ESAIM: M2AN 51 (2017) 949–964
DOI: 10.1051/m2an/2016043 **ESAIM: Mathematical Modelling and Numerical Analysis**
www.esaim-m2an.org

RECONSTRUCTION OF POLYGONAL INCLUSIONS IN A HEAT CONDUCTIVE BODY FROM DYNAMICAL BOUNDARY DATA

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Abstract. In this paper, we consider a reconstruction problem of small and polygonal heat-conducting inhomogeneities from dynamic boundary measurements on part of the boundary and for finite interval in time. Our identification procedure is based on asymptotic method combined with appropriate averaging of the partial dynamic boundary measurements. Our approach is expected to lead to an effective computational identification algorithms.

Mathematics Subject Classification. 35R30, 80A23.

Received June 22, 2015. Revised April 19, 2016. Accepted June 1st, 2016.

1. Introduction

This work deals with an inverse boundary value problem arising from the equation of heat conduction. We identify locations and certain properties of the shapes of small polygonal conductivity defects in the heat equation from partial (on accessible part of the boundary) dynamic boundary measurements and for finite interval in time.

Notice that reconstruction methods that allow partial boundary data are very interesting because, in most experimental settings, one does not have access to measurements on the whole boundary. The inverse heat conduction problem arises in most thermal manufacturing processes of solids and has recently attracted much attention.

Let Ω be a bounded domain of $\mathbb{R}^d$, $d = 2, 3$ with smooth boundary $\partial\Omega$. We suppose that Ω contains a finite number of polygonal inhomogeneities D_j, $j = 1, \ldots, m$. We denote $D = \bigcup_{j=1}^m D_j$ such that $\overline{D} \subset \Omega$ and $\Omega \backslash \overline{D}$ is connected (see Fig. 1). We suppose that the inhomogeneities satisfy:

$$\overline{D}_j \cap \overline{D}_l = \emptyset, \text{ for } j \neq l, \text{ and } 0 < \sup_{x,y \in D_j} \operatorname{dist}(x, y) \leq \operatorname{dist}(D_j, \partial\Omega);\ \forall\ j = 1, \ldots, m. \tag{1.1}$$

The fact that D_j is a polygonal domain means that the boundary of D_j is the union of a finite number of line segments. The vertices $s_i^{(j)}$ are the ends of the edges, where $i = 1, \ldots, p_j$ and p_j means the number of edges of D_j.

Keywords and phrases. Inverse initial boundary value problem, parabolic equation, thermal imaging, polygonal inclusion, reconstruction.

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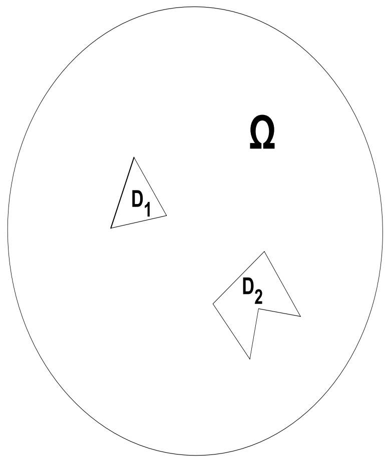

FIGURE 1. Domain with polygonal inhomogeneities.

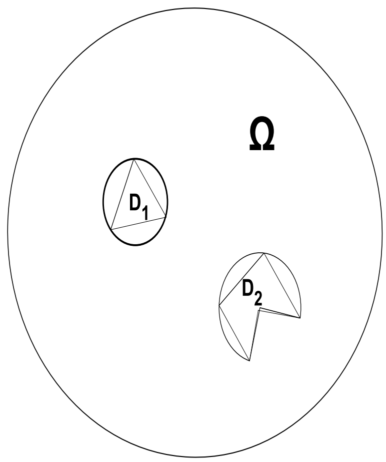

FIGURE 2. Approximation of the polygonal inclusion by smooth inhomogeneity.

Let us denote by $\mathcal{C}_j := \{s_i^{(j)};\ 1 \leq i \leq p_j\} \subset \partial \mathbb{D}_j$, $(1 \leq j \leq m)$, the set of the vertices associated to the polygonal inhomogeneity D_j.

Let $\mathbb{D}_j = z_j + \alpha B_j$ be such that $\partial \mathbb{D}_j$ is *circumscribed* to D_j (for a simple example, one can see Fig. 2), where $B_j \subset \mathbb{R}^d$ is a bounded and smooth domain containing no corners.

Evidently, we have

$$\mathcal{C}_j \subset \partial \mathbb{D}_j, \text{ for all } j = 1, \dots m. \tag{1.2}$$

Next, let z_j be the center of the new nonpolygonal inhomogeneity $\mathbb{D}_j$ and α is the common magnitude of the diameter of these new inhomogeneities containing no corners. We denote $\mathbb{D}_\alpha = \bigcup_{j=1}^m \mathbb{D}_j$.

We now have the following remark.

Remark 1.1. Suppose that the polygonal inclusions D_j satisfy the hypothesis (1.1) and (1.2) for all $j = 1, \dots, m$. We can assume that the points $z_j \in \Omega, j = 1 \dots m$, that determines the location of the inhomogeneities $\mathbb{D}_j$ satisfy

$$|z_j - z_l| \geq d_0 > 0, \quad \forall j \neq l, \text{ and } \operatorname{dist}(z_j, \partial \Omega) \geq d_0 > 0, \quad \forall j = 1 \dots m. \tag{1.3}$$

Let c_j denote the heat conductivity of the jth polygonal inhomogeneity D_j. We introduce the piecewise constant conductivity:

$$c_\alpha(x) = \begin{cases} c_0, \text{ if } x \in \Omega\backslash\overline{D} \\ c_j, \text{ if } x \in D_j. \end{cases}$$

Let u and u_α be the solutions of the heat equation without and with inhomogeneities respectively:

$$\begin{cases} \partial_t u - c_0 \Delta u = 0, & (x,t) \in \Omega \times [0,T] \\ u(x,0) = \varphi(x), & x \in \Omega \\ u(x,t)|_{\partial\Omega\times[0,T]} = f(x,t) \end{cases} \tag{1.4}$$

$$\begin{cases} \partial_t u_\alpha - (\nabla \cdot c_\alpha \nabla) u_\alpha = 0, & (x,t) \in \Omega \times [0,T] \\ u_\alpha(x,0) = \varphi(x), & x \in \Omega \\ u_\alpha(x,t)|_{\partial\Omega\times[0,T]} = f(x,t) \end{cases} \tag{1.5}$$

where $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$ and $f \in \mathcal{C}^\infty([0,T] \times \mathcal{C}^\infty(\partial\Omega))$.

It is classical to prove that the transmission problem of the heat equation (1.5) has a unique weak solution $u_\alpha \in L^2\big([0,T]; H^2(\Omega)\big) \cap H^1\big([0,T]; L^2(\Omega)\big)$ (see, for example, [13] p. 92 or [20]).

In the rest of this paper we will focus our attention to approximate the polygonal inhomogeneity D_j by the spherical one $\mathbb{D}_j$. Since the inhomogeneity $\mathbb{D}_j$ is not polygonal and is smooth enough, then we will try to use some results that we have developed in [4].

Now, we define ν_j to be the outward unit normal to $\partial(z_j + \alpha B_j)$ for $j = 1, \ldots, m$ and let $\Gamma \subset \partial\Omega$ be a given part of the boundary $\partial\Omega$.

Inverse problem. *Reconstruct unknown polygonal inclusion D_j satisfying* (1.1) *and* (1.2) *from only knowledge of boundary measurements of*

$$\frac{\partial u_\alpha}{\partial \nu} \quad \text{on } \Gamma \times (0,T),$$

i.e.*, on the part Γ of the boundary $\partial\Omega$ and on the finite interval in time* $(0,T)$.

More precisely, the aim of this paper is to identify the location and certain properties of the shapes of the polygonal inclusion D_j from some information concerning $\mathbb{D}_j$, that we will recuperate by using the asymptotic method.

Throughout this paper we suppose that $\mathcal{C}_j \subset \partial\mathbb{D}_j$ $j = 1, \ldots, m$. In the presence of $\mathbb{D}_j$, we develop an asymptotic method based on appropriate averaging, using particular background solutions as weights. These particular solutions are constructed by a control method as it has been done in the original work [31].

The procedure used to prove our central results is different from the ones used in [1, 8, 12, 14, 16–18, 21–24]. By means of specific test functions our first result can be read as an approximation to the Fourier transformation of the delta distributions at the centers of the inhomogeneities $\mathbb{D}_j$ and this was suggested as an idea for a numerical reconstruction algorithm.

The first basic step in the design of our reconstruction method is the derivation of an asymptotic formula for $\frac{\partial u_\alpha}{\partial \nu_j}|_{\partial\mathbb{D}_j^+}$ in terms of the reference solution u, the location z_j of the imperfection $\mathbb{D}_j := z_j + \alpha B_j$ and the geometry of B_j.

The second basic step of this investigation is to summon major information to build D_j from that found concerning $\mathbb{D}_j$. We will use some classical geometric results if the unknown polygonal inclusion is a triangle and for other regular polygonal inclusions one can apply similar methods developed here to identify them.

The above inverse boundary value problem is related to non-destructive testing where one looks for anomalous materials inside a known material. As an example one can mention the monitoring of a blast furnace used in iron making: the corroded thickness of the accreted refractory wall based on temperature and heat flux measurement on the accessible part of the furnace wall [33].

A similar approach may be applied to Stokes equations with small polygonal inhomogeneities of different parameters. This will be discussed in a forthcoming paper. The elastodynamic inverse problem will also be considered.

There are lots of works on inverse problem of heat conductivity [4, 6, 10, 12, 15–18, 24, 29, 34].

In general, the determination of conductivity shapes from information of boundary measurements has received an enormous deal of attention (see for example [7, 11, 32]) the reconstruction of polygonal inclusions within dynamics is much less investigated. In this context, one can refer to the series of works developed by Liu and Zou [22, 23]. These works are concerned with the inverse electromagnetic scattering by polyhedral obstacles. To the best of our knowledge, the present paper is the first attempt to design an effective method to determine the location and the size of small and polygonal heat-conducting imperfections inside a homogeneous medium from the dynamical measurements on part of the boundary. Our method is quite similar to the ideas used (in the time-independent case) by Sylvester and Uhlmann in their important work [29] on uniqueness of the three-dimensional inverse conductivity problem. It is also closely related to ideas used by Yamamoto in the original works [30, 31] on inverse source hyperbolic problems, and by Rakesh and Symes [28]. For discussions on other interesting inverse source hyperbolic problems, the reader is referred for example to Puel and Yamamoto [26, 27], Bruckner and Yamamoto [5].

The paper is organized as follows. In Section 2, we present an asymptotic expansion for the resulting heat flux associated to temperature distribution, which will be useful for our future results. In Section 3, we describe our identification procedure for $\mathbb{D}_j$ and as consequence we identify the polygonal inclusion by using some well-known geometrical results. In the last section, we give some numerical examples.

2. Asymptotic formula and identification procedure

In this section, we begin by citing an asymptotic formula for the heat flux $\frac{\partial u_\alpha}{\partial \nu_j}$ on the boundary $\partial \mathbb{D}_j$ for $j = 1, \ldots, m$.

Proposition 2.1. *Suppose that we have hypothesis (1.1)–(1.3). Then for $y \in \partial B_j$ we have:*

$$\frac{\partial u_\alpha}{\partial \nu_j}\Big|_{\partial(\mathbb{D}_j)^+}(z_j + \alpha y) = \nu_j.\nabla(u(z_j, t)) + \left(\frac{c_0}{c_j} - 1\right)\frac{\partial \Phi_j}{\partial \nu_j}\Big|_+(y).\nabla u(z_j, t) + o(1), \tag{2.1}$$

where ν_j means the outward unit normal to $\partial(z_j + \alpha B_j)$, and Φ_j is the solution of the problem

$$\begin{cases} \Delta \Phi_j = 0, \ \ in \ \mathbb{D}_j \cup \mathbb{R}^d \backslash \overline{\mathbb{D}_j}, \\ \Phi_j \ \ is \ continuous \ across \ \partial \mathbb{D}_j, \\ \frac{c_0}{c_j}\frac{\partial \Phi_j}{\partial \nu_j}|_+ - \frac{\partial \Phi_j}{\partial \nu_j}|_- = -\nu_j, \\ \lim_{|y| \to +\infty} |\Phi_j(y)| = 0. \end{cases} \tag{2.2}$$

The term $o(1)$ is uniform in $y \in \partial B_j$ and $t \in (0, T)$ and depends on the shape of $\{B_j\}_{j=1}^m$ and Ω, the constants d_0, T, $\{c_j\}_{j=1}^m$, the data φ and f, but is otherwise independent of the points $\{z_j\}_{j=1}^m$.

Proposition 2.1 may be proven by using the *matching conditions* for a single inhomogeneity, and by iteration we can deduce the desired result. For detailed proof the reader can see Proposition 3.1 in [4].

Let $\beta(x) \in \mathcal{C}_0^\infty(\Omega)$ be a cutoff function such that $\beta(x) \equiv 1$ in a subdomain Ω' of Ω that contains the inhomogeneities $\mathbb{D}_\alpha$. Therefore, Ω' contains $\cup_{j=1}^m D_j$. For an arbitrary $\eta \in \mathbb{R}^d$, we assume that we are in possession of the boundary measurements of

$$\frac{\partial u_\alpha}{\partial \nu} \quad \text{on } \Gamma \times (0, T)$$

for

$$\varphi(x) = \varphi_\eta(x) = \frac{J_0\left(\frac{z}{a|\eta|}|x|\right)}{zJ_1(z)}, \quad f(x,t) = f_\eta(x,t) = \frac{J_0\left(\frac{z}{a|\eta|}|x|\right)}{zJ_1(z)} \mathrm{e}^{-\left(\frac{z}{a|\eta|}\right)^2 c_0 t};$$

where z is the positive zero of the Bessel function of the first kind $J_0(x)$ ([3], pp. 37–39), and $a = 1/2 \max\{d(x,y) : x, y \in \Omega\}$.

This particular choice of data φ and f implies that the background solution u of the heat equation (1.4) in the absence of any inhomogeneity is given by

$$u(x,t) = u_\eta(x,t) = \frac{J_0\left(\frac{z}{a|\eta|}|x|\right)}{zJ_1(z)} \mathrm{e}^{-\left(\frac{z}{a|\eta|}\right)^2 c_0 t} \quad \text{in } \Omega \times (0,T).$$

Next, we consider the function $v_{\alpha,\eta} \in \mathcal{C}^0(0,T; L^2(\Omega)) \cap \mathcal{C}^1(0,T; H^{-1}(\Omega))$ satisfying:

$$\begin{aligned} \partial_t v_{\alpha,\eta} - c_0 \Delta v_{\alpha,\eta} &= 0, \quad (x,t) \in \Omega \times [0,T] \\ v_{\alpha,\eta}(x,0) &= \sum_{j=1}^m i\left(1 - \frac{c_o}{c_j}\right)\eta.\left(\nu_j + \left(\frac{c_0}{c_j} - 1\right)\frac{\partial \Phi_j}{\partial \nu_j}|_+\right) \mathrm{e}^{i\eta.z_j}\delta_{\partial \mathbb{D}_j}, x \in \Omega \\ v_{\alpha,\eta}|_{\partial\Omega\times[0,T]} &= 0, \\ \frac{\partial v_{\alpha,\eta}}{\partial t}(x,0) &= 0, \quad x \in \partial\Omega. \end{aligned} \tag{2.3}$$

When $\frac{\partial \Phi_j}{\partial \nu_j}|_+(y)\delta_{\partial\mathbb{D}_j} \in H^{-1}(\Omega)$, for $j = 1, \ldots, m$, the existence and uniqueness of a solution $v_{\alpha,\eta}$ can be established by transposition, see [20] and [19] (Thm. 4.2, p. 46). For a closely discussions, we can also refer to [2].

For $\eta \in \mathbb{R}^d$, introduce the function $g_\eta \in H^1_0(0,T; L^2(\Gamma))$ such that the unique weak solution w_η in $\mathcal{C}^0(0,T; L^2(\Omega)) \cap \mathcal{C}^1(0,T; H^{-1}(\Omega))$ of the following problem

$$(\partial_t + c_0\Delta)w_\eta = 0 \quad \text{in } \Omega \times (0,T),$$

$$w_\eta|_{t=0} = \beta(x)\mathrm{e}^{i\eta\cdot x} \in H^1_0(\Omega), \tag{2.4}$$

$$w_\eta|_{\Gamma\times(0,T)} = g_\eta, \tag{2.5}$$

$$w_\eta|_{\partial\Omega\setminus\overline{\Gamma}\times(0,T)} = 0,$$

satisfies $w_\eta(T) = 0$.

On the other hand, we will use the following proposition.

Proposition 2.2. *Suppose that we have all assumptions (1.1)–(1.3). For any $\eta \in \mathbb{R}^d$, we have the following result:*

$$\sum_{j=1}^m i\left(1 - \frac{c_0}{c_j}\right)\mathrm{e}^{i\eta.z_j}\eta.\int_{\partial\mathbb{D}_j}\left(\nu_j + \left(\frac{c_0}{c_j} - 1\right)\frac{\partial\Phi_j}{\partial\nu_j}|_+(y)\right)\mathrm{e}^{i\eta.y}\mathrm{d}s_j(y) = -c_0\int_0^T\int_\Gamma \frac{\partial v_{\alpha,\eta}}{\partial\nu}g_\eta,$$

where g_η is given in (2.5).

From the previous proposition, we can deduce the following one for $d = 2$.

Proposition 2.3. *Suppose that we have the (1.1)–(1.3), and let $d = 2$. For any $\eta \in \mathbb{R}^2$ we have:*

$$\alpha^2\sum_{j=1}^m\left(1 - \frac{c_0}{c_j}\right)\mathrm{e}^{2i\eta z_j}\eta.\int_{\partial B_j}\left(\nu_j + \left(\frac{c_0}{c_j} - 1\right)\frac{\partial\Phi}{\partial\nu_j}|_+(y)\right)\eta.y\,\mathrm{d}s_j(y) = c_0\int_0^T\int_\Gamma g_\eta\frac{\partial v_{\alpha,\eta}}{\partial\nu} + o\left(\alpha^2\right).$$

Therefore, from Propositions 2.1 and 2.3, we can deduce the following theorem which gives the procedure to identify the locations of the inhomogeneities $\mathbb{D}_j$ for $j = 1, \ldots, m$, and consequently the reconstruction of D_j.

Theorem 2.4. *Suppose that we have all assumptions* (1.1)–(1.3)*. Let* $\eta \in \mathbb{R}^2$ *and let* u_α *be the unique solution of* (1.5)*. Then:*

$$\begin{aligned}\int_0^T \int_\Gamma \left[\theta_\eta \left(\frac{\partial u_\alpha}{\partial \nu} - \frac{\partial u}{\partial \nu}\right) + \partial_t \theta_\eta \partial_t \left(\frac{\partial u_\alpha}{\partial \nu} - \frac{\partial u}{\partial \nu}\right)\right] &= -\int_0^T \int_\Gamma \mathrm{e}^{i\sqrt{c_0}|\eta|} \partial_t \left[\mathrm{e}^{-i\sqrt{c_0}|\eta|} g_\eta\right] \left(\frac{\partial u_\alpha}{\partial \nu} - \frac{\partial u}{\partial \nu}\right) \\ &= \alpha^2 \sum_{j=1}^m \left(\frac{1}{c_0} - \frac{1}{c_j}\right) \mathrm{e}^{2i\eta.z_j} M_j \eta.\eta + o\left(\alpha^2\right). \end{aligned} \tag{2.6}$$

Here M_j *is the polarisation tensor of* B_j*, defined by:*

$$(M_j)_{k,l} = e_k. \left(\int_{\partial B_j} \left(\nu_j + \left(\frac{c_0}{c_j} - 1\right) \frac{\partial \Phi_j}{\partial \nu_j}\bigg|_+ (y)\right) y.e_l \mathrm{d}s_j(y)\right).$$

with $(e_i)_{1\le i\le 2}$ *being an orthonormal basis of* $\mathbb{R}^2$*, and* θ_η *is the solution of the following problem as a function of* η*:*

$$\begin{cases} \partial_t \theta_\eta(x,t) + \int_t^T \mathrm{e}^{-i\sqrt{c_0}|\eta|(s-t)} (\theta_\eta(x,s) - i\sqrt{c_0}|\eta| \partial_t \theta_\eta(x,s))\, \mathrm{d}s = g_\eta(x,t), & (x,t) \in \partial\Omega \times (0,T); \\ \theta_\eta(x,0) = 0, & x \in \partial\Omega; \\ \partial\theta_\eta(x,T) = 0, & x \in \partial\Omega. \end{cases} \tag{2.7}$$

Proof. Firstly, we set

$$I = \int_0^T \int_\Gamma \partial_t \theta_\eta \partial_t \left(\frac{\partial u_\alpha}{\partial \nu} - \frac{\partial u}{\partial \nu}\right).$$

Noting that $\left(\frac{\partial u_\alpha}{\partial \nu} - \frac{\partial u}{\partial \nu}\right)|_{t=0} = 0$, and integrating by parts, we get:

$$I = -\int_0^T \int_\Gamma \partial_t^2 \theta_\eta \left(\frac{\partial u_\alpha}{\partial \nu} - \frac{\partial u}{\partial \nu}\right) = -\int_0^T \int_\Gamma \left[\mathrm{e}^{i\sqrt{c_0}|\eta|t} \partial_t \left(\mathrm{e}^{-i\sqrt{c_0}|\eta|t} g_\eta\right) + \theta_\eta\right] \left(\frac{\partial u_\alpha}{\partial \nu} - \frac{\partial u}{\partial \nu}\right).$$

Using the Volterra equation, we get:

$$\int_0^T \int_\Gamma \left[\theta_\eta \left(\frac{\partial u_\alpha}{\partial \nu} - \frac{\partial u}{\partial \nu}\right) + \partial_t \theta_\eta \partial_t \left(\frac{\partial u_\alpha}{\partial \nu} - \frac{\partial u}{\partial \nu}\right)\right] = -\int_0^T \int_\Gamma \left[\mathrm{e}^{i\sqrt{c_0}|\eta|t} \partial_t \left(\mathrm{e}^{-i\sqrt{c_0}|\eta|t} g_\eta\right)\right] \left(\frac{\partial u_\alpha}{\partial \nu} - \frac{\partial u}{\partial \nu}\right).$$

To prove the second equality, we consider the following function:

$$\widetilde{u}_{\alpha,\eta}(x,t) = u(x,t) + \int_0^T \mathrm{e}^{-i\sqrt{c_0}|\eta|s} v_{\alpha,\eta}(x, t-s) \mathrm{d}s.$$

Using the function $\widetilde{u}_{\alpha,\eta}(x,t)$, we obtain:

$$\begin{aligned}\int_0^T \int_\Gamma \left[\theta_\eta \left(\frac{\partial u_\alpha}{\partial \nu} - \frac{\partial u}{\partial \nu}\right) + \partial_t \theta_\eta \partial_t \left(\frac{\partial u_\alpha}{\partial \nu} - \frac{\partial u}{\partial \nu}\right)\right] &= \int_0^T \int_\Gamma \left[\theta_\eta \left(\frac{\partial u_\alpha}{\partial \nu} - \frac{\partial \widetilde{u}_{\alpha,\eta}}{\partial \nu}\right) + \partial_t \theta_\eta \partial_t \left(\frac{\partial u_\alpha}{\partial \nu} - \frac{\partial \widetilde{u}_{\alpha,\eta}}{\partial \nu}\right)\right] \\ &+ \int_0^T \int_\Gamma \left[\theta_\eta \int_0^t \mathrm{e}^{i\sqrt{c_0}|\eta|s} \partial_\nu v_{\alpha,\eta}(x, t-s) \mathrm{d}s + \partial_t \theta_\eta \partial_t \int_0^t \mathrm{e}^{-i\sqrt{c_0}|\eta|s} \partial_\nu v_{\alpha,\eta}(x, t-s) \mathrm{d}s\right].\end{aligned}$$

On the other hand, we have

$$\partial_t\theta(x,t) + \int_t^T \mathrm{e}^{-i\sqrt{c_0}|\eta|(s-t)}(\theta_\eta(x,s) - i\sqrt{c_0}|\eta|\partial_t\theta_\eta(x,s))\mathrm{d}s = g_\eta(x,t),$$

and by performing a variable change, we get

$$\partial_t \int_0^t \mathrm{e}^{-i\sqrt{c_0}|\eta|s}\partial_\nu v_{\alpha,\eta}(x,t-s)\mathrm{d}s = -i\sqrt{c_0}|\eta|\mathrm{e}^{-i\sqrt{c_0}|\eta|t}\int_0^t \mathrm{e}^{i\sqrt{c_0}|\eta|s}\partial_\nu v_{\alpha,\eta}(x,z)\mathrm{d}z + \partial_\nu v_{\alpha,\eta}(x,t).$$

Therefore,

$$\begin{aligned}
&\int_0^T\int_\Gamma \left[\theta_\eta \int_0^t \mathrm{e}^{i\sqrt{c_0}|\eta|s}\partial_\nu v_{\alpha,\eta}(x,t-s)\mathrm{d}s + \partial_t\theta_\eta\partial_t\int_0^t \mathrm{e}^{-i\sqrt{c_0}|\eta|s}\partial_\nu v_{\alpha,\eta}(x,t-s)\mathrm{d}s\right]\\
&= \int_0^T\int_\Gamma \left[\theta_\eta \int_0^t \mathrm{e}^{i\sqrt{c_0}|\eta|s}\partial_\nu v_{\alpha,\eta}(x,t-s)\mathrm{d}s\right.\\
&\quad \left.+\partial_t\theta_\eta\left(-i\sqrt{c_0}|\eta|\mathrm{e}^{-i\sqrt{c_0}|\eta|t}\int_0^t \mathrm{e}^{i\sqrt{c_0}|\eta|s}\partial_\nu v_{\alpha,\eta}(x,z)\mathrm{d}z + \partial_\nu v_{\alpha,\eta}(x,t)\right)\right]\\
&= \int_0^T\int_\Gamma \partial_\nu v_{\alpha,\eta}(x,t)\left[\partial_t\theta_\eta + \int_t^T(\theta_\eta(z) - i\sqrt{c_0}|\eta|\partial_t\theta_\eta(z))\mathrm{e}^{i\sqrt{c_0}|\eta|(t-z)}\mathrm{d}z\right]\mathrm{d}t\\
&= \int_0^T\int_\Gamma g_\eta(x,t)\partial_\nu v_{\alpha,\eta}(x,t)\mathrm{d}t.
\end{aligned}$$

Then,

$$\begin{aligned}
\int_0^T\int_\Gamma\left[\theta_\eta\left(\frac{\partial u_\alpha}{\partial\nu} - \frac{\partial u}{\partial\nu}\right) + \partial_t\theta_\eta\partial_t\left(\frac{\partial u_\alpha}{\partial\nu} - \frac{\partial u}{\partial\nu}\right)\right] &= \int_0^T\int_\Gamma\left[\theta_\eta\left(\frac{\partial u_\alpha}{\partial\nu} - \frac{\partial \widetilde{u}_{\alpha,\eta}}{\partial\nu}\right) + \partial_t\theta_\eta\partial_t\left(\frac{\partial u_\alpha}{\partial\nu} - \frac{\partial \widetilde{u}_{\alpha,\eta}}{\partial\nu}\right)\right]\\
&\quad + \int_0^T\int_\Gamma g_\eta(x,t)\partial_\nu v_{\alpha,\eta}(x,t)\mathrm{d}t.
\end{aligned}$$

From Proposition 2.3, we get:

$$\begin{aligned}
&\int_0^T\int_\Gamma\left[\theta_\eta\left(\frac{\partial u_\alpha}{\partial\nu} - \frac{\partial u}{\partial\nu}\right) + \partial_t\theta_\eta\partial_t\left(\frac{\partial u_\alpha}{\partial\nu} - \frac{\partial u}{\partial\nu}\right)\right]\\
&\quad = \int_0^T\int_\Gamma\left[\theta_\eta\left(\frac{\partial u_\alpha}{\partial\nu} - \frac{\partial \widetilde{u}_{\alpha,\eta}}{\partial\nu}\right) + \partial_t\theta_\eta\partial_t\left(\frac{\partial u_\alpha}{\partial\nu} - \frac{\partial \widetilde{u}_{\alpha,\eta}}{\partial\nu}\right)\right]\\
&\quad\quad + \alpha^2\sum_{j=1}^m\left(\frac{1}{c_0} - \frac{1}{c_j}\right)\mathrm{e}^{2i\eta z_j}\eta.\int_{\partial B_j}\left(\nu_j + \left(\frac{c_0}{c_j} - 1\right)\frac{\partial\Phi}{\partial\nu_j}|_+(y)\right)\eta.y\,\mathrm{d}s_j(y) + o(\alpha^2)\\
&\quad = \int_0^T\int_\Gamma\left[\theta_\eta\left(\frac{\partial u_\alpha}{\partial\nu} - \frac{\partial \widetilde{u}_{\alpha,\eta}}{\partial\nu}\right) + \partial_t\theta_\eta\partial_t\left(\frac{\partial u_\alpha}{\partial\nu} - \frac{\partial \widetilde{u}_{\alpha,\eta}}{\partial\nu}\right)\right]\\
&\quad\quad + \alpha^2\sum_{j=1}^m\left(\frac{1}{c_0} - \frac{1}{c_j}\right)\mathrm{e}^{2i\eta z_j}\eta.M_j(\eta) + o\left(\alpha^2\right).
\end{aligned}$$

Eventually, to obtain the final result we should prove

$$\int_0^T\int_\Gamma\left[\theta_\eta\left(\frac{\partial u_\alpha}{\partial\nu} - \frac{\partial \widetilde{u}_{\alpha,\eta}}{\partial\nu}\right) + \partial_t\theta_\eta\partial_t\left(\frac{\partial u_\alpha}{\partial\nu} - \frac{\partial \widetilde{u}_{\alpha,\eta}}{\partial\nu}\right)\right] = o(\alpha^2).$$

On the other hand, by using a variable change, we get

$$\partial_t \int_0^t \mathrm{e}^{-i\sqrt{c_0}|\eta|s} v_{\alpha,\eta}(x,t-s)\mathrm{d}s = i\sqrt{c_0}|\eta|\mathrm{e}^{-i\sqrt{c_0}|\eta|t} \int_0^t \mathrm{e}^{i\sqrt{c_0}|\eta|s} v_{\alpha,\eta}(x,s)\mathrm{d}s + v_{\alpha,\eta}(x,t)$$

which implies that,

$$\begin{aligned}(\partial_t - c_0\Delta)\left(\int_0^t \mathrm{e}^{-i\sqrt{c_0}|\eta|s} v_{\alpha,\eta}(x,t-s)\mathrm{d}s\right) &= i\sqrt{c_0}|\eta|\mathrm{e}^{-i\sqrt{c_0}|\eta|t}\\ &\quad\times \int_0^t \mathrm{e}^{i\sqrt{c_0}|\eta|s} v_{\alpha,\eta}(x,s)\mathrm{d}s + v_{\alpha,\eta}(x,t)\\ &\quad - c_0 \int_0^t \mathrm{e}^{-i\sqrt{c_0}|\eta|s} \Delta v_{\alpha,\eta}(x,t-s)\mathrm{d}s.\end{aligned}$$

By identity (2.1), we obtain:

$$\begin{cases}(\partial_t - c_0\Delta)\left(\int_0^t \mathrm{e}^{-i\sqrt{c_0}|\eta|s} v_{\alpha,\eta}(x,t-s)\mathrm{d}s\right) = \alpha\sum_{j=1}^m i\left(1-\frac{c_o}{c_j}\right)\eta\left(\nu_j + \left(\frac{c_0}{c_j}-1\right)\frac{\partial\Phi_j}{\partial\nu_j}|_+\right)\mathrm{e}^{i\eta.z_j}\delta_{\partial\mathbb{D}_j}\\ \int_0^t \mathrm{e}^{-i\sqrt{c_0}|\eta|s} v_{\alpha,\eta}(x,t-s)\mathrm{d}s|_{t=0} = 0\\ \int_0^t \mathrm{e}^{-i\sqrt{c_0}|\eta|s} v_{\alpha,\eta}(x,t-s)\mathrm{d}s|_{\partial\Omega\times(0,T)} = 0.\end{cases}$$

Using these equalities and by means of Proposition 2.1, we can demonstrate the following relations:

$$\begin{cases}(\partial_t - c_0\Delta)(u_\alpha - \widetilde{u}_{\alpha,\eta}) = o(\alpha^2), & (x,t)\in\Omega\times[0,T]\\ (u_\alpha - \widetilde{u}_{\alpha,\eta})|_{t=0} = 0, & x\in\Omega\\ (u_\alpha - \widetilde{u}_{\alpha,\eta}) = 0, & (x,t)\in\partial\Omega\times(0,T).\end{cases} \tag{2.8}$$

Next, let $\widehat{\varphi}(x) = \displaystyle\int_0^T \varphi(x,t)\theta(t)\mathrm{d}t$ where $\theta\in\mathcal{C}_0^\infty(]0,T[)$. Then,

$$\begin{aligned}(\nabla\cdot c_\alpha\nabla)\left(\widehat{u}_\alpha - \widehat{\widetilde{u}}_{\alpha,\eta}\right) &= (\nabla\cdot\, c_\alpha\nabla)\int_0^T (u_\alpha - \widetilde{u}_{\alpha,\eta})\theta(t)\mathrm{d}t\\ &= \int_0^T (\partial_t(u_\alpha - \widetilde{u}_{\alpha,\eta}) + o(\alpha^2))\theta(t)\mathrm{d}t = o(\alpha^2).\end{aligned}$$

Hence, using Proposition 2.1 in [4], we obtain:

$$\|\nabla(\widehat{u}_\alpha - \widehat{\widetilde{u}}_{\alpha,\eta})\| + \|\widehat{u}_\alpha - \widehat{\widetilde{u}}_{\alpha,\eta}\| \le \lambda\alpha,$$

which implies that

$$\widehat{u}_\alpha - \widehat{\widetilde{u}}_{\alpha,\eta} = o(\alpha^2).$$

Then we obtain,

$$\|\partial_t(\widehat{u}_\alpha - \widehat{\widetilde{u}}_{\alpha,\eta})\| = o(\alpha^2).$$

Our aim now is to prove that

$$\|\frac{\partial}{\partial\nu}(\widehat{u}_\alpha - \widehat{\widetilde{u}}_{\alpha,\eta})\|_{L^2(\Gamma)} = o(\alpha^2), \quad \forall\theta\in\mathcal{C}_0^\infty([0,T]).$$

Recalling that

$$\left\|\frac{\partial}{\partial\nu}\left(\widehat{u}_\alpha - \widehat{\widetilde{u}}_{\alpha,\eta}\right)\right\|^2_{L^2(\Gamma)} = \int_\Gamma \left[\nabla\left(\int_0^T (u_\alpha - \widetilde{u}_{\alpha,\eta})(x,t)\theta(t)\right).\nu\right]^2.$$

Therefore, by using the divergence theorem, the previous integral becomes:

$$\int_{\Omega}\left[\nabla\cdot\left(\nabla\int_0^T\left(u_\alpha-\widetilde{u}_{\alpha,\eta}\right)(x,t)\theta(t)\mathrm{d}t\right]^2=\int_{\Omega}\left[\int_0^T\Delta\left(u_\alpha-\widetilde{u}_{\alpha,\eta}\right)\theta(t)\mathrm{d}t\right]^2$$
$$\leq\frac{1}{c_0^2}\left(\left\|\partial_t\left(\widehat{u}_\alpha-\widehat{\widetilde{u}}_{\alpha,\eta}\right)\right\|^2+o(\alpha^2)\right)=o(\alpha^2).$$

Thus,

$$\left\|\frac{\partial}{\partial\nu}\left(\widehat{u}_\alpha-\widehat{\widetilde{u}}_{\alpha,\eta}\right)\right\|_{L^2(\Gamma)}=o(\alpha^2),\quad\forall\theta\in\mathcal{C}_0^\infty([0,T]),$$

which implies that,

$$\left\|\frac{\partial}{\partial\nu}\left(u_\alpha-\widetilde{u}_{\alpha,\eta}\right)\right\|_{L^2(\Gamma)}=o(\alpha^2).$$

Therefore,

$$\int_0^T\int_\Gamma\left[\theta_\eta\left(\frac{\partial u_\alpha}{\partial\nu}-\frac{\partial\widetilde{u}_{\alpha,\eta}}{\partial\nu}\right)+\partial_t\theta_\eta\partial_t\left(\frac{\partial u_\alpha}{\partial\nu}-\frac{\partial\widetilde{u}_{\alpha,\eta}}{\partial\nu}\right)\right]=o(\alpha^2).$$

Then, the desired proof is achieved. □

2.1. Locations of the centers

First of all, we will neglect the asymptotic remainder of the asymptotic formula (2.6) given in Theorem 2.4. As described in [4], we will use the fact that the function $\mathrm{e}^{2i\eta.z_j}$ is the Fourier transform of the Dirac function δ_{-2z_j}. Let us denote:

$$\Lambda_\alpha(\eta)=\int_0^T\int_\Gamma\left[\theta_\eta\left(\frac{\partial u_\alpha}{\partial\nu}-\frac{\partial u}{\partial\nu}\right)+\partial_t\theta_\eta\partial_t\left(\frac{\partial u_\alpha}{\partial\nu}-\frac{\partial u}{\partial\nu}\right)\right].\tag{2.9}$$

To obtain the locations of the centers $z_j,\ j=1,\dots,m$ we need to apply an inversion Fourier transform to the function $\Lambda_\alpha(\eta)$. We have to recall also that the function $\mathrm{e}^{2\mathrm{i}\eta\cdot z_j}$ is exactly the Fourier transform (up to a multiplicative constant) of the Dirac function δ_{-2z_j} (a point mass located at $-2z_j$), where the set of the points z_j, $j=1,\dots,m$ represents the centers of the inhomogeneities to be detected.

Furthermore, if we construct numerically the term $\Lambda_\alpha(\eta)$ then, by applying the IFFT (Inverse Fast Fourier Transforms) algorithm to $\Lambda_\alpha(\eta)$, we obtain a linear combination of the Dirac functions δ_{-2z_j}. So, after rescaling, we obtain the total collection of the points $z_j,\ j=1,\dots,m$. However, it has to be noticed that the number of different values of the variable $\eta\in\mathbb{R}^2$, which is considered as the set of the frequencies for the Inverse Fourier Transform, is very important for the total computation time for the direct problem, and also gives the final detection precision. Thus, any numerical reconstruction for the underlined inverse problem may be stable.

From Shannon's theorem the following two principal facts follow:

- If the domain which contains the inclusions is a square with dimension M, then the function $\Lambda_\alpha(\eta)$ has to be sampled with the step size $\Delta\eta=\frac{1}{M}$.
- If we sample in the frequency domain $|\eta|<\eta_{\max}$, then the reconstruction resolution will not be less than $\delta=\frac{1}{2\eta_{\max}}$.

In order to construct a suitable algorithm to find the location of the inclusions, we should begin by identifying the control function g_η on Γ. In the next section, we will give the method used to find the function g_η.

2.2. Determination of the control g_η

Let g_η be a function such that w_η be the unique solution of the following problem:

$$\begin{cases} (\partial_t + ic_0\Delta)w_\eta = 0, \text{ in } \Omega \times (0,T), \\ w_\eta|_{t=0} = \beta \mathrm{e}^{i\eta.x} \in H_0^1(\Omega), \\ w_\eta|_{\Gamma\times(0,T)} = g_\eta, \\ w_\eta|_{\partial\Omega\setminus\overline{\Gamma}\times(0,T)} = 0, \\ w_\eta(T) = 0. \end{cases} \tag{2.10}$$

There are several works concerned with the controllability problems. The key point in these investigations is the well-known Hilbert Uniqueness Method (HUM) [19]. On the other hand, in this paragraph we use the decomposition method [9]. This method is based on the decomposition of the function w_η in two simple functions u_η and v_η such that $u_\eta + v_\eta$ be the unique weak solution of (2.10). Explicitly we can write $u_\eta(x,t) = \frac{1}{2}\beta(\eta.x + c_0|\eta|t).\eta^\perp \mathrm{e}^{\frac{i}{2}(\eta.x+c_0|\eta|t)}$ and $v_\eta(x,t) = \frac{1}{2}\beta(\eta.x - c_0|\eta|t).\eta^\perp \mathrm{e}^{\frac{i}{2}(\eta.x-c_0|\eta|t)}$. We can prove easily that the function $w_\eta = u_\eta + v_\eta$ solves the problem (2.10). Therefore, it is not difficult to determine the function g_η by using the calculated term of w_η.

2.3. Numerical algorithm

In this section, we will give the algorithm used to find the locations of the centers z_j, $j = 1,\dots,m$. Using Theorem 2.4, the steps of the reconstruction of the inhomogeneities are as follows:

(1) Solve the problems (1.4) and (1.5) to find u and u_α.
(2) Apply the decomposition method to find the control g_η.
(3) Compute $\Lambda_\alpha(\eta)$ by using its expression defined in (2.9).
(4) Apply the IFFT to find the locations of the centers z_j of the inhomogeneities $\mathbb{D}_j$, $j = 1,\dots,m$.
(5) Calculate the polarization tensors $(M_j)_{j=1}^m$ of B_j, to find the position of the hypotenuse of each polygonal inhomogeneity D_j with respect to $(e_i)_{1\le i\le 2}$.
(6) Apply the law of sines to find the position of the other vertex.

2.4. Determination of the inhomogeneity D_j

In this section, we are convinced that the centers z_j of the circumscribed circle $\partial\mathbb{D}_j$ to the triangular inclusion D_j are well determined according to the mentioned above. Now suppose that the number of edges of D_j is $p_j := 3$, and suppose that we have the assumption (1.2). Then, $\mathbb{D}_j$ is the circumcircle of the triangle D_j.

Moreover, to compute the length of any side of D_j, one can use the law of sines to remark that (see Fig. 3).

Remark 2.5.

(i) If D_j is a right triangle, then the circumcenter z_j is at the center of the hypotenuse. As a consequence, the length of the hypotenuse is exactly the diameter of $\mathbb{D}_j$.
(ii) The length of any side of D_j can be computed as the diameter of $\mathbb{D}_j$, multiplied by the sine of the opposite angle.

The following main result holds.

Theorem 2.6. *Suppose that the assumptions (1.1)–(1.3) are satisfied. Let u and u_α be the solutions of (1.4) and (1.5) respectively, and let Λ_α be given by (2.9). If the polygonal inclusion D_j is a right triangle, then its area can be given by the following relation:*

$$\Lambda_\alpha(e_k) = \pi\alpha \sum_{j=1}^m \frac{\lambda_j}{\sin(2\varphi_j)}|D_j| + o(\alpha^2), \tag{2.11}$$

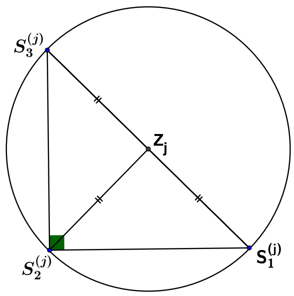

FIGURE 3. The circumcenter of a right triangle ($D_j := \Delta S_1^{(j)} S_2^{(j)} S_3^{(j)}$) is at the center of the hypotenuse.

where λ_j *is a constant depending on the ratio* $\frac{c_0}{c_j}$, $0 < \varphi_j < \pi/2$ *is an inscribed angle, and* $(e_i)_{1 \leq i \leq 2}$ *is an orthonormal basis of* $\mathbb{R}^2$.

Proof. To justify our results, we directly use Theorem 2.4. So, by using definition (2.9) we can write:

$$\Lambda_\alpha(\eta) = \alpha^2 \sum_{j=1}^{m} \left(\frac{1}{c_0} - \frac{1}{c_j} \right) \mathrm{e}^{2i\eta . z_j} M_j(\eta).\eta + o(\alpha^2), \text{ for any } \eta \in \mathbb{R}^d. \tag{2.12}$$

On the other hand, we recall from [7] that if B_j is a ball in $\mathbb{R}^d$ its polarization tensor M_j has the following explicit expression:

$$M_j = M_j(c_0, c_j; B_j) = \zeta_j |B_j| I_d,$$

where I_d is the identity matrix and $\zeta_j = \zeta_j(c_0/c_j)$. Therefore, (2.12) becomes

$$\Lambda_\alpha(\eta) = \alpha^2 \sum_{j=1}^{m} (a_j \zeta_j |\eta|^2) |B_j| + o(\alpha^2) \text{ for any } \eta \in \mathbb{R}^2, \tag{2.13}$$

where the constant a_j depends on z_j which now is known according to Theorem 2.4 and Section 2.1.

Now, if D_j is a right triangle with $\mathcal{C}_j := \{s_i^{(j)};\ 1 \leq i \leq 3\}$ $(1 \leq j \leq m)$ the set of vertices, then, one can compute, by using the classical law of sines and Remark 2.5, that

$$|D_j| = \frac{\sin(2\varphi_j)}{\pi} |\mathbb{D}_j|, \tag{2.14}$$

where $\varphi_j \equiv \widehat{s_i^{(j)}} (mod\ \pi)$ and $\widehat{s_i^{(j)}}$ is the opposite angle.

Moreover, since $|\mathbb{D}_j| = \alpha |B_j|$ and relation (2.13) is valid for all $\eta \in \mathbb{R}^2$, we can insert (2.14) into (2.13) to get that

$$\Lambda_\alpha(e_k) = \alpha \sum_{j=1}^{m} \frac{\lambda_j}{\sin(2\varphi_j)} \pi |D_j| + o(\alpha^2),$$

where we have taken $\eta = e_k$; $\quad 1 \leq k \leq 2$ and λ_j depends on ζ_j. □

In the following result, we state that if one knows the ratio of areas for two successive triangular inclusions $\frac{|D_j|}{|D_{j-1}|} = \beta_j$, then we can determine more precisely the area of each one.

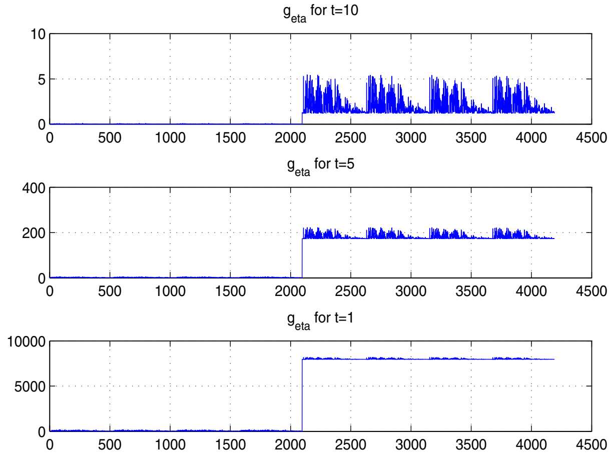

FIGURE 4. The function g_η at three different times.

Corollary 2.7. *Suppose that all the assumptions of Theorem 2.6 are satisfied. Let* $\beta_j = |D_j|/|D_{j-1}|$, $j = 1,\dots,m$. *Then we have*

$$|D_j| = \frac{|D_m|}{\Pi_{s=j+1}^{m}\beta_s}$$

and

$$|D_m| \approx \frac{\Lambda_\alpha(e_k)/\alpha}{\pi\sum_{j=1}^{m}\frac{\lambda_j}{\Pi_{s=j+1}^{m}\beta_s}\sin(2\varphi_j)}, \tag{2.15}$$

where λ_j *is given in* (2.11) *and the product term* $\Pi_{s=j+1}^{m}\beta_s := \beta_{j+1}\cdot\beta_{j+2}\dots\beta_m$.

3. Numerical examples

In this section, we choose Ω to be the circle centered in the origin, its radius is $r = 1$ and the thermal conductivity is $c_0 = 1$. In the rest of this section we will try to detect one inhomogeneity and then two inhomogeneities.

In all the examples of this section, steps 1, 2, 3 and 4 of the algorithm, given in the paragraph 2.3, were done using Matlab. After obtaing locations of the centers, we use FreeFem++ to apply steps 5 and 6.

In order to find the locations of the inhomogeneities, we begin by determining the function $g_\eta(x,t)$. Using the decomposition method, we obtain the following graphs.

Now, we can use the above algorithm to detect a single inhomogeneity.

3.1. Detection of a one inhomogeneity

The aim of this section is to find the location of one polygonal inhomogeneity which has the form of a triangle and whose thermal conductivity is $c = 0.5$. Then, the graphs of the density u_α and the heat flux $\partial_n u_\alpha$ for $t = 2$ are given by the Figures 5 and 6, respectively.

Using these values, the detection result is giving by Figures 7 and 8.

In the table below, we have a comparison between the coordinates of the vertices of the real inhomogeneity and the numerical one.

From Table 1, one may see that the error is small and our numerical result are stable.

To go more with the performance of this procedure, we will try to detect two inhomogeneities in the next paragraph.

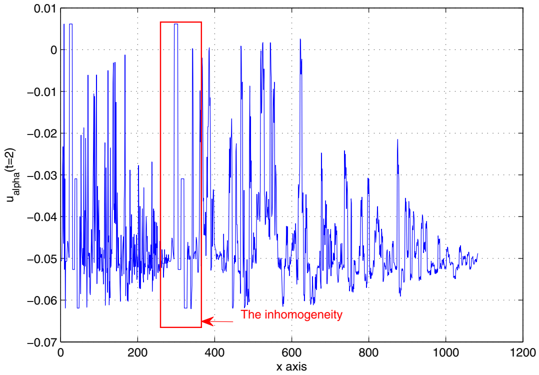

FIGURE 5. The density $u_\alpha(t=2)$.

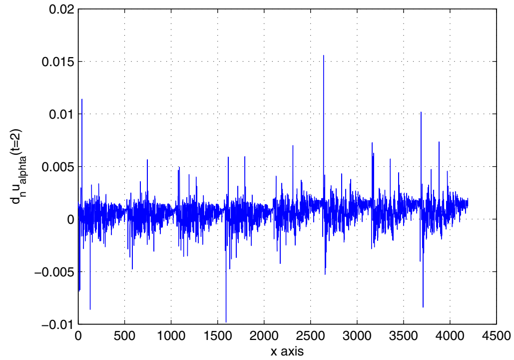

FIGURE 6. The heat flux $\frac{\partial u_\alpha}{\partial n}(t=2)$.

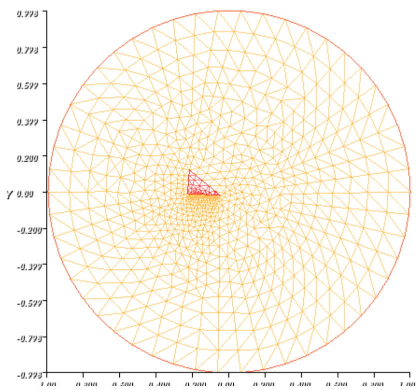

FIGURE 7. The real position of the inhomogeneity.

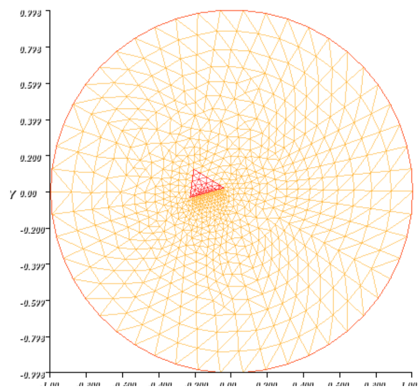

FIGURE 8. The numerical position of the inhomogeneity.

TABLE 1. Comparison between the real and the detected inhomogeneity.

Mesh	1490 triangles		
	1st vertex	2nd vertex	3rd vertex
Real vertices	(–0.22,0.12)	(–0.05;–0.02)	(–0.23,–0.01)
Detected vertices	(–0.2098,0.1222)	(–0.04277,0.02037)	(–0.2301,–0.03259)
Norm of the error	10^{-2}	4×10^{-2}	2×10^{-2}

3.2. Detection of two inhomogeneities

In this paragraph, we choose two inhomogeneities, the first is a triangle and the second is a pentagon. Their thermal conductivities are $c_1 = 0.5$ and $c_2 = 0.1$ respectively. The following figures represent the density and the heat flux associated to the present configuration.

On the other hand, we apply our algorithm to obtain the two figures below containing the real and the detected cavities.

In the next table, we give the exact coordinates of the vertices of the two real inhomogeneities and the coordinates of the detected ones after applying our method.

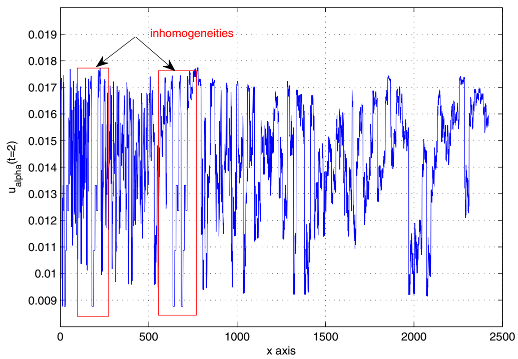

FIGURE 9. The density $u_\alpha(t=2)$.

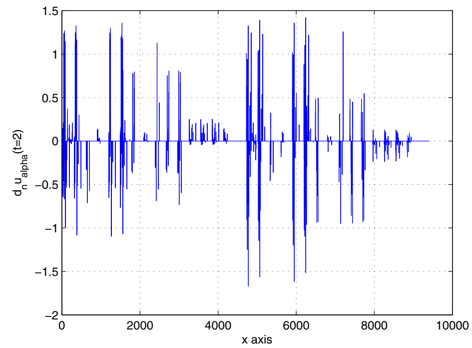

FIGURE 10. The heat flux $\frac{\partial u_\alpha}{\partial n}(t=2)$.

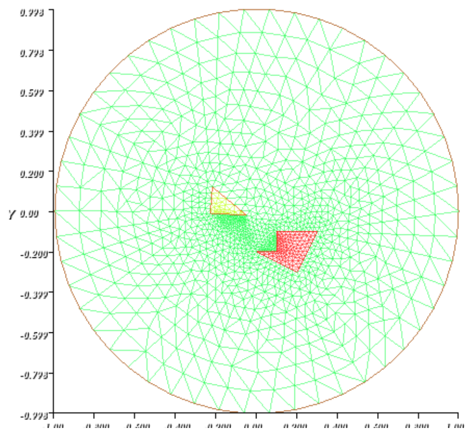

FIGURE 11. The real locationss of the inhomogeneities.

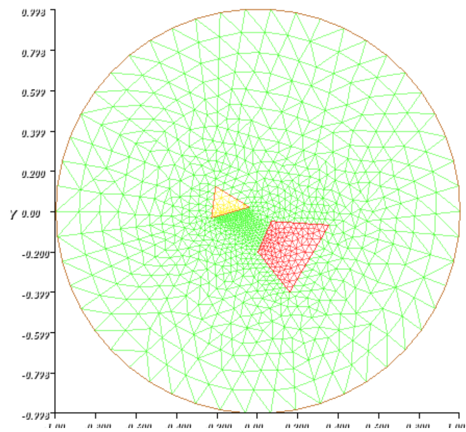

FIGURE 12. The numerical locations of the inhomogeneities.

TABLE 2. Comparison between the real and the detected inhomogeneities.

Mesh	2079 triangles				
	1st vertex	2nd vertex	3rd vertex	4th vertex	5th vertex
Real vertices	(0.1,–0.1)	(0.1,–0.2)	(0,–0.2)	(0.2,–0.3)	(0.3,–0.1)
Detected vertices	(0.06571,–0.05011)		(0.–0.201)	(0.157,–0.3998)	(0.3509,–0.0687)
Norm of the error	6×10^{-2}		10^{-3}	10^{-1}	5×10^{-2}

Table 2 shows that our algorithm can be applied not only for the inhomogeneities whose form is triangular, but also for other inhomogeneities containing right angles.

4. CONCLUSION

The reconstruction of unknown inhomogeneities becomes a very interesting problem and its applications take places in many fields such as biology, physics, and others. The reconstruction of polygonal cavities was studied by Ikehata using the enclosure method for several classes of PDEs [14, 16]. The asymptotic method is often used for inhomogeneities with smooth boundaries [1, 7, 32]. In this paper we have proved that the asymptotic method is also available for the polygonal case. In the last part of this paper we have developed some numerical applications to prove the performance and the stability of the numerical reconstruction method. In a forthcoming paper we will focus our attention to identify more complicated polygonal inclusions.

Acknowledgements. The authors wish to thank the Editor and the Referees for their valuable criticisms and suggestions, leading to the present improved version of our paper.

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