.1.

Remark 6.2. Since $a(s),\ b(s),\ c(s)$ satisfy Assumption 6.1, we know, thanks to Theorem 5.4, that they are of the form (5.1) and therefore correspond to Lorentz materials. However, the results of this section can be extended to the case when $a(s)^{-1},\ b(s)^{-1},\ c(s)$ are of the form (5.1) with infinite number of terms ($n=\infty$), *cf. e.g.* ([41], Thm. 1, p. 308). A numerical illustration to this statement can be found in [11].

Such systems can be written in the form (5.4a)−(5.4e), with $a,\ b,\ c$ given in (3.8). Thus, we can write the dispersion relation for the time-domain system (5.4a)−(5.4e), see report [11]:

$$F(\omega,\mathbf{k})=\left(\omega^{-1}\tilde{a}(\omega)k_x^2+\omega^{-1}\tilde{b}(\omega)k_y^2-\omega\tilde{c}(\omega)\right)\prod_{\ell=0}^{n_a}(\omega^2-\omega_{a\ell}^2)\prod_{\ell=0}^{n_b}(\omega^2-\omega_{b\ell}^2)\prod_{\ell=0}^{n_c}(\omega^2-\omega_{c\ell}^2)\omega^2=0. \tag{6.1}$$

Here $(\omega_{a\ell})_{\ell=0}^{n_a}$, $(\omega_{b\ell})_{\ell=0}^{n_b}$, $(\omega_{c\ell})_{\ell=0}^{n_c}$ are the poles of correspondingly $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$.

Crucially, the analysis in [9] applies only to non-dissipative systems, as defined below.

Definition 6.3 [9]**.** A system is called non-dissipative if all modes $\omega(\mathbf{k})$ of its dispersion relation are real for all $\mathbf{k}=(k_x,\ k_y)\in\mathbb{R}^2$.

Proposition 6.4 (Non-dissipativity of (5.4a)−(5.4e))**.** *Under Assumption 6.1, all modes $\omega(\mathbf{k})$ of (6.1) are real.*

Proof. The solutions of the dispersion relation (6.1) are either poles of $\tilde{a},\ \tilde{b},\ \tilde{c}$, vanish, or are the solutions of

$$\tilde{a}(\omega)k_x^2+\tilde{b}(\omega)k_y^2-\omega^2\tilde{c}(\omega)=0. \tag{6.2}$$

Thanks to Lemma 5.3 and passivity of $a^{-1},\ b^{-1},\ c$, the poles of $\tilde{a},\ \tilde{b},\ \tilde{c}$ are real.

It remains to show that $\omega(\mathbf{k})$ solving (6.2) are real. According to Proposition 3.18, all solutions of (6.2) satisfy $\operatorname{Im}\omega(\mathbf{k})\leq 0$. Additionally, since $a,\ b,\ c$ are even, if ω solves (6.2), so does $-\omega$. Therefore, $\omega(\mathbf{k})\in\mathbb{R}$. □

The PML change of variables (4.6) leads to the system (5.6)−(5.7) with $\sigma(x)\equiv\sigma$, with the dispersion relation [11]

$$\begin{aligned}F_\sigma(\sigma,\omega,\mathbf{k})=&\left(\omega^{-1}\tilde{a}(\omega)\left(1-\frac{\sigma\tilde{\psi}(\omega)}{i\omega}\right)^{-2}k_x^2+\omega^{-1}\tilde{b}(\omega)k_y^2-\omega\tilde{c}(\omega)\right)\times\prod_{\ell}(\omega^2-\omega_{c\ell}^2)\\&\times\prod_{\ell}(\omega^2-\omega_{\psi\ell}^2)^2\prod_{\ell}(\omega^2-\omega_{a\ell}^2)\prod_{\ell}(\omega^2-\omega_{b\ell}^2)\left(1+\sigma\tilde{\psi}(\omega)(-i\omega)^{-1}\right)^2\omega^4=0.\end{aligned} \tag{6.3}$$

Here $(\omega_{\psi\ell})_{\ell=0}^{n_\psi}$ are the poles of $\tilde{\psi}$. As in Section 3.3, one studies the stability of the PML system by examining the modes of the corresponding dispersion relation.

Definition 6.5 (Def. 3.6 in [9])**.** A PML system (5.6)−(5.7) obtained with the change of variables (4.6) is called uniformly stable if all solutions $\omega(\mathbf{k},\sigma)$ of its dispersion relation (6.3) satisfy: $\operatorname{Im}\omega(\mathbf{k},\sigma)\leq 0$, for all $\mathbf{k}\in\mathbb{R}^2$ and $\sigma\geq 0$.

To formulate the necessary PML stability condition, let us recall the concepts of phase and group velocities.

Definition 6.6. A phase velocity of a mode $\omega(\mathbf{k})$ is defined by $\mathbf{v}_p(\omega(\mathbf{k})) = \frac{\omega(\mathbf{k})}{|\mathbf{k}|}\frac{\mathbf{k}}{|\mathbf{k}|}$, and the group velocity by $\mathbf{v}_g(\omega(\mathbf{k})) = \nabla_{\mathbf{k}}\omega(\mathbf{k})$, provided that this derivative is well-defined.

Finally, the choice of the function ψ in [9] is restricted to the class of functions $\tilde{\psi}(\omega) = h(\omega)$, where

$$h(\omega) = 1 + \sum_{\ell=0}^{n} \frac{h_\ell}{\omega^2 - \omega_\ell^2}, \qquad h_\ell \in \mathbb{R},\ \omega_\ell \in \mathbb{R}\setminus\{0\}, \qquad \ell = 0,\dots,n. \tag{6.4}$$

In [9] the authors have classified the solutions of the dispersion relation (6.3) and provided the necessary conditions of the uniform stability for each of the classes of the modes. We will discuss these classes afterwards.

Theorem 6.7 (Props. 3.10, 3.12 in [9]; necessary PML stability condition)**.** *Given* $\tilde{\psi}(\omega) = 1 + \sum\limits_{\ell=0}^{n_\psi} \frac{\psi_\ell}{\omega^2-\omega_{\psi\ell}^2}$, *with* $\omega_{\psi\ell} \in \mathbb{R}\setminus\{0\}$, $\psi_\ell \in \mathbb{R}$, $\ell = 0,\dots,n_\psi$, *let a PML system corresponding to the dispersion relation* (6.3) *be uniformly stable. Then*

(N1) *the coefficients of the expansion of* $\tilde{\psi}(\omega)$ *satisfy* $\psi_\ell > 0$, $\ell = 0,\dots,n_\psi$. *Additionally,* $\tilde{\psi}(0) \geq 0$.
(N2) *for all the solutions* $\omega_j(\mathbf{k})$, $j = 1,\dots,N$, *of the original dispersion relation (6.1), it holds that*

$$\tilde{\psi}(\omega_j(\mathbf{k}))\mathbf{v}_{g,x}(\omega_j(\mathbf{k}))\mathbf{v}_{p,x}(\omega_j(\mathbf{k})) \geq 0,\ \mathbf{k} \in \mathbb{R}^2.$$

Here the subscript x indexes the x-component of a vector.

Let us re-interpret Theorem 6.7. All the modes of (6.3) belong to one of the following classes:

(1) they are *non-propagating* (*i.e.* independent of $\mathbf{k}$). Such modes can be real $\omega(\mathbf{k}) = \text{const} \in \mathbb{R}$ or solve

$$1 - (i\omega)^{-1}\tilde{\psi}(\omega)\sigma = 0.$$

The stability of the modes that solve the above equation is ensured by the condition (N1), or, in our terms, *cf.* Theorem 5.6, by the condition that $\operatorname{Re}(\bar{s}\psi(s)) > 0$, $s \in \mathbb{C}_+$.

(2) they are *propagating*, *i.e.* solve, for some $k_x,\ k_y \in \mathbb{R}$:

$$\omega^{-1}\tilde{a}(\omega)\left(1 - \frac{\sigma\tilde{\psi}(\omega)}{i\omega}\right)^{-2} k_x^2 + \omega^{-1}\tilde{b}(\omega)k_y^2 - \omega\tilde{c}(\omega) = 0. \tag{6.5}$$

A necessary condition of the stability of the propagating modes is given by the condition (N2).

Let us rewrite (N2) with the use of the implicit function theorem. Clearly, this condition is of interest for the modes that depend on $\mathbf{k}$ non-trivially, *i.e.* modes that solve, *cf.* dispersion relation (6.1):

$$\mathcal{F}(\omega,\mathbf{k}) = \omega^{-1}\tilde{a}(\omega)k_x^2 + \omega^{-1}\tilde{b}(\omega)k_y^2 - \omega\tilde{c}(\omega) = 0. \tag{6.6}$$

We denote the set of the propagating modes of the original system by

$$\Omega_p := \left\{\omega \in \mathbb{R} :\ \mathcal{F}(\omega,\mathbf{k}) = 0 \text{ for some } \mathbf{k} \in \mathbb{R}^2\right\}. \tag{6.7}$$

Then, given $\mathcal{F}(\omega,\mathbf{k})$ as in (6.6), we deduce with the help of the implicit function theorem:

$$\partial_{k_x}\omega(k_x,k_y) = -\left(\partial_{k_x}\mathcal{F}(\omega,\mathbf{k})\right)\left(\partial_\omega\mathcal{F}(\omega,\mathbf{k})\right)^{-1} = -2k_x\omega^{-1}\tilde{a}(\omega)\left(k_x^2(\omega^{-1}\tilde{a}(\omega))' + (\omega^{-1}\tilde{b}(\omega))'k_y^2 - (\omega\tilde{c}(\omega))'\right)^{-1}.$$

The condition (N2) in Theorem 6.7 can be rewritten in the form: for all $\omega \in \Omega_p$,

$$-k_x^2\tilde{\psi}(\omega)\tilde{a}(\omega)\left(k_x^2(\omega^{-1}\tilde{a}(\omega))' + k_y^2(\omega^{-1}\tilde{b}(\omega))' - (\omega\tilde{c}(\omega))'\right)^{-1} \geq 0, \tag{6.8}$$

provided that the above expression is well-defined.

We will need to reformulate the above condition in a simpler form. First of all, we will simplify (6.7). Given a rational function $r(z)$ satisfying Assumption 5.1, let us introduce a set $\mathcal{D}_r = \{\omega \in \mathbb{R} :\ \tilde{r}(\omega) = 0 \text{ or } \tilde{r}(\omega)^{-1} = 0\}$.

Lemma 6.8. *Let $a(s),\ b(s),\ c(s)$ satisfy Assumption 5.1. Then Ω_p defined in (6.7) can be represented as a union of the following (possibly intersecting) sets:*

$$\Omega_p = \Omega_p^{\tilde{a}\tilde{c}>0} \cup \Omega_p^{\tilde{a}\tilde{b}<0} \cup \mathcal{Z}_p,$$

$$\Omega_p^{\tilde{a}\tilde{c}>0} = \{\omega \in \mathbb{R} \setminus (\mathcal{D}_a \cup \mathcal{D}_c) :\ \tilde{a}(\omega)\tilde{c}(\omega) > 0\}, \tag{6.9}$$

$$\Omega_p^{\tilde{a}\tilde{b}<0} = \left\{\omega \in \mathbb{R} \setminus (\mathcal{D}_a \cup \mathcal{D}_b) :\ \tilde{a}(\omega)\tilde{b}(\omega) < 0\right\}, \tag{6.10}$$

$$\mathcal{Z}_p = \left\{\omega \in \mathcal{D}_a \cup \mathcal{D}_b \cup \mathcal{D}_c :\ \text{ for some } k_x,\ k_y \in \mathbb{R},\ \text{it holds } \omega^{-1}\tilde{a}(\omega)k_x^2 + \omega^{-1}\tilde{b}(\omega)k_y^2 - \omega\tilde{c}(\omega) = 0\right\}.$$

Proof. First we show $\Omega_p \subset \Omega_p^{\tilde{a}\tilde{c}>0} \cup \Omega_p^{\tilde{a}\tilde{b}<0} \cup \mathcal{Z}_p$. Let $\omega \notin \mathcal{Z}_p$. Then the modes $\omega \in \Omega_p$ satisfy

$$k_x^2 = -\frac{\tilde{b}(\omega)}{\tilde{a}(\omega)}k_y^2 + \omega^2\frac{\tilde{c}(\omega)}{\tilde{a}(\omega)}\cdot$$

This equation has at least one real solution k_x if and only if there exists $k_y \in \mathbb{R}$ s.t.

$$-\tilde{b}(\omega)\tilde{a}(\omega)^{-1}k_y^2 + \omega^2\tilde{c}(\omega)\tilde{a}(\omega)^{-1} \geq 0. \tag{6.11}$$

The above inequality has no real solutions k_y if $\tilde{b}(\omega)\tilde{a}(\omega)^{-1} \geq 0$ and $\tilde{c}(\omega)\tilde{a}(\omega)^{-1} < 0$. Hence, the inequality (6.11) has a real solution k_y if and only if $\omega \notin \mathcal{Z}_p$ satisfies at least one of the following conditions:

- $\tilde{b}(\omega)\tilde{a}(\omega)^{-1} < 0$, *i.e.* $\omega \in \Omega_p^{\tilde{a}\tilde{b}<0}$;
- $\tilde{c}(\omega)\tilde{a}(\omega)^{-1} > 0$, *i.e.* $\omega \in \Omega_p^{\tilde{a}\tilde{c}>0}$.

The inclusion $\Omega_p^{\tilde{a}\tilde{c}>0} \cup \Omega_p^{\tilde{a}\tilde{b}<0} \cup \mathcal{Z}_p \subset \Omega_p$ can be proved similarly. ☐

With the above we can reformulate the necessary stability condition (N2).

Lemma 6.9 (Necessary stability condition (N2))**.** *Let $a(s),\ b(s),\ c(s)$ satisfy Assumption 6.1, and let $\psi(s)$ satisfy Assumption 5.1 and $\operatorname{Re}(\bar{s}\psi(s)) > 0,\ s \in \mathbb{C}_+$. Let for $\omega \in \Omega_p$ the stability condition (6.8) hold true, provided that the latter expression is well-defined. Then for all $\omega \in \Omega_p^{\tilde{a}\tilde{c}>0} \cup \Omega_p^{\tilde{a}\tilde{b}<0}$ it holds*

$$\tilde{\psi}(\omega)\tilde{a}(\omega) \geq 0. \tag{6.12}$$

Proof. Consider the denominator of the left-hand side of (6.8):

$$\mathcal{H} := k_x^2(\omega^{-1}\tilde{a}(\omega))' + k_y^2(\omega^{-1}\tilde{b}(\omega))' - (\omega\tilde{c}(\omega))'.$$

Thanks to Lemma 5.5 about the sign of the derivatives of passive functions (applied to $c(s)$) and the analogical Lemma 5.7 for reciprocals of passive functions (applied to $a(s),\ b(s)$), for all $\omega \notin \mathcal{Z}_p$, it holds

$$(\omega^{-1}\tilde{a}(\omega))' < 0, \qquad (\omega^{-1}\tilde{b}(\omega))' < 0, \qquad (\omega\tilde{c}(\omega))' > 0,$$

Therefore, the denominator $\mathcal{H} < 0$ for all $k_x,\ k_y \in \mathbb{R}$ and $\omega \notin \mathcal{Z}_p$. We can thus rewrite the inequality (6.8) as

$$\tilde{\psi}(\omega)\tilde{a}(\omega) \geq 0.$$ ☐

6.2. Equivalence of the necessary and sufficient stability conditions

In this section we show that the necessary stability condition of the PML in Lemma 6.9 is equivalent to the sufficient stability condition of Lemma 4.3. To formulate it, we need the following assumption.

Assumption 6.10. The functions $a(s),\ b(s),\ c(s)$ satisfy Assumption 6.1, and $\psi(s)$ satisfies Assumption 5.1 and $\operatorname{Re}(\bar{s}\psi(s)) > 0,\ s \in \mathbb{C}_+$.

Theorem 6.11 (Equivalence of the necessary and sufficient stability conditions)**.**
Let $a(s),\ b(s),\ c(s),\ \psi(s)$ satisfy Assumption 6.10 and let

$$A(u,v) = a(s)(\partial_x u, \partial_x v) + b(s)(\partial_y u, \partial_y v) + s^2 c(s)(u,v)$$

be the corresponding passive sesquilinear form. Then the following two conditions are equivalent:

(NSC1) *for all $\omega \in \Omega_p^{\tilde{a}\tilde{b}<0} \cup \Omega_p^{\tilde{a}\tilde{c}>0}$ (see (6.9, 6.10)), it holds that $\tilde{\psi}(\omega)\tilde{a}(\omega) \geq 0$.*
(NSC2) *for all $\sigma \geq 0$, the sesquilinear form*

$$\tilde{A}_\sigma(u,v) = \frac{\psi(s)}{a(s)}\left(a(s)\left(1+\frac{\sigma\psi(s)}{s}\right)^{-2}(\partial_x u, \partial_x v) + b(s)(\partial_y u, \partial_y v) + s^2 c(s)(u,v)\right),\ s \in \mathbb{C}_+,$$

is passive in the sense of Definition 3.11, and thus, bounded and coercive (due to Prop. 3.12).

Proof. The proof (NSC2) $\Longrightarrow$ (NSC1) follows from the results of [9]. Consider one of the factors of the dispersion relation (6.3) given by (6.5). Thanks to Proposition 3.18, the passivity of the sesquilinear form $\tilde{A}_\sigma(u,v)$ implies that all the solutions $\omega(\mathbf{k}),\ \mathbf{k} \in \mathbb{R}^2$, of (6.5) have a non-positive imaginary part, *i.e.* $\operatorname{Im}\omega(\mathbf{k}) \leq 0$. In [9], Proposition 3.12, the following implication was shown:

$$\text{all solutions } \omega(\mathbf{k}),\ \mathbf{k} \in \mathbb{R}^2, \text{ of (6.5) are s.t. } \operatorname{Im}\omega(\mathbf{k}) \leq 0 \implies \text{for } \omega \in \Omega_p,\quad \text{(6.8) holds true.}$$

Thanks to Lemma 6.9, from the above (NSC1) follows.

To show the implication (NSC1) $\Longrightarrow$ (NSC2), we use Lemma 4.3. It is sufficient to prove that for $\psi(s)$ satisfying the sign conditions (NSC1), it holds

$$\operatorname{Re}\left(sc(s)a(s)^{-1}\psi(s)\right) > 0,\ s \in \mathbb{C}_+, \qquad \text{(NSC2)-(a)}$$
$$\operatorname{Re}\left(\bar{s}b(s)a(s)^{-1}\psi(s)\right) > 0,\ s \in \mathbb{C}_+. \qquad \text{(NSC2)-(b)}$$

The proof of the above relies on several technical results, which we will formulate and prove in Section 6.3. □

We readily obtain the following corollary for a specific class of anisotropic systems (3.6) with $c(s) = b(s)^{-1}$. A particular representative of this class is a uniaxial cold plasma model given in (2.7), with $b(s) = 1$.

Corollary 6.12. *Let $a(s), \psi(s), b(s)$ satisfy Assumption 6.10. Then the following two conditions are equivalent:*
(1) *for all $\sigma \geq 0$, the sesquilinear form*

$$\tilde{A}_\sigma(u,v) = \frac{\psi(s)}{a(s)}\left(a(s)\left(1+\frac{\sigma\psi(s)}{s}\right)^{-2}(\partial_x u, \partial_x v) + b(s)(\partial_y u, \partial_y v) + s^2 b(s)^{-1}(u,v)\right),\ s \in \mathbb{C}_+,$$

is passive in the sense of Definition 3.11.
(2) $\psi(s) = a(s)$.

Proof. Notice that the above sesquilinear form is as in Theorem 6.11 with $\tilde{c}(\omega) = \tilde{b}(\omega)^{-1}$. Using Theorem 6.11, we see that in points $\omega \in \mathbb{R}$ s.t. $\tilde{a}(\omega)\tilde{c}(\omega) = \tilde{a}(\omega)\tilde{b}(\omega)^{-1} > 0$, the functions $\tilde{\psi}(\omega)$ and $\tilde{a}(\omega)$ should be of the same sign. The same should hold in points where $\tilde{a}(\omega)\tilde{b}(\omega) < 0$. Thus, the poles and zeros of $\psi(s)$ and $a(s)$ coincide (not necessarily with their multiplicity). Since all the poles and zeros of $\psi(s)$ are simple (with a possible exception of $s = 0$), see also ($\overline{M}2$) in Lemma 5.7, the assertion follows immediately. □

Remark 6.13. From Theorem 4.8, there always exists $\tilde{\psi}$ which satisfies (NSC1) of Theorem 6.11, namely $\psi(s) = a(s)$, and thus, results in a stable PML.

6.3. Proof of Theorem 6.11

To prove Theorem 6.11, we proceed as follows. First, let us notice that the condition (NSC1) of Theorem 6.11 is equivalent to the following two conditions:

$$\text{For all } \omega \in \Omega_p^{\tilde{a}\tilde{c}>0}: \ \tilde{\psi}(\omega)\tilde{a}(\omega) \geq 0, \tag{NSC1)-(a}$$

$$\text{For all } \omega \in \Omega_p^{\tilde{a}\tilde{b}<0}: \ \tilde{\psi}(\omega)\tilde{a}(\omega) \geq 0. \tag{NSC1)-(b}$$

We will first demonstrate that (NSC1)-(a) implies the passivity condition (NSC2)-(a). This is done in two steps:

(1) rewrite the condition (NSC2)-(a) in terms of equivalent conditions on the signs of $\tilde{\psi}$ in poles (zeros) of functions $\tilde{a}$, $\tilde{c}$ (Lem. 6.14);
(2) show that if $\tilde{\psi}$ satisfies (NSC1)-(a), then above conditions on the signs of $\tilde{\psi}$ follows. This is explained in Lemma 6.15, whose proof relies largely on Lemmas 6.17, 6.19 and Corollary 6.18.

To show that (NSC1)-(b) implies (NSC2)-(b), we will make use of a logical argument (Lem. 6.20).

6.3.1. Reformulation of the condition (NSC2)-(a)

In the following lemma we reformulate the passivity condition (NSC2)-(a) in a more convenient form.

Lemma 6.14. *Let $a(s)$, $c(s)$, $\psi(s)$ satisfy Assumption 6.10. Then (NSC2)-(a) holds if and only if the following limits exist and satisfy*:

$$\begin{aligned}
&(a) \text{ in all poles } \left(\omega_{c_j}\right)_{j=0}^{n_c} \text{ of } \tilde{c}(\omega), \text{ it holds that } \lim_{\omega\to\omega_{c_j}} \tilde{a}(\omega)^{-1}\tilde{\psi}(\omega) \geq 0.\\
&(b) \text{ in all zeros } \left(\omega_{a_j}\right)_{j=0}^{n_a} \text{ of } \tilde{a}(\omega), \text{ it holds that } \lim_{\omega\to\omega_{a_j}} \tilde{\psi}(\omega)\tilde{c}(\omega) \geq 0.\\
&(c) \text{ in all poles } \left(\omega_{\psi_j}\right)_{j=0}^{n_\psi} \text{ of } \tilde{\psi}(\omega), \text{ it holds that } \lim_{\omega\to\omega_{\psi_j}} \tilde{a}(\omega)^{-1}\tilde{c}(\omega) \leq 0.
\end{aligned}$$

Proof. By Theorem 5.4, the function $c(s)a(s)^{-1}\psi(s)$ is passive if and only if its partial fraction expansion reads:

$$\tilde{c}(\omega)\tilde{a}(\omega)^{-1}\tilde{\psi}(\omega) = 1 - \sum_{\ell=0}^{N} \frac{p_\ell}{\omega^2 - \omega_\ell^2}, \qquad p_\ell > 0, \ \omega_\ell \in \mathbb{R}, \ \ell = 0, \dots, N.$$

Thus, (NSC2)-(a) is equivalent to the following. In each of the poles $\omega = \omega_k$, $k = 0, \dots, N$, of $a(s)^{-1}c(s)\psi(s)$ the following limit exists and satisfies

$$\lim_{\omega\to\omega_k} (\omega^2 - \omega_k^2)\tilde{c}(\omega)\tilde{a}(\omega)^{-1}\tilde{\psi}(\omega) = -p_k < 0. \tag{6.13}$$

Let us demonstrate that the above condition implies the inequalities $(a)-(c)$. Since $(\omega_k)_{k=0}^N$ is a subset of poles of $\tilde{c}$, $\tilde{a}^{-1}$ and $\tilde{\psi}$, we separately consider (6.13) for the poles of each of these functions. Let ω_{c_j} be a pole of $\tilde{c}(\omega)$. The function $\tilde{c}(\omega)$, thanks to Theorem 5.4, is of the form

$$\tilde{c}(\omega) = 1 - \sum_{\ell=0}^{n_c} \frac{c_\ell}{\omega^2 - \omega_{c_\ell}^2},$$

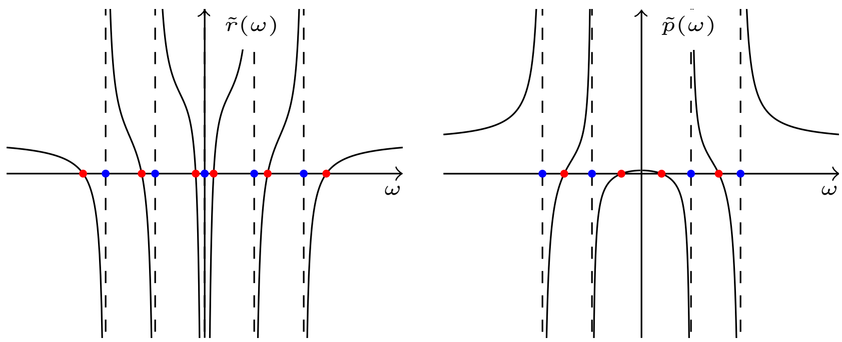


FIGURE 5. In the left figure we depict $\tilde{r}(\omega)$, s.t. $\mathrm{Re}\,(sr(s)) > 0,\ s \in \mathbb{C}_+$ and in the right one we plot $\tilde{p}(\omega)$, s.t. $\mathrm{Re}\,(\bar{s}p(s)) > 0,\ s \in \mathbb{C}_+$. The functions $r(s),\ p(s)$ satisfy Assumption 5.1.

where $c_\ell > 0$ for $\ell = 0, \dots, n_c$. Let us compute the limit (6.13) substituting ω_k by ω_{c_j}:

$$\lim_{\omega\to\omega_{c_j}} (\omega^2 - \omega_{c_j}^2)\tilde{c}(\omega)\tilde{a}(\omega)^{-1}\tilde{\psi}(\omega) = \lim_{\omega\to\omega_{c_j}} \Big((\omega^2 - \omega_{c_j}^2)\tilde{c}(\omega)\Big) \lim_{\omega\to\omega_{c_j}} \tilde{a}(\omega)^{-1}\tilde{\psi}(\omega) = -c_j \lim_{\omega\to\omega_{c_j}} \tilde{a}(\omega)^{-1}\tilde{\psi}(\omega) \leq 0.$$

The strict inequality holds if and only if ω_{c_j} is a pole of $\tilde{c}\tilde{a}^{-1}\tilde{\psi}$, *cf.* (6.13). The above limit vanishes if and only if $\tilde{c}\tilde{a}^{-1}\tilde{\psi}$ has no pole in ω_{c_j}. The other inequalities in the statement of the lemma are obtained similarly.

The proof of the implication $(a)-(c) \implies$ (6.13) is almost verbatim the same. $\square$

6.3.2. Necessary stability condition (NSC1)-(a) implies passivity of $c(s)a(s)^{-1}\psi(s)$ (NSC2)-(a)

Let us first formulate the main result of this section, namely that (NSC1)-(a) implies (NSC2)-(a).

Lemma 6.15. *Let $a(s),\ \psi(s),\ c(s)$ satisfy Assumption 6.10. Assume that (NSC1)-(a) holds. Then (NSC2)-(a) holds true as well.*

Proof. We would like to show that (NSC1)-(a) implies inequalities (a)–(c) of Lemma 6.14, which, in turn, implies (NSC2)-(a). Let us look at the corresponding cases:

- in poles ω_{c_j} of $\tilde{c}(\omega)$, we must show that the following limit exists and satisfies $\lim\limits_{\omega\to\omega_{c_j}} \tilde{a}(\omega)^{-1}\tilde{\psi}(\omega) \geq 0$. This follows by a direction application of Lemma 6.17, see below, with $\alpha = a$, $\gamma = c$ and $\phi = \psi$.
- Corollary 6.18 shows that in zeros $\omega_{a_j}$of $\tilde{a}(\omega)$ the following limit exists and satisfies $\lim\limits_{\omega\to\omega_{a_j}} \tilde{c}(\omega)\tilde{\psi}(\omega) \geq 0$.
- Lemma 6.19 shows that in poles ω_{ψ_j} of $\tilde{\psi}$ the following limit exists and satisfies $\lim\limits_{\omega\to\omega_{\psi_j}} \tilde{c}(\omega)\tilde{a}(\omega)^{-1} \leq 0$. $\square$

Remark 6.16. The necessary condition in the above lemma is also sufficient, see Appendix C.

Before formulating the results mentioned in the proof of Lemma 6.15, let us recall the properties of passive functions and their reciprocals crucial for proving the result of Theorem 6.11, see also Figure 5 for illustration. For a passive rational function $r(s)$ satisfying Assumption 5.1, it holds, thanks to Lemma 5.5 and Theorem 5.4:

$(P1)$ if $\omega = 0$ is a pole of $\tilde{r}$, then $\tilde{r}(\omega) < 0$ in a sufficiently small vicinity of $\omega = 0$. Otherwise $\tilde{r}(0) > 0$.
$(P2)$ in a positive pole, the function $\tilde{r}$ changes its sign from positive to negative.
$(P3)$ in a positive zero, the function $\tilde{r}$ changes its sign from negative to positive.

Similarly, for a rational function $p(s)$ satisfying Assumption 5.1, for which $\operatorname{Re}(\bar{s}p(s)) > 0$, $s \in \mathbb{C}_+$, it holds, thanks to Lemma 5.7 and Theorem 5.6:

($\overline{P}1$) if $\omega = 0$ is a zero of $\tilde{p}$, then $\tilde{p}(\omega) < 0$ in a sufficiently small vicinity of $\omega = 0$. Otherwise $\tilde{p}(0) > 0$.
($\overline{P}2$) in a positive pole, the function $\tilde{p}$ changes its sign from negative to positive.
($\overline{P}3$) in a positive zero, the function $\tilde{p}$ changes its sign from positive to negative.

Now let us state the auxiliary results used in the proof of Lemma 6.15.

Lemma 6.17. *Let $\alpha(s)$, $\phi(s)$, $\gamma(s)$ satisfy Assumption 5.1, and $\operatorname{Re}(\bar{s}\alpha(s)) > 0$, $\operatorname{Re}(\bar{s}\phi(s)) > 0$, and $\operatorname{Re}(s\gamma(s)) > 0$, for all $s \in \mathbb{C}_+$. Assume that for all $\omega \in \mathbb{R} \setminus (\mathcal{D}_\gamma \cup \mathcal{D}_\alpha)$ it holds*

$$\tilde{\gamma}(\omega)\tilde{\alpha}(\omega) > 0 \implies \tilde{\phi}(\omega)\tilde{\alpha}(\omega) \geq 0. \tag{6.14}$$

Then in all the poles $\left(\omega_{\gamma_j}\right)_{j=0}^{n_\gamma}$ of $\tilde{\gamma}(\omega)$ the following limit exists and satisfies

$$0 \leq \lim_{\omega \to \omega_{\gamma_j}} \tilde{\alpha}(\omega)^{-1}\tilde{\phi}(\omega) < +\infty. \tag{6.15}$$

Proof. Let $\omega_{\gamma_j} \geq 0$ be a pole of $\tilde{\gamma}(\omega)$ (negative poles are treated similarly). Then the following cases are possible:

- $\omega_{\gamma_j} = 0$. According to Theorem 5.6 applied to $\alpha(s)$, only one of the following can hold true:
  (1) either $\tilde{\alpha}(0) = 0$. Then, for sufficiently small $\omega > 0$:
    - $\tilde{\alpha}(\omega)^{-1} < 0$, see ($\overline{P}1$),
    - $\tilde{\gamma}(\omega) < 0$, see ($P1$).

    Thus, in a small vicinity of zero, $\tilde{\alpha}\tilde{\gamma} > 0$, and with (6.14) it follows that there $\tilde{\phi}(\omega) \leq 0$, which is possible only if $\tilde{\phi}(0) = 0$, see ($\overline{P}1$). Thus, $\tilde{\phi}$, $\tilde{\alpha}$ both have a zero of order 2 in $\omega = 0$, see ($\overline{M}2$) of Lemma 5.7. Therefore, the following limit exists and satisfies: $\lim\limits_{\omega\to 0} \tilde{\phi}(\omega)\tilde{\alpha}(\omega)^{-1} > 0$.

  (2) or $\tilde{\alpha}(0)^{-1} > 0$. Applying Theorem 5.6 to $\phi(s)$, we notice $\tilde{\phi}(0) \geq 0$, hence (6.15) holds true.
- $\omega_{\gamma_j} > 0$. The following cases are possible:
  (1) $0 < \tilde{\alpha}(\omega_{\gamma_j}) < \infty$ ($-\infty < \tilde{\alpha}(\omega_{\gamma_j}) < 0$). Due to ($P2$), the inequality $\tilde{\gamma}(\omega)\tilde{\alpha}(\omega) > 0$ holds for all $\omega = \omega_{\gamma_j} - \delta$ ($\omega = \omega_{\gamma_j} + \delta$), with $\delta > 0$ being sufficiently small. As for $\tilde{\phi}$, there are two possibilities:
    - $\tilde{\phi}(\omega)$ does not have a pole in ω_{γ_j}. Then, due to (6.14), the limit (6.15) exists and satisfies the inequality (6.15) (this includes the case $\tilde{\phi}(\omega_{\gamma_j}) = 0$).
    - $\tilde{\phi}$ has a pole in ω_{γ_j}. This is impossible, since in this case (6.14) would require that $\tilde{\phi}(\omega) > 0$ ($\tilde{\phi}(\omega) < 0$) for $\omega = \omega_{\gamma_j} - \delta$ ($\omega = \omega_{\gamma_j} + \delta$), for $\delta > 0$; this would be a contradiction to the fact that in its pole $\tilde{\phi}(\omega)$ changes its sign from negative to positive, see ($\overline{P}2$).

  (2) $\tilde{\alpha}(\omega)$ has a zero in ω_{γ_j}. Then in ω_{γ_j} $\tilde{\alpha}(\omega)$ changes its sign from positive to negative, see ($\overline{P}3$), and so does $\tilde{\gamma}$, see ($P2$). Thus, in the vicinity of ω_{γ_j}, the product $\tilde{\alpha}(\omega)^{-1}\tilde{\gamma}(\omega) > 0$. Therefore, (6.14) requires that $\tilde{\phi}(\omega) \geq 0$ there. Thus $\tilde{\phi}$ changes the sign in ω_{γ_j} from positive to negative, which is possible only if $\tilde{\phi}(\omega_{\gamma_j}) = 0$, see ($\overline{P}3$). Since nonvanishing zeros of $\tilde{\alpha}, \tilde{\psi}$ are simple, (6.15) holds true.
  (3) $\tilde{\alpha}$ has a pole in ω_{γ_j}. Then there are two possibilities:
    - $\tilde{\phi}$ is finite in ω_{γ_j}. This immediately implies that the limit (6.15) vanishes.
    - $\tilde{\phi}$ has a pole in ω_{γ_j}. Due to ($\overline{P}2$), in ω_{γ_j} both functions $\tilde{\phi}$, $\tilde{\alpha}$ change their signs from negative to positive (and hence their product is positive in the vicinity of the pole). This, combined with the simplicity of poles of $\tilde{\alpha}$, $\tilde{\phi}$, shows that (6.15) holds true. □

The following result is a direct corollary of the above.

Corollary 6.18. *Let $a(s)$, $\psi(s)$, $c(s)$ satisfy Assumption 6.10. Assume that (NSC1)-(a) holds true. Then in all the zeros $\left(\omega_{a_j}\right)_{j=0}^{n_a}$ of $\tilde{a}(\omega)$ the following limit exists and satisfies*

$$0 \leq \lim_{\omega \to \omega_{a_j}} \tilde{c}(\omega)\tilde{\psi}(\omega) < +\infty. \tag{6.16}$$

Proof. The condition (NSC1)-(a) can be rewritten as follows: for all $\omega \in \mathbb{R} \setminus (\mathcal{D}_a \cup \mathcal{D}_c)$, it holds: $\tilde{c}(\omega)^{-1}\tilde{a}(\omega)^{-1} > 0 \implies \tilde{\psi}(\omega)\tilde{c}(\omega)^{-1} \geq 0$. Then $\alpha(s) = c(s)^{-1}$, $\gamma(s) = a(s)^{-1}$ and $\phi(s) = \psi(s)$ satisfy conditions of Lemma 6.17 (see (3.3)), from which the result is immediately obtained. ☐

Finally, what remains to show is the following.

Lemma 6.19. *Let $a(s)$, $\psi(s)$, $c(s)$ satisfy Assumption 6.10. Assume that (NSC1)-(a) holds. Then in all the poles $\left(\omega_{\psi_j}\right)_{j=0}^{n_\psi}$ of $\tilde{\psi}(\omega)$, the following limit exists and satisfies*

$$-\infty < \lim_{\omega \to \omega_{\psi_j}} \tilde{c}(\omega)\tilde{a}(\omega)^{-1} \leq 0. \tag{6.17}$$

Proof. Recall that $\omega_{\psi_j} = 0$ cannot be a pole of $\tilde{\psi}$, see Theorem 5.6. Hence we consider the case $\omega_{\psi_j} > 0$ (negative values are treated similarly). We thus look at the following cases:

(1) let $-\infty < \tilde{c}(\omega_{\psi_j}) < 0$ $(0 < \tilde{c}(\omega_{\psi_j}) < \infty)$. As for $\tilde{a}(\omega)$, it satisfies either of the following:
- $\tilde{a}(\omega)$ is finite and does not change its sign in ω_{ψ_j}. Assume by contradiction that (6.17) does not hold true, *i.e.* $\tilde{c}(\omega_{\psi_j})\tilde{a}(\omega_{\psi_j})^{-1} > 0$. By continuity this holds in a vicinity of ω_{ψ_j}. Then, due to (NSC1)-(a), $\tilde{a}(\omega)\tilde{\psi}(\omega) \geq 0$ in the vicinity of ω_{ψ_j}. We arrive at the contradiction, since $\tilde{\psi}$ changes its sign in ω_{ψ_j} but not $\tilde{a}$. Hence, necessarily, $\tilde{c}(\omega_{\psi_j})\tilde{a}(\omega_{\psi_j})^{-1} \leq 0$.
- ω_{ψ_j} is a pole of $\tilde{a}(\omega)$. Then obviously the limit (6.17) vanishes.
- $\tilde{a}(\omega_{\psi_j}) = 0$: due to ($\overline{P}3$), $\tilde{a}$ changes its sign from positive to negative in ω_{ψ_j}, and the inequality $\tilde{c}(\omega)\tilde{a}(\omega) > 0$ holds for all $\omega = \omega_{\gamma_j} + \delta$ $(\omega = \omega_{\gamma_j} - \delta)$, with $\delta > 0$ being sufficiently small. Then, due to (NSC1)-(a), the product $\tilde{a}(\omega)\tilde{\psi}(\omega) \geq 0$ for such values ω, which is impossible, since $\tilde{\psi}$ changes in ω_{ψ_j} its sign from negative to positive and $\tilde{a}$ from positive to negative, see ($\overline{P}3$) and ($\overline{P}2$).

(2) $\tilde{c}(\omega_{\psi_j}) = 0$. Then either is possible for $\tilde{a}$:
- $\tilde{a}(\omega)$ is finite or has a pole in ω_{ψ_j}. Obviously, the limit (6.17) vanishes.
- if $\tilde{a}(\omega_{\psi_j}) = 0$, then ($P3$) and ($\overline{P}2$) imply that $\tilde{a}(\omega)\tilde{c}(\omega) < 0$ in a vicinity of ω_{ψ_j}. Combined with the fact that positive zeros and poles of $\tilde{a}$ and $\tilde{c}$ are simple, we obtain the existence of the limit.

(3) $\tilde{c}$ has a pole in ω_{ψ_j}. As shown in Lemma 6.17, with $\alpha = a$, $\phi = \psi$, $\gamma = c$, in this case $\tilde{\psi}$ can have a pole in ω_{ψ_j} only if $\tilde{a}$ has a pole in ω_{ψ_j}. The existence of the limit follows from the fact that ω_{ψ_j} is a simple pole of $\tilde{a}$, $\tilde{c}$, and the sign of the limit follows from the character of the sign change of $\tilde{c}$ and $\tilde{a}$ in ω_{ψ_j}, see ($P2$) and ($\overline{P}2$). ☐

6.3.3. *Necessary stability condition (NSC1)-(b) implies passivity of $b(s)^{-1}a(s)\psi(s)^{-1}$ (NSC2)-(b)*

Now our goal is to connect the passivity of $b(s)^{-1}a(s)\psi(s)^{-1}$ and the necessary stability condition. We will not make use of lemmas similar to Lemma 6.17, but rather use some trivial logic arguments to show that Lemma 6.15 implies the validity of the following result.

Lemma 6.20. *Let $a(s)$, $\psi(s)$, $b(s)$ satisfy Assumption 6.10. If (NSC1)-(b) holds true, then so does (NSC2)-(b).*

Proof. Given a, b, ψ, let us introduce the following new unknowns:

$$c_n := \psi^{-1},\ a_n := b,\ \psi_n := a. \tag{6.18}$$

Notice that $c_n(s)$ is passive, see (3.3). Due to Lemma 6.15, the first statement below implies the second one:

(1) for all $\omega \in \mathbb{R} \setminus (\mathcal{D}_{a_n} \cup \mathcal{D}_{c_n})$ it holds that

$$\tilde{c}_n(\omega)\tilde{a}_n(\omega) > 0 \implies \tilde{\psi}_n(\omega)\tilde{a}_n(\omega) \geq 0, \tag{6.19}$$

(2) $\operatorname{Re}(sc_n(s)a_n(s)^{-1}\psi_n(s)) > 0$ in $\mathbb{C}_+$, or, equivalently, see (3.3), and the notation (6.18), $\operatorname{Re}\left(\bar{s}b(s)\psi(s)a(s)^{-1}\right) > 0$ in $\mathbb{C}_+$.

With the new notation (6.18), the expression (6.19) reads

$$\tilde{\psi}(\omega)^{-1}\tilde{b}(\omega) > 0 \implies \tilde{a}(\omega)\tilde{b}(\omega) \geq 0, \qquad \omega \in \mathbb{R} \setminus (\mathcal{D}_\psi \cup \mathcal{D}_b),$$

However, the above is equivalent to

$$\tilde{a}(\omega)\tilde{b}(\omega) < 0 \implies \tilde{\psi}(\omega)^{-1}\tilde{b}(\omega)^{-1} \leq 0, \qquad \omega \in \mathbb{R} \setminus (\mathcal{D}_\psi \cup \mathcal{D}_b),$$

or, alternatively, due to the continuity of all the functions in ω,

$$\tilde{a}(\omega)\tilde{b}(\omega) < 0 \implies \tilde{\psi}(\omega)\tilde{a}(\omega) \leq 0, \qquad \omega \in \mathbb{R} \setminus (\mathcal{D}_a \cup \mathcal{D}_b).$$

From this we obtain the desired statement. □

Remark 6.21. As before, the necessary condition in the above lemma is also sufficient, see Remark 6.16.

6.4. Numerical experiments

In this section we will numerically verify Theorem 6.11, and study its applicability to materials with losses.

Example 6.22 (Anisotropic Lorentz material. Numerical verification of Thm. 6.11)**.** We start with the system (2.4), where we choose the parameters as follows, see Remark 3.6,

$$\tilde{a}(\omega) = \tilde{\varepsilon}_2(\omega)^{-1} = \frac{\omega^2}{\omega^2 - \omega_p^2}, \qquad \tilde{b}(\omega) = \tilde{\varepsilon}_1(\omega)^{-1} = \frac{\omega^2 - \frac{\omega_p^2}{4}}{\omega^2 - 4\omega_p^2}, \qquad \tilde{c}(\omega) = \tilde{\mu}(\omega) = 1, \qquad \omega_p = 5. \tag{6.20}$$

By Theorem 6.11, stable PML in x is generated by $\psi(s)$, s.t. $\psi(s)^{-1}$ is passive and $\tilde{\psi}(\omega)\tilde{a}(\omega) \geq 0$ for ω from

$$\Omega_p^{\tilde{a}\tilde{b}<0} \cup \Omega^{\tilde{a}\tilde{c}>0} = (-\infty,\ -\omega_p) \cup \left(-\frac{\omega_p}{2},\ 0\right) \cup \left(0, \frac{\omega_p}{2}\right) \cup (\omega_p,\ +\infty).$$

Let us consider the following possible choices of $\psi(s)$:

$$\psi_1(s) = 1, \qquad \tilde{\psi}_2(\omega) = \frac{\omega^2}{\omega^2 - \frac{\omega_p^2}{4}}, \qquad \tilde{\psi}_3(\omega) = \frac{\omega^2\left(\omega^2 - \frac{\omega_p^2}{4}\right)}{\left(\omega^2 - \frac{\omega_p^2}{8}\right)\left(\omega^2 - \omega_p^2\right)}, \qquad \psi_4(s) = \varepsilon_2(s)^{-1}. \tag{6.21}$$

Notice that $\operatorname{Re}\left(\bar{s}\psi_j(s)\right) > 0,\ j = 1, \ldots, 4$, in $\mathbb{C}_+$ thanks to $(\overline{M}2)$ of Lemma 5.7. We expect the PMLs to be unstable for $\psi(s) = \psi_1(s)$ (classical PMLs) and stable otherwise (since $\tilde{\psi}_j\tilde{a} \geq 0$, $j = 2, 3, 4$, in $\Omega_p^{\tilde{a}\tilde{b}<0} \cup \Omega^{\tilde{a}\tilde{c}>0}$). In particular, ψ_4 corresponds to the PML of Theorem 4.8. The parameters of the numerical experiment are given in Table 4. As before, in y-direction we use zero Dirichlet boundary conditions. The results of these experiments are shown in Figure 6. The new PMLs are stable, unlike the classical PMLs. The rate of the decrease of the solution norm for the PMLs with $\psi(s) = \psi_2(s)$, $\psi(s) = \psi_3(s)$, $\psi(s) = \psi_4(s)$ is almost indistinguishable on the scale used in the figure.

TABLE 4. Parameters for the experiment with dielectric permittivity and magnetic permeability (6.20). By $H(x)$ we denote the Heaviside function. See Section 4.1.2 for notation.

L_x	L_y	L_x^σ	L_y^σ	$\sigma_x(x)$	$\sigma_y(y)$	Δx	Δt	$f(t,x,y)$
8	12	2	0	$20x^2$	0	0.025	0.0125	$(t-2)\mathrm{e}^{-10(t-2)^2-1000(x-3.8)^2-2z^2}H(3.98-x)$

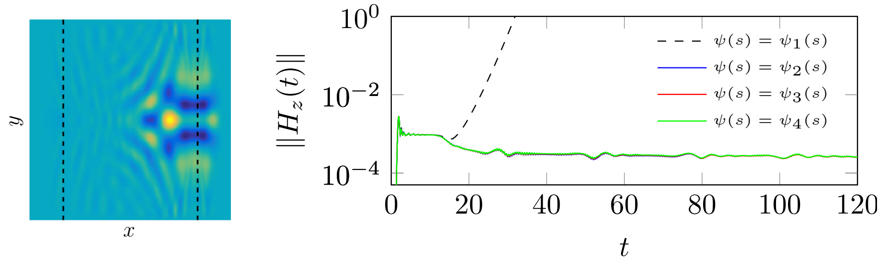


FIGURE 6. Left: H_z of Example 6.22 at $t = 60$ computed with $\psi(s) = \psi_2(s)$, see (6.21). The boundary between the physical domain and the PML is marked in black. Right: The dependence of $\|H_z\|_{L^2}$, computed with different $\psi(s)$, on time.

Example 6.23 (Anisotropic Lorentz materials with loss)**.** Another interesting question is whether a perfectly matched layer which we have devised for the Lorentz materials (2.8) in this section can be applied to Lorentz materials *with losses*, more precisely, when the dielectric permittivity/permeability are of the form

$$r(s) = 1 + \sum_{\ell=0}^{n} \frac{r_\ell}{s^2 + 2\nu_\ell s + \omega_\ell^2}, \qquad r_\ell > 0,\ \omega_\ell \geq 0,\ \nu_\ell \geq 0.$$

It is easy to check that $\operatorname{Re}(sr(s)) > 0$, $s \in \mathbb{C}_+$; this shows that all the roots of $r(s)$ lie in $\mathbb{C}\backslash\mathbb{C}_+$. Moreover, all poles of $r(s)$ also lie in $\mathbb{C}\backslash\mathbb{C}_+$. To see this, notice that the poles of $r(s)$ are given by $s_\ell^\pm = -\nu_\ell \pm \sqrt{\nu_\ell^2 - \omega_\ell^2}$, $\ell = 0, \ldots, n$. If $\nu_\ell \geq \omega_\ell$, then $s_\ell^\pm \leq 0$; otherwise $\operatorname{Re} s_\ell^\pm = -\nu_\ell$.

We consider the problem (2.4) with the following parameters:

$$\varepsilon_1(s) = 1, \qquad \varepsilon_2(s) = 1 + \frac{\varepsilon_{21}}{s^2 + 2\nu s + \omega_p^2}, \qquad \mu(s) = 1, \qquad \varepsilon_{21} = 12, \qquad \omega_p = 2. \tag{6.22}$$

Although these parameters do not satisfy Assumption 5.1, we would like to verify whether it's possible to use for it the PML in x-direction, which was constructed for the analogical non-dissipative case, namely $\psi(s) = \varepsilon_2(s)^{-1}$ from (6.22) with $\nu = 0$. The necessary stability conditions of the PMLs in [9] do not cover this case, and hence we would like to check it numerically. We test the following choices of $\psi(s) = \psi_j(s)$:

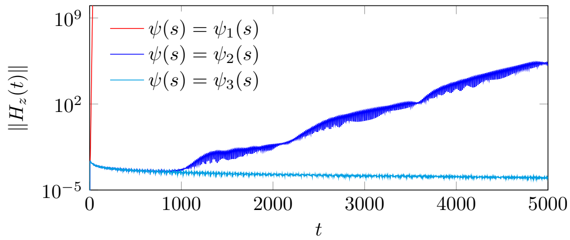
$$\psi_1(s) = 1, \qquad \psi_2(s) = \left(1 + \frac{\varepsilon_{21}}{s^2 + \omega_p^2}\right)^{-1}, \qquad \psi_3(s) = \left(1 + \frac{\varepsilon_{21}}{s^2 + 2\nu s + \omega_p^2}\right)^{-1}. \tag{6.23}$$

The choice $\psi(s) = \psi_2(s)$ corresponds to a stable PML in the case $\nu = 0$, whereas the choice $\psi(s) = \psi_3(s)$ is a stable PML of Theorem 4.8. The parameters of the experiment are presented in Table 5.

The results of the experiment are shown in Figure 7. The choice $\psi(s) = \psi_3(s)$ results in a stable solution, thus confirming numerically the result of Theorem 4.8 (even though here $\sigma(x) \neq \text{const}$). The choice $\psi(s) = \psi_2(s)$, which one would hope to be stable, results in instabilities, which seem to develop at long time. However, the larger the absorption ν is, the faster the instability occurs. The classical PMLs develop instabilities almost immediately. For other examples involving the application of the PML of Theorem 6.11 to non-local materials we refer an interested reader to [11].

TABLE 5. Parameters for the experiment of Example 6.23 with the dielectric permittivity and magnetic permeability (6.22). By $H(x)$ we denote the Heaviside function. The notation can be found in the beginning of Section 4.1.2.

L_x	L_y	L_x^σ	L_y^σ	$\sigma_x(x)$	$\sigma_y(y)$	Δx	Δt	$f(t,x,y)$
8	12	2	0	$50x^2$	0	0.05	0.025	$\mathrm{e}^{-1000(x-3.8)^2-300y^2-300z^2} H(3.98-x)\mathrm{e}^{-15(t-1.6)^2}$



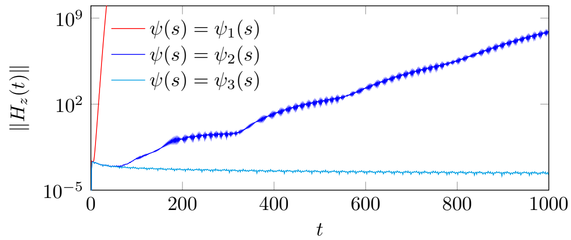


FIGURE 7. The dependence of the L^2-norm of the solution H_z (computed inside the domain and the PML) to the problem of Example 6.23 on time, computed with different PMLs. In the left plot $\nu = 0.01$ in (6.22), and in the right plot $\nu = 0.1$.

7. Conclusions and open questions

7.1. Open questions: Stable PMLs in 3D

It is natural to ask whether the suggested PML technique and the analysis can be extended to a more broad class of problems, in particular, to the wave propagation in 3D. Naively, one would expect that in the simplest case of the Maxwell equations with a diagonal tensor of the dielectric permittivity and $\mu(s) = 1$, it is sufficient to study the well-posedness of the PML (4.6) applied to the acoustic dispersive 3D wave equation (written in the Laplace domain)

$$a(s)\partial_x^2 u + b(s)\partial_y^2 u + c(s)\partial_z^2 u - s^2 d(s)u = 0. \tag{7.1}$$

This is, however, not the case. One should rather consider the vector equation

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} + s^2\underline{\underline{\varepsilon}}(s)\mathbf{E} = 0,$$

with $\underline{\underline{\varepsilon}}$ being a diagonal matrix. This leads to several difficulties, both analytic and conceptual. First of all, when $\underline{\underline{\varepsilon}} = \mathrm{Id}$, it is well-known that the standard PMLs are stable. However, proving the coercivity of the corresponding sesquilinear form with the PML in $\mathbb{C}_+$, as we have done in 2D, is no longer trivial, because the corresponding spatial operator looses its self-adjoint nature. On the other hand, it is still possible to prove very special inf-sup conditions. This is a subject of future research.

Second, it is not always possible to stabilize the PML for 3D dispersive problems using the change of variables as in this article. In particular, in [8], one considers the 3D wave propagation in cold strongly magnetized plasmas. There $\underline{\underline{\varepsilon}}(s) = \mathrm{diag}(1,\ 1,\ 1+\frac{\omega_p^2}{s^2})$, with $\omega_p \in \mathbb{R}$ being a plasma frequency, and $\mu(s) = 1$. However, even in this very simple case the system cannot be reduced to the form (7.1). Moreover, one can demonstrate that the dispersion relation is not of the form

$$a(s)k_x^2 + b(s)k_y^2 + c(s)k_z^2 + s^2 d(s) = 0, \tag{7.2}$$

but is a product of two such terms. The first term corresponds to isotropic non-dispersive waves (*i.e.* $a = b = c = d = 1$ in (7.2)), which are absorbed by the classical PMLs (*i.e.* the choice $\psi(s) = 1$ would be stable). Whereas the second term is of the form (7.2) with $a(s) = b(s) = (1 + \frac{\omega_p^2}{s^2})^{-1}$ and $c(s) = d(s) = 1$, for which the choice $\psi(s) = a(s) = (1+\frac{\omega_p^2}{s^2})^{-1}$ is stable. Nevertheless, since both kinds of waves are present simultaneously for the same frequency, none of these choices is suitable. It is possible to show that there exists no $\psi(s)$ satisfying assumptions of Section 6 that would lead to a stable 3D PML system in this case. The construction of stable PMLs for cold plasmas requires a more elaborate treatment, and constitutes the subject of [8]. However, the method of this article serves as an important component of the technique suggested in [8].

To summarize, the new PML change of variables (4.6) does not provide a general method for the stabilization of the PMLs for a 3D anisotropic dispersive Maxwell system, however, serves as a component for stabilizing the PMLs for such systems.

7.2. Conclusions

In this work we have shown how to construct stable perfectly matched layers for a class of anisotropic dispersive models described in the Laplace domain by the wave equation with frequency-dependent coefficients

$$a(s)\partial_x^2 u + b(s)\partial_y^2 u - s^2 c(s) = 0,\ s \in \mathbb{C}_+,$$

where the coefficients $a(s),\ b(s),\ c(s)$ are analytic in $\mathbb{C}_+$, and satisfy $\operatorname{Re}(\bar{s}a(s)) > 0,\ \operatorname{Re}(\bar{s}b(s)) > 0$, $\operatorname{Re}(sc(s)) > 0$, for $s \in \mathbb{C}_+$. Following [9], in order to construct the PML in one direction (assuming that the PML layer is located in a half-plane $x > 0$), we suggest to use the following change of variables:

$$x \to x + \frac{\psi(s)}{s}\int\limits_0^x \sigma(x')\mathrm{d}x',\ x > 0,$$

where $\psi(s)$ satisfies $\operatorname{Re}(\bar{s}\psi(s)) > 0,\ s \in \mathbb{C}_+$. This article provides choices of $\psi(s)$ that would stabilize the PMLs. Based on the Laplace domain analysis in the free space for $\sigma = \text{const}$, we claim that the following choices of $\psi(s)$ would result in stable PMLs in the x-direction:

(1) for arbitrary passive models: $\psi(s) = a(s)$;
(2) for isotropic passive models (where $a(s) = b(s)$): $\psi(s) = \left(\alpha c(s) + (1-\alpha)a^{-1}(s)\right)^{-1}$, $0 \le \alpha \le 1$;
(3) in the case when $a(s)^{-1},\ b(s)^{-1},\ c(s)$ correspond to generalized Lorentz models, one can choose $\psi(s)$, s.t. $\psi(s)^{-1}$ is Lorentz and which would satisfy the following condition for all $\omega \in \mathbb{R}$:

$$\tilde{a}(\omega)\tilde{b}(\omega) < 0 \text{ or } \tilde{a}(\omega)\tilde{c}(\omega) > 0 \implies \tilde{a}(\omega)\tilde{\psi}(\omega) \ge 0.$$

Moreover, if $b(s) = c(s)^{-1}$ (2D uniaxial cold plasma model (2.7)), the only possible choice among passive $\psi(s)^{-1}$ satisfying Assumption 5.1 is $\psi(s) = a(s)$.

We confirm the obtained results with the help of numerical experiments, including examples with dissipation. Indeed, there are many open questions remaining which are the subject of the future work. In particular, they include the construction of stable PMLs for 3D anisotropic Maxwell's equations, even in the simplest case of a diagonal tensor of dielectric permittivity (where, as we discussed earlier, the instabilities cannot be overcome by the use of a special frequency-dependent change of variables). Another important question is a construction of stable PMLs for nonpassive materials, which is also a subject of the future research.

Appendix A. Proof of Lemma 3.4

The upper bound in this lemma can be obtained using ([41], Thm. 8', p. 18), and the lower bound can be viewed as a minor improvement of ([46], Thm. 3) (we consider a more general class of functions, however, the main idea of the proof is basically the same).

Let us first obtain the upper bound, which we derive using the methods of [20]. This will serve as a basis to compute the lower bound for $\operatorname{Im} f_c(z)$ as well. The main idea is to construct a function on the unit circle to which the Schwarz's lemma ([20], Chap. 4, Thm. 1) can be easily applied. Let us set $h(s) = sc(s)$. Taking $s_0 \in \mathbb{C}_+$, we define the Möbius transformation ([20], p. 43,44)

$$r := \frac{s - s_0}{s + \overline{s_0}}, \qquad |r| < 1,\ s \in \mathbb{C}_+,$$
$$g\left(\frac{s - s_0}{s + \overline{s_0}}\right) := \frac{h(s) - h(s_0)}{h(s) + \overline{h(s_0)}}.$$

The function $g(r)$ is analytic inside the unit circle $|r|<1$; $g(0)=0$ and $|g(r)|<1$, due to passivity of c. Then, thanks to the Schwarz's lemma, $|g(r)|\leq |r|$ inside the unit circle. Hence,

$$\left|\frac{h(s)-h(s_0)}{h(s)+\overline{h(s_0)}}\right| \leq \left|\frac{s-s_0}{s+\overline{s_0}}\right|. \tag{A.1}$$

(1) The proof of the upper bound on $|h(s)|$. From (A.1) it follows

$$|h(s)h(s_0)^{-1}-1| \leq \left|\frac{s-s_0}{s+\overline{s_0}}\right| \left|h(s)h(s_0)^{-1}-\overline{h(s_0)}h(s_0)^{-1}\right| \leq \left|\frac{s-s_0}{s+\overline{s_0}}\right| \left(\left|h(s)h(s_0)^{-1}\right|+1\right).$$

Using $|h(s)h(s_0)^{-1}-1|\geq |h(s)h(s_0)|^{-1}-1$, and recalling that $r=\left|\frac{s-s_0}{s+\overline{s_0}}\right|$, and $|r|<1$, we get the following bound

$$|h(s)| \leq |h(s_0)|\,(1+r)\,(1-r)^{-1} = |h(s_0)|\,(1+r)^2\left(1-r^2\right)^{-1} \leq 4|h(s_0)|\left(1-r^2\right)^{-1}. \tag{A.2}$$

Notice that the function $(1-r^2)^{-1}$ grows in r; hence, let us obtain the upper bound on r in terms of $\operatorname{Re} s$ and $|s|$:

$$\left|\frac{s-s_0}{s+\overline{s_0}}\right|^2 = \frac{|s|^2-2\operatorname{Re}(s\overline{s_0})+|s_0|^2}{|s|^2+2\operatorname{Re}(ss_0)+|s_0|^2} = 1-\frac{2\operatorname{Re}(ss_0)+2\operatorname{Re}(s\overline{s_0})}{|s|^2+2\operatorname{Re}(ss_0)+|s_0|^2} = 1-\frac{4\operatorname{Re} s\operatorname{Re} s_0}{|s|^2+2\operatorname{Re}(ss_0)+|s_0|^2}. \tag{A.3}$$

The lower bound for

$$\frac{\operatorname{Re} s\operatorname{Re} s_0}{|s+\overline{s_0}|^2} \geq \frac{1}{2}\frac{\operatorname{Re} s\operatorname{Re} s_0}{|s|^2+|s_0|^2}. \tag{A.4}$$

Therefore, using (A.2), we obtain

$$|h(s)| \leq 2|h(s_0)|(\operatorname{Re} s\operatorname{Re} s_0)^{-1}\left(|s|^2+|s_0|^2\right) \leq C'\left(|s|^2+|s_0|^2\right)(\operatorname{Re} s)^{-1},$$

for some $C'>0$. From this we immediately obtain, for some $C>0$,

$$|sc(s)| \leq C'|s|^2\left(1+|s_0|^2|s|^{-2}\right)(\operatorname{Re} s)^{-1} \leq C|s|^2(\operatorname{Re} s)^{-1}\max\left(1,(\operatorname{Re} s)^{-2}\right),\ s\in\mathbb{C}_+.$$

The same (up to a constant) bound can be obtained with the help of ([41], Thm. 8', p. 18).

(2) The proof of the lower bound on $\operatorname{Re} h(s)$, $s\in\mathbb{C}_+$. We rewrite (A.1), taking the square of both sides:

$$\frac{|h(s)|^2-2\operatorname{Re}(\overline{h(s_0)}h(s))+|h(s_0)|^2}{|h(s)|^2+2\operatorname{Re}(h(s_0)h(s))+|h(s_0)|^2} = 1-\frac{4\operatorname{Re} h(s)\operatorname{Re} h(s_0)}{|h(s)+\overline{h(s_0)}|^2} \leq 1-\frac{4\operatorname{Re} s\operatorname{Re} s_0}{|s|^2+2\operatorname{Re}(s_0 s)+|s_0|^2},$$

where the last inequality is obtained with the help of (A.3). This gives, using (A.4),

$$\frac{\operatorname{Re} h(s)\operatorname{Re} h(s_0)}{|h(s)+\overline{h(s_0)}|^2} \geq \frac{1}{2}\frac{\operatorname{Re} s\operatorname{Re} s_0}{|s|^2+|s_0|^2}.$$

Next, notice that $|h(s)+\overline{h(s_0)}|^2 \geq \left(\operatorname{Re} h(s)+\operatorname{Re}\overline{h(s_0)}\right)^2 \geq (\operatorname{Re} h(s_0))^2$. Using $|s|^2+|s_0|^2 \leq C_0\max(|s|^2,1)$, $C_0>0$, we obtain the desired bound:

$$\operatorname{Re} h(s) \geq C\min(|s|^{-2},1)\operatorname{Re} s \geq C|s|^{-2}\operatorname{Re} s\min(1,(\operatorname{Re} s)^2).$$

Appendix B. Proof of Corollary 4.6

First, let us prove the result for arbitrary materials. Notice that for any $a(s)$, $\psi(s)$ satisfying the conditions of the corollary, the bound (4.9) holds with the following parameters, see Corollary 3.5,

$$\left|\frac{\psi(s)}{a(s)}\right| = \left|\frac{\bar{s}\psi(s)}{\bar{s}a(s)}\right| \leq |s||\psi(s)| \left(\operatorname{Re}\left(\bar{s}a(s)\right)\right)^{-1} \leq C|s|^4 \left(\operatorname{Re} s\right)^{-2} \max(1, (\operatorname{Re} s)^{-4}),\ C>0.$$

For the Lorentz materials, let us bound $a(s)^{-1}$ which has an expansion (2.8), *i.e.* $a(s)^{-1} = 1 + \sum\limits_{\ell=0}^{n_a} \frac{a_\ell}{s^2+\omega_\ell^2}$, with $a_\ell > 0$ and $\omega_\ell \in \mathbb{R}$. Notice that for all $\omega \in \mathbb{R}$ and $s \in \mathbb{C}_+$, it holds that

$$\left|\frac{1}{s^2+\omega^2}\right| = \frac{1}{|s-i\omega||s+i\omega|} \leq \frac{1}{|s|\operatorname{Re} s}. \tag{B.1}$$

Then, for some $C, \tilde{C} > 0$, the modulus $|a(s)^{-1}| \leq 1 + \frac{C}{|s|\operatorname{Re} s} \leq \tilde{C} \max\left(1, (\operatorname{Re} s)^{-2}\right)$. Thanks to the inequality $\operatorname{Re}(s\psi(s)^{-1}) > \operatorname{Re} s$, *cf.* Theorem 5.4, we obtain

$$\left|\frac{\psi(s)}{a(s)}\right| \leq \left|\frac{sa(s)^{-1}}{s\psi(s)^{-1}}\right| \leq \tilde{C}|s| \max\left(1, (\operatorname{Re} s)^{-2}\right) \left(\operatorname{Re} s\psi(s)^{-1}\right)^{-1} \leq \tilde{C}|s| \max\left(1, (\operatorname{Re} s)^{-3}\right).$$

Appendix C. Proof of equivalence in Lemma 6.15

Lemma C.1. *Let $a(s)$, $\psi(s)$, $c(s)$ satisfy Assumption 6.10. Assume (NSC2)-(a) holds true, i.e. $c(s)a(s)^{-1}\psi(s)$ is passive. Then (NSC1)-(a) holds, i.e. for all $\omega \in \Omega_p^{\tilde{a}\tilde{c}>0}$ the product $\tilde{\psi}(\omega)\tilde{a}(\omega) \geq 0$.*

Proof. First of all, in $\omega = 0$, due to Theorem 5.6, it holds that $\tilde{\psi}(0)\tilde{a}(0) \geq 0$. Due to the equivalence of the positivity of the derivative $(\omega\tilde{f}(\omega))' > 0$, $\omega \in \mathbb{R}$, and the property $\operatorname{Re}(sf(s)) > 0$ in $\mathbb{C}_+$ (see [11], or Lem. 5.5 for a part of the equivalence result), the following derivative is strictly positive for $\omega \in \mathbb{R}$ with the exception of poles of $\tilde{c}\tilde{\psi}\tilde{a}^{-1}$:

$$\left(\omega\tilde{c}(\omega)\tilde{\psi}(\omega)\tilde{a}(\omega)^{-1}\right)' = (\omega\tilde{c}(\omega))'\tilde{\psi}(\omega)\tilde{a}(\omega)^{-1} + \omega\tilde{c}(\omega)\tilde{\psi}(\omega)'\tilde{a}(\omega)^{-1} + \omega\tilde{c}(\omega)\tilde{\psi}(\omega)(\tilde{a}(\omega)^{-1})' > 0. \tag{C.1}$$

Take ω s.t. $\tilde{c}(\omega)\tilde{a}(\omega) > 0$ and assume that $\tilde{a}(\omega)\tilde{\psi}(\omega) < 0$. Using Lemma 5.5 about the signs of the derivatives of the passive functions, the equation (3.3), which shows the passivity of $a(s)^{-1}$, as well as the fact that $(\tilde{a}(\omega)^{-1})' > 0$, $(\tilde{\psi}(\omega))' \leq 0$ outside of the poles of these functions (this can be verified by a direct computation, see Theorems 5.4, 5.6), we obtain that the above is strictly negative, and thus the contradiction. □

Remark C.2. One could try proving Lemma 6.15 by examining the sign of the derivative (C.1); however, when $\tilde{c}(\omega)\tilde{a}(\omega) < 0$, and $\tilde{\psi}(\omega)\tilde{a}(\omega)$ is of any sign, one does not see immediately that the expression (C.1) is strictly positive.

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