NUMERICAL SIMULATION OF WAVE PROPAGATION IN INHOMOGENEOUS MEDIA USING GENERALIZED PLANE WAVES

LISE-MARIE IMBERT-GÉRARD[1] AND PETER MONK[2]

Abstract. The Trefftz Discontinuous Galerkin (TDG) method is a technique for approximating the Helmholtz equation (or other linear wave equations) using piecewise defined local solutions of the equation to approximate the global solution. When coefficients in the equation (for example, the refractive index) are piecewise constant it is common to use plane waves on each element. However when the coefficients are smooth functions of position, plane waves are no longer directly applicable. In this paper we show how Generalized Plane Waves (GPWs) can be used in a modified TDG scheme to approximate the solution for piecewise smooth coefficients in two dimensions. GPWs are approximate solutions to the equation that reduce to plane waves when the medium through which the wave propagates is constant. We shall show how to modify the TDG sesquilinear form to allow us to prove convergence of the GPW based version. The new scheme retains the high order convergence of the original TDG scheme (when the solution is smooth) and also retains the same number of degrees of freedom per element (corresponding to the directions of the GPWs). Unfortunately it looses the advantage that only skeleton integrals need to be performed. Besides proving convergence, we provide numerical examples to test our theory.

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1. INTRODUCTION

The Trefftz Discontinuous Galerkin (TDG) method proposed in [7] is a mesh based method for approximating solutions of the Helmholtz equation. This method generalizes the Ultra Weak Variational Formulation (UWVF) of the same problem [4, 5] by allowing different weighting strategies on penalty terms in the TDG method. Error analysis [3, 7–10] and computational experience [11] show that the method can be an efficient way of approximating solutions of the Helmholtz equation. It has also become clear that the method works best in an hp-mode (see [10]) where large elements are used away from boundaries in the computational domain, together with larger numbers of plane waves.

However because of the use of simple Trefftz functions (usually plane waves element by element), it has to be assumed that the coefficients in the governing partial differential equation are piecewise constant. Of course smoothly varying coefficient functions could be first approximated by a piecewise constant function

and then the resulting perturbed Helmholtz equation could be solved by TDG or UWVF. But this would require small elements and hence defeat some of the potential advantages of using large elements in a Trefftz based scheme.

To circumvent the difficulty with smoothly varying coefficients, we propose to use approximate solutions of the underlying partial differential equation constructed element by element where the coefficients are variable. In this work we use the Generalized Plane Waves (GPW) of [13, 14] as a basis for the TDG type scheme. The basic derivation and approximation theory for GPWs can be found in [13, 14]. In this paper the novelty is to use these elements in a provably convergent numerical scheme. In order to do this we have to prove new approximation results for how well the equation is satisfied (Thm. 3.3) as well estimates for the approximation of piecewise linear functions by GPWs (starting with the basic Lem. 3.1 through Cor. 3.1).

Note that smoothly varying coefficients arise in the simulation of electromagnetic wave propagation in tokamaks where the permittivity is spatially variable and may even become negative. Indeed the original design of GPWs in [13, 14] was motivated precisely by this application.

To describe the setting for applying GPWs in more detail let us consider the following two dimensional model problem from [9]. Given a bounded Lipschitz polyhedron $\Omega \subset \mathbb{R}^2$ that is star shaped with respect to the origin and a larger Lipschitz polyhedron Ω_R containing $\overline{\Omega_D}$, we define the computational domain to be the annulus $\Omega = \Omega_R \setminus \overline{\Omega_D}$. The two boundaries of Ω are $\Gamma_D = \partial \Omega_D$ and $\Gamma_R = \partial \Omega_R$ (so that $\mathrm{dist}(\Gamma_D, \Gamma_R) > 0$) and we use a normal $\mathbf{n}$ that is outwards from Ω. Because we shall use some regularity results from [9] we need to assume that Ω_R is star-like with respect to a ball of radius $\gamma_R d_\Omega$ centered at the origin where $\gamma_R > 0$ and d_Ω is the diameter of Ω.

Suppose we are given a wave number $\kappa > 0$. In addition given a strictly positive, piecewise smooth and bounded real function $\epsilon \in L^\infty(\Omega)$ and another function $g \in L^2(\Gamma_R)$, we want to approximate the solution u of

$$\Delta u + \kappa^2 \epsilon u = 0 \qquad \text{in} \qquad \Omega, \tag{1.1}$$

$$u = 0 \qquad \text{on} \qquad \Gamma_D, \tag{1.2}$$

$$\partial_n u + \mathrm{i}\kappa u = \mathrm{i}\kappa g \qquad \text{on} \qquad \Gamma_R. \tag{1.3}$$

As pointed out in [9] this is a model problem for scattering (for example of an s-polarized electromagnetic wave from a perfect conductor embedded in a dielectric in two dimensions). The impedance boundary condition is then a simple radiation boundary condition.

As we shall detail shortly, if we assume that the function ϵ is analytic on each element, we can approximate it by a power series. With this in hand we shall give details of a recursive algorithm for generating the coefficients of basis functions on each element that satisfy (1.1) to high accuracy. These are constructed so that if the coefficient ϵ is constant on an element, the resulting basis function is just a plane wave. Using these generalized plane waves we can prove convergence of a modified TDG scheme. The resulting discrete problem obtains high order convergence for smooth solutions as is the case for the standard TDG or UWVF. We only consider h-convergence in this paper, and the analysis only pertains to two dimensional domains.

As we shall see, the main disadvantage of the use of GPWs in the TDG method is the need to integrate over elements in the grid. We would prefer to use them in a generalized UWVF avoiding this integration. However, although numerical experiments are encouraging [12], we do not have a theoretical justification of this approach.

The paper proceeds as follows. In the next section we briefly outline our modification of the basic TDG method. Then in Section 3 we show how to construct GPWs and then obtain two new error estimates for these functions that will underlie our error analysis. We also show that piecewise linear functions can be approximated element by element using the GPW functions. In Section 4 we derive error estimates for the new TDG scheme with GPW basis functions. Finally in Section 5 we give some basic numerical tests of the new algorithm.

2. THE PLANE WAVE DISCONTINUOUS GALERKIN METHOD

Even with a variable coefficient ϵ, the choice of domain Ω and the conditions on ϵ guarantee that if $g \in H^s(\Gamma_R)$ for some sufficiently small $s > 0$ (depending on the interior angles of Γ_R) then $u \in H^{3/2+s}(\Omega)$ ([9], Thm. 2.3).

This regularity will allow us to develop consistent fluxes for u and its derivatives. Unfortunately, because ϵ is variable, the dependence on κ of the continuity constants for estimates in this paper is not easy to track. Therefore we note now that constants in the analysis will depend in an unspecified way on κ.

As is usual for DG schemes, we start with a mesh and continue to define the method using definitions from [9]. Suppose we cover Ω by a finite element mesh $\mathcal{T}_h$ of regular triangular elements K of maximum diameter h (in fact more general domains can easily be allowed). The diameter of an element K is denoted h_K. To simplify the presentation we have used parameter choices corresponding to the UWVF later in the paper, so we assume that the mesh is also quasi-uniform. Thus there exists a constant $\tau > 0$ such that

$$\frac{h}{h_K} \leq \tau$$

for all elements $K \in \mathcal{T}_h$ and all sufficiently small $h > 0$.

Let $\mathbf{n}_K$ denote the unit outward normal to element K. Let K and K' denote two elements in $\mathcal{T}_h$ meeting at an edge e. Then, on e, we make the standard definitions of the average value and jump of functions across e of a function u and a vector field σ:

$$\{\!\!\{u\}\!\!\} = \frac{u|_K + u|_{K'}}{2}, \qquad \{\!\!\{\sigma\}\!\!\} = \frac{\sigma|_K + \sigma|_{K'}}{2},$$
$$[\![u]\!] = u|_K \mathbf{n}_K + u|_{K'} \mathbf{n}_{K'}, \qquad [\![\sigma]\!] = \sigma|_K \cdot \mathbf{n}_K + \sigma|_{K'} \cdot \mathbf{n}_{K'}.$$

We denote by $\mathcal{E}_h$ the set of all edges in the mesh. Then let

- $\mathcal{E}_I$ denote the set of all edges in the mesh interior to Ω,
- $\mathcal{E}_D$ is the set of all boundary edges on Γ_D,
- $\mathcal{E}_R$ is the set of all boundary edges on Γ_R.

We also need three positive penalty parameters that are functions of position on the skeleton of the mesh: α, β and δ. At this point these are simply assumed to be positive functions of position on $\mathcal{E}_h$ and will be given in more detail shortly. Using the above defined jumps and average values, we are lead to consider the following standard sesquilinear form for TDG [6, 9, 15]:

$$A_h(u, v) = \int_\Omega \left(\nabla_h u \cdot \nabla_h \overline{v} - \kappa^2 \epsilon u \overline{v} \right) \mathrm{d}A - \int_{\mathcal{E}_I} \left(\{\!\!\{\nabla_h u\}\!\!\} \cdot [\![\overline{v}]\!] + [\![u]\!] \cdot \{\!\!\{\nabla_h \overline{v}\}\!\!\} \right) \mathrm{d}s$$
$$- \frac{1}{\mathrm{i}\kappa} \int_{\mathcal{E}_I} \beta [\![\nabla_h u]\!] [\![\nabla_h \overline{v}]\!] \, \mathrm{d}s + \mathrm{i}\kappa \int_{\mathcal{E}_I} \alpha [\![u]\!] \cdot [\![\overline{v}]\!] \, \mathrm{d}s - \int_{\mathcal{E}_R} \delta u \partial_n \overline{v} \, \mathrm{d}s$$
$$- \int_{\mathcal{E}_R} \delta \partial_n u \overline{v} \, \mathrm{d}s - \frac{1}{\mathrm{i}\kappa} \int_{\mathcal{E}_R} \delta \partial_n u \partial_n \overline{v} \, \mathrm{d}s + \mathrm{i}\kappa \int_{\mathcal{E}_D} \alpha u \overline{v} \, \mathrm{d}s$$
$$+ \mathrm{i}\kappa \int_{\mathcal{E}_R} (1 - \delta) u \overline{v} \, \mathrm{d}s - \int_{\mathcal{E}_D} (\partial_n u \overline{v} + u \partial_n \overline{v}) \, \mathrm{d}s. \tag{2.1}$$

Here ∇_h is the piecewise defined gradient and $\partial_n u = \nabla_h u \cdot n$ element by element. In addition the right hand side is given by

$$F(v) = -\frac{1}{\mathrm{i}\kappa} \int_{\mathcal{E}_R} \delta g \, \partial_n \overline{v} \, \mathrm{d}s + \int_{\mathcal{E}_R} (1 - \delta) g \overline{v} \, \mathrm{d}s$$

By virtue of the regularity of the solution of (1.1)−(1.3) noted above, it satisfies

$$A_h(u, v) = F(v)$$

for all sufficiently smooth test functions v (for example piecewise H^2 is sufficient).

Now suppose we wish to discretize the problem. Let $V_h \subset \Pi_{K \in T_h} H^2(K)$ be a finite dimensional space. If V_h is chosen to consist of piecewise smooth solutions of (1.1), we have the standard TDG and seek an approximate $u_h \in V_h$ that satisfies

$$A_h(u_h, v) = F(v) \quad \text{for all } v \in V_h.$$

For piecewise constant media, the space V_h can be chosen in many ways. One choice uses Bessel functions (good for conditioning but bad for computational speed because of the need for quadrature), another, more standard choice, uses plane waves (typically worse conditioned but easier to use since integrals can be computed in closed form) [16].

However if $\epsilon(x, y)$ is non-constant on an element we cannot use simple solutions of the Helmholtz equation. In this paper we assume that $\epsilon(x, y)$ is a smooth function on each element (but may be discontinuous between elements). Then, as we shall shortly describe, the space V_h can be constructed using "Generalized Plane Waves" (GPWs) that approximately satisfy the Helmholtz equation. However we have been unable to prove convergence for the standard TDG method in this case, and instead add a stabilizing term to the sesquilinear form so define

$$B_h(u, v) = A_h(u, v) + \frac{\mathrm{i}}{\kappa^2} \int_\Omega \gamma (\Delta u + \kappa^2 \epsilon u) \overline{(\Delta v + \kappa^2 \epsilon v)} \, \mathrm{d}A.$$

Here $\gamma > 0$ is a new penalty parameter that is a piecewise constant function of position on the mesh. We note that the new term vanishes on elements with constant material coefficients allowing plane waves to be used and the method reduces to the standard TDG. Now we seek $u_h \in V_h$ such that

$$B_h(u_h, v) = F(v) \quad \forall v \in V_h. \tag{2.2}$$

In the next section we describe how to construct GPWs element by element and hence complete the specification of the method.

3. GENERALIZED PLANE WAVES

In this section we focus specifically on GPWs. We will use the following notation:

(1) the parameter p is the dimension of this local approximation space (the number of GPWs per element),
(2) the parameter q is the order of approximation of the Taylor expansion for each GPW.

Firstly we describe the design process, including an explicit algorithm to build a local set of GPWs on a given element of the mesh $\mathcal{T}_h$. Secondly we turn to interpolation of a solution of (1.1) and prove error estimates. We provide various interpolation properties of such a set of local GPWs on a given element of the mesh, and derive a global interpolation property on the whole domain Ω by piecewise GPWs. Thirdly we prove a result on approximation of the space of bi-variate polynomials of degree 1 by GPWs. This result will be useful for the error analysis.

The design procedure together with Theorem 3.1 were already presented in [13], but are repeated here for the sake of completeness.

3.1. Design and interpolation properties

GPWs have been introduced in [12, 14]. They generalize the use of classical plane waves, as exact solutions of an equation with piecewise constant coefficients, to the case of variable coefficients. The GPWs are not exact solutions of (1.1) but approximately solve the equation element by element. Their design process is based on a Taylor expansion and ensures that the homogeneous equation is locally satisfied up to a given order on each element K of the mesh.

On a given element K, consider the centroid (x_K, y_K). A GPW on K is a function $\varphi = \mathrm{e}^P$ where

$$P(x, y) = \sum_{i=0}^{\mathrm{d}_P} \sum_{j=0}^{\mathrm{d}_P - i} \lambda_{i,j} (x - x_K)^i (y - y_K)^j, \tag{3.1}$$

d_P being the total degree of the polynomial P. A GPW is designed to be an approximate solution of the Helmholtz equation: the polynomial coefficients $\{\lambda_{i,j}, 0 \leq i+j \leq d_P\}$ are computed from the Taylor expansion of the variable coefficient ϵ in order for the function $\varphi = e^P$ to satisfy

$$[\Delta + \kappa^2 \epsilon] e^{P(x,y)} = O\left(\|(x,y) - (x_K, y_K)\|^q\right). \tag{3.2}$$

The parameter q is the order of approximation of the equation. Canceling all the terms of order less than q in the Taylor expansion (3.2) is equivalent to a non linear system of $q(q+1)/2$ non linear equations. The unknowns of this system are the $(d_P + 1)(d_P + 2)/2$ coefficients of P. Setting simultaneously $d_P = q + 1$ and giving the values of the $2q + 3$ coefficients $\{\lambda_{i,j}, i \in \{0,1\}, 0 \leq j \leq q + 1 - i\}$ leads to a unique solution of the non linear system. This solution is explicitly expressed as

$$\forall (i,j) \text{ s.t. } 0 \leq i+j \leq q-1, \lambda_{i+2,j} = \frac{1}{(i+2)(i+1)}\Bigg(- \kappa^2 \frac{\partial_x^i \partial_y^j \epsilon (x_K, y_K)}{i!j!} - (j+2)(j+1)\lambda_{i,j+2}$$

$$- \sum_{k=0}^{i} \sum_{l=0}^{j} (i-k+1)(k+1)\lambda_{i-k+1,j-l}\lambda_{k+1,l}$$

$$- \sum_{k=0}^{j} \sum_{l=0}^{i} (j-k+1)(k+1)\lambda_{i-l,j-k+1}\lambda_{l,k+1} \Bigg), \tag{3.3}$$

where $\partial_x = \partial/\partial_x$ and $\partial_y = \partial/\partial_y$. More precisely, as defined in [13], a GPW at (x_K, y_K) corresponds to the following normalization:

- $\lambda_{0,0} = 0$,
- $(\lambda_{1,0}, \lambda_{0,1}) = N(\cos\theta, \sin\theta)$, for some $N \in \mathbb{C}$ and $\theta \in \mathbb{R}$,
- $\lambda_{i,j} = 0$ for $i \in \{0,1\}$ and $1 < i+j \leq q+1$.

A local set of linearly independent GPWs is then obtained for a given value of N by considering p equi-spaced directions $\theta_l = 2\pi(l-1)/p$ for $1 \leq l \leq p$. The interpolation properties of this set of functions are the main topic of [13]. The main result of that paper provides a sufficient condition on the parameters p and q to achieve a high order interpolation of a smooth solution of (1.1) by GPWs, as well as a high order interpolation of its gradient. We will denote by $GPW_\kappa^{p,q}(K)$ the space spanned by the p GPWs corresponding to $\theta_l = 2\pi(l-1)/p$ for $1 \leq l \leq p$ and $N = \sqrt{-\kappa^2 \epsilon(x_K, y_K)}$, or $N = i\kappa\sqrt{\epsilon(x_K, y_K)}$. Let $\mathcal{C}^k(S)$ denote the set of functions with k continuous derivatives on a set S. As a reminder, with the present notation, the interpolation result stated and proved in [13] reads:

Theorem 3.1. *Consider* $K \in \mathcal{T}_h$ *together with* $n \in \mathbb{N}^*$. *Assume that* $q \geq n+1$, $p = 2n+1$ *and* $\mathbf{g}_K = (x_K, y_K) \in K$ *is the centroid of* K. *Finally suppose the coefficient* $\epsilon \in \mathcal{C}^{q-1}(K)$. *Consider a solution* u *of scalar wave equation* (1.1), *satisfying* $u \in \mathcal{C}^{n+1}(K)$. *Then there is a function* $u_a \in GPW_\kappa^{p,q}(K)$ *implicitly depending on* ϵ *and its derivatives, and a constant* $C(\kappa, K, n)$, *implicitly depending on* ϵ *and its derivatives as well, such that: for all* $\mathbf{m} \in K$

$$\begin{cases} |u(\mathbf{m}) - u_a(\mathbf{m})| \leq C(\kappa, K, n) |\mathbf{m} - \mathbf{g}_K|^{n+1} \|u\|_{\mathcal{C}^{n+1}(K)}, \\ \|\nabla u(\mathbf{m}) - \nabla u_a(\mathbf{m})\| \leq C(\kappa, K, n) |\mathbf{m} - \mathbf{g}_K|^n \|u\|_{\mathcal{C}^{n+1}(K)}. \end{cases} \tag{3.4}$$

The following interpolation property of higher order derivatives stems directly from the proof of the previous theorem.

Theorem 3.2. *Consider* $K \in \mathcal{T}_h$ *together with* $n \in \mathbb{N}$ *such that* $n > 0$. *Assume that* $q \geq n+1$, $p = 2n+1$ *and* $\mathbf{g}_K = (x_K, y_K) \in K$ *is the centroid of* K. *Finally suppose the coefficient* $\epsilon \in \mathcal{C}^{q-1}(K)$. *Consider a solution*

u of scalar wave equation (1.1), *satisfying* $u \in \mathcal{C}^{n+1}(K)$. *Then the function* $u_a \in GPW_\kappa^{p,q}(K)$ *and the constant* $C(\kappa, K, n)$ *provided by Theorem* 3.1 *also satisfies: for all* $\mathbf{m} \in K$ *and all* j *such that* $0 \leq j \leq k$

$$\left| \partial_x^j \partial_y^{k-j} u\left(\mathbf{m}\right) - \partial_x^j \partial_y^{k-j} u_a\left(\mathbf{m}\right) \right| \leq C(\kappa, K, n) \frac{(n+1)!}{(n+1-k)!} \left|\mathbf{m} - \mathbf{g}_K\right|^{n+1-k} \|u\|_{\mathcal{C}^{n+1}(K)}, \tag{3.5}$$

where $k \leq n$. *Moreover there is a constant* $\mathfrak{C}(\kappa, K, n)$ *such that for all* $\mathbf{m} \in K$

$$\left| [\Delta + \kappa^2 \epsilon] u_a\left(\mathbf{m}\right) \right| \leq \mathfrak{C}(\kappa, K, n) \left|\mathbf{m} - \mathbf{g}_K\right|^q \|u\|_{\mathcal{C}^{n+1}(K)}. \tag{3.6}$$

Proof. The interpolation property of GPWs for any derivative of u directly stems from the Taylor expansion of $u - u_a$, exactly as for the gradient and this proves (3.5).

The design of GPWs directly yields that for all l such that $1 \leq l \leq p$ the corresponding GPW satisfies

$$\left| [\Delta + \kappa^2 \epsilon] \varphi_l \right| \leq C_l \left|\mathbf{m} - \mathbf{g}_K\right|^q.$$

Moreover $u_a = \sum\limits_{l=1}^{2n+1} \mathsf{X}_l \varphi_l$ and it was already noticed in [13] that $|\mathsf{X}_l| \leq C(\kappa, K, n) \|u\|_{\mathcal{C}^{n+1}}$. As a result

$$\left| [\Delta + \kappa^2 \epsilon] u_a\left(\mathbf{m}\right) \right| = \left| \sum\limits_{l=1}^{2n+1} \mathsf{X}_l [\Delta + \kappa^2 \epsilon] \varphi_l \right| \leq C(\kappa, K, n) \|u\|_{\mathcal{C}^{n+1}} \left|\mathbf{m} - \mathbf{g}_K\right|^q \sum\limits_{l=1}^{2n+1} C_l$$

and so (3.6) holds $\mathfrak{C}(\kappa, K, n) = C(\kappa, K, n) \sum\limits_{l=1}^{2n+1} C_l$. $\qquad\square$

The last step is to build, from the local function spaces $GPW_\kappa^{p,q}(K)$, a set of GPWs on the whole domain Ω: the GPWs space V_h is naturally defined as $\prod_{K \in T_h} GPW_\kappa^{p,q}(K)$. Note that p and q could vary from element to element but are here considered fixed on every elements.

As a result, we have the following estimates on the whole domain Ω:

Theorem 3.3. *Consider* $n \in \mathbb{N}^*$. *Assume that* $q \geq n+1$ *and* $p = 2n+1$. *Finally suppose the coefficient* $\epsilon \in \mathcal{C}^{q-1}(K)$. *Suppose that* u *is a solution of scalar wave equation* (1.1) *which belongs to* $\mathcal{C}^{n+1}(\Omega)$. *Then the function* $v_h \in V_h = \prod_{K \in T_h} GPW_\kappa^{p,q}(K)$, *provided element by element by Theorem* 3.1, *satisfies: there exists a constant* C *independent of* h *such that*

$$\|[\Delta + \kappa^2 \epsilon](u - v_h)\|_{L^2(\Omega)} \leq C\, area(\Omega)^{1/2} \left(\max_{K \in T_h} h_K \right)^q \|u\|_{\mathcal{C}^{n+1}(\Omega)}$$

$$\|\nabla(u - v_h)\|_{L^2(\Omega)} \leq C\, area(\Omega)^{1/2} \left(\max_{K \in T_h} h_K \right)^n \|u\|_{\mathcal{C}^{n+1}(\Omega)}$$

where h_K *is the radius of* K.

Remark 3.4. If the order of approximation q were allowed to vary from element to element, the local number of basis functions $p(K) = 2q(K) - 1$ would provide the following estimate:

$$\|[\Delta + \kappa^2 \epsilon](u - v_h)\|_{L^2(\Omega)} \leq C\, \mathrm{area}(\Omega)^{1/2} \max_{K \in T_h} \left(h_K^{q(K)} \right) \|u\|_{\mathcal{C}^{n+1}(\Omega)}.$$

3.2. Approximation of linear polynomials

The result ([7], Lem. 3.10) addresses the approximation of bi-variate polynomials of degree 1 by classical plane waves, and here we are interested in the approximation of bi-variate polynomials of degree 1 by GPWs. This result is needed to apply the h-based analysis of [7] or [15]. Even though we use triangular elements in the numerical algorithm, the next lemma uses a square reference element (as in [7]). This is needed because affine transformations do not preserve solutions of the Helmholtz equation, and the connection to triangular elements is explained in the proof of Corollary 3.8.

Lemma 3.5. *Consider $\hat{K} \in [0,1]^2$ the reference element. Suppose $n \in \mathbb{N}$ is such that $n \geq 2$. For $p = 2n+1$ and $q \geq n+1$ there is a constant C independent of κ (but not of p) such that*

$$\inf_{v \in GPW_\kappa^{p,q}(\hat{K})} \|f - v\|_{0,\hat{K}} \leq C\kappa^2 |\epsilon(x_{\hat{K}}, y_{\hat{K}})| \|f\|_{0,\hat{K}}, \ \forall f \in \mathcal{P}_1(\mathbb{R}^2).$$

Remark 3.6. The proof strongly relies on the fact that the GPW space is designed with the normalization $(\lambda_{1,0}^k, \lambda_{0,1}^k) = \mathrm{i}\kappa\sqrt{\epsilon(x_{\hat{K}}, y_{\hat{K}})}(\cos\theta_k, \sin\theta_k)$, for equi-spaced angles θ_k, for $1 \leq k \leq p$.

Proof. For the sake of clarity we define $\tilde{\kappa} = \kappa\sqrt{\epsilon(x_{\hat{K}}, y_{\hat{K}})}$. Consider

$$b_j := (\mathrm{i}\tilde{\kappa})^{-[j/2]} \sum_{k=1}^{p} \alpha_k^{(j)} \varphi_k$$

where the φ_ks are the GPWs, $\alpha_k^{(j)} = (\mathsf{M}_p)^{-1}$, $p = 2n+1$ and $\mathsf{M}_p \in \mathbb{R}^{p,p}$ is defined for $1 \leq k, l \leq p$ by

$$(\mathsf{M}_p)_{kl} := \begin{cases} 1 \text{ for } l = 1 \\ \cos\left(\dfrac{l}{2}\theta_k\right) \text{ for } l \text{ even} \\ \sin\left(\dfrac{l-1}{2}\theta_k\right) \text{ for } l \geq 3 \text{ odd} \end{cases}$$

We know that

$$\sum_{k=1}^{p} \alpha_k^{(j)} \varphi_k = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=1}^{p} \alpha_k^{(j)} (P_k(x,y))^n,$$

and, if $\hat{\mathbf{x}} = (\hat{x}, \hat{y}) = (x - x_{\hat{K}}, y - y_{\hat{K}})$, then

$$
\begin{aligned}
(P_k(x,y))^n &= \left(\begin{pmatrix} \lambda_{1,0}^k \\ \lambda_{0,1}^k \end{pmatrix} \cdot \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \tilde{\kappa}^2 f_{k,q}(\mathbf{x}) \right)^n \quad \text{see Lemma 3.7} \\
&= \left(\begin{pmatrix} \lambda_{1,0}^k \\ \lambda_{0,1}^k \end{pmatrix} \cdot \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \right)^n + \tilde{\kappa}^n \sum_{j=0}^{n-1} \binom{n}{j} \tilde{\kappa}^{n-2j} \left(\begin{pmatrix} \lambda_{1,0}^k \\ \lambda_{0,1}^k \end{pmatrix} \cdot \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \right)^j f_{k,q}(\mathbf{x})^{n-j} \\
&= \left(\begin{pmatrix} \lambda_{1,0}^k \\ \lambda_{0,1}^k \end{pmatrix} \cdot \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \right)^n + \tilde{\kappa}^{n+1} g_{k,q}(\mathbf{x}),
\end{aligned}
$$

the function $g_{k,q}$, defined as $g_{k,q}(\mathbf{x}) = \sum_{j=0}^{n-1} \binom{n}{j} \tilde{\kappa}^{n-1-2j} (\begin{pmatrix} \lambda_{1,0}^k \\ \lambda_{0,1}^k \end{pmatrix} \cdot \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix})^j f_{k,q}(\mathbf{x})^{n-j}$, being uniformly bounded for $\mathbf{x} = (x,y) \in \hat{K}$ as $\tilde{\kappa} \to 0$.

We assume that $(\lambda_{1,0}^k, \lambda_{0,1}^k) = \mathrm{i}\tilde{\kappa}(\cos\theta_k, \sin\theta_k)$. Define

$$K_0^n(\mathbf{x}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\begin{pmatrix} \lambda_{1,0}^k \\ \lambda_{0,1}^k \end{pmatrix} \cdot \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \right)^n \mathrm{d}\theta \quad \text{and } \forall 1 \leq j \leq n:$$

$$K_{2j}^n(\mathbf{x}) = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\begin{pmatrix} \lambda_{1,0}^k \\ \lambda_{0,1}^k \end{pmatrix} \cdot \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \right)^n \cos(j\theta)\mathrm{d}\theta, \ K_{2j+1}^n(\mathbf{x}) = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\begin{pmatrix} \lambda_{1,0}^k \\ \lambda_{0,1}^k \end{pmatrix} \cdot \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \right)^n \sin(j\theta)\mathrm{d}\theta.$$

The leading order term of $(P_k(x,y))^n$ as $\tilde{\kappa} \to 0$ is $\left(\begin{pmatrix} \lambda_{1,0}^k \\ \lambda_{0,1}^k \end{pmatrix} \cdot \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \right)^n$. As shown in [7], it can be written $\left(\begin{pmatrix} \lambda_{1,0}^k \\ \lambda_{0,1}^k \end{pmatrix} \cdot \right.$

$\left. \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \right)^n = (\mathrm{i}\tilde{\kappa})^n \sum_{l=1}^{2n+1} K_l^n(\mathbf{x})\mathsf{M}_{lk}$ so that

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=1}^{p} \alpha_k^{(j)} \left(\begin{pmatrix} \lambda_{1,0}^k \\ \lambda_{0,1}^k \end{pmatrix} \cdot \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \right)^n = \sum_{n=0}^{\infty} \frac{(\mathrm{i}\tilde{\kappa})^n}{n!} \sum_{l=1}^{2n+1} K_l^n(\mathbf{x}) \sum_{k=1}^{p} \alpha_k^{(j)} \mathsf{M}_{lk}$$
$$= \sum_{n=0}^{m} \frac{(\mathrm{i}\tilde{\kappa})^n}{n!} \sum_{l=1}^{2n+1} K_l^n(\mathbf{x}) \sum_{k=1}^{p} \alpha_k^{(j)} \mathsf{M}_{lk} + \sum_{n=m+1}^{\infty} \frac{(\mathrm{i}\tilde{\kappa})^n}{n!} \sum_{l=1}^{2n+1} K_l^n(\mathbf{x}) \sum_{k=1}^{p} \alpha_k^{(j)} \mathsf{M}_{lk},$$

and since $K_j^n = 0$ if $[j/2] > n$ and $\sum_{k=1}^{p} \alpha_k^{(j)} \mathsf{M}_{lk} = \delta_{jl}$ for $1 \le l, j \le p$, it yields

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=1}^{p} \alpha_k^{(j)} \left(\begin{pmatrix} \lambda_{1,0}^k \\ \lambda_{0,1}^k \end{pmatrix} \cdot \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \right)^n = \sum_{n=[j/2]}^{m} \frac{(\mathrm{i}\tilde{\kappa})^n}{n!} \left(K_j^n(\mathbf{x}) + \sum_{l=p+1}^{2n+1} K_l^n(\mathbf{x}) \sum_{k=1}^{p} \alpha_k^{(j)} \mathsf{M}_{lk} \right)$$
$$+ \sum_{n=m+1}^{\infty} \frac{(\mathrm{i}\tilde{\kappa})^n}{n!} \left(K_j^n(\mathbf{x}) + \sum_{l=p+1}^{2n+1} K_l^n(\mathbf{x}) \sum_{k=1}^{p} \alpha_k^{(j)} \mathsf{M}_{lk} \right)$$
$$= \sum_{n=[j/2]}^{m} \frac{(\mathrm{i}\tilde{\kappa})^n}{n!} K_j^n(\mathbf{x}) + \tilde{\kappa}^{m+1} R_j(\tilde{\kappa}, \mathbf{x})$$

since $l/2 \ge m+1 \Rightarrow [K_l^n = 0$ for $n \le m]$, and where the remainder function R_j defined by

$$R_j(\kappa, \mathbf{x}) = \frac{1}{\tilde{\kappa}^{m+1}} \sum_{n=m+1}^{\infty} \frac{(\mathrm{i}\tilde{\kappa})^n}{n!} \left(K_j^n(\mathbf{x}) + \sum_{l=p+1}^{2n+1} K_l^n(\mathbf{x}) \sum_{k=1}^{p} \alpha_k^{(j)} \mathsf{M}_{lk} \right)$$

is uniformly bounded on $\hat{K}$. So

$$\sum_{k=1}^{p} \alpha_k^{(j)} \varphi_k = \sum_{n=[j/2]}^{m} \left(\frac{(\mathrm{i}\tilde{\kappa})^n}{n!} K_j^n(\mathbf{x}) + \tilde{\kappa}^{n+1} g_{k,q}(\mathbf{x}) \right) + \tilde{\kappa}^{m+1} R_j(\tilde{\kappa}, \mathbf{x})$$

Since $b_j(\mathbf{x}) = (\mathrm{i}\tilde{\kappa})^{-[j/2]} \sum_{k=1}^{p} \alpha_k^{(j)} \varphi_k$ it clearly shows that

$$\lim_{\tilde{\kappa} \to 0} b_j(\mathbf{x}) = \frac{1}{[j/2]!} K_j^{[j/2]}(\mathbf{x}).$$

As a consequence, the definition of $K_j^{[j/2]}$ combined with $((\lambda_{1,0}^k, \lambda_{0,1}^k) = \mathrm{i}\tilde{\kappa}(\cos\theta_k, \sin\theta_k))$ lead to

$$b_1(\mathbf{x}) = 1 + O(\tilde{\kappa}^2), \ b_2(\mathbf{x}) = x - x_K + O(\tilde{\kappa}^2), \ b_3(\mathbf{x}) = y - y_K + O(\tilde{\kappa}^2). \qquad \square$$

To complete the proof of Lemma 3.5, we need to prove the following result.

Lemma 3.7. *Suppose $n \in \mathbb{N}$ is such that $n \ge 2$. For $p = 2n + 1$ and $q \ge n + 1$, consider the basis of p functions $\varphi_k \in GPW_\kappa^{p,q}\left(\hat{K}\right)$ approximating (1.1) at the point $\mathbf{g}_{\hat{K}} = (x_{\hat{K}}, y_{\hat{K}})$ at order q, and, for all k*

such that $1 \leq k \leq p$, *the corresponding polynomials* $P_k(x,y) = \sum\limits_{0 \leq i+j \leq q+1} \lambda_{i,j}^k (x - x_{\hat{K}})^i (y - y_{\hat{K}})^j$ *satisfying* $\varphi_k = \exp P_k$. *These polynomials satisfy*

$$P_k(x,y) = \begin{pmatrix} \lambda_{1,0}^k \\ \lambda_{0,1}^k \end{pmatrix} \cdot \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \kappa^2 \epsilon(x_K, y_K) f_{k,q}(\mathbf{x})$$

where $\mathbf{x} = (x,y)$, $(\hat{x}, \hat{y}) = (x - x_{\hat{K}}, y - y_{\hat{K}})$ *and the remainder function* $f_{k,q}$ *is uniformly bounded on* $\hat{K}$.

Proof. The normalization $(\lambda_{1,0}^k, \lambda_{0,1}^k) = i\kappa \sqrt{\epsilon(x_K, y_K)}(\cos \theta_k, \sin \theta_k)$ implies that $\lambda_{2,0}^k = 0$ so that the induction formula (3.3) reads:

$$\begin{cases} \lambda_{2,j}^k = -\dfrac{1}{2} \dfrac{\kappa^2 \partial_y^j \epsilon(x_K, y_K)}{j!} \\[2mm] \lambda_{3,j}^k = -\dfrac{1}{6} \left(\dfrac{\kappa^2 \partial_x \partial_y^j \epsilon(x_K, y_K)}{j!} + 4\lambda_{2,j}^k \lambda_{1,0}^k \right) \\[3mm] \lambda_{i+2,j}^k = -\dfrac{1}{(i+2)(i+1)} \left(\dfrac{\kappa^2 \partial_x^i \partial_y^j \epsilon(x_K, y_K)}{i!j!} + (j+2)(j+1)\lambda_{i,j+2}^k \lambda_{1,0}^k \quad \forall i > 1, \right. \\[3mm] \qquad\qquad + \sum\limits_{k=1}^{i-1} \sum\limits_{l=0}^{j} (i-k+1)(k+1)\lambda_{i-k+1,j-l}^k \lambda_{k+1,l}^k \\[2mm] \qquad\qquad \left. + \sum\limits_{l=1}^{i-1} \sum\limits_{k=0}^{j} (j-k+1)(k+1)\lambda_{i-l,j-k+1}^k \lambda_{l,k+1}^k \right) \end{cases} \qquad (3.7)$$

This clearly completes the proof by induction. $\qquad\qquad\square$

More precisely, the result needed is the following result that corresponds to Lemma 3.12 from [7]. We state it here to specify the GPWs parameters, and provide no more than a sketch of the proof since it relies on Lemma 3.5 but not specifically on the basis function set.

Corollary 3.8. *Suppose that* $n \in \mathbb{N}$ *is such that* $n \geq 2$. *For* $p = 2n + 1$ *and* $q \geq n + 1$, *suppose* w_h^c *is a linear function on a triangle* K *then there is a GPW function* $w_h \in GPW_\kappa^{p,q}(K)$ *such that*

$$\|w_h^c - w_h\|_{L^2(K)} \leq C h_K^2 \|w_h^c\|_{L^2(K)}.$$

Proof. By translation and dilation by $1/h_K$ we can map an element K to an element $\tilde{K} \subset \hat{K} = (0,1)^2$. Let $\hat{w}^c$ denote the transformed polynomial and note that $\hat{w}_h^c \in \mathcal{P}_1(\mathbb{R}^2)$. Let $\hat{P}$ the $L^2(\hat{K})$-projection onto the plane wave space $GPW_{\hat{\kappa}}^{p,q}(\hat{K})$ where $\hat{\kappa} = h_K \kappa$. Applying Lemma 3.5, we get

$$\|(I - \hat{P})\hat{w}_h^c\|_{0,\hat{K}} \leq C \hat{\kappa}^2 |\epsilon(x_{\hat{K}}, y_{\hat{K}})| \|\hat{w}_h^c\|_{0,\hat{K}}. \qquad (3.8)$$

The conclusion then follows by transforming back to K using the fact that the transformation from a triangle K to the reference triangle $\tilde{K}$ changes the frequency into $\hat{\kappa} = h_K \kappa$ and the bound $\|\hat{w}_h^c\|_{0,\hat{K}} \leq C \|\hat{w}_h^c\|_{0,\tilde{K}}$ with C independent of $\tilde{K}$ which holds because $\hat{w}_h^c$ is a linear polynomial and the mesh is regular. $\qquad\square$

4. Error estimates

In this section we start by establishing a straightforward error estimate using the coercivity and boundedness of the sesquilinear form $B_h(\cdot, \cdot)$, and then prove convergence in the global L^2 norm.

We define the obvious modification of the DG norm from [9] by

$$\|u\|_{DG}^2 = \frac{1}{\kappa}\|\beta^{1/2}[\![\nabla_h u]\!]\|_{L^2(\mathcal{E}_I)}^2 + \kappa\|\alpha^{1/2}[\![u]\!]\|_{L^2(\mathcal{E}_I)}^2 + \frac{1}{\kappa}\|\delta^{1/2}\partial_n u\|_{L^2(\mathcal{E}_R)}^2$$

$$+\kappa\|(1-\delta)^{1/2}u\|_{L^2(\mathcal{E}_R)}^2 + \kappa\|\alpha^{1/2}u\|_{L^2(\mathcal{E}_D)}^2 + \frac{1}{\kappa^2}\|\gamma^{1/2}(\Delta_h u + \kappa^2\epsilon u)\|_{L^2(\Omega)}^2$$

where Δ_h is the Laplacian defined piecewise element by element. Obviously this is a semi-norm, but due to the new term it is also a norm even on functions that do not exactly satisfy the Helmholtz equation.

Lemma 4.1. *The semi-norm $\|\cdot\|_{DG}$ is a norm.*

Proof. Suppose $\|u\|_{DG} = 0$ then u satisfies $\Delta u + \kappa^2\epsilon u = 0$ element-wise, and the normal derivatives and function values have no jump across interior edges. So $\Delta u + \kappa^2\epsilon u = 0$ in Ω. In addition the Cauchy data vanishes and so $u = 0$ in Ω. Hence $\|\cdot\|_{DG}$ is a norm. $\qquad\square$

We also need a new DG+ norm:

$$\|u\|_{DG+}^2 = \|u\|_{DG}^2 + \kappa\|\beta^{-1/2}\{\!\!\{u\}\!\!\}\|_{L^2(\mathcal{E}_I)}^2 + \frac{1}{\kappa}\|\alpha^{-1/2}\{\!\!\{\nabla_h u\}\!\!\}\|_{L^2(\mathcal{E}_I)}^2 + \kappa\|\delta^{-1/2}u\|_{L^2(\mathcal{E}_R)}^2$$

$$+\frac{1}{\kappa}\|\alpha^{-1}\partial_n u\|_{L^2(\mathcal{E}_D)}^2 + \kappa^2\|\gamma^{-1/2}u\|_{L^2(\Omega)}.$$

The following estimates hold:

Lemma 4.2. *Under the assumption that $\alpha > 0$, $\beta > 0$, $1 > \delta > 0$ and $\gamma > 0$ in the generalized TDG, and provided u is such that $\|u\|_{DG}$ is finite,*

$$\Im(B_h(u,u)) \geq \|u\|_{DG}^2.$$

Provided $\|u\|_{DG+}$ and $\|v\|_{DG}$ are finite, there exists a constant C independent of κ, u and v such that

$$|B_h(u,v)| \leq C\|u\|_{DG+}\|v\|_{DG}.$$

Proof. The coercivity estimate follows from in the usual way by considering $\Im(B_h(u,u))$ and using the assumption that ϵ is real [8].

To obtain the desired continuity, we integrate the term $\nabla_h u \cdot \nabla_h v$ term in the definition of $A_h(\cdot,\cdot)$ by parts to obtain, for any $u, v \in V_h$,

$$A_h(u,v) = -\int_\Omega u(\overline{\Delta_h v + \kappa^2\epsilon v})\,\mathrm{d}A + \int_{\mathcal{E}_I}\{\!\!\{u\}\!\!\}[\![\nabla_h\overline{v}]\!]\,\mathrm{d}s - \int_{\mathcal{E}_I}\{\!\!\{\nabla_h u\}\!\!\}[\![\overline{v}]\!]\,\mathrm{d}s$$

$$+\int_{\mathcal{E}_R}(1-\delta)u\partial_n\overline{v}\,\mathrm{d}s - \frac{1}{\mathrm{i}\kappa}\int_{\mathcal{E}_I}\beta[\![\nabla_h u]\!][\![\nabla_h\overline{v}]\!]\,\mathrm{d}s$$

$$+\mathrm{i}\kappa\int_{\mathcal{E}_I}\alpha[\![u]\!][\![\overline{v}]\!]\,\mathrm{d}s - \frac{1}{\mathrm{i}\kappa}\int_{\mathcal{E}_R}\delta\partial_n u\partial_n\overline{v}\,\mathrm{d}s$$

$$+\mathrm{i}\kappa\int_{\mathcal{E}_R}(1-\delta)u\overline{v}\,\mathrm{d}s - \int_{\mathcal{E}_R}\delta\partial_n u\,\overline{v}\,\mathrm{d}s$$

$$-\int_{\mathcal{E}_D}\partial_n u\overline{v} + \mathrm{i}\kappa\int_{\mathcal{E}_D}\alpha u\overline{v}. \tag{4.1}$$

The result now follows from the definition of $B_h(u,v)$ and the Cauchy−Schwarz inequality. $\qquad\square$

The following result is now a standard consequence of the above estimates [6,9]:

Lemma 4.3. *There is a unique solution $u_h \in V_h$ that satisfies* (2.2), *and the following error estimate holds with constant C independent of κ, u, and u_h:*

$$\|u - u_h\|_{DG} \leq C\|u - v\|_{DG+} \quad \text{for all } v \in V_h. \tag{4.2}$$

To obtain an order estimate, we need to make specific choices of the parameters α, β, δ and γ. There are several choices in the literature depending on the precise setting of the problem (see for example [3,7,9,10]). In this paper we shall make the classical UWVF choice [3]:

$$\alpha = \beta = \delta = 1/2, \tag{4.3}$$

so that we can use results from [15]. Then we choose for γ

$$\gamma = \gamma_0 h_K^r$$

where γ_0 is constant and $r \geq 0$. We shall examine the role of r later.

Using the estimates from Section 3 we can then prove the following error estimate.

Theorem 4.4. *Suppose $n \in \mathbb{N}$ is such that $n \geq 2$ and consider $p = 2n+1$ and $q \geq n+1$. Suppose V_h is formed from qth order GPWs element by element using p directions. Then the solution $u_h \in V_h$ of* (2.2) *exists for all $h > 0$ independent of κ and it satisfies the following estimate with constant C independent of κ and u_h:*

$$\|u - u_h\|_{DG} \leq C(h^{n-1/2} + h^{q+r/2}).$$

Here C depends on the $\|u\|_{\mathcal{C}^{\max(n+1,q)}(\Omega)}$ norm of u.

Remark 4.5. Since we need $q \geq n+1$ in the GPW theory, we see that the choice $q = n+1$ guarantees that the approximation of the Helmholtz equation is high enough order.

Proof. We pick $v \in V_h$ in equation (4.2) element by element to be the approximation by GPWs denoted by u_a in Theorem 3.3. We now need to estimate each term in $\|u - v\|_{DG+}$. Using Theorem 3.3 the volume term in the DG norm reads

$$\|\gamma^{1/2}(\Delta_h(u-v) + \kappa^2\epsilon(u-v))\|_{L^2(\Omega)} = \|\gamma^{1/2}(\Delta_h v + \kappa^2\epsilon v)\|_{L^2(\Omega)} \leq Ch^{q+r/2}\|u\|_{\mathcal{C}^q}. \tag{4.4}$$

The remaining terms can be estimated using Theorem 3.2. For example

$$\|\alpha^{-1/2}\{\!\!\{\nabla_h(u-v)\}\!\!\}\|_{L^2(\mathcal{E}_I)}^2 \leq C\sum_K \|\alpha^{-1/2}\nabla_h(u-v)\|_{L^2(\partial K)}^2$$

$$\leq C\sum_K \max_{e \in \partial K} \alpha^{-1}(e)\|\nabla_h(u-v)\|_{L^2(\partial K)}^2$$

$$\leq C\sum_K \max_{e \in \partial K} \alpha^{-1}(e)\left[h_K^{-1}\|\nabla_h(u-v)\|_{L^2(K)}^2 + h_K\|\nabla_h\nabla_h(u-v)\|_{L^2(K)}^2\right]$$

where we have used the standard trace estimate on ∂K. Using Theorem 3.2 with $k = 2$ and Theorem 3.1 we obtain

$$\|\alpha^{-1/2}\{\!\!\{\nabla_h(u-v)\}\!\!\}\|_{L^2(\mathcal{E}_I)}^2 \leq C\sum_K \max_{e \in \partial K} \alpha^{-1}(e)h_K^{2n-1}h_K^2\|u\|_{\mathcal{C}^{n+1}(K)}^2$$

$$\leq C\max_{e \in \mathcal{E}_I} \alpha^{-1}(e)h^{2n-1}\|u\|_{\mathcal{C}^{n+1}(\Omega)}^2.$$

Of course under our assumptions $\alpha = 1/2$. The remaining terms are estimated in the same way. $\qquad\square$

We now use the standard duality approach to prove an $L^2(\Omega)$ norm estimate on the error.

Theorem 4.6. *Suppose we choose $r = 3$ in the penalty parameter γ, $p = 2n + 1$, $n \geq 2$ and $q = n + 1$. Then there exists a constant C depending on κ but independent of h such that*

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^s \|u - u_h\|_{DG}$$

for some s with $0 < s \leq 1/2$ depending on Ω (given in [9], Thm. 2.3).

Under best possible conditions we then have the following convergence estimate:

Corollary 4.7. *Suppose u is a smooth solution of the Helmholtz equation, that $r = 3$, $p = 2n + 1$, $n \geq 2$ and $q = n + 1$ then*

$$\|u - u_h\|_{L^2(\Omega)} \leq Ch^{n+s-1/2}$$

Remark 4.8. Since $s \leq 1/2$ the maximum rate of convergence predicted for the method assuming a smooth solution and best regularity is $O(h^n)$.

Proof. Define the dual variable $z \in H^1(\Omega)$ to satisfy

$$\Delta z + \kappa^2 \epsilon z = u - u_h \qquad \text{in} \qquad \Omega,$$
$$\partial_n z - \mathrm{i}\kappa z = 0 \qquad \text{on} \qquad \Gamma_R,$$
$$z = 0 \text{ on } \Gamma_D.$$

Under the geometric assumptions on the domain given in the introduction, it can be shown that $z \in H^{3/2+s}(\Omega)$, $s > 0$, [9] and so z is sufficiently regular that

$$A_h(\xi, z) = \int_\Omega \xi \overline{(u - u_h)} \, \mathrm{d}A$$

for all test function ξ that are H^2 piecewise smooth. This follows from (4.1). Hence, by the definition of B_h

$$B_h(\xi, z) = \frac{1}{\kappa^2} \int_\Omega \gamma(\Delta_h \xi + \kappa^2 \epsilon \xi)\overline{(u - u_h)} \, \mathrm{d}A + \int_\Omega \xi \overline{(u - u_h)} \, \mathrm{d}A$$

so choosing $\xi = u - u_h$ and letting $z_h \in V_h$ be arbitrary

$$\|u - u_h\|^2_{L^2(\Omega)} = B(u - u_h, z - z_h) - \frac{1}{\kappa^2} \int_\Omega \gamma(\Delta_h(u - u_h) + \kappa^2 \epsilon(u - u_h))\overline{(u - u_h)} \, \mathrm{d}A.$$

The second term on the right hand side can be estimated using the Cauchy−Schwarz and arithmetic-geometric mean inequality to give

$$\left| \int_\Omega \gamma(\Delta_h(u - u_h) + \kappa^2 \epsilon(u - u_h))\overline{(u - u_h)} \, \mathrm{d}A \right| \leq \frac{1}{k}\|u - u_h\|_{DG}\|\gamma^{1/2}(u - u_h)\|_{L^2(\Omega)}$$

$$\leq \frac{\gamma_{\max}}{2\kappa^2}\|u - u_h\|^2_{DG} + \frac{1}{2}\|u - u_h\|^2_{L^2(\Omega)},$$

where $\gamma_{\max} = \max_{x \in \Omega} \gamma = O(h^r)$.

To estimate $B_h(u - u_h, z - z_h)$ we integrate the grad-grad term in $A_h(u, v)$ by parts onto u to obtain

$$A_h(u, v) = -\int_\Omega (\Delta_h u + \kappa^2 \epsilon u)\overline{v} \, \mathrm{d}A + \int_{\mathcal{E}_I} ([\![\nabla_h u]\!]\{\!\{\overline{v}\}\!\} - [\![u]\!] \cdot \{\!\{\nabla_h \overline{v}\}\!\}) \, \mathrm{d}s - \frac{1}{\mathrm{i}\kappa} \int_{\mathcal{E}_I} \beta[\![\nabla_h u]\!][\![\nabla_h \overline{v}]\!] \, \mathrm{d}s$$

$$+ \mathrm{i}\kappa \int_{\mathcal{E}_I} \alpha[\![u]\!] \cdot [\![\overline{v}]\!] \, \mathrm{d}s - \int_{\mathcal{E}_R} \frac{\delta}{\mathrm{i}\kappa}(\mathrm{i}\kappa u - \partial_n u)\partial_n \overline{v} \, \mathrm{d}s$$

$$+ \int_{\mathcal{E}_R} (1 - \delta)(\partial_n u - \mathrm{i}\kappa u)\overline{v} \, \mathrm{d}s + \int_{\mathcal{E}_D} u(\mathrm{i}\kappa\alpha\overline{v} - \partial_n \overline{v}) \, \mathrm{d}s. \tag{4.5}$$

Using this in the definition of $B_h(u, v)$ shows that

$$|B_h(u, v)| \leq C\|u\|_{DG}\|v\|_{DG+}$$

where C is independent of u and v so that we have the estimate

$$\|u - u_h\|_{L^2(\Omega)}^2 \leq C\|u - u_h\|_{DG}\|z - z_h\|_{DG+} + \frac{\gamma_{\max}}{2\kappa^2}\|u - u_h\|_{DG}^2. \tag{4.6}$$

It is now necessary to choose z_h. Following the proof of ([15], Thm. 5.6), let z_h^c denote the continuous piecewise linear finite element interpolant of z. We choose $z_h \in V_h$ to be the GPW approximation of z_h^c constructed in Lem. 3.8). Of course

$$\|z - z_h\|_{DG+} \leq \|z - z_h^c\|_{DG+} + \|z_h^c - z_h\|_{DG+}$$

and it remains to estimate each term. Estimates from the proof of ([15], Thm. 5.6) show that on each interior edge in the mesh

$$\|\alpha^{-1/2}\{\!\!\{\nabla_h(z - z_h^c)\}\!\!\}\|_{L^2(e)} \leq C \sum_{j=1}^{2} h_{K_j}^s |z|_{H^{3/2+s}(K_j)}$$

$$\|\beta^{-1/2}\{\!\!\{z - z_h^c\}\!\!\}\|_{L^2(e)} \leq C \sum_{j=1}^{2} h_{K_j}^{1+s} |z|_{H^{3/2+s}(K_j)},$$

with corresponding results for the jumps in the above quantities and for boundary terms. In addition

$$\|\gamma^{1/2}(\Delta_h(z - z_h^c) + \kappa^2\epsilon(z - z_h^c)\|_{L^2(\Omega)} = \|\gamma^{1/2}(u - u_h - (\Delta_h z_h^c + \kappa^2\epsilon z_h^c))\|_{L^2(\Omega)}$$
$$\leq Ch^{r/2}\|u - u_h\|_{L^2(\Omega)} + \|\gamma^{1/2}(\Delta_h z_h^c + \kappa^2\epsilon z_h^c)\|_{L^2(\Omega)}.$$

On an element K we can use the regularity of the mesh to establish local inverse estimates and prove:

$$\|\gamma^{1/2}\Delta_h z_h^c\|_{L^2(K)} \leq Ch_K^{r/2-1}\|z_h^c\|_{H^1(K)} \leq Ch_K^{r/2-1}(\|z_h^c - z\|_{H^1(K)} + \|z\|_{H^1(K)}) \leq Ch_K^{r/2-1}\|z\|_{H^{3/2+s}(K)}.$$

Proceeding similarly for the lower order term, we conclude that provided $r/2 > 1$ we have

$$\|\gamma^{1/2}(\Delta_h(z - z_h^c) + \kappa^2\epsilon(z - z_h^c))\|_{L^2(\Omega)} \leq Ch^{r/2-1}\|z\|_{H^{3/2+s}(\Omega)}.$$

In addition we must estimate

$$\|\gamma^{-1/2}(z - z_h^c)\|_{L^2(\Omega)}^2 = \sum_K \int_K \gamma^{-1}(z - z_h^c)^2 \, \mathrm{d}A \leq C \sum_K h_K^{3+2s-r}\|z\|_{H^{3/2+s}(K)}^2$$
$$\leq Ch^{3+2s-r}\|z\|_{H^{3/2+s}(\Omega)}^2.$$

Taken together, if $3 + 2s \geq r \geq 2$ we have

$$\|z - z_h^c\|_{DG+} \leq C(h^{r/2-1} + h^{3/2+s-r/2})\|z\|_{H^{3/2+s}(\Omega)}$$

A good choice is then $r = 3$ since in that case $r/2 - 1 \geq s$ and using the *a priori* estimate for z from ([9], Thm. 2.3)

$$\|z - z_h^c\|_{DG+} \leq Ch^s\|u - u_h\|_{L^2(\Omega)}.$$

It now remains to estimate $\|z_h^c - z_h\|_{DG+}$. As we have seen there are two troublesome terms: $\|\gamma^{-1/2}(z_h^c - z_h)\|_{L^2(\Omega)}^2$ and $\|\gamma^{1/2}(\Delta_h(z_h^c - z_h) + \kappa^2\epsilon(z_h^c - z_h))\|_{L^2(\Omega)}$ with the remaining terms following using Lemma 3.5 as in [15]. Using first a local inverse estimate, then Lemma 3.5,

$$\|\gamma^{1/2}(\Delta_h(z_h^c - z_h) + \kappa^2\epsilon(z_h^c - z_h))\|_{L^2(K)} \leq Ch_K^{r/2-2}\|z_h^c - z_h\|_{L^2(K)} \leq Ch_K^{r/2}\|z_h^c\|_{L^2(K)}$$

so that, squaring and adding, and using the *a priori* estimate for z from ([9], Thm. 2.3)

$$\|\gamma^{1/2}(\Delta_h(z_h^c - z_h) + \kappa^2\epsilon(z_h^c - z_h))\|_{L^2(\Omega)} \leq Ch_K^{r/2}\|z_h^c\|_{L^2(\Omega)}$$
$$\leq Ch_K^{r/2}(\|z - z_h^c\|_{L^2(K)} + \|z\|_{L^2(K)})$$
$$\leq Ch^{r/2}\|u - u_h\|_{L^2(\Omega)}.$$

To estimate the global L^2 term, again using Lemma 3.5,

$$\|\gamma^{-1/2}(z_h^c - z_h)\|_{L^2(K)}^2 \leq Ch_K^{-r/2}\|(z_h^c - z_h)\|_{L^2(K)}^2 \leq Ch_K^{2-r/2}\|z_h^c\|_{L^2(K)}^2$$

Adding over all elements and using the *a priori* estimate for z from ([9], Thm. 2.3)

$$\|\gamma^{-1/2}(z_h^c - z_h)\|_{L^2(\Omega)}^2 \leq Ch^{2-r/2}\|u - u_h\|_{L^2(K)}^2$$

We have thus proved that when $r = 3$ and noting $0 < s < 1/2$ we have

$$\|z - z_h\| \leq Ch^s\|u - u_h\|_{L^2(\Omega)}.$$

so we conclude the desired result using this result in (4.6). $\qquad\square$

5. NUMERICAL TESTS

We now test the GPW based TDG method on two test problems with a known solution: Airy waves (linear variation in ϵ) and Weber Waves (quadratic variation in ϵ). In Figure 2 we plot the relative $L^2(\Omega)$ error in the computed solution against a parameter labeled C/h. This is computed, using the total number of degrees of freedom N_{dof} and the number of directions per element $p = 2n + 1$, as $C/h = \sqrt{N_{\text{dof}}/p}$. We choose this parameter since, in our theorems, convergence is in terms of mesh size rather than total number of degrees of freedom.

Our basic convergence results Lemma 4.3 and Theorem 4.6 are proved within the geometric assumptions stated in the introduction (from [9]) allowing corners on the scatterer or external domain. In the upcoming examples the domain is convex and so full regularity ensues with $s = 1/2$ in the L^2 norm estimate. The convergence rates predicted from our theory in Corollary 4.7 then depend on the regularity of the computed solution which in our examples is smooth. Thus we can expect the full order $O(h^n)$ convergence rate in the L^2 norm. If reentrant corners were present the convergence rate of the GPW solution would be limited by the smoothness of the solution and the appropriate choice of s. The construction of GPW's near singularities is not considered that case here.

5.1. Airy Waves

The simplest example of a spatially dependent refractive index is $\epsilon(x, y) = -y$ on the domain $[-1, 1] \times [-1, 1]$. We can then choose Dirichlet boundary data (for our theory we need an impedance boundary condition, but the same result holds in the case provided κ is not an eigenvalue of the domain) such that the exact solution is

$$u(x, y) = Ai(\kappa^{2/3}y)$$

where $Ai(r)$ is the Airy function as shown in the left panel of Figure 1.
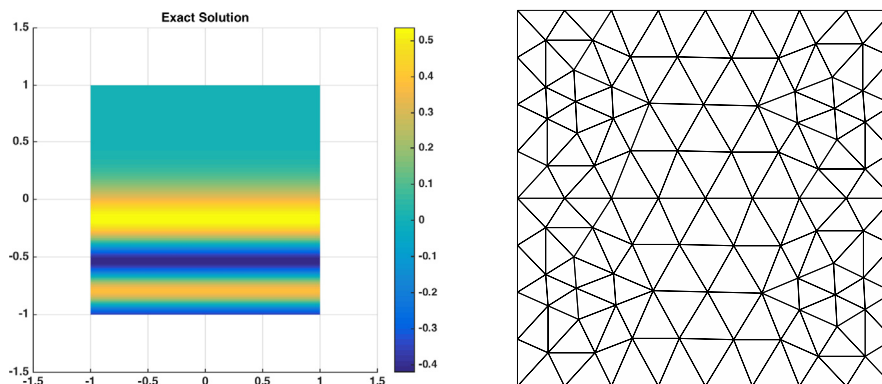

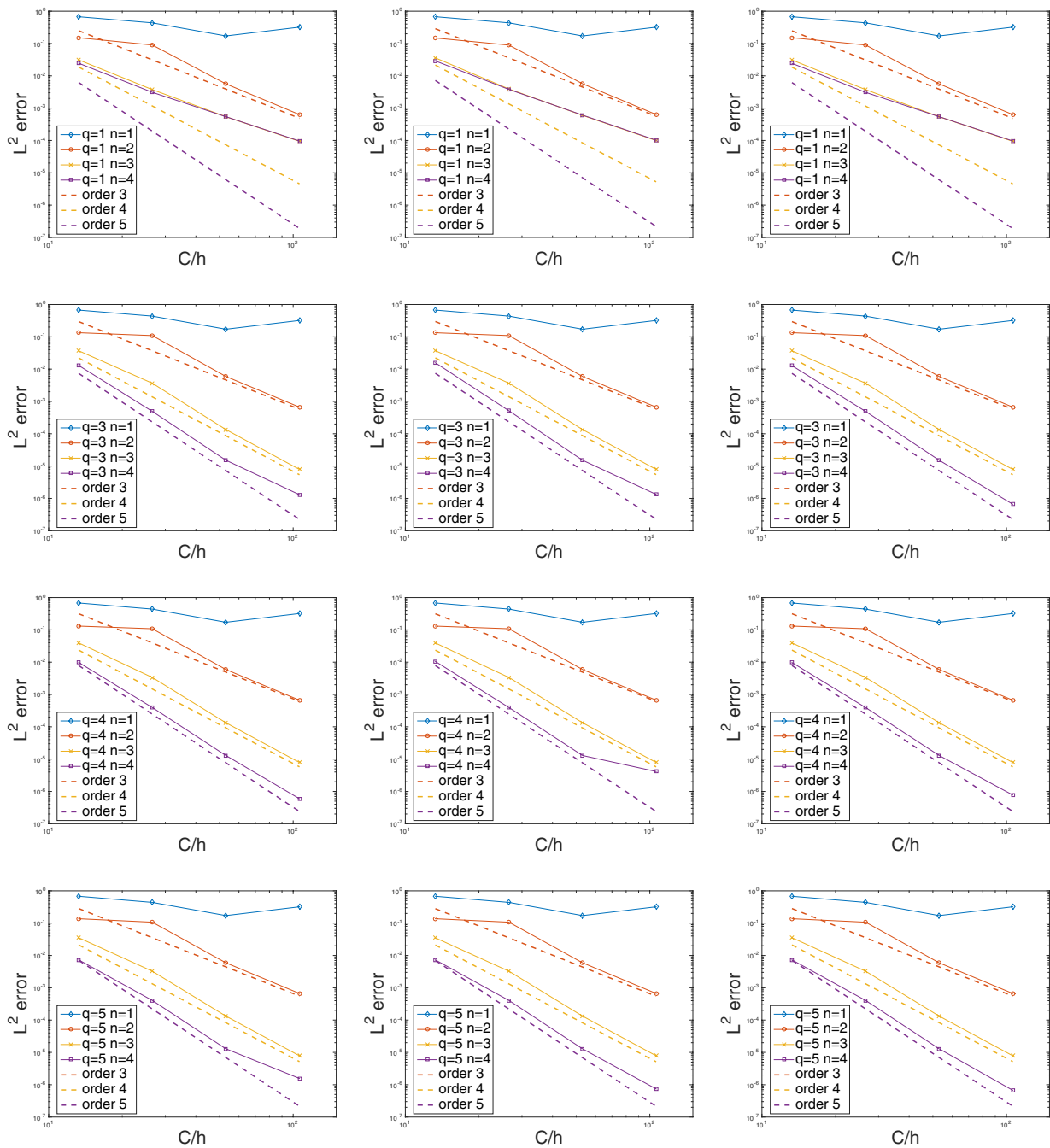FIGURE 1. *Left:* Exact Airy function solution. *Right:* Initial mesh.

FIGURE 2. Airy Waves. $L^2(\Omega)$ norm convergence when $\gamma = h^3$ (*left panels*), $\gamma = h$ (*middle panels*) and $\gamma = 0$ (*right panels*). The results are sorted with respect to the value of q: $q = 1$ on the first row, 3 on the second row, 4 on the third row and 5 on the last row, while values of n vary from 1 to 4. The dashed lines in each figure are reference lines showing $O(h^3)$, $O(h^4)$, and $O(h^5)$ convergence. Reference lines and axes limits are identical on all panels.

TABLE 1. Predicted and measured convergence rate for plane wave propagation when $\gamma = h^3$ (see Fig. 2 left column of panels). Our convergence results are stated assuming $q = n + 1$ and here $s = 1/2$. For $q < n + 1$ we state "No Theory". For $q > n + 1$ we predict the same convergence rate as for $q = n + 1$.

q	n	Predicted Convergence Rate	Measured Convergence Rate
1	1	No theory	No convergence
1	2	No theory	3
1	3	No theory	3
1	4	No theory	3
3	1	No theory	No convergence
3	2	2	3
3	3	No theory	4
3	4	No theory	5
4	1	No theory	No convergence
4	2	2	3
4	3	3	4
4	4	No theory	5
5	1	No theory	No convergence
5	2	2	3
5	3	3	4
5	4	4	5

This solution is oscillatory for $y < 0$ and exponentially decaying for $y > 0$. In all the experiments, we make the choice

$$\alpha = \beta = \delta = 1/2,$$

The initial mesh for the experiments is shown Figure 1 right panel.

Starting with the mesh in Figure 1 and using uniform refinement, we have computed the error in approximating the Airy function solution for $\kappa = 15$, and for different values of γ. The order of approximation of the Helmholtz equation q is set to 1, 3, 4, and 5 and the corresponding results are displayed in separated graphs.

5.1.1. The case $\gamma = h^3$

We focus here on the case $\gamma = h^3$, corresponding to the hypothesis of Theorem 4.6. Our goal is to demonstrate that an appropriate choice of n and q can produce high order convergence. Indeed our theory predicts that we should choose $q = n + 1$, $n \geq 2$ and expect $O(h^n)$ convergence in the $L^2(\Omega)$ norm since the Airy function solution is smooth and the domain is convex (see Cor. 4.7). Results are shown in Figure 2.

The left panels of Figure 2 demonstrate the need for GPWs in order to obtain high order convergence. When $q = 1$ (first row), the GPWs are plane waves and we see no obvious convergence when $n = 1$ (three plane waves per element), but at most third order convergence for $n > 1$. This suggests that one approach using an h-refinement strategy is to use simple plane waves with $n = 3$ to obtain third order convergence under mesh refinement (it appears that $n = 3$ offers a useful improvement in accuracy over $n = 2$ even if the order of convergence is the same). To obtain fourth or higher order convergence we need true GPWs with $q > 1$.

The case $n = 1$ is also interesting. Regardless of q we do not see obvious convergence when $n = 1$, whereas for a constant medium the plane wave basis with $n = 1$ gives $O(h^2)$ convergence [4]. The variable refractive index seems to require $n > 1$.

For $n = 2$ regardless of the choice $q = 1, \ldots, 5$ we see $O(h^3)$ convergence, and for $n = 3$ we get $O(h^4)$ convergence provided $q > 1$, while if $q = 1$ we get $O(h^2)$ convergence. Finally for $n = 4$ we only have $O(h^3)$ convergence when $q = 1$, but $O(h^5)$ convergence for $q > 1$ (with some deterioration on the finest mesh when $q = 3$ or $q = 5$). These results are summarized in Table 1. This deterioration may be due to the usual conditioning problem experienced by plane wave type methods since when $n = 4$ the condition number of the system
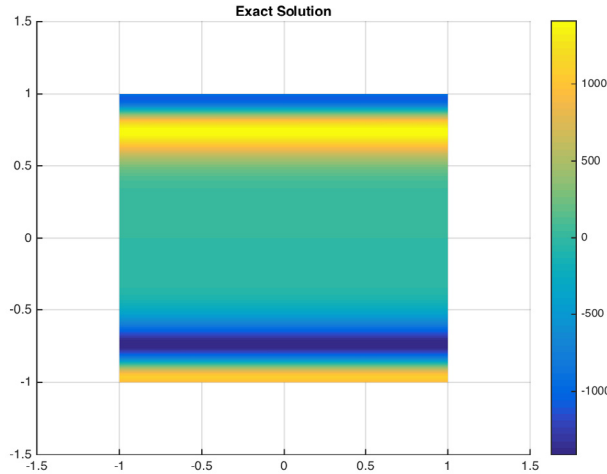
FIGURE 3. Exact solution for Weber's equation with $a = 5$.

is roughly 10^{20} which may impact convergence. The last cases $n = 3, 4$ confirms that q must increase as n increases although apparently more slowly than we predict. In addition, with an adequate choice of q we appear to see $O(h^{n+1})$ convergence for $n > 1$. This is the same order as has been found experimentally using $2n + 1$ plane waves when ϵ is constant! [4]. Even in that case, an optimal error analysis in that case is also elusive [3].

5.1.2. The case $\gamma = h$.

In this section we describe numerical results obtained by increasing the parameter γ from h^3 to $\gamma = h$, and we compare convergence rates obtained for different combinations of the order of approximation of the equation by the basis functions, q, and the number of basis functions per element, $p = 2n + 1$. This choice for γ violates the hypothesis of Theorem 4.6, but should result in greater stability.

The middle column of panels of Figure 2 display series of results for several choices of q, while n varies from 1 to 4. The results are broadly similar to those in the left panels of the same figure, although the case $n = 4$ shows a slowing of convergence on fine meshes for $q = 3$ and $q = 4$. In this case it appears that $q = n + 1$ is indeed a good choice.

These convergence studies emphasize the fact that the three choices $n = q$, $n = q - 1$, and $n = q + 1$ seem to result in approximately the same rate of convergence, suggesting that, in practice, $q = n - 1$ would be the best choice for a fixed value of n. The best rates of convergence obtained are 3 for $n = 2$, 4 for $n = 3$, and 5 for $n = 4$. Although the convergence rates are similar to those when $\gamma = h^3$, the accuracy attained on a given mesh is slightly worse. This suggests that choosing γ larger than $O(h^3)$ is not useful (other tests, not shown, with $\gamma = 1$ and $\gamma = 10^3$ show similar results but even worse error at a particular mesh).

5.1.3. The case $\gamma = 0$.

Our theoretical analysis requires that $\gamma > 0$ even to obtain convergence in the DG norm but this term requires integration over the interior of all the elements (unlike the standard PWDG or UWVF) and we would prefer to drop it. In addition we saw that $\gamma = O(h^3)$ gives better results than $\gamma = O(h)$ so we want to test if an even smaller penalty is better. In the right column of panels of Figure 2 we show results when $\gamma = 0$.

Overall the results are similar to previous results. Provided q is chosen large enough, we can obtain $O(h^{n+1})$ convergence. In fact the mesh now seems slightly more stable!
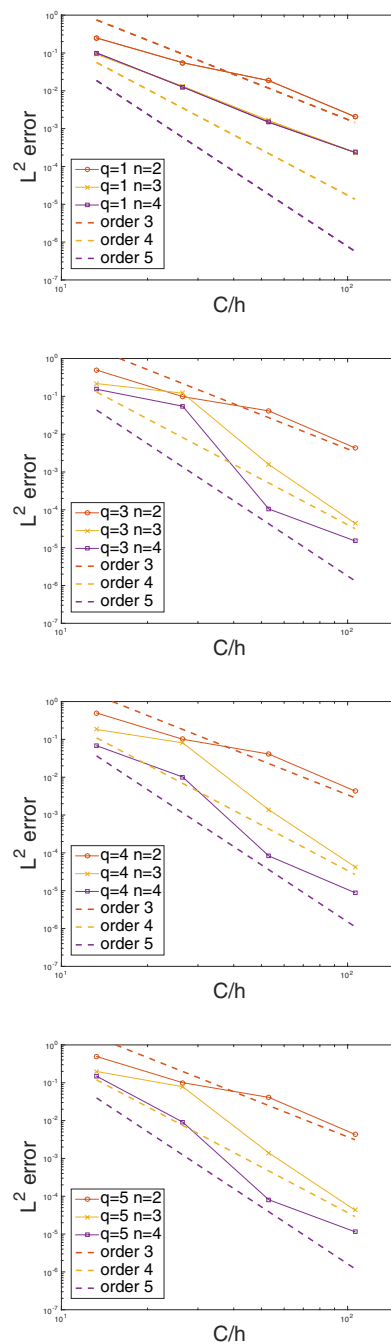
FIGURE 4. Analogues of Figure 2 for the Weber wave example. $L^2(\Omega)$ norm convergence when $\gamma = h^3$. The results are sorted with respect to the value of q: $q = 1$ on the first row, 3 on the second row, 4 on the third row and 5 on the last row, while values of n vary from 1 to 4. The dashed lines in each figure are reference lines showing $O(h^3)$, $O(h^4)$, and $O(h^5)$ convergence. Reference lines and axes limits are identical on all panels.

5.2. Weber waves

In this section we approximate what we term Weber waves. These are solutions of the following problem

$$\Delta u + \kappa^2 \left(\frac{x_2^2}{4} - \frac{a}{\kappa} \right) u = 0$$

in the domain $\Omega = [-1,1]^2$ subject to

$$u(x_2, y_2) = P_o(\sqrt{\kappa} x_2, a) \quad \text{on} \quad \partial\Omega$$

where $w(x_2) = P_o(x_2, a)$ is the odd solution of Weber's differential equation

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x_2^2} + \left(\frac{x_2^2}{4} - a \right) = 0$$

defined in [2] and implemented in [1]. We choose $a = 5$ and $\kappa = 50$ which gives the solution in Figure 3. For this choice of a, κ and domain, the solution is evanescent for $|x_2| < \sqrt{(2/5)}$ and oscillatory otherwise. So this example again tests how well GPWs can approximate both traveling and evanescent solutions.

Results are shown in Figure 4. Broadly the same picture emerges for the Weber example as for the Airy example. We see $O(h^{n+1})$ convergence (this is not completely clear when $n = 4$) provided q is large enough.

6. CONCLUSION

We have provided a modification to the TDG approach that allows the approximation of solutions of the Helmholtz equation in which the refractive index is piecewise smooth using Generalized Plane Waves. The resulting numerical scheme maintains one advantage of TDG: the number of degrees of freedom per element increases linearly with the order of approximation of the method. But the method looses one advantage of pure TDG: there is now a need to perform numerical integration element by element. This is required because we introduce a new stabilization term, and also because the GPW basis functions are not exact solutions of the adjoint problem.

Theory suggests a choice of parameters that balances polynomial degree with the number of GPWs in the basis element by element. This is examined in detail using Airy's equation to provide an exact solution, and substantiated further by using Weber's example. In the Airy case we have also studied if our new stabilization term is necessary: the numerical results in this one simple case suggest that it can be ignored, but much more testing (for example with less smooth solutions with curved wavefronts) and theoretical backup would be needed to confirm this. Our testing also suggests that our predicted choice of polynomial degree $q = n + 1$ may be excessive.

In summary, we have achieved a first theoretical convergence result for GPWs in a TDG setting. Our numerical investigations suggest that the theory is not optimal so far, but do show examples where GPWs can provide accurate solutions to wave propagation problems in which the coefficients are smooth functions of position.

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