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EXISTENCE AND UNIQUENESS OF A SOLUTION FOR A FIELD/CIRCUIT COUPLED PROBLEM *

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Abstract. In this paper we show unique solvability of an abstract coupled problem which originates from a field/circuit coupled problem. The coupled problem arises in particular from modified nodal analysis equations linked with an eddy current problem *via* solid conductor model. The proof technique in the paper relies on Rothe's method and the theory of monotone operator. We also provide error estimates for time discretization.

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1. Introduction

The growing complexity of electrical devices calls for more refined mathematical models for their simulation. This can give rise to various coupled problems. A field/circuit coupled model, for instance, consists of lumped-element circuit equations coupled with a distributed field model. Mathematically, it is a coupling between a differential-algebraic system of equations and a partial differential equation model. Such problems have attracted some attention.

In [18] the author focuses on the time-transient simulation of device/circuit coupled problems using multiscale models. A coupled system of circuit and Maxwell's equations was also studied in [2]. In [20] the author studies the coupled device/circuit problem for a semiconductor device and shows the existence and uniqueness of a solution together with a perturbation result using a Galerkin approach in an abstract setting. A similar approach is employed in the work [12], where a system of nonlinear partial differential-algebraic equations is studied. A time-harmonic case for a field-circuit coupling has been studied in [14].

In this paper, we study an abstract nonlinear parabolic coupled problem of the form

$$\begin{aligned} Su'(t) + Au(t) + SKy(t) &= 0 \\ (My)'(t) + K^*Su'(t) + B(t)y + K^*SKy(t) &= g(t), \end{aligned} \tag{1.1}$$

which arises from of a certain type of field/circuit coupled problem. Here, the first equation can represent the eddy current approximation of Maxwell's equations. The second equation then corresponds to a circuit

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sub-problem coming from the so-called modified nodal analysis. Both sub-problems are linked together *via* the solid-conductor model for field/circuit coupling.

We prove the unique solvability of the problem (1.1) and provide error estimates for its time discretization, which is the highlight of this study. Our proof-technique relies on the theory of monotone operators [13, 21, 23], which has already been applied to more standard evolution problems, *cf. e.g.* [15–17]. The paper is organized as follows. Section 2 is devoted to a short presentation of the field/circuit coupled problem. In Section 3 we prove unique solvability of the resultant abstract coupled problem in appropriate functional spaces. The abstract formulation will allow us to keep the focus on the monotone structure of the coupled problem and ease the otherwise lengthy notation. In Section 4 we derive some error estimates for time discretization.

2. Field/circuit coupled problem

2.1. Electric network model

We will briefly go over an electric network model as presented in [6, 20]. Let us consider an electric network with the nodes $1, \dots, n$ and branches $1, \dots, b$. The (reduced) incidence matrix $\mathcal{A} \in \mathbb{R}^{n\times b}$ between the nodes and edges is defined by its elements

$$\mathcal{A} = (A_{ij})_{i=1,\dots,n}^{j=1,\dots,b}, \qquad A_{ij} = \begin{cases} 1 & \text{if branch } j \text{ leaves the node } i \\ -1 & \text{if branch } j \text{ enters the node } i \\ 0 & \text{otherwise.} \end{cases}$$

In the reduced incidence matrix (again denoted as) $\mathcal{A}$, one node is selected as the ground node and its row is skipped. The reduced matrix $\mathcal{A}$ can be split into the incidence matrices associated with capacitive branches, inductive branches, resistive branches, branches of voltage sources and branches of current sources, respectively, *i.e.* $\mathcal{A} = (A_C\, A_R\, A_L\, A_V\, A_I)$. Let $i = (i_C, i_R, i_L, i_V, i_I)^T$ denote the vector of all branch currents. Kirchhoff's current law states that

$$A_C i_C + A_R i_R + A_L i_L + A_V i_V + A_I i_I = 0. \tag{2.1}$$

One can then apply the constitutive equations for the branch currents $i = i(v)$ and Kirchhoff's voltage law $v = \mathcal{A}^T e$, where v denotes the vector of all branch voltages and e denotes the vector of all node potentials. This eventually yields the modified nodal analysis equations

$$\begin{aligned} A_C \frac{\mathrm{d}q_C(A_C^T e)}{\mathrm{d}t} + A_R g_R(A_R^T e, t) + A_L i_L + A_V i_V &= -A_I i_s(t) \\ \frac{\mathrm{d}\phi_L(i_L, t)}{\mathrm{d}t} - A_L^T e = 0 \quad \text{and} \quad A_V^T e &= v_s(t) \end{aligned} \tag{2.2}$$

with the initial data for e and i_L. Natural conditions for an field/circuit model have been formulated in ([20], p. 16). They are referred as (A1) Smoothness, (A2) Local Passivity, and (A3) Consistency. (A1) describes the continuity and differentiability of q_C, g_R and ϕ_L. (A2) reflects the positive definiteness of $\frac{\partial q_C(v,t)}{\partial v}$, $\frac{\partial g_R(v,t)}{\partial v}$ and $\frac{\partial \phi_L(j,t)}{\partial j}$. Assumption (A3) is necessary for a consistent model description to guarantee the unique solvability.

The vector functions q_C and ϕ_L are continuously differentiable and strongly monotone in the first arguments. These properties follows from the smoothness and the local passivity hypotheses ([20], p. 16). The function g_R is Lipschtiz continuous and strongly monotone in the first argument and continuous in the second one, see (A1) and (A2) at ([20], p. 16). The dimension of the above differential-algebraic system is $n1 + n_L + n_V$, where n_L is the number of inductive branches and by n_V the number of voltage source branches. Adding certain topological assumptions on the network, one can prove the unique solvability of the above problem for the given data i_s and v_s, see for instance [12].

2.2. Eddy current problem with nonlocal voltage excitation

In this subsection we discuss a generalized eddy current problem with nonlocal voltage excitation. We state its magnetic potential formulation for a simple case with a single conductor. We have drawn the model from the article [9]. We refer the reader to this article for a detailed treatment and an in-depth discussion of current and voltage excitations for the eddy current model.

Let us begin with the following eddy current model

$$\begin{aligned} \mathbf{curl}\,\boldsymbol{h} &= \sigma\boldsymbol{e} \quad \text{in } \Omega & \qquad \mathbf{curl}\,\boldsymbol{e} &= -\partial_t\boldsymbol{b} \quad \text{in } \Omega \\ \operatorname{div}\boldsymbol{b} &= 0 \quad \text{in } \Omega & \qquad \boldsymbol{h} &= \nu(|\boldsymbol{b}|)\boldsymbol{b} \quad \text{in } \Omega, \end{aligned}$$

where $\boldsymbol{e}$ and $\boldsymbol{h}$ stand for the electric and magnetic field, respectively, and $\boldsymbol{b}$ denotes the magnetic induction. The electric conductivity $\sigma = \sigma(\boldsymbol{x})$ is bounded and strictly positive in Ω_C and it equals zero in Ω_I. The reluctivity ν is a positive constant in Ω_I and strictly positive and bounded function in Ω_C. The function $s \mapsto \nu(s)s$ is assumed to be strictly monotonically increasing with the Gâteaux potential (see [21]) Φ_ν, *i.e.* $\frac{\mathrm{d}\Phi_\nu}{\mathrm{d}s} = \nu(s)s$, and it describes the nonlinear response of material to a magnetic field. Please note that the monotone behavior of $\nu(s)s$ implies the strong monotonicity of $\nu(|\boldsymbol{b}|)\boldsymbol{b}$. The presumptions on $\nu(s)s$ are reflected in Assumption 3.2(ii).

A typical example of such a field could be $\nu(|\boldsymbol{b}|)\boldsymbol{b} = \boldsymbol{b} + \beta(|\boldsymbol{b}|)\boldsymbol{b}$, where the real function β obeys:

$$0 \le \beta(s) \le C, \quad 0 \le (\beta(s)s)' \le C.$$

One can easily check in this situation that

$$\begin{aligned} (\nu(|\boldsymbol{x}|)\boldsymbol{x} - \nu(|\boldsymbol{y}|)\boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y}) &= |\boldsymbol{x} - \boldsymbol{y}|^2 + (\beta(|\boldsymbol{x}|)\boldsymbol{x} - \beta(|\boldsymbol{y}|)\boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y}) \\ &= |\boldsymbol{x} - \boldsymbol{y}|^2 + \beta(|\boldsymbol{x}|)\,|\boldsymbol{x}|^2 + \beta(|\boldsymbol{y}|)\,|\boldsymbol{y}|^2 - \beta(|\boldsymbol{x}|)\boldsymbol{x}\cdot\boldsymbol{y} - \beta(|\boldsymbol{y}|)\boldsymbol{y}\cdot\boldsymbol{x} \\ &\ge |\boldsymbol{x} - \boldsymbol{y}|^2 + \beta(|\boldsymbol{x}|)\,|\boldsymbol{x}|^2 + \beta(|\boldsymbol{y}|)\,|\boldsymbol{y}|^2 - \beta(|\boldsymbol{x}|)\,|\boldsymbol{x}|\,|\boldsymbol{y}| - \beta(|\boldsymbol{y}|)\,|\boldsymbol{x}|\,|\boldsymbol{y}| \\ &= |\boldsymbol{x} - \boldsymbol{y}|^2 + (\beta(|\boldsymbol{x}|)\,|\boldsymbol{x}| - \beta(|\boldsymbol{y}|)\,|\boldsymbol{y}|)\,(|\boldsymbol{x}| - |\boldsymbol{y}|) \\ &\ge |\boldsymbol{x} - \boldsymbol{y}|^2, \end{aligned}$$

which proves the strong monotonicity of $\nu(|\boldsymbol{b}|)\boldsymbol{b}$. The Gâteaux potential in this case is

$$\Phi_\nu(\boldsymbol{x}) = \int_0^{|\boldsymbol{x}|} \nu(s)s\,\mathrm{d}s = \int_0^{|\boldsymbol{x}|} (1 + \beta(s))\,s\,\mathrm{d}s = \frac{1}{2}\,|\boldsymbol{x}|^2 + \int_0^{|\boldsymbol{x}|} \beta(s)s\,\mathrm{d}s.$$

Indeed for the Gâteaux derivative [21] in the direction $\boldsymbol{y}$ we have

$$\operatorname{grad}\,\Phi_\nu(\boldsymbol{x}) \cdot \boldsymbol{y} = \lim_{t\to 0} \frac{\Phi_\nu(\boldsymbol{x} + t\boldsymbol{y}) - \Phi_\nu(\boldsymbol{x})}{t} = \boldsymbol{x}\cdot\boldsymbol{y} + \beta(|\boldsymbol{x}|)\boldsymbol{x}\cdot\boldsymbol{y} = \nu(|\boldsymbol{x}|)\boldsymbol{x}\cdot\boldsymbol{y}.$$

Let us denote the time frame by $[0, T]$. The domain $\Omega \subset \mathbb{R}^3$ is a bounded domain with outward unit normal vector field $\boldsymbol{n}$ on its smooth boundary $\partial\Omega$. The domain Ω will consist of two disjoint sub-domains Ω_C and Ω_I, $\overline{\Omega}_C \cup \overline{\Omega}_I = \overline{\Omega}$, where Ω_C represents the conductors and Ω_I the insulating air region. Figure 1 shows two simplest topologies under consideration. In both cases the nonlocal excitation is supplied through by voltages imposed at contacts Σ.

Since $\boldsymbol{b}$ is divergence free, we can write

$$\boldsymbol{b} = \mathbf{curl}\,\boldsymbol{a} \quad \text{and} \quad \boldsymbol{e} = -\partial_t\boldsymbol{a} - \mathbf{grad}\,\phi$$

for the magnetic potential $\boldsymbol{a}$ and the scalar potential ϕ. We recall that the potential $\boldsymbol{a}$ is not unique (neither is ϕ) and an additional gauging condition is required, *e.g.* Coulomb's gauge $\operatorname{div}\boldsymbol{a} = 0$, [7]. The scalar potential can be used to introduce a non-local voltage excitation. One can write the electric field as the sum

$$\boldsymbol{e} = -\partial_t\boldsymbol{a} - v\boldsymbol{p},$$

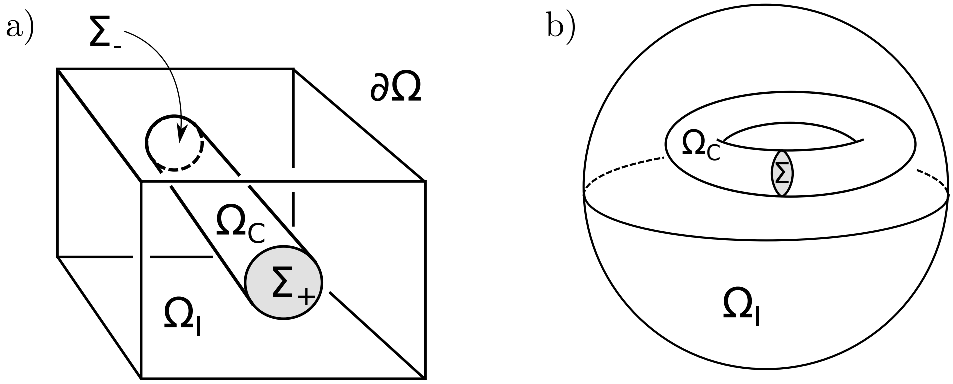

FIGURE 1. Case (a): conductor Ω_C touching $\partial\Omega$ with $\partial\Omega_C \cap \partial\Omega = \Sigma_+ \cup \Sigma_- = \Sigma$. Case (b): conducting loop away from $\partial\Omega$ with cutting surface Σ.

where $v = v(t)$ is a given voltage and $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{x}) \in \mathbf{L}^2(\Omega)$ is a known vector function such that $\boldsymbol{p} = \mathbf{0}$ in Ω_I. In case (a), $\boldsymbol{p} = \nabla\theta$ for the function $\theta \in H^1(\Omega_C)$ with $\Sigma_+ = 1$ and $\Sigma_- = 0$. In case (b), one chooses (a representative of the first co-homology space for $\boldsymbol{p}$, *i.e.*) $\boldsymbol{p} = \nabla\theta$ where $\theta \in H^1(\Omega_C \setminus \Sigma)$ and $[\theta]_\Sigma = 1$. Note that there is no curl-free extension of $\boldsymbol{p}|_{\Omega_C}$ to $\boldsymbol{H}(\mathbf{curl};\Omega)$ neither in case (a) nor in case (b).

The resultant boundary value problem for $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x}, t)$ reads as

$$
\begin{aligned}
\sigma\partial_t\boldsymbol{a} + \mathbf{curl}\,(\nu(|\mathbf{curl}\,\boldsymbol{a}|)\mathbf{curl}\,\boldsymbol{a}) &= -v\sigma\boldsymbol{p} &\quad& \text{in } \Omega\\
\boldsymbol{a}\times\boldsymbol{n} &= \mathbf{0} && \text{on } \partial\Omega
\end{aligned}
\tag{2.3}
$$

with the divergence-free initial condition $\boldsymbol{a}(0) = \boldsymbol{a}_0(\boldsymbol{x})$. For the given voltage v in the above problem, the associated electric current $i = i(t)$ can be recovered using the power balance formula $p = iv$ and Poynting's theorem. One obtains

$$
i = \int_{\Omega_C} [\sigma\partial_t\boldsymbol{a}\cdot\boldsymbol{p} + v\sigma\boldsymbol{p}\cdot\boldsymbol{p}]\,\mathrm{d}\boldsymbol{x}. \tag{2.4}
$$

More complicated topologies are treated analogously, see *e.g.* [5].

It is easy to see that the parabolic problem (2.3) is degenerate. The conductivity σ vanishes in the subdomain Ω_I, where we are left with an elliptic problem which requires the additional divergence-free condition on $\boldsymbol{a}$. The authors in [3] proved the unique solvability of this problem, see also [11]. Their main idea is to restrict the eddy current problem (2.3) only to the conducting domain Ω_C with making use of the harmonic extension from Ω_C to "degenerate" domain Ω_I.

This leads to the variational formulation[2] for $\boldsymbol{a} \in L^2\left((0,T), \boldsymbol{H_0}(\mathbf{curl};\Omega_C)\right) \cap C\left([0,T], \boldsymbol{L}^2\left(\Omega_C\right)\right)$ with $\partial_t\boldsymbol{a} \in L^2\left((0,T), \boldsymbol{L}^2\left(\Omega_C\right)\right)$

$$
\begin{aligned}
\int_{\Omega_C} [\sigma\partial_t\boldsymbol{a}\cdot\boldsymbol{\varphi} + \nu(|\mathbf{curl}\,\boldsymbol{a}|)\mathbf{curl}\,\boldsymbol{a}\cdot\mathbf{curl}\,\boldsymbol{\varphi}]\,\mathrm{d}\boldsymbol{x} + \int_{\Omega_I} \mathbf{curl}\,\mathcal{H}(\boldsymbol{a})\cdot\mathbf{curl}\,\mathcal{H}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x}\\
= \int_{\Omega_C} -v\sigma\boldsymbol{p}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} \qquad \text{for all } \boldsymbol{\varphi} \in \boldsymbol{H_0}(\mathbf{curl};\Omega_C),
\end{aligned}
\tag{2.5}
$$

where the function space $\boldsymbol{H_0}(\mathbf{curl};\Omega_C)$ is a standard Sobolev space of square-integrable vector functions and their curls with vanishing tangential trace. The mapping $\mathcal{H} : \boldsymbol{H}(\mathbf{curl};\Omega_C) \to \boldsymbol{H}(\mathbf{curl};\Omega_I)$ is defined as the

[2]Please note that the variational spaces in an abstract network formulation will change, see Theorem 3.3.

solution $\boldsymbol{a}_I$ of the problem

$$\begin{gathered}\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{a}_I = \mathbf{0} \quad \text{and} \quad \operatorname{div}\boldsymbol{a}_I = 0 \quad \text{in } \Omega_I,\\ \boldsymbol{a}_I \times \boldsymbol{n} = \boldsymbol{a}_C \times \boldsymbol{n} \quad \text{on } \partial\Omega_I/\partial\Omega \quad \text{and} \quad \boldsymbol{a}_I \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \partial\Omega\end{gathered} \tag{2.6}$$

for a given $\boldsymbol{a}_C$. Unique solvability needs certain topology assumption, *cf.* [1].

2.3. The coupled problem

To obtain the field/circuit coupled problem, one simply adds the field-sub-problem currents i_F given by (2.4) into the current balance equation (2.1) (or equivalently to (2.2))

$$A_C i_C + A_R i_R + A_L i_L + A_V i_V + A_I i_I + A_F i_F = 0,$$

and set $v = A_F^T e$ in (2.3). This coupling is also called a solid-conductor model, see [19].

In presence of no voltage sources, we can use the following (operator) notation for (2.2)

$$y = \begin{pmatrix} e \\ i_L \end{pmatrix},\ M = \begin{pmatrix} A_C q_C(A_C^T \bullet, t) & 0 \\ 0 & \phi_L(\bullet, t) \end{pmatrix},\ B = \begin{pmatrix} A_R g_R(A_R^T \bullet, t) & A_L \\ -A_L^T & 0 \end{pmatrix},$$

and $g = (-A_I i_s, 0)^T$. This leads to

$$My = \begin{pmatrix} A_C q_C(A_C^T e, t) \\ \phi_L(i_L, t) \end{pmatrix}, \qquad By = \begin{pmatrix} A_R g_R(A_R^T e, t) + A_L i_L \\ -A_L^T e \end{pmatrix}.$$

The vector functions q_C and ϕ_L are continuously differentiable and strongly monotone in the first arguments. This is reflected in Assumption 3.2(iii). The properties of g_R are reflected in Assumption 3.2(iv). We next introduce the coupling operator K as

$$\begin{aligned} &K : \mathbb{R}^{n+n_L} \to \boldsymbol{L}^2(\Omega), \quad y \mapsto A_F^T y \boldsymbol{p} \qquad \text{with}\\ &K^* : \boldsymbol{L}^2(\Omega) \to \mathbb{R}^{n+n_L}, \quad \boldsymbol{u} \mapsto A_F \int_{\Omega_C} \boldsymbol{p}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x},\end{aligned} \tag{2.7}$$

where $\boldsymbol{L}^2(\Omega)$ is the space of square-integrable vector functions on Ω. The problem (2.3) can be rewritten in a similar manner so that we can derive that the coupled field/circuit problem has indeed the form of the abstract problem (1.1).

In the next sections we will analyze the abstract model as it provides an elegant way to study existence of a solution and convergence of numerical approximations. We will introduce the vector spaces V and H which will correspond to $\boldsymbol{H}_{\boldsymbol{0}}(\mathbf{curl};\Omega)$ and $\boldsymbol{L}^2(\Omega)$ respectively. The operator S will correspond to the multiplication by σ. The assumptions on the operator A will reflect the properties of the second and third term in the equation (2.5) which can be written as one.

Given standard assumptions on the electric network topology, the equations (2.2) will be a system of ordinary differential equations or (more realistically) first order differential-algebraic equations (see [12]). In the later case the system can be still rewritten into an ODE form if g_R is strongly monotone in the first argument. The assumptions on M, B and K will reflect theses properties.

3. Abstract coupled problem

In this section, we use the Rothe method and the theory of monotone operators (*e.g.* [16]) to show that the abstract problem (1.1) has a unique solution under certain standard assumptions.

We first introduce some notation and at the same time summarize basic assumptions about the function spaces, see ([22], Chap. 23) for more details. We use standard notation for the scalar product $x^T y$ and the associated Euclidean norm $|x| = \sqrt{x^T x}$, where $x, y \in \mathbb{R}^N$. We work with the evolution triple in the following sense:

Assumption 3.1. Assume that:

(i) V is a real, separable and reflexive Banach space with the norm $\|u\|_V$ for $u \in V$. The space V^* is its dual space with the norm $\|v\|_{V^*}$ and the dual pairing $\langle v, u\rangle$ for $v \in V^*$ and $u \in V$.
(ii) H is a real, separable Hilbert space with the scalar product (h, k) for $h, k \in H$ and the induced norm $\|h\| := \sqrt{(h,h)}$.
(iii) V is dense in H and
$$\|v\|_V = \|v\| + |v|_V \quad \text{for all} \quad v \in V,$$
where $|v|_V$ is a seminorm on V.

The space H is identified with its dual H^* by the Riesz theorem. We also identify $h \in H$ with the functional $h \in V^*$
$$\langle h, v\rangle = (h, v) \text{ for all } v \in V \quad \text{with } \|h\|_{V^*} \le \|h\|, \tag{3.1}$$
and so in this sense $H \subseteq V^*$. The same identification will be often used for $u \in V \subset H$. We make use of the standard parabolic space $W^{1,2}((0,T), V, H)$ equipped with the norm
$$\|u\|_{W^{1,2}((0,T),V,H)} = \left(\int_0^T \|u(t)\|_V^2 \,\mathrm{d}t + \int_0^T \|u'(t)\|_{V^*}^2 \,\mathrm{d}t\right)^{1/2},$$
which is continuously embedded into the space $C([0,T], H)$ and
$$\max_{t\in[0,T]} \|u\| \le \text{const} \|u\|_{W^{1,2}((0,T),V,H)}. \tag{3.2}$$
The formula
$$\langle u'(t), v\rangle = \frac{\mathrm{d}}{\mathrm{d}t}(u(t), v) \tag{3.3}$$
holds true in weak sense for given $u \in W^{1,2}((0,T), V)$ and $v \in V$. In particular, we have the integration by parts formula
$$(u(T), v(T)) - (u(0), v(0)) = \langle u', v\rangle_{(0,T)} + \langle v', u\rangle_{(0,T)}, \tag{3.4}$$
where $\langle \cdot, \cdot\rangle_{(0,T)}$ denotes the dual pairing between $L^2((0,T), V^*)$ and $L^2((0,T), V)$.
As it is usual, C will stand for a generic positive constant later on in the estimates.

Here follows the assumptions on the operators and the data.

Assumption 3.2. Assume the following:

(i) The operator $S : H \to H$ is bounded linear, self-adjoint and strongly positive[3].
(ii) The operator $A : V \to V^*$ is hemicontinuous, monotone and moreover there exists a constant $c_1 > 0$ such that
$$\langle Au - Av, u - v\rangle \ge c_1 |u - v|_V^2 \qquad \text{for all } u, v \in V.$$
It satisfies the growth estimate
$$\|Av\|_{V^*} \le C(1 + |v|_V) \qquad \text{for all } v \in V.$$
$A(0) = 0$ and there exists the potential $P_A : V \to \mathbb{R}$ such that $P_A' = A$, *i.e.* A is the Gâteaux derivative of P_A, see [21].

[3]Then we can write $S = (S^{1/2})^* S^{1/2}$ and there exists S^{-1}.

(iii) The vector function $M : \mathbb{R}^N \times [0,T] \to \mathbb{R}^N$ is continuously differentiable, and strongly monotone in the first argument[4], *i.e.*

$$(x-y)^T(M(x,t)-M(y,t)) \geq c_1|x-y|^2 \qquad \text{for all } x,y \in \mathbb{R}^N \text{ and } t \in [0,T].$$

$M(0,t) = 0$ and there exists the potential $P_M : V \times [0,T] \to \mathbb{R}$ such that $P_M' = M$, *i.e.* M is the Gâteaux derivative of P_M with respect to the first variable and the second variable t is taken as a parameter.

(iv) The vector function $B : \mathbb{R}^N \times [0,T] \to \mathbb{R}^N$ is Lipschitz continuous in the first argument[5], continuous in the second one, and $B(0,t) = 0$.

(v) The operator $K : \mathbb{R}^N \to H$ is bounded linear with the adjoint $K^* : H \to \mathbb{R}^N$ associated *via* the scalar product identity

$$(u, Ky) = y^T K^* u$$

for any $y \in \mathbb{R}^N$ and $u \in H$.

(vi) The data $u_0 \in V$, $y_0 \in \mathbb{R}^N$ and $g \in C([0,T],\mathbb{R}^N)$ are given.

The monotone behavior of A in natural for a single conductor. The coupling of a single device to a network is sparse and it gives rise to a solid-conductor model, see [19]. Certain topological assumptions on the network ensure the positiveness of S and the monotonicity of M. We recall that the above operators S, A and K are time independent. One can always suppose that $P_A(0) = 0$ and $P_M(0) = 0$.

It follows then from the Hadamard lemma that

$$P_A(u) = P_A(u) - P_A(0) = \int_0^1 \langle A(\theta u), u\rangle \, \mathrm{d}\theta = \int_0^1 \frac{\langle A(\theta u), \theta u\rangle}{\theta} \, \mathrm{d}\theta \geq \frac{c_1}{2}|u|_V^2. \tag{3.5}$$

Furthermore, it is easy to see that

$$x^T M'(y)x = \lim_{h\to 0} \left[\frac{y+hx-y}{h}\right]^T \frac{[M(y+hx)-M(y)]}{h} \geq c_1|x|^2,$$

and so by a similar argument as above

$$y^T M(y) - P_M(y) \geq \frac{c_1}{2}|y|^2. \tag{3.6}$$

The Lipschitz continuity assumption on B directly provides the growth estimate $|B(u,t)| \leq C|u|$ for $t \in [0,T]$.

We will state now the main result of this section.

Theorem 3.3. *Let Assumptions 3.1 and 3.2 hold. Then there exists a unique solution* $(u,y) \in (W^{1,2}((0,T),V,H) \cap C^1([0,T],H)) \times C([0,T],\mathbb{R}^N)$ *of the problem*

$$\left.\begin{aligned} Su'(t) + Au(t) + SKy(t) &= 0 \quad \text{in } V^* \\ (My)'(t) + By(t) + K^*SKy(t) + K^*Su'(t) &= g(t) \end{aligned}\right\} \quad \text{on } (0,T) \tag{3.7}$$

with the initial data $u(0) = u_0$ *and* $y(0) = y_0$

Remark 3.4. The first equation in (3.7) should be understood as the variational problem

$$(Su', v) + \langle Au, v\rangle = -(SKy, v) \quad \text{for a.e. } t \in (0,T) \text{ and for any } v \in V.$$

However, it will be shown from the continuous data g that the time derivative u' in (3.7) is strong and both equations indeed hold not only for almost all $t \in (0,T)$, but on the whole time interval.

[4] We will regularly shorten the notation $M(y,t)$ to $M(y)$ or even My.

[5] We will regularly shorten the notation $B(y,t)$ to By.

Proof of uniqueness. Assume that there are two solutions (u_i, y_i) for $i = 1, 2$. We subtract the associated problems (3.7) for index $i = 1, 2$ from each other. We apply then the first equation to $u_1 - u_2 \in V$ and use (3.3) to get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|S^{1/2}(u_1 - u_2)\right\|^2 + \langle Au_1 - Au_2, u_1 - u_2\rangle = -(u_1 - u_2, SK(y_1 - y_2)),$$

for almost all $t \in (0, T)$. The integration in time yields

$$\|u_1(t) - u_2(t)\|^2 + \int_0^t |u_1 - u_2|_V^2 \le C \int_0^t \left(\|u_1 - u_2\|^2 + |y_1 - y_2|^2\right), \tag{3.8}$$

where we have used the strong positivity of S, monotonicity of A and the Young inequality with the boundedness of the operator SK.

Similarly, we integrate the second equation in time and multiply by $y_1 - y_2$ to see that

$$(y_1 - y_2)^T \left(My_1 - My_2 + K^*S(u_1 - u_2) + \int_0^t [By_1 - By_2 + K^*SK(y_1 - y_2)]\right) = 0,$$

and so the assumptions show

$$c_1|y_1(t) - y_2(t)|^2 \le C|y_1(t) - y_2(t)| \left(\|u_1(t) - u_2(t)\| + \int_0^t |y_1 - y_2|\right).$$

It follows thus from the Grönwall Lemma [4] that

$$|y_1(t) - y_2(t)| \le C\left(\|u_1(t) - u_2(t)\| + \int_0^t \|u_1 - u_2\|\right) \qquad \text{for any } t \in [0, T].$$

We use the above result in (3.8) to conclude again by the Grönwall argument that $u_1 = u_2$ in $W^{1,2}((0, T), V, H)$. This implies in return that also $y_1 = y_2$. Thus both solutions are identical. □

The rest of this section is devoted to the proof of existence of the solution.

3.1. Time discretization

Let the grid points $t_i = i\tau$ for $i = 0, \ldots, n$ with $\tau = T/n, n \in \mathbb{N}$ be a discretization of the time interval $(0, T)$. We adopt the standard notation for time discretized functions and its backward time difference,

$$u_i = u(t_i) \qquad \text{and} \qquad \delta u_i = \frac{u_i - u_{i-1}}{\tau},$$

and discretize the problem (3.7) in time using the backward Euler method

$$\begin{aligned} S\delta u_i + Au_i + SKy_i &= 0 \\ \delta M_i + By_i + K^*SKy_i + K^*S\delta u_i &= g_i \end{aligned} \qquad \text{for } i = 1, \ldots, n, \tag{3.9}$$

where $M_i = M(y_i)$.

Lemma 3.5 (Unique solvability of the discrete problem)**.** *Given Assumptions 3.1 and 3.2, there exists $n_0 \in \mathbb{N}$ such that for all integers $n > n_0$ the system (3.9) has a unique solution $(u_i, y_i) \in V \times \mathbb{R}^N$, $i = 1, \ldots, n$.*

Proof. We multiply the second equation in (3.9) by τ and consider the equivalent system

$$\begin{aligned} S\frac{u_i}{\tau} + Au_i + SKy_i &= S\frac{u_{i-1}}{\tau} \\ M_i + \tau By_i + \tau K^*SKy_i + K^*Su_i &= \tau g_i + M_{i-1} + K^*Su_i \end{aligned} \tag{3.10}$$

for $i = 1, \ldots, n$. Set $\mathcal{V} = V \times \mathbb{R}^N$. The left-hand side of (3.10) defines the operator

$$\mathcal{A}_\tau : \mathcal{V} \to \mathcal{V}^* \quad (u, y) \mapsto \left(S\frac{u}{\tau} + Au + SKy, My + \tau By + \tau K^* SKy + K^* Su \right).$$

The operator $\mathcal{A}_\tau$ is strongly monotone. Indeed, we can derive step by step by the monotonicity assumptions and the ε-Young inequality that

$$\begin{gathered} \langle \mathcal{A}_\tau(u, y) - \mathcal{A}_\tau(v, x), (u - v, y - x) \rangle_{\mathcal{L}(\mathcal{V}, \mathcal{V}^*)} \\ = \\ \frac{1}{\tau} \left\| S^{1/2}(u - v) \right\|^2 + \langle Au - Av, u - v \rangle + (SK(y - x), u - v) \\ +(y - x)^T (My - Mx) + \tau(y - x)^T (By - Bx) \\ +\tau \left\| S^{1/2} K(y - x) \right\|^2 + (SK(y - x), u - v) \\ \geq \\ \frac{1}{\tau} \left\| S^{1/2}(u - v) \right\|^2 + c_1 |u - v|_V^2 + c_1 |y - x|^2 + \tau \left\| S^{1/2} K(y - x) \right\|^2 \\ -\varepsilon |y - x|^2 - C_\varepsilon \left\| S^{1/2}(u - v) \right\|^2 - C\tau |y - x|^2 \\ \geq \\ C_0 \left(|y - x|^2 + \|u - v\|^2 + |u - v|_V^2 \right) \end{gathered}$$

for sufficiently small $\varepsilon > 0$ and sufficiently large $n > n_0$.

One can easily verify that the operator $\mathcal{A}_\tau$ is hemicontinuous and so according to the monotone operator theory (see [23], Thm. 26.A), there exists a unique solution (u_i, y_i) of the operator problem (3.10), for any $i = 1, \ldots, n$, which was to be proved. $\square$

Lemma 3.6 (First *a priori* estimate)**.** *If (u_i, y_i) is the solution from Lemma 3.5, then there exists $C > 0$ independent on n such that*

$$\max_{i=1,\ldots,n} (|y_i| + |u_i|_V) < C.$$

Proof. We apply the first equation in (3.9) to δu_i, multiply the second one by y_i^T and add them up to get

$$\left\| S^{1/2} \delta u_i \right\|^2 + \langle Au_i, \delta u_i \rangle + 2(SKy_i, \delta u_i) + y_i^T \delta M_i + y_i^T By_i + \left\| S^{1/2} Ky_i \right\|^2 = y_i^T g_i.$$

Obviously,

$$-|2(SKy_i, \delta u_i)| + \left\| S^{1/2} Ky_i \right\|^2 + \left\| S^{1/2} \delta u_i \right\|^2 \geq 0$$

and so we can write

$$\sum_{i=1}^{j} \tau \langle Au_i, \delta u_i \rangle + \sum_{i=1}^{j} \tau y_i^T \delta M_i \leq \sum_{i=1}^{j} \tau y_i^T g_i + C\tau \sum_{i=1}^{j} |y_i|^2. \tag{3.11}$$

Convexity of the functional P_A follows from the monotonicity of the operator A (*cf.* [21], Thm. 5.1). Therefore we have

$$\sum_{i=1}^{j} \tau \langle Au_i, \delta u_i \rangle \geq \sum_{i=1}^{j} P_A(u_i) - P_A(u_{i-1}) = P_A(u_j) - P_A(u_0). \tag{3.12}$$

The summation by parts yields in a similar way

$$\sum_{i=1}^{j} \tau y_i^T \delta M_i = \left[y_j^T M_j - y_0^T M_0\right] - \sum_{i=1}^{j} \tau \delta y_i^T M_{i-1} \geq y_j^T M_j - P_M(y_j) - \left[y_0^T M_0 - P_M(y_0)\right].$$

The right hand side of (3.11) can be estimated from above by the Young inequality. Gathering all the results and using (3.5) with (3.6) shows that

$$|y_j|^2 + |u_j|_V^2 \leq C + C\tau \sum_{i=1}^{j} |y_i|^2.$$

We use the discrete Grönwall inequality ([8], Lem. 5.1) and take maximum over $j = 1, \ldots, n$ to conclude the proof. □

Lemma 3.7 (Second *a priori* estimate)**.** *If (u_i, y_i) is the solution from Lemma 3.5, then there exists $C > 0$ independent on n such that*

(i)

$$\max_{i=1,\ldots,n} \|u_i\| + \sum_{i=1}^{n} \tau |u_i|_V^2 < C,$$

(ii)

$$\sum_{i=1}^{n} \tau \left\|\delta u_i\right\|^2 + \sum_{i=1}^{n} \tau^2 |\delta u_i|_V^2 + \sum_{i=1}^{n} \tau |\delta y_i|^2 < C.$$

Proof.

(i) Applying the first equation in (3.9) to τu_i gives

$$\tau(S\delta u_i, u_i) + \tau \langle A u_i, u_i \rangle = \tau(-SKy_i, u_i).$$

We add it up for $i = 1, \ldots, n$, use the Abel summation and monotonicity of A with $A(0) = 0$ to obtain

$$\frac{1}{2}\left(\left\|S^{1/2}u_j\right\|^2 - \left\|S^{1/2}u_0\right\|^2 + \sum_{i=1}^{j}\left\|S^{1/2}(u_i - u_{i-1})\right\|^2\right) + \sum_{i=1}^{j} \tau c_1 |u_i|_V^2 \leq \sum_{i=1}^{j} \tau(-SKy_i, u_i),$$

and so by the Young inequality

$$\frac{1}{2}\left\|S^{1/2}u_j\right\|^2 + \sum_{i=1}^{j} \tau c_1 |u_i|_V^2 \leq \frac{1}{2}\left\|S^{1/2}u_0\right\|^2 + \sum_{i=1}^{j} \tau \frac{\left\|S^{1/2}Ky_i\right\|^2 + \left\|S^{1/2}u_i\right\|^2}{2}.$$

We use then the discrete Grönwall argument together with Lemma 3.6 and the assumptions on S. Taking maximum over $j = 1, \ldots, n$ concludes the proof.

(ii) We apply the first equation in (3.9) to $\tau\delta u_i$, then we add it up for $i = 1, \ldots, n$ and use the Young inequality to obtain

$$\sum_{i=1}^{n} \tau \left\|S^{1/2}\delta u_i\right\|^2 + \sum_{i=1}^{n} \tau \langle A u_i \pm A u_{i-1}, \delta u_i \rangle \leq \sum_{i=1}^{n} \tau (SKy_i, \delta u_i). \tag{3.13}$$

It follows from the Young inequality that and estimate on Lemma 3.6 that

$$\sum_{i=1}^{n} \tau (SKy_i, \delta u_i)^2 \leq C + \frac{1}{2}\sum_{i=1}^{n} \tau \left\|S^{1/2}\delta u_i\right\|^2.$$

Combining the growth estimate for A –see Assumption 3.2(ii)– with (3.5)

$$|P_A(u)| = \left|\int_0^1 \langle A(\theta u), u\rangle \, \mathrm{d}\theta\right| \leq \int_0^1 \|A(\theta u)\|_{V^*} \|u\|_V \, \mathrm{d}\theta \leq C \int_0^1 (1 + |\theta u|_V) \|u\|_V \, \mathrm{d}\theta \leq C (1 + |u|_V) \|u\|_V \, .$$

Using this together with the convexity of P_A and Lemmas 3.6 and 3.7 we deduce that

$$\sum_{i=1}^{n} \tau \langle A u_{i-1}, \delta u_i\rangle \leq \sum_{i=1}^{n} P_A(u_i) - P_A(u_{i-1}) = P_A(u_n) - P_A(u_0) \leq C,$$

and so we can move this term to the right hand side of (3.13). The strong positivity of S and strong monotonicity of A finally leads to

$$\sum_{i=1}^{n} \tau \|\delta u_i\|^2 + \sum_{i=1}^{n} \tau^2 |\delta u_i|_V^2 \leq C. \tag{3.14}$$

We now multiply the second equation in (3.9) by $\tau \delta y_i$ to get

$$\tau c_1 |\delta y_i|^2 \leq \tau \delta y_i^T (-B y_i - K^* S K y_i - K^* S \delta u_i + g_i),$$

where we have already used the monotonicity of M. A clever use of the ε-Young inequality together with Lemma 3.6 and the assumption s on B, S and K shows for the right hand side that

$$\tau \delta y_i^T (-B y_i - K^* S K y_i - K^* S \delta u_i + g_i) \leq \tau \frac{c_1}{2} |\delta y_i|^2 + \tau C \left(1 + \|\delta u_i\|^2 + |g_i|^2\right)$$

The estimate (3.14) and the assumption on g yield

$$\frac{c_1}{2} \sum_{i=1}^{n} \tau |\delta y_i|^2 \leq \sum_{i=1}^{n} \tau C \left(1 + \|\delta u_i\|^2 + |g_i|^2\right) \leq C,$$

which finishes the proof. □

3.2. Rothe's method and existence of a solution

Let us introduce the piecewise-linear-in-time functions u_n and y_n

$$\begin{aligned} u_n(t) &= u_{i-1} + \delta u_i (t - t_{i-1}) \quad \text{for } t \in (t_{i-1}, t_i], \qquad u_n(0) = u_0, \\ y_n(t) &= y_{i-1} + \delta y_i (t - t_{i-1}) \quad \text{for } t \in (t_{i-1}, t_i], \qquad y_n(0) = u_0 \end{aligned} \tag{3.15}$$

and the piecewise-constant-in-time functions $\bar{u}_n$

$$\bar{u}_n(t) = u_i \quad \text{and} \quad \bar{y}_n(t) = y_i \quad \text{for } t \in (t_{i-1}, t_i], \tag{3.16}$$

where (u_i, y_i) is the solution of (3.9). The functions u_n and y_n have the right-hand derivative $u_n'(t) = \delta u_i$ and $y_n'(t) = \delta y_i$ for $t \in (t_{i-1}, t_i]$. It holds that

$$\bar{u}_n - u_n = \delta u_i (t_i - t) \quad \text{and} \quad \bar{y}_n - y_n = \delta y_i (t_i - t) \quad \text{on} \quad (t_{i-1}, t_i]. \tag{3.17}$$

The discrete system (3.9) reads in this notation as

$$\begin{aligned} S u_n' + A \bar{u}_n + S K \bar{y}_n &= 0 \\ M_n' + B \bar{y}_n + K^* S K \bar{y}_n + K^* S u_n' &= \bar{g}_n. \end{aligned} \tag{3.18}$$

We will prove that the sequence of Rothe's functions $\{(u_n, y_n)\}$ converges to the unique solution (u, y) of the original problem (3.7) for $n \to \infty$.

Lemma 3.8 (Convergence)**.** *Let Assumptions 3.1 and 3.2 hold. Assume that u_n and y_n solve the problem (3.18). Then there exists a subsequence*

(i) $u_n \rightharpoonup u$ *in* $W^{1,2}((0,T);V,H)$ *and* $u_n \to u$ *in* $C([0,T],H)$,
(ii) $\bar{u}_n \rightharpoonup u$ *in* $L^2(0,T),V)$,
(iii) $A\bar{u}_n \rightharpoonup w$ *in* $L^2(0,T),V^*)$,
(iv) $y_n \to y$ *and* $\bar{y}_n \to y$ *uniformly for all* $t \in [0,T]$.

Proof. We remark that the uniqueness of the solution justifies keeping the same subscript n for all subsequences we will choose.

(i) and (ii): Lemmas 3.6 and 3.7 imply that

$$\int_0^T \left[\|\bar{u}_n\|_V^2 + \|u_n\|_V^2 + \|u_n'\|^2 \right] \mathrm{d}t \le C.$$

and so $u_n \rightharpoonup u$ and $\bar{u}_n \rightharpoonup \bar{u}$ in the above Sobolev spaces, since every bounded sequence in a reflexive space has a weakly convergent subsequence. Moreover, it follows from (3.17)

$$\int_0^T \|u_n - \bar{u}_n\|_V^2 \,\mathrm{d}t = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|\delta u_i (t - t_{i-1})\|_V^2 \,\mathrm{d}t \le \sum_{i=2}^n \tau^3 \|\delta u_i\|_V^2 \le C\tau,$$

which clearly forces $u = \bar{u}$. The convergence of u_n to u follows from the continuous embedding (3.2), see also ([15], Lem. 1.3.13).

(iii): The growth estimate on A yields

$$\int_0^T \|A\bar{u}_n\|_{V^*}^2 \,\mathrm{d}t < C,$$

hence the weak compactness argument as above provides the assertion.

(iv): It is easy to see from Lemmas 3.6 and 3.7 that the sequence $\{y_n\}$ is equi-bounded and equi-continuous. The uniform convergence $y_n \to y$ is then direct consequence of Arzela–Ascoli theorem (*e.g.* [10], Thm. 1.5.3). Moreover, we see from (3.17) that

$$|\bar{y}_n(t) - y_n(t)| \le (\tau |\delta y_i|^2) \tau^{1/2} \le C\tau^{1/2} \quad \text{for } t \in (t_{i-1}, t_i], \tag{3.19}$$

and so the uniform convergence for $\bar{y}_n$ to y follows from the triangle inequality. □

We can take $n \to \infty$ in (3.18) and apply Lemma 3.8 to obtain

$$\begin{aligned} Su' + w + SKy &= 0 \\ (My)' + By + K^*SKy + K^*Su' &= g. \end{aligned} \tag{3.20}$$

in the weak sense.

We now use the Minty–Browder trick to show that $w = Au$, compare with ([23], Lem. 30.6).

Lemma 3.9 (Monotonicity trick)**.** *Let Assumptions 3.1 and 3.2 hold. Assume that u_n and y_n solve the problem (3.18). Then*

(i) $\limsup_{n\to\infty} \langle A\bar{u}_n, \bar{u}_n \rangle_{(0,T)} = \langle w, u \rangle_{(0,T)}$
(ii) $w = A(u)$.

Proof.

(i) Integration by parts (3.4) applied to (3.18) yields

$$\frac{1}{2}\left(\left\|S^{1/2}u_n(T)\right\|^2 - \left\|S^{1/2}u_n(0)\right\|^2\right) = \left\langle S^{1/2}u_n', S^{1/2}u_n\right\rangle_{(0,T)} = \langle -SK\bar{y}_n - A\bar{u}_n, u_n\rangle_{(0,T)},$$

and hence

$$\langle A\bar{u}_n, u_n\rangle_{(0,T)} = \langle -SK\bar{y}_n, u_n\rangle_{(0,T)} + \frac{1}{2}\left(\left\|S^{1/2}u_n(0)\right\|^2 - \left\|S^{1/2}u_n(T)\right\|^2\right).$$

Taking the limit $n \to \infty$ and then using the first equation in (3.20) leads to

$$\begin{aligned}\limsup_{n\to\infty} \langle A\bar{u}_n, \bar{u}_n\rangle_{(0,T)} &= \limsup_{n\to\infty}\left(\langle A\bar{u}_n, \bar{u}_n - u_n\rangle_{(0,T)} + \langle A\bar{u}_n, u_n\rangle_{(0,T)}\right)\\ &= \langle -SKy, u\rangle_{(0,T)} + \frac{1}{2}\left(\left\|S^{1/2}u_0\right\|^2 - \left\|S^{1/2}u(T)\right\|^2\right)\\ &= \langle w, u\rangle_{(0,T)},\end{aligned}$$

where we have used the fact that $\bar{u}_n - u_n \rightharpoonup 0$ in $L^2(0,T), V)$.

(ii) We start with

$$\langle A\bar{u}_n - Av, \bar{u}_n - v\rangle_{(0,T)} \geq 0,$$

and take the limit $n \to \infty$. With help of the result (i), we obtain

$$\langle w - Av, u - v\rangle_{(0,T)} \geq 0$$

for any function $v \in L^2((0,T), V)$. If $v = u - \varepsilon h$ with $\varepsilon > 0$, we have

$$-\varepsilon \langle w - A(u - \varepsilon h), h\rangle_{(0,T)} \geq 0.$$

Therefrom

$$\langle w - Au, h\rangle_{(0,T)} \leq 0 \quad \text{as well as} \quad \langle w - Au, h\rangle_{(0,T)} \geq 0$$

for any $h \in L^2((0,T), V)$, which implies the assertion. $\square$

Finally, we note that the data g in (3.7) and the solution (u, y) are well defined for all $t \in [0,T]$ and so is in return its time derivative (u', y'), which means that (3.7) holds true for all $t \in [0,T]$. We have thus proved Theorem 3.3.

4. Time error estimates

This section deals with the convergence rate of Rothe's method.

Theorem 4.1. *Let u be the solution from Theorem 3.3 and (u_n, y_n) be its Rothe approximation defined in (3.15). Then there is a constant $C > 0$ such that*

$$\max_{t\in[0,T]}\left(\|u(t) - u_n(t)\|^2 + \|y(t) - y_n(t)\|^2\right) + \int_0^T |u - \bar{u}_n|_V^2 \,\mathrm{d}t \leq C\tau.$$

Proof. We follow a standard technique and consider the difference between the second equations in (3.7) and (3.18)

$$(My - M_n)' + By - B\bar{y}_n + K^*S(u' - u_n') + K^*SK(y - \bar{y}_n) = g - \bar{g}_n.$$

We integrate it time and multiply it by the difference $(y - y_n)^T$ to derive that

$$c_1|y(t) - y_n(t)|^2 \leq C|y(t) - y_n(t)|\left(\|(u(t) - u_n(t))\| + \tau + \int_0^t |y - \bar{y}_n| + \int_0^t |g - \bar{g}_n|\right)$$

by the monotonicity of M and the Lipschitz continuity of K^*S and $B + K^*SK$. Let us estimate the last two terms above. The triangle inequality and Lemma 3.7 give

$$\int_0^t |y \pm y_n - \bar{y}_n| \le \int_0^t |y - y_n| + T^{1/2} \left(\int_0^T |y_n - \bar{y}_n|^2 \right)^{1/2} \le \int_0^t |y - y_n| + C\tau.$$

The 1-mean continuity argument for the continuous function g on a compact interval shows that

$$\int_0^t |g - \bar{g}_n| \le \int_0^T |g - \bar{g}_n| \le C\tau. \tag{4.1}$$

Collecting both results leads to

$$|y(t) - y_n(t)| \le C \left(\|(u(t) - u_n(t))\| + \tau + \int_0^t |y - y_n| \right),$$

hence the Grönwall lemma yields

$$|y(t) - y_n(t)| \le C \left(\|u(t) - u_n(t)\| + \int_0^t \|u - u_n\| + \tau \right). \tag{4.2}$$

Let us now consider the difference between the first equations in (3.7) and (3.18)

$$u' - u_n' + Au - A\bar{u}_n + SK(y - \bar{y}_n) = 0 \quad \text{ on } (0, T). \tag{4.3}$$

We apply it to the difference $u - \bar{u}_n$ and integrate in time to obtain

$$\int_0^t (u' - u_n', u \pm u_n - \bar{u}_n) + \int_0^t \langle Au - A\bar{u}_n, u - \bar{u}_n \rangle + \int_0^t (SK(y - \bar{y}_n), u - \bar{u}_n) = 0,$$

and so by the monotonicity of A

$$\frac{1}{2} \|u(t) - u_n(t)\|^2 + \int_0^t c_1 |u - \bar{u}_n|_V^2 \le \int_0^t -(u' - u_n', u_n - \bar{u}_n) - \int_0^t (SK(y - \bar{y}_n), u - \bar{u}_n).$$

We deduce from Lemma 3.8 that

$$\begin{aligned} - \int_0^t (u' - u_n', u_n - \bar{u}_n) \, \mathrm{d}s &\le \left[\|u'\|_{L^2((0,T),H)} + \|u_n'\|_{L^2((0,T),H)} \right] \|u_n - \bar{u}_n\|_{L^2((0,T),H)} \\ &\le C\tau. \end{aligned} \tag{4.4}$$

It follows next from the assumptions and (4.2) that

$$\begin{aligned} (SK(y - \bar{y}_n), u_n - \bar{u}_n) &\le C|y - \bar{y}_n| \, \|u_n - \bar{u}_n\| \le C \left(|y - y_n| + |y_n - \bar{y}_n| \right) \|u_n - \bar{u}_n\| \\ &\le C \left(\|u(t) - u_n(t)\| + \int_0^t \|u - u_n\| + \tau + |\partial_t y_n| \tau \right) \|\partial_t u_n\| \, \tau, \end{aligned}$$

and subsequently

$$- \int_0^t (SK(y - \bar{y}_n), u_n - \bar{u}_n) \le C \left(\tau^2 + \int_0^t \|u - u_n\|^2 \right).$$

Combining both results and using the Grönwall lemma, we get

$$\max_{t \in [0,T]} \|u(t) - u_n(t)\|^2 + \int_0^T |u - \bar{u}_n|_V^2 \, \mathrm{d}t \le C\tau,$$

which implies the same convergence rate for y_n, see (4.2). □

The sub-linear convergence in Theorem 4.1 results from the estimate 4.4. To obtain the linear convergence of the Rothe scheme, we will additionally need the Lipschitz continuity assumption on A and g.

Assumption 4.2. Assume the following:

(i) The operator $A : V^* \to V$ is Lipschitz continuous in the sense that

$$\|Au - Av\|_{V^*} \leq C\,\|u - v\|_V \qquad \text{for all } v, u \text{ in } V.$$

(ii) The data $g \in C([0,T], \mathbb{R}^N)$ is Lipschitz continuous.

We will first need the following *a priori* estimate to prove it.

Lemma 4.3 (Third *a priori* estimate)**.** *Let u_i be the solution from Lemma 3.5 and let Assumption 4.2 hold. Assume that $u_0 \in D(A)$. Then there exists $C > 0$ independent of n such that*

$$\max_{i=1,\ldots,n} \|\delta u_i\| + \tau \sum_{i=1}^{n} |\delta u_i|_V^2 \leq C.$$

Proof. Consider the following operator equation

$$S\delta^2 u_i + \frac{Au_i - Au_{i-1}}{\tau} = -SK\delta y_i \qquad \text{for } i = 1, \ldots, n, \tag{4.5}$$

which is in fact the difference between two consecutive first equations in (3.9). For $i = 1$ we set $\delta u_0 = Au_0 + f_0$. We will proceed in the same fashion as in the proof of Lemma 3.7. Applying (4.5) to $\tau\delta u_i$ gives

$$\tau(S\delta^2 u_i, \delta u_i) + \frac{\langle Au_i - Au_{i-1}, u_i - u_{i-1}\rangle}{\tau} = (-\tau SK\delta y_i, \delta u_i).$$

We add it up for $i = 1, \ldots, n$, use the Abel summation and monotonicity of A to obtain

$$\frac{1}{2}\left(\left\|S^{1/2}\delta u_j\right\|^2 - \left\|S^{1/2}\delta u_0\right\|^2 + \sum_{i=1}^{j}\left\|S^{1/2}(\delta u_i - \delta u_{i-1})\right\|^2\right) + \sum_{i=1}^{j}\tau c_1|\delta u_i|_V^2 \leq \sum_{i=1}^{j}\tau(-SK\delta y_i, \delta u_i),$$

which becomes

$$\|\delta u_j\|^2 + \sum_{i=1}^{j}\tau|\delta u_i|_V^2 \leq C + C\sum_{i=1}^{j}\tau\left(\|\delta y_i\|^2 + \|\delta u_i\|^2\right).$$

To conclude the proof, we use Lemma 3.7, apply the discrete Grönwall argument and take maximum over $j = 1, \ldots, n$. □

Theorem 4.4. *Let u be the solution from Theorem 3.3 and (u_n, y_n) be its Rothe approximation defined in (3.15) and let Assumption 4.2 hold. Then there is a constant $C > 0$ such that*

$$\max_{t\in[0,T]}\left(\|u(t) - u_n(t)\|^2 + \|y(t) - y_n(t)\|^2\right) + \int_0^T |u - \bar{u}_n|_V^2 \,\mathrm{d}t \leq C\tau^2.$$

Proof. In the view of the estimate, (4.2), it is sufficient to show the linear convergence of u_n. We consider again the difference (4.3). We apply it to $u - u_n$ and integrate in time to obtain

$$\frac{1}{2}\|u(t) - u_n(t)\|^2 + \int_0^t \langle Au \pm Au_n - A\bar{u}_n, u - u_n\rangle + \int_0^t (SK(y - \bar{y}_n), u - u_n) = 0.$$

and so by the monotonicity of A

$$\frac{1}{2}\left\|u(t)-u_n(t)\right\|^2 + c_1\int_0^t |u-u_n|_V^2 \le \int_0^t \langle SK(y-\bar{y}_n), u-u_n\rangle - \int_0^t \langle Au_n - A\bar{u}_n, u-u_n\rangle \,.$$

Let us estimate the right-hand side. By Lemma 3.7 and (4.2)

$$\begin{aligned}\int_0^t \langle SK(y-\bar{y}_n), u-u_n\rangle &\le C\int_0^t \left\|(y\pm y_n-\bar{y}_n)\right\|^2 + \left\|u-u_n\right\|^2\\ &\le C\tau^2 + C\int_0^t \left\|u-u_n\right\|^2.\end{aligned}$$

The $c_1/2$-Young inequality for the second term yields

$$-\int_0^t \langle Au_n - A\bar{u}_n, u-u_n\rangle \le \int_0^t \left(\frac{2}{c_1}\left\|Au_n - A\bar{u}_n\right\|_{V^*}^2 + \frac{c_1}{2}\left\|u-u_n\right\|_V^2\right),$$

and subsequently by the Lipschitz continuity of A and Lemma 4.3 we have

$$\int_0^t \left\|Au_n - A\bar{u}_n\right\|_{V^*}^2 \le \int_0^T C\left\|u_n-\bar{u}_n\right\|_V^2 \le C\sum_{i=1}^n \tau^3\left\|\delta u_i\right\|_V^2 \le C\tau^2.$$

Collecting all the results shows that

$$\left\|u(t)-u_n(t)\right\|^2 + \int_0^t |u-u_n|_V^2 \,\mathrm{d}s \le C\tau^2 + C\int_0^t \left\|u-u_n\right\|^2 \,\mathrm{d}s.$$

We finally apply the Grönwall argument and take maximum over $t\in[0,T]$ to complete the proof. □

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