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ON THE MACH-UNIFORMITY OF THE LAGRANGE-PROJECTION SCHEME*

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Abstract. In the present work, we show that the Implicit-Explicit Lagrange-projection scheme applied to the isentropic Euler equations, presented in Coquel *et al.*'s paper (*Math. Comp.* **79** (2010) 1493–1533), is *asymptotic preserving* regarding the Mach number, *i.e.*, it is asymptotically stable in ℓ_∞-norm with unrestrictive CFL condition for all-Mach flows, and asymptotically consistent which means that it gives a consistent discretization to the incompressible Euler equations in the limit, *e.g.*, it preserves the incompressible limit as to satisfy the *div*-free condition and the analogues of continuous-level asymptotic expansion for the density. This consistency analysis has been done formally as well as rigorously. Moreover, we prove that the scheme is positivity-preserving and entropy-admissible under some Mach-uniform restrictions. The analysis is similar to what has been presented in the original paper, but with the emphasis on the uniformity regarding the Mach number, which is crucial for a scheme to be useful in the low-Mach regime. We then extend the modified (but similar) analysis to the shallow water equations with topography and get similar stability and consistency results.

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1. Introduction

Singular limits of conservation laws (or more generally PDEs), may present severe difficulties to be treated either in analysis or numerics. The main issue is that the type of the equations changes in the limit [41], *e.g.*, when the Mach number approaches zero for the Euler equations. This limit is singular, since the sound speed (the characteristic speed) goes to the infinity and the PDE changes to be hyperbolic-elliptic, in the so-called incompressible limit. So, there are difficulties to show convergence of the solution of the compressible Euler equations to the incompressible ones (see [34, 41]). Tackling this problem numerically is complicated as well, since as the eigenvalues of the flux Jacobian blow up, the time step should tend to zero due to the Courant–Friedrichs–Lewy (CFL) condition, which leads to very small time steps and thus huge computational cost. Also it has been shown that in the general case, the usual numerical schemes lose their accuracy in the limit for under-resolved mesh sizes; see [18, 20, 25, 26, 45–47].

Keywords and phrases. All-Mach number scheme, Lagrange-projection scheme, asymptotic preserving scheme, stability analysis.

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$$
\begin{tikzcd}
\mathcal{M}^{\epsilon}_{\Delta} \arrow[r, "\epsilon \to 0"] \arrow[d, "\Delta \to 0"'] & \mathcal{M}^{0}_{\Delta} \arrow[d, "\Delta \to 0"] \\
\mathcal{M}^{\epsilon} \arrow[r, "\epsilon \to 0"'] & \mathcal{M}^{0}
\end{tikzcd}
$$

FIGURE 1. Illustration of Asymptotic Preserving schemes.

Throughout this paper, we assume that at least at the continuous level, the *solution* of compressible flow equations with the Mach number ϵ, converges to the *solution* of the limit equations, as $\epsilon \to 0$, and try to show that the counterpart of such a convergence also exists at the discrete level. This is in fact the idea of Asymptotic Preserving (AP) schemes, which has been introduced by Jin in [30,31] for relaxation systems; see also [32] for a general review and [37] for older works. Figure 1 illustrates this definition; $\mathcal{M}^{\epsilon}$ stands for a continuous physical model with the (singular) perturbation parameter $\epsilon \in (0, 1]$, and $\mathcal{M}^{\epsilon}_{\Delta}$ is a discrete-level model which provides a consistent discretization of $\mathcal{M}^{\epsilon}$. As in [32], if $\mathcal{M}^{0}_{\Delta}$ is a *suitable* and *efficient* scheme for $\mathcal{M}^{0}$, then the scheme is called to be AP.

However, note that there are several definitions of AP schemes in different contexts owing to different interpretations of a *suitable* and *efficient* scheme. So we define an AP scheme for the framework of this article more precisely.

Definition 1.1 (AP schemes)**.** A scheme is called to be AP, provided that it fulfills the following conditions:

(i) It gives a consistent discretization of $\mathcal{M}^{\epsilon}$ for all ϵ, in particular for the limit problem $\mathcal{M}^{0}$.
(ii) It is efficient uniformly in ϵ, *e.g.*, the CFL condition should be uniform in ϵ and the implicit step should be solved efficiently for all $\epsilon > 0$.
(iii) It is stable in some suitable sense, uniformly in ϵ.

For brevity, we call these properties respectively Asymptotic Consistency (AC), Asymptotic Efficiency (AEf), and Asymptotic Stability (AS).

The AP property has been studied widely and several AP (in our terminology AC and AEf) schemes have been developed for the Euler or shallow water equations; see [3,14,17,27,44]. The bottom line of these schemes is a mixed implicit-explicit (IMEX) approach, stems from the more general *operator splitting* method; to split the flux (or its Jacobian) into two parts and treat one part explicitly in time and the other one implicitly in time. This approach is definitely necessary to find schemes with ϵ-uniform CFL conditions. But as mentioned in [16], it is not sufficient at all to claim for asymptotic stability; see for example [1] where it is shown that for an explicit-explicit splitting with the Lax–Wendroff scheme, even if both split parts are stable in terms of CFL condition, the resulting scheme is unconditionally unstable in L_2-norm using von Neumann stability analysis. On the other hand, it is shown in [27] that IMEX schemes are L_2-stable, as long as each step is L_2-stable. So, there is a huge gap between these two cases. Note that using IMEX splitting schemes makes the analysis more delicate compared to explicit splittings; see [7,8] for some results on the Lagrange-projection scheme (which is also the topic of this paper) and [48,51] for a motivating study on linear systems.

Recently, inspired by the Arbitrary Lagrangian-Eulerian (ALE) approach, there have been some works devoted to the so-called Lagrange-projection scheme, which have resulted in some rigorous stability results; see [7, 8, 10, 13] for example. ALE nowadays is a classic approach in mechanics, trying to benefit from the advantages of Eulerian and Lagrangian formulations simultaneously; see [21] for a nice introduction, and this is the heart of the Lagrange-projection scheme, as we will see later on in Section 2. However, most of these works do not take the asymptotic limit into account. For example in [13], a rigorous numerical analysis has been done for a two-phase model including positivity of the density and entropy stability, but with no concern about the incompressible limit. Another study was carried out later for balance laws by Chalons *et al.* [7], in particular for the Euler equations with friction. Moreover, in [8], the Lagrange-projection scheme has been analyzed for the two-dimensional Euler equations, to construct an all-Mach scheme (the scheme with Mach-uniform consistency). So the focus is on the accuracy problems in the low-Mach regime, which one expects to see for Godunov-type schemes (of which Lagrange-projection scheme is a member), and to cure them by a careful look at the truncation error. In fact, it has been shown in [8] that the truncation error of the two-dimensional Lagrange-projection scheme blows up in the low Mach regime, *i.e.*, it behaves as $\mathcal{O}(\frac{\Delta x}{Ma})$ where Ma stands for the Mach number. The authors of [8] could show that the truncation error can be made uniform regarding the Mach number for a particular modification of the scheme, namely by multiplying the dissipation involved in the discretization of the pressure terms by an $\mathcal{O}(Ma)$ term. Although this is a promising step, it is not clear if this uniform accuracy in terms of the truncation error is equivalent to the asymptotic consistency, due to the lack of convergence analysis of the scheme. Very recently in [9], the authors have extended the Lagrange-projection framework to the one-dimensional shallow water equations with particular attention to the well-balancing and the validity of the entropy inequality.

On the other hand, it is well-known that Godunov-type schemes show no accuracy problem for low-Mach one-dimensional problems as long as the initial condition is *well-prepared* (see Def. 3.2 and Refs. [24,36,43]). The reader can consult with [8,18,20,46,47] for more details. This accuracy of Godunov-type schemes motivates the present paper, whose goal is to investigate the results of [13] regarding the Mach number. In this paper, we study the issue of consistency and stability of one-dimensional IMEX Lagrange-projection scheme, or the so-called LP-IMEX scheme as been proposed in [7], in the incompressible limit. In particular, we show that the stability conditions in [13] are uniform in the Mach number provided that the initial condition is well-prepared (this eliminates spurious initial layers). So, all the stability properties in [13] hold without any restriction regarding the Mach number. Also we show that the solution is asymptotically consistent for well-prepared initial data (see Thm. 3.3). Indeed, these estimates imply convergence of a sub-sequence for fixed grids, as ϵ tends to zero (see Appendix A). The study has also been extended to the one-dimensional shallow water equations with topography, where the source term presents an additional difficulty to prove asymptotic consistency and well-balancing. Also note that in Sections 3.2 and 4.2 we prove ϵ-uniform bounds for the (implicit) solution, which justifies the asymptotic expansions used throughout the paper.

The paper is organized as follows. In Section 2 we introduce the splitting along with a brief introduction to the ALE formalism and relaxation schemes. Then, in Section 3, we introduce the IMEX Lagrange-projection scheme with a specific relaxation approximation and then discuss the numerical analysis of the scheme. We prove the formal asymptotic consistency, positivity preserving, stability and entropy stability, all under a non-restrictive CFL condition. Then we show that the formal asymptotic consistency is in fact rigorous. In Section 4 we show similar results for the shallow water equations with a non-flat bottom topography. We then conclude the discussion with some possible extensions and future works. Appendix A provides some results about the consequences of entropy stability on the stability of the solution in the limit $\epsilon \to 0$.

2. Lagrange-projection scheme: continuous PDE level

In this section, we introduce the splitting to be used, let us call it *ALE splitting*, which is inspired by the classical Lagrange-projection scheme (see [22]). In fact, for the isentropic Euler equations, one natural way to split the waves is to split them into acoustic and transport waves. The high speed acoustic waves are formulated in the Lagrangian framework and slow transport waves in the Eulerian one. The idea is similar as for the Lagrange-projection scheme, which consists of solving Riemann problems for the acoustic system in the Lagrangian formulation and then projecting the computed solution onto the fixed Eulerian grid (which is equivalent to the transport system). In this way, the scheme handles Riemann problems in the Lagrangian coordinates which is much easier than the Eulerian one, and takes advantage of using a fixed grid (see [22], Chap. III, Sect. 2.5). It is in this regard that the ALE splitting (as well as the Lagrange-projection scheme) can be understood (see [13]) in the framework of ALE, to write the equations in the *referential* coordinates χ which are necessarily neither spatial (Eulerian) x nor material (Lagrangian) X. The referential frame has a relative velocity v seen from the spatial frame, which is arbitrarily chosen. Note that the Lagrange-projection scheme is a special case of ALE, in which the velocity v is chosen such that after completing each step, the domain is the same as the fixed Eulerian one. We refer the reader to ([13], Sect. 3.3) for more details.

Now, consider system of the isentropic Euler equations in $\mathbb{T}(\Omega) \times \mathbb{R}_+$ where $\mathbb{T}(\Omega)$ denotes a one-dimensional torus:

$$\begin{aligned} \partial_t \varrho + \partial_x(\varrho u) &= 0, \\ \partial_t(\varrho u) + \partial_x\left(\varrho u^2 + p(\varrho)\right) &= 0, \end{aligned} \tag{2.1}$$

with given $\varrho_0(x) := \varrho(x,0)$ and $u_0(x) := u(x,0)$ respectively as the initial density and velocity. In (2.1) $p(\varrho) := \kappa\varrho^\gamma$ with $\kappa > 0$ and $\gamma > 1$ is the isentropic pressure law. As an entropy function, we choose the total energy of the solution ϱE which can be shown to be strictly convex with respect to the conservative variables. The total energy density is written as $E = \mathcal{E} + \frac{u^2}{2}$ where $\mathcal{E}(\varrho) := \frac{\kappa}{\gamma-1}\varrho^{\gamma-1}$ is the internal energy density (see [39]).

Remark 2.1. We stick to this general isentropic pressure function $p(\varrho) = \kappa\varrho^\gamma$ except in Section 4, where we pick $k = \frac{1}{2}$ and $\gamma = 2$ to investigate the shallow water equations. We also only consider periodic boundary conditions for the sake of simplicity of the presentation. However, we expect that with a bit of effort and by changing some of the arguments particularly for the asymptotic consistency analysis (see Sects. 3.1 and 4.1), one can generalize the present study to other boundary conditions such as open boundary condition; see also [13] for some interesting results for the case of coupling boundary conditions.

The ALE splitting splits the original system, (2.1) into the following acoustic and transport sub-systems

$$\begin{aligned} \partial_t \varrho + \varrho\partial_x u &= 0, \\ \partial_t(\varrho u) + \varrho u \partial_x u + \partial_x p &= 0, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \partial_t \varrho + u\partial_x \varrho &= 0, \\ \partial_t(\varrho u) + u\partial_x(\varrho u) &= 0, \end{aligned} \tag{2.3}$$

and one solves them successively. Simply by using the Taylor expansion it can be seen that this splitting is in general (globally) first-order accurate in time. We refer the reader to [28] for more details about the operator splitting methods. Note that the transport part is simply a transport of conservative variables $(\varrho, \varrho u)$ with the velocity field u.

2.1. Lagrange step

In the Lagrangian coordinates, the frame moves with the velocity field. So, what an observer sees is the acoustic part (2.2). It is not difficult to show that it can also be written as

$$\partial_t \tau - \partial_z u = 0, \tag{2.4}$$

$$\partial_t u + \partial_z p = 0, \tag{2.5}$$

where τ is specific volume (the reciprocal of ϱ) and $\mathrm{d}z := \varrho \mathrm{d}x$ is the mass coordinate. This is exactly the classical form of the isentropic Euler equations in the Lagrangian framework. To obtain the *non-dimensionalized* equations, we set

$$\widehat{\tau} := \tau\varrho_\circ, \quad \widehat{u} := \frac{u}{u_\circ}, \quad \widehat{p} := \frac{p}{p_\circ}, \quad p_\circ := \varrho_\circ c_\circ^2/\gamma,$$

where $c_\circ$ is the characteristic sound speed, defined as $c_\circ := \sqrt{\gamma p_\circ/\varrho_\circ}$, and $\varrho_\circ$, $p_\circ$ and $u_\circ$ are characteristic density, pressure and velocity. Also we denote the Mach number as the ratio of the characteristic speed to the sound speed, $Ma := \frac{u_\circ}{c_\circ}$. Thus, after suppressing hats, the equations become

$$\partial_t \tau - \partial_z u = 0, \tag{2.6}$$

$$\partial_t u + \frac{1}{\epsilon^2}\partial_z p_z = 0, \tag{2.7}$$

where $\epsilon := \sqrt{\gamma} Ma$. From now on and for simplicity, we call ϵ the Mach number, though it is different from Ma by the factor $\sqrt{\gamma}$.

To solve this Cauchy initial value problem, we relax the system so that all characteristic fields get linearly degenerate, which is easy to solve the Riemann problem for. We actually substitute the source of genuine nonlinearity $p(\varrho)$ with some variable π, called relaxation pressure and add another equation for π. This is the heart of so-called *relaxation schemes*; we refer the reader to [5, 11, 33, 40] for more details. Like [13], we employ the Suliciu relaxation system [5, 12], which yields the following non-dimensionalized equations:

$$\partial_t \tau - \partial_z u = 0, \tag{2.8}$$

$$\partial_t u + \partial_z \Pi = 0, \tag{2.9}$$

$$\partial_t \Pi + \alpha^2 \partial_z u = \Lambda(p - \pi), \tag{2.10}$$

with the definitions

$$\Pi_\epsilon := \Pi := \frac{\pi}{\epsilon^2}, \qquad \alpha_\epsilon := \alpha := \frac{a}{\epsilon}, \qquad \Lambda_\epsilon := \Lambda := \frac{\lambda}{\epsilon^2},$$

where $a = \mathcal{O}(1)$ is a constant to be specified and λ is the relaxation parameter.

At least formally, one can observe that in the asymptotic regime $\lambda \to \infty$, π tends to p and the original system will be recovered. Also one can easily check that the relaxation system only has linearly-degenerate characteristic fields. To use the feature of linear degeneracy, at first we solve the problem out of equilibrium, setting $\lambda = 0$, and then we project the out-of-equilibrium solution to the equilibrium manifold, *cf.* [13].

In order to prevent the instabilities to happen for this relaxation system, or in other words to enforce the dissipativity of the Chapman–Enskog expansion (see [11, 40]), the parameter α must be chosen sufficiently large according to the so-called *sub-characteristic* or *Whitham stability* condition

$$\alpha^2 > \frac{\max(-p_\tau)}{\epsilon^2}, \tag{2.11}$$

see [6] for the proof.

Remark 2.2. Considering $\epsilon \ll 1$, the sub-characteristic condition (2.11) justifies the choice $\alpha = a/\epsilon$. However, one can choose α differently, for example as a/ϵ^2, which is more diffusive compared to the current α.

Since the relaxation system with $\lambda = 0$ is strictly hyperbolic with eigenvalues given by $0, \pm a$ (compared to exact eigenvalues $0, \pm c$ for the original system) the sub-characteristic condition requires the eigenvalues of the original system to lie between the eigenvalues of the relaxation system, which means that information propagates faster in the relaxation model. Also linear degeneracy of the fields allows us to analytically solve the Riemann problem when $\lambda = 0$. This property justifies by itself the introduction of the proposed relaxation model [8].

For $\lambda = 0$, one can simply put the relaxation system (2.8)–(2.10) into an equivalent diagonal form like ([7], Eq. (12))

$$\tau_t - u_z = 0, \tag{2.12}$$

$$\overrightarrow{w}_t + \alpha \overrightarrow{w}_z = 0, \quad \overrightarrow{w} := \Pi + \alpha u = \frac{\pi}{\epsilon^2} + \frac{a}{\epsilon} u, \tag{2.13}$$

$$\overleftarrow{w}_t - \alpha \overleftarrow{w}_z = 0, \quad \overleftarrow{w} := \Pi - \alpha u = \frac{\pi}{\epsilon^2} - \frac{a}{\epsilon} u. \tag{2.14}$$

Note that $\overrightarrow{w}$ and $\overleftarrow{w}$ are two of Riemann invariants of the relaxation system; the third one is $\mathscr{I} := \Pi + \alpha^2\tau$. So, instead of (2.12) one can use $\mathscr{I}_t = 0$.

Remark 2.3. Note that the naturally-split systems (2.2) and (2.3) are not conservative if they are written in the Eulerian coordinates. As shown in [13], changing the coordinates to the Lagrangian one not only helps solving Riemann problems, but also provides a conservative formulation to circumvent the complications coming with non-conservative products (see [15] for example).

2.2. Projection step

This step is in fact equivalent to remapping the updated solution onto the Eulerian grid so that the referential and spatial (Eulerian) coordinate coincide at the end of each step. Following the notation in [13], the projection step can be summarized as

$$\partial_t \mathbb{U} + u\partial_x \mathbb{U} = 0, \tag{2.15}$$

where $\mathbb{U} := (\varrho, \varrho u)^T$ stands for the conservative variables.

Like the acoustic part, the transport part (2.3) or (2.15) can be written in the Lagrangian coordinates which provides a conservative form; for further details the reader can consult with [13].

3. Lagrange-projection scheme for the isentropic Euler equations

As mentioned above, for linearly-degenerate systems, it is straightforward to solve the Riemann problem. Since along each characteristic line one of the Riemann invariants remains constant, there is only one set of symmetric scalar linear advection equations to be solved for $\overrightarrow{w}$ and $\overleftarrow{w}$, while $\mathscr{I}$ does not change at all.

At the beginning of the Lagrange (acoustic) step from n to $n\dagger$, the Eulerian and Lagrangian coordinates coincide with each other, also the solution of the relaxation system is at equilibrium such that $p_j^n = \pi_j^n$. The implicit Lagrange step reads

$$\tau_j^{n\dagger} = \tau_j^n + \frac{\Delta t}{\Delta z_j^n}\left(\widetilde{u}_{j+\frac{1}{2}}^{n\dagger} - \widetilde{u}_{j-\frac{1}{2}}^{n\dagger}\right). \tag{3.1}$$

$$\overrightarrow{w}_j^{n\dagger} = \overrightarrow{w}_j^n - \frac{a\Delta t}{\epsilon\Delta z_j^n}\left(\overrightarrow{w}_j^{n\dagger} - \overrightarrow{w}_{j-1}^{n\dagger}\right), \tag{3.2}$$

$$\overleftarrow{w}_j^{n\dagger} = \overleftarrow{w}_j^n + \frac{a\Delta t}{\epsilon\Delta z_j^n}\left(\overleftarrow{w}_{j+1}^{n\dagger} - \overleftarrow{w}_j^{n\dagger}\right), \tag{3.3}$$

where $\Delta z_j^n := \varrho_j^n \Delta x$ and $j \in \mathbb{S}$ denotes cell indices when $\mathbb{S}$ is a periodic set (the discretization of Ω). The interface velocity $\widetilde{u}_{j+\frac{1}{2}}^{n\dagger}$ comes from solving a simple Riemann problem for the relaxation system with characteristics 0, $\pm\frac{a}{\epsilon}$ (see [13]), and reads

$$\widetilde{u}_{j+\frac{1}{2}}^{n\dagger} := \frac{u_j^{n\dagger} + u_{j+1}^{n\dagger}}{2} - \frac{1}{2a\epsilon}\left(\pi_{j+1}^{n\dagger} - \pi_j^{n\dagger}\right). \tag{3.4}$$

Note that there are several (equivalent) variants of the scheme (3.1)-(3.3), in different coordinates or with/without using the Riemann invariants; see [13] for further details.

In the next step, the explicit projection step from $n\dagger$ to $n+1$, we map updated values onto the fixed Eulerian grid. There are 4 cases based on the upwind direction ([7], Eq. (34)), which can be summarized as

$$\mathbb{U}_j^{n+1} = \mathbb{U}_j^{n\dagger} + \frac{\Delta t}{\Delta x}\left[(\widetilde{u}_{j-\frac{1}{2}}^{n\dagger})^+ \mathbb{U}_{j-1}^{n\dagger} + \left((\widetilde{u}_{j+\frac{1}{2}}^{n\dagger})^- - (\widetilde{u}_{j-\frac{1}{2}}^{n\dagger})^+\right)\mathbb{U}_j^{n\dagger} - (\widetilde{u}_{j+\frac{1}{2}}^{n\dagger})^- \mathbb{U}_{j+1}^{n\dagger}\right], \tag{3.5}$$

with the definitions $\cdot^+ := \frac{\cdot+|\cdot|}{2}$ and $\cdot^- := \frac{\cdot-|\cdot|}{2}$.

Adding these two steps to each other is what we call the LP-IMEX scheme.

3.1. Numerical analysis of the LP-IMEX scheme

Considering the LP-IMEX scheme introduced in the previous section, one can obtain some stability results, gathered in Theorem 3.3. But firstly and for future reference, let us define the formal incompressible limit of the isentropic Euler equations and the so-called well-prepared initial datum, with the following asymptotic (or Poincaré) expansions for density and velocity

$$\begin{aligned}\varrho(x,t) &= \varrho_{(0)} + \epsilon\varrho_{(1)} + \epsilon^2\varrho_{(2)},\\ u(x,t) &= u_{(0)} + \epsilon u_{(1)} + \epsilon^2 u_{(2)}.\end{aligned} \tag{3.6}$$

Definition 3.1. The formal incompressible limit of the isentropic Euler equations (2.1) reads

$$\begin{aligned}&\varrho_{(0)} = \text{const.}, \qquad \varrho_{(1)} = \text{const.},\\ &\operatorname{div} u_{(0)} = 0,\\ &\partial_t u_{(0)} + \partial_x\left(u_{(0)}^2 + p_{(2)}\right) = 0.\end{aligned}$$

Definition 3.2. For the isentropic Euler equations (2.1), the well-prepared initial datum is defined as (see [20, 34, 35, 42])

$$\varrho_0^{\mathrm{WP}}(x) := \varrho_{(0)} + \epsilon^2\varrho_{(2)}(x), \tag{3.7}$$

$$u_0^{\mathrm{WP}}(x) := u_{(0)} + \epsilon u_{(1)}(x), \tag{3.8}$$

with constant $\varrho_{(0)}$ and $u_{(0)}$.

Theorem 3.3. *The Lagrange-projection scheme (3.1)–(3.3) and (3.5) with a well-prepared initial datum, satisfies the following properties:*

(i) *It can be expressed in the locally conservative form.*

(ii) *The scheme is AC,* i.e., *it gives a consistent discretization of the incompressible Euler equations in terms of Definition 3.1.*

(iii) *Under some non-ϵ-restrictive CFL constraint (3.21), the scheme is positivity preserving,* i.e., *$\varrho_j^n > 0$ provided that $\varrho_j^0 > 0$ for all $j \in \mathbb{S}$. Moreover the density is bounded away from zero,* i.e., *there exists some $\varrho_{\mathrm{LB}}^n > 0$ such that $\varrho_j^n \geq \varrho_{\mathrm{LB}}^n$ for all $j \in \mathbb{S}$.*

(iv) *Under the CFL constraint (3.21) and the sub-characteristic condition (3.33), the solution fulfills the local (cell) entropy (energy) inequality,* i.e.

$$\frac{(\varrho E)_j^{n+1} - (\varrho E)_j^n}{\Delta t} + \frac{\left(\varrho E\widetilde{u} + \dfrac{\widetilde{\pi}\widetilde{u}}{\epsilon^2}\right)_{j+\frac{1}{2}}^{n\dagger} - \left(\varrho E\widetilde{u} + \dfrac{\widetilde{\pi}\widetilde{u}}{\epsilon^2}\right)_{j-\frac{1}{2}}^{n\dagger}}{\Delta x} \leq 0, \tag{3.9}$$

which is consistent with the energy inequality accompanied with the isentropic Euler equations (2.1) [39]

$$\partial_t\left(\varrho E\right) + \partial_x\left(\left(\varrho E + \frac{p}{\epsilon^2}\right)u\right) \leq 0 \quad \textit{in } \mathcal{D}'(\Omega_T).$$

(v) *Under the CFL constraint (3.21) and the sub-characteristic condition (3.33), the computed density, momentum and velocity are stable,* i.e., *bounded in ℓ_∞-norm, uniformly in ϵ.*

We analyze the properties of this scheme in the subsequent subsections. Note that the locally conservative form of the scheme is proved in [13] and skipped here.

Remark 3.4. Throughout this section and the subsequent one, it is very natural to ask about the order of magnitudes of quantities (in terms of ϵ). For now, we only do the analysis formally. That is to say, we only take the explicit ϵ into account and assume that all other quantities are $\mathcal{O}(1)$. But in a separate section, Section 3.2, we justify this assumption.

3.1.1. Proof of asymptotic consistency (ii)

At first, we show that the solution is close to the incompressible limit in the sense of Definition 3.1, *i.e.*, constant density up to the second order of asymptotic expansions, and a *div*-free (solenoidal) zeroth-order velocity component. Then, using these results, we prove that the scheme provides a consistent discretization of the PDE in the limit $\epsilon \to 0$. Thus, the asymptotic consistency in the sense of Definition 1.1 holds.

Considering Definitions 3.1 and 3.2, we consider a well-prepared solution at the step n, *i.e.*

$$\varrho_j^n = \varrho_{0c}^n + \epsilon^2 \varrho_{(2)j}^n, \tag{3.10}$$

$$p_j^n = \pi_{0c}^n + \epsilon^2 \pi_{(2)j}^n, \tag{3.11}$$

$$u_j^n = u_{0c}^n + \epsilon u_{(1)j}^n, \tag{3.12}$$

where $\varrho_{0c}^n, \pi_{0c}^n$ and u_{0c}^n are constant values. Here, we want to show that the scheme (3.1)–(3.3) preserves the well-preparedness of the solution at the step n to the intermediate step $n\dagger$ and then to the next step $n+1$.

For the Lagrange step and after substituting the asymptotic expansion (3.6) into the scheme, we start with the $\mathcal{O}(1/\epsilon)$ terms in the τ-update (3.1), which yield

$$\pi_{(0)j+1}^{n\dagger} - 2\pi_{(0)j}^{n\dagger} + \pi_{(0)j-1}^{n\dagger} = 0.$$

So $\pi_{(0)j}^{n\dagger}$ is a linear sequence over $j \in \mathbb{S}$ and due to periodicity $\pi_{(0)j}^{n\dagger} = \pi_{(0)}^{n\dagger}$ which is constant in space.

Since π and ϱ are two independent variables at this level, we cannot conclude immediately that the same is true for the density. But one can establish their relation by combining (3.1)–(3.3) to find the update for the relaxation pressure as

$$\varrho_j^n \frac{\pi_j^{n\dagger} - \pi_j^n}{\Delta t} + \frac{a^2}{\Delta x}\left(\widetilde{u}_{j+\frac{1}{2}}^{n\dagger} - \widetilde{u}_{j-\frac{1}{2}}^{n\dagger}\right) = 0. \tag{3.13}$$

Then using the τ-update (3.1), it yields

$$a^2\left(\tau_j^{n\dagger} - \tau_j^n\right) + \left(\pi_j^{n\dagger} - \pi_j^n\right) = 0. \tag{3.14}$$

From (3.14) it is clear that

$$a^2\big(\varrho_j^{n\dagger} - \varrho_j^n\big) = \varrho_j^n \varrho_j^{n\dagger}\big(\pi_j^{n\dagger} - \pi_j^n\big). \tag{3.15}$$

So, the leading order part is

$$a^2\big(\varrho_{(0)j}^{n\dagger} - \varrho_{(0)j}^n\big) = \varrho_{(0)j}^n \varrho_{(0)j}^{n\dagger}\big(\pi_{(0)j}^{n\dagger} - \pi_{(0)j}^n\big),$$

which gives that

$$\varrho_{(0)j}^{n\dagger}\Big(a^2 - \varrho_{0c}^n\big(\pi_{(0)}^{n\dagger} - \pi_{0c}^n\big)\Big) = a^2 \varrho_{0c}^n \Longrightarrow \varrho_{(0)j}^{n\dagger} = \varrho_{(0)}^{n\dagger} \text{ const. in space.}$$

Then, due to periodicity and by a spatial summation on (3.1), it can be obtained that $\varrho_{(0)j}^{n\dagger}$ is constant in time as well, *i.e.*, $\varrho_{(0)j}^{n\dagger} = \varrho_{0c}^n$. Also from the update for the relaxation pressure, (3.13), and again due to periodicity, the numerical fluxes cancel out with each other and it turns out that $\pi_{(0)j}^{n\dagger} = \pi_{0c}^n$, constant in both time and space.

Next, we continue with the $\overrightarrow{w}$-update (3.2) (there is no difference between $\overrightarrow{w}$ and $\overleftarrow{w}$ in this regard):

$$\varrho_j^n\left(\frac{\pi_j^{n\dagger}}{\epsilon^2} + \frac{a}{\epsilon}u_j^{n\dagger}\right) = \varrho_j^n\left(\frac{\pi_j^n}{\epsilon^2} + \frac{a}{\epsilon}u_j^n\right) - \frac{a\Delta t}{\epsilon^2 \Delta x}\left(\frac{\pi_j^{n\dagger} - \pi_{j-1}^{n\dagger}}{\epsilon} + a\left(u_j^{n\dagger} - u_{j-1}^{n\dagger}\right)\right).$$

So if one balances $\mathcal{O}(1/\epsilon^2)$ terms, one obtains

$$\varrho_{0c}^n \pi_{(0)j}^{n\dagger} = \varrho_{0c}^n \pi_{0c}^n - \frac{a\Delta t}{\Delta x}\Big(\pi_{(1)j}^{n\dagger} - \pi_{(1)j-1}^{n\dagger} + a\big(u_{(0)j}^{n\dagger} - u_{(0)j-1}^{n\dagger}\big)\Big),$$

which yields

$$\pi_{(1)j}^{n\dagger} - \pi_{(1)j-1}^{n\dagger} + a\big(u_{(0)j}^{n\dagger} - u_{(0)j-1}^{n\dagger}\big) = 0. \tag{3.16}$$

So, there is the possibility that both $\pi_{(1)j}^{n\dagger}$ and $u_{(0)j}^{n\dagger}$ be constant in space. To show it, note that from $\mathcal{O}(1)$ terms in (3.1), one gets

$$\begin{aligned}\varrho_{(0)j}^{n} =& \varrho_{(0)j}^{n\dagger}\left(1 + \frac{\Delta t}{2a\Delta x}\Big(a\big(u_{(0)j+1}^{n\dagger} - u_{(0)j-1}^{n\dagger}\big) - \big(\pi_{(1)j-1}^{n\dagger} - 2\pi_{(1)j}^{n\dagger} + \pi_{(1)j+1}^{n\dagger}\big)\Big)\right)\\ &- \frac{\Delta t}{2a\Delta x}\varrho_{(1)j}^{n\dagger}\big(\pi_{(0)j-1}^{n\dagger} - 2\pi_{(0)j}^{n\dagger} + \pi_{(0)j+1}^{n\dagger}\big).\end{aligned}$$

So,

$$a\big(u_{(0)j+1}^{n\dagger} - u_{(0)j-1}^{n\dagger}\big) - \big(\pi_{(1)j-1}^{n\dagger} - 2\pi_{(1)j}^{n\dagger} + \pi_{(1)j+1}^{n\dagger}\big) = 0. \tag{3.17}$$

Combining (3.17) and (3.16) yields that $\pi_{(1)j}^{n\dagger} = \pi_{(1)}^{n\dagger}$ and $u_{(0)j}^{n\dagger} = u_{(0)}^{n\dagger}$.

Similar to the leading order, one can show that $\pi_{(1)j}^{n\dagger}$ and $\varrho_{(1)j}^{n\dagger}$ are constant in time and space, *i.e.*, $\pi_{(1)j}^{n\dagger} = \pi_{1c}^{n}$ and $\varrho_{(1)j}^{n\dagger} = \varrho_{1c}^{n}$. Hence, the solution of the Lagrange step is close to the incompressible limit.

For the projection step (3.5), we show asymptotic consistency for $\widetilde{u}_{j-\frac{1}{2}}^{n\dagger} < 0$ and $\widetilde{u}_{j+\frac{1}{2}}^{n\dagger} < 0$. The other cases can be analyzed in a very similar way. In this case and for the density, it can be seen that

$$\varrho_j^{n+1} = \varrho_j^{n\dagger} - \frac{\Delta t}{2a\Delta x}\big(\varrho_{j+1}^{n\dagger} - \varrho_j^{n\dagger}\big)\Big(-\frac{\pi_{j+1}^{n\dagger} - \pi_j^{n\dagger}}{\epsilon} + a\big(u_{j+1}^{n\dagger} + u_j^{n\dagger}\big)\Big).$$

So the leading order terms give

$$\begin{aligned}\varrho_{(0)j}^{n+1} = \varrho_{(0)j}^{n\dagger} - \frac{\Delta t}{2a\Delta x}\Big[&-\big(\varrho_{(0)j+1}^{n\dagger} - \varrho_{(0)j}^{n\dagger}\big)\big(\pi_{(1)j+1}^{n\dagger} - \pi_{(1)j}^{n\dagger}\big)\\ &-\big(\varrho_{(1)j+1}^{n\dagger} - \varrho_{(1)j}^{n\dagger}\big)\big(\pi_{(0)j+1}^{n\dagger} - \pi_{(0)j}^{n\dagger}\big)\\ &+a\big(\varrho_{(0)j+1}^{n\dagger} - \varrho_{(0)j}^{n\dagger}\big)\big(u_{(0)j+1}^{n\dagger} + u_{(0)j}^{n\dagger}\big)\Big],\end{aligned}$$

thus the leading order of the computed density is constant, *i.e.*, $\varrho_{(0)j}^{n+1} = \varrho_{(0)j}^{n\dagger} = \varrho_{0c}^{n}$. Similarly, one can find that the first order components are also constant in time and space; if they do not exist in the initial condition, so at the time t_{n+1} there is no $\mathcal{O}(\epsilon)$ density (or pressure) fluctuation:

$$\begin{aligned}\varrho_{(1)j}^{n+1} = \varrho_{(1)j}^{n\dagger} - \frac{\Delta t}{2a\Delta x}\Big[&-\big(\varrho_{(0)j+1}^{n\dagger} - \varrho_{(0)j}^{n\dagger}\big)\big(\pi_{(2)j+1}^{n\dagger} - \pi_{(2)j}^{n\dagger}\big)\\ &-\big(\varrho_{(1)j+1}^{n\dagger} - \varrho_{(1)j}^{n\dagger}\big)\big(\pi_{(1)j+1}^{n\dagger} - \pi_{(1)j}^{n\dagger}\big)\\ &-\big(\varrho_{(2)j+1}^{n\dagger} - \varrho_{(2)j}^{n\dagger}\big)\big(\pi_{(0)j+1}^{n\dagger} - \pi_{(0)j}^{n\dagger}\big)\\ &+a\big(\varrho_{(0)j+1}^{n\dagger} - \varrho_{(0)j}^{n\dagger}\big)\big(u_{(1)j+1}^{n\dagger} + u_{(1)j}^{n\dagger}\big)\\ &+a\big(\varrho_{(1)j+1}^{n\dagger} - \varrho_{(1)j}^{n\dagger}\big)\big(u_{(0)j+1}^{n\dagger} + u_{(0)j}^{n\dagger}\big)\Big],\end{aligned}$$

and $\varrho_{(1)j}^{n+1} = \varrho_{(1)j}^{n\dagger} = \varrho_{1c}^{n} = 0$.

To show the *div*-free condition, one can consider $\mathcal{O}(1)$ terms of the momentum update in (3.5):

$$\begin{aligned}\varrho^{n+1}_{(0)j}u^{n+1}_{(0)j} = \varrho^{n\dagger}_{(0)j}u^{n\dagger}_{(0)j} - \frac{\Delta t}{2a\Delta x}\Big[&- \big(\varrho^{n\dagger}_{(0)j+1}u^{n\dagger}_{(0)j+1} - \varrho^{n\dagger}_{(0)j}u^{n\dagger}_{(0)j}\big)\big(\pi^{n\dagger}_{(1)j+1} - \pi^{n\dagger}_{(1)j}\big)\\ &- \big(\varrho^{n\dagger}_{(1)j+1}u^{n\dagger}_{(0)j+1} - \varrho^{n\dagger}_{(1)j}u^{n\dagger}_{(0)j}\big)\big(\pi^{n\dagger}_{(0)j+1} - \pi^{n\dagger}_{(0)j}\big)\\ &- \big(\varrho^{n\dagger}_{(0)j+1}u^{n\dagger}_{(1)j+1} - \varrho^{n\dagger}_{(0)j}u^{n\dagger}_{(1)j}\big)\big(\pi^{n\dagger}_{(0)j+1} - \pi^{n\dagger}_{(0)j}\big)\\ &+ a\big(\varrho^{n\dagger}_{(0)j+1}u^{n\dagger}_{(0)j+1} - \varrho^{n\dagger}_{(0)j}u^{n\dagger}_{(0)j}\big)\big(u^{n\dagger}_{(0)j+1} + u^{n\dagger}_{(0)j}\big)\Big].\end{aligned}$$

Thus $u^{n+1}_{(0)j} = u^{n\dagger}_{(0)j} = u^n_{0c}$, and the leading order component of the velocity field is constant (solenoidal). Hence, combining the results for the Lagrange and projection steps together, it is obvious that the limit conditions are satisfied.

To prove asymptotic consistency in the sense of Definition 1.1, it remains to show the consistency of the discretization in the limit. For the Lagrange step the consistency holds if the velocity update

$$\frac{u^{n\dagger}_j - u^n_j}{\Delta t} + \frac{1}{2\Delta z^n_j \epsilon^2}\left(\pi^{n\dagger}_{j+1} - \pi^{n\dagger}_{j-1}\right) - \frac{a/\epsilon}{2\Delta z^n_j}\left(u^{n\dagger}_{j+1} - 2u^{n\dagger}_j + u^{n\dagger}_{j-1}\right) = 0, \tag{3.18}$$

is a consistent discretization of $\partial_t u + \frac{1}{\epsilon^2}\partial_z \pi = 0$ in the limit, when (3.18) gives

$$\frac{u^{n\dagger}_{(0)j} - u^n_{(0)j}}{\Delta t} + \frac{1}{2\Delta z^n_{(0)j}}\left(\pi^{n\dagger}_{(2)j+1} - \pi^{n\dagger}_{(2)j-1}\right) - \frac{a}{2\Delta z^n_{(0)j}}\left(u^{n\dagger}_{(1)j+1} - 2u^{n\dagger}_{(1)j} + u^{n\dagger}_{(1)j-1}\right) = 0. \tag{3.19}$$

It is clear that (3.19) is the Rusanov scheme applied to $\partial_t u_{(0)} + \partial_z \pi_{(2)} = 0$, so the Lagrange step is AC.

To show the consistency of the discretization in the limit also for the projection step, comparing (2.15) and (3.5), it is sufficient to confirm that $\widetilde{u}^{n\dagger}_{(0)j+\frac{1}{2}}$ is consistent with $u_{(0)}$. This is in fact the case, due to the definition of $\widetilde{u}^{n\dagger}_{j+\frac{1}{2}}$ in (3.4) and the asymptotic behavior of $u^{n\dagger}_{(0)j}$ and $\pi^{n\dagger}_{(1)j}$, namely that $u^{n\dagger}_{(0)j}$ and $\pi^{n\dagger}_{(1)j}$ are constant in space. So, the projection step (3.5) is a consistent discretization of (2.15) and the scheme is AC in the sense of Definition 1.1.

3.1.2. Proof of density positivity (iii)

In this section, we show that the density is positive under a time step condition which is not restrictive for small ϵ. From ([13], Eq. (2.25a)) we first define the *local acoustic CFL ratio* μ_j, and the *local apparent propagation factor* e_j as

$$\mu_j := \frac{a\Delta t}{\Delta z^n_j}, \qquad e_j := \frac{\mu_j/\epsilon}{1 + \mu_j/\epsilon}.$$

Then, one can write (3.2) as

$$\overrightarrow{w}^{n\dagger}_j = e_j \, \overrightarrow{w}^{n\dagger}_{j-1} + (1 - e_j)\, \overrightarrow{w}^{n}_j \, .$$

Since $0 < e_j < 1$ (which can be satisfied for all $\epsilon > 0$ uniformly), the updates for $\overrightarrow{w}_j$ and $\overleftarrow{w}_j$ are monotone, *i.e.*, no new extremum can be generated. To show it for $\overrightarrow{w}^{n\dagger}$, assume that i is the index of the maximum value of $\overrightarrow{w}^{n\dagger}_j$, that is $\overrightarrow{w}^{n\dagger}_i \geq \overrightarrow{w}^{n\dagger}_j$ for all $j \in \mathbb{S}$. So,

$$\overrightarrow{w}^{n\dagger}_i \leq e_i \, \overrightarrow{w}^{n\dagger}_i + (1 - e_i)\, \overrightarrow{w}^{n}_i \, .$$

Thus $\overrightarrow{w}_i^{n\dagger} \leq \overrightarrow{w}_i^{n}$ and then $\max_{j\in\mathbb{S}} \overrightarrow{w}_j^{n\dagger} \leq \max_{j\in\mathbb{S}} \overrightarrow{w}_j^{n}$. So, it is bounded from above. The proofs for the lower-bound and $\overleftarrow{w}^{n\dagger}$ are likewise. Hence, defining the upper-bounds $\overrightarrow{M}^n$ and $\overleftarrow{M}^n$ and the lower-bounds $\overrightarrow{m}^n$ and $\overleftarrow{m}^n$ for $\overrightarrow{w}^n$ and $\overleftarrow{w}^n$, one can write

$$\begin{aligned}\overrightarrow{m}^n &\leq \overrightarrow{w}_j^{n\dagger} \leq \overrightarrow{M}^n,\\ \overleftarrow{m}^n &\leq \overleftarrow{w}_j^{n\dagger} \leq \overleftarrow{M}^n .\end{aligned} \tag{3.20}$$

Having (3.20), one can show the following theorem.

Theorem 3.5. *For some Δt satisfying*

$$\frac{\Delta t}{\Delta x} \leq \frac{2a/\epsilon}{\left(\overrightarrow{M}^n - \overleftarrow{m}^n\right)^+ - \left(\overrightarrow{m}^n - \overleftarrow{M}^n\right)^-}, \tag{3.21}$$

the LP-IMEX scheme preserves the positivity of density provided that $\varrho_j^0 > 0$ for all $j \in \mathbb{S}$.

Proof. Along the lines of [13], for the Lagrange step to satisfy positivity, one gets from the τ-update (3.1) that

$$\frac{\Delta t}{\Delta x}\left(\widetilde{u}_{j-\frac{1}{2}}^{n\dagger} - \widetilde{u}_{j+\frac{1}{2}}^{n\dagger}\right) < 1, \tag{3.22}$$

which ensures $\varrho_j^{n\dagger} > 0$ for all $j \in \mathbb{S}$. But on the other hand, Δt should be such that the projection step is a convex combination, thus

$$\frac{\Delta t}{\Delta x}\left(\left(\widetilde{u}_{j-\frac{1}{2}}^{n\dagger}\right)^+ - \left(\widetilde{u}_{j+\frac{1}{2}}^{n\dagger}\right)^-\right) < 1. \tag{3.23}$$

Between (3.22) and (3.23) the stronger condition should be chosen, which is (3.23). Then, based on the definition of $\widetilde{u}^{n\dagger}$, we express Δt in terms of $\overrightarrow{M}$, $\overleftarrow{M}$, $\overrightarrow{m}$ and $\overleftarrow{m}$, which concludes the proof. ☐

The next goal is to show that this bound for the time step is uniform in ϵ, *i.e.*, it does not vanish as the Mach number goes to zero. One can pose the following corollary.

Corollary 3.6. *For well-prepared initial data, the time step restriction (3.21) is uniform in ϵ.*

Proof. Recall that asymptotic consistency implies that with a well-prepared initial datum and for $\epsilon \ll 1$, the density (and thus the pressure) has a constant part as the zeroth-order term in the asymptotic expansion. So the differences $\overrightarrow{M}^n - \overleftarrow{m}^n$ and $\overrightarrow{m}^n - \overleftarrow{M}^n$ are not of $\mathcal{O}(1/\epsilon^2)$ but $\mathcal{O}(1/\epsilon)$, thus the CFL constraint (3.21) is uniform in ϵ. In other words, the solution can be expanded w.r.t. ϵ, so

$$\begin{aligned}\overrightarrow{M}^n &\leq \frac{p_0^n}{\epsilon^2} + \max_{j\in\mathbb{S}} p_{2,j}^n + \frac{a}{\epsilon}\left(u_0^n + \epsilon \max_{j\in\mathbb{S}}(u_{1,j}^n)\right), & \overrightarrow{m}^n &\geq \frac{p_0^n}{\epsilon^2} + \min_{j\in\mathbb{S}} p_{2,j}^n + \frac{a}{\epsilon}\left(u_0^n + \epsilon \min_{j\in\mathbb{S}}(u_{1,j}^n)\right),\\ \overleftarrow{M}^n &\leq \frac{p_0^n}{\epsilon^2} + \max_{j\in\mathbb{S}} p_{2,j}^n - \frac{a}{\epsilon}\left(u_0^n + \epsilon \max_{j\in\mathbb{S}}(u_{1,j}^n)\right), & \overleftarrow{m}^n &\geq \frac{p_0^n}{\epsilon^2} + \min_{j\in\mathbb{S}} p_{2,j}^n - \frac{a}{\epsilon}\left(u_0^n + \epsilon \min_{j\in\mathbb{S}}(u_{1,j}^n)\right).\end{aligned}$$

Thus,

$$\begin{aligned}\overrightarrow{M}^n - \overleftarrow{m}^n &\leq \frac{a}{\epsilon}\left(2u_0^n + \epsilon\left(\max_{j\in\mathbb{S}}(u_{1,j}^n) + \min_{j\in\mathbb{S}}(u_{1,j}^n)\right)\right) + \left(\max_{j\in\mathbb{S}} p_{2,j}^n - \min_{j\in\mathbb{S}} p_{2,j}^n\right),\\ \overrightarrow{m}^n - \overleftarrow{M}^n &\leq \frac{a}{\epsilon}\left(2u_0^n - \epsilon\left(\max_{j\in\mathbb{S}}(u_{1,j}^n) + \min_{j\in\mathbb{S}}(u_{1,j}^n)\right)\right) - \left(\max_{j\in\mathbb{S}} p_{2,j}^n - \min_{j\in\mathbb{S}} p_{2,j}^n\right),\end{aligned}$$

and one gets

$$\lim_{\epsilon\to 0}\left[\frac{2a/\epsilon}{\left(\overrightarrow{M}^n-\overleftarrow{m}^n\right)^+-\left(\overrightarrow{m}^n-\overleftarrow{M}^n\right)^-}\right]\geq\frac{2a/\epsilon}{\mathcal{O}(\frac{1}{\epsilon})+\mathcal{O}(1)}\geq C. \tag{3.24}$$

Hence, there is an $\mathcal{O}(1)$ constant which bounds (3.21) from below, *i.e.*, the condition (3.21) is uniform in ϵ. □

The following lemma shows that the density is also bounded from below for a finite time.

Lemma 3.7. *Under the condition* (3.21)*, the computed density* $\{\varrho_j^{n+1}\}_{j\in\mathbb{S}}$ *is bounded away from zero in a finite time, where the lower-bound is given by*

$$\varrho_{\mathrm{LB}}^{n+1}:=\frac{\min_{j\in\mathbb{S}}\varrho_j^n}{1+\dfrac{\Delta t\epsilon}{2a\Delta x}\left[\left(\overrightarrow{M}^n+\overleftarrow{M}^n\right)-\left(\overrightarrow{m}^n+\overleftarrow{m}^n\right)\right]}>0. \tag{3.25}$$

Proof. From the τ-update (3.1) and $\widetilde{u}^{n\dagger}_{j+\frac{1}{2}}=\frac{\epsilon}{2a}\left(\overrightarrow{w}^{n\dagger}_j-\overleftarrow{w}^{n\dagger}_{j+1}\right)$, one can get

$$\varrho_j^n=\varrho_j^{n\dagger}\left(1+\frac{\epsilon\Delta t}{2a\Delta x}\left(\overrightarrow{w}^{n\dagger}_j-\overleftarrow{w}^{n\dagger}_{j+1}-\overrightarrow{w}^{n\dagger}_{j-1}+\overleftarrow{w}^{n\dagger}_j\right)\right).$$

So, to find the minimum value of the computed density, one should determine the maximum value of the right-hand side. Due to (3.20) and under the condition (3.22), it can be seen that

$$\varrho^{n\dagger}\geq\frac{\varrho_j^n}{1+\dfrac{\Delta t\epsilon}{2a\Delta x}\left[\left(\overrightarrow{M}^n+\overleftarrow{M}^n\right)-\left(\overrightarrow{m}^n+\overleftarrow{m}^n\right)\right]}\cdot$$

Thus, since the projection step is a convex combination under the condition (3.21), the lower-bound is obtained as (3.25). □

3.1.3. Proof of local energy inequality (iv)

We show that the solution of the scheme satisfies the energy inequality under an ϵ-independent time step restriction. For the Lagrange step, based on ([13], Thm. 2.3), we define the entropy function for the symmetric advection problem, (2.13)–(2.14), as

$$\eta(\overrightarrow{w},\overleftarrow{w}):=s(\overrightarrow{w})+s(\overleftarrow{w}),\quad s(w):=\frac{\epsilon^2w^2}{4a^2}\cdot$$

So, it can be rewritten as

$$\eta(\overrightarrow{w},\overleftarrow{w})=\frac{1}{2}\left(u^2+\frac{\pi^2}{\epsilon^2a^2}\right)=E-\frac{\mathcal{E}}{\epsilon^2}+\frac{\pi^2}{2a^2\epsilon^2}, \tag{3.26}$$

since after non-dimensionalization, one gets $E=\frac{\mathcal{E}}{\epsilon^2}+\frac{u^2}{2}$ where $\mathcal{E}(\varrho)=\frac{\kappa}{\gamma-1}\varrho^{\gamma-1}$. For later use, we should mention that such a definition of internal energy fulfills the Weyl's assumptions as defined below.

Definition 3.8. The Weyl's assumptions for the internal energy function are (see [13, 50])

$$\mathcal{E}>0,\quad\mathcal{E}_\tau=-p<0,\quad\mathcal{E}_{\tau\tau}>0,\quad\mathcal{E}_{\tau\tau\tau}<0.$$

We also define an entropy flux function $\psi(\overrightarrow{w},\overleftarrow{w})$ as

$$\psi(\overrightarrow{w},\overleftarrow{w}) := \frac{a}{\epsilon^2}\left(s(\overrightarrow{w}) - s(\overleftarrow{w})\right) = \frac{\pi u}{\epsilon^2}\cdot \tag{3.27}$$

Then, the cell entropy inequality reads

$$\frac{\eta_j^{n\dagger} - \eta_j^n}{\Delta t} + \frac{\psi_{j+\frac{1}{2}}^{n\dagger} - \psi_{j-\frac{1}{2}}^{n\dagger}}{\Delta z_j^n} \leq 0. \tag{3.28}$$

Substituting (3.26) and (3.27), one can relate the entropy inequality for the symmetric advection problem to the energy inequality for the isentropic Euler equations, *i.e.*

$$\varrho_j^n \frac{E_j^{n\dagger} - E_j^n}{\Delta t} + \frac{\left(\frac{\pi u}{\epsilon^2}\right)_{j+\frac{1}{2}}^{n\dagger} - \left(\frac{\pi u}{\epsilon^2}\right)_{j-\frac{1}{2}}^{n\dagger}}{\Delta x} \leq \frac{\varrho_j^n}{\epsilon^2} \underbrace{\left[\mathcal{E}_j^{n\dagger} - \mathcal{E}_j^n - \frac{\left(\pi_j^{n\dagger}\right)^2 - \left(\pi_j^n\right)^2}{2a^2}\right]}_{=:\mathcal{R}_j^{n\dagger}}\cdot \tag{3.29}$$

Then, to prove entropy stability of the scheme, one should show that the entropy residual $\mathcal{R}_j^{n\dagger}$ is non-positive. Considering $\pi_j^n = p_j^n$, we rewrite $\mathcal{R}_j^{n\dagger}$ as

$$\mathcal{R}_j^{n\dagger} := \mathcal{E}_j^{n\dagger} - \mathcal{E}_j^n - \frac{p_j^n}{2a^2}\left(\pi_j^{n\dagger} - p_j^n\right) - \frac{\left(\pi_j^{n\dagger} - p_j^n\right)^2}{2a^2}$$
$$(\text{due to (3.14)}) = \mathcal{E}_j^{n\dagger} - \mathcal{E}_j^n + p_j^n\left(\tau_j^{n\dagger} - \tau_j^n\right) - \frac{a^2}{2}\left(\tau_j^{n\dagger} - \tau_j^n\right)^2.$$

On the other hand, from a Taylor expansion with integral remainder, one gets

$$\mathcal{E}_j^{n\dagger} = \mathcal{E}_j^n + \mathcal{E}_\tau|_{x_j,t_n}\left(\tau_j^{n\dagger} - \tau_j^n\right) + \int_{\tau_j^n}^{\tau_j^{n\dagger}} \mathcal{E}_{\tau\tau}(\xi)\left(\tau_j^{n\dagger} - \xi\right)\mathrm{d}\xi.$$

Then, Weyl's assumptions and a change of variables in the integral (re-parameterization) yield that

$$\mathcal{E}_j^{n\dagger} = \mathcal{E}_j^n - p_j^n\left(\tau_j^{n\dagger} - \tau_j^n\right) + \left(\tau_j^{n\dagger} - \tau_j^n\right)^2 \int_0^1 \mathcal{E}_{\tau\tau}(\tau_j^{n+\frac{1}{2}})(1-\zeta)\mathrm{d}\zeta, \tag{3.30}$$

where $\tau_j^{n+\frac{1}{2}} := \zeta\tau_j^{n\dagger} + (1-\zeta)\tau_j^n$. So, for the entropy residual to be non-positive, one gets

$$\mathcal{R}_j^{n\dagger} = \left(\tau_j^{n\dagger} - \tau_j^n\right)^2 \int_0^1 \left(\mathcal{E}_{\tau\tau}(\tau_j^{n+\frac{1}{2}}) - a^2\right)(1-\zeta)\mathrm{d}\zeta \tag{3.31}$$

$$= \left(\tau_j^{n\dagger} - \tau_j^n\right)^2 \int_0^1 \left(-p_\tau(\tau_j^{n+\frac{1}{2}}) - a^2\right)(1-\zeta)\mathrm{d}\zeta \leq 0, \tag{3.32}$$

and a sufficient condition would be to set the integrand to be negative. Since $p_\tau = -\kappa\gamma\varrho^{1+\gamma}$ it yields

$$a^2 \geq \kappa\gamma \max_{j\in\mathbb{S}} \max_{0\leq\zeta\leq1}\left(\left(\varrho_j^{n+\frac{1}{2}}(\zeta)\right)^{\gamma+1}\right) = \kappa\gamma\max\left(\|\varrho^{n\dagger}\|_{\ell_\infty}^{\gamma+1}, \|\varrho^n\|_{\ell_\infty}^{\gamma+1}\right), \tag{3.33}$$

which satisfies the sub-characteristic condition.

For the projection step, it is clear that due to Jensen's inequality the energy inequality holds as

$$(\varrho E)_j^{n+1} \leq \varrho_j^n E_j^{n\dagger} - \frac{\Delta t}{\Delta x}\left((\varrho E\widetilde{u})_{j+\frac{1}{2}}^{n\dagger} - (\varrho E\widetilde{u})_{j-\frac{1}{2}}^{n\dagger}\right). \tag{3.34}$$

Combining (3.29) and (3.34) we get the energy inequality (3.9) under the ϵ-uniform time restriction (3.21) and the sub-characteristic condition (3.33).

3.1.4. Proof of ℓ_∞-stability (v)

In this section, we prove the stability of the LP-IMEX scheme in the ℓ_∞-norm.

Lemma 3.9. *For the well-prepared initial data, the computed density, momentum and velocity are stable in ℓ_∞-norm uniformly in ϵ.*

Proof. As we have shown in Appendix A for fixed ϵ, the entropy stability is enough to conclude the ℓ_∞ stability provided that the density is shown to be positive. Thus, the density, velocity and so the momentum are stable. For the proof of ϵ-uniformity of these results, we refer the reader to Appendix A. □

3.2. Rigorous analysis of asymptotic consistency

The proofs of asymptotic consistency for numerical schemes are often based on the formal asymptotic expansion as we presented in Section 3.1, that is, using a well-prepared initial datum one analyzes the update as $\epsilon \to 0$. The analysis is rather formal; one usually does not show how the variables change in terms of ϵ, but inserts an asymptotic expansion into the scheme and does the formal asymptotic analysis, assuming implicitly that all the variables are $\mathcal{O}(1)$ in terms of ϵ. In this part, we show that it is possible for the LP-IMEX scheme to go further and show asymptotic consistency more rigorously.

In this approach, the main point is to study the implicit step to check how the *unique* updated solution behaves as $\epsilon \to 0$. Once we show that this solution does not blow up for the limit, one can combine it with the update and show asymptotic consistency, *e.g.*, by studying the kernel of the solution operator of the implicit step. Such an analysis can be done by combining the formal analysis in Section 3.1 with Theorem 3.10 below. The approach we present here to justify the formal analysis is akin to what has been used by Bispen in [2], in the context of the Finite Volume Evolution Galerkin (FVEG) scheme [3].

Note that for the scheme written in the form of (3.1)–(3.3), $\overrightarrow{w}^{n\dagger}$ and $\overleftarrow{w}^{n\dagger}$ should be computed implicitly, and then $\tau^{n\dagger}$ is obtained explicitly. Now, let us define $\mathcal{N} := |\mathbb{S}|$ and consider the matrix $\overrightarrow{J}$ as the $\mathcal{N} \times \mathcal{N}$ coefficient matrix for the implicit update of $\overrightarrow{w}$, *i.e.*

$$\overrightarrow{J}\, \underline{\overrightarrow{w}}^{n\dagger} = \underline{\overrightarrow{y}}, \tag{3.35}$$

with

$$\overrightarrow{J} := \frac{a\Delta t}{\epsilon} \begin{bmatrix} 1 + \dfrac{\epsilon \Delta z_1^n}{a\Delta t} & 0 & \cdots & -1 \\ -1 & 1 + \dfrac{\epsilon \Delta z_2^n}{a\Delta t} & 0 & \cdots \\ 0 & \ddots & \ddots & 0 \\ 0 & \ddots & -1 & 1 + \dfrac{\epsilon \Delta z_{\mathcal{N}}^n}{a\Delta t} \end{bmatrix}, \tag{3.36}$$

with vector $\underline{\overrightarrow{y}}$ specified in the case of isentropic Euler equations by (3.2), as $\underline{\overrightarrow{y}} := \underline{\Delta z^n} \odot \underline{\overrightarrow{w}}^n$, where $\odot$ denotes the entry-wise or Hadamard product. Likewise we denote $\overleftarrow{J}$ and $\underline{\overleftarrow{y}}$; but, we will only present the proofs in this section and Section 4 for the former since they are very similar for the latter. Now, we are in a position to pose the following theorem.

Theorem 3.10. *Consider the matrix $\overrightarrow{J}$ as defined in (3.36). Then*

(i) *$\overrightarrow{J}$ is non-singular for all $\epsilon > 0$;*

(ii) *$\lim_{\epsilon\to 0} \| \overrightarrow{J}^{-1} \|$ is bounded for any natural matrix norm.*

Proof. Regarding part (i), it is clear that matrix $\overrightarrow{J}$ is strictly diagonally dominant (SDD), and it is a classical result that SDD matrices are non-singular for instance (see [29], Thm. 6.1.10) for instance. This is enough to show the non-singularity of $\overrightarrow{J}$, and so to conclude that the solution of the implicit step, $\underline{\overrightarrow{w}}^{n\dagger}$, is unique.

For point (ii), the infinity-norm of the inverse of an SDD matrix $M \in \mathbb{R}^{n\times n}$ can be bounded as [49]

$$\|M^{-1}\|_\infty \leq \max_{1\leq i\leq n} \frac{1}{\Delta_i(M)}, \qquad \Delta_i(M) := |M_{ii}| - \sum_{j\neq i} |M_{ij}|. \tag{3.37}$$

For $\overrightarrow{J}$, one can find $\Delta_i(\overrightarrow{J}_\mu) = \Delta z_i^n > 0$. So there is an ϵ-uniform bound for the infinity-norm of matrix inverse. Since $\mathcal{N}$ is fixed, all matrix norms are equivalent which implies the point (ii). □

From Theorem 3.10 one can immediately conclude that a unique solution $\underline{\overrightarrow{w}}^{n\dagger}$ (thus a unique numerical solution for the whole scheme) exists, which has the same order as $\underline{\overrightarrow{w}}^n$ in terms of ϵ. Also by Theorem 3.10 and (3.35), one can see that the leading order of $\overrightarrow{w}^{n\dagger}$, thus (due to similar results for $\overleftarrow{w}$) $\pi_{(0)}^{n\dagger}$ are constant. Employing (3.14) one can confirm that $\varrho^{n\dagger} = \mathcal{O}(1)$. Showing the boundedness of $u_{(0)}^{n\dagger}$ needs more work. From (3.35) one can write the update for $u^{n\dagger}$ as

$$\left(\Delta z_j^n + \frac{a\Delta t}{\epsilon}\right) u_j^{n\dagger} - \frac{a\Delta t}{2\epsilon} u_{j-1}^{n\dagger} - \frac{a\Delta t}{2\epsilon} u_{j+1}^{n\dagger} = u_j^n \Delta z_j^n - \frac{\Delta t}{2\epsilon^2}\left(\pi_{j+1}^{n\dagger} - \pi_{j-1}^{n\dagger}\right), \tag{3.38}$$

which can be recast as a linear system of equations with the companion matrix H defined as

$$H := \begin{bmatrix} \Delta z_1^n + \frac{a\Delta t}{\epsilon} & -\frac{a\Delta t}{2\epsilon} & 0 & \cdots & -\frac{a\Delta t}{2\epsilon} \\ -\frac{a\Delta t}{2\epsilon} & \Delta z_2^n + \frac{a\Delta t}{\epsilon} & -\frac{a\Delta t}{2\epsilon} & 0 & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{3.39}$$

Since H is SDD, one can show (as in Thm. 3.10) that H^{-1} exists and is bounded, *i.e.*, $\lim_{\epsilon\to 0} \|H^{-1}\| < \infty$. Moreover, the rigorous results of this section implies that $\pi_{(0)}^{n\dagger}$ is constant; so, the right-hand side of (3.38) is $\mathcal{O}(1/\epsilon)$. On the other hand, one can show that the inverse of matrix H is a constant matrix (up to $\mathcal{O}(\epsilon)$); we refer the reader to Lemma 4.4 for a very similar proof. Thus $u_{(0)}^{n\dagger} = \mathcal{O}(1)$ due to periodic boundary conditions.

The boundedness in terms of ϵ makes the asymptotic consistency analysis in Section 3.1 rigorous and shows the behavior of the quantities in terms of ϵ. Since the projection step is explicit, its asymptotic consistency can be simply studied as in Section 3.1.

Remark 3.11. This approach proves asymptotic consistency rigorously, *i.e.*, the solution moves to the limit as $\epsilon \to 0$. This is the result that makes the uniformity proofs of the previous sections valid and rigorous, as been mentioned earlier in Remark 3.4.

To summarize, the scheme is AC and AS, and since $\overrightarrow{J}$ and $\overleftarrow{J}$ can be inverted simply due to their structure and that the time step is ϵ-uniform, it is also AEf. Thus the scheme is AP in the sense of Definition 1.1.

4. Lagrange-projection scheme for the shallow water equations with topography

In this section, we show that similar stability arguments work for the LP-IMEX scheme applied to the non-dimensionalized shallow water equations with non-flat bottom:

$$\begin{aligned} \partial_t h + \partial_x q &= 0, \\ \partial_t q + \partial_x \left(\frac{q^2}{h} + \frac{p(h)}{\epsilon^2}\right) &= -\frac{h b_x}{\epsilon^2}, \end{aligned} \tag{4.1}$$

where h and $q := hu$ stand respectively for the water height and the momentum and ϵ denotes the Froude number, defined as the ratio of the characteristic speed to the speed of gravity waves. Also $b(x)$ is the bottom function, and the pressure function is chosen as before but with $k = \frac{1}{2}$ and $\gamma = 2$, *i.e.*, $p(h) = \frac{h^2}{2}$. Note that for this shallow water model to be valid, the bottom slope b_x should be small enough such that $\tan\theta \approx \theta$ where $\tan\theta$ is the bottom slope; see [4] for details.

We omit the details of the splitting and numerical scheme, and refer the reader to consult with ([7], Sect. 3.2.2) and [10]; although the considered source terms are not exactly the same, the structure is similar. Also very recently [9] has tailored the scheme to the shallow water equations. We only need to mention that the transport sub-system is exactly like the the homogeneous case (2.3) with the conservative variable $\mathbb{U} := (h, hu)^T$ in (2.15), while the acoustic sub-system includes the source term in addition. So the relaxation system reads (see also [9])

$$
\begin{aligned}
&\tau_t - u_z = 0,\\
&\overrightarrow{w}_t + \alpha \overrightarrow{w}_z = -\frac{\alpha}{\epsilon^2}\frac{b_z}{\tau},\\
&\overleftarrow{w}_t - \alpha \overleftarrow{w}_z = \frac{\alpha}{\epsilon^2}\frac{b_z}{\tau},
\end{aligned}
\tag{4.2}
$$

with the same $\alpha = \frac{a}{\epsilon}$ as in Section 2 for the isentropic Euler equations.

Using this splitting, one can see that the projection step is like (3.5). Also motivated by ([7], Sect. 5; See also [9]), the Lagrange step of the scheme can be written as

$$
\tau_j^{n\dagger} = \tau_j^n + \frac{\Delta t}{\Delta z_j^n}\left(\widetilde{u}_{j+\frac{1}{2}}^{n\dagger} - \widetilde{u}_{j-\frac{1}{2}}^{n\dagger}\right). \tag{4.3}
$$

$$
\overrightarrow{w}_j^{n\dagger} = \overrightarrow{w}_j^{n} - \frac{a\Delta t}{\epsilon \Delta z_j^n}\left(\overrightarrow{w}_j^{n\dagger} - \overrightarrow{w}_{j-1}^{n\dagger}\right) - \frac{\Delta t a}{\epsilon^3}\frac{\Delta z_{j-\frac{1}{2}}^n}{\Delta z_j^n} b_{x,j-\frac{1}{2}}, \tag{4.4}
$$

$$
\overleftarrow{w}_j^{n\dagger} = \overleftarrow{w}_j^{n} + \frac{a\Delta t}{\epsilon \Delta z_j^n}\left(\overleftarrow{w}_{j+1}^{n\dagger} - \overleftarrow{w}_{j}^{n\dagger}\right) + \frac{\Delta t a}{\epsilon^3}\frac{\Delta z_{j+\frac{1}{2}}^n}{\Delta z_j^n} b_{x,j+\frac{1}{2}}, \tag{4.5}
$$

where $\Delta z_{j+\frac{1}{2}}^n := \frac{\Delta z_j^n + \Delta z_{j+1}^n}{2}$, $b_{x,j+\frac{1}{2}} := \frac{b_{j+1} - b_j}{\Delta x}$ is the one-sided discretization of the bottom function, and the interface velocity is defined as

$$
\widetilde{u}_{j+\frac{1}{2}}^{n\dagger} := \frac{u_j^{n\dagger} + u_{j+1}^{n\dagger}}{2} - \frac{1}{2a\epsilon}\left(\pi_{j+1}^{n\dagger} - \pi_j^{n\dagger}\right) - \frac{1}{2a\epsilon}\Delta z_{j+\frac{1}{2}}^n b_{x,j+\frac{1}{2}}. \tag{4.6}
$$

Notice that this choice of $b_{x,j+\frac{1}{2}}$ provides the C-property (or well-balancing, which is to preserve the steady states at the discrete level, $cf.$ [5] for details) for the lake at rest (LaR) equilibrium state which is defined as the set

$$
\mathcal{U}_{LaR}^{\Delta} := \left\{ \mathbb{U}_j = \begin{bmatrix} h_j \\ q_j \end{bmatrix} \mid h_j = \eta - b_j, u_j = 0, \forall j \in \mathbb{S} \right\},
$$

with zero velocity and flat water surface, where $\eta := h + b$ is the water surface.

The basic difference of scheme (4.3)–(4.5) (LP-IMEX for the shallow water equations) with (3.1)–(3.3) (LP-IMEX for the isentropic Euler equations) is the source term discretization in the right-hand side, which on the one hand requires refining the arguments for asymptotic consistency (see Sects. 4.1.1 and 4.2 below), and on the other hand does not allow for the conservative form of the discrete entropy inequality. We refer the reader to consult with [9] where the authors could show the entropy inequality in the *non-conservative* form.

4.1. Numerical analysis of the LP-IMEX scheme

Before we discuss the stability results for the case of shallow water equations, let us define the formal zero-Froude limit system and the well-prepared initial datum, with the following asymptotic expansions for height and momentum

$$
\begin{aligned}
h(x,t) &= h_{(0)} + \epsilon h_{(1)} + \epsilon^2 h_{(2)},\\
q(x,t) &= q_{(0)} + \epsilon q_{(1)} + \epsilon^2 q_{(2)}.
\end{aligned}
$$

Definition 4.1. The formal zero-Froude limit of the shallow water equations (lake equations) (4.1) reads

$$\begin{aligned}
&\eta_{(0)} = h_{(0)} + b = \text{const.}, \qquad h_{(1)} = \text{const.},\\
&\operatorname{div} q_{(0)} = 0,\\
&\partial_t q_{(0)} + \partial_x \left(\frac{q_{(0)}^2}{h_{(0)}} + p_{(2)} \right) = -h_{(2)} b_x.
\end{aligned}$$

Definition 4.2. For the shallow water equations (4.1), the well-prepared initial datum is defined as

$$\begin{aligned}
h_0^{\mathrm{WP}}(x) &:= h_{(0)}(x) + \epsilon^2 h_{(2)}(x),\\
q_0^{\mathrm{WP}}(x) &:= q_{(0)} + \epsilon q_{(1)}(x),
\end{aligned} \tag{4.7}$$

where $h_{(0)}(x) = \eta_{(0)} - b(x)$ with constant $\eta_{(0)}$ and $q_{(0)}$.

The following theorem summarizes the results for the case of shallow water equations.

Theorem 4.3. *The LP-IMEX scheme (4.3)–(4.5) and (3.5), applied to the case of shallow water equations with a well-prepared initial datum, satisfies the following properties:*

(i) *It can be expressed in the locally conservative form for the density.*
(ii) *The scheme is AC,* i.e., *it gives a consistent discretization of the lake equations in terms of Definition 4.1.*
(iii) *Under some non-ϵ-restrictive CFL constraint (3.23), the scheme is positivity preserving,* i.e., $h_j^n > 0$ *provided that* $h_j^0 > 0$ *for all* $j \in \mathbb{S}$.
(iv) *The scheme preserves the Lake at Rest (LaR) equilibrium state, so it is well-balanced.*

Now, we go through the proof of parts (ii), (iii) and (iv) of the theorem briefly.

4.1.1. Proof of asymptotic consistency (ii)

Because of Definition 4.1, the argument for the asymptotic consistency should consider two important differences compared with Section 3. The shallow water equations with a non-flat bottom have different limit as $\epsilon \to 0$: the limit density (or height) is no longer constant, but the surface elevation is constant. Also rather than *div*-free velocity field, the momentum field should be solenoidal. Since the Lagrangian formulation does not consider these two differences into account, it is a bit complicated to check if the scheme drives the solution towards the limit manifold or not; however, one can check the consistency of the discretization rather readily. Note that as before, we start with a well-prepared initial datum in the sense of Definition 4.2.

For the Lagrange step and using the τ-update (4.3) as well as (4.6), one obtains that

$$\left(\pi_{(0)j+1}^{n\dagger} - \pi_{(0)j}^{n\dagger} \right) + \Delta z_{(0)j+\frac{1}{2}}^{n} b_{x,j+\frac{1}{2}} = 0. \tag{4.8}$$

To show the consistency of the discretization in the limit, using (4.8) and (4.6), one can show the consistency of the interface velocity in the limit. Having that, it is straightforward to show that the Lagrange step (4.3)–(4.5) is consistent with (4.2) in the limit. For the projection step, consistency of discretization of the interface velocity concludes the argument.

4.1.2. Proof of height positivity (iii)

The proof is very similar to Section 3.1, but compared to that, there is an additional contribution due to the source terms. It is not difficult to show that due to (4.8), the condition (3.23) – with the interface velocity as defined in (4.6) – can be fulfilled uniformly in ϵ. So under the non-restrictive CFL condition (3.23), the scheme is positivity preserving.

4.1.3. Proof of well-balancing (C-property) (vi)

The Lake at Rest (LaR) is a very important equilibrium state which every scheme for the shallow water equations has to preserve. It simply describes a steady water with flat surface and zero velocity. The failure in satisfying LaR at the discrete level leads to spurious oscillations. To show that the scheme is well-balanced, *i.e.*, it preserves LaR, at first we show that the scheme *may* have such a solution if one starts with a LaR initial datum. Then, we argue that since the solution of the scheme is unique, then this should be the only case which can happen.

Since the projection step is a convex combination of $\mathbb{U}_j^{n\dagger}$ under the CFL constraint (3.23), one can confirm that to have a discrete solution at the LaR equilibrium, it is sufficient and necessary that $\underline{\mathbb{U}}^{n\dagger} \in \mathcal{U}_{LaR}^{\Delta}$ and $\widetilde{u}_{j+\frac{1}{2}}^{n\dagger} = 0$ for all $j \in \mathbb{S}$. Then, we can check if such a state is compatible with the scheme, *i.e.*, if the scheme may have such a solution. It is clear for the projection step, but let us clarify it for the Lagrange step.

For the Lagrange step, we have three equations. The τ-update is compatible with zero interface velocity and steady density. From the discussion on the asymptotic consistency, we have the τ-π relation (3.14) which clearly shows that for a steady density, the relaxation pressure would be also steady, so the solution remains at the equilibrium. It only remains to show the compatibility condition for the velocity after the Lagrange update, that is to show $u^{n\dagger} = u^n \equiv 0$, and also to complete the loop by confirming the compatibility of the zero interface velocity. One can write the update for the velocity as

$$u_j^{n\dagger} = u_j^n - \frac{\Delta t}{2\epsilon^2 \Delta z_j^n}\left(\pi_{j+1}^{n\dagger} - \pi_{j-1}^{n\dagger} + h_{j-\frac{1}{2}}^{n\dagger}(b_j - b_{j-1}) + h_{j+\frac{1}{2}}^{n\dagger}(b_{j+1} - b_j)\right). \tag{4.9}$$

Since $\pi^{n\dagger} = \pi^n$ from the arguments above, one can confirm that $u^{n\dagger} \equiv 0$ is compatible. Since the velocity is zero and the relaxation pressure is at equilibrium, the definition of the interface velocity (4.6) makes the loop complete.

Up to now, we have shown the compatibility of such a solution at equilibrium. By Theorem 3.10, the existence and uniqueness of a solution (which should be the well-balanced solution) is known, thus the well-balancing is concluded.

4.2. Rigorous analysis of asymptotic consistency

For the case of non-flat bottom, Theorem 3.10 also applies. But, since the right-hand side vector $\underline{\vec{y}}$ has also a contribution from the bottom topography as

$$\underline{\vec{y}} = \underline{\Delta z^n} \odot \underline{\vec{w}}^n - \frac{a\Delta t}{2\epsilon^3}\underline{\Delta z_-^n} \odot \underline{\Delta b_-},$$

where $\underline{\Delta z_-^n}$ and $\underline{\Delta b_-}$ are the vectors of $\Delta z_{j-\frac{1}{2}}^n$ and $(b_j - b_{j-1})$ respectively, one can see that $\|\underline{\vec{y}}\| = \mathcal{O}(1/\epsilon^3)$. So, using the boundedness of $\| \vec{J}^{-1} \|$ to rigorously show asymptotic consistency is futile. Instead, one has to study the structure of $\vec{J}^{-1}$, which proposes the following lemma.

Lemma 4.4. *Denote* $\vec{J}' := \frac{\epsilon}{a\Delta t}\vec{J}$. *Then,*

(i) *Denote the adjugate matrix of* $\vec{J}'$ *by* $\mathrm{adj}(\vec{J}')$, *and the all-ones matrix of size* $\mathcal{N}$ *by* $\mathbf{1}_{\mathcal{N}}$. *Then*

$$\mathrm{adj}(\vec{J}') = (1 + \mathcal{O}(\epsilon))\,\mathbf{1}_{\mathcal{N}},$$

(ii) $\det(\vec{J}') = \mathcal{O}(\epsilon)$.

Proof. It is known that the inverse of a circulant matrix is also circulant [23]. So, it is enough if we show that the entries of the first column of $\mathrm{adj}(\vec{J}')$ are $1 + \mathcal{O}(\epsilon)$, which correspond to the first row of the cofactor matrix.

We denote $\chi_j := \frac{\epsilon \Delta z_j^n}{a \Delta t}$ and for simplicity of the notation, we assume that $\chi_j = \chi$ is constant; the proof is similar for the non-constant case. For the cofactor matrix, one can see that the entry of the first row and j-th column is

$$\mathrm{cof}(\overrightarrow{J}')_{1j} = (-1)^{j+1} \det \begin{bmatrix} K_1 & O_{(j-1)\times(\mathcal{N}-j)} \\ O_{(\mathcal{N}-j)\times(j-1)} & K_2 \end{bmatrix},$$

where $O_{q\times r}$ is the zero matrix of size $q \times r$, and K_1 and K_2 are defined as

$$K_1 := \begin{bmatrix} -1 & 1+\chi & 0 & \cdots & 0 \\ 0 & -1 & 1+\chi & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & -1 \end{bmatrix}_{(j-1)\times(j-1)}, \qquad K_2 := \begin{bmatrix} 1+\chi & 0 & \cdots & 0 \\ -1 & 1+\chi & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & -1 & 1+\chi \end{bmatrix}_{(\mathcal{N}-j)\times(\mathcal{N}-j)}.$$

Then, it is clear that

$$\mathrm{cof}(\overrightarrow{J}')_{1j} = (-1)^{2j}(1+\chi)^{\mathcal{N}-j} = (1+\chi)^{\mathcal{N}-j}.$$

Hence, all the entries of $\mathrm{cof}(\overrightarrow{J}')$ and so $\mathrm{adj}(\overrightarrow{J}')$ are $1+\mathcal{O}(\epsilon)$. As we mentioned, the proof for the scheme (4.3)–(4.5) with non-constant χ_j is similar.

For the point (ii), at first we show that the circulant matrix $\delta_+ := \overrightarrow{J}'|_{\chi_j=0}$, *i.e.*

$$\delta_+ := \mathbf{Circ}(1, \underbrace{0, \dots, 0}_{\mathcal{N}-2 \text{ times}}, -1),$$

has rank $\mathcal{N} - 1$. To show that, we recall the explicit relation for the eigenvalues of a circulant matrix ([23], Eq. (3.7)), which for δ_+ gives $1 - \omega_j^{\mathcal{N}-1}$ as the eigenvalues for $j = 0, 1, \dots, \mathcal{N} - 1$, where $\omega_j := e^{\frac{2\pi i j}{\mathcal{N}}}$ is the $\mathcal{N}$th-root of unity. So, for the eigenvalues to be zero, $e^{\frac{2\pi i j \mathcal{N}}{\mathcal{N}}} = 1$, which is only possible for $j = 0$, so there is exactly one zero eigenvalue, and the rank of δ_+ is $\mathcal{N} - 1$. From this result, and by assuming χ_j to be constant, among the eigenvalues of $\overrightarrow{J}'$ denoted by λ_j for $j = 1, 2, \dots, \mathcal{N}$, there is exactly one which has been shifted to $\mathcal{O}(\epsilon)$, while all other eigenvalues are $\mathcal{O}(1)$; thus

$$\det(\overrightarrow{J}') = \prod_{j=1}^{\mathcal{N}} \lambda_j = \mathcal{O}(\epsilon). \tag{4.10}$$

For the case of non-constant χ_j, one can simply use the result of part (i) to compute the determinant directly as $\prod_{j=1}^{\mathcal{N}}(1+\chi_j) - 1$ which is $\mathcal{O}(\epsilon)$. □

Lemma 4.4 implies that the implicit operator is bounded as $\epsilon \to 0$ (which is the same result as Theorem 3.10 but with different arguments) and $\overrightarrow{J}'$ is almost constant up to some deviations of order $\mathcal{O}(\epsilon)$. Using this and periodic boundary conditions, one can prove the following theorem.

Theorem 4.5. *The norm of the updated solution* $\overrightarrow{\underline{w}}^{n\dagger}$, *is at most as large as* $\|\overrightarrow{\underline{w}}^n\| = \mathcal{O}(1/\epsilon^2)$, *for periodic or compactly supported bottom topography function.*

Proof. From Lemma 4.4, the leading part of $\overrightarrow{J}'^{-1}$ is constant. For the statement of the theorem to be true, we should show that such a structure implies $\|\overrightarrow{J}'^{-1}\overrightarrow{\underline{y}}\| = \mathcal{O}(1/\epsilon^2)$. In other words, it filters the $\mathcal{O}(1/\epsilon^3)$ part of the vector $\underline{\overrightarrow{y}}$; we denote it as $\underline{\overrightarrow{y}}^* := -\frac{a\Delta t}{2\epsilon^3}\underline{\Delta z^n_-} \odot \underline{\Delta b_-}$. Manipulating $\underline{\overrightarrow{y}}^*$, one can find that

$$\begin{aligned}
y^*_j &= -\frac{a\Delta t}{2\epsilon^3}\left(\varrho^n_j + \varrho^n_{j-1}\right)(b_j - b_{j-1}) \\
&= -\frac{a\Delta t}{2\epsilon^3}\left(2\eta^n_{(0)j} - b_j - b_{j-1} + \mathcal{O}(\epsilon^2)\right)(b_j - b_{j-1}) \\
&= -\frac{a\Delta t}{2\epsilon^3}\left(2\eta^n_{(0)c} - b_j - b_{j-1}\right)(b_j - b_{j-1}) + \mathcal{O}(1/\epsilon^2),
\end{aligned}$$

due to the well-preparedness of initial datum.

Now, it is enough to show that $\underline{\overrightarrow{y}}^*$ belongs to the kernel of the leading order part of $\overrightarrow{J}'^{-1}$ since the leading order of adj$(\overrightarrow{J}'^{-1})$ is $\mathbf{1}_{\mathcal{N}}$. That is to show $\|\overrightarrow{J}'^{-1}\underline{\overrightarrow{y}}^*\| = \mathcal{O}(1/\epsilon^2)$, which can be done simply by making a spatial summation and using the boundary condition, *i.e.*,

$$\sum_j y^*_j = -\frac{a\Delta t}{2\epsilon^3}\sum_j\left[2\eta^n_{(0)c}(b_j - b_{j-1}) - \left(b^2_j - b^2_{j-1}\right)\right] = 0.$$

Hence, the $\mathcal{O}(1/\epsilon^3)$ terms vanish and one is left with the contributions of order $\mathcal{O}(1/\epsilon^2)$. □

Note that the similar results of Lemma 4.4 and Theorem 4.5, hold for $\overleftarrow{J}$ and $\overleftarrow{w}$ respectively. This implies the following corollary.

Corollary 4.6. *The asymptotic consistency analysis for the LP-IMEX scheme,* (4.3)–(4.5) *and* (3.5) *is rigorous,* i.e.*, the asymptotic expansion is justified.*

Proof. Due to Theorem 4.5, the implicit step preserves the order of $\|\underline{\overrightarrow{w}}\|$. It gives $\pi^{n+1} = \mathcal{O}(1)$, thus recovers (4.8). As in Section 3.2, one can show the boundedness of $\varrho^{n\dagger}$ using the τ-π relation (3.14). Also due to (4.8), the proof of the boundedness of $u^{n\dagger}$ is similar as Section 3.2 since the coefficient matrix H is exactly like (3.39). This concludes the rigorous asymptotic consistency of the implicit step, and also the whole scheme since the explicit step has already been shown to be asymptotically consistent. □

5. Conclusion and future works

In this paper, we have investigated the stability results of the LP-IMEX scheme for the one-dimensional isentropic Euler equations, which has been presented in [13], regarding the uniformity in terms of the Mach number with well-prepared initial data. We have shown that the scheme is asymptotically consistent (not only formally but also rigorously), also have obtained a Mach-uniform time step restriction which provides entropy inequality and density positivity, as well as stability of the solution in ℓ_∞-norm. Also, we have extended the analysis to the shallow water equations with a non-flat bottom as an important example of balance laws.

The natural next step would be to extend this analysis to the full Euler equations or multiple space dimensions, which are formidable tasks, particularly the latter as has been discussed to some extent in [8, 19]. Also along the lines of [38], it is of interest to prove the convergence of the scheme to the unique entropy solution by the compensated compactness approach.

Appendix A. Entropy (energy) stability in the zero-mach limit

In this section, we discuss the implications of entropy stability for $\epsilon \to 0$, aiming to show the stability of the solution and, due to the compactness, its strong convergence to some limit. Note that this is not the classical convergence result for $\Delta x \to 0$. Here the main objective is to discuss the stability region which entropy stability provides. The entropy stability also provides some *a priori* informations about the asymptotic consistency, although it cannot recover the limit completely since it does not use any detail neither from the splitting nor the spatial discretization and time integration. It is worth mentioning that this analysis assumes the positivity of density and energy inequality, so it is not limited to the LP-IMEX scheme.

The procedure is as follows: firstly, we recall that positivity and energy inequality gives boundedness of the density and velocity, but not directly in the limit $\epsilon \to 0$. We then show this boundedness as $\epsilon \to 0$, thus the existence of a converging subsequence due to the compactness. Then, we show that the limit density is the incompressible limit solution.

Consider the entropy (energy) function $\mathcal{J} := \varrho E$ and assume a fixed grid of size $\mathcal{N}$. Having the discrete entropy (energy) inequality ((3.9) for instance), we make a spatial summation to get the *global entropy (energy) inequality*

$$\sum_j \mathcal{J}(\underline{\mathbb{U}}_j^{n+1}) \leq \sum_j \mathcal{J}(\underline{\mathbb{U}}_j^{n}) \Longrightarrow \sum_j \mathcal{J}(\underline{\mathbb{U}}_j^{n+1}) \leq \sum_j \mathcal{J}(\underline{\mathbb{U}}_j^{0}) \leq C_\epsilon < \infty. \tag{A.1}$$

If in addition we assume positivity, $\mathcal{J}(\mathbb{U}) = \frac{1}{2}\frac{(\varrho u)^2}{\varrho} + \frac{\kappa/\epsilon^2}{\gamma-1}\varrho^\gamma$ is positive. Thus

$$0 < \mathcal{J}(\underline{\mathbb{U}}_j^{n+1}) \leq C_\epsilon \qquad \forall j \in \mathbb{S}.$$

One immediate result, for fixed ϵ, is the ℓ_∞-boundedness, *i.e.*, $\underline{\varrho}, \underline{u} \in \ell_\infty(\mathbb{S})$. So the energy inequality accompanied with positivity, provides a stability region Ξ_ϵ^0 which depends on the initial condition as well as ϵ. But how does the stability region Ξ_ϵ^0 change if rather than a fixed ϵ, we consider $\epsilon \to 0$? If one keeps the grid fixed and considers $\epsilon \to 0$, the boundedness of the density is rather clear, either due to positivity or due to the boundedness of the energy. But it is not straightforward to conclude the boundedness of the velocity. This is the first question we want to answer in this section.

Also note that the boundedness of the density sequence w.r.t. ϵ provides strong convergence. In other words, solutions with positive density, owing to the compactness of the space, have a converging sequence of vectors $\{\varrho^{\epsilon_k,n}\}_{k\in\mathbb{N}}$ for any step n ($\epsilon_k \to 0$ is a sequence approaching the incompressible limit) that converges strongly to *some limit* $\underline{\varrho}^{\epsilon_\infty,n}$, *i.e.*

$$\lim_{k\to\infty} \left\| \underline{\varrho}^{\epsilon_k,n} - \underline{\varrho}^{\epsilon_\infty,n} \right\| = 0.$$

But it is not clear whether or not the limit is in the space of incompressible solutions. To determine whether the limit is the correct limit, is the second question we discuss in this section.

In what follows, with the help of energy inequality we show that the computed density by the scheme actually converges to its incompressible limit. We then show that the same assumptions are not enough to prove asymptotic consistency of the velocity, *i.e.*, the *div*-free condition; nonetheless the boundedness of the velocity sequence, so its convergence to some limit, can be obtained. We discuss the results in the following lemma.

Lemma A.1. *Consider $\mathcal{N}$-vector sequence $\left\{\underline{\varrho}^{\epsilon_k,n}\right\}_{k\in\mathbb{N}}$, accompanied with a well-prepared initial datum, as the discrete density solution of the isentropic Euler equations* (2.1)*. Assume that the solution sequence satisfies density positivity and energy inequality. Then as $\epsilon \to 0$ the sequence is bounded and approaches the incompressible limit $\underline{\varrho}^{0,n}$ with the rate of $\mathcal{O}(\epsilon)$. Moreover, the velocity sequence $\{\underline{u}^{\epsilon_k,n}\}_{k\in\mathbb{N}}$ is bounded in ℓ_∞.*

Remark A.2. Both from formal asymptotic expansion and rigorous analysis [35], one expects to see the convergence of density to its incompressible limit with $\mathcal{O}(\epsilon^2)$ rate. So, the convergence rate of Lemma A.1 is not optimal. We see that exactly due to this issue, the asymptotic consistency of the velocity cannot be obtained by these assumptions.

Proof. Consider a well-prepared initial datum as $\varrho_i^{\epsilon,0} := \varrho^{0,0} + \delta_i^{\epsilon,0}$ with $\delta_i^{\epsilon,0} = \mathcal{O}(\epsilon^2)$. Then write the density at the step n as $\varrho_i^{\epsilon,n} := \varrho^{0,0} + \delta_i^{\epsilon,n}$. Due to conservation of scheme $\|\varrho^{\epsilon,n}\|_{\ell_1} = \left\|\varrho^{\epsilon,0}\right\|_{\ell_1}$, one can simply get

$$\sum_i \delta_i^{\epsilon,n} = \sum_i \delta_i^{\epsilon,0} = \mathcal{N}\mathcal{O}(\epsilon^2). \tag{A.2}$$

So, it seems that in the limit the density, in general, oscillates around a constant state. But this does not give us convergence since the perturbations have no sign. By the global energy inequality (as in (A.1)) one can see that

$$\sum_i \left(\varrho^{0,0}+\delta_i^{\epsilon,n}\right)^2 \le \sum_i \left(\varrho^{0,0}+\delta_i^{\epsilon,0}\right)^2 + C_0\epsilon^2, \qquad C_0 := \sum_i \left(\varrho^{0,0}+\delta_i^{\epsilon,0}\right)\left(u^{0,0}+\mu_i^{\epsilon,0}\right)^2, \tag{A.3}$$

where $\mu_i^{\epsilon,0} = \mathcal{O}(\epsilon)$ to fulfill the well-preparedness of initial datum. Then, combining (A.2) and (A.3) yields

$$\|\underline{\delta}^{\epsilon,n}\|_{\ell_2}^2 = \left\|\underline{\delta}^{0,n}\right\|_{\ell_2}^2 + C_0\epsilon^2 = \mathcal{O}(\epsilon^2),$$

which shows that each component converges to the incompressible limit with – at least – $\mathcal{O}(\epsilon)$ rate. Although this rate is smaller that what one would expect, this is the best can be obtained by these assumptions. Furthermore, by straightforward calculations, one can show that

$$\left\|\underline{\varrho}^{\epsilon,n}\right\|_{\ell_2}^2 - \left\|\underline{\varrho}^{\epsilon,0}\right\|_{\ell_2}^2 = \|\underline{\delta}^{\epsilon,n}\|_{\ell_2}^2 + \mathcal{O}(\epsilon^4) = \mathcal{O}(\epsilon^2)$$

and then by the complete energy inequality, not (A.3), one can obtain

$$\left\|\underline{\left(\varrho u^2\right)^{\epsilon,n}}\right\|_{\ell_1} - \left\|\underline{\left(\varrho u^2\right)^{\epsilon,0}}\right\|_{\ell_1} \le \mathcal{O}(1),$$

thus $\left\|\underline{\left(\varrho u^2\right)^{\epsilon,n}}\right\|_{\ell_1}$ is bounded, and since the density converges to the incompressible limit uniformly which is away from zero, the velocity is bounded as well. □

Remark A.3. It is not difficult to show, with the additional assumption of $\|\underline{\delta}^{\epsilon,n}\|_{\ell_2} = \mathcal{O}(\epsilon^2)$, that $\left\|\underline{\mu}^{\epsilon,n}\right\|_{\ell_2} = \mathcal{O}(\epsilon)$. Thus the asymptotic consistency would be obtained completely.

The similar analysis can be done for the case of shallow water equations. Here we state the main result and sketch of the proof.

Lemma A.4. *Consider $\mathcal{N}$-vector sequence $\{\underline{h}^{\epsilon_k,n}\}_{k\in\mathbb{N}}$, accompanied with a well-prepared initial datum, as the discrete height solution of the shallow water equations (4.1). Assume that the solution sequence satisfies height positivity and energy inequality. Then as $\epsilon \to 0$ the sequence is bounded and approaches the zero-Froude limit $\underline{h}^{0,n}$ with the rate of $\mathcal{O}(\epsilon)$. Moreover, the velocity sequence $\{\underline{u}^{\epsilon_k,n}\}_{k\in\mathbb{N}}$ is bounded in ℓ_∞.*

Proof. Consider the well-prepared initial datum as $h_i^{\epsilon,0} := \eta - b_i + \delta_i^{\epsilon,0}$ with $\delta_i^{\epsilon,0} = \mathcal{O}(\epsilon)$. Then write the height at the step n as $h_i^{\epsilon,n} := \eta - b_i + \delta_i^{\epsilon,n}$. Using mass conservation, one can find $\sum_i \delta_i^{\epsilon,n} = \sum_i \delta_i^{\epsilon,o}$ and from the global energy inequality

$$\sum_i \left(\left(\eta - b_i + \delta_i^{\epsilon,n}\right)^2 + 2b_i\left(\eta - b_i + \delta_i^{\epsilon,n}\right)\right) \le \sum_i \left(\left(\eta - b_i + \delta_i^{\epsilon,0}\right)^2 + 2b_i\left(\eta - b_i + \delta_i^{\epsilon,0}\right)\right) + C_0\epsilon^2,$$

with $C_0 := \sum_i \left(h^{0,0}+\delta_i^{\epsilon,0}\right)\left(u^{0,0}+\mu_i^{\epsilon,0}\right)^2$, one concludes that $\|\underline{\delta}^{\epsilon,n}\|_{\ell_2} = \mathcal{O}(\epsilon)$. Analogous arguments as for Lemma A.1 concludes the boundedness of the velocity sequence. □

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