

## STABILITY ANALYSIS AND ERROR ESTIMATES OF LOCAL DISCONTINUOUS GALERKIN METHODS WITH IMPLICIT-EXPLICIT TIME-MARCHING FOR THE TIME-DEPENDENT FOURTH ORDER PDES

HAIJIN WANG<sup>1</sup>, QIANG ZHANG<sup>2</sup> AND CHI-WANG SHU<sup>3</sup>

**Abstract.** The main purpose of this paper is to give stability analysis and error estimates of the local discontinuous Galerkin (LDG) methods coupled with three specific implicit-explicit (IMEX) Runge–Kutta time discretization methods up to third order accuracy, for solving one-dimensional time-dependent linear fourth order partial differential equations. In the time discretization, all the lower order derivative terms are treated explicitly and the fourth order derivative term is treated implicitly. By the aid of energy analysis, we show that the IMEX-LDG schemes are unconditionally energy stable, in the sense that the time step  $\tau$  is only required to be upper-bounded by a constant which is independent of the mesh size  $h$ . The optimal error estimate is also derived by the aid of the elliptic projection and the adjoint argument. Numerical experiments are given to verify that the corresponding IMEX-LDG schemes can achieve optimal error accuracy.

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### 1. INTRODUCTION

We will carry out a fully-discrete numerical analysis for one dimensional time-dependent fourth order problems, which have broad applications, such as phase separation in binary mixtures and interface dynamics in multi-phase fluids. In order to alleviate the stringent time step restriction of explicit time discretization for high order partial differential equations (PDE), we consider a class of implicit-explicit (IMEX) time discretization [3] which treats all the lower order derivative terms explicitly and the highest order derivative term implicitly. For the spatial discretization we adopt the standard local discontinuous Galerkin (LDG) method.

The LDG method was introduced by Cockburn and Shu for solving convection-diffusion problems in [8], motivated by the work of Bassi and Rebay [4] for solving compressible Navier-Stokes equations. This scheme shares some of the advantages of the Runge–Kutta discontinuous Galerkin (RKDG) schemes for solving hyperbolic conservation laws [9], such as high order accuracy, flexibility of  $h$ - $p$  adaptivity, flexibility on complex

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<sup>1</sup> College of Science, Nanjing University of Posts and Telecommunications, Nanjing 210023, Jiangsu Province, P.R. China. [hjwang@njupt.edu.cn](mailto:hjwang@njupt.edu.cn)

<sup>2</sup> Department of Mathematics, Nanjing University, Nanjing 210093, Jiangsu Province, P.R. China. [qzh@nju.edu.cn](mailto:qzh@nju.edu.cn)

<sup>3</sup> Division of Applied Mathematics, Brown University, Providence, RI 02912, USA. [shu@dam.brown.edu](mailto:shu@dam.brown.edu)

geometry, and so on. Besides, it is locally solvable, that is, the auxiliary variables approximating the gradient of the solution can be locally eliminated [6, 21].

LDG methods have been successfully designed for solving high order problems, such as for the KdV type equations [23, 24], the nonlinear dispersive problems [13], time-dependent fourth order problems [10, 24], the Cahn-Hilliard equation [19], the Kuramoto–Sivashinsky type equation [22], the fifth order dispersive equations [20, 24], *etc.* For more knowledge about LDG methods for solving high order problems, please refer to the review article [21] and the reference therein. These methods satisfy the cell entropy inequality [21] and consequently are  $L^2$  stable in the semi-discrete framework. However, as far as the authors know, there is little work on the fully discrete analysis of the LDG scheme for solving high order problems. On the other hand, the efficient time discretization is an important issue in practical computations. It is well known that explicit time discretization will suffer from stringent time step restriction for time-dependent problems with high order derivatives, namely  $\tau = \mathcal{O}(h^p)$ , where  $\tau$  and  $h$  are time step and mesh size, respectively, and  $p$  is the order of the PDE. There are several efficient time discretization methods which can be used to solve such problems, for example, the semi-implicit spectral deferred correction (SDC) method, the additive Runge–Kutta (ARK) method and the exponential time differencing (ETD) method, and so on; see [18] for more details.

In [15–17], we studied different types of IMEX time discretization coupled with LDG spatial discretization for solving one and multi-dimensional convection-diffusion problems with linear or nonlinear convection terms. We showed that the corresponding schemes are unconditional stable in the sense that the time step  $\tau$  is only required to be upper bounded by a positive constant which is independent of the mesh size  $h$ , and only depends on the coefficients of the convection and diffusion terms. The IMEX-LDG schemes were also applied and analyzed for the drift-diffusion model of one dimensional semiconductor devices in [14]. In [11], the authors applied a kind of IMEX-LDG methods for solving highly nonlinear PDEs. These results suggest that the IMEX-LDG methods are efficient for the convection-diffusion type problems.

In this paper, we will study three specific IMEX Runge–Kutta (RK) schemes from first to the third order given in [3] coupling with LDG spatial discretization for solving the time-dependent fourth order problems. When we use IMEX time discretization, we treat the linear fourth order derivative term implicitly and the remaining potential nonlinear lower order derivative terms explicitly. By the aid of energy analysis, we show that the corresponding IMEX-LDG schemes are all energy stable, provided that the time step  $\tau$  is upper bounded by a positive constant which is not depending on the mesh size  $h$ . The optimal error estimates will be obtained by adopting the technique used in [10], *i.e.*, the so called elliptic projection and the adjoint argument.

This work is a continuation of our previous work [15]. Even though similar conclusion as that for convection-diffusion problems can be obtained, the analysis is not a trivial generalization. For the fourth order problems, the energy norm only contains the  $L^2$  norm of the solution and the  $L^2$  norm of the second order derivative, which brings some difficulties in estimating the first order and third order derivative terms, *i.e.*, the convection and the dispersion terms. To overcome these difficulties, we build up and resort to the discrete version of the Sobolev interpolation relationship, and the discrete version of the Poincaré inequality, which links the numerical solution of the first order derivative and the numerical solution of the second order derivative. With the help of these techniques and the technique adopted in [15], we obtain stronger stability results in some special cases. Furthermore, it is worth mentioning that, the explicit discretization of the dispersion term makes the stability result containing the information of  $\|p^0\|$  on the right hand side, where  $p^0$  is the initial numerical approximation of the second order derivative. Our analysis for the third order Runge–Kutta IMEX schemes considered in [15] does not seem to work, hence we consider a different third order scheme [3] which has four stages for both the explicit and implicit parts.

The paper is organized as follows. In Section 2 we present the semi-discrete LDG scheme for the model problem and give some preliminary results. Section 3 is devoted to the presentation of several IMEX Runge–Kutta schemes, and to the proof for the linear stability of the corresponding IMEX-LDG schemes. We will take the first order scheme as an example to show the optimal error estimate in Section 4. Several numerical results are presented in Section 5 to verify the stability and accuracy of the schemes. Finally, we give concluding remarks and some technical proof in Section 6 and in the appendix, respectively.

## 2. THE SEMI-DISCRETE LDG METHOD AND ITS STABILITY ANALYSIS

### 2.1. The semi-discrete LDG scheme

In this subsection we present the definition of semi-discrete LDG schemes for the linear time-dependent fourth order equation

$$U_t + c_1 U_x + c_2 U_{xx} + c_3 U_{xxx} + c_4 U_{xxxx} = 0, \quad (x, t) \in Q_T = (a, b) \times (0, T], \quad (2.1a)$$

$$U(x, 0) = U_0(x), \quad x \in \Omega = (a, b), \quad (2.1b)$$

with periodic boundary condition, where  $c_1, c_2, c_3$  and  $c_4 > 0$  are arbitrary constants. Without loss of generality, we assume  $c_1 > 0$  and  $c_3 > 0$  in this paper, but we do not require the sign of  $c_2$  to be either positive or negative. The initial solution  $U_0(x)$  is assumed to be in  $L^2(\Omega)$ .

Let  $Q = U_x$ ,  $P = Q_x$  and  $R = P_x$ , then (2.1) is equivalent to the following first-order differential system

$$U_t + c_1 U_x + c_2 Q_x + c_3 P_x + c_4 R_x = 0, \quad (2.2a)$$

$$R - P_x = 0, \quad (2.2b)$$

$$P - Q_x = 0, \quad (2.2c)$$

$$Q - U_x = 0, \quad (2.2d)$$

with the same initial condition (2.1b) and boundary condition.

Let  $\mathcal{T}_h = \{I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})\}_{j=1}^N$  be the partition of  $\Omega$ , where  $x_{\frac{1}{2}} = a$  and  $x_{N+\frac{1}{2}} = b$  are the two boundary endpoints. Denote the cell length as  $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$  for  $j = 1, \dots, N$ , and define  $h = \max_j h_j$ . We assume  $\mathcal{T}_h$  is quasi-uniform in this paper, that is, there exists a positive constant  $\rho$  such that for all  $j$  there holds  $h_j/h \geq \rho$ , as  $h$  goes to zero.

Associated with this mesh, we define the discontinuous finite element space

$$V_h = \{v \in L^2(\Omega) : v|_{I_j} \in \mathcal{P}_k(I_j), \forall j = 1, \dots, N\}, \quad (2.3)$$

where  $\mathcal{P}_k(I_j)$  denotes the space of polynomials in  $I_j$  of degree at most  $k \geq 1$ . Note that the functions in this space are allowed to have discontinuities across element interfaces. At each element interface point, for any piecewise function  $p$ , there are two traces along the right-hand and left-hand, denoted by  $p^+$  and  $p^-$ , respectively. As usual, the jump is denoted by  $\llbracket p \rrbracket = p^+ - p^-$ .

The semi-discrete LDG scheme is defined as follows: for any  $t > 0$ , find the numerical solution  $\mathbf{w}(t) := (u(t), q(t), p(t), r(t)) \in \mathbf{V}_h = V_h \times V_h \times V_h \times V_h$  (where the argument  $x$  is omitted), such that the variational forms (in the below, we drop the argument  $t$  if there is no confusion)

$$(u_t, v)_j = c_1 \mathcal{H}_j^-(u, v) + c_2 \mathcal{H}_j^+(q, v) + c_3 \mathcal{H}_j^+(p, v) + c_4 \mathcal{H}_j^+(r, v), \quad (2.4a)$$

$$(r, \rho)_j = -\mathcal{H}_j^-(p, \rho), \quad (2.4b)$$

$$(p, \phi)_j = -\mathcal{H}_j^+(q, \phi), \quad (2.4c)$$

$$(q, \psi)_j = -\mathcal{H}_j^-(u, \psi), \quad (2.4d)$$

hold in each cell  $I_j$ ,  $j = 1, 2, \dots, N$ , for any test functions  $\mathbf{z} = (v, \rho, \phi, \psi) \in \mathbf{V}_h$ . Here  $(\cdot, \cdot)_j$  is the usual inner product in  $L^2(I_j)$ , and

$$\mathcal{H}_j^\pm(v, w) = (v, w_x)_j - v_{j+\frac{1}{2}}^\pm w_{j+\frac{1}{2}}^- + v_{j-\frac{1}{2}}^\pm w_{j-\frac{1}{2}}^+. \quad (2.5)$$

Here we would like to point out that, we take periodic boundary conditions for all the four variables, *i.e.*, we let  $\varsigma_{\frac{1}{2}}^- = \varsigma_{N+\frac{1}{2}}^-$  and  $\varsigma_{N+\frac{1}{2}}^+ = \varsigma_{\frac{1}{2}}^+$  for  $\varsigma = u, q, p, r$ .

The initial condition  $u(x, 0)$  can be taken as any approximation of the given initial solution  $U_0(x)$ , for example, the standard  $L^2$  projection of  $U_0(x)$ , *i.e.*,  $\int_{I_j} (U_0(x) - u(x, 0))v(x)dx = 0$  for arbitrary  $v \in \mathcal{P}_k(I_j)$ , for  $j = 1, \dots, N$ . We have now defined the semi-discrete LDG scheme.

**Remark 2.1.** In (2.4), we adopt the upwind numerical flux ( $u^-$  if  $c_1 > 0$  and  $u^+$  if  $c_1 < 0$ ) for the first order term and the downwind numerical flux ( $p^+$  if  $c_3 > 0$  and  $p^-$  if  $c_3 < 0$ ) for the third order term, this is a special choice of monotone numerical fluxes considered in [22]. The alternating numerical flux [24] is used for the discretization of the fourth order term. To illustrate this clearly, we denote  $\hat{r}$  as the numerical flux for  $r$  in (2.4a), and  $\hat{p}, \hat{q}, \hat{u}$  are numerical fluxes for  $p, q, u$  in (2.4b), (2.4c), (2.4d), respectively. The choice of these numerical fluxes depends on the sign of  $c_3$ . Under our assumption  $c_3 > 0$ , we take  $\hat{r} = r^+, \hat{p} = p^-, \hat{q} = q^+, \hat{u} = u^-$ . If  $c_3 < 0$ , the signs should be opposite. In addition, the numerical flux for the second order term does not depend on the sign of  $c_2$ , but for simplicity we take it the same as the one taken in (2.4c).

Denote by  $(v, w) = \sum_{j=1}^N (v, w)_j$  the inner product in  $L^2(\Omega)$ , and summing up the variational formulations (2.4) over  $j = 1, 2, \dots, N$ , we can write the above semi-discrete LDG scheme in the global form: for any  $t > 0$ , find the numerical solution  $\mathbf{w} = (u, q, p, r) \in \mathbf{V}_h$  such that the variation equations

$$(u_t, v) = \mathcal{H}(\boldsymbol{\varphi}; v) + \mathcal{L}(r, v), \tag{2.6a}$$

$$(r, \rho) = -\mathcal{H}^-(p, \rho), \tag{2.6b}$$

$$(p, \phi) = -\mathcal{H}^+(q, \phi), \tag{2.6c}$$

$$(q, \psi) = -\mathcal{H}^-(u, \psi), \tag{2.6d}$$

hold for any  $\mathbf{z} = (v, \rho, \phi, \psi) \in \mathbf{V}_h$ . Here  $\mathcal{H}^\pm = \sum_{j=1}^N \mathcal{H}_j^\pm$ ,  $\boldsymbol{\varphi} = (u, q, p)$  and

$$\mathcal{H}(\boldsymbol{\varphi}; v) = c_1 \mathcal{H}^-(u, v) + c_2 \mathcal{H}^+(q, v) + c_3 \mathcal{H}^+(p, v), \tag{2.6e}$$

$$\mathcal{L}(r, v) = c_4 \mathcal{H}^+(r, v), \tag{2.6f}$$

where  $\mathcal{H}(\cdot; \cdot)$  and  $\mathcal{L}(\cdot, \cdot)$  represent the discretization for lower order terms and the highest order term, respectively.

## 2.2. Preliminaries

In this subsection, we first present some notations and norms which will be used throughout this paper, and then we will present some properties of the LDG spatial discretizations.

### 2.2.1. Notations and norms

We use the standard norms and semi-norms in Sobolev spaces. For example, for any integer  $s \geq 0$ , we denote  $H^s(D)$  as the space equipped with the norm  $\|\cdot\|_{H^s(D)}$ , in which the function itself and the derivatives up to the  $s$ th order are all in  $L^2(D)$ . In particular,  $H^0(D) = L^2(D)$  and the associated  $L^2$ -norm is denoted as  $\|\cdot\|_D$  for the simplicity of notations. If  $D = \Omega$ , we omit the subscript  $\Omega$  for convenience.

We also would like to define the so called ‘‘jump semi-norm’’

$$[v]^2 = \sum_{j=1}^N [v]_{j-\frac{1}{2}}^2, \tag{2.7}$$

for arbitrary  $v \in V_h$ .

In addition, throughout this paper we use  $\mu$  which is independent of  $h$ , to denote the inverse constant. That is to say, for any function  $v \in V_h$ ,

$$\|v_x\| \leq \mu h^{-1} \|v\|, \quad \|v\|_{\partial\mathcal{T}_h} \leq \sqrt{\mu h^{-1}} \|v\|, \tag{2.8}$$

where  $\|v_x\| = \left[ \sum_{j=1}^N \|v_x\|_{I_j}^2 \right]^{1/2}$ , and  $\|v\|_{\partial\mathcal{T}_h} = \left[ \sum_{j=1}^N (v_{j-\frac{1}{2}}^+)^2 + (v_{j+\frac{1}{2}}^-)^2 \right]^{1/2}$  is the  $L^2$ -norm on the element interfaces.

### 2.2.2. The properties of the LDG spatial discretization

We will first present several properties of the operators  $\mathcal{H}^\pm$  defined in Section 2.1. Lemma 2.2 describes the skew symmetry properties of the operators, and Lemma 2.4 gives the boundedness of the operators. The proofs are trivial so we omit them to save space, for readers who are interested in the details, we refer to [25].

**Lemma 2.2.** *For any  $w, v \in V_h$ , there hold the equalities*

$$\mathcal{H}^\pm(v, v) = \pm \frac{1}{2} \|v\|^2, \quad (2.9)$$

$$\mathcal{H}^-(w, v) = -\mathcal{H}^+(v, w). \quad (2.10)$$

**Corollary 2.3.** *Suppose  $\mathbf{w} = (u, q, p, r) \in \mathbf{V}_h$  satisfy (2.6b)–(2.6d), then*

$$\mathcal{L}(r, u) = -c_4 \|p\|^2. \quad (2.11)$$

**Lemma 2.4.** *For any  $w, v \in V_h$ , there hold the following inequalities*

$$|\mathcal{H}^\pm(w, v)| \leq \left( \|w_x\| + \sqrt{\mu h^{-1}} \|w\| \right) \|v\|, \quad (2.12a)$$

$$|\mathcal{H}^\pm(w, v)| \leq \left( \|v_x\| + \sqrt{\mu h^{-1}} \|v\| \right) \|w\|. \quad (2.12b)$$

The next lemma establishes the important relationships between the primal variable and the auxiliary variables, which plays a key role in obtaining the stability of the IMEX-LDG scheme in the next section.

**Lemma 2.5.** *Suppose  $\mathbf{w} = (u, q, p, r) \in \mathbf{V}_h$  satisfy (2.6b)–(2.6d), then there exists a positive constant  $C_\mu$  independent of  $h$  but maybe depending on the inverse constant  $\mu$ , such that*

$$\|u_x\| + \sqrt{\mu h^{-1}} \|u\| \leq C_\mu \|q\|, \quad (2.13)$$

$$\|q_x\| + \sqrt{\mu h^{-1}} \|q\| \leq C_\mu \|p\|, \quad (2.14)$$

$$\|q\|^2 \leq \|u\| \|p\|. \quad (2.15)$$

*Proof.* We refer the readers to [15] for the proof of (2.13), and the proof for (2.14) is similar. Furthermore, by taking  $\psi = q$  in (2.6d) and then following from (2.10) and (2.6c) we get

$$\|q\|^2 = -\mathcal{H}^-(u, q) = \mathcal{H}^+(q, u) = -(p, u). \quad (2.16)$$

Hence (2.15) is obtained by the Cauchy–Schwarz inequality.  $\square$

**Remark 2.6.** (2.15) can be viewed as the discrete version of the Sobolev interpolation inequality [1]

$$|w|_{H^1(\Omega)} \leq C \|w\|_{L^2(\Omega)}^{1/2} |w|_{H^2(\Omega)}^{1/2}. \quad (2.17)$$

**Lemma 2.7.** *If  $v \in V_h$  satisfies  $\int_\Omega v dx = 0$  and the periodic boundary condition, then we have the following discrete version of Poincaré inequality*

$$\|v\| \leq C_p (\|v_x\| + \sqrt{\mu h^{-1}} \|v\|), \quad (2.18)$$

where  $C_p$  denotes the Poincaré constant which is independent of  $v$ .

*Proof.* Following [2], we define  $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$  by  $-\phi_{xx} = v$ . Then from [12], there exists a constant  $C_0$  independent of  $v$  such that  $\|\phi\|_{H^2} \leq C_0\|v\|$ . As a result, integrating by parts on each cell we get

$$\begin{aligned} \|v\|^2 &= \sum_{j=1}^N \int_{I_j} -v\phi_{xx} dx = \sum_{j=1}^N \left\{ \int_{I_j} v_x \phi_x dx - v_{j+\frac{1}{2}}^-(\phi_x)_{j+\frac{1}{2}} + v_{j-\frac{1}{2}}^+(\phi_x)_{j-\frac{1}{2}} \right\} \\ &= \sum_{j=1}^N \int_{I_j} v_x \phi_x dx + \sum_{j=2}^N \left[ [v]_{j-\frac{1}{2}}(\phi_x)_{j-\frac{1}{2}} - v_{N+\frac{1}{2}}^-(\phi_x)_{N+\frac{1}{2}} + v_{\frac{1}{2}}^+(\phi_x)_{\frac{1}{2}} \right]. \end{aligned}$$

Since  $\int_{\Omega} v dx = 0$ , we have  $\phi_x(a) = \phi_x(b)$ . And by the periodic boundary condition of  $v$ , we obtain

$$\|v\|^2 = \sum_{j=1}^N \left\{ \int_{I_j} v_x \phi_x dx + [v]_{j-\frac{1}{2}}(\phi_x)_{j-\frac{1}{2}} \right\}. \tag{2.19}$$

Hence by the Cauchy–Schwarz inequality, the trace inequality [1]

$$\|\phi_x\|_{\partial T_h}^2 \leq C\|\phi_x\| \|\phi_{xx}\| \leq C\|\phi\|_{H^2}^2$$

and the elliptic regularity, we have

$$\|v\|^2 \leq \|v_x\| \|\phi_x\| + [v] \|\phi_x\|_{\partial T_h} \leq C(\|v_x\| + [v]) \|\phi\|_{H^2} \leq C_p(\|v_x\| + \sqrt{\mu h^{-1}}[v]) \|v\|.$$

Thus we are led to (2.18) by dividing  $\|v\|$  on both sides of the above inequality. □

**Corollary 2.8.** *Suppose  $q, p \in V_h$  satisfy (2.6c) and (2.6d), then*

$$\|q\| \leq C_p C_{\mu} \|p\|. \tag{2.20}$$

*Proof.* It is easy to verify that  $q$  satisfies the condition of Lemma 2.7. Firstly,  $q$  satisfies the periodic boundary condition according to the definition of the scheme. Secondly, by taking the test function  $v = 1$  in (2.6d), we get

$$\int_{\Omega} q dx = \sum_{j=1}^N (u_{j+\frac{1}{2}}^- - u_{j-\frac{1}{2}}^-) = 0$$

due to the periodic boundary condition of  $u$ . As a result, by Lemma 2.7 and the property (2.14), we have  $\|q\| \leq C_p(\|q_x\| + \sqrt{\mu h^{-1}}[q]) \leq C_p C_{\mu} \|p\|$ . □

### 2.3. Stability of the semi-discrete LDG scheme

**Theorem 2.9.** *Let  $\mathbf{w} = (u, q, p, r) \in \mathbf{V}_h$  be the solution of the scheme (2.6), then we have*

$$\|u(t)\|^2 + 2 \int_0^t (c_4 \|p(s)\|^2 + |c_2| \|q(s)\|^2) ds \leq \|u(0)\|^2, \tag{2.21}$$

if  $c_2 \leq 0$ . And

$$\|u(t)\|^2 + \int_0^t c_4 \|p(s)\|^2 ds \leq e^{C_0 t} \|u(0)\|^2, \tag{2.22}$$

if  $c_2 > 0$ , where  $C_0 = \frac{c_2^2}{c_4}$ .

*Proof.* Taking  $v = u$  in (2.6a), we get

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 = \mathcal{H}(\varphi, u) + \mathcal{L}(r, u). \quad (2.23)$$

Then owing to (2.11) we get

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + c_4 \|p\|^2 = c_1 \mathcal{H}^-(u, u) + c_2 \mathcal{H}^+(q, u) + c_3 \mathcal{H}^+(p, u). \quad (2.24)$$

According to Lemma 2.2 and (2.6) we have

$$\begin{aligned} c_1 \mathcal{H}^-(u, u) &\leq 0, \\ c_2 \mathcal{H}^+(q, u) &= -c_2 \mathcal{H}^-(u, q) = c_2 \|q\|^2, \\ c_3 \mathcal{H}^+(p, u) &= -c_3 \mathcal{H}^-(u, p) = c_3(q, p) = c_3(p, q) = -c_3 \mathcal{H}^+(q, q) \leq 0. \end{aligned} \quad (2.25)$$

Hence, if  $c_2 \leq 0$  then

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + c_4 \|p\|^2 + |c_2| \|q\|^2 \leq 0. \quad (2.26)$$

Integrating over  $[0, t]$  we get (2.21).

If  $c_2 > 0$ , then by (2.15) and the Young's inequality we have

$$c_2 \|q\|^2 \leq c_2 \|u\| \|p\| \leq \frac{c_4}{2} \|p\|^2 + \frac{c_2^2}{2c_4} \|u\|^2. \quad (2.27)$$

So from (2.24) and (2.25) we get

$$\frac{d}{dt} \|u\|^2 + c_4 \|p\|^2 \leq \frac{c_2^2}{c_4} \|u\|^2. \quad (2.28)$$

Hence by Gronwall's inequality we obtain (2.22).  $\square$

**Remark 2.10.** In the case  $c_2 > 0$ , we can get a stronger result

$$\|u(t)\|^2 + \int_0^t c_4 \|p(s)\|^2 ds \leq \|u(0)\|^2, \quad (2.29)$$

if we assume  $c_2 \leq \frac{c_4}{2C_p^2 C_\mu^2}$ . Since

$$c_2 \|q\|^2 \leq c_2 C_p^2 C_\mu^2 \|p\|^2 \quad (2.30)$$

owing to corollary 2.8, from (2.24) and (2.25) we get

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + c_4 \|p\|^2 \leq c_2 C_p^2 C_\mu^2 \|p\|^2. \quad (2.31)$$

Thus if  $c_2 C_p^2 C_\mu^2 \leq \frac{1}{2} c_4$ , i.e.,  $c_2 \leq \frac{c_4}{2C_p^2 C_\mu^2}$ , then we get

$$\frac{d}{dt} \|u\|^2 + c_4 \|p\|^2 \leq 0. \quad (2.32)$$

Hence integrating over  $[0, t]$  yields (2.29).

### 3. THE IMEX RK FULLY DISCRETE SCHEMES AND THEIR STABILITY ANALYSIS

Let  $\{t^n = n\tau\}_{n=0}^M$  be the uniform partition of the time interval  $[0, T]$ , with time step  $\tau$ . The time step could actually change from step to step, but in this paper we take the time step as a constant for simplicity. Given  $u^n$ , we would like to find the numerical solution at the next time level  $t^{n+1}$ , maybe through several intermediate stages  $t^{n,\ell}$ , by the IMEX RK methods given in [3]. For simplicity of notations, we denote  $\varphi^{n,\ell} = (u^{n,\ell}, q^{n,\ell}, p^{n,\ell})$  in this paper.

### 3.1. First order scheme

The LDG scheme with the first order IMEX time-marching scheme is given in the following form:

$$(u^{n+1} - u^n, v) = \tau [\mathcal{H}(\varphi^n; v) + \mathcal{L}(r^{n+1}, v)], \tag{3.1a}$$

$$(r^{n,\ell}, \rho) = -\mathcal{H}^-(p^{n,\ell}, \rho), \tag{3.1b}$$

$$(p^{n,\ell}, \phi) = -\mathcal{H}^+(q^{n,\ell}, \phi), \tag{3.1c}$$

$$(q^{n,\ell}, \psi) = -\mathcal{H}^-(u^{n,\ell}, \psi), \quad \text{for } \ell = 0, 1 \tag{3.1d}$$

for any function  $(v, \rho, \phi, \psi) \in \mathbf{V}_h$ . Here  $w^{n,0} = w^n$  and  $w^{n,1} = w^{n+1}$ .

We first give the stability result for general coefficients  $c_1, c_2, c_3$  and  $c_4$ .

**Proposition 3.1.** *There exists a positive constant  $\tau_0$  independent of  $h$ , such that if  $\tau \leq \tau_0$ , then the solution of scheme (3.1) satisfies*

$$\|u^n\|^2 + c_4\tau\|p^n\|^2 \leq e^{C_0n\tau}(\|u^0\|^2 + c_4\tau\|p^0\|^2), \quad \forall n, \tag{3.2}$$

where  $C_0$  is a positive constant independent of  $h$  and  $\tau$ .

*Proof.* Taking  $v = u^{n+1}$  in (3.1a), by (2.11) we get

$$(u^{n+1} - u^n, u^{n+1}) + c_4\tau\|p^{n+1}\|^2 = R_a + R_b + R_c, \tag{3.3}$$

where

$$R_a = c_1\tau\mathcal{H}^-(u^n, u^{n+1}), \quad R_b = c_2\tau\mathcal{H}^+(q^n, u^{n+1}), \quad R_c = c_3\tau\mathcal{H}^+(p^n, u^{n+1}). \tag{3.4}$$

Noting that

$$(u^{n+1} - u^n, u^{n+1}) = \frac{1}{2}\|u^{n+1}\|^2 + \frac{1}{2}\|u^{n+1} - u^n\|^2 - \frac{1}{2}\|u^n\|^2.$$

Thus (3.3) is equivalent to

$$LHS = \frac{1}{2}\|u^{n+1}\|^2 - \frac{1}{2}\|u^n\|^2 + \frac{1}{2}\|u^{n+1} - u^n\|^2 + c_4\tau\|p^{n+1}\|^2 = R_a + R_b + R_c. \tag{3.5}$$

To investigate the effect of the explicit discretization of the lower derivative terms, we will estimate the three terms  $R_a, R_b$  and  $R_c$  separately.

**Estimate for  $R_a$ .** By adding and subtracting a term  $c_1\tau\mathcal{H}^-(u^{n+1}, u^{n+1})$ , we obtain

$$\begin{aligned} R_a &= c_1\tau\mathcal{H}^-(u^{n+1}, u^{n+1}) - c_1\tau\mathcal{H}^-(u^{n+1} - u^n, u^{n+1}) \\ &= -\frac{c_1}{2}\tau\|u^{n+1}\|^2 - c_1\tau\mathcal{H}^-(u^{n+1} - u^n, u^{n+1}), \end{aligned}$$

where the last equality holds by the property (2.9). Thus by (2.12b), we have

$$R_a \leq c_1\tau|\mathcal{H}^-(u^{n+1} - u^n, u^{n+1})| \leq c_1\tau \left( \|u_x^{n+1}\| + \sqrt{\mu h^{-1}}\|u^{n+1}\| \right) \|u^{n+1} - u^n\|. \tag{3.6}$$

Exploiting (2.13), the Young's inequality and (2.20) successively gives

$$\begin{aligned} R_a &\leq c_1C_\mu\tau\|q^{n+1}\|\|u^{n+1} - u^n\| \\ &\leq \frac{\varepsilon}{2C_p^2C_\mu^2}\tau\|q^{n+1}\|^2 + \frac{c_1^2C_p^2C_\mu^4}{2\varepsilon}\tau\|u^{n+1} - u^n\|^2 \\ &\leq \frac{\varepsilon}{2}\tau\|p^{n+1}\|^2 + \frac{c_1^2C_p^2C_\mu^4}{2\varepsilon}\tau\|u^{n+1} - u^n\|^2. \end{aligned} \tag{3.7}$$



**Estimate for  $R_b$ .** Owing to (2.10) and (3.1d), (3.1c) we get

$$\begin{aligned}\mathcal{H}^+(q^n, u^{n+1}) &= -\mathcal{H}^-(u^{n+1}, q^n) = (q^{n+1}, q^n) = (q^n, q^{n+1}) = -\mathcal{H}^-(u^n, q^{n+1}) \\ &= \mathcal{H}^+(q^{n+1}, u^n) = -(p^{n+1}, u^n).\end{aligned}$$

Hence

$$R_b = -c_2\tau(p^{n+1}, u^n). \quad (3.8)$$

Then a simple use of the Cauchy–Schwarz inequality and the Young’s inequality leads to

$$R_b \leq |c_2|\tau\|p^{n+1}\|\|u^n\| \leq \frac{\varepsilon}{4}\tau\|p^{n+1}\|^2 + \frac{c_2^2}{\varepsilon}\tau\|u^n\|^2. \quad (3.9)$$

**Estimate for  $R_c$ .** From (2.10) and (3.1d) we get

$$\mathcal{H}^+(p^n, u^{n+1}) = -\mathcal{H}^-(u^{n+1}, p^n) = (q^{n+1}, p^n). \quad (3.10)$$

Hence a simple use of the Cauchy–Schwarz inequality and the Young’s inequality yields

$$R_c \leq c_3\tau\|q^{n+1}\|\|p^n\| \leq \varepsilon\tau\|p^n\|^2 + \frac{c_3^2}{4\varepsilon}\tau\|q^{n+1}\|^2. \quad (3.11)$$

Again using (2.15), the Young’s inequality and the triangle inequality successively, we have

$$R_c \leq \varepsilon\tau\|p^n\|^2 + \frac{\varepsilon}{4}\tau\|p^{n+1}\|^2 + \frac{c_3^4}{8\varepsilon^3}\tau(\|u^{n+1} - u^n\|^2 + \|u^n\|^2). \quad (3.12)$$

Consequently, combining (3.5), (3.7), (3.9) and (3.12) we have

$$LHS \leq \varepsilon\tau(\|p^{n+1}\|^2 + \|p^n\|^2) + \frac{C_0}{2}\tau\|u^n\|^2 + \frac{C_1}{2}\tau\|u^{n+1} - u^n\|^2,$$

where  $C_0 = 2(\frac{c_2^2}{\varepsilon} + \frac{c_3^4}{8\varepsilon^3})$  and  $C_1 = 2(\frac{c_1^2 C_p^2 C_\mu^4}{2\varepsilon} + \frac{c_3^4}{8\varepsilon^3})$ . Therefore, if we take  $\varepsilon = \frac{c_4}{2}$  and let  $C_1\tau \leq 1$ , i.e.,  $\tau \leq \frac{1}{C_1}$ , then

$$(\|u^{n+1}\|^2 + c_4\tau\|p^{n+1}\|^2) - (\|u^n\|^2 + c_4\tau\|p^n\|^2) \leq C_0\tau\|u^n\|^2 \leq C_0\tau(\|u^n\|^2 + c_4\tau\|p^n\|^2). \quad (3.13)$$

Hence (3.2) follows by the discrete Gronwall’s inequality.  $\square$

Next we give some stronger conclusions in some special cases.

**Proposition 3.2.**

(i) In the case  $c_3 = 0$ , under the condition of Proposition 3.1, the solution of scheme (3.1) satisfies

$$\|u^n\|^2 + c_4\tau \sum_{m=1}^n \|p^m\|^2 \leq e^{C_0 n\tau} \|u^0\|^2, \quad \forall n, \quad (3.14)$$

where  $C_0$  is a positive constant independent of  $h$  and  $\tau$ .

(ii) If  $c_3 = 0$  and  $c_2 \leq \frac{c_4}{4C_p^2 C_\mu^2}$ , we have

$$\|u^n\|^2 + c_4\tau \sum_{m=1}^n \|p^m\|^2 \leq \|u^0\|^2, \quad \forall n. \quad (3.15)$$

(iii) If  $0 < c_3 \leq \frac{c_4}{2C_p C_\mu}$  and  $c_2 \leq \frac{c_4}{4C_p^2 C_\mu^2}$ , then we can get

$$\|u^n\|^2 + \frac{c_4}{2}\tau\|p^n\|^2 \leq \|u^0\|^2 + \frac{c_4}{2}\tau\|p^0\|^2, \quad \forall n. \tag{3.16}$$

*Proof.*

(i) In the case  $c_3 = 0$ , we have  $R_c = 0$  in the proof of Proposition 3.1. Thus, according to (3.5), (3.7) and (3.9), we get

$$LHS \leq \frac{3\varepsilon}{4}\tau\|p^{n+1}\|^2 + \frac{c_2^2}{\varepsilon}\tau\|u^n\|^2 + \frac{c_1^2 C_p^2 C_\mu^4}{2\varepsilon}\tau\|u^{n+1} - u^n\|^2.$$

Therefore, if we take  $\varepsilon = \frac{2c_4}{3}$  and let  $\frac{c_1^2 C_p^2 C_\mu^4}{\varepsilon}\tau \leq 1$ , i.e.,  $\tau \leq \frac{\varepsilon}{c_1^2 C_p^2 C_\mu^4}$ , then

$$\|u^{n+1}\|^2 - \|u^n\|^2 + c_4\tau\|p^{n+1}\|^2 \leq C_0\tau\|u^n\|^2, \tag{3.17}$$

where  $C_0 = \frac{2c_2^2}{\varepsilon}$ . Hence (3.14) follows by the discrete Gronwall's inequality.

(ii) We infer from (3.8) and (2.16) that

$$R_b = c_2\tau(p^{n+1}, u^{n+1} - u^n) - c_2\tau(p^{n+1}, u^{n+1}) = c_2\tau(p^{n+1}, u^{n+1} - u^n) + c_2\tau\|q^{n+1}\|^2.$$

Hence, by Cauchy-Schwarz inequality, the Young's inequality and (2.20) we have

$$\begin{aligned} R_b &\leq \frac{\varepsilon}{2}\tau\|p^{n+1}\|^2 + \frac{c_2^2}{2\varepsilon}\tau\|u^{n+1} - u^n\|^2 + c_2\tau\|q^{n+1}\|^2 \\ &\leq \left(\frac{\varepsilon}{2} + c_2C_p^2C_\mu^2\right)\tau\|p^{n+1}\|^2 + \frac{c_2^2}{2\varepsilon}\tau\|u^{n+1} - u^n\|^2, \end{aligned} \tag{3.18}$$

if  $c_2 > 0$ . Thus, from (3.5), (3.7) and (3.18), we get

$$LHS \leq (\varepsilon + c_2C_p^2C_\mu^2)\tau\|p^{n+1}\|^2 + \frac{c_1^2 C_p^2 C_\mu^4 + c_2^2}{2\varepsilon}\tau\|u^{n+1} - u^n\|^2.$$

So if we take  $\varepsilon = \frac{c_4}{4}$  and let  $c_2 \leq \frac{c_4}{4C_p^2 C_\mu^2}$ , and let  $\frac{c_1^2 C_p^2 C_\mu^4 + c_2^2}{\varepsilon}\tau \leq 1$ , i.e.,  $\tau \leq \frac{\varepsilon}{c_1^2 C_p^2 C_\mu^4 + c_2^2}$ , then

$$\|u^{n+1}\|^2 - \|u^n\|^2 + c_4\tau\|p^{n+1}\|^2 \leq 0, \tag{3.19}$$

which leads to the strong conclusion (3.15). The conclusion holds obviously for  $c_2 \leq 0$ .

(iii) From (3.11), using (2.20) yields

$$R_c \leq \varepsilon\tau\|p^n\|^2 + \frac{c_3^2}{4\varepsilon}C_p^2C_\mu^2\tau\|p^{n+1}\|^2. \tag{3.20}$$

Thus, from (3.5), (3.7), (3.18) and (3.20), we obtain

$$LHS \leq \varepsilon\tau\|p^n\|^2 + \left(\varepsilon + c_2C_p^2C_\mu^2 + \frac{c_3^2}{4\varepsilon}C_p^2C_\mu^2\right)\tau\|p^{n+1}\|^2 + \frac{c_1^2 C_p^2 C_\mu^4 + c_2^2}{2\varepsilon}\tau\|u^{n+1} - u^n\|^2.$$

So if we take  $\varepsilon = \frac{c_4}{4}$  and let  $c_2 \leq \frac{c_4}{4C_p^2 C_\mu^2}$  and  $c_3 \leq \frac{c_4}{2C_p C_\mu}$ , and let  $\tau \leq \frac{\varepsilon}{c_1^2 C_p^2 C_\mu^4 + c_2^2}$ , then

$$\|u^{n+1}\|^2 + \frac{c_4}{2}\tau\|p^{n+1}\|^2 \leq \|u^n\|^2 + \frac{c_4}{2}\tau\|p^n\|^2, \tag{3.21}$$

which leads to the strong conclusion (3.16). □

**Remark 3.3.** From the proof of the above two propositions, we see that:

- (a) If  $c_2$  and  $c_3$  are absent, then we can get the strong stability result (3.15). The time step  $\tau$  is not required to be dependent of the mesh size  $h$ , but it depends on the coefficients of  $c_1$  and  $c_4$ . The key techniques are the discrete version of Poincaré inequality (2.20) in addition to the technique adopted in [15], *i.e.*, the relationship (2.13) and the stability term  $\|u^{n+1} - u^n\|$  provided by the implicit discretization of the highest order derivative term.
- (b) If  $c_2 > \frac{c_4}{4C_p^2 C_\mu^2}$ , then we can only get exponential stability, which coincides with the feature of the corresponding PDE.
- (c) In the case  $c_3 \neq 0$ , we could not get the strong stability in the  $L^2$  norm of  $u$  alone, but in the “energy norm” which contains the information of the  $L^2$  norms of both  $u$  and  $p$ . This is because the terms involving  $p^n$  generated by the explicit discretization of the third order derivative term can not be dealt with in the same way as the treatment for the terms involving  $u^n$  generated by the discretization of the first order derivative term, as we do not have the time difference term  $\|p^{n+1} - p^n\|$  on the left side of the energy equation. However, numerical experiments show that strong stability in the  $L^2$  norm of  $u$  can be obtained, but the maximum value  $\tau_0$  to ensure the stability of the scheme is rather sensitive to the coefficient of the third order derivative term.

### 3.2. Second order scheme

The LDG scheme with the second order IMEX time marching scheme [3] is given as:

$$(u^{n,1}, v) = (u^n, v) + \gamma\tau\mathcal{H}(\varphi^n; v) + \gamma\tau\mathcal{L}(r^{n,1}, v), \quad (3.22a)$$

$$(u^{n+1}, v) = (u^n, v) + \delta\tau\mathcal{H}(\varphi^n; v) + (1 - \delta)\tau\mathcal{H}(\varphi^{n,1}; v) \\ + (1 - \gamma)\tau\mathcal{L}(r^{n,1}, v) + \gamma\tau\mathcal{L}(r^{n+1}, v), \quad (3.22b)$$

for any function  $v \in V_h$ , where  $\gamma = 1 - \frac{\sqrt{2}}{2}$ ,  $\delta = 1 - \frac{1}{2\gamma}$ . And (3.1b)–(3.1d) hold for  $\ell = 0, 1, 2$ , where  $w^{n,0} = w^n$  and  $w^{n,2} = w^{n+1}$ .

**Proposition 3.4.** *Under the condition of Proposition 3.1, the solution of the scheme (3.22) satisfies (3.2).*

*Proof.* From (3.22a) and (3.22b), and noting that  $\delta - \gamma = -1$ , we get

$$(u^{n,1} - u^n, v) = \gamma\tau\mathcal{H}(\varphi^n; v) + \gamma\tau\mathcal{L}(r^{n,1}, v), \quad (3.23a)$$

$$(u^{n+1} - u^{n,1}, v) = -\tau\mathcal{H}(\varphi^n; v) + (1 - \delta)\tau\mathcal{H}(\varphi^{n,1}; v) \\ + (1 - 2\gamma)\tau\mathcal{L}(r^{n,1}, v) + \gamma\tau\mathcal{L}(r^{n+1}, v). \quad (3.23b)$$

By taking  $v = u^{n,1}, u^{n+1}$  in (3.23a) and (3.23b), respectively, and adding them together, we obtain

$$LHS = \frac{1}{2}\|u^{n+1}\|^2 - \frac{1}{2}\|u^n\|^2 + \frac{1}{2}\|u^{n+1} - u^{n,1}\|^2 + \frac{1}{2}\|u^{n,1} - u^n\|^2 = R_1 + R_2,$$

where

$$R_1 = \gamma\tau\mathcal{H}(\varphi^n; u^{n,1}) - \tau\mathcal{H}(\varphi^n; u^{n+1}) + (1 - \delta)\tau\mathcal{H}(\varphi^{n,1}; u^{n+1}), \\ R_2 = \gamma\tau\mathcal{L}(r^{n,1}, u^{n,1}) + (1 - 2\gamma)\tau\mathcal{L}(r^{n,1}, u^{n+1}) + \gamma\tau\mathcal{L}(r^{n+1}, u^{n+1}) \\ = c_4 [-\gamma\tau\|p^{n,1}\|^2 - (1 - 2\gamma)\tau(p^{n,1}, p^{n+1}) - \gamma\tau\|p^{n+1}\|^2].$$

To obtain  $R_2$ , we have used the property (2.11) and the similar property

$$\mathcal{H}^+(r_1, u_2) = -\mathcal{H}^-(u_2, r_1) = (q_2, r_1) = (r_1, q_2) = -\mathcal{H}^-(p_1, q_2) = \mathcal{H}^+(q_2, p_1) \\ = -(p_2, p_1), \quad (3.24)$$

for any pairs  $(u_1, q_1, p_1, r_1)$  and  $(u_2, q_2, p_2, r_2)$ , owing to (2.10) and (2.6).

In order to estimate  $R_1$ , we divide it into three parts  $R_{11}, R_{12}, R_{13}$ , where

$$\begin{aligned} R_{11} &= c_1\tau [\gamma\mathcal{H}^-(u^n, u^{n,1}) - \mathcal{H}^-(u^n, u^{n+1}) + (1 - \delta)\mathcal{H}^-(u^{n,1}, u^{n+1})], \\ R_{12} &= c_2\tau [\gamma\mathcal{H}^+(q^n, u^{n,1}) - \mathcal{H}^+(q^n, u^{n+1}) + (1 - \delta)\mathcal{H}^+(q^{n,1}, u^{n+1})], \\ R_{13} &= c_3\tau [\gamma\mathcal{H}^+(p^n, u^{n,1}) - \mathcal{H}^+(p^n, u^{n+1}) + (1 - \delta)\mathcal{H}^+(p^{n,1}, u^{n+1})]. \end{aligned}$$

Next we would like to estimate them one by one. First we rewrite  $R_{11}$  in the following equivalent form:

$$\begin{aligned} R_{11} &= c_1\gamma\tau\mathcal{H}^-(u^{n,1}, u^{n,1}) + c_1(1 - \gamma)\tau\mathcal{H}^-(u^{n+1}, u^{n+1}) - c_1\gamma\tau\mathcal{H}^-(u^{n,1} - u^n, u^{n,1}) \\ &\quad - c_1(1 - \gamma)\tau\mathcal{H}^-(u^{n+1} - u^{n,1}, u^{n+1}) + c_1\tau\mathcal{H}^-(u^{n,1} - u^n, u^{n+1}). \end{aligned}$$

By the property (2.9) we have

$$\begin{aligned} R_{11} &= -\frac{c_1}{2}\gamma\tau\|u^{n,1}\|^2 - \frac{c_1}{2}(1 - \gamma)\tau\|u^{n+1}\|^2 - c_1\gamma\tau\mathcal{H}^-(u^{n,1} - u^n, u^{n,1}) \\ &\quad - c_1(1 - \gamma)\tau\mathcal{H}^-(u^{n+1} - u^{n,1}, u^{n+1}) + c_1\tau\mathcal{H}^-(u^{n,1} - u^n, u^{n+1}). \end{aligned}$$

Proceeding the same argument as the estimate for  $R_a$  in the proof of Proposition 3.1, we can derive

$$R_{11} \leq \varepsilon\tau(\|p^{n,1}\|^2 + \|p^{n+1}\|^2) + c_1^2C_p^2C_\mu^4C_{\gamma,\varepsilon}\tau(\|u^{n,1} - u^n\|^2 + \|u^{n+1} - u^{n,1}\|^2), \tag{3.25}$$

for arbitrary  $\varepsilon > 0$ , here and below we use  $C_{\gamma,\varepsilon}$  to denote a positive constant depending on  $\gamma$  and  $\varepsilon$ , it may have different values in each occurrence.

Along the similar analysis as the estimate for  $R_b$  in the proof of Proposition 3.1, we get

$$R_{12} = -c_2\tau [\gamma(p^{n,1}, u^n) - (p^{n+1}, u^n) + (1 - \delta)(p^{n+1}, u^{n,1})]. \tag{3.26}$$

Thus a simple use of the Cauchy–Schwarz inequality, the Young’s inequality and the triangle inequality leads to

$$R_{12} \leq \varepsilon\tau(\|p^{n,1}\|^2 + \|p^{n+1}\|^2) + c_2^2C_{\gamma,\varepsilon}\tau(\|u^n\|^2 + \|u^{n,1} - u^n\|^2). \tag{3.27}$$

Similar to the estimate of  $R_c$  in the proof of Proposition 3.1, we have

$$R_{13} = c_3 [\gamma\tau(q^{n,1}, p^n) - \tau(q^{n+1}, p^n) + (1 - \delta)\tau(q^{n+1}, p^{n,1})]. \tag{3.28}$$

Hence by Cauchy–Schwarz inequality, the Young’s inequality and (2.15) we can get

$$R_{13} \leq \varepsilon\tau(\|p^n\|^2 + \|p^{n,1}\|^2) + c_3^2C_{\gamma,\varepsilon}\tau(\|p^{n,1}\|\|u^{n,1}\| + \|p^{n+1}\|\|u^{n+1}\|).$$

Using the Young’s inequality again and the triangle inequality gives

$$R_{13} \leq \varepsilon\tau(\|p^n\|^2 + 2\|p^{n,1}\|^2 + \|p^{n+1}\|^2) + c_3^4C_{\gamma,\varepsilon}\tau(\|u^n\|^2 + \|u^{n,1} - u^n\|^2 + \|u^{n+1} - u^{n,1}\|^2). \tag{3.29}$$

Finally, combining the above estimates we obtain

$$LHS + \varepsilon\tau(\|p^{n+1}\|^2 - \|p^n\|^2) + S \leq \frac{1}{2}C_0\tau\|u^n\|^2 + \frac{1}{2}C_1\tau(\|u^{n,1} - u^n\|^2 + \|u^{n+1} - u^{n,1}\|^2),$$

where  $C_0 = 2(c_2^2C_{\gamma,\varepsilon} + c_3^4C_{\gamma,\varepsilon})$ ,  $C_1 = C_0 + 2c_1^2C_p^2C_\mu^4C_{\gamma,\varepsilon}$  and

$$S = (c_4\gamma - 4\varepsilon)\tau\|p^{n,1}\|^2 + c_4(1 - 2\gamma)\tau(p^{n,1}, p^{n+1}) + (c_4\gamma - 4\varepsilon)\tau\|p^{n+1}\|^2.$$

TABLE 1. The coefficients of  $a_{\ell i}$  and  $\hat{a}_{\ell i}$ , the left is  $a_{\ell i}$ , the right is  $\hat{a}_{\ell i}$ .

$\ell \backslash i$	0	1	2	3	1	2	3	4
1	1/2	0	0	0	1/2	0	0	0
2	11/18	1/18	0	0	1/6	1/2	0	0
3	5/6	-5/6	1/2	0	-1/2	1/2	1/2	0
4	1/4	7/4	3/4	-4/7	3/2	-3/2	1/2	1/2

We denote by  $\mathbf{x}^\top = (p^{n,1}, p^{n+1})$ , then  $S = \tau \int_\Omega \mathbf{x}^\top \mathbb{M} \mathbf{x} dx$ , with

$$\mathbb{M} = \begin{pmatrix} c_4\gamma - 4\varepsilon & c_4(\frac{1}{2} - \gamma) \\ c_4(\frac{1}{2} - \gamma) & c_4\gamma - 4\varepsilon \end{pmatrix}.$$

It is easy to check that  $\mathbb{M}$  is positive definite if we take  $\varepsilon = \frac{c_4\gamma}{16}$ , so  $S > 0$ , which implies that

$$\frac{1}{2}\|u^{n+1}\|^2 - \frac{1}{2}\|u^n\|^2 + \frac{c_4\gamma}{16}\tau(\|p^{n+1}\|^2 - \|p^n\|^2) \leq \frac{1}{2}C_0\tau\|u^n\|^2,$$

if we let  $C_1\tau \leq 1$ , i.e.,  $\tau \leq \frac{1}{C_1}$ . Hence

$$(\|u^{n+1}\|^2 + \frac{c_4\gamma}{8}\tau\|p^{n+1}\|^2) - (\|u^n\|^2 + \frac{c_4\gamma}{8}\tau\|p^n\|^2) \leq C_0\tau\|u^n\|^2 \leq C_0\tau(\|u^n\|^2 + \frac{c_4\gamma}{8}\tau\|p^n\|^2).$$

As a result, by the discrete Gronwall's inequality we have

$$\|u^n\|^2 + \frac{c_4\gamma}{8}\tau\|p^n\|^2 \leq e^{C_0n\tau}(\|u^0\|^2 + \frac{c_4\gamma}{8}\tau\|p^0\|^2) \leq e^{C_0n\tau}(\|u^0\|^2 + c_4\tau\|p^0\|^2).$$

Furthermore, we can easily get (3.2) with a different  $C_0$ . □

### 3.3. Third order scheme

The LDG scheme with the third order IMEX time marching scheme [3] reads:

$$(u^{n,\ell}, v) = (u^n, v) + \tau \sum_{i=0}^3 a_{\ell i} \mathcal{H}(\varphi^{n,i}; v) + \tau \sum_{i=1}^4 \hat{a}_{\ell i} \mathcal{L}(r^{n,i}, v), \quad \text{for } \ell = 1, 2, 3, 4, \tag{3.30a}$$

and (3.1b)–(3.1d) hold for  $\ell = 0, 1, 2, 3, 4$ . Here and below,  $w^{n,0} = w^n$  and  $w^{n,4} = w^{n+1}$ . The coefficients are given in Table 1.

For the convenience of analysis, we would like to denote

$$\mathbb{D}_1 w^n = w^{n,1}, \quad \mathbb{D}_2 w^n = 2w^{n,2} - 3w^{n,1}, \quad \mathbb{D}_3 w^n = w^{n,3} - 2w^{n,2} + 2w^{n,1}, \quad \mathbb{D}_4 w^n = w^{n+1}.$$

By introducing a series of notations

$$\begin{aligned} \mathbb{E}_1 w^n &= w^{n,1} - w^n & \mathbb{E}_2 w^n &= \mathbb{D}_2 w^n + w^n \\ \mathbb{E}_3 w^n &= \mathbb{D}_3 w^n - w^{n,1} & \mathbb{E}_4 w^n &= 2w^{n+1} + \mathbb{D}_2 w^n - \mathbb{D}_3 w^n \\ \mathbb{E}_{41} w^n &= w^{n+1} + \mathbb{D}_2 w^n & \mathbb{E}_{42} w^n &= w^{n+1} - \mathbb{D}_3 w^n \end{aligned}$$

for arbitrary  $w$ , we rewrite the above scheme into the following compact form. In the following we denote  $\Theta^n = (\varphi^n, \varphi^{n,1}, \varphi^{n,2}, \varphi^{n,3})$  and  $\mathbf{r}^n = (r^{n,1}, r^{n,2}, r^{n,3}, r^{n+1})$ .

$$(\mathbb{E}_\ell w^n, v) = \Phi_\ell(\Theta^n, v) + \Psi_\ell(\mathbf{r}^n, v), \quad \text{for } \ell = 1, 2, 3, 4 \tag{3.31}$$

TABLE 2. The coefficients of  $c_{\ell i}$  and  $d_{\ell i}$ , the left is  $c_{\ell i}$ , the right is  $d_{\ell i}$ .

$\ell \backslash i$	0	1	2	3	1	2	3	4
1	1/2	0	0	0	1/2	0	0	0
2	-5/18	1/9	0	0	1/3	1/2	0	0
3	1/9	-17/18	1/2	0	-7/12	1/4	1/2	0
4	-7/18	41/9	1	-7/2	-1/12	-1/4	1/2	1

where

$$\Phi_\ell(\Theta^n, v) = \tau \sum_{i=0}^3 c_{\ell i} \mathcal{H}(\varphi^{n,i}, v), \quad \Psi_\ell(\mathbf{r}^n, v) = \tau \sum_{i=1}^4 d_{\ell i} \mathcal{L}(\mathbb{D}_i r^n, v), \tag{3.32}$$

and the coefficients are listed in Table 2.

**Proposition 3.5.** *Under the condition of Proposition 3.1, the solution of the scheme (3.30) satisfies (3.2).*

*Proof.* By taking  $v = \mathbb{D}_\ell u^n$  in (3.31), for  $\ell = 1, 2, 3, 4$ , respectively, we can derive:

$$\frac{1}{2} \|u^{n,1}\|^2 + \frac{1}{2} \|\mathbb{E}_1 u^n\|^2 - \frac{1}{2} \|u^n\|^2 = \Phi_1(\Theta^n, \mathbb{D}_1 u^n) + \Psi_1(\mathbf{r}^n, \mathbb{D}_1 u^n), \tag{3.33a}$$

$$\frac{1}{2} \|\mathbb{D}_2 u^n\|^2 + \frac{1}{2} \|\mathbb{E}_2 u^n\|^2 - \frac{1}{2} \|u^n\|^2 = \Phi_2(\Theta^n, \mathbb{D}_2 u^n) + \Psi_2(\mathbf{r}^n, \mathbb{D}_2 u^n), \tag{3.33b}$$

$$\frac{1}{2} \|\mathbb{D}_3 u^n\|^2 + \frac{1}{2} \|\mathbb{E}_3 u^n\|^2 - \frac{1}{2} \|u^{n,1}\|^2 = \Phi_3(\Theta^n, \mathbb{D}_3 u^n) + \Psi_3(\mathbf{r}^n, \mathbb{D}_3 u^n), \tag{3.33c}$$

$$\|u^{n+1}\|^2 + \frac{1}{2} \|\mathbb{E}_{41} u^n\|^2 + \frac{1}{2} \|\mathbb{E}_{42} u^n\|^2 - \frac{1}{2} \|\mathbb{D}_2 u^n\|^2 - \frac{1}{2} \|\mathbb{D}_3 u^n\|^2 = \Phi_4(\Theta^n, \mathbb{D}_4 u^n) + \Psi_4(\mathbf{r}^n, \mathbb{D}_4 u^n). \tag{3.33d}$$

Adding (3.33) together leads to

$$\|u^{n+1}\|^2 - \|u^n\|^2 + \mathcal{S} = \mathcal{T}_1 + \mathcal{T}_2, \tag{3.34}$$

where

$$\mathcal{S} = \frac{1}{2} \left[ \sum_{\ell=1}^3 \|\mathbb{E}_\ell u^n\|^2 + \|\mathbb{E}_{41} u^n\|^2 + \|\mathbb{E}_{42} u^n\|^2 \right], \tag{3.35a}$$

and

$$\mathcal{T}_1 = \sum_{\ell=1}^4 \Phi_\ell(\Theta^n, \mathbb{D}_\ell u^n), \quad \mathcal{T}_2 = \sum_{\ell=1}^4 \Psi_\ell(\mathbf{r}^n, \mathbb{D}_\ell u^n). \tag{3.35b}$$

We will first consider the term  $\mathcal{T}_2$ . By the properties (2.11) and (3.24) we have

$$\begin{aligned} \mathcal{T}_2 = c_4 \left[ -\frac{1}{2} \tau \|\mathbb{D}_1 p^n\|^2 - \frac{1}{2} \tau \|\mathbb{D}_2 p^n\|^2 - \frac{1}{2} \tau \|\mathbb{D}_3 p^n\|^2 - \tau \|\mathbb{D}_4 p^n\|^2 \right. \\ \left. - \frac{1}{3} \tau (\mathbb{D}_2 p^n, \mathbb{D}_1 p^n) + \frac{7}{12} \tau (\mathbb{D}_3 p^n, \mathbb{D}_1 p^n) - \frac{1}{4} \tau (\mathbb{D}_3 p^n, \mathbb{D}_2 p^n) \right. \\ \left. + \frac{1}{12} \tau (\mathbb{D}_4 p^n, \mathbb{D}_1 p^n) + \frac{1}{4} \tau (\mathbb{D}_4 p^n, \mathbb{D}_2 p^n) - \frac{1}{2} \tau (\mathbb{D}_4 p^n, \mathbb{D}_3 p^n) \right]. \end{aligned} \tag{3.36}$$

We denote by  $\mathbf{w}^\top = (\mathbb{D}_1 p^n, \mathbb{D}_2 p^n, \mathbb{D}_3 p^n, \mathbb{D}_4 p^n)$ , then

$$\mathcal{T}_2 = -c_4 \tau \int_{\Omega} \mathbf{w}^\top \mathbb{A} \mathbf{w} \, dx, \tag{3.37}$$

where

$$\mathbb{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} & -\frac{7}{24} & -\frac{1}{24} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{8} & -\frac{1}{8} \\ -\frac{7}{24} & \frac{1}{8} & \frac{1}{2} & \frac{1}{4} \\ -\frac{1}{24} & -\frac{1}{8} & \frac{1}{4} & 1 \end{pmatrix}. \tag{3.38}$$

It can be verified that  $\mathbb{A}$  is positive definite by verifying the principal minor determinants of  $\mathbb{A}$  are all positive, so  $\mathcal{T}_2 < 0$ . Towards the goal of estimating the term  $\mathcal{T}_1$ , we would like to divide  $\Phi_i$  into three corresponding parts as the same division as (2.6e), namely,

$$\Phi_i(\Theta^n, v) = c_1\Phi_i^u(\mathbf{u}^n, v) + c_2\Phi_i^q(\mathbf{q}^n, v) + c_3\Phi_i^p(\mathbf{p}^n, v), \quad \text{for } i = 1, 2, 3, 4, \tag{3.39}$$

where  $\mathbf{u}^n = (u^n, u^{n,1}, u^{n,2}, u^{n,3})$ ,  $\mathbf{q}^n = (q^n, q^{n,1}, q^{n,2}, q^{n,3})$  and  $\mathbf{p}^n = (p^n, p^{n,1}, p^{n,2}, p^{n,3})$ . Based on this division, we also would like to divide

$$\mathcal{T}_1 = c_1\mathcal{T}^u + c_2\mathcal{T}^q + c_3\mathcal{T}^p, \tag{3.40}$$

where  $\mathcal{T}^u = \sum_{\ell=1}^4 \Phi_\ell^u(\mathbf{u}^n, \mathbb{D}_\ell u^n)$ ,  $\mathcal{T}^q = \sum_{\ell=1}^4 \Phi_\ell^q(\mathbf{q}^n, \mathbb{D}_\ell u^n)$ , and  $\mathcal{T}^p = \sum_{\ell=1}^4 \Phi_\ell^p(\mathbf{p}^n, \mathbb{D}_\ell u^n)$ . In the following we will estimate them one by one.

First we estimate  $\mathcal{T}^u$ . After some simple algebra manipulation we can get

$$\begin{aligned} \mathcal{T}^u &= 2\tau\mathcal{H}^-(\mathbb{D}_1 u^n, \mathbb{D}_1 u^n) + \tau \left[ \sum_{\ell=1}^3 \alpha_\ell \mathcal{H}^-(\mathbb{D}_1 u^n, \mathbb{E}_\ell u^n) + \frac{5}{3} \mathcal{H}^-(\mathbb{D}_1 u^n, \mathbb{E}_{41} u^n) \right] \\ &\quad + \tau \sum_{k=1}^3 \sum_{\ell=1}^4 \beta_{k\ell} \mathcal{H}^-(\mathbb{E}_k u^n, \mathbb{D}_\ell u^n), \end{aligned}$$

where  $\alpha_1 = \alpha_2 = -11/6, \alpha_3 = -1/3, \beta_{11} = -1/2, \beta_{12} = 5/18, \beta_{13} = 5/36, \beta_{14} = -47/18, \beta_{23} = 1/4, \beta_{24} = -3, \beta_{34} = -7/2$ , and the remaining coefficients are 0.

Hence by exploiting (2.9), Lemma 2.4 and (2.13) successively, we have

$$|\mathcal{T}^u| \leq CC_\mu\tau \sum_{\ell=1}^4 \|\mathbb{D}_\ell q^n\| T,$$

where

$$T = \|\mathbb{E}_1 u^n\| + \|\mathbb{E}_2 u^n\| + \|\mathbb{E}_3 u^n\| + \|\mathbb{E}_{41} u^n\|. \tag{3.41}$$

Then a simple use of the Young’s inequality and (2.20) leads to

$$|c_1\mathcal{T}^u| \leq \frac{\varepsilon}{C_p^2 C_\mu^2} \tau \sum_{\ell=1}^4 \|\mathbb{D}_\ell q^n\|^2 + C_\varepsilon c_1^2 C_p^2 C_\mu^4 \tau T^2 \leq \varepsilon\tau \sum_{\ell=1}^4 \|\mathbb{D}_\ell p^n\|^2 + C_\varepsilon c_1^2 C_p^2 C_\mu^4 \tau T^2, \tag{3.42}$$

for arbitrary positive constant  $\varepsilon$ , here and below  $C_\varepsilon$  is a positive constant only depending on  $\varepsilon$ , it may have different value in each occurrence.

Along the similar argument as in the proof of Proposition 3.1, we can get

$$\mathcal{T}^q = -\tau \sum_{\ell=1}^4 \sum_{i=0}^3 c_{\ell i}(\mathbb{D}_\ell p^n, u^{n,i}), \quad \mathcal{T}^p = \tau \sum_{\ell=1}^4 \sum_{i=0}^3 c_{\ell i}(\mathbb{D}_\ell q^n, p^{n,i}),$$

where  $c_{\ell i}$  are listed in Table 2. Then by a simple use of the Cauchy–Schwarz inequality, the Young’s inequality and the triangle inequality, we get

$$|c_2\mathcal{T}^q| \leq C|c_2|\tau \sum_{\ell=0}^3 \|u^{n,\ell}\| \sum_{\ell=1}^4 \|\mathbb{D}_\ell p^n\| \leq \varepsilon\tau \sum_{\ell=1}^4 \|\mathbb{D}_\ell p^n\|^2 + c_2^2 C_\varepsilon \tau (\|u^n\|^2 + T^2), \tag{3.43}$$

and

$$|c_3 \mathcal{T}^p| \leq C c_3 \tau \sum_{\ell=0}^3 \|p^{n,\ell}\| \sum_{\ell=1}^4 \|\mathbb{D}_\ell q^n\| \leq \varepsilon \tau (\|p^n\|^2 + \sum_{\ell=1}^3 \|\mathbb{D}_\ell p^n\|^2) + c_3^2 C_\varepsilon \tau \sum_{\ell=1}^4 \|\mathbb{D}_\ell q^n\|^2.$$

Then by exploiting (2.15), the Young's inequality and the triangle inequality we get

$$|c_3 \mathcal{T}^p| \leq \varepsilon \tau (\|p^n\|^2 + \|p^{n+1}\|^2) + 2\varepsilon \tau \sum_{\ell=1}^3 \|\mathbb{D}_\ell p^n\|^2 + c_3^4 C_\varepsilon^2 \tau (\|u^n\|^2 + T^2). \quad (3.44)$$

Combining the above estimates, we can derive

$$\|u^{n+1}\|^2 - \|u^n\|^2 + \varepsilon \tau (\|p^{n+1}\|^2 - \|p^n\|^2) + \mathcal{S} \leq -\tau \int_{\Omega} \mathbf{w}^\top (c_4 \mathbb{A} - 4\varepsilon \mathbb{I}) \mathbf{w} \, dx + C_0 \tau \|u^n\|^2 + C_1 \tau T^2, \quad (3.45)$$

where  $C_0 = c_2^2 C_\varepsilon + c_3^4 C_\varepsilon^2$  and  $C_1 = C_0 + C_\varepsilon c_1^2 C_p^2 C_\mu^4$ ,  $\mathbb{I}$  is the identity matrix.

It can be verified that the matrix  $c_4 \mathbb{A} - 4\varepsilon \mathbb{I}$  is positive definite if  $\varepsilon \leq \frac{c_4}{64}$ . Noting that  $C_1 \tau T^2 \leq 8C_1 \tau \mathcal{S}$ , so if we let  $\tau \leq \frac{1}{8C_1}$  and take  $\varepsilon = \frac{c_4}{64}$ , then we have

$$(\|u^{n+1}\|^2 + \frac{c_4 \tau}{64} \|p^{n+1}\|^2) - (\|u^n\|^2 + \frac{c_4 \tau}{64} \|p^n\|^2) \leq C_0 \tau \|u^n\|^2 \leq C_0 \tau (\|u^n\|^2 + \frac{c_4 \tau}{64} \|p^n\|^2). \quad (3.46)$$

Hence we can get (3.2) by the discrete Gronwall's inequality.  $\square$

**Remark 3.6.** The conclusions of Proposition 3.2 also hold true for the second order scheme (3.22) and the third order scheme (3.30), we omit the details to save space.

#### 4. ERROR ESTIMATES

In this section, we are going to obtain the optimal error estimates for the IMEX-LDG schemes introduced in Section 3, for solving time-dependent linear fourth order problem (2.1). To this end, we would like to assume that the exact solution of (2.1) is sufficiently smooth, for example, for  $s$ th order fully discrete IMEX-LDG schemes (3.1), (3.22) or (3.30), we assume

$$U(x, t) \in L^\infty(0, T; H^{k+4}), \quad D_t U(x, t) \in L^\infty(0, T; H^{k+1}), \quad \text{and} \quad D_t^{s+1} U(x, t) \in L^\infty(0, T; L^2), \quad (4.1)$$

where  $D_t^\ell U$  means the  $\ell$ th order time derivative of  $U$ .

We give the main results in the following theorem.

**Theorem 4.1.** *Let  $U(x, t)$  be the exact solution of (2.1), satisfying the smoothness assumption (4.1), and let  $u^n \in V_h$  be the solution of the  $s$ th order fully discrete IMEX-LDG schemes (3.1), (3.22) or (3.30). Then there exist positive constants  $h_0, \tau_0$ , such that if  $h \leq h_0$  and  $\tau \leq \tau_0$ , but  $\tau$  is independent of  $h$ , then*

$$\max_{n\tau \leq T} \|U(x, t^n) - u^n\| \leq C(h^{k+1} + \tau^s), \quad (4.2)$$

for  $s = 1, 2, 3$ , where  $T$  is the final computing time and the bounding constant  $C > 0$  is independent of  $h$  and  $\tau$ .

In the following, we take the first order scheme (3.1) as an example to prove Theorem 4.1, the procedure for the second and third order schemes are similar, so we omit the details to save space. The elliptic projection [10] plays an important role in obtaining the optimal error estimates.



#### 4.1. Elliptic projection

For any function  $U, Q = U_x, P = Q_x, R = P_x$ , the elliptic projection is the unique solution  $(U_h, Q_h, P_h, R_h) \in \mathbf{V}_h$  such that, for any  $(v, \rho, \phi, \psi) \in \mathbf{V}_h$ ,

$$\begin{aligned}\mathcal{L}(R_h, v) &= \mathcal{L}(R, v), \\ (R_h, \rho) &= -\mathcal{H}^-(P_h, \rho), \\ (P_h, \phi) &= -\mathcal{H}^+(Q_h, \phi), \\ (Q_h, \psi) &= -\mathcal{H}^-(U_h, \psi).\end{aligned}\tag{4.3a}$$

Furthermore, since in the elliptic problems with periodic boundary conditions,  $U_h$  is determined up to additive constants, following [10] we make the assumption

$$(U - U_h, 1) = 0,\tag{4.3b}$$

to ensure (4.3a) is well-defined. To obtain the approximation property for the elliptic projection, we would like to follow [10] to consider the adjoint problem

$$\theta = \sigma_x, \quad \omega = \theta_x, \quad \zeta = \omega_x, \quad z = \zeta_x.\tag{4.4}$$

And we assume the following elliptic regularity [5]: there exists a positive constant  $C_{er}$  such that

$$\|\zeta\|_{H^1(\Omega)} + \|\omega\|_{H^2(\Omega)} + \|\theta\|_{H^3(\Omega)} + \|\sigma\|_{H^4(\Omega)} \leq C_{er} \|z\|_{L^2(\Omega)}.\tag{4.5}$$

Along the similar argument as in [10], and by the aid of the Gauss-Radau projection to be defined in (A.1), we can obtain the following lemma.

**Lemma 4.2.** *For any function  $U, Q = U_x, P = Q_x, R = P_x$  with the regularity*

$$\|U\|_{H^{k+1}(\Omega)} + \|Q\|_{H^{k+1}(\Omega)} + \|P\|_{H^{k+1}(\Omega)} + \|R\|_{H^{k+1}(\Omega)} \leq C,\tag{4.6}$$

let  $U_h, Q_h, P_h, R_h \in V_h$  be the elliptic projection (4.3), we have

$$\|U_h - U\| + \|Q_h - Q\| + \|P_h - P\| + \|R_h - R\| \leq Ch^{k+1},\tag{4.7}$$

where  $C$  is independent of  $h$  and  $\tau$ , but is depending on the regularity of the function  $U$  and the elliptic regularity constant  $C_{er}$  defined in (4.5).

We will put the proof of this lemma in the appendix.

#### 4.2. “Reference” functions and error division

Following [15], we define the first order “reference” function of (2.1) as follows: let  $U^{(0)} = U$  be the exact solution of (2.1), then we define  $U^{(1)}$  as the solution of the following first order IMEX time discrete problem:

$$U^{(1)} = U^{(0)} - \tau(c_1 U_x^{(0)} + c_2 Q_x^{(0)} + c_3 P_x^{(0)} + c_4 R_x^{(1)}),\tag{4.8a}$$

where

$$Q^{(\ell)} = U_x^{(\ell)},\tag{4.8b}$$

$$P^{(\ell)} = Q_x^{(\ell)},\tag{4.8c}$$

$$R^{(\ell)} = P_x^{(\ell)},\tag{4.8d}$$

for  $\ell = 0, 1$ . Then for any index  $n$  and  $\ell$  under consideration, the “reference” function defined at each stage time level is defined as

$$(U^{n,\ell}, Q^{n,\ell}, P^{n,\ell}, R^{n,\ell}) = (U^{(\ell)}(x, t^n), Q^{(\ell)}(x, t^n), P^{(\ell)}(x, t^n), R^{(\ell)}(x, t^n)).$$

In what follows, we would like to denote

$$(e_u^{n,\ell}, e_q^{n,\ell}, e_p^{n,\ell}, e_r^{n,\ell}) = (U^{n,\ell} - u^{n,\ell}, Q^{n,\ell} - q^{n,\ell}, P^{n,\ell} - p^{n,\ell}, R^{n,\ell} - r^{n,\ell}), \quad \text{for } \ell = 0, 1. \quad (4.9)$$

And based on the elliptic projection (4.3a), we divide the error  $e_\varsigma^{n,\ell}$  in the form

$$e_\varsigma^{n,\ell} = \xi_\varsigma^{n,\ell} - \eta_\varsigma^{n,\ell}, \quad (4.10)$$

for  $\varsigma = u, q, p, r$ , where

$$\xi_u^{n,\ell} = U_h^{n,\ell} - u^{n,\ell}, \quad \eta_u^{n,\ell} = U_h^{n,\ell} - U^{n,\ell}, \quad (4.11)$$

with  $U_h^{n,\ell}$  being the elliptic projection of the “reference” function  $U^{n,\ell}$  defined in (4.8). Similarly for  $\varsigma = q, p, r$ .

According to Lemma 4.2 and the smoothness regularity (4.1), we have

$$\|\eta_u^n\| + \|\eta_q^n\| + \|\eta_p^n\| + \|\eta_r^n\| \leq Ch^{k+1}, \quad (4.12)$$

and

$$\|\eta_u^{n+1} - \eta_u^n\| \leq Ch^{k+1}\tau. \quad (4.13)$$

In addition, from the proof of Lemma 4.2 (see Appendix A), we can also get

$$\|\eta_u^n\|_{\partial\mathcal{T}_h} + \|\eta_q^n\|_{\partial\mathcal{T}_h} + \|\eta_p^n\|_{\partial\mathcal{T}_h} + \|\eta_r^n\|_{\partial\mathcal{T}_h} \leq Ch^{k+1/2}$$

by the triangle inequality and the trace inverse inequality (2.8).

### 4.3. Energy estimate for $\xi_u$

In this subsection we will focus on the estimate for  $\xi_u$ . First we would like to build up the error equation. Thanks to the regularity (4.1), we can verify that

$$(U^{n+1} - U^n, v) = \tau [\mathcal{H}(\Psi^n; v) + \mathcal{L}(R^{n+1}, v)] + (\zeta^n, v), \quad (4.14)$$

where  $\Psi^n = (U^n, Q^n, P^n)$ , and  $\zeta^n = \mathcal{O}(\tau^2)$ . Hence subtracting (3.1a) from (4.14) we get

$$(e_u^{n+1} - e_u^n, v) = \tau[\mathcal{H}(e_\psi^n; v) + \mathcal{L}(e_r^{n+1}, v)] + (\zeta^n, v).$$

That is

$$\begin{aligned} (\xi_u^{n+1} - \xi_u^n, v) &= (\eta_u^{n+1} - \eta_u^n + \zeta^n, v) + \tau[\mathcal{H}(\xi_\psi^n; v) + \mathcal{L}(\xi_r^{n+1}, v)] \\ &\quad - \tau[\mathcal{H}(\eta_\psi^n; v) + \mathcal{L}(\eta_r^{n+1}, v)]. \end{aligned} \quad (4.15)$$

Here  $e_\psi^n = (e_u^n, e_q^n, e_p^n)$ ,  $\xi_\psi^n = (\xi_u^n, \xi_q^n, \xi_p^n)$  and  $\eta_\psi^n = (\eta_u^n, \eta_q^n, \eta_p^n)$ .

On the other hand, from the regularity property (4.1), we can verify that the “reference” functions (4.8) satisfy the following variational forms: for any  $\rho, \phi, \psi \in V_h$ ,

$$\begin{aligned} (R^{n,\ell}, \rho) &= -\mathcal{H}^-(P^{n,\ell}, \rho), \\ (P^{n,\ell}, \phi) &= -\mathcal{H}^+(Q^{n,\ell}, \phi), \\ (Q^{n,\ell}, \psi) &= -\mathcal{H}^-(U^{n,\ell}, \psi), \quad \text{for } \ell = 0, 1. \end{aligned} \quad (4.16)$$

Then from the definition of the elliptic projection (4.3), the projection error satisfies

$$0 = \mathcal{L}(\eta_r^{n+1}, v), \quad (4.17a)$$

$$(\eta_r^{n,\ell}, \rho) = -\mathcal{H}^-(\eta_p^{n,\ell}, \rho), \quad (4.17b)$$

$$(\eta_p^{n,\ell}, \phi) = -\mathcal{H}^+(\eta_q^{n,\ell}, \phi), \quad (4.17c)$$

$$(\eta_q^{n,\ell}, \psi) = -\mathcal{H}^-(\eta_u^{n,\ell}, \psi), \quad \text{for } \ell = 0, 1. \quad (4.17d)$$

Substituting (4.17a), (4.17c) and (4.17d) into (4.15) leads to

$$\begin{aligned} (\xi_u^{n+1} - \xi_u^n, v) &= (\eta_u^{n+1} - \eta_u^n + \zeta^n, v) + \tau[c_1(\eta_q^n, v) + c_2(\eta_p^n, v) + c_3\mathcal{H}^+(\eta_p^n, v)] \\ &\quad + \tau[\mathcal{H}(\xi_\psi^n; v) + \mathcal{L}(\xi_r^{n+1}, v)]. \end{aligned} \quad (4.18a)$$

And from (4.3a) and (3.1) we have

$$(\xi_r^{n,\ell}, \rho) = -\mathcal{H}^-(\xi_p^{n,\ell}, \rho), \quad (4.18b)$$

$$(\xi_p^{n,\ell}, \phi) = -\mathcal{H}^+(\xi_q^{n,\ell}, \phi), \quad (4.18c)$$

$$(\xi_q^{n,\ell}, \psi) = -\mathcal{H}^-(\xi_u^{n,\ell}, \psi), \quad \text{for } \ell = 0, 1. \quad (4.18d)$$

Till now we have established the error equation for  $\xi_u$ . In the following, we will estimate it by the energy method. Taking  $v = \xi_u^{n+1}$  in (4.18a), we get

$$\frac{1}{2}\|\xi_u^{n+1}\|^2 + \frac{1}{2}\|\xi_u^{n+1} - \xi_u^n\|^2 - \frac{1}{2}\|\xi_u^n\|^2 + c_4\tau\|\xi_p^{n+1}\|^2 = T_p + T_s, \quad (4.19)$$

where

$$T_p = (\eta_u^{n+1} - \eta_u^n + \zeta^n, \xi_u^{n+1}) + \tau[c_1(\eta_q^n, \xi_u^{n+1}) + c_2(\eta_p^n, \xi_u^{n+1}) + c_3\mathcal{H}^+(\eta_p^n, \xi_u^{n+1})], \quad (4.20)$$

and

$$T_s = \tau\mathcal{H}(\xi_\psi^n; \xi_u^{n+1}). \quad (4.21)$$

In (4.19), we have used the property

$$\mathcal{L}(\xi_r, \xi_u) = -c_4\|\xi_p\|^2, \quad (4.22)$$

which is similar to (2.11).

Next we will estimate  $T_p$  and  $T_s$ , let's begin with the term  $V \doteq c_3\tau\mathcal{H}^+(\eta_p^n, \xi_u^{n+1})$  in  $T_p$ . We can prove that

$$|V| \leq Cc_3h^{k+1}\tau(\|\xi_u^{n+1}\|_x + \sqrt{\mu h^{-1}}\|\xi_u^{n+1}\|)$$

similarly as (2.12b), where we need to use the property  $\|\eta_p^n\| + h^{1/2}\|\eta_p^n\|_{\partial\mathcal{T}_h} \leq Ch^{k+1}$ . Furthermore, we can also prove the property

$$\|\xi_u^{n+1}\|_x + \sqrt{\mu h^{-1}}\|\xi_u^{n+1}\| \leq C_\mu\|\xi_q^{n+1}\| \leq C_pC_\mu^2\|\xi_p^{n+1}\|,$$

along the proof line of (2.13) and Corollary 2.8. Hence

$$|V| \leq Cc_3C_pC_\mu^2h^{k+1}\tau\|\xi_p^{n+1}\|.$$

Then a simple use of the Cauchy-Schwarz inequality, the approximation properties (4.12), (4.13) and the Young's inequality yields

$$\begin{aligned} |T_p| &\leq \varepsilon\tau\|\xi_u^{n+1}\|^2 + \varepsilon\tau\|\xi_p^{n+1}\|^2 + C(h^{2k+2}\tau + \tau^3) \\ &\leq 2\varepsilon\tau(\|\xi_u^n\|^2 + \|\xi_u^{n+1} - \xi_u^n\|^2) + \varepsilon\tau\|\xi_p^{n+1}\|^2 + C(h^{2k+2}\tau + \tau^3), \end{aligned} \quad (4.23)$$

TABLE 3. The maximum value of  $\tau_0$  such that the schemes are strongly stable in the  $L^2$  norm of  $u$ .

		Second order ( $k = 1$ )		Third order ( $k = 2$ )	
$c_4 \backslash c_1$		2	4	2	4
	$c_4$				
1		0.938	0.354	1.321	0.536
	2	1.207	0.469	1.96	0.660
$c_4 \backslash c_2$		-2	-4	-2	-4
	$c_4$				
1		0.380	0.090	1.604	0.632
	2	0.796	0.190	5.572	0.802
$c_4 \backslash c_3$		2	4	2	4
	$c_4$				
1		0.0078	0.0004	0.123	0.0073
	2	0.0627	0.0032	1.290	0.0616

for arbitrary positive  $\varepsilon$ , where the triangle inequality is used in the last inequality.

Proceeding as in the proof of Proposition 3.1 for  $R_a, R_b, R_c$ , we get

$$|T_s| \leq \varepsilon\tau(\|\xi_p^{n+1}\|^2 + \|\xi_p^n\|^2) + \frac{C'_0}{2}\tau\|\xi_u^n\|^2 + \frac{C_1}{2}\tau\|\xi_u^{n+1} - \xi_u^n\|^2, \tag{4.24}$$

where  $C'_0 = 2(\frac{c_2^2}{\varepsilon} + \frac{c_3^4}{8\varepsilon^3})$  and  $C_1 = 2(\frac{c_1^2 C_p^2 C_\mu^4}{2\varepsilon} + \frac{c_3^4}{8\varepsilon^3})$ . Therefore, if we take  $\varepsilon = \frac{c_4}{3}$  and let  $(C_1 + 4\varepsilon)\tau \leq 1$ , i.e.,  $\tau \leq \frac{1}{C_1 + 4\varepsilon}$ , then

$$(\|\xi_u^{n+1}\|^2 + \frac{2}{3}c_4\tau\|\xi_p^{n+1}\|^2) - (\|\xi_u^n\|^2 + \frac{2}{3}c_4\tau\|\xi_p^n\|^2) \leq C_0\tau\|\xi_u^n\|^2 + C(h^{2k+2}\tau + \tau^3), \tag{4.25}$$

where  $C_0 = C'_0 + 4\varepsilon$ . Hence by the discrete Gronwall's inequality we have

$$\|\xi_u^n\| \leq C(h^{k+1} + \tau), \quad \forall n\tau \leq T. \tag{4.26}$$

Finally by (4.12), (4.26) and the triangle inequality, we obtain (4.2) with  $s = 1$  in Theorem 4.1.

### 5. NUMERICAL EXPERIMENTS

The purpose of this section is to numerically test the stability and error accuracy for the second and third order Runge–Kutta IMEX-LDG schemes (3.22) and (3.30), for solving linear and nonlinear time-dependent fourth order problems. We will first consider the equation (2.1) with the exact solution

$$U(x, t) = e^{(c_2 - c_4)t} \sin(x - (c_1 - c_3)t), \tag{5.1}$$

in the interval  $[-\pi, \pi]$ , with periodic boundary condition.

Firstly we test the stability of the schemes, we will test the effect of explicit discretization of the lower order derivative terms separately, for example, when we test the effect of the first order derivative term, we let  $c_2 = c_3 = 0$ . In Table 3 we list the maximum value of  $\tau_0$  such that the schemes are strongly stable in the  $L^2$  norm of  $u$ , that is,  $\|u^n\|$  is non-increasing for every time step during the evolution. Since the schemes are exponentially stable for some positive  $c_2$ , we only test the case  $c_2 < 0$ . The results show that, if  $c_1, c_2$  and  $c_3$  are larger in magnitude, then  $\tau_0$  is smaller, and if  $c_4$  is larger,  $\tau_0$  is larger. We can also observe that the coefficient  $c_3$  affects  $\tau_0$  most sensitively. In addition,  $\tau_0$  is independent of the mesh size because it does not change if we refine the mesh.

TABLE 4. The second and third order Runge–Kutta IMEX-LDG scheme for solving (2.1).

scheme	$N$	$c_2 = 1$				$c_2 = -1$				
		$L^\infty$ error	$L^\infty$ order	$L^2$ error	$L^2$ order	$L^\infty$ error	$L^\infty$ order	$L^2$ error	$L^2$ order	
(3.22)	40	3.99E-03	–	2.83E-03	–	1.84E-03	–	2.30E-03	–	
	80	1.01E-03	1.98	6.78E-04	2.06	4.36E-04	2.07	5.34E-04	2.11	
	with	160	2.55E-04	1.99	1.68E-04	2.01	1.07E-04	2.03	1.29E-04	2.05
	$k = 1$	320	6.41E-05	1.99	4.19E-05	2.00	2.65E-05	2.01	3.20E-05	2.01
	640	1.73E-05	1.89	1.07E-05	1.97	6.74E-06	1.97	8.21E-06	1.96	
(3.30)	10	3.91E-03	–	2.17E-03	–	4.51E-03	–	8.03E-03	–	
	20	5.12E-04	2.93	2.74E-04	2.99	7.34E-04	2.62	1.30E-03	2.63	
	with	40	6.44E-05	2.99	3.42E-05	3.00	9.33E-05	2.98	1.65E-04	2.98
	$k = 2$	80	8.07E-06	3.00	4.28E-06	3.00	1.21E-05	2.95	2.14E-05	2.95
	160	1.01E-06	3.00	5.36E-07	3.00	1.55E-06	2.96	2.75E-06	2.96	

TABLE 5. The second and third order Runge–Kutta IMEX-LDG schemes for solving (5.2).

$N$	second order, $k = 1, \tau = 0.0004$				third order, $k = 2, \tau = 0.001$			
	$L^\infty$ error	$L^\infty$ order	$L^2$ error	$L^2$ order	$L^\infty$ error	$L^\infty$ order	$L^2$ error	$L^2$ order
100	6.54E-01	–	8.36E-01	–	5.56E-02	–	2.19E-02	–
200	1.36E-01	2.27	1.28E-01	2.70	7.22E-03	2.95	2.67E-03	3.04
400	3.90E-02	1.80	2.36E-02	2.44	9.38E-04	2.95	3.33E-04	3.00
800	1.06E-02	1.88	5.18E-03	2.19	1.20E-04	2.97	4.17E-05	3.00
1600	2.76E-03	1.94	1.24E-03	2.06	1.54E-05	2.96	5.61E-06	2.90

In Table 4, we list the  $L^\infty$  and  $L^2$  errors and orders of accuracy for the IMEX-LDG schemes (3.22) and (3.30) for solving (2.1) on uniform meshes. In this experiment, we take  $c_1 = c_3 = c_4 = 1$  for simplicity. We take  $\tau = h$  and the final computing time  $T = 1$  in all the tests, where  $h = 2\pi/N$  with  $N$  the number of elements. We can clearly observe the designed orders of accuracy from the table.

Next we would like to consider the generalized Kuramoto–Sivashinsky equation considered in [22]

$$U_t + UU_x + U_{xx} + \sigma U_{xxx} + U_{xxxx} = 0, \quad (5.2)$$

with the exact solution

$$U(x, t) = 15 - 15 \left( \tanh\left(\frac{1}{2}\kappa\right) + \tanh^2\left(\frac{1}{2}\kappa\right) - \tanh^3\left(\frac{1}{2}\kappa\right) \right), \quad (5.3)$$

for  $\sigma = 4$ , where  $\kappa = x - c_0 t$  with  $c_0$  the speed of the wave, here  $c_0 = 6$ . The computational domain is  $[-30, 30]$ , and the final computing time is  $T = 1$ . In Table 5, we present the  $L^\infty$  and  $L^2$  errors and orders of accuracy for the second and third order IMEX-LDG schemes for solving (5.2) on uniform meshes. In this experiment, we take small time step in order to ensure the stability of the schemes. From the table, we can observe the optimal order of accuracy for the two schemes when solving nonlinear problems.

## 6. CONCLUDING REMARKS

Three specific implicit-explicit Runge–Kutta time marching methods coupled with the LDG schemes are considered for solving one dimensional time-dependent linear fourth order equations. We have performed the stability analysis for these IMEX-LDG methods, which shows that the schemes are stable under the time step restriction  $\tau \leq \tau_0$ , where the constant  $\tau_0$  is independent of the mesh size  $h$ , but is dependent of the coefficients of  $c_1, c_2, c_3$  and  $c_4$ . The optimal error estimates for the linear problem was obtained by the aid of the elliptic

projection and adjoint argument. The analysis in this paper can be easily extended to multi-dimensional time-dependent fourth order problems. The numerical experiments indicates that the IMEX-LDG schemes can achieve optimal error accuracy. In our future work, we would like to study the corresponding analysis for nonlinear fourth order problems.

APPENDIX A.

In the appendix, we will give the proof for Lemma 4.2. Before doing that, we first introduce two special projections, namely, the Gauss-Radau projections, denoted by  $\pi_h^-$  and  $\pi_h^+$ , respectively. For any function  $p \in H^1(\mathcal{T}_h) = \{\phi \in L^2(\Omega) : \phi|_{I_j} \in H^1(I_j), \forall j = 1, \dots, N\}$ , the projection  $\pi_h^\pm p$  is defined as the unique element in  $V_h$  such that, in each element  $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$

$$(\pi_h^- p - p, v)_{I_j} = 0, \quad \forall v \in \mathcal{P}_{k-1}(I_j), \quad (\pi_h^- p)_{j+\frac{1}{2}}^- = p_{j+\frac{1}{2}}^-; \tag{A.1a}$$

$$(\pi_h^+ p - p, v)_{I_j} = 0, \quad \forall v \in \mathcal{P}_{k-1}(I_j), \quad (\pi_h^+ p)_{j-\frac{1}{2}}^+ = p_{j-\frac{1}{2}}^+. \tag{A.1b}$$

By a standard scaling argument [7], it is easy to obtain the following approximation property

$$\|w - \pi_h^\pm w\| + h\|(w - \pi_h^\pm w)_x\| + h^{1/2}\|w - \pi_h^\pm w\|_{\partial\mathcal{T}_h} \leq Ch^{\min(k+1,s)}\|w\|_{H^s(\Omega)}, \tag{A.2}$$

for any  $w \in H^s(\Omega)$ , where the bounding constant  $C > 0$  is independent of  $h$ .

Besides, from the definition of  $\mathcal{H}^\pm$  we have

$$\mathcal{H}^\pm(\pi_h^\pm w - w, v) = 0, \tag{A.3}$$

for any  $w \in H^1(\mathcal{T}_h)$  and  $v \in V_h$ .

Based on the Gauss-Radau projection, we can divide the error  $\eta_\varsigma$  ( $\varsigma = u, q, p, r$ ) into

$$\begin{aligned} \eta_u &= U - \pi_h^- U + \pi_h^- \eta_u, & \eta_q &= Q - \pi_h^+ Q + \pi_h^+ \eta_q, \\ \eta_p &= P - \pi_h^- P + \pi_h^- \eta_p, & \eta_r &= R - \pi_h^+ R + \pi_h^+ \eta_r. \end{aligned}$$

Then from (4.17) and by (A.3) we get

$$0 = \mathcal{H}^+(\pi_h^+ \eta_r, v), \tag{A.4a}$$

$$(\eta_r, \rho) = -\mathcal{H}^-(\pi_h^- \eta_p, \rho), \tag{A.4b}$$

$$(\eta_p, \phi) = -\mathcal{H}^+(\pi_h^+ \eta_q, \phi), \tag{A.4c}$$

$$(\eta_q, \psi) = -\mathcal{H}^-(\pi_h^- \eta_u, \psi). \tag{A.4d}$$

First taking  $\rho = \pi_h^+ \eta_r$  in (A.4b) and by (2.10) and (A.4a) we have

$$(\eta_r, \pi_h^+ \eta_r) = -\mathcal{H}^-(\pi_h^- \eta_p, \pi_h^+ \eta_r) = \mathcal{H}^+(\pi_h^+ \eta_r, \pi_h^- \eta_p) = 0.$$

Hence

$$\|\pi_h^+ \eta_r\|^2 = (\pi_h^+ \eta_r, \pi_h^+ \eta_r - \eta_r) = (\pi_h^+ \eta_r, \pi_h^+ R - R).$$

The simple use of Cauchy-Schwarz inequality and (A.2) leads to

$$\|\pi_h^+ \eta_r\| \leq Ch^{k+1}. \tag{A.5}$$

And hence

$$\|\eta_r\| \leq Ch^{k+1}. \tag{A.6}$$

Next taking  $\phi = \pi_h^- \eta_p$  in (A.4c) and by (2.10) and (A.4b) we get

$$(\eta_p, \pi_h^- \eta_p) = -\mathcal{H}^+(\pi_h^+ \eta_q, \pi_h^- \eta_p) = \mathcal{H}^-(\pi_h^- \eta_p, \pi_h^+ \eta_q) = -(\eta_r, \pi_h^+ \eta_q).$$

Then

$$\|\pi_h^- \eta_p\|^2 = (\eta_p, \pi_h^- \eta_p) + (\pi_h^- \eta_p - \eta_p, \pi_h^- \eta_p) = -(\eta_r, \pi_h^+ \eta_q) + (\pi_h^- P - P, \pi_h^- \eta_p).$$

Hence by Cauchy–Schwarz inequality, the Young’s inequality, (A.2) and (A.6) we can get

$$\|\pi_h^- \eta_p\|^2 \lesssim \|\pi_h^+ \eta_q\|^2 + h^{2k+2}, \quad (\text{A.7})$$

here and below the notation  $a \lesssim b$  means that, there exists a positive constant  $C$  such that  $a \leq Cb$ . Hence by the triangle inequality we get

$$\|\eta_p\| \lesssim \|\pi_h^+ \eta_q\| + h^{k+1}. \quad (\text{A.8})$$

Finally, we take  $\psi = \pi_h^+ \eta_q$  in (A.4d), then by (2.10) and (A.4c) we get

$$(\eta_q, \pi_h^+ \eta_q) = -\mathcal{H}^-(\pi_h^- \eta_u, \pi_h^+ \eta_q) = \mathcal{H}^+(\pi_h^+ \eta_q, \pi_h^- \eta_u) = -(\eta_p, \pi_h^- \eta_u).$$

Then

$$\|\pi_h^+ \eta_q\|^2 = (\eta_q, \pi_h^+ \eta_q) + (\pi_h^+ \eta_q - \eta_q, \pi_h^+ \eta_q) = -(\eta_p, \pi_h^- \eta_u) + (\pi_h^+ Q - Q, \pi_h^+ \eta_q).$$

Hence by Cauchy–Schwarz inequality, the Young’s inequality, (A.2) and (A.8) we can get

$$\|\pi_h^+ \eta_q\|^2 \lesssim \|\pi_h^- \eta_u\|^2 + h^{2k+2}, \quad (\text{A.9})$$

and hence

$$\|\pi_h^+ \eta_q\| \lesssim \|\pi_h^- \eta_u\| + h^{k+1}. \quad (\text{A.10})$$

So from (A.8) and by the triangle inequality we have

$$\|\eta_p\| \lesssim \|\pi_h^- \eta_u\| + h^{k+1}, \quad \text{and} \quad \|\eta_q\| \lesssim \|\pi_h^- \eta_u\| + h^{k+1}. \quad (\text{A.11})$$

In what follows, we will give the estimate for  $\eta_u$  by the aid of the adjoint problem (4.4). To this end, we give the following lemma.

**Lemma A.1.** *Given  $z \in L^2(\Omega)$ , we have*

$$\begin{aligned} (\pi_h^- \eta_u, z) &= (\eta_q, \zeta - \pi_h^+ \zeta) - (Q - \pi_h^+ Q, (\omega - \pi_h^- \omega)_x) \\ &\quad - (\eta_p, \omega - \pi_h^- \omega) + (P - \pi_h^- P, (\theta - \pi_h^+ \theta)_x) \\ &\quad + (\eta_r, \theta - \pi_h^+ \theta) - (R - \pi_h^+ R, (\sigma - \pi_h^- \sigma)_x), \end{aligned} \quad (\text{A.12})$$

where  $\theta, \omega, \zeta$  is the solution of the elliptic problem (4.4).

*Proof.* First by the adjoint problem (4.4) we have

$$(\pi_h^- \eta_u, z) = (\pi_h^- \eta_u, \zeta_x) = \sum_{j=1}^N \{(\pi_h^- \eta_u, (\zeta - \pi_h^+ \zeta)_x)_j + (\pi_h^- \eta_u, (\pi_h^+ \zeta)_x)_j\}.$$

Then integrating by parts and by the definition of  $\pi_h^+$  we have

$$\begin{aligned}
 (\pi_h^- \eta_u, z) &= \sum_{j=1}^N \{(\zeta - \pi_h^+ \zeta)_{j+\frac{1}{2}}^- (\pi_h^- \eta_u)_{j+\frac{1}{2}}^- + (\pi_h^- \eta_u, (\pi_h^+ \zeta)_j)_j\} \\
 &= \mathcal{H}^-(\pi_h^- \eta_u, \pi_h^+ \zeta) + \sum_{j=1}^N \{-(\pi_h^- \eta_u)_{j-\frac{1}{2}}^- (\pi_h^+ \zeta)_{j-\frac{1}{2}}^+ + \zeta_{j+\frac{1}{2}}^- (\pi_h^- \eta_u)_{j+\frac{1}{2}}^-\} \\
 &= -(\eta_q, \pi_h^+ \zeta) + \sum_{j=1}^N (\pi_h^- \eta_u)_{j-\frac{1}{2}}^- [(\zeta - \pi_h^+ \zeta)_{j-\frac{1}{2}}^+ - \llbracket \zeta \rrbracket_{j-\frac{1}{2}}] \\
 &= -(\eta_q, \zeta) + (\eta_q, \zeta - \pi_h^+ \zeta), \tag{A.13}
 \end{aligned}$$

where the third line holds by (A.4d) and the periodic boundary condition, the last equation holds by the definition of  $\pi_h^+$  and the fact that  $\llbracket \zeta \rrbracket = 0$ . Next

$$\begin{aligned}
 -(\eta_q, \zeta) &= -(\eta_q, \omega_x) = -(Q - \pi_h^+ Q + \pi_h^+ \eta_q, \omega_x) \\
 &= -(Q - \pi_h^+ Q, (\omega - \pi_h^- \omega)_x) - (\pi_h^+ \eta_q, \omega_x). \tag{A.14}
 \end{aligned}$$

Similarly as (A.13) we can get

$$-(\pi_h^+ \eta_q, \omega_x) = -\mathcal{H}^+(\pi_h^+ \eta_q, \pi_h^- \omega) = (\eta_p, \pi_h^- \omega) = (\eta_p, \omega) - (\eta_p, \omega - \pi_h^- \omega). \tag{A.15}$$

Also similarly

$$\begin{aligned}
 (\eta_p, \omega) &= (\eta_p, \theta_x) = (P - \pi_h^- P + \pi_h^- \eta_p, \theta_x) \\
 &= (P - \pi_h^- P, (\theta - \pi_h^+ \theta)_x) + (\pi_h^- \eta_p, \theta_x). \tag{A.16}
 \end{aligned}$$

$$(\pi_h^- \eta_p, \theta_x) = \mathcal{H}^-(\pi_h^- \eta_p, \pi_h^+ \theta) = -(\eta_r, \pi_h^+ \theta) = -(\eta_r, \theta) + (\eta_r, \theta - \pi_h^+ \theta). \tag{A.17}$$

And

$$-(\eta_r, \theta) = -(\eta_r, \sigma_x) = -(R - \pi_h^+ R + \pi_h^+ \eta_r, \sigma_x) = -(R - \pi_h^+ R, (\sigma - \pi_h^- \sigma)_x) \tag{A.18}$$

since

$$(\pi_h^+ \eta_r, \sigma_x) = \mathcal{H}^+(\pi_h^+ \eta_r, \pi_h^- \sigma) = 0. \tag{A.19}$$

Hence we get (A.12) by combining the above estimates. □

At the end, taking  $z = \pi_h^- \eta_u$  in (A.12) and by the Cauchy–Schwarz inequality and (A.2) we get

$$\begin{aligned}
 \|\pi_h^- \eta_u\|^2 &\leq C[\|\eta_q\| h^{\min\{1, k+1\}} \|\zeta\|_{H^1} + \|\eta_p\| h^{\min\{2, k+1\}} \|\omega\|_{H^2} + \|\eta_r\| h^{\min\{3, k+1\}} \|\theta\|_{H^3}] \\
 &\quad + Ch^{k+1} [h^{\min\{1, k\}} \|\omega\|_{H^2} + h^{\min\{2, k\}} \|\theta\|_{H^3} + h^{\min\{3, k\}} \|\sigma\|_{H^4}] \\
 &\leq Ch(\|\eta_q\| + \|\eta_p\| + \|\eta_r\| + h^{k+1})(\|\zeta\|_{H^1} + \|\omega\|_{H^2} + \|\theta\|_{H^3} + \|\sigma\|_{H^4}) \\
 &\leq CC_{er} h(\|\eta_q\| + \|\eta_p\| + \|\eta_r\| + h^{k+1}) \|\pi_h^- \eta_u\|,
 \end{aligned}$$

if  $k \geq 1$ , where the elliptic regularity (4.5) is used in the last inequality. Then from (A.6), (A.11) and Young’s inequality we have

$$\|\pi_h^- \eta_u\|^2 \leq Ch \|\pi_h^- \eta_u\|^2 + \varepsilon \|\pi_h^- \eta_u\|^2 + C_\varepsilon h^{2(k+2)}, \tag{A.20}$$

for arbitrary  $\varepsilon$ . Hence if  $h \leq h_0$  such that  $Ch \leq \frac{1}{2}$  and  $\varepsilon$  is small enough, then

$$\|\pi_h^- \eta_u\| \leq Ch^{k+2}. \tag{A.21}$$

As a result we obtain (4.7) by the triangle inequality and (A.2).



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