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DOMAIN DECOMPOSITION PRECONDITIONERS FOR THE DISCONTINUOUS PETROV–GALERKIN METHOD *

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Abstract. In this paper, we design some efficient domain decomposition preconditioners for the discontinuous Petrov–Galerkin (DPG) method. Due to the special properties of the DPG method, the boundary condition becomes crucial in both of its application and analysis. We mainly focus on one of the boundary conditions: the Robin boundary condition, which actually appears in some useful model problems like the Helmholtz equation. We first design a two-level additive Schwarz preconditioner for the Poisson equation with a Robin boundary condition and give a rigorous condition number estimate for the preconditioned algebraic system. Moreover we also construct an additive Schwarz preconditioner for solving the Helmholtz equation. Numerical results show that the condition number of the preconditioned system is independent of wavenumber ω and mesh size h.

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1. Introduction

In recent years, the discontinuous Petrov-Galerkin (DPG) method become a very popular numerical tool for solving partial differential equations (PDEs). Different from the standard discontinuous Galerkin (DG) methods, the DPG method is based on the so-called "ultra-weak" variational formulation. In [11], Demkowicz and Gopalakrishnan firstly introduced the idea of optimal test space and viewed the DPG method as the minimization of the residual in a dual norm. Furthermore, by using the concept of the *optimal test norm* and its equivalent norm, one may analyze the DPG method in a norm of interest. In [19], using an approximated test space, Gopalakrishnan and Qiu introduced a practical DPG method for solving the Laplace equation and linear elasticity. In this paper, we shall adopt their approximated test space both in the analysis and in the numerical experiments.

Nowadays this method has been successfully applied to various problems. For instance, application to the Laplace equation was developed in [12]. A new type of DPG method for the Poisson's equation, which avoids reformulating the problem as a first order system, was designed in [13]. The DPG method for the singularly

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perturbed problems was discussed in [14]. There also exist the DPG methods for the Helmholtz equation [15], the Friedrichs' systems [4], and the Stokes problem [27]. Recently, a hp-adaptive DPG method was designed for the convection-dominated diffusion problem in [16]. Meanwhile, in [6], a general *a posteriori* error analysis for the DPG method was established. Moreover, the DPG method was also applied to some nonlinear problems, like the Burgers and compressible Navier-Stokes equations [7, 8, 28].

Although there are lots of works on applying the DPG method to a wide range of PDEs in the literature, only few works [2, 20] relate to the fast solvers of the algebraic systems resulting from the DPG method. In the authors' opinion, one reason may be that the complicated structure of DPG, especially the optimal test norm, is not easy to deal with. Another reason is that the boundary condition of the original PDEs, which becomes essential for the DPG method, also brings many difficulties.

In this paper, we shall consider domain decomposition (DD) methods for the DPG method for solving the Poisson equation and the Helmholtz equation. A pioneering work in this area was proposed by Barker *et al.* in [2], where, with the help of two extension operators, a one-level overlapping Schwarz preconditioner was designed for the Poisson equation. This DD preconditioner was shown to be efficient, however this DD preconditioner cannot be directly extended to two-level case. As a result, the convergence of this preconditioned system may become slow when the overlap of the subdomains becomes small. Moreover this preconditioner was only designed for a Dirichlet boundary problem and is not readily applicable to other boundary conditions like the Robin boundary condition. For the Helmholtz equation, in [20], Gopalakrishnan and *Schöberl* designed a fast solver for the DPG method. Numerical results showed that their fast solver was independent of the wavenumber ω.

For the Laplace equation, through introducing a special coarse space, in this paper we shall extend the one-level domain decomposition preconditioner in [2] to two-level case. Furthermore under a new framework, we may prove that our two-level solver is efficient for various kinds of boundary conditions. We give a rigorous condition number estimate for the case of the Robin boundary condition. There are two reasons why we focus on the model problem with Robin boundary condition. First, similar to many least-square type methods, the DPG method treats all boundary conditions as essential boundary conditions. The essential boundary condition makes a big difference both in the analysis and in application for least-square type methods. Furthermore the case of Robin boundary condition becomes the most difficult case for the theoretical analysis of our DD method. Actually, the Robin boundary condition makes a coupling of the trace spaces of the velocity field and the pressure field, thus we need to derive analysis in this coupled space. To overcome this difficulty, we shall construct some special discrete and continuous Helmholtz decompositions in this paper. More details may be found in Section 4. The second reason is that Robin boundary condition is an important boundary condition for some useful model problems like the Helmholtz equation. In this case, the Robin boundary condition is also called the Sommerfeld condition. We also construct a one-level additive Schwarz preconditioner for the Helmholtz equation. Numerical results show that the number of iteration given by the preconditioned CG is independent of wavenumber ω and mesh size h. Moreover our DD fast solver is easy to parallelize. This parallelization property is very important for solving the Helmholtz equation with high wave number.

The remaining part of this paper is organized as follows: the DD preconditioner for the Poisson equation is introduced in Sections 2–4. Section 2 is an introduction of the corresponding model problem. In Section 3, we shall give a framework for the DD method. We shall estimate the condition number of the preconditioned system in Section 4. For the Helmholtz equation, we shall introduce a one-level preconditioner in Section 5. Numerical results for the both cases shall be given in Section 6.

2. The discontinuous Petrov–Galerkin (DPG) method

Let $\Omega \in \mathbb{R}^d$ $(d = 2, 3)$ be a convex polyhedral domain with a Lipschitz continuous boundary and Ω_h be a geometrically conforming, shape regular triangulation consisting of simplicial elements. Element of Ω_h is denoted by K, $\overline{\Omega} = \{\overline{K} : K \in \Omega_h\}$. The diameter of each element K is $O(h)$. We denote by $\partial\Omega_h$ the collection of all the element boundaries, *i.e.* $\partial\Omega_h := \bigcup\{\partial K : K \in \Omega_h\}$.

Although it is not difficult to extend our analysis to some more complicated problems, for simplicity, we only consider the following model problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} + u = 0 & \text{on } \partial\Omega. \end{cases}$$

The ultra-weak variation formulation of the DPG method may be defined as: find $\mathcal{U} = (\vec{\sigma}, u, \hat{u}, \hat{\sigma_n}) \in U$ such that

$$b\,(\mathcal{U}, \mathcal{V}) := b\,((\vec{\sigma}, u,\ \hat{u}, \hat{\sigma_n}), (\vec{\tau}, v)) = (f, v)_\Omega \quad \forall\ \ \mathcal{V} = (\vec{\tau}, v)) \in V, \tag{2.1}$$

where

$$\begin{aligned} b\,((\vec{\sigma}, u, \hat{u}, \hat{\sigma_n}), (\vec{\tau}, v)) := (\vec{\sigma}, \vec{\tau})_{\Omega_h} - (u, \vec{\nabla} \cdot \vec{\tau})_{\Omega_h} - (\vec{\sigma}, \vec{\nabla} v)_{\Omega_h} \\ + \langle v, \hat{\sigma_n} \rangle_{\partial\Omega_h} + \langle \hat{u}, \vec{\tau} \cdot \vec{n} \rangle_{\partial\Omega_h}, \end{aligned}$$

here $(u, v)_{\Omega_h} = \sum_{K \in \Omega_h} (u, v)_K$ and $\langle \hat{u}, \hat{v} \rangle_{\partial\Omega_h} = \sum_{K \in \Omega_h} \langle \hat{u}, \hat{v} \rangle_{\partial K}$, $(\cdot, \cdot)_D$ denotes the $L^2(D)$ inner product, $\langle \hat{u}, \cdot \rangle_{\partial K}$ denotes the action of a linear functional $\hat{u} \in H^{-\frac{1}{2}}(\partial K)$, $\vec{n}$ stands for the outer unit normal of K.

To deal with the Robin boundary condition, define spaces S and Q as

$$\begin{aligned} S &:= \{(\vec{\sigma}, u) \in H(\mathrm{div}; \Omega) \times H^1(\Omega) : (\vec{\sigma} \cdot \vec{n} - u)|_{\partial\Omega} = 0\}, \\ Q &:= \{(\hat{u}, \hat{\sigma}_n) : \exists (\vec{\sigma}, u) \in S \quad \text{such that} \quad tr_{\partial\Omega_h}(\vec{\sigma}, u) = (\hat{\sigma}_n, \hat{u})\}, \end{aligned}$$

here $tr_{\partial\Omega_h}$ stands for the traces of every element K, $tr_{\partial\Omega_h}(\vec{\sigma}, u) = (\hat{\sigma}_n, \hat{u})$ means that

$$\vec{\sigma} \cdot \vec{n}|_{\partial K} = \hat{\sigma}_n|_{\partial K} \text{ and } u|_{\partial K} = \hat{u}|_{\partial K} \quad \forall\ \ K \in \Omega_h.$$

For the convenience of our proof, we view Q as a subspace of $H^{\frac{1}{2}}(\partial\Omega_h) \times H^{-\frac{1}{2}}(\partial\Omega_h)$, where

$$\begin{aligned} H^{\frac{1}{2}}(\partial\Omega_h) &:= \ \{\eta : \exists u \in H^1(\Omega) \text{ such that } u|_{\partial K} = \eta|_{\partial K} \ \ \forall\ \ K \in \Omega_h\}, \\ H^{-\frac{1}{2}}(\partial\Omega_h) &:= \ \{\eta \in \prod_{K \in \Omega_h} H^{-\frac{1}{2}}(\partial K) : \exists\ \vec{\sigma} \in H(\mathrm{div}; \Omega) \text{ such that} \\ &\qquad \vec{\sigma} \cdot \vec{n}|_{\partial K} = \eta|_{\partial K} \ \ \forall\ \ K \in \Omega_h\}. \end{aligned}$$

The norms in $H^{\frac{1}{2}}(\partial\Omega_h)$ and $H^{-\frac{1}{2}}(\partial\Omega_h)$ are defined as:

$$\|\hat{u}\|_{H^{\frac{1}{2}}(\partial\Omega_h)} = \ \inf\{\|u\|_{H^1(\Omega)} : u \in H^1(\Omega),\ u|_{\partial K} = \hat{u}|_{\partial K} \ \ \forall \ \ \ K \in \Omega_h\}, \tag{2.2}$$

$$\|\hat{\sigma}_n\|_{H^{-\frac{1}{2}}(\partial\Omega_h)} = \ \inf\{\|\vec{\sigma}\|_{H(\mathrm{div};\Omega)} : \vec{\sigma} \in H(\mathrm{div}; \Omega),\ \vec{\sigma} \cdot \vec{n}|_{\partial K} = \hat{\sigma}_n|_{\partial K} \ \ \forall \ \ \ K \in \Omega_h\}. \tag{2.3}$$

We also define the "broken" spaces $H(\mathrm{div}; \Omega_h)$ and $H^1(\Omega_h)$ as:

$$H(\mathrm{div}; \Omega_h) := \prod_{K \in \Omega_h} H(\mathrm{div}; K),\ H^1(\Omega_h) := \prod_{K \in \Omega_h} H^1(K),$$

which are Hilbert spaces with inner products

$$\begin{aligned} (\vec{\sigma}, \vec{\tau})_{H(\mathrm{div};\Omega_h)} &:= \sum_{K \in \Omega_h} ((\vec{\sigma}, \vec{\tau})_K + (\mathrm{div}\vec{\sigma}, \mathrm{div}\vec{\tau})_K), \\ (u, v)_{H^1(\Omega_h)} &:= \sum_{K \in \Omega_h} ((u, v)_K + (\vec{\nabla} u, \vec{\nabla} v)_K). \end{aligned}$$

With these notations, we may set the trial space U and the test space V in weak formulation (2.1) by

$$U := (L^2(\Omega))^d \times L^2(\Omega) \times Q,\ V := H(\mathrm{div};\Omega_h) \times H^1(\Omega_h),$$

and the inner product in V is defined as:

$$((\vec{\sigma}, u), (\vec{\tau}, v))_V := (\vec{\sigma}, \vec{\tau})_{H(\mathrm{div};\Omega_h)} + (u, v)_{H^1(\Omega_h)}.$$

Remark 2.1. We may find by the definition of Q that the Robin boundary condition becomes an essential boundary condition in the trace space Q. For the case of other boundary conditions, the restrictions are also needed in Q. For example, the restriction of Q should be $\hat{u} = 0$ on $\partial\Omega$ for the homogeneous Dirichlet boundary condition, and $\hat{\sigma}_n = 0$ *on* $\partial\Omega$ for the homogeneous Neumann boundary condition.

With the help of the weak formulation (2.1), we may derive the ideal DPG method, which is to calculate an approximation of $\mathcal{U}$ in a trial space $U_h(\subseteq U)$. The test space may be chosen as $TU_h(\subseteq V)$, here T is the *trial-to-test* operator defined as:

$$T:\ U_h \to V,\ (T\mathcal{U}_h, \mathcal{V})_V = b(\mathcal{U}_h, \mathcal{V}) \quad \forall\ \ \mathcal{V} \in V.$$

However, in practical computation, in most cases we cannot calculate such test functions in V. Instead, according to [19], we may define a discrete *trial-to-test* operator T_h and do the calculation in a subspace U_h and an *enriched space* V_h. One proper choice of the discrete spaces U_h and V_h may be

$$\begin{aligned} Q_h &:= \ (\widetilde{P}^{m+1}(\partial\Omega_h) \times P^m(\partial\Omega_h)) \ \cap\ Q, \\ U_h &:= \prod_{K\in\Omega_h} (P^m(K))^d \times \prod_{K\in\Omega_h} P^m(K) \times Q_h \\ V_h &:= \prod_{K\in\Omega_h} (P^{m+d}(K))^d \times \prod_{K\in\Omega_h} P^{m+d}(K), \end{aligned}$$

where $P^m(K)$ is the space of polynomials in K with total degree $\leq m$. Spaces $\widetilde{P}^m(\partial\Omega_h)$ and $P^m(\partial\Omega_h)$ are defined by

$$\begin{aligned} \widetilde{P}^m(\partial\Omega_h) &:= \{p \in \prod_{F\in\mathcal{F}_h} P^m(F) :\ p \text{ is continuous on } \partial\Omega_h\}, \\ P^m(\partial\Omega_h) &:= \prod_{F\in\mathcal{F}_h} P^m(F), \end{aligned}$$

here $P^m(F)$ is the space of piecewise polynomials on each face F with total degree $\leq m$ and $\mathcal{F}_h$ is the set of faces in triangulation Ω_h:

$$\mathcal{F}_h := \{F : \exists K \in \Omega_h \text{ such that } F \text{ is an face of } K\}.$$

It is worth mentioning that, in each element K, $\prod_{F\in K} P^m(F)$ may be viewed as the trace space of Raviart−Thomas element space $RT^{m+1}(K)$ (*cf.* [25,26]).

The discrete *trial-to-test* operator T_h may be defined as:

$$T_h:\ U_h \to V_h,\ (T_h\mathcal{U}_h, \mathcal{V}_h)_V = b(\mathcal{U}_h, \mathcal{V}_h) \quad \forall\ \ \mathcal{V}_h \in V_h,$$

then the test space of this practical DPG is T_hU_h, which is a subspace of V_h. Now we may derive the DPG form for the model problem: find $\mathcal{U}_h \in U_h$ such that

$$b\,(\mathcal{U}_h, \mathcal{V}_h) = (f, v_h) \quad \forall\ \ \mathcal{V} = (\vec{\tau}_h, v_h) \in T_hU_h. \tag{2.4}$$

Note that if $\mathcal{V} = T\mathcal{U}'$, the bilinear form $b(\mathcal{U}, \mathcal{V})$ may be viewed as $b(\mathcal{U}, \mathcal{V}) = (T\mathcal{U}, T\mathcal{U}')_V$. Moreover, since the DPG system is uniquely solvable ($cf.$ [12]), we may define a positive definite form $a(\mathcal{U}, \mathcal{U}') := b(\mathcal{U}, T\mathcal{U}')$.

For practical DPG, we may also define $a_h(\mathcal{U}_h, \mathcal{U}_h') := b(\mathcal{U}_h, T_h\mathcal{U}_h')$, thus practical DPG method also produces a symmetric positive definite system.

According to (2.1) and the definition of our test space, the variational form of the DPG method for our model problem may be rewritten as: find $\mathcal{U}_h \in U_h$ such that

$$a_h(\mathcal{U}_h, \mathcal{U}_h') = \langle \widetilde{f}, \mathcal{U}_h' \rangle \quad \forall\ \mathcal{U}_h' \in U_h, \tag{2.5}$$

where $\widetilde{f}$ belongs to U_h' (the dual space of U_h), $\widetilde{f}$ is defined as:

$$\langle \widetilde{f}, \mathcal{U}_h' \rangle = (f, v_h) \quad \forall\ (\vec{\tau}_h, v_h) = T_h\mathcal{U}_h'.$$

Actually (2.5) is the discrete system of our model problem.

In the following parts of our paper, for simplicity we use $A \lesssim B$ and $A \gtrsim B$ instead of $A \le CB$ and $A \ge CB$, where C is a constant which only depends on shape of Ω_h and polynomial degree m. $A \approx B$ means that $A \lesssim B$ and $A \gtrsim B$.

Lemma 2.2 is an equivalent result given in [12, 19], which shall play an essential role in our analysis.

Lemma 2.2. *For all* $\mathcal{U} = (\vec{\sigma}, u, \hat{u}, \hat{\sigma}_n) \in U$, $\mathcal{U}_h = (\vec{\sigma}_h, u_h, \hat{u}_h, \hat{\sigma}_{n,h}) \in U_h$, *we have*

$$\begin{aligned} a(\mathcal{U}, \mathcal{U}) &\approx \|\vec{\sigma}\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} + \|\hat{u}\|^2_{H^{\frac{1}{2}}(\partial\Omega_h)} + \|\hat{\sigma}_n\|^2_{H^{-\frac{1}{2}}(\partial\Omega_h)}, \\ a_h(\mathcal{U}_h, \mathcal{U}_h) &\approx \|\vec{\sigma}_h\|^2_{L^2(\Omega)} + \|u_h\|^2_{L^2(\Omega)} + \|\hat{u}_h\|^2_{H^{\frac{1}{2}}(\partial\Omega_h)} + \|\hat{\sigma}_{n,h}\|^2_{H^{-\frac{1}{2}}(\partial\Omega_h)}. \end{aligned}$$

3. Domain decomposition algorithm

Let $\{\Omega_i\}_{i=1}^N$ be a family of subdomains of Ω. The triangulation of Ω_i is aligned with Ω_h. Define the triangulation of $\{\Omega_i\}_{i=1}^N$ by $\{\Omega_{i,h}\}_{i=1}^N$. Similar to the function spaces defined in Ω and Ω_h, we may define function spaces on Ω_i and $\Omega_{i,h}$, such as $H^1(\Omega_{i,h})$, $H(\mathrm{div}; \Omega_{i,h})$, $H^{\frac{1}{2}}(\partial\Omega_{i,h})$ and $H^{-\frac{1}{2}}(\partial\Omega_{i,h})$. The overlap of $\{\Omega_i\}_{i=1}^N$ is measured by δ such that there exists a partition of unity $\{\theta_i\}_{i=1}^N \in (W^{1,\infty}(\Omega))^N$ which satisfies

$$\mathrm{supp}(\theta_i) \subseteq \Omega_i,\ \sum_{i=1}^N \theta_i = 1 \text{ in } \Omega,\ 0 \le \theta_{\mathrm{i}} \le 1, \text{ and } \|\nabla\theta_{\mathrm{i}}\|_{\mathrm{L}^\infty(\Omega_{\mathrm{i}})} \lesssim \frac{1}{\delta}. \tag{3.1}$$

According to Chapter 3 in [30], we may also obtain a modified partition of unity by interpolating $\{\theta_i\}_{i=1}^N$ on the fine triangulation Ω_h and the above properties still hold. A proper choice of such interpolation is the nodal piecewise linear interpolation, which makes the modified partition of unity piecewise linear and continuous. We shall use this modified partition of unity instead of the original one in the proofs that follow, and still denote it by $\{\theta_i\}_{i=1}^N$. Moreover we shall also limit the intersections between the subdomains $\{\Omega_i\}_{i=1}^N$, *i.e.*,

Assumption 3.1. The partition $\{\Omega_i\}_{i=1}^N$ can be colored using at most N^c colors, in such a way that subdomains with the same color are disjoint.

Next, we shall define a shape regular coarse triangulation Ω_H, which is also aligned with Ω_h. We assume that Ω_h is a refinement of Ω_H. Each element $K_H \in \Omega_H$ has a diameter of $O(H)$. Similarly, we may define the spaces like $H^1(\Omega_H)$ and $H(\mathrm{div}; \Omega_H)$ on the coarse triangulation. Since Ω_h is a refinement of Ω_H, those spaces defined on Ω_H are subspaces of the corresponding spaces defined on Ω_h (for example $H^1(\Omega_H) \subseteq H^1(\Omega_h)$). The coarse space U_H for our domain decomposition preconditioner shall be based on the triangulation Ω_H. For the convenience of indexing, we also denote $\Omega_0 := \Omega_H$.

Let $\{U_i\}_{i=0}^N$ be a family of spaces defined in $\{\Omega_i\}_{i=0}^N$, which are restrictions of U_h on $\{\Omega_i\}_{i=0}^N$. All $(\vec{\sigma}_i, u_i, \hat{u}_i, \hat{\sigma}_{n,i}) \in U_i$ should satisfy

$$\hat{u}_i = 0,\ \hat{\sigma}_{n,i} = 0, \text{ on } \partial\Omega_i \setminus \partial\Omega.$$

Define the inner product in $\{U_i\}_{i=0}^N$ as:

$$a_i(\mathcal{U}_i, \mathcal{U}_i') = a_h(R_{h,i}^T \mathcal{U}_i, R_{h,i}^T \mathcal{U}_i') \quad \forall\ \mathcal{U}_i, \mathcal{U}_i' \in U_i,$$

here we choose $\{R_{h,i}^T : U_i \to U_h\}_{i=1}^N$ as the trivial extension operator. Since $\{\Omega_{i,h}\}_{i=0}^N$ are all aligned with Ω_h, we use exact solvers in all subspaces in our DD algorithm.

To construct our preconditioner, define operator $\widetilde{P}_i : U \to U_i$ by

$$a_i(\widetilde{P}_i \mathcal{U}, \mathcal{U}_i') = a_h(\mathcal{U}, R_{h,i}^T \mathcal{U}_i') \quad \forall\ \mathcal{U} \in U,\ \mathcal{U}_i' \in U_i.$$

Moreover, define $P_i := R_{h,i}^T \widetilde{P}_i$, then $P_i : U \to R_{h,i}^T U_i$ satisfies

$$a_h(P_i \mathcal{U}, R_{h,i}^T \mathcal{U}_i') = a_h(\mathcal{U}, R_{h,i}^T \mathcal{U}_i') \quad \forall\ \mathcal{U} \in U,\ \mathcal{U}_i' \in U_i.$$

Now we may construct an additive Schwarz preconditioner for our model problem (2.5). Actually the preconditioned system may be written in an operator form:

$$\sum_{i=0}^N P_i \mathcal{U} = \sum_{i=0}^N \widetilde{f}_i,$$

where $\widetilde{f}_i$ is defined as:

$$\widetilde{f}_i \in R_{h,i}^T U_i,\ a_h(\widetilde{f}_i, R_{h,i}^T \mathcal{U}_i) = \langle \widetilde{f}, R_{h,i}^T \mathcal{U}_i \rangle \quad \forall\ \mathcal{U}_i \in U_i.$$

Following [30], we may use the abstract theory of Schwarz methods to estimate the condition number of the above preconditioned system. Since we use an exact solver in each subspace, we only need to show that the following two assumptions are satisfied:

(A1) (Stable space decomposition). For all $\mathcal{U}_h \in U_h$, there exists $\mathcal{U}_i \in U_i,\ i = 0, \ldots, N$ such that

$$\mathcal{U}_h = \sum_{i=0}^N R_{h,i}^T \mathcal{U}_i,\ \sum_{i=0}^N a_i(\mathcal{U}_i, \mathcal{U}_i) \lesssim C_0 a_h(\mathcal{U}, \mathcal{U}).$$

(A2) For all $\mathcal{U}_i \in U_i, \mathcal{U}_j \in U_j,\ i, j = 1, \ldots, N$, there exists ϵ_{ij} such that

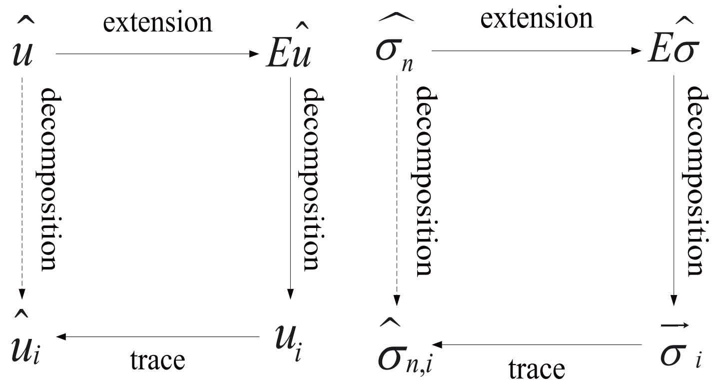
$$a_h(R_{h,i}^T \mathcal{U}_i, R_{h,j}^T \mathcal{U}_j) \le \epsilon_{ij} a_h(R_{h,i}^T \mathcal{U}_i, R_{h,i}^T \mathcal{U}_i)^{\frac{1}{2}} a_h(R_{h,j}^T \mathcal{U}_j, R_{h,j}^T \mathcal{U}_j)^{\frac{1}{2}}.$$

We denote the spectral radius of $\mathcal{E}_p = \{\epsilon_{ij}\}$ by $\rho(\mathcal{E}_p)$.

It is known [30] that given assumptions (A1) and (A2), condition number κ of the preconditioned system satisfies $\kappa(\sum_{i=0}^N P_i) \lesssim C_0(1 + \rho(\mathcal{E}_p))$.

In the following, we first verify assumption (A1). Instead of inner product $a_h(\cdot,\cdot)$, we shall verify the stable space decomposition in the equivalent norm given in Lemma 2.2, *i.e.*, we shall verify that for all $\mathcal{U}_h = (\vec{\sigma}, u, \hat{u}, \hat{\sigma}_n) \in U_h$, there exists $\mathcal{U}_i = (\vec{\sigma}_i, u_i, \hat{u}_i, \hat{\sigma}_{n,i}) \in U_i,\ i = 0, \ldots, N$ such that $\mathcal{U}_h = \sum_{i=0}^N R_{h,i}^T \mathcal{U}_i$ and

$$\sum_{i=0}^N \left(\|\vec{\sigma}_i\|_{L^2(\Omega_i)}^2 + \|u_i\|_{L^2(\Omega_i)}^2 + \|\hat{u}_i\|_{H^{\frac{1}{2}}(\partial\Omega_{i,h})}^2 + \|\hat{\sigma}_{n,i}\|_{H^{-\frac{1}{2}}(\partial\Omega_{i,h})}^2 \right)$$
$$\lesssim C_0 \left(\|\vec{\sigma}\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 + \|\hat{u}\|_{H^{\frac{1}{2}}(\partial\Omega_h)}^2 + \|\hat{\sigma}_n\|_{H^{-\frac{1}{2}}(\partial\Omega_h)}^2 \right).$$



(a) *Decomposition of* $\hat{u}$ (b) *Decomposition of* $\hat{\sigma}_n$

FIGURE 1. Decompositions of the quotient spaces.

In our case, actually $C_0 = N^c(1 + \frac{H^2}{\delta^2})$, however we still use C_0 in this section for simplicity and one may check for details in Section 4.

Note that the equivalent norm only consists of L^2 norms and two quotient norms. Due to the definition of our subspaces, the stable space decomposition for the L^2 spaces is trivial, we only need to construct a stable decomposition for the quotient space $Q_h \subseteq H^{\frac{1}{2}}(\partial\Omega_h) \times H^{-\frac{1}{2}}(\partial\Omega_h)$. To derive such decomposition, we shall extend functions in such quotient space into a discrete subspace $W \subseteq H^1(\Omega) \times H(\mathrm{div};\Omega)$, which is defined as follows: let $\widetilde{P}^{m+1}(\Omega_h)$ be the space of continuous polynomials for triangulation Ω_h with total degree $\leq (m+1)$, and define $RT^{m+1}(\Omega_h)$ by the $(m+1)$th order Raviart–Thomas space for Ω_h (*cf.* [25, 26]). Since $\widetilde{P}^{m+1}(\Omega_h) \subseteq H^1(\Omega)$ and $RT^{m+1}(\Omega_h) \subseteq H(\mathrm{div};\Omega)$, according to the Robin boundary condition, we define space W as

$$W := \Big\{(u,\vec{\sigma}) \in \widetilde{P}^{m+1}(\Omega_h) \times RT^{m+1}(\Omega_h)) : \vec{\sigma}\cdot\vec{n} = u \text{ on } \partial\Omega\Big\}.$$

For simplicity, we denote $\hat{W} := Q_h$. We may similarly define spaces in the subdomains, such as W_0, $\{W_i\}_{i=1}^N$, $\hat{W}_0$ and $\{\hat{W}_i\}_{i=1}^N$ by restricting W and $\hat{W}$ to the corresponding subspaces. Essential boundary conditions should also be satisfied on $\partial\Omega$. Moreover functions in $\hat{W}_i$ and W_i should vanish on $\partial\Omega_i \setminus \partial\Omega$, which allows us to define trivial extension operators $\{R_i^T : W_i \to W\}_{i=0}^N$ and $\{\hat{R}_i^T : \hat{W}_i \to \hat{W}\}_{i=0}^N$. An example of this space setting may be found in Section 4.

Now we may derive the stable decomposition for $\hat{W}$ with the help of two extension operators E and $\vec{E}$. First we extend $(\hat{u}, \hat{\sigma}_n)$ to $(E\hat{u}, \vec{E}\hat{\sigma}_n)$, which belongs to W, then decompose $(E\hat{u}, \vec{E}\hat{\sigma}_n)$ into $\{(u_i, \vec{\sigma}_i)\}_{i=0}^N$, which belongs to $\{W_i\}_{i=0}^N$. The traces of $\{(u_i, \vec{\sigma}_i)\}_{i=0}^N$ may become the stable decomposition component in the quotient spaces $\{\hat{W}_i\}_{i=0}^N$. We have shown the procedures in Figure 1.

Now we introduce the extension operators E and $\vec{E}$. We shall first define these extension operators in an element $K \in \Omega_h$. Actually, for each element K, these extension operators have already been constructed in [2]. As shown in Lemmas 1 and 2 of [2], the extension operators shall satisfy the following two Lemmas.

Lemma 3.2 (Lem. 1 of [2])**.** *There exists an extension operator* $E_K \colon \widetilde{P}^{m+1}(\partial K) \to \widetilde{P}^{m+1}(K)$ *such that for all* $\hat{u} \in \widetilde{P}^{m+1}(\partial K)$,

$$\|\hat{u}\|^2_{H^{\frac{1}{2}}(\partial K)} \approx \|E_K\hat{u}\|^2_{H^1(K)} \approx h\|\hat{u}\|^2_{L^2(\partial K)} + h\sum_{F\in\partial K} |\hat{u}|^2_{H^1(F)},$$

$$\|E_K\hat{u}\|^2_{L^2(K)} \approx h\|\hat{u}\|^2_{L^2(\partial K)},$$

here $F \in \partial K$ *stands for the faces of element* K.

Lemma 3.3 (Lem. 2 of [2]). *There exists an extension operator* $\vec{E}_K$: $P^m(\partial K) \to RT^{m+1}(K)$ *such that for all* $\hat{\sigma}_n \in P^m(\partial K)$,

$$\|\hat{\sigma}_n\|^2_{H^{-\frac{1}{2}}(\partial K)} \approx \|\vec{E}_K\hat{\sigma}_n\|^2_{H(\mathrm{div};K)} \approx h\|\hat{\sigma}_n\|^2_{L^2(\partial K)} + h^{-d}\left(\int_{\partial K}\hat{\sigma}_n \mathrm{d}S\right)^2,$$
$$\|\vec{E}_K\hat{\sigma}_n\|^2_{L^2(K)} \approx h\|\hat{\sigma}_n\|^2_{L^2(\partial K)}.$$

According to the definitions of these operators in [2], the two operators $\vec{E}_K$ and E_K only consists of operators like averaging and L^2 projection, thus they are linear operators. Moreover, in the case of $m = 1$, according to [2], $\vec{E}_K$ extends functions in $P^1(\partial K)$ to a subspace of $RT^2(K)$, which is $\{\vec{\sigma} \in RT^2(K) : \mathrm{div}\vec{\sigma}$ is constant in $K\}$. According to [3], introducing the first order Brezzi−Douglas−Marini (BDM) element space as:

$$BDM^1(\Omega_H) := \left\{\vec{\sigma} \in \prod_{K\in\Omega_H}(P^1(K))^2 : \ \vec{\sigma}\cdot\vec{n} \text{ is continuous accoss } \partial K \ \ \forall \quad K \in \Omega_H\right\},$$

we may see that space $\{\vec{\sigma} \in RT^2(K) : \mathrm{div}\vec{\sigma}$ is constant in $K\}$ is actually the BDM^1 space in K. This means that when $m = 1$, the extension operator $\vec{E}_K$ may be defined from the space $P^1(\partial K)$ to $BDM^1(K)$. Thus we may reduce the coarse space from $RT^2(\Omega_H)$ to $BDM^1(\Omega_H)$, more details may be found in Section 4.

For all $\hat{u} \in \widetilde{P}^{m+1}(\partial\Omega_h)$, $\hat{\sigma}_n \in P^m(\partial\Omega_h)$, we may define $E : \widetilde{P}^{m+1}(\partial\Omega_h) \to \widetilde{P}^{m+1}(\Omega_h)$ and $\vec{E} : P^m(\partial\Omega_h) \to RT^{m+1}(\Omega_h)$ as:

$$(E\hat{u})|_K := E_K(\hat{u}|_{\partial K}), \ (\vec{E}\hat{\sigma}_n)|_K := \vec{E}_K(\hat{\sigma}_n|_{\partial K}) \quad \forall \ \ K \in \Omega_h.$$

Summing up the inequalities in Lemma 3.2, we may derive that

$$\|\hat{u}\|^2_{H^{\frac{1}{2}}(\partial\Omega_h)} \approx \|E\hat{u}\|^2_{H^1(\Omega)} \approx h\sum_{K\in\Omega_h}\|\hat{u}\|^2_{L^2(\partial K)} + h\sum_{F\in F_h}|\hat{u}|^2_{H^1(F)}, \tag{3.2}$$
$$\|E_K\hat{u}\|^2_{L^2(\Omega)} \approx h\sum_{K\in\Omega_h}\|\hat{u}\|^2_{L^2(\partial K)}. \tag{3.3}$$

Similarly we may derive from Lemma 3.3 that

$$\|\hat{\sigma}_n\|^2_{H^{-\frac{1}{2}}(\partial\Omega_h)} \approx \|\vec{E}\hat{\sigma}_n\|^2_{H(\mathrm{div};\Omega)} \approx h\textstyle\sum_{K\in\Omega_h}\|\hat{\sigma}_n\|^2_{L^2(\partial K)} + h^{-d}\textstyle\sum_{K\in\Omega_h}\left(\displaystyle\int_{\partial K}\hat{\sigma}_n\mathrm{d}S\right)^2, \tag{3.4}$$
$$\|\vec{E}\hat{\sigma}_n\|^2_{L^2(\Omega)} \approx h\textstyle\sum_{K\in\Omega_h}\|\hat{\sigma}_n\|^2_{L^2(\partial K)}. \tag{3.5}$$

In the same way, we may also denote the corresponding extension operators for triangulation Ω_H (resp. $\{\Omega_{i,h}\}_{i=1}^N$) by $\vec{E}_H$ (resp. $\{\vec{E}_i\}_{i=1}^N$) and E_H (resp. $\{E_i\}_{i=1}^N$). Since the extension operators do not change the numerical trace $\hat{u}$ and the numerical flux $\hat{\sigma}_n$, the coupling condition on $\partial\Omega$ is trivially kept. Now we are prepared to derive the the stable decomposition in the quotient spaces.

Theorem 3.4. *Assume that* W *has a stable space decomposition* $\{W_i\}_{i=0}^n$, *which means that for all* $(u,\vec{\sigma}) \in W$, *there exist* $(u_i,\vec{\sigma}_i) \in W_i$, $i = 0,\ldots,N$ *such that*

$$\sum_{i=0}^N R_i^T(u_i,\vec{\sigma}_i) = (u,\vec{\sigma}),$$
$$\sum_{i=0}^N\left(\|\vec{\sigma}_i\|^2_{H(\mathrm{div};\Omega_i)} + \|u_i\|^2_{H^1(\Omega_i)}\right) \lesssim C_0\left(\|\vec{\sigma}\|^2_{H(\mathrm{div};\Omega)} + \|u\|^2_{H^1(\Omega)}\right).$$

Then $\hat{W}$ also has a stable space decomposition $\{\hat{W}_i\}_{i=0}^N$, which means that for all $(\hat{u}, \hat{\sigma}_n) \in \hat{W}$, there exist $(\hat{u}_i, \hat{\sigma}_{n,i}) \in \hat{W}_i$ such that

$$\sum_{i=0}^N \hat{R}_i^T(\hat{u}_i, \hat{\sigma}_{n,i}) = (\hat{u}, \hat{\sigma}_n),$$

$$\sum_{i=0}^N \left(\|\hat{\sigma}_{n,i}\|^2_{H^{-\frac{1}{2}}(\partial\Omega_{i,h})} + \|\hat{u}_i\|^2_{H^{\frac{1}{2}}(\partial\Omega_{i,h})} \right) \lesssim C_0 \left(\|\hat{\sigma}_n\|^2_{H^{-\frac{1}{2}}(\partial\Omega_h)} + \|\hat{u}\|^2_{H^{\frac{1}{2}}(\partial\Omega_h)} \right).$$

Proof. For all $(\hat{u}, \hat{\sigma}_n) \in \hat{W}$, we extend $(\hat{u}, \hat{\sigma}_n)$ to $(E\hat{u}, \vec{E}\hat{\sigma}_n) \in W$, by the assumption of this theorem, there exists $(\vec{\sigma}_i, u_i) \in U_i$ such that

$$\sum_{i=0}^N R_i^T(u_i, \vec{\sigma}_i) = (E\hat{u}, \vec{E}\hat{\sigma}_n),$$

$$\sum_{i=0}^N \left(\|\vec{\sigma}_i\|^2_{H(\mathrm{div};\Omega_i)} + \|u_i\|^2_{H^1(\Omega_i)} \right) \lesssim C_0 \left(\|\vec{E}\hat{\sigma}_n\|^2_{H(\mathrm{div};\Omega)} + \|E\hat{u}\|^2_{H^1(\Omega)} \right).$$

Since the trace space of W_i is $\hat{W}_i$, we may define $(\hat{u}_i, \hat{\sigma}_{n,i}) \in \hat{W}_i$ as the trace of $(u_i, \vec{\sigma}_i)$ in W_i, then $(\hat{u}_i, \hat{\sigma}_{n,i})$ satisfies

$$\sum_{i=0}^N \hat{R}_i^T(\hat{u}_i, \hat{\sigma}_{n,i}) = (\hat{u}, \hat{\sigma}_n),$$

and

$$\begin{aligned}\sum_{i=0}^N \|\hat{\sigma}_{n,i}\|^2_{H^{-\frac{1}{2}}(\partial\Omega_h)} + \|\hat{u}_i\|^2_{H^{\frac{1}{2}}(\partial\Omega_h)} &\lesssim \sum_{i=0}^N \left(\|\vec{\sigma}_i\|^2_{H(\mathrm{div};\Omega_i)} + \|u_i\|^2_{H^1(\Omega_i)} \right) \\ &\lesssim C_0 \left(\|\vec{E}\hat{\sigma}_n\|^2_{H(\mathrm{div};\Omega)} + \|E\hat{u}\|^2_{H^1(\Omega)} \right) \\ &\lesssim C_0 \left(\|\hat{\sigma}_n\|^2_{H^{-\frac{1}{2}}(\partial\Omega_h)} + \|\hat{u}\|^2_{H^{\frac{1}{2}}(\partial\Omega_h)} \right),\end{aligned}$$

here the first inequality follows from the definition of the quotient norms and the last inequality holds from (3.2) and (3.4). □

4. Condition number estimate

In this section we shall derive condition number estimate for the above model problem. Theorem 3.4 tells us that Assumption (A1) in the DPG system may be verified if we are able to find a stable space decomposition for $W \subseteq H^1(\Omega) \times H(\mathrm{div};\Omega)$. In this section we shall construct a stable subspace decomposition for the model problem with Robin boundary condition. As shown in the above sections, due to the Robin boundary condition, the space in which we need to derive a stable subspace decomposition is coupled on the boundary. In this paper, we only verify assumptions (A1) and (A2) when $m = 1$, and $d = 2$. For 3D case, some comments shall be given when there are significant differences to the 2D case.

When $m = 1$ and $d = 2$, according to Section 3, the quotient spaces $\hat{W}$, $\hat{W}_i$ and $\hat{W}_0$ are defined as

$$\begin{aligned}\hat{W} &:= \{(\hat{u}, \hat{\sigma}_n) \in \widetilde{P}^2(\partial\Omega_h) \times P^1(\partial\Omega_h) :\ \hat{u} = \hat{\sigma}_n \text{ on } \partial\Omega\},\\ \hat{W}_i &:= \{(\hat{u}_i, \hat{\sigma}_{n,i}) \in \widetilde{P}^2(\partial\Omega_{i,h}) \times P^1(\partial\Omega_{i,h}) : \hat{u}_i = \hat{\sigma}_{n,i} \text{ on } \partial\Omega_i \cap \partial\Omega,\\ &\qquad (\hat{u}_i, \hat{\sigma}_{n,i}) = 0 \text{ on } \partial\Omega_i \setminus \partial\Omega\},\ i = 1, \dots, N,\\ \hat{W}_0 = \hat{W}_H &:= \{(\hat{u}, \hat{\sigma}_n) \in \widetilde{P}^1(\partial\Omega_h) \times P^1(\partial\Omega_h) :\ \exists(\vec{\sigma}, u) \in W_0 \text{ such that}\\ &\qquad \hat{u}|_{\partial K} = u|_{\partial K},\ \vec{\sigma}\cdot\vec{n}|_{\partial K} = \hat{\sigma}_n|_{\partial K}\ \forall \quad K \in \Omega_h\}.\end{aligned}$$

In order to derive the stable decomposition, we shall also introduce the following spaces on Ω and $\{\Omega_i\}_{i=1}^N$.

$$\begin{aligned}W &:= \{(u,\vec{\sigma}) \in \widetilde{P}^2(\Omega_h) \times RT^2(\Omega_h) :\ u = \vec{\sigma}\cdot\vec{n} \text{ on } \partial\Omega\},\\ W_i &:= \{(u_i,\vec{\sigma}_i) \in \widetilde{P}^2(\Omega_{i,h}) \times RT^2(\Omega_{i,h}) :\ u_i = \vec{\sigma}_i\cdot\vec{n} \text{ on } \partial\Omega_i\cap\partial\Omega,\\ &\qquad (u_i,\vec{\sigma}_i) = 0 \text{ on } \partial\Omega_i\setminus\partial\Omega\},\ i=1,\ldots,N,\\ W_0 := W_H &:= \{(u_H,\vec{\sigma}_H) \in \widetilde{P}^1(\Omega_H) \times BDM^1(\Omega_H) :\ u_H = \vec{\sigma}_H\cdot\vec{n} \text{ on } \partial\Omega\}.\end{aligned}$$

Before constructing stable decompositions for spaces $\hat{W}$ and W, we first present Lemma 4.1, which states some well-known results for the stable subspace decompositions for the finite element approximation spaces. One may check for the case of $H^1(\Omega)$ in [17], the case of $H(\mathrm{div};\Omega)$ in [1] and the case of $H(\mathrm{curl};\Omega)$ in [22].

Define $R_{i,H^1}^T : H^1(\Omega_i) \to H^1(\Omega)$, $\vec{R}_{i,\mathrm{curl}}^T : H(\mathrm{curl};\Omega_i) \to H(\mathrm{curl};\Omega)$ and $\vec{R}_{i,\mathrm{div}}^T : H(\mathrm{div};\Omega_i) \to H(\mathrm{div};\Omega)$ as the corresponding trivial extension operators, then we have

Lemma 4.1. *There are stable decompositions for the discrete spaces $\widetilde{P}^m(\Omega_h) \subseteq H^1(\Omega)$, $ND^m(\Omega_h) \subseteq H(\mathrm{curl};\Omega)$ and $RT^m(\Omega_h) \subseteq H(\mathrm{div};\Omega)$, such that*

(1). *For all $u \in \widetilde{P}^m(\Omega_h) \subseteq H^1(\Omega)$, there exist $u_i \in \widetilde{P}^m(\Omega_{i,h})$, $i=1,\ldots,N$, $u_0 \in \widetilde{P}^1(\Omega_H)$ such that u_i vanish on $\partial\Omega_i\setminus\partial\Omega$, and*

$$u = \sum_{i=0}^N R_{i,H^1}^T u_i,\ \sum_{i=0}^N \|u_i\|_{H^1(\Omega_i)}^2 \lesssim N^c\left(1+\frac{H}{\delta}\right)\|u\|_{H^1(\Omega)}^2.$$

(2). *For all $\vec{\sigma} \in ND^m(\Omega_h) \subseteq H(\mathrm{curl};\Omega)$, there exist $\vec{\sigma}_i \in ND^m(\Omega_{i,h})$, $i=1,\ldots,N$, $\vec{\sigma}_0 \in ND^1(\Omega_H)$ such that $\vec{\sigma}_i$ vanish on $\partial\Omega_i\setminus\partial\Omega$, and*

$$\vec{\sigma} = \sum_{i=0}^N \vec{R}_{i,\mathrm{curl}}^T\vec{\sigma}_i,\quad \sum_{i=0}^N \|\vec{\sigma}_i\|_{H(\mathrm{curl};\Omega_i)}^2 \lesssim N^c\left(1+\frac{H^2}{\delta^2}\right)\|\vec{\sigma}\|_{H(\mathrm{curl};\Omega)}^2.$$

(3). *For all $\vec{\sigma} \in RT^m(\Omega_h) \subseteq H(\mathrm{div};\Omega)$, there exist $\vec{\sigma}_i \in RT^m(\Omega_{i,h})$, $i=1,\ldots,N$, $\vec{\sigma}_0 \in RT^1(\Omega_H)$ such that $\vec{\sigma}_i$ vanish on $\partial\Omega_i\setminus\partial\Omega$, and*

$$\vec{\sigma} = \sum_{i=0}^N \vec{R}_{i,\mathrm{div}}^T\vec{\sigma}_i,\quad \sum_{i=0}^N \|\vec{\sigma}_i\|_{H(\mathrm{div};\Omega_i)}^2 \lesssim N^c\left(1+\frac{H^2}{\delta^2}\right)\|\vec{\sigma}\|_{H(\mathrm{div};\Omega)}^2,$$

here $ND^m(\Omega_h)$ is the mth order *Nédélec* element space (*cf.* [24]) defined in Ω_h.

Moreover, we consider the boundary condition and define

$$\begin{aligned}H_0(\mathrm{div};\Omega) &:= \{\vec{\sigma} \in H(\mathrm{div};\Omega) :\ \vec{\sigma}\cdot\vec{n} = 0 \textit{ on } \partial\Omega\},\\ H_0(\mathrm{curl};\Omega) &:= \{\vec{\sigma} \in H(\mathrm{curl};\Omega) :\ \vec{\sigma}\times\vec{n} = 0 \textit{ on } \partial\Omega\},\\ \widetilde{P}_0^m(\Omega_h) &:= \widetilde{P}^m(\Omega_h)\cap H_0^1(\Omega),\\ RT_0^m(\Omega_h) &:= RT^m(\Omega_h)\cap H_0(\mathrm{div};\Omega),\\ ND_0^m(\Omega_h) &:= ND^m(\Omega_h)\cap H_0(\mathrm{curl};\Omega).\end{aligned}$$

According to [1,17,22], same results in Lemma 4.1 are also true for the spaces $\widetilde{P}_0^m(\Omega_h) \subseteq H_0^1(\Omega)$, $ND_0^m(\Omega_h) \subseteq H_0(\mathrm{curl};\Omega)$ and $RT_0^m(\Omega_h) \subseteq H_0(\mathrm{div};\Omega)$.

By Theorem 3.4, we know that our goal is to prove the following decomposition: For all $(u,\vec{\sigma}) \in W$, there exist $(u_i,\vec{\sigma}_i) \in W_i$, $i=0,\ldots,N$, such that

$$(u,\vec{\sigma}) = R_i^T\sum_{i=0}^N (u_i,\vec{\sigma}_i),$$

$$\sum_{i=0}^N\left(\|\vec{\sigma}_i\|_{H(\mathrm{div};\Omega_i)}^2 + \|u_i\|_{H^1(\Omega_i)}^2\right) \lesssim N^c\left(1+\frac{H^2}{\delta^2}\right)\left(\|\vec{\sigma}\|_{H(\mathrm{div};\Omega)}^2 + \|u\|_{H^1(\Omega)}^2\right).$$

First of all, we shall give two Lemmas on continuous and discrete Helmholtz decomposition, which shall play a very important role in the proofs that follow.

Lemma 4.2 (Continuous Helmholtz decomposition)**.** *If $\vec{\sigma} \in H(\mathrm{div};\Omega)$ satisfies $\vec{\sigma}\cdot\vec{n} \in H^{\frac{1}{2}}(\partial\Omega)$, we may decompose $\vec{\sigma}$ into $\vec{\sigma} = \vec{\nabla}p + \vec{\nabla}\times\psi$, where $p \in H^2(\Omega)$, $\psi \in H_0^1(\Omega)$, and*

$$
\begin{aligned}
\|p\|_{H^2(\Omega)} &\lesssim \|\vec{\nabla}\cdot\vec{\sigma}\|_{L^2(\Omega)} + \|\vec{\sigma}\cdot\vec{n}\|_{H^{\frac{1}{2}}(\partial\Omega)},\\
\|\vec{\nabla}\times\psi\|_{L^2(\Omega)} &\lesssim \|\vec{\sigma}\|_{H(\mathrm{div};\Omega)} + \|\vec{\sigma}\cdot\vec{n}\|_{H^{\frac{1}{2}}(\partial\Omega)}.
\end{aligned}
$$

Proof. Let p solve $(\vec{\nabla}p, \vec{\nabla}q) = (\vec{\sigma}, \vec{\nabla}q)$ $\forall q \in H^1(\Omega)$, it is equivalent to say that p is the weak solution of the following problem (*cf.*, *e.g.*, p. 216, Eq. (1.43) in [10])

$$
\begin{cases}
-\Delta p = \vec{\nabla}\cdot\vec{\sigma} & \text{in } \Omega,\\
\dfrac{\partial p}{\partial n} = \vec{\sigma}\cdot\vec{n} & \text{on } \partial\Omega.
\end{cases}
$$

Recall that Ω is convex, by a regularity result of the Poisson equation with a Neumann boundary condition (*cf.*, Cor. 23.5 in [9]), we know that there exists p in the quotient space $H^2(\Omega)\setminus\mathbb{R}$ such that

$$
\|p\|_{H^2(\Omega)} \lesssim \|\vec{\nabla}\cdot\vec{\sigma}\|_{L^2(\Omega)} + \|\vec{\sigma}\cdot\vec{n}\|_{H^{\frac{1}{2}}(\partial\Omega)}.
$$

Since $(\vec{\sigma}-\vec{\nabla}p)\cdot\vec{n} = 0$ on $\partial\Omega$ and $\vec{\nabla}\cdot(\vec{\sigma}-\vec{\nabla}p) = 0$ weakly in Ω, we have that $\vec{\sigma}-\vec{\nabla}p$ is in the kernel of the div operator in $H_0(\mathrm{div};\Omega)$. Lemma A.25 in [30] says that if $d = 2$, then $\{\vec{\sigma} \in H_0(\mathrm{div};\Omega) : \mathrm{div}\vec{\sigma} = 0\} = \vec{\nabla}\times H_0^1(\Omega)$, thus there exists a $\psi \in H_0^1(\Omega)$ such that $\vec{\sigma}-\vec{\nabla}p = \vec{\nabla}\times\psi$. By the triangle inequality, we prove that another inequality in Lemma 4.2 also holds. □

Remark 4.3. For 3D case, we may also get a similar result in Lemma 4.2, the only difference is that $\psi \in (H_0^1(\Omega))^3$. In this case we may use Lemma A.27 in [30], which says that $\{\vec{\sigma} \in H_0(\mathrm{div};\Omega) : \mathrm{div}\vec{\sigma} = 0\} = \vec{\nabla}\times(H_0^1(\Omega))^3$ when $d = 3$.

Let $\Pi_h^{RT^{m+1}}$ be the interpolation operator in $RT^{m+1}(\Omega_h)$ (*cf.* [18,24] for details), we may introduce a discrete Helmholtz decomposition.

Lemma 4.4 (Discrete Helmholtz decomposition)**.** *If $\vec{\sigma} \in RT^{m+1}(\Omega_h)$ satisfies that $\vec{\sigma}\cdot\vec{n}$ is continuous on $\partial\Omega$,* i.e.*, $\vec{\sigma}\cdot\vec{n} \in H^{\frac{1}{2}}(\partial\Omega)$, we may decompose $\vec{\sigma}$ into*

$$
\vec{\sigma} = \Pi_h^{RT^{m+1}}(\vec{\nabla}p) + \vec{\nabla}\times\psi_h,
$$

where

$$
p \in H^2(\Omega),\ \Pi_h^{RT^{m+1}}(\vec{\nabla}p) \in RT^{m+1}(\Omega_h),\ \psi_h \in \widetilde{P}_0^{m+1}(\Omega_h).
$$

Moreover we have

$$
\begin{aligned}
\|\Pi_h^{RT^{m+1}}(\vec{\nabla}p)\|_{L^2(\Omega)} &\lesssim \|\vec{\nabla}\cdot\vec{\sigma}\|_{L^2(\Omega)} + \|\vec{\sigma}\cdot\vec{n}\|_{H^{\frac{1}{2}}(\partial\Omega)},\\
\|\vec{\nabla}\times\psi_h\|_{L^2(\Omega)} &\lesssim \|\vec{\sigma}\|_{H(\mathrm{div};\Omega)} + \|\vec{\sigma}\cdot\vec{n}\|_{H^{\frac{1}{2}}(\partial\Omega)},\\
\|\vec{\nabla}p - \Pi_h^{RT^{m+1}}(\vec{\nabla}p)\|_{L^2(\Omega)} &\lesssim h|\vec{\nabla}p|_{H^1(\Omega)}.
\end{aligned}
$$

Proof. Note that $\vec{\sigma} \in H(\mathrm{div};\Omega)$ and $\vec{\sigma}\cdot\vec{n} \in H^{\frac{1}{2}}(\partial\Omega)$, so that we may use the continuous decomposition $\vec{\sigma} = \vec{\nabla} p + \vec{\nabla}\times\psi$ introduced in Lemma 4.2. Moreover, since $\vec{\nabla} p\cdot\vec{n} = \vec{\sigma}\cdot\vec{n}$ is a mth order polynomial on $\partial\Omega$, by the definition of RT interpolation in [25], we know that the RT interpolation reproduces such $\vec{\nabla} p\cdot\vec{n}$ on the boundary. Meanwhile we have

$$\Pi_h^{RT^{m+1}}(\vec{\nabla} p)\cdot\vec{n} = \vec{\nabla} p\cdot\vec{n} = \vec{\sigma}\cdot\vec{n} \text{ on } \partial\Omega.$$

By a commuting-diagram property (Thm. 2.3 of [21]), we also have

$$\vec{\nabla}\cdot\Pi_h^{RT^{m+1}}(\vec{\nabla} p) = \Pi^h_{\widetilde{P}^m}(\vec{\nabla}\cdot\vec{\nabla} p) = \Pi^h_{\widetilde{P}^m}(\vec{\nabla}\cdot\vec{\sigma}) = \vec{\nabla}\cdot\vec{\sigma},$$

here $\Pi^h_{\widetilde{P}^m}$ is the L^2 projection into $\widetilde{P}^m(\Omega_h)$, thus the last equality holds for all $\vec{\sigma}\in RT^{m+1}(\Omega_h)$ since $\nabla\cdot RT^{m+1}(\Omega_h)$ belongs to $\widetilde{P}^m(\Omega_h)$.

Moreover, since $\mathrm{div}(\vec{\sigma} - \Pi_h^{RT^{m+1}}(\vec{\nabla} p)) = 0$, Theorem 2.36 of [21] tells us that for 2D case, $\{\vec{\sigma}\in RT_0^{m+1}(\Omega_h) : \mathrm{div}\vec{\sigma} = 0\} = \vec{\nabla}\times\widetilde{P}_0^{m+1}(\Omega_h)$, thus there exists a $\psi_h\in\widetilde{P}_0^{m+1}(\Omega_h)$ such that $\vec{\nabla}\times\psi_h := \vec{\sigma} - \Pi_h^{RT^{m+1}}(\vec{\nabla} p)$.

Since $\vec{\nabla}\cdot\vec{\nabla} p = \vec{\nabla}\cdot\vec{\sigma}\in\widetilde{P}^m(\Omega_h)$, by an error estimate of RT interpolation (Lem. 10.11 of [30]), we may get

$$\|\Pi_h^{RT^{m+1}}(\vec{\nabla} p) - \vec{\nabla} p\|_{L^2(\Omega_h)} \lesssim h|\vec{\nabla} p|_{H^1(\Omega_h)},$$

which, combining Lemma 4.2 together, we may obtain

$$\begin{aligned}\|\Pi_h^{RT^{m+1}}(\vec{\nabla} p)\|_{L^2(\Omega_h)} &\lesssim \|\vec{\nabla} p\|_{L^2(\Omega_h)} + \|\Pi_h^{RT^{m+1}}(\vec{\nabla} p) - \vec{\nabla} p\|_{L^2(\Omega_h)}\\ &\lesssim \|\vec{\nabla}\cdot\vec{\sigma}\|_{L^2(\Omega)} + \|\vec{\sigma}\cdot\vec{n}\|_{H^{\frac{1}{2}}(\partial\Omega)}.\end{aligned}$$

□

Remark 4.5. For 3D case, we may also derive a similar result. The only difference is that if $d = 3$, then $\{\vec{\sigma}\in RT_0^{m+1}(\Omega_h) : \mathrm{div}\vec{\sigma} = 0\} = \vec{\nabla}\times ND_0^{m+1}(\Omega_h)$ (Thm. 2.36 of [21]), thus ψ_h belongs to $ND_0^{m+1}(\Omega_h)$.

Define $H(\mathrm{curl}0;\Omega) := \{\vec{\sigma}\in H(\mathrm{curl};\Omega) : \mathrm{curl}\vec{\sigma} = 0\}$, we introduce an imbedding property which shall be used in the proofs that follow.

Lemma 4.6. *If $\vec{\sigma}\in H(\mathrm{div};\Omega)\cap H(\mathrm{curl}0;\Omega)$, $\vec{\sigma}\cdot\vec{n}\in H^{\frac{1}{2}}(\partial\Omega)$, then $\vec{\sigma}\in(H^1(\Omega))^2$, and*

$$\|\vec{\sigma}\|^2_{H^1(\Omega)} \lesssim \|\vec{\nabla}\cdot\vec{\sigma}\|^2_{L^2(\Omega)} + \|\vec{\sigma}\cdot\vec{n}\|^2_{H^{\frac{1}{2}}(\partial\Omega)}.$$

Proof. According to ([10], Cor. 1, p. 212), for all $\vec{\sigma}\in H(\mathrm{div};\Omega)\cap H(\mathrm{curl}0;\Omega)$, we have

$$\|\vec{\sigma}\|^2_{H^1(\Omega)} \lesssim \|\vec{\sigma}\|^2_{L^2(\Omega)} + \|\vec{\nabla}\cdot\vec{\sigma}\|^2_{L^2(\Omega)} + \|\vec{\sigma}\cdot\vec{n}\|^2_{H^{\frac{1}{2}}(\partial\Omega)}. \tag{4.1}$$

Next we shall give an estimation for $\|\vec{\sigma}\|_{L^2(\Omega)}$, according to Lemma 10.10 of [30], for all $\vec{\sigma}\in H_0(\mathrm{div};\Omega)\cap H(\mathrm{curl}0;\Omega)$,

$$\|\vec{\sigma}\|_{L^2(\Omega)} \lesssim \|\vec{\nabla}\cdot\vec{\sigma}\|^2_{L^2(\Omega)}.$$

Similar to the proof of ([10], Cor. 1, p. 12), by constructing a stable trace lifting from $H^{\frac{1}{2}}(\partial\Omega)$ to $H(\mathrm{div};\Omega)$, we may extend this conclusion from $H_0(\mathrm{div};\Omega)\cap H(\mathrm{curl}0;\Omega)$ to $H(\mathrm{div};\Omega)\cap H(\mathrm{curl}0;\Omega)$ such that

$$\|\vec{\sigma}\|_{L^2(\Omega)} \lesssim \|\vec{\nabla}\cdot\vec{\sigma}\|^2_{L^2(\Omega)} + \|\vec{\sigma}\cdot\vec{n}\|^2_{H^{\frac{1}{2}}(\partial\Omega)}. \tag{4.2}$$

Then the result of the lemma may be obtained by combining (4.1) and (4.2). □

The next Lemma gives a stable decomposition for one of the components in the discrete Helmholtz decomposition, *i.e.*, $\vec{\nabla}\times\psi_h$.

Lemma 4.7. *For all $\psi_h \in \widetilde{P}^2_0(\Omega_h)$, there exists $\psi_i \in \widetilde{P}^2_0(\Omega_{i,h})$, $i = 1, \ldots, N$, $\psi_0 \in \widetilde{P}^1_0(\Omega_H)$ such that $\psi_h = \sum_{i=0}^N \psi_i$, moreover*

$$\|\vec{\nabla} \times \psi_0\|_{L^2(\Omega)} \lesssim \|\vec{\nabla} \times \psi_h\|_{L^2(\Omega)}, \tag{4.3}$$

$$\sum_{i=0}^N \|\vec{\nabla} \times \psi_i\|^2_{H(\mathrm{div};\Omega_i)} = \sum_{i=0}^N \|\vec{\nabla} \times \psi_i\|^2_{L^2(\Omega_i)} \lesssim N^c \left(1 + \frac{H}{\delta}\right) \|\vec{\nabla} \times \psi_h\|^2_{L^2(\Omega_i)} \tag{4.4}$$

Proof. Since we only consider the 2D case, $\vec{\nabla} \times u = \gamma(\vec{\nabla} u)$, where $\gamma(a, b) = (b, -a)$. This decomposition may be derived by constructing a stable decomposition in $H^1_0(\Omega)$. □

Remark 4.8. For 3D case, a stable decomposition for $\psi_h \in ND^2_0(\Omega)$ may be found in [23]. Term $N^c(1 + \frac{H}{\delta})$ in (4.4) should be replaced by $N^c(1 + \frac{H^2}{\delta^2})$.

4.1. Component in coarse space

In this section, we shall construct a $(u_0, \vec{\sigma}_0) \in W_0$, which is a part of the stable decomposition $(u, \vec{\sigma}) = \sum_{i=0}^N R_i^T(u_i, \vec{\sigma}_i)$. To do this, we first introduce the Scott–Zhang interpolation operator, which is given in [29], then give some extension operators. All these operators help us construct $\vec{\sigma}_0$ and u_0.

According to [29], the Scott–Zhang interpolation operator Π_h^{SZ} is defined by

$$\Pi_h^{SZ} u := \sum_{x_i} (\Pi_i u)\phi_i, \tag{4.5}$$

where the sum is taken over all x_i, each x_i stands for a vertex of $K \in \Omega_h$. The function ϕ_i denotes the (linear) basis function nodal at x_i and $\Pi_i u$ is defined by $\Pi_i u := \int_{F_{ij}} u\theta_i \mathrm{d}S$, here F_{ij} is an edge with vertices x_i and x_j, which is not uniquely chosen, but if $x_i \in \partial\Omega$, we choose $F_{ij} \subseteq \partial\Omega$. Function θ_i satisfies $\int_{F_{ij}} \theta_i \phi_j \mathrm{d}S = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Lemma 4.9 is a well-known property of Scott–Zhang interpolation (*cf.* [29] for details).

Lemma 4.9. *Let $\Pi_h^{SZ} : H^1(\Omega) \to \widetilde{P}^1(\Omega_h)$ denote the Scott–Zhang operator defined in (4.5), then for each $u \in H^1(\Omega)$, we have*

$$\|u - \Pi_h^{SZ} u\|_{L^2(\Omega)} \lesssim h|u|_{H^1(\Omega)}, \ \|\Pi_h^{SZ} u\|_{H^1(\Omega)} \lesssim \|u\|_{H^1(\Omega)},$$

moreover, for all $u_h \in \widetilde{P}^1(\Omega_h)$, we have $\Pi_h^{SZ} u_h = u_h$.

For a vector space $(H^1(\Omega_h))^2$, we apply the Scott–Zhang interpolation in each dimension respectively, denote by $\vec{\Pi}_h^{SZ}$ the corresponding interpolation operator. Denote the corresponding interpolation operators in $H^1(\Omega_H)$ and $(H^1(\Omega_H))^2$ by Π_H^{SZ} and $\vec{\Pi}_H^{SZ}$.

Define $P^1(\partial\Omega \cap \partial\Omega_H) := \prod_{F \in \partial\Omega\cap\partial\Omega_H} P^1(F)$, where F stands for the element edges in Ω_H. The next lemma introduces a trace extension operator $\vec{\mathcal{E}}_H : P^1(\partial\Omega \cap \partial\Omega_H) \to BDM^1(\Omega_H)$, which ensures the coupling condition on $\partial\Omega$ in our coarse subspace.

Lemma 4.10. *For all $\hat{\sigma}_{n,H} \in P^1(\partial\Omega \cap \partial\Omega_H)$, there exists an extension operator $\vec{\mathcal{E}}_H : P^1(\partial\Omega \cap \partial\Omega_H) \to BDM^1(\Omega_H)$ such that*

$$\|\vec{\mathcal{E}}_H \hat{\sigma}_{n,H}\|_{H(\mathrm{div};\Omega)} \lesssim H^{-1} \|\hat{\sigma}_{n,H}\|_{L^2(\partial\Omega)}, \tag{4.6}$$

$$\|\vec{\mathcal{E}}_H \hat{\sigma}_{n,H}\|_{L^2(\Omega)} \lesssim H \|\hat{\sigma}_{n,H}\|_{L^2(\partial\Omega)}. \tag{4.7}$$

Proof. Define a trivial extension operator $R_H^{vec}: P^1(\partial\Omega\cap\partial\Omega_H)\to P^1(\partial\Omega_H)$ such that

$$R_H^{vec}\hat\sigma_n=\hat\sigma_n \text{ on } \partial\Omega,\ \text{else } R_H^{vec}\hat\sigma_n=0,$$

then the extension operator $\vec{\mathcal{E}}_H$ may be defined as $\vec{\mathcal{E}}_H:=R_H^{vec}\vec E_H$, where $\vec E_H$ is the extension operator introduced in Section 3. In each element $K\in\Omega_H$, according to Lemma 3.3, we have

$$\begin{aligned}\|\vec E_H R_H^{vec}\hat\sigma_{n,H}\|^2_{H(\mathrm{div};K)} &\lesssim H\|R_H^{vec}\hat\sigma_{n,H}\|^2_{L^2(\partial K_H)}+H^{-2}\left(\int_{\partial K_H}R_H^{vec}\hat\sigma_{n,H}\mathrm{d}S\right)^2\\ &\lesssim H^{-1}\|R_H^{vec}\hat\sigma_{n,H}\|^2_{L^2(\partial K_H)} &&\text{(inverse inequality)}\\ &\lesssim H^{-1}\|\hat\sigma_{n,H}\|^2_{L^2(\partial K_H\cap\partial\Omega)}, &&\text{(trivial extension)}\end{aligned}$$

and

$$\|\vec E_H R_H\hat\sigma_{n,H}\|^2_{L^2(K_H)}\lesssim H\|R_H\hat\sigma_{n,H}\|^2_{L^2(\partial K_H)}\lesssim H\|\hat\sigma_{n,H}\|^2_{L^2(\partial K_H\cap\partial\Omega)}.$$

Note that $\|\hat\sigma_{n,H}\|_{L^2(\partial K_H\cap\partial\Omega)}$ vanishes when K has no edges on $\partial\Omega$, we may complete the proof by summing up the above two inequalities for all $K\in\Omega_H$. $\square$

Next we may construct the coarse space components for $(u,\vec\sigma)\in\widetilde P^2(\Omega_h)\times RT^2(\Omega_h)$. By Lemma 4.4, we know

$$\vec\sigma=\Pi_h^{RT^2}(\vec\nabla p)+\vec\nabla\times\psi_h.$$

Define the coarse space components $(u_0,\vec\sigma_0)$ as

$$\begin{cases}\vec\sigma_0:=\vec\Pi_H^{SZ}(\vec\nabla p)+\vec\nabla\times\psi_0+\vec{\mathcal{E}}_H(\Pi_H^{SZ}u-\vec\Pi_H^{SZ}(\vec\nabla p)\cdot\vec n),\\ u_0:=\Pi_H^{SZ}u.\end{cases}\tag{4.8}$$

Lemma 4.2 tells us that $p\in H^2(\Omega)$, thus $\vec\nabla p\in(H^1(\Omega))^2$ and the Scott–Zhang interpolation of $\vec\nabla p$ is well-defined. The term $\vec{\mathcal{E}}_H(\Pi_H^{SZ}u-\vec\Pi_H^{SZ}(\vec\nabla p)\cdot\vec n)$ is a fixing term which ensures $u_0=\vec\sigma_0\cdot\vec n$ on $\partial\Omega$. For the above definitions, we may prove Lemma 4.11.

Lemma 4.11. *For the coarse space components $(u_0,\vec\sigma_0)\in W$, it holds that*

$$\|\vec\sigma_0\|^2_{H(\mathrm{div};\Omega)}\lesssim\|\vec\sigma\|^2_{H(\mathrm{div};\Omega)}+\|u\|^2_{H^1(\Omega)},\tag{4.9}$$

$$\|u_0\|^2_{H^1(\Omega)}\lesssim\|u\|^2_{H^1(\Omega)},\tag{4.10}$$

and

$$\|\vec{\mathcal{E}}_H(\Pi_H^{SZ}u-\vec\Pi_H^{SZ}(\vec\nabla p)\cdot\vec n)\|^2_{L^2(\Omega)}\lesssim H^2(\|\vec\sigma\|^2_{H(\mathrm{div};\Omega)}+\|u\|^2_{H^1(\Omega)}),\tag{4.11}$$

$$\|\Pi^{RT^2}(\vec\nabla p)-\vec\Pi_H^{SZ}(\vec\nabla p)\|^2_{L^2(\Omega)}\lesssim H^2(\|\vec\sigma\|^2_{H(\mathrm{div};\Omega)}+\|u\|^2_{H^1(\Omega)}),\tag{4.12}$$

$$\|u-u_0\|^2\lesssim H^2|u|^2_{H^1(\Omega)}.\tag{4.13}$$

Proof. Combining Lemmas 4.9 and 4.2, we may derive

$$\|\vec\nabla\cdot\vec\Pi_H^{SZ}(\vec\nabla p)\|^2_{H(\mathrm{div};\Omega)}\lesssim\|\vec\nabla p\|^2_{H^1(\Omega)}\lesssim\|\vec\sigma\|^2_{H(\mathrm{div};\Omega)}+\|\vec\sigma\cdot\vec n\|^2_{H^{\frac12}(\partial\Omega)},$$

and

$$\|u_0\|^2_{H^1(\Omega)}\|=\|\Pi_H^{SZ}u\|^2_{H^1(\Omega)}\lesssim\|u\|^2_{H^1(\Omega)}.$$

Moreover, by (4.3), we have

$$\|\vec{\nabla} \times \psi_0\|^2_{H(\mathrm{div};\Omega)} = \|\vec{\nabla} \times \psi_0\|^2_{L^2(\Omega)} \lesssim \|\vec{\nabla} \times \psi\|^2_{L^2(\Omega)} \lesssim \|\vec{\sigma}\|^2_{H(\mathrm{div};\Omega)} + \|\vec{\sigma} \cdot \vec{n}\|^2_{H^{\frac{1}{2}}(\partial\Omega)}.$$

For the "fixing term" $\vec{\mathcal{E}}_H(\Pi_H^{SZ} u - \vec{\Pi}_H^{SZ}(\vec{\nabla} p) \cdot \vec{n})$, with the help of Lemma 4.10, we may obtain

$$\|\vec{E}_H R_H^{vec}((\Pi_H^{SZ} u - \vec{\Pi}_H^{SZ}(\vec{\nabla} p) \cdot \vec{n}))\|^2_{H(\mathrm{div};\Omega)} \lesssim H^{-1} \|\Pi_H^{SZ} u - \vec{\Pi}_H^{SZ}(\vec{\nabla} p) \cdot \vec{n}\|^2_{L^2(\partial\Omega)}.$$

Meanwhile, by the coupling condition $\vec{\sigma} \cdot \vec{n} = u$ on $\partial\Omega$, we have

$$\begin{aligned} H^{-1}\|\Pi_H^{SZ} u - \vec{\Pi}_H^{SZ}(\vec{\nabla} p) \cdot \vec{n}\|^2_{L^2(\partial\Omega)} &\lesssim H^{-1}\|\Pi_H^{SZ} u - u\|^2_{L^2(\partial\Omega)} + H^{-1}\|\vec{\nabla} p \cdot \vec{n} - \vec{\Pi}_H^{SZ}(\vec{\nabla} p) \cdot \vec{n}\|^2_{L^2(\partial\Omega)} \\ &\lesssim H^{-1}\|\Pi_H^{SZ} u - u\|^2_{L^2(\partial\Omega)} + H^{-1}\|\vec{\nabla} p - \vec{\Pi}_H^{SZ}(\vec{\nabla} p)\|^2_{(L^2(\partial\Omega))^2}. \end{aligned}$$

Using trace inequality $\|u\|^2_{L^2(\partial\Omega)} \lesssim \|u\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)}$, Lemma 4.2 and Scott–Zhang error estimate, we have

$$H^{-1}\|\Pi_H^{SZ} u - u\|^2_{L^2(\partial\Omega)} \lesssim H^{-1}\|u - \Pi_H^{SZ}(u)\|_{L^2(\Omega)} \|u - \Pi_H^{SZ}(u)\|_{H^1(\Omega)} \lesssim \|u\|^2_{H^1(\Omega)},$$

and

$$\begin{aligned} H^{-1}\|\vec{\nabla} p - \vec{\Pi}_H^{SZ}(\vec{\nabla} p)\|^2_{(L^2(\partial\Omega))^2} &\lesssim H^{-1}\|\vec{\nabla} p - \vec{\Pi}_H^{SZ}(\vec{\nabla} p)\|_{(L^2(\Omega))^2} \|\vec{\nabla} p - \vec{\Pi}_H^{SZ}(\vec{\nabla} p)\|_{(H^1(\Omega))^2} \\ &\lesssim \|\vec{\nabla} p\|^2_{H^1(\Omega)} \lesssim \|\vec{\nabla} \cdot \vec{\sigma}\|^2_{L^2(\Omega)} + \|\vec{\sigma} \cdot \vec{n}\|^2_{H^{\frac{1}{2}}(\partial\Omega)}. \end{aligned}$$

By the Robin boundary condition and the trace theorem, we get

$$\|\vec{\sigma} \cdot \vec{n}\|^2_{H^{\frac{1}{2}}(\partial\Omega)} = \|u\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \lesssim \|u\|^2_{H^1(\Omega)}.$$

Combining the above inequalities, we may derive the stability estimates of $(u_0, \vec{\sigma}_0)$, which are (4.9) and (4.10).

Using Lemma 3.3, we may also prove that

$$\begin{aligned} \|\vec{E}_H R_H((\Pi_H^{SZ} u - \vec{\Pi}_H^{SZ}(\vec{\nabla} p) \cdot \vec{n}))\|^2_{L^2(K_H)} &\lesssim H \|R_H(\Pi_H^{SZ} u - \vec{\Pi}_H^{SZ}(\vec{\nabla} p) \cdot \vec{n})\|^2_{L^2(\partial K_H)} \\ &\lesssim H \|\Pi_H^{SZ} u - \vec{\Pi}_H^{SZ}(\vec{\nabla} p) \cdot \vec{n}\|^2_{L^2(\partial K_H \cap \partial\Omega)} \\ &\lesssim H^2 (\|\vec{\sigma}\|^2_{H(\mathrm{div};\Omega)} + \|u\|^2_{H^1(\Omega)}), \end{aligned}$$

which is (4.11).

Using Lemmas 4.9 and 4.2, together with the triangle inequality, we may prove that

$$\|\vec{\nabla} p - \vec{\Pi}_H^{SZ}(\vec{\nabla} p)\|^2_{L^2(\Omega)} \lesssim H^2 |\vec{\nabla} p|^2_{H^1(\Omega)} \lesssim H^2 (\|\vec{\sigma}\|^2_{H(\mathrm{div};\Omega)} + \|u\|^2_{H^1(\Omega)}),$$

and

$$\begin{aligned} \|\Pi_h^{RT^2}(\vec{\nabla} p) - \vec{\Pi}_H^{SZ}(\vec{\nabla} p)\|^2_{L^2(\Omega)} &\lesssim \|\Pi_h^{RT^2}(\vec{\nabla} p) - \vec{\nabla} p\|^2_{L^2(\Omega)} + \|\vec{\nabla} p - \vec{\Pi}_H^{SZ}(\vec{\nabla} p)\|^2_{L^2(\Omega)} \\ &\lesssim H^2 (\|\vec{\sigma}\|^2_{H(\mathrm{div};\Omega)} + \|u\|^2_{H^1(\Omega)}), \end{aligned}$$

which is (4.12). Equation (4.13) is a direct consequence of Lemma 4.9.

Finally by the definition of our coarse triangulation components, we have $u_0 = \vec{\sigma}_0 \cdot \vec{n} = \Pi_H^{SZ} u$ on $\partial\Omega$, moreover since Ω_h is a refinement of Ω_H, we have $(u_0, \vec{\sigma}_0) \in W$. $\square$

4.2. Components in fine subspaces

In this section, we shall construct $(u_i, \vec{\sigma}_i) \in W_i$, $i = 1, \dots, N$, which are the parts of the stable decomposition $(u, \vec{\sigma}) = \sum_{i=0}^{N} R_i^T(u_i, \vec{\sigma}_i)$. To help us obtain these components, first we shall define two extension operators, then we shall introduce two interpolation operators.

Let F denote element edges of $\Omega_{i,h}$, define

$$P^2(\partial\Omega \cap \partial\Omega_{i,h}) := \prod_{F \in \Omega_{i,h} \cap \partial\Omega} P^2(F),$$
$$\widetilde{P}^2(\partial\Omega \cap \partial\Omega_{i,h}) := \{\hat{u} \in P^2(\partial\Omega \cap \partial\Omega_{i,h}) : \hat{u} \text{ is continuous along } \partial\Omega\}.$$

In the following we shall introduce two extension operators on the fine triangulation and give their properties.

Lemma 4.12. *For all $\hat{\sigma}_{n,i} \in P^2(\partial\Omega \cap \partial\Omega_{i,h})$, there exists an extension operator $\vec{\mathcal{E}}_{i,h} : P^2(\partial\Omega \cap \partial\Omega_{i,h}) \to RT^3(\Omega_{i,h})$ such that*

$$\|\vec{\mathcal{E}}_{i,h}\hat{\sigma}_{n,i}\|_{H(\mathrm{div};\Omega_i)} \lesssim h^{-1}\|\hat{\sigma}_{n,i}\|_{L^2(\partial\Omega_i \cap \partial\Omega)}, \tag{4.14}$$
$$\|\vec{\mathcal{E}}_{i,h}\hat{\sigma}_{n,i}\|_{L^2(\Omega_i)} \lesssim h\|\hat{\sigma}_{n,i}\|_{L^2(\partial\Omega_i \cap \partial\Omega)}. \tag{4.15}$$

Proof. Similar to the proof of Lemma 4.10, define a trivial extension operator $R_{i,h}^{vec} : P^2(\partial\Omega \cap \partial\Omega_{i,h}) \to P^2(\partial\Omega_{i,h})$ such that

$$R_{i,h}^{vec}\hat{\sigma}_n = \hat{\sigma}_n \text{ on } \partial\Omega \cap \Omega_{i,h}, \text{ else } R_{i,h}^{vec}\hat{\sigma}_n = 0.$$

Then the extension operator $\vec{\mathcal{E}}_{i,h}$ may be defined as $\vec{\mathcal{E}}_{i,h} := \vec{E}_i R_{i,h}^{vec}$, where $\vec{E}_i$ is the extension operator in $\Omega_{i,h}$ introduced in Section 3. Properties of $\vec{\mathcal{E}}_{i,h}$ may be derived by the same procedure in the proof of Lemma 4.10. □

Lemma 4.13. *For all $\hat{u}_i \in \widetilde{P}^2(\partial\Omega \cap \partial\Omega_{i,h})$, there exists an extension operator $\mathcal{E}_{i,h} : \widetilde{P}^2(\partial\Omega \cap \partial\Omega_{i,h}) \to \widetilde{P}^2(\Omega_{i,h}))$ such that*

$$\|\mathcal{E}_{i,h}\hat{u}_i\|^2_{H(\mathrm{div};\Omega_i)} \lesssim h^{-1}\|\hat{u}_i\|^2_{L^2(\partial\Omega_i \cap \partial\Omega)}, \tag{4.16}$$
$$\|\mathcal{E}_{i,h}\hat{u}_i\|^2_{L^2(\Omega_i)} \lesssim h\|\hat{u}_i\|^2_{L^2(\partial\Omega_i \cap \partial\Omega)}. \tag{4.17}$$

Proof. Similarly we shall first define an extension operator $R_{i,h}^{sca} : \widetilde{P}^2(\partial\Omega \cap \partial\Omega_{i,h}) \to \widetilde{P}^2(\partial\Omega_{i,h})$. In each triangle K, let $N(K)$ be the set of nodal points of the P^2 Lagrange finite element associated with K (actually $N(K)$ only contains the vertices of K and the midpoints of the edges of K). Since each function in $\widetilde{P}^2(\partial K)$ may be determined by the nodal values on $N(K)$, define $N(\partial\Omega_{i,h}) := \bigcup_{K \in \Omega_{i,h}} N(K)$, we may define $R_{i,h}^{sca}\hat{u}$ by:

$$\forall x \in N(\partial\Omega_{i,h}), \quad (R_{i,h}^{sca}\hat{u})(x) = \begin{cases} \hat{u}(x) & x \in \partial\Omega_i \cap \partial\Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{E}_{i,h}$ may be defined as $\mathcal{E}_{i,h} := E_i R_{i,h}^{sca}$, here $E_i : \widetilde{P}^2(\partial\Omega_{i,h}) \to \widetilde{P}^2(\Omega_{i,h})$ is the extension operator introduced in Lemma 3.2. In each $K \in \Omega_{i,h}$, due to the inverse inequality and norm equivalence in the finite dimension spaces, it holds that

$$\begin{aligned}\|\mathcal{E}_{i,h}\hat{u}\|_{H^1(K)}^2 &\lesssim h\|R_{i,h}^{sca}\hat{u}\|_{L^2(\partial K)}^2 + h\sum_{F\in\partial K}|R_i^{sca}\hat{u}|_{H^1(F)}^2\\ &\lesssim \|R_i^{sca}\hat{u}\|_{L^2(\partial K)}^2 && \text{(inverse inequality)}\\ &\lesssim h^{-1}\sum_{x\in N(K)} h(R_i^{sca}\hat{u})(x)^2 && \text{(norm equivalence)}\\ &= h^{-1}\sum_{x\in N(K)\cap\partial\Omega} h\hat{u}(x)^2.\end{aligned}$$

Summing up the above inequality over all $K\in\Omega_{i,h}$, we have

$$\begin{aligned}\|\mathcal{E}_{i,h}\hat{u}\|_{H^1(\Omega)}^2 &\lesssim h^{-1}\sum_{x\in N(\partial\Omega_{i,h})} h\hat{u}(x)^2\\ &\lesssim h^{-1}\|\hat{u}_i\|_{L^2(\partial\Omega_i\cap\partial\Omega)}^2. && \text{(norm equivalence)}\end{aligned}$$

Similarly, we may get (4.17). □

Next we introduce two interpolation operators which keep the space coupling condition.

Lemma 4.14. *For all $(u,\vec{\sigma})\in\widetilde{P}^3(\Omega_{i,h})\times RT^3(\Omega_{i,h})$ which satisfies $\vec{\sigma}\cdot\vec{n}=u$ on $\partial\Omega_i\cap\partial\Omega$, there exist interpolation operators Π_i^h, $\vec{\Pi}_i^h$ such that $(\Pi_i^h u,\vec{\Pi}_i^h\vec{\sigma})\in W_i$, moreover*

$$\|\vec{\Pi}_h^i\vec{\sigma}\|_{H(\mathrm{div};\Omega_i)}^2 \lesssim \|\vec{\sigma}\|_{H(\mathrm{div};\Omega_i)}^2 + \|u\|_{H^1(\Omega_i)}^2, \tag{4.18}$$

$$\|\Pi_h^i u\|_{H^1(\Omega_i)}^2 \lesssim \|u\|_{H^1(\Omega_i)}^2. \tag{4.19}$$

Proof. Define $\Pi_h^{\widetilde{P}^m}$, $m=1,2$ as the nodal interpolation into $\widetilde{P}^m(\Omega_h)$. When applying $\Pi_m^{\widetilde{P}^2}$ to a function u that belongs to a proper finite dimension space (*e.g.* $u\in\widetilde{P}^3(\Omega_{i,h})$), by an interpolation error estimate and the inverse inequality, we have

$$\|u-\Pi_h^{\widetilde{P}^m}(u)\|_{L^2(\Omega_i)}^2 \lesssim h^4\sum_{K\in\Omega_{i,h}}\|u\|_{H^2(K)}^2 \lesssim h^2\|u\|_{H^1(\Omega_i)}^2.$$

Define the interpolation operators as

$$\begin{cases}\Pi_h^i u := \Pi_h^{\widetilde{P}^2}(u-\mathcal{E}_{i,h}(u-\Pi_h^{\widetilde{P}^1}u)),\\ \vec{\Pi}_h^i\vec{\sigma} := \Pi_h^{RT^2}(\vec{\sigma}-\vec{\mathcal{E}}_{i,h}(\vec{\sigma}\cdot\vec{n}-\Pi_h^{\widetilde{P}^1}(\vec{\sigma}\cdot\vec{n})).\end{cases}$$

Note that although $\vec{\sigma}\cdot\vec{n}$ is not continuous on $\partial\Omega_{h,i}$, we only define $\Pi_h^{\widetilde{P}^1}(\vec{\sigma}\cdot\vec{n})$ on $\partial\Omega_i\cap\partial\Omega_h$. By the Robin boundary condition, $\vec{\sigma}\cdot\vec{n}=u$ on $\partial\Omega_i\cap\partial\Omega_h$, we may find that the interpolation operator $\vec{\Pi}_h^i$ is well-defined. By the stability of $\Pi_h^{RT^2}$ interpolation, we have

$$\begin{aligned}\|\vec{\Pi}_h^i\vec{\sigma}\|_{H(\mathrm{div};\Omega_i)}^2 &= \|\Pi_h^{RT^2}(\vec{\sigma}-\vec{\mathcal{E}}_{i,h}(\vec{\sigma}\cdot\vec{n}-\Pi_h^{\widetilde{P}^1}(\vec{\sigma}\cdot\vec{n}))\|_{H(\mathrm{div};\Omega_i)}^2\\ &\lesssim \|\vec{\sigma}-\vec{\mathcal{E}}_{i,h}(\vec{\sigma}\cdot\vec{n}-\Pi_h^{\widetilde{P}^1}(\vec{\sigma}\cdot\vec{n}))\|_{H(\mathrm{div};\Omega_i)}^2\\ &\lesssim \|\vec{\sigma}\|_{H(\mathrm{div};\Omega_i)}^2 + \|\vec{\mathcal{E}}_{i,h}(\vec{\sigma}\cdot\vec{n}-\Pi_h^{\widetilde{P}^1}(\vec{\sigma}\cdot\vec{n}))\|_{H(\mathrm{div};\Omega_i)}^2.\end{aligned} \tag{4.20}$$

Since $\vec{\sigma}\cdot\vec{n} = u$ on $\partial\Omega_i \cap \partial\Omega$, according to (4.14) and the trace inequality, we may obtain

$$\begin{aligned}
\|\vec{\mathcal{E}}_{i,h}(\vec{\sigma}\cdot\vec{n} - \Pi_h^{\widetilde{P}^1}(\vec{\sigma}\cdot\vec{n}))\|_{H(\mathrm{div};\Omega_i)}^2 &\lesssim h^{-1}\|\vec{\sigma}\cdot\vec{n} - \Pi_h^{\widetilde{P}^1}(\vec{\sigma}\cdot\vec{n})\|_{L^2(\partial\Omega_i\cap\partial\Omega_i)}^2 \\
&= h^{-1}\|u - \Pi_h^{\widetilde{P}^1}(u)\|_{L^2(\partial K\cap\partial\Omega)}^2 \\
&\lesssim h^{-1}\|u - \Pi_h^{\widetilde{P}^1}(u)\|_{L^2(\Omega_i)}\|u - \Pi_h^{\widetilde{P}^1}(u)\|_{H^1(\Omega_i)} \\
&\lesssim \|u\|_{H^1(\Omega_i)}^2.
\end{aligned}$$

With the help of (4.16), stability (4.19) may be derived similarly. Moreover, by definition we may find that $\Pi_h^i u = (\vec{\Pi}_h^i\vec{\sigma})\cdot\vec{n} = \Pi_h^{\widetilde{P}^1} u$ on $\partial\Omega_i\cap\partial\Omega$, thus $(\Pi_h^i u, \vec{\Pi}_h^i\vec{\sigma}) \in W_i$ $\square$

Now we may define the components in the fine subspaces, using (4.8), (3.1) and (4.4), we may define the fine subspace components $(u_i, \vec{\sigma}_i)$ as

$$\begin{cases} \vec{\sigma}_i := \vec{\Pi}_i^h(\theta_i(\Pi_h^{RT^2}(\vec{\nabla}p) - Gp_H)) + \vec{\nabla}\times\psi_i, \\ u_i := \Pi_i^h(\theta_i(u-u_0)), \end{cases} \tag{4.21}$$

here $Gp_H := \vec{\Pi}_H^{SZ}(\vec{\nabla}p) + \vec{\mathcal{E}}_H(\Pi_H^{SZ}u - \vec{\Pi}_H^{SZ}(\vec{\nabla}p)\cdot\vec{n})$ and $\{\theta_i\}_{i=1}^n$ is the modified partition of unit introduced in Section 3. The following lemma gives the stability estimates of the fine subspace components.

Lemma 4.15. *For $(u_i, \vec{\sigma}_i)$, it holds that $(u_i, \vec{\sigma}_i) \in W_i$, moreover*

$$\|\vec{\sigma}_i\|_{H(\mathrm{div};\Omega_i)}^2 \lesssim \left(1 + \frac{H^2}{\delta^2}\right)(\|\vec{\sigma}\|_{H(\mathrm{div};\Omega_i)}^2 + \|u\|_{H^1(\Omega_i)}^2) \tag{4.22}$$

$$\|u_i\|_{H^1(\Omega_i)}^2 \lesssim \left(1 + \frac{H^2}{\delta^2}\right)\|u\|_{H^1(\Omega_i)}^2 \tag{4.23}$$

Proof. Since $\vec{\nabla}\times\psi_i \in H_0(\mathrm{div};\Omega_i)$, by Lemma 4.14, we have $(u_i, \vec{\sigma}_i) \in W_i$. By the triangle inequality we know

$$\|\vec{\sigma}_i\|_{H(\mathrm{div};\Omega_i)}^2 \lesssim \|\vec{\Pi}_i^h(\theta_i(\Pi_h^{RT^2}(\vec{\nabla}p) - Gp_H))\|_{H(\mathrm{div};\Omega_i)}^2 + \|\vec{\nabla}\times\psi_i\|_{H(\mathrm{div};\Omega_i)}^2.$$

By inequality (3.1) (4.18), Lemmas 4.7 and 4.11, we may derive

$$\begin{aligned}
\|\vec{\Pi}_i^h(\theta_i(\Pi_h^{RT^2}(\vec{\nabla}p) - Gp_H))\|_{H(\mathrm{div};\Omega_i)}^2 &\lesssim \|\theta_i(\Pi_h^{RT^2}(\vec{\nabla}p) - Gp_H)\|_{H(\mathrm{div};\Omega_i)}^2 && \text{(inequality (4.18))} \\
&\lesssim \|\vec{\nabla}\theta_i\|_{L^\infty(\Omega_i)}^2\|\Pi_h^{RT^2}(\vec{\nabla}p) - Gp_H\|_{L^2(\Omega_i)}^2 + \|\mathrm{div}(\Pi_h^{RT^2}(\vec{\nabla}p) - Gp_H)\|_{L^2(\Omega_i)}^2 \\
&\lesssim \left(1 + \frac{H^2}{\delta^2}\right)(\|\vec{\sigma}\|_{H(\mathrm{div};\Omega_i)}^2 + \|u\|_{H^1(\Omega_i)}^2), && \text{(Lem. 4.11, Eq. (3.1))}
\end{aligned}$$

and

$$\|\vec{\nabla}\times\psi_i\|_{H(\mathrm{div};\Omega_i)}^2 \lesssim \left(1 + \frac{H}{\delta}\right)\left(\|\vec{\sigma}\|_{H(\mathrm{div};\Omega_i)}^2 + \|u\|_{H^1(\Omega_i)}^2\right). \tag{Lem. 4.4}$$

By inequality (3.1) (4.19) and Lemma 4.11, we may obtain

$$\begin{aligned}
\|u_i\|_{H^1(\Omega_i)}^2 &= \|\Pi_i^h(\theta_i(u-u_0))\|_{H^1(\Omega_i)}^2 \\
&\lesssim \|\theta_i(u-u_0)\|_{H^1(\Omega_i)}^2 && \text{(inequality (4.19))} \\
&\lesssim \|\vec{\nabla}\theta_i\|_{L^\infty(\Omega_i)}^2\|u-u_0\|_{L^2(\Omega_i)}^2 + \|\vec{\nabla}(u-u_0)\|_{L^2(\Omega_i)}^2 \\
&\lesssim \left(1 + \frac{H^2}{\delta^2}\right)\|u\|_{H^1(\Omega_i)}^2. && \text{(Lem. 4.11, Eq. (3.1))}
\end{aligned}$$

$\square$

Now we are in a position to give our main result.

Theorem 4.16. *For all* $(u, \vec{\sigma}) \in W$*, there exists a stable decomposition* $\{(u_i, \vec{\sigma_i}) \in W_i\}_{i=0}^N$ *such that*

$$\vec{\sigma} = \sum_{i=0}^{N} \vec{\sigma}_i, \quad u = \sum_{i=0}^{N} u_i, \tag{4.24}$$

and

$$\sum_{i=0}^{N} \left(\|\vec{\sigma}_i\|^2_{H(\mathrm{div};\Omega_i)} + \|u_i\|^2_{H^1(\Omega_i)} \right) \lesssim \left(1 + \frac{H^2}{\delta^2}\right) N^c \left(\|\vec{\sigma}\|^2_{H(\mathrm{div};\Omega)} + \|u\|^2_{H^1(\Omega)} \right). \tag{4.25}$$

Proof. According to Assumption 3.1, we find that each point $x (\in \Omega)$ belongs to at most N^c subdomains. Summing up inequalities (4.9), (4.10), (4.22), and (4.23) over each subdomain, we may derive (4.25).

Now we shall check if (4.24) is true. Note that since $\vec{\sigma} \cdot \vec{n} \in P^1(\partial\Omega \cap \partial\Omega_h)$, $u \in \widetilde{P}^2(\partial\Omega \cap \partial\Omega_h)$ and $\hat{\sigma} \cdot \vec{n} = u$ on $\partial\Omega$, both $\hat{\sigma} \cdot \vec{n}$ and u belong to the intersection of $P^1(\partial\Omega \cap \partial\Omega_h)$ and $\widetilde{P}^2(\partial\Omega \cap \partial\Omega_h)$, *i.e.*, $\widetilde{P}^1(\partial\Omega \cap \partial\Omega_h)$. Similarly, we may also find that $(\Pi_h^{RT^2}(\vec{\nabla} p) - Gp_H) \cdot \vec{n} \in \widetilde{P}^1(\partial\Omega \cap \partial\Omega_h)$ and $u - u_0 \in \widetilde{P}^1(\partial\Omega \cap \partial\Omega_h)$.

By the definition of $\vec{\sigma}_i$, we may derive

$$\begin{aligned}
\sum_{i=1}^{N} \vec{\sigma}_i &= \sum_{i=1}^{N} (\vec{\Pi}_i^h (\theta_i (\Pi_h^{RT^2}(\vec{\nabla} p) - Gp_H)) + \vec{\nabla} \times \psi_i \\
&= \vec{\Pi}_i^h \left(\sum_{i=1}^{N} \theta_i (\Pi_h^{RT^2}(\vec{\nabla} p) - Gp_H) \right) + \sum_{i=1}^{N} \vec{\nabla} \times \psi_i \\
&= \vec{\Pi}_i^h \left((\Pi_h^{RT^2}(\vec{\nabla} p) - Gp_H) \right) + \sum_{i=1}^{N} \vec{\nabla} \times \psi_i.
\end{aligned}$$

Since $\Pi_h^{RT^2}(\vec{\nabla} p) - Gp_H \in RT_h^2(\Omega_h)$, and $(\Pi_h^{RT^2}(\vec{\nabla} p) - Gp_H) \cdot \vec{n} \in \widetilde{P}^1(\partial\Omega \cap \partial\Omega_h)$, we have

$$\begin{aligned}
\vec{\Pi}_i^h ((\Pi_h^{RT^2}(\vec{\nabla} p) - Gp_H)) &= \Pi_h^{RT^2}((\Pi_h^{RT^2}(\vec{\nabla} p) - Gp_H) - \mathcal{E}_{i,h}((\Pi_h^{RT^2}(\vec{\nabla} p) - Gp_H) \cdot \vec{n} \\
&\quad - \Pi_h^{\widetilde{P}^1}((\Pi_h^{RT^2}(\vec{\nabla} p) - Gp_H) \cdot \vec{n})) \\
&= \Pi_h^{RT^2}(\Pi_h^{RT^2}(\vec{\nabla} p) - Gp_H) = \Pi_h^{RT^2}(\vec{\nabla} p) - Gp_H.
\end{aligned}$$

Using the discrete Helmholtz decomposition (Lem. 4.4), we may get

$$\begin{aligned}
\sum_{i=0}^{N} \vec{\sigma}_i &= \Pi_h^{RT^2}(\vec{\nabla} p) - Gp_H + Gp_H + \sum_{i=0}^{N} \vec{\nabla} \times \psi_i \\
&= \Pi_h^{RT^2}(\vec{\nabla} p) + \vec{\nabla} \times \psi_h = \vec{\sigma}.
\end{aligned}$$

Similarly,

$$\sum_{i=1}^{N} u_i = \sum_{i=1}^{N} \Pi_i^h (\theta_i (u - u_0)) = \Pi_i^h \left(\sum_{i=1}^{N} \theta_i (u - u_0) \right) = \Pi_i^h (u - u_0).$$

Meanwhile $u - u_0 \in \widetilde{P}_h^2(\Omega_h)$ and $u - u_0 \in \widetilde{P}^1(\partial\Omega \cap \partial\Omega_h)$, we have

$$\begin{aligned}
\Pi_i^h (u - u_0) &= \Pi_h^{\widetilde{P}^2}((u - u_0) - \mathcal{E}_{i,h}(u - \Pi_h^{\widetilde{P}^1}(u - u_0))) \\
&= \Pi_h^{\widetilde{P}^2}(u - u_0) = u - u_0.
\end{aligned}$$

Then we obtain $\sum_{i=0}^{N} u_i = u$. $\square$

Next we shall verify assumption (A2).

It is easy to check that if $\operatorname{supp}\{\mathcal{U}_h\} \subseteq \Omega_u \subseteq \Omega$, then $\operatorname{supp}\{T_h\mathcal{U}_h\} \subseteq \Omega_u$. Actually, since $(T_h\mathcal{U}_h, \mathcal{V}_h)_V = b(\mathcal{U}_h, \mathcal{V}_h) = 0$ for all $\mathcal{V}_h \in V_h$ which satisfy $\operatorname{supp}\{\mathcal{V}_h\} \cap \Omega_u = \emptyset$, and since the function $T\mathcal{U}_h$ $(\in V_h)$ is discontinuous, we may find that $\operatorname{supp}\{T_h\mathcal{U}_h\} \subseteq \Omega_u$. Thus if Ω_i and Ω_j do not overlap, then

$$(R_{h,i}^T T^h \mathcal{U}_i, R_{h,j}^T T^h \mathcal{U}_j)_V = 0, \ \forall \ \ \mathcal{U}_i \in U_i, \ \mathcal{U}_j \in U_j.$$

Assumption 3.1 also ensures that if $\operatorname{supp}\{\mathcal{U}_i\} \subseteq \Omega_i$, $\operatorname{supp}\{\mathcal{U}_j\} \subseteq \Omega_j$, Ω_i and Ω_j are of different colors, then we have $(R_{h,i}^T T^h \mathcal{U}_i, R_{h,j}^T T^h \mathcal{U}_j)_V = 0$, thus we may deduce $\rho(\mathcal{E}_p) \lesssim N^c$ in assumption (A2).

Finally we get

Theorem 4.17. *The condition number κ of the preconditioned algebraic system, which is given by our two-level additive Schwarz preconditioning method for the DPG method, is bounded by $\kappa \lesssim (N^c)^2(1+\frac{H^2}{\delta^2})$.*

Remark 4.18. Our analysis in this paper is for the Robin boundary condition case, however, by our framework introduced in Section 3, we may also deal with other boundary conditions, such as Dirichlet and Neumann boundary conditions. These boundary conditions also give essential boundary conditions, but there is no coupling between the space of the numerical trace $\hat{u}$ and the space of the numerical flux $\hat{\sigma}_n$, thus we may extend the trace space into a proper subspace of $H^1(\Omega)\times H(\mathrm{div};\Omega)$, and construct the stable decomposition in $H^1(\Omega)$ and $H(\mathrm{div};\Omega)$ respectively. These decompositions are well-known results in *e.g.* [17, 23]. Following our framework introduced in Therorem 3.4, we know that the analysis of such boundary cases become trivial.

5. One-level preconditioner for the Helmholtz equation

In this section we design a one-level additive Schwarz preconditioner for the Helmholtz equation. To avoid using complex notations, we still use the same notations as above section, such as $(\vec{\sigma}, u)$.

The Helmholtz equation we want to solve is

$$-\Delta u - \omega^2 u = f \quad \text{ in } \Omega$$

with a homogeneous Robin boundary condition

$$\frac{\partial u}{\partial n} + i\omega u = 0 \quad \text{ on } \partial\Omega,$$

here we use the same geometry of Ω as Section 2, $i=\sqrt{-1}$.

Following [15], we consider the following first order systems:

$$\begin{cases} i\omega\vec{\sigma} + \vec{\nabla} u = \vec{0} & \text{in } \Omega, \\ i\omega u + \vec{\nabla}\cdot\vec{\sigma} = f & \text{in } \Omega, \\ \vec{\sigma}\cdot\vec{n} - u = 0 & \text{on } \partial\Omega. \end{cases}$$

Define $\overline{u}$ as the complex conjugation of u, the ultra-weak form of DPG reads: find $\mathcal{U} = (\vec{\sigma}, u, \hat{\sigma_n}, \hat{u}) \in U$ such that

$$b(\mathcal{U}, \mathcal{V}) := b((\vec{\sigma}, u, \hat{\sigma_n}, \hat{u}), (\vec{\tau}, v)) = (f, v) \quad \forall \ \ \mathcal{V} = (\vec{\tau}, v)) \in V,$$

where

$$\begin{aligned} b((\vec{\sigma}, u, \hat{\sigma_n}, \hat{u}), (\vec{\tau}, v)) &:= i\omega(\vec{\sigma}, \vec{\tau})_{\Omega_h} - (u, \vec{\nabla}\cdot\vec{\tau})_{\Omega_h} + \langle \hat{u}, \overline{\vec{\tau}\cdot\vec{n}}\rangle_{\partial\Omega_h} \\ &\quad + i\omega(u, v)_{\Omega_h} - (\vec{\sigma}, \vec{\nabla} v)_{\Omega_h} + \langle \overline{v}, \hat{\sigma_n}\rangle_{\partial\Omega_h}. \end{aligned}$$

Different from Section 2, here $(u, v)_{\Omega_h} := \sum_{K\in\Omega_h}(u, v)_K$, $(\cdot,\cdot)_D$ denotes the (sesquilinear) $L^2(D)$ inner product.

Define spaces S and Q as:

$$S := \{(\vec{\sigma}, u) \in H(\mathrm{div}; \Omega) \times H^1(\Omega) : \vec{\sigma} \cdot \vec{n} - u = 0\},$$
$$Q := \{(\hat{\sigma}_n, \hat{u}) : \exists (\vec{\sigma}, u) \in S \text{ such that } (\hat{\sigma}_n, \hat{u}) = \mathrm{tr}_{\partial\Omega_h}((\vec{\sigma}, u))\}$$

with norms

$$\|(\vec{\sigma}, u)\|_S^2 = \|\vec{\sigma}\|_\Omega^2 + \|u\|_\Omega^2 + \|i\omega\vec{\sigma} + \vec{\nabla}u\|_{\Omega_h}^2 + \|i\omega u + \vec{\nabla} \cdot \vec{\sigma}\|_{\Omega_h}^2,$$
$$\|(\hat{\sigma}_n, \hat{u})\|_Q = \inf\{\|(\vec{\sigma}, u)\|_S : \ \forall \ (\vec{\sigma}, u) \text{ such that } \mathrm{tr}_{\partial\Omega_h}(\vec{\sigma}, u) = (\hat{\sigma}_n, \hat{u})\},$$

then spaces U and V may be defined as:

$$U := (L^2(\Omega))^2 \times L^2(\Omega) \times Q,$$
$$V := H(\mathrm{div}; \Omega_h) \times H^1(\Omega_h).$$

The norms on U and V are defined by

$$\|(\vec{\sigma}, u, \hat{\sigma}_n, \hat{u})\|_U^2 = \|\vec{\sigma}\|_\Omega^2 + \|u\|_\Omega^2 + \|(\hat{\sigma}_n, \hat{u})\|_Q^2,$$
$$\|(\vec{\tau}, v)\|_V^2 = \|\vec{\tau}\|_\Omega^2 + \|v\|_\Omega^2 + \|i\omega\vec{\tau} + \vec{\nabla}v\|_{\Omega_h}^2 + \|i\omega v + \vec{\nabla} \cdot \vec{\tau}\|_{\Omega_h}^2$$

The discrete spaces U_h and V_h may be chosen as:

$$Q_h := (P^1(\partial\Omega_h) \times \widetilde{P}^2(\partial\Omega_h)) \cap Q,$$
$$U_h := \prod_{K\in\Omega_h} (P^1(K))^2 \times \prod_{K\in\Omega_h} P^1(K) \times Q_h,$$
$$V_h := \prod_{K\in\Omega_h} (P^3(K))^2 \times \prod_{K\in\Omega_h} P^3(K).$$

Using the same scheme introduced in Section 2, we may construct the test space TU and T^hU_h similarly and derive the corresponding inner product $a(\cdot,\cdot)$, $a_h(\cdot,\cdot)$ for the Helmholtz equation. We use the same partition $\{\Omega_i\}_{i=1}^N$ as Section 2, along with the triangulation $\{\Omega_{i,h}\}_{i=1}^N$, we may define subspaces $\{U_i\}_{i=0}^N$ as a family of spaces, which are the restriction of U_h in $\{\Omega_i\}_{i=1}^N$, the numerical flux and trace $(\hat{\sigma}_{n,i}, \hat{u}_i)$ should satisfy

$$\hat{u}_i - \hat{\sigma}_{n,i} = 0 \text{ on } \partial\Omega_i \cap \partial\Omega, \ \hat{u}_i = 0, \ \hat{\sigma}_{n,i} = 0 \text{ on } \partial\Omega_i \setminus \partial\Omega.$$

We may obtain the preconditioned system as

$$\sum_{i=1}^N P_i\mathcal{U} = \sum_{i=1}^N \widetilde{f}_i.$$

The procedure to derive the system is similar to the case of the Poisson equation (cf. Sect. 3), the only difference is that we do not have a coarse space. Though there is no rigorous analysis for this case, numerical results in the next section show that the preconditioner is efficient, and the condition number of the preconditioned system is independent of mesh size h and wavenumber ω.

6. Numerical experiments

We first solve the Poisson problem in a unit square $(0,1)^2$, with exact solution $u = x(1-x)y(1-y)$ for both the Dirichlet and Robin boundary conditions. In our numerical experiments, we use a uniform triangulation consisting of right triangles oriented so that the hypotenuses have slope -1, and use the Lagrange nodal basis. Our code is developed from a MATLAB package named Finite Element Frameworks (cf. [5]).

TABLE 1. Number of iterations for the Laplace equation with Dirichlet boundary condition, 64 subdomains, $\delta = 1/16$.

h	No. coarse space	$H = 1/2$	$H = 1/4$
1/32	49	39	33
1/64	49	38	33
1/128	49	38	33

TABLE 2. Number of iterations for the Laplace equation with Robin boundary condition, 64 subdomains, $\delta = 1/16$.

h	No. coarse space	$H = 1/2$	$H = 1/4$
1/32	57	41	35
1/64	56	41	34
1/128	54	40	33

TABLE 3. Number of iterations for the Laplace equation with Dirichlet boundary condition, 64 subdomains, $H/\delta = 4$.

	No. coarse space, $h = 1/128$	$h = 1/64$	$h = 1/128$
$H = 1/2$, $\delta = 1/8$	31	30	30
$H = 1/4$, $\delta = 1/16$	49	33	33
$H = 1/8$, $\delta = 1/32$	57	33	33

For the Dirichlet boundary condition case, the trial and test spaces are chosen as:

$$
\begin{aligned}
U_h &:= \prod_{K\in\Omega_h} (P^1(K))^2 \times \prod_{K\in\Omega_h} P^1(K) \times \widetilde{P}_0^2(\partial\Omega_h) \times P^1(\partial\Omega_h),\\
V_h &:= \prod_{K\in\Omega_h} (P^3(K))^2 \times \prod_{K\in\Omega_h} P^3(K),
\end{aligned}
$$

For the Robin boundary condition case, the trial and test spaces are chosen as:

$$
\begin{aligned}
U_h &:= \prod_{K\in\Omega_h} (P^1(K))^2 \times \prod_{K\in\Omega_h} P^1(K) \times \hat{W},\\
V_h &:= \prod_{K\in\Omega_h} (P^3(K))^2 \times \prod_{K\in\Omega_h} P^3(K),
\end{aligned}
$$

Since we use preconditioned CG in our algorithm, the iteration stops when the residual measured in the l^2 norm is smaller than 10^{-10}. Different subdomain size, coarse triangulation size and overlapping δ are considered in our test. We compare the number of iterations with and without a coarse triangulation for both cases in Tables 1 and 2. When $\frac{H}{\delta}$ is fixed, number of iterations are given in Table 3, we may see that the convergence rate of the one-level additive Schwarz preconditioner deteriorates when δ becomes small, but our two-level additive Schwarz preconditioner is scalable, that is what we expect from our theoretical analysis. Moreover, by the authors' observation, the calculating time is significantly shortened when a coarse space is added into the preconditioned system.

Tables 4–7 show some results for the one-level additive Schwarz preconditioner for the Helmholtz equation. We solve the Helmholtz equation with a Robin boundary condition on a unit square $(0,1)^2$, the triangulation is the same as the Laplace case. We use Lagrange nodal basis in trial and test spaces U_h and V_h, which are defined in Section 5. The exact solution of the equation is set to be $\sin(\omega(\frac{3}{5}x + \frac{4}{5}y))$, which is part of the plane wave

TABLE 4. Number of iterations for the Helmholtz equation without preconditioner.

	$h = 1/4$	$h = 1/8$	$h = 1/16$	$h = 1/32$
$\omega = \pi$	538	1210	2274	4326
$\omega = 2\pi$	750	1540	2990	5633
$\omega = 4\pi$	**847**	2364	4577	7307
$\omega = 8\pi$	**338**	**1573**	5381	10626

TABLE 5. Number of iterations for the Helmholtz equation, 4 subdomains, $\delta = 1/4$.

	$h = 1/4$	$h = 1/8$	$h = 1/16$	$h = 1/32$
$\omega = \pi$	15	15	15	14
$\omega = 2\pi$	14	14	14	14
$\omega = 4\pi$	**11**	13	15	15
$\omega = 8\pi$	**10**	**9**	11	11

TABLE 6. Number of iterations for the Helmholtz equation, 4 subdomains, $\delta = 1/8$.

	$h = 1/8$	$h = 1/16$	$h = 1/32$
$\omega = \pi$	20	20	20
$\omega = 2\pi$	22	23	20
$\omega = 4\pi$	18	21	22
$\omega = 8\pi$	**10**	15	20

TABLE 7. Number of iterations for the Helmholtz equation, 16 subdomains, $\delta = 1/4$.

	$h = 1/8$	$h = 1/16$	$h = 1/32$
$\omega = \pi$	43	38	38
$\omega = 2\pi$	44	41	40
$\omega = 4\pi$	41	38	40
$\omega = 8\pi$	**21**	28	37

$\mathrm{e}^{i\omega(\frac{3}{5}x+\frac{4}{5}y)}$. We also use preconditioned CG and iteration stops until the residual measured in the l^2 norm is smaller than 10^{-8}. We first show the number of CG iteration without using our preconditioner in Table 4. We give the number of CG iteration with our preconditioner in Tables 5–7. Different number of subdomains and different δ are chosen in Tables 5–7. We may find that if our preconditioner is not used, as the wavenumber ω increases and as the mesh size h decreases, the steps of iterations increases heavily, thus the condition number of the original system is sensitive to ω and h. However when the additive Schwarz preconditioner is applied, the number of iteration only depends on the overlap δ and the number of subdomains, but it is independent of ω and h.

We should mention that although the DPG method is stable for all ω and h, we should still suggest $\omega h \leq \frac{\pi}{2}$ so that our numerical solution may resolve the wave. Those numerical results that do not satisfies $\omega h \leq \frac{\pi}{2}$ are written in bold.

7. CONCLUSIONS

In this paper we constructed a two-level additive Schwarz preconditioners for the DPG method for solving the Poission equation and gave a rigorous condition number estimate. Furthermore we designed a one-level additive preconditioner for the Helmholtz equation. Numerical tests have shown that our preconditioners for both PDE systems perform very well.

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