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A PRIORI DIFFUSION-UNIFORM ERROR ESTIMATES FOR NONLINEAR SINGULARLY PERTURBED PROBLEMS: BDF2, MIDPOINT AND TIME DG *

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Abstract. This work deals with a nonlinear nonstationary semilinear singularly perturbed convection-diffusion problem. We discretize this problem by the discontinuous Galerkin method in space and by the midpoint rule, BDF2 and quadrature variant of discontinuous Galerkin in time. We present *a priori* error estimates for these three schemes that are uniform with respect to the diffusion coefficient going to zero and valid even in the purely convective case.

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1. Introduction

The discontinuous Galerkin (DG) finite element method developed by Reed and Hill in [19] is a popular numerical method for the solution of advective and convective problems. The method uses high order piecewise polynomial approximations on a triangulation which are generally discontinuous between elements, unlike the standard conforming finite element method. The discontinuous nature of the approximation is natural for problems where discontinuities or sharp gradients and boundary layers occur in the solution, *e.g.* nonlinear convective problems or singular perturbations thereof.

Among the basic goals of numerical analysis is to prove *a priori* error estimates for the given problem and numerical method. For partial differential equations such techniques are usually based on some form of ellipticity/monotonicity in some part of the equation considered. The other terms are then dominated by this 'nice' part. In our case, for convection-diffusion problems, the convective terms are dominated by the elliptic diffusion terms, which, after the application of Gronwall's inequality leads to error estimates that blow up exponentially with respect to the diffusion parameter $\varepsilon \to 0$. Moreover this technique cannot be applied for purely convective problems, where the elliptic/monotone term is missing.

The fact that the DG scheme performs well for small or vanishing diffusion ε and even for the purely convective case is well known. When applied to smooth solutions, we know from practice that the error does not blow up exponentially, but rather stays bounded with respect to $\varepsilon \to 0$. Many numerical experiments confirming this

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can be found throughout the literature for various discretizations in time and varying ε, very small ε and $\varepsilon = 0$. For example, for the implicit-explicit (IMEX) variants of the backward difference formulas applied to the DG scheme, such results are contained in the papers [7,9]. In [9], the experiments are especially interesting as they are performed on general, very unusual grids, *e.g.* based on nonconvex quadrilaterals. For a combination of IMEX and time-DG, results are presented in the paper [22]. For explicit schemes and small ε, such results can be found in [8]. A comparison of small ε and $\varepsilon = 0$ and other similar numerical experiments, we refer to the recent book [6]. For purely convective problems, *i.e.* $\varepsilon = 0$, namely for the Euler equations, such results are obtained *e.g.* in [3,11]. Other works include for example [5,13]. In the presented paper we prove these observed results theoretically.

We will follow the ideas of Zhang and Shu [24], who developed a technique for *a priori* analysis of explicit time stepping DG schemes for convective problems. The technique is based on a specific estimate of the convective form which leads to the following: If the error is of the order $O(h^{(1+d)/2})$, where d is the spatial dimension of the computational domain, then we can prove the error estimate of the order $O(h^{p+1/2})$, where p is the spatial approximation order. A bootstrapping argument using mathematical induction is then applied to remove the *a priori* $O(h^{(1+d)/2})$ assumption. The argument works for explicit schemes under the assumption $p > (1 + d)/2$.

In [16], the technique of Zhang and Shu was extended to the space semidiscretized DG and to the implicit Euler scheme. There it is proved that for implicit schemes, more information about the discrete solution is necessary to perform the bootstrapping argument. In [16], this difficulty is overcome by constructing a suitable continuation of the discrete solution with respect to time. The error analysis is then performed for the continued discrete solution, which implies error estimates for the original discrete solution. In the presented paper, we generalize these ideas to the BDF2, midpoint and quadrature version of time DG schemes. Specifically, we construct suitable continuations for these three schemes and then apply the induction argument presented in [16]. The quadrature time-DG scheme is especially interesting, since the continuation then depends on two variables, one of which is used for the induction argument, while the other represents the time variable of the original problem. In this case, the construction of the continuation is not complicated, however proving its necessary properties needed in the analysis is rather technical. Moreover, we were able to carry out the analysis only for the scheme where a quadrature formula in time is applied to the nonlinear terms. We do not view this as a limitation, since for practical computations, one must apply some form of quadrature to these terms anyway in order to evaluate them.

The structure of the paper is as follows. In Sections 2 and 3, we introduce the continuous problem, its spatial discretization by the DG method and the three considered time discretizations. In Section 4 we review the basic tools for our analysis, such as the basic estimate of the convective terms. Sections 5 and 6 deal with the analysis of the BDF2 and midpoint rules. We prove $O(h^{p+1/2} + \varepsilon h^p + \tau^2)$ error estimates in the $L^\infty(L^2)$-norm with the constant in the estimate independent of ε, h and τ. The estimates are derived under the $\tau = O(h)$ and $p > 1 + d/2$ conditions. For $\varepsilon = 0$, we obtain the weaker condition $p > (1 + d)/2$ and the estimate of order $O(h^{p+1/2} + \tau^2)$.

Finally, Section 7 deals with the quadrature variant of the time-DG scheme. Under the same assumptions as for the BDF2 and midpoint schemes, we prove estimates of the order $O(h^{p+1/2} + \varepsilon h^p + \tau^{q+1})$, where q is the approximation order in time.

2. Continuous problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded polyhedral domain and $T > 0$. We set $Q_T = \Omega \times (0, T)$. Let us consider the following problem: Find $u : Q_T \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial t} + \nabla \cdot f(u) - \varepsilon\, \Delta u = g \quad \text{in } Q_T, \tag{2.1}$$
$$u\big|_{\partial\Omega\times(0,T)} = 0,$$
$$u(x, 0) = u^0(x), \quad x \in \Omega.$$

Here $f = (f_1, \ldots, f_d)$, $f_s \in C^2(\mathbb{R}) \cap W^{2,\infty}(\Omega)$, $f_s(0) = 0$, $s = 1, \ldots, d$ represents convective terms, $\varepsilon \geq 0$, $g \in C([0,T]; L^2(\Omega))$ and $u^0 \in L^2(\Omega)$ is an initial condition. We assume that the weak solution of (2.1) is sufficiently regular and we will specify the exact assumptions on the smoothness of the weak solution for each time discretization method individually.

We note that in [16], mixed Dirichlet–Neumann boundary conditions are treated along with only locally Lipschitz nonlinearities $f_s \in C^2(\mathbb{R})$. This is also possible in our context, however we stay in the simpler setting to avoid too many technicalities.

To simplify the notation, we use $(\cdot,\cdot)$ to denote the L^2 scalar product and $\|\cdot\|$ for the L^2 norm. To further simplify notation, we shall drop the argument Ω in Sobolev norms, *e.g.* $\|\cdot\|_{H^{p+1}}$ denotes the $H^{p+1}(\Omega)$-norm. We will also denote the Bochner norms over the whole interval $(0,T)$ in concise form, *e.g.* $\|u\|_{L^\infty(H^{p+1})}$ denotes the $L^\infty(0,T;H^{p+1}(\Omega))$-norm.

3. Discrete problem

3.1. Space discretization

Let $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ be a system of partitions of $\overline{\Omega}$ into a finite number of closed d-dimensional simplices K with mutually disjoint interiors. Let $\mathcal{F}_h$ the system of all faces (edges in 2D) of $\mathcal{T}_h$ and let $\mathcal{F}_h^I$ be the set of interior edges and $\mathcal{F}_h^B$ the set of boundary edges. For each $\Gamma \in \mathcal{F}_h$ we fix a unit normal $\mathbf{n}_\Gamma$, which for $\Gamma \in \mathcal{F}_h^B$ has the same orientation as the outer normal to Ω. For each $\Gamma \in \mathcal{F}_h^I$ there exist two neighbours $K_\Gamma^{(L)}, K_\Gamma^{(R)} \in \mathcal{T}_h$ such that $\mathbf{n}_\Gamma$ is the outer normal to $K_\Gamma^{(L)}$. For v piecewise defined on $\mathcal{T}_h$ and $\Gamma \in \mathcal{F}_h^I$ we introduce $v|_\Gamma^{(L)}$ as the trace of $v|_{K_\Gamma^{(L)}}$ on Γ, $v|_\Gamma^{(R)}$ as the trace of $v|_{K_\Gamma^{(R)}}$ on Γ, $\langle v\rangle_\Gamma = \frac{1}{2}\big(v|_\Gamma^{(L)} + v|_\Gamma^{(R)}\big)$ and $[v]_\Gamma = v|_\Gamma^{(L)} - v|_\Gamma^{(R)}$. On $\partial\Omega$, we define $v|_\Gamma^{(L)}$ as the trace of $v|_{K_\Gamma^{(L)}}$, *i.e.* on the element adjacent to Γ and $v|_\Gamma^{(R)} = 0$ corresponds to the homogeneous Dirichlet boundary conditions. If $[\cdot]_\Gamma, \langle\cdot\rangle_\Gamma, v|_\Gamma^{(L)}, v|_\Gamma^{(R)}$ appear in an integral over $\Gamma \in \mathcal{F}_h$, we omit the subscript Γ. Let

$$S_h = \{w;\ w|_K \in P_p(K),\ \forall K \in \mathcal{T}_h\}$$

denote the space of discontinuous piecewise polynomial functions of degree p on each $K \in \mathcal{T}_h$. We say that the function $u_h \in C^1(0,T;S_h)$ is the *semi-discrete approximate solution* of (2.1) if it satisfies the equation

$$\left(\frac{\partial u_h}{\partial t}(t), w\right) + \varepsilon A_h(u_h(t), w) + b_h(u_h(t), w) = \ell_h(w)\,(t) \quad \forall\, w \in S_h,\ \forall\, t \in [0,T],$$

and $(u_h(0), w) = (u^0, w)\ \forall\, w \in S_h$. Here the following forms are used: The *convective form*

$$b_h(v,\varphi) = -\sum_{K\in\mathcal{T}_h}\int_K f(v)\cdot\nabla\varphi\,\mathrm{d}x + \int_{\mathcal{F}_h^I} H(v^{(L)}, v^{(R)}, \mathbf{n})[\varphi]\mathrm{d}S + \int_{\mathcal{F}_h^B} H(v^{(L)}, v^{(R)}, \mathbf{n})\varphi^{(L)}\mathrm{d}S,$$

the diffusion terms are defined as

$$A_h(v,\varphi) = a_h(v,\varphi) + J_h(v,\varphi),$$

where the bilinear *diffusion form* corresponding to the *symmetric interior penalty Galerkin* (SIPG) is

$$a_h(v,\varphi) = \sum_{K\in\mathcal{T}_h}\int_K \nabla v\cdot\varphi\,\mathrm{d}x - \int_{\mathcal{F}_h^I}\langle\nabla v\rangle\cdot\mathbf{n}[\varphi]\mathrm{d}S - \int_{\mathcal{F}_h^I}\langle\nabla\varphi\rangle\cdot\mathbf{n}[v]\mathrm{d}S - \int_{\mathcal{F}_h^B}\nabla v\cdot\mathbf{n}\varphi\,\mathrm{d}S - \int_{\mathcal{F}_h^B}\nabla\varphi\cdot\mathbf{n}v\,\mathrm{d}S$$

and the *interior and boundary penalty jump terms* are defined by

$$J_h(v,\varphi) = \int_{\mathcal{F}_h^I}\sigma[v][\varphi]\mathrm{d}S + \int_{\mathcal{F}_h^B}\sigma v\varphi\,\mathrm{d}S.$$

Here the parameter σ is constant on every edge and defined by $\sigma|_\Gamma = C_W/|\Gamma|$ for all $\Gamma \in \mathcal{F}_h$, where $C_W > 0$ is a constant, which must be chosen large enough to ensure coercivity of the form A_h (*cf. e.g.* [10]).

Finally, we have the *right-hand side form*:

$$l_h(\varphi)(t) = \int_\Omega g(t)\varphi \mathrm{d}x.$$

As stated earlier, this is the case of homogeneous Dirichlet boundary conditions, for general mixed Dirichlet–Neumann conditions A_h has a more complicated form (*cf.* [16]), which we do not consider for simplicity.

We assume the numerical fluxes H in the convective form b_h to be Lipschitz continuous, conservative and consistent. Moreover, we assume that the numerical fluxes are E-fluxes:

$$(H(v,w,n) - f(q)\cdot n)(v-w) \geq 0, \quad \forall v, w \in \mathbb{R},\ \forall q \text{ between } v \text{ and } w,$$

where $n \in \mathbb{R}^d$ is an unit vector, *cf. e.g.* [2,18] for details.

We find that a sufficiently regular weak solution of (2.1) satisfies the identity

$$\left(\frac{\partial u}{\partial t}(t), w\right) + \varepsilon A_h(u(t), w) + b_h(u(t), w) = \ell_h(w)\,(t) \tag{3.1}$$

for all $w \in S_h$ and all $t \in (0,T)$.

Throughout this paper, we assume the mesh system $\{\mathcal{T}_h\}_{h\in(0,h_0)}$ to be shape regular, satisfying the inverse assumption [4].

3.2. Time discretization

For simplicity we assume a uniform time partition $t_m = m\tau$, $m = 0,\dots,r$ with time intervals $I_m = (t_{m-1}, t_m)$ and with the time step $\tau = T/r = |I_m|$. To simplify the notation, we set $v^m = v(t_m)$.

3.2.1. BDF2

Definition 3.1. The set of functions $U^m \in S_h,\ m = 0,\dots,r$ is an approximate solution of problem (2.1) obtained by the BDF2-DG scheme if for all $w \in S_h$

$$\left(\frac{3}{2}U^m - 2U^{m-1} + \frac{1}{2}U^{m-2}, w\right) + \tau\varepsilon A_h(U^m, w) + \tau b_h(U^m, w) = \tau\ell_h(w)(t_m) \tag{3.2}$$

for $m \geq 2$. For $m = 1$ we define U^1 by

$$(U^1 - U^0, w) + \tau\varepsilon A_h(U^1, w) + \tau b_h(U^1, w) = \tau\ell_h(w)(t_m), \quad \forall w \in S_h. \tag{3.3}$$

The initial condition $U^0 \in S_h$ is the $L^2(\Omega)$-projection of u^0 onto S_h, *i.e.*

$$(U^0, w) = (u^0, w), \quad \forall w \in S_h. \tag{3.4}$$

Remark 3.2. Since the BDF2 is a 2-step method, we need to specify two initial values U^0 and U^1 to start the method. The value U^0 can be obtained by L^2 projection of initial condition u^0 and U^1 can be obtained by one step of the implicit Euler method. In this case the resulting scheme does not lose its second order of accuracy in time.

3.2.2. Midpoint rule

Definition 3.3. The set of functions $U^m \in S_h,\ m = 0,\dots,r$ is an approximate solution of problem (2.1) obtained by the midpoint-DG scheme if

$$(U^m - U^{m-1}, w) + \frac{\tau\varepsilon}{2}A_h(U^m + U^{m-1}, w) + \tau b_h\left(\frac{U^m + U^{m-1}}{2}, w\right) = \tau\ell_h(w)(t_{m-1} + \tau/2), \quad \forall w \in S_h, \tag{3.5}$$

where U^0 is the initial condition obtained by (3.4).

3.2.3. Discontinuous Galerkin method in time

We define the space

$$S_h^\tau = \{v \in L^2(0,T;S_h) : v|_{I_m} = \sum_{j=0}^{q} v_j^{(m)} t^j,\ v_j^{(m)} \in S_h,\ m = 1,\dots,r\},$$

which represents the space of piecewise polynomials up to degree p in space and up to degree q in time. For the functions from such a space we need to define one-sided values at nodes of the time partition:

$$v_\pm^m = v(t_m\pm) = \lim_{t\to t_m\pm} v(t)$$

and the jumps

$$\{v\}_m = v_+^m - v_-^m.$$

Definition 3.4. The function $U \in S_h^\tau$ is an approximate solution of problem (2.1) obtained by the space-time discontinuous Galerkin scheme if for all $w \in S_h^\tau$,

$$\int_{I_m} (U', w) + \varepsilon A_h(U,w) + b_h(U,w)\mathrm{d}t + \left(\{U\}_{m-1}, w_+^{m-1}\right) = \int_{I_m} \ell_h(w)(t)\mathrm{d}t,$$

for all $m = 1,\dots,r$. Here $U_-^0 := U^0$ is the initial condition obtained by (3.4).

Let us define the Radau quadrature on each interval I_m:

$$\int_{I_m} \Phi(t)\mathrm{d}t \approx Q_\tau^m[\Phi] := \tau \sum_{i=0}^{q} \omega_i \Phi(t_{m-1} + \tau\psi_i),$$

where ψ_i are Radau quadrature nodes in $[0,1]$ with $\psi_q = 1$. Such a quadrature has algebraic order $2q$ and the quadrature weights are positive and satisfy

$$\sum_{i=0}^{q} \omega_i = 1.$$

When we apply Radau quadrature to the integrals in Definition 3.4 we obtain the *quadrature version of the time-DG scheme*.

Definition 3.5. The function $U \in S_h^\tau$ is an approximate solution of problem (2.1) obtained by the quadrature time discontinuous Galerkin (QT-DG) scheme if for all $w \in S_h^\tau$

$$\int_{I_m} (U', w) + \varepsilon A_h(U,w)\mathrm{d}t + Q_\tau^m[b_h(U,w)] + (\{U\}_{m-1}, w_+^{m-1}) = Q_\tau^m[\ell_h(w)(t)], \tag{3.6}$$

for all $m = 1,\dots,r$. Here $U_-^0 := U^0$, the initial condition obtained by (3.4).

Remark 3.6. We note that the first integral in (3.6) does not need to be approximated by quadrature. Due to the linearity of the terms (U', w) and $A_h(U,w)$ w.r.t. both arguments, these terms are a polynomial of degree at most $2q$ on each I_m and can therefore be integrated exactly by Radau quadrature. However, due to the nonlinearity of the convective fluxes f_s, the term $b_h(U,w)$ cannot be, in general, integrated analytically w.r.t. time and quadrature must be applied in practice. The same holds for the right-hand side form $l_h(w)(t)$ containing the general function g.

Remark 3.7. The numerical solution U from Definition 3.4 or 3.5 is constructed on each I_m independently, inductively for $m = 1,\dots,r$, with only U_-^{m-1} coming from the previous time interval I_{m-1} or the initial condition U^0.

4. Auxiliary results

We denote the energy norm $|||w|||^2 := A_h(w,w)$ for all $w \in S_h$. Note that the inverse inequality takes the following form $|||w||| \le Ch^{-1}\|w\|$ for $w \in S_h$. Let Π be the $L^2(\Omega)$-orthogonal projection on S_h.

Throughout this work we denote by C a generic constant independent of h, τ, t and the diffusion coefficient ε.

Lemma 4.1. *Let $u \in W^{1,\infty}(H^{p+1})$. Then*

$$\|\Pi u(t) - u(t)\| \le Ch^{p+1}|u(t)|_{H^{p+1}}, \tag{4.1}$$
$$\|\Pi u'(t) - u'(t)\| \le Ch^{p+1}|u'(t)|_{H^{p+1}}, \tag{4.2}$$
$$\big((\Pi u - u)(s_1) - (\Pi u - u)(s_2), w\big) \le C|s_1 - s_2|h^{p+1}\|u\|_{W^{1,\infty}(H^{p+1})}\|w\|, \tag{4.3}$$

for all $w \in S_h$ and $s_1, s_2, t \in [0,T]$.

Proof. Estimates (4.1) and (4.2) are a standard estimate for the $L^2(\Omega)$-projection approximation. Estimate (4.3) can be found *e.g.* in [9]. □

We summarize the properties of the forms A_h and b_h.

Lemma 4.2. *Let $u \in H^{p+1}(\Omega)$. Then*

$$A_h(v,w) \le C|||v|||\,|||w|||, \quad \forall v, w \in S_h, \tag{4.4}$$
$$A_h(\Pi u - u, w) \le Ch^p|||w|||, \quad \forall w \in S_h. \tag{4.5}$$

Proof. The proof of (4.4) and (4.5) can be done in a similar way as in ([8], Lem. 9). □

Lemma 4.3. *Let $u \in H^{p+1}(\Omega) \cap W^{1,\infty}(\Omega)$. Then*

$$b_h(v,w) - b_h(\bar v, w) \le C\|v - \bar v\|\,|||w|||, \quad \forall v, \bar v, w \in S_h, \tag{4.6}$$
$$b_h(v, v - \Pi u) - b_h(u, v - \Pi u) \le C\left(1 + \frac{\|v-u\|_\infty^2}{h^2}\right)(h^{2p+1} + \|v - \Pi u\|^2), \quad \forall v \in S_h. \tag{4.7}$$

Proof. The proof of (4.6) can be found in [8]. The proof of (4.7) is essentially the same as that of ([16], Lem. 5.1), however there the statement and proof are written for the specific choice $v := u_h, \xi := u_h - \Pi u$. □

In the following analyses, it will be important to eliminate the unpleasant term $\|e(t)\|_\infty^2/h^2$ in (4.7), where $e = u_h - u$. This is possible if we know *a priori* that $\|e(t)\|_\infty = O(h)$. Since we are concerned in $L^2(\Omega)$-estimates, we want to reformulate this in terms of the $L^2(\Omega)$-norm. The following result is proven in [16], we include the proof here for convenience:

Lemma 4.4. *Let $p \ge d/2$ and u satisfy the regularity assumptions (5.1). Then*

$$\|e(t)\| \le h^{1+d/2} \quad \Longrightarrow \quad \|e(t)\|_\infty \le Ch,$$

where C is independent of h, ε, t.

Proof. We write the error as $e(t) = \eta(t) + \xi(t)$, where $\eta = \Pi u - u$ and $\xi = u_h - \Pi u \in S_h$. Due to standard approximation properties of Π and the inverse inequality between the L^∞ and L^2-norms, we have

$$\begin{aligned}\|e(t)\|_\infty &\le \|\eta(t)\|_\infty + \|\xi(t)\|_\infty \le Ch|u(t)|_{W^{1,\infty}} + Ch^{-d/2}\|\xi(t)\| \\ &\le Ch + Ch^{-d/2}\|e(t)\| + Ch^{-d/2}\|\eta(t)\| \le Ch + Ch^{p+1-d/2}|u(t)|_{H^{p+1}} \le Ch.\end{aligned}$$ □

5. Error estimates for BDF2

We want to estimate the error $e_h^m = U^m - u^m$, where the values of U^m are obtained by the BDF2-DG method. To do so, we construct a suitable continuation $U(t)$ (*i.e.* continuous function with respect to time) such that $U(t_m) = U^m$. Then we can also generalize the error as $e_h = U - u$. Our aim is to investigate the generalized error at arbitrary time $t \in (0, T)$ and prove a suitable *a priori* error bound. Then the error bound for the BDF2-DG method is a trivial consequence of the more general error estimate. For the purpose of analysis of the BDF2-DG scheme we assume following regularity

$$u \in W^{1,\infty}(H^{p+1}) \cap L^\infty(W^{1,\infty}) \cap W^{3,\infty}(L^2). \tag{5.1}$$

Definition 5.1. We define the continued approximate solution $U : [0, T] \to S_h$ of problem (2.1) obtained by the BDF2-DG scheme in the following way: Let $m \geq 2$ and $s \in [0, \tau]$, we seek $U(t_{m-1} + s) \in S_h$ such that

$$\begin{aligned}\Big(\frac{\tau+2s}{\tau+s}U(t_{m-1}+s) - \frac{\tau+s}{\tau}U^{m-1} + \frac{s^2}{\tau^2+\tau s}U^{m-2}, w\Big) + s\varepsilon A_h(U(t_{m-1}+s), w) + sb_h(U(t_{m-1}+s), w)\\ = s\ell_h(w)(t_{m-1}+s), \quad \forall w \in S_h.\end{aligned} \tag{5.2}$$

This defines U on I_m for $m \geq 2$. For $m = 1$ we define U on I_1 by seeking $U(s) \in S_h$ such that

$$(U(s) - U^0, w) + s\varepsilon A_h(U(s), w) + sb_h(U(s), w) = s\ell(w)(s), \quad \forall w \in S_h. \tag{5.3}$$

Remark 5.2. Equation (5.3) was already used for general m in [16] to define the continuation of the implicit Euler scheme. It represents the implicit Euler method with a variable time step s. By taking $s = 0$, we get $U(0) = U^0$, while setting $s = \tau$, we get $U(\tau) = U(t_1) = U^1$ and it can be proven that between these two values, $U(\cdot)$ changes continuously.

The motivation for (5.2) is similar. This equation is in fact the backward difference formula with variable time step, *cf*. [14]. Setting $s = 0$, we get $U(t_{m-1}) = U^{m-1}$, while setting $s = \tau$, we recover the original BDF2-DG scheme (3.2), hence $U(t_m) = U^m$. Similarly as in [16], we shall prove that between the $s = 0$ and $s = \tau$, $U(\cdot)$ changes continuously.

Lemma 5.3. *There exist constants $C_1, C_2 > 0$ independent of h, τ, t, ε, such that the following holds. Let $h \in (0, h_0)$ and $\tau \in [0, \tau_0)$, where $\tau_0 = \max\{C_1\varepsilon, C_2 h\}$. Then U, the continued solution from Definition 5.1 exists, is uniquely determined, $\|U(t)\|$ is uniformly bounded with respect to $t \in [0, T]$, $U(t_m) = U^m$ for all $m = 0, \ldots, r$ and $\|U(t)\|$ depends continuously on t.*

Proof. For $m = 1$, it is already proven in [16] that the resulting solution U is continuous on I_1 and $U(0) = U^0, U(t_1) = U(\tau) = U^1$. Therefore it is sufficient to consider the case $m \geq 2$.

(i) **Existence**: Let $m \geq 2$ and $s \in [0, \tau]$, we consider U on I_m. We denote the left- and right-hand sides from (5.2):

$$\begin{aligned}B_s(v, w) &= \frac{\tau+2s}{\tau+s}(v, w) + s\varepsilon A_h(v, w) + sb_h(v, w),\\ L_s^m(w) &= \left(\frac{\tau+s}{\tau}U^{m-1} - \frac{s^2}{\tau^2+\tau s}U^{m-2}, w\right) + s\ell_h(w)(t_{m-1}+s).\end{aligned}$$

We will show that B_s is strictly monotone and Lipschitz continuous on S_h equipped with the $L^2(\Omega)$-scalar product. Existence and uniqueness then follows from the nonlinear Lax–Milgram lemma, *cf*. [23].

Monotonicity: using the ellipticity of A_h, the boundedness of b_h and the inverse inequality, we get

$$\begin{aligned}B_s(v, v-w) - B_s(w, v-w) &\geq \frac{\tau+2s}{\tau+s}\|v-w\|^2 + s\varepsilon |\!|\!| v-w |\!|\!|^2 - Cs\|v-w\| |\!|\!| v-w |\!|\!|\\ &\geq \left(1 - \frac{Cs}{h}\right)\|v-w\|^2 = M\|v-w\|^2, \quad \forall v, w \in S_h,\end{aligned}$$

for s, τ sufficiently small with respect to h.

On the other hand, we may estimate using Young's inequality:

$$\begin{aligned} B_s(v, v-w) - B_s(w, v-w) &\geq \frac{\tau+2s}{\tau+s}\|v-w\|^2 + s\varepsilon |\!|\!| v-w |\!|\!|^2 - Cs\|v-w\| |\!|\!| v-w |\!|\!| \\ &\geq \|v-w\|^2 + s\varepsilon |\!|\!| v-w |\!|\!|^2 - s\varepsilon |\!|\!| v-w |\!|\!|^2 - \frac{C^2 s}{4\varepsilon}\|v-w\|^2 \\ &\geq \left(1 - \frac{C^2 s}{4\varepsilon}\right)\|v-w\|^2 = M\|v-w\|^2, \quad \forall v, w \in S_h, \end{aligned} \tag{5.4}$$

in this case we get the condition s, τ sufficiently small with respect to ε.

Lipschitz continuity: We shall show that B_s is Lipschitz continuous:

$$\begin{aligned} B_s(v, w) - B_s(\bar v, w) &\leq \frac{3}{2}\|v-\bar v\|\,\|w\| + Cs\varepsilon |\!|\!| v-\bar v |\!|\!|\, |\!|\!| w |\!|\!| + Cs\|v-\bar v\|\, |\!|\!| w |\!|\!| \\ &\leq \Big(\frac{3}{2} + \frac{Cs\varepsilon}{h^2} + \frac{Cs}{h}\Big)\|v-\bar v\|\,\|w\| = L\|v-\bar v\|\,\|w\|. \end{aligned}$$

Since the right-hand side L_s^m is a linear functional on the finite-dimensional space S_h, it is also bounded and by the nonlinear Lax–Milgram lemma we obtain the existence and uniqueness of the continued discrete solution and classical discrete solution, respectively. Finally, we obtain the uniform boundedness of $\|U(t)\|$ w.r.t. $t \in I_m$, since the nonlinear Lax–Milgram lemma gives us $\|U(t)\| \leq C\|L_s^m\|_{\mathcal{L}(L^2(\Omega),\mathbb{R})}$, which can be bounded independent of s similarly as in [16].

Since we have existence and uniqueness, we see that $U(t_m) = U^m$ by setting $s = \tau$ in (5.2).

(ii) **Continuity**: Now we show that the continued discrete solution is continuous with respect to time. Let $m > 1$ and $t, \bar t \in (t_{m-1}, t_m]$ and $s = t - t_{m-1},\ \bar s = \bar t - t_{m-1}$. Then by monotonicity,

$$\begin{aligned} M\|U(t) - U(\bar t)\|^2 &\leq B_t(U(t), U(t) - U(\bar t)) - B_t(U(\bar t), U(t) - U(\bar t)) \\ &= L_t^m(U(t) - U(\bar t)) - L_{\bar t}^m(U(t) - U(\bar t)) + B_{\bar t}(U(\bar t), U(t) - U(\bar t)) - B_t(U(\bar t), U(t) - U(\bar t)). \end{aligned} \tag{5.5}$$

We estimate the B and L terms individually.

$$\begin{aligned} &|B_{\bar t}(U(\bar t), U(t) - U(\bar t)) - B_t(U(\bar t), U(t) - U(\bar t))| \\ &\quad\leq \Big|\frac{\tau+2\bar s}{\tau+\bar s} - \frac{\tau+2s}{\tau+s}\Big|\|U(\bar t)\|\|U(t) - U(\bar t)\| + |\bar s - s|\varepsilon A_h(U(\bar t), U(t) - U(\bar t)) + |\bar s - s| b_h(U(\bar t), U(t) - U(\bar t)) \\ &\quad\leq |\bar s - s|\big(\tau + \frac{C\varepsilon}{h^2} + + \frac{C}{h}\big)\|U(\bar t)\|\|U(t) - U(\bar t)\|. \end{aligned} \tag{5.6}$$

Similarly we get

$$\begin{aligned} &|L_t^m(U(t) - U(\bar t)) - L_{\bar t}^m(U(t) - U(\bar t))| \leq \Big(\Big|\frac{\tau+\bar s}{\tau} - \frac{\tau+s}{\tau}\Big|\|U^{m-1}\| + \Big|\frac{\bar s^2}{\tau^2+\tau\bar s} - \frac{s^2}{\tau^2+\tau s}\Big|\|U^{m-2}\|\Big)\|U(t) - U(\bar t)\| \\ &\quad + |s\ell_h(U(t) - U(\bar t))(t) - \bar s\ell_h(U(t) - U(\bar t))(\bar t)| \\ &\leq |\bar s - s|\big(\tau^{-1}\|U^{m-1}\| + 3\tau^{-1}\|U^{m-2}\|\big)\|U(t) - U(\bar t)\| + |s\ell_h(U(t) - U(\bar t))(t) - \bar s\ell_h(U(t) - U(\bar t))(\bar t)|. \end{aligned} \tag{5.7}$$

Assuming $|\bar s - s| = |\bar t - t| \to 0$, we get the limit for the terms on the last row

$$|s\ell_h(U(t) - U(\bar t))(t) - \bar s\ell_h(U(t) - U(\bar t))(\bar t)| \leq \underbrace{|s - \bar s|}_{\to 0}\underbrace{|\ell_h(U(t) - U(\bar t))(t)|}_{\text{bounded}} + \bar s|(\underbrace{g(t) - g(\bar t)}_{\to 0}, \underbrace{U(t) - U(\bar t)}_{\text{bounded}})| \to 0,$$

since $\|U(t)\|, \|U(\bar t)\|$ are uniformly bounded with respect to $t, \bar t \in (t_{m-1}, t_m]$.

From now it is possible to see that the terms in (5.6) and (5.7) tend to zero as $|t - \bar t|$ tends to zero. Together with (5.5) we get

$$\|U(t) - U(\bar t)\| \to 0 \text{ as } |\bar s - s| = |\bar t - t| \to 0.$$

Now, we prove the continuity at t_{m-1}, *i.e.* $U(t_{m-1}+s) \to U^{m-1}$ as s tends to 0+. Since $\frac{\tau+2s}{\tau+s} \to 1$, $\frac{\tau+s}{\tau} \to 1$, $\frac{s^2}{\tau^2+\tau s} \to 0$ and the terms $A_h(U(t_{m-1}+s), w)$, $b_h(U(t_{m-1}+s), w)$ and $\ell_h(w)$ are bounded, we get from (5.2)

$$\underbrace{\frac{\tau+2s}{\tau+s}(U(t_{m-1}+s),w)}_{\to(U(t_{m-1}+s),w)} - \underbrace{\frac{\tau+s}{\tau}(U^{m-1},w)}_{\to(U^{m-1},w)} + \underbrace{\frac{s^2}{\tau^2+\tau s}(U^{m-2},w)}_{\to 0}$$
$$+ \underbrace{s\varepsilon A_h(U(t_{m-1}+s),w) + s b_h(U(t_{m-1}+s),w)}_{\to 0} = \underbrace{s\ell_h(w)(t_{m-1}+s)}_{\to 0},$$

i.e. continuity at t_{m-1}.

It remains to prove continuity of $U(\cdot)$ on I_1. In the case of computing initial condition by (3.3), we can continuate the solution on $I_1 = [0, \tau]$ by

$$(U(s) - U^0, w) + s\varepsilon A_h(U(s), w) + s b_h(U(s), w) = s\ell(w)(s).$$

It is already proved in [16] that such a continuation is continuous on $[0, \tau]$. □

Due to the regularity assumptions (5.1) the exact solution $u \in C([0,T]; L^2(\Omega))$ and therefore uniformly continuous on the closed interval $[0, T]$. Therefore, by Lemma 5.3, the error $e_h = U(t) - u(t)$ is also uniformly continuous. We divide the error $e_h = \xi + \eta$, where $\xi = U - \Pi u$ and $\eta = \Pi u - u$.

Lemma 5.4. *Let u satisfy regularity assumptions* (5.1). *Let $s \in (0, \tau]$. Then*

$$\left(\frac{\tau+2s}{\tau+s}u(t_{m-1}+s) - \frac{\tau+s}{\tau}u^{m-1} + \frac{s^2}{\tau^2+\tau s}u^{m-2} - su'(t_{m-1}+s), w\right) \le Cs\tau^2 \|u\|_{W^{3,\infty}(L^2)}\|w\|, \tag{5.8}$$

$$\left(u(s) - u^0 - su'(s), w\right) \le Cs\tau\|u\|_{W^{2,\infty}(L^2)}\|w\|, \tag{5.9}$$

$$\left(\frac{\tau+2s}{\tau+s}\eta(t_{m-1}+s) - \frac{\tau+s}{\tau}\eta^{m-1} + \frac{s^2}{\tau^2+\tau s}\eta^{m-2}, w\right) \le Csh^{p+1}\|u\|_{W^{1,\infty}(H^{p+1})}\|w\|. \tag{5.10}$$

Proof. Let us denote $y = t_{m-1} + s$. Since

$$\frac{\tau+s}{\tau} - \frac{s^2}{\tau^2+\tau s} = \frac{\tau+2s}{\tau+s}, \qquad \frac{\tau s+s^2}{\tau} - \frac{\tau s^2+s^3}{\tau^2+\tau s} = s, \qquad \frac{\tau s^2+s^3}{2\tau} - \frac{s^2(\tau+s)^2}{2\tau^2+2\tau s} = 0,$$

we can formally rewrite

$$\begin{aligned}\frac{\tau+2s}{\tau+s}u(y) - \frac{\tau+s}{\tau}u^{m-1} + \frac{s^2}{\tau^2+\tau s}u^{m-2} - su'(y) &= \frac{\tau+s}{\tau}\left(u(y) - su'(y) + \frac{s^2}{2}u''(y) - u^{m-1}\right)\\ &\quad - \frac{s^2}{\tau^2+\tau s}\left(u(y) - (\tau+s)u'(y) + \frac{(\tau+s)^2}{2}u''(y) - u^{m-2}\right)\end{aligned} \tag{5.11}$$

and

$$\frac{\tau+2s}{\tau+s}\eta(y) - \frac{\tau+s}{\tau}\eta^{m-1} + \frac{s^2}{\tau^2+\tau s}\eta^{m-2} = \frac{\tau+s}{\tau}(\eta(y) - \eta^{m-1}) - \frac{s^2}{\tau^2+\tau s}(\eta(y) - \eta^{m-2}). \tag{5.12}$$

Then it is simple to see

$$\begin{aligned}\frac{\tau+s}{\tau}\left(u(y) - su'(y) + \frac{s^2}{2}u''(y) - u^{m-1}, w\right) &= \frac{\tau+s}{\tau}\int_{t_{m-1}}^{y}\int_{z_1}^{y}\int_{z_2}^{y}(u'''(z_3), w)\mathrm{d}z_3\mathrm{d}z_2\mathrm{d}z_1\\ \le \frac{\tau+s}{\tau}\|u\|_{W^{3,\infty}(L^2)}\|w\|\int_{t_{m-1}}^{y}\int_{z_1}^{y}\int_{z_2}^{y}1\mathrm{d}z_3\mathrm{d}z_2\mathrm{d}z_1 &= \frac{\tau+s}{\tau}\frac{s^3}{6}\|u\|_{W^{3,\infty}(L^2)}\|w\| \le Cs\tau^2\|u\|_{W^{3,\infty}(L^2)}\|w\|\end{aligned}$$

and

$$\frac{s^2}{\tau^2+\tau s}\left(u(y)-(\tau+s)u'(y)+\frac{(\tau+s)^2}{2}u''(y)-u^{m-2},w\right)=\frac{s^2}{\tau^2+\tau s}\int_{t_{m-2}}^{y}\int_{z_1}^{y}\int_{z_2}^{y}(u'''(z_3),w)\mathrm{d}z_3\mathrm{d}z_2\mathrm{d}z_1$$
$$\le \frac{s^2}{\tau^2+\tau s}\|u\|_{W^{3,\infty}(L^2)}\|w\|\int_{t_{m-2}}^{y}\int_{z_1}^{y}\int_{z_2}^{y}1\mathrm{d}z_3\mathrm{d}z_2\mathrm{d}z_1$$
$$=\frac{s^2}{\tau(\tau+s)}\frac{(\tau+s)^3}{6}\|u\|_{W^{3,\infty}(L^2)}\|w\|\le Cs\tau^2\|u\|_{W^{3,\infty}(L^2)}\|w\|,$$

which proves (5.8). The proof of (5.9) follows from

$$(u(s)-u^0-su'(s),w)=-\int_0^s\int_{z_1}^s(u''(z_2),w)\mathrm{d}z_2\mathrm{d}z_1\le\frac{1}{2}s^2\|u\|_{W^{2,\infty}(L^2)}\|w\|.$$

The proof of (5.10) follows directly from (5.12) and Lemma 4.1. □

In the proof of the error estimate of Lemma 5.7, we will need to estimate the BDF coefficients at $U(t_{m-1}+s)$, U^{m-1}, U^{m-2} in (5.2). For this purpose we define the sequence $\{\gamma_j\}_{j=0}^{\infty}$ by

$$\begin{aligned}\gamma_0&=\frac{\tau+s}{\tau+2s},\\ \frac{3}{2}\gamma_1-\frac{\tau+s}{\tau}\gamma_0&=0,\\ \frac{3}{2}\gamma_2-2\gamma_1+\frac{s^2}{\tau^2+\tau s}\gamma_0&=0,\\ \frac{3}{2}\gamma_{j+2}-2\gamma_{j+1}+\frac{1}{2}\gamma_j&=0,\quad\forall j=1,2,3,\dots\end{aligned}\tag{5.13}$$

Lemma 5.5. *Let the sequence $\{\gamma_j\}_{j=0}^{\infty}$ be defined by (5.13). Then such a sequence is positive and bounded,* i.e. *$0<\gamma_j<\gamma_\infty$ for all $j=0,1,\dots$ for some $\gamma_\infty\in\mathbb{R}$. Moreover,*

$$\gamma_1-2\frac{s^2}{\tau^2+\tau s}\gamma_0>0\tag{5.14}$$

and for $j\ge1$ the sequence γ_j is increasing.

Proof. Let us calculate the initial values for γ_j. γ_0 is defined already by (5.13).

$$\begin{aligned}\gamma_1&=\frac{2}{3}\frac{(\tau+s)^2}{\tau(\tau+2s)},\\ \gamma_2&=\frac{8}{9}\frac{(\tau+s)^2}{\tau(\tau+2s)}-\frac{2}{3}\frac{s^2}{\tau(\tau+2s)}.\end{aligned}$$

From this (5.14) immediately follows. For $j=1,2,\dots$, γ_j are defined by a difference equation with the initial condition γ_1 and γ_2 and with the solution

$$\gamma_j=\left(\frac{(\tau+s)^2}{\tau(\tau+2s)}-\frac{s^2}{\tau(\tau+2s)}\right)+\left(\frac{1}{3}\frac{s^2}{\tau(\tau+2s)}-\frac{1}{9}\frac{(\tau+s)^2}{\tau(\tau+2s)}\right)\left(\frac{1}{3}\right)^{j-2}.$$

Since

$$\frac{(\tau+s)^2}{\tau(\tau+2s)} - \frac{s^2}{\tau(\tau+2s)} > 0,$$
$$\frac{1}{3}\frac{s^2}{\tau(\tau+2s)} - \frac{1}{9}\frac{(\tau+s)^2}{\tau(\tau+2s)} < 0,$$

we can see that the sequence γ_j is increasing, positive and bounded. □

Let us start with the result on the initial condition defined by (3.3).

Lemma 5.6. *Let $p > d/2$. Let $s \in (0,\tau]$. If $\|e(t)\| \le h^{1+d/2}$ for $t \in [0,s]$, then*

$$\sup_{t\in[0,s]} \|e(t)\|^2 \le \tilde{C}_T^2 (h^{2p+1} + \varepsilon h^{2p} + \tau^4),$$

where the constant $\tilde{C}_T$ is independent of h, τ, ε.

Proof. Since $U^0 = \Pi u^0$ we can see that $\|e^0\| \le Ch^{p+1}$. Multiplying (3.1) for $t = s$ by s, subtracting from (5.3) and adding several terms we get

$$\begin{aligned}\big(\xi(s) - \xi^0, w\big) + s\varepsilon A_h(\xi(s), w) &= \big(su'(s) - u(s) + u^0, w\big) - \big(\eta(s) - \eta^0, w\big)\\ &\quad + s\big(b_h\,(u(s), w) - b_h\,(U(s), w)\,\big) - s\varepsilon A_h(\eta(s), w).\end{aligned}$$

Setting $w = 2\xi(s)$ and using Lemmas 4.1, 4.2, 4.3 and 5.4, we get

$$\begin{aligned}\|\xi(s)\|^2 - \|\xi^0\|^2 + \|\xi(s) - \xi^0\|^2 + s\varepsilon|\!|\!|\xi(s)|\!|\!|^2 &\le C\tau\left(1 + \frac{\|e(s)\|_\infty^2}{h^2}\right)(h^{2p+1} + \|\xi(s)\|^2)\\ &\quad + C\tau^4 + Ch^{2p+2} + C\varepsilon h^{2p} + \frac{1}{2}\|\xi(s)\|^2.\end{aligned}$$

Using the assumptions and Lemma 4.4, we can get rid of the unpleasant term $\|e(s)\|_\infty^2/h^2$ and we get

$$\|\xi(s)\|^2 \le C(\|\xi^0\|^2 + h^{2p+1} + \varepsilon h^{2p} + \tau^4).$$

The proof is completed by taking similar estimates for η and the triangle inequality to estimate $e(s)$. □

Now, we extend Lemma 5.6 to the rest of $[0,T]$ by analyzing the BDF scheme (5.2).

Lemma 5.7. *Let $p > d/2$. Let $n > 0$ and $s \in (0,\tau]$. If $\|e(t)\| \le h^{1+d/2}$ for $t \in [0, t_{n-1}+s]$, then*

$$\sup_{t\in[0,t_{n-1}+s]} \|e(t)\|^2 \le C_T^2 (h^{2p+1} + \varepsilon h^{2p} + \tau^4),$$

where the constant C_T is independent of h, τ, ε.

Proof. To simplify the relations we set $y = t_{n-1} + s$. Multiplying (3.1) for $t = y$ by s, subtracting from (5.2) and adding several terms we get

$$\begin{aligned}&\left(\frac{\tau+2s}{\tau+s}\xi(y) - \frac{\tau+s}{\tau}\xi^{n-1} + \frac{s^2}{\tau^2+\tau s}\xi^{n-2}, w\right) + s\varepsilon A_h(\xi(y), w)\\ &\quad = \left(su'(y) - \frac{\tau+2s}{\tau+s}u(y) + \frac{\tau+s}{\tau}u^{n-1} - \frac{s^2}{\tau^2+\tau s}u^{n-2}, w\right) - \left(\frac{\tau+2s}{\tau+s}\eta(y) - \frac{\tau+s}{\tau}\eta^{n-1} + \frac{s^2}{\tau^2+\tau s}\eta^{n-2}, w\right)\\ &\qquad + s\,(b_h\,(u(y), w) - b_h\,(U(y), w)) - s\varepsilon A_h(\eta(y)), w).\end{aligned}$$

For lower time levels $m \leq n-1$ we obtain analogically

$$\left(\frac{3}{2}\xi^m - 2\xi^{m-1} + \frac{1}{2}\xi^{m-2}, w\right) + \tau\varepsilon A_h(\xi^m, w) = \left(\tau u'(t_m) - \frac{3}{2}u^m + 2u^{m-1} - \frac{1}{2}u^{m-2}, w\right)$$
$$- \left(\frac{3}{2}\eta^m - 2\eta^{m-1} + \frac{1}{2}\eta^{m-2}, w\right) + \tau\left(b_h\left(u^m, w\right) - b_h\left(U^m, w\right)\right) - \tau\varepsilon A_h(\eta^m, w).$$

Setting $w = 2\xi(y)$ we obtain on the left-hand side using the fact $s \in (0, \tau]$

$$2\left(\frac{\tau+2s}{\tau+s}\xi(y) - \frac{\tau+s}{\tau}\xi^{n-1} + \frac{s^2}{\tau^2+\tau s}\xi^{n-2}, \xi(y)\right) + 2s\varepsilon A_h(\xi(y), \xi(y))$$
$$= 2\frac{\tau+s}{\tau}(\xi(y) - \xi^{n-1}, \xi(y)) - 2\frac{s^2}{\tau^2+\tau s}(\xi(y) - \xi^{n-2}, \xi(y)) + 2s\varepsilon|||\xi(y)|||^2$$
$$= \frac{\tau+s}{\tau}(\|\xi(y)\|^2 - \|\xi^{n-1}\|^2 + \|\xi(y) - \xi^{n-1}\|^2) - \frac{s^2}{\tau^2+\tau s}(\|\xi(y)\|^2 - \|\xi^{n-2}\|^2 + \|\xi(y) - \xi^{n-2}\|^2) + 2s\varepsilon|||\xi(y)|||^2$$
$$\geq \frac{\tau+2s}{\tau+s}\|\xi(y)\|^2 - \frac{\tau+s}{\tau}\|\xi^{n-1}\|^2 + \frac{s^2}{\tau^2+\tau s}\|\xi^{n-2}\|^2 + \frac{\tau+s}{\tau}\|\xi(y) - \xi^{n-1}\|^2$$
$$- 2\frac{s^2}{\tau^2+\tau s}\|\xi(y) - \xi^{n-1}\|^2 - 2\frac{s^2}{\tau^2+\tau s}\|\xi^{n-1} - \xi^{n-2}\|^2 + 2s\varepsilon|||\xi(y)|||^2$$
$$\geq \frac{\tau+2s}{\tau+s}\|\xi(y)\|^2 - \frac{\tau+s}{\tau}\|\xi^{n-1}\|^2 + \frac{s^2}{\tau^2+\tau s}\|\xi^{n-2}\|^2 + \frac{s}{\tau}\|\xi(y) - \xi^{n-1}\|^2$$
$$- 2\frac{s^2}{\tau^2+\tau s}\|\xi^{n-1} - \xi^{n-2}\|^2 + 2s\varepsilon|||\xi(y)|||^2.$$

Setting $s = \tau$ (*i.e.* with $w = 2\xi^m$), the relations simplify to the usual

$$2\left(\frac{3}{2}\xi^m - 2\xi^{m-1} + \frac{1}{2}\xi^{m-2}, \xi^m\right) + 2\tau\varepsilon A_h(\xi^m, \xi^m)$$
$$\geq \frac{3}{2}\|\xi^m\|^2 - 2\|\xi^{m-1}\|^2 + \frac{1}{2}\|\xi^{m-2}\|^2 + \|\xi^m - \xi^{m-1}\|^2 - \|\xi^{m-1} - \xi^{m-2}\|^2 + 2\tau\varepsilon|||\xi^m|||^2.$$

Using Lemmas 4.1, 4.2, 4.3 and 5.4 to estimate the right-hand side terms, we get

$$\frac{\tau+2s}{\tau+s}\|\xi(y)\|^2 - \frac{\tau+s}{\tau}\|\xi^{n-1}\|^2 + \frac{s^2}{\tau^2+\tau s}\|\xi^{n-2}\|^2 - 2\frac{s^2}{\tau^2+\tau s}\|\xi^{n-1} - \xi^{n-2}\|^2$$
$$\leq Cs\left(1 + \frac{\|e(y)\|_\infty^2}{h^2}\right)(\varepsilon h^{2p} + h^{2p+1} + \tau^4 + \|\xi(y)\|^2)$$

and

$$\frac{3}{2}\|\xi^m\|^2 - 2\|\xi^{m-1}\|^2 + \frac{1}{2}\|\xi^{m-2}\|^2 + \|\xi^m - \xi^{m-1}\|^2 - \|\xi^{m-1} - \xi^{m-2}\|^2$$
$$\leq C\tau\left(1 + \frac{\|e^m\|_\infty^2}{h^2}\right)(\varepsilon h^{2p} + h^{2p+1} + \tau^4 + \|\xi^m\|^2).$$

Using the assumptions and Lemma 4.4, we can eliminate the terms $\|e(y)\|_\infty^2/h^2$ and $\|e^m\|_\infty^2/h^2$:

$$\frac{\tau+2s}{\tau+s}\|\xi(y)\|^2 - \frac{\tau+s}{\tau}\|\xi^{n-1}\|^2 + \frac{s^2}{\tau^2+\tau s}\|\xi^{n-2}\|^2 - 2\frac{s^2}{\tau^2+\tau s}\|\xi^{n-1} - \xi^{n-2}\|^2$$
$$\leq Cs(\varepsilon h^{2p} + h^{2p+1} + \tau^4 + \|\xi(y)\|^2), \tag{5.15}$$
$$\frac{3}{2}\|\xi^m\|^2 - 2\|\xi^{m-1}\|^2 + \frac{1}{2}\|\xi^{m-2}\|^2 + \|\xi^m - \xi^{m-1}\|^2 - \|\xi^{m-1} - \xi^{m-2}\|^2$$
$$\leq C\tau(\varepsilon h^{2p} + h^{2p+1} + \tau^4 + \|\xi^m\|^2). \tag{5.16}$$

Multiplying (5.15) by γ_0 and (5.16) by γ_{n-m} for $m = 2, \dots, n-1$, where the sequence $\{\gamma_j\}_{j=0}^{\infty}$ is defined by (5.13), and by summing all these inequalities together, we get by Lemma 5.5

$$\|\xi(y)\|^2 \le Cs\gamma_0\|\xi(y)\|^2 + C\gamma_\infty(\|\xi^1\|^2 + \|\xi^0\|^2) + C\tau\gamma_\infty \sum_{j=2}^{n-1} \|\xi^j\|^2 + C\gamma_\infty y(\varepsilon h^{2p} + h^{2p+1} + \tau^4).$$

Analogically we can obtain a similar result for $\|\xi^m\|$ for $m = 2, \dots, n-1$. Only this time the sequence for $\{\gamma_j\}_{j=0}^{\infty}$ used for multiplying the equations is modified by taking (5.13) with $s = \tau$:

$$\|\xi^m\|^2 \le Cs\gamma_0\|\xi^m\|^2 + C\gamma_\infty(\|\xi^1\|^2 + \|\xi^0\|^2) + C\tau\gamma_\infty \sum_{j=2}^{m-1} \|\xi^j\|^2 + C\gamma_\infty t_m(\varepsilon h^{2p} + h^{2p+1} + \tau^4).$$

Since $\|\xi^1\|^2$ and $\|\xi^0\|^2$ are bounded according to Lemma 5.6, we obtain the result using the discrete Gronwall lemma. □

Now we get rid of the *a priori* assumption $\|e(t)\| \le h^{1+d/2}$ from Lemmas 5.6 and 5.7.

Theorem 5.8. *Let $p > 1 + d/2$. Let τ_0 be defined as in Lemma 5.3. Let $h \in (0, h_0)$ and $\tau_1 \in (0, \tau_0)$ be such that*

$$C_T^2(h^{2p+1} + \varepsilon h^{2p} + \tau^4) \le \frac{1}{4}h^{2+d}, \tag{5.17}$$

where C_T is the constant from Lemma 5.7 independent of h, τ, ε. Then the error of the BDF2-DG scheme satisfies

$$\sup_{t\in[0,T]} \|e(t)\|^2 \le C_T^2(h^{2p+1} + \varepsilon h^{2p} + \tau^4). \tag{5.18}$$

Proof. We will follow the idea of continuous mathematical induction from [16]. Since the proof essentially follows the same pattern therein, we only give a brief description without details.

For time $t = 0$ it is easy to see that the error estimate holds, because the error is in fact the error of L^2 projection in initial data, which is sufficiently small under the assumptions of the theorem. Let us assume that the error estimate (5.18) holds on the interval $[0, s]$ for some $s \in [0, T]$. According to the assumption (5.17) we can see that the error can be estimated by $\|e(t)\| \le \frac{1}{2}h^{1+d/2}$, $t \in [0, s]$. Since the error $e(\cdot)$ is continuous (even uniformly continuous) with respect to time, we know that there exists some $\delta > 0$ such that $\|e(t)\| \le h^{1+d/2}$, $t \in [0, s+\delta]$ and we can see that it is possible to use Lemma 5.7 on the larger interval $[0, s+\delta]$, which guarantees the error estimate (5.18) on $[0, s+\delta]$. Since the error is *uniformly* continuous in time, we have a fixed $\delta > 0$ independent of s during the induction process and using the argument repeatedly we obtain the result up to $s = T$. □

Remark 5.9. The condition (5.17) can be essentially split into two parts, *e.g.* $C_T^2(h^{2p+1} + \varepsilon h^{2p}) \le \frac{1}{8}h^{2+d}$ and $C_T^2\tau^4 \le \frac{1}{8}h^{2+d}$. The first condition can be satisfied for sufficiently small h only if $p > 1 + d/2$. The second condition is satisfied only if the CFL-like condition $\tau = O(h^{1/2+d/4})$ holds. Of course, we still need the continued error $e(\cdot)$ to exist uniquely and be continuous in time, for which we need $\tau = O(\max\{\varepsilon, h\})$ by Lemma 5.3.

Remark 5.10. We note that if $\varepsilon = 0$, we obtain the improved estimate $O(h^{p+1/2} + \tau^2)$ under the weaker condition $p > (1 + d)/2$. This is also the case for Theorems 6.6 and 7.15 for the midpoint rule and QT-DG scheme.

The reader might ask why such an elaborate construction of the continuation as (5.2) is used, why not use *e.g.* some simple interpolation in time. In the proof of the estimates we proceed by induction from one time node to the next. Starting from the error of the initial condition, we want to prove that if the error e^{m-1}

at t_{m-1} is of the desired order, *e.g.* $O(h^{p+1/2})$, then so is e^m. The estimates of the convective terms allow us to do this if we know *a priori* that $\|e^m\| = O(h^{1+d/2})$. But in [16] it is proven, given the presented estimates, that the implication $\|e^{m-1}\| = O(h^{p+1/2}) \Longrightarrow \|e^m\| = O(h^{1+d/2})$ does not hold for implicit schemes (the proof is for the backward Euler scheme, however exactly the same reasoning holds *e.g.* for the BDF2 scheme). The proposed solution is to work with a continuous in time variant of the error, not discrete, and the continuity will help us go from t_{m-1} to t_m *via* suitably small intermediate steps while satisfying the necessary assumption $\|e\| = O(h^{1+d/2})$ along the way simply by continuity. Therefore the three requirements on the continuation are that $e(t)$ is continuous w.r.t. t, that it coincides with e^{m-1} and e^m at t_{m-1}, t_m and that it has the same order of approximation in time as the analyzed scheme for all t. If we used *e.g.* a simple Lagrange interpolation of $e^m, e^{m-1}, \ldots$ or $U^m, U^{m-1}, \ldots$, in order to prove anything about this interpolation between t_{m-1}, t_m we would first need to know the behavior of the interpolated function at the interpolation nodes, including the last one: t_m. In other words, we would need to have estimates for e^m in advance, which we do not, we only have estimates for e^{m-1} and earlier. In our approach, for $t \in (t_{m-1}, t_m)$ the continuation is constructed only from U^{m-1}, U^{m-2} without any knowledge of U^m or e^m. We start from t_{m-1} and by varying the time step in a variable time step BDF2 scheme, we go continuously from t_{m-1} to t_m and obtain U^m at t_m in a natural way. The estimates for the continuation are therefore obtained only from estimates of e^{m-1}, e^{m-2} (which we have from the induction assumption) while having the advantage of continuity in time to help control the *a priori* assumption $\|e\| = O(h^{1+d/2})$. To work with the Lagrange interpolation of the error in time, we would need to know not only the behavior at e^{m-1}, but also at e^m as stated earlier. Another possibility, to use some form of extrapolation from $t_{m-1}, t_{m-2}, \ldots$ would also not work, since at t_m we would not obtain U^m and therefore would not be estimating the BDF2 scheme but some different extrapolated solution.

Moreover, since our continuation is constructed using the BDF2 scheme itself (albeit with variable coefficients), the analysis of its properties is done using tools that would be used anyway. Perhaps a slightly simpler form than (5.2) could be possible, but given the presented reasoning, in the end it must be some variation on the BDF2 scheme itself, not simple interpolation.

6. Error estimates for the Midpoint rule

In this section, we investigate the error estimates of the approximate solution U^m, $m = 0, \ldots, r$ obtained by the method (3.5). As in the case of the BDF2-DG scheme, we construct a continuous extension of the discrete solution similar to Definition 5.1. For the purpose of analysis of the midpoint-DG scheme we assume the following regularity

$$u \in W^{1,\infty}(H^{p+1}) \cap W^{2,\infty}\big(H^2 \cap W_0^{1,\infty}\big) \cap W^{3,\infty}(L^2) \tag{6.1}$$

Definition 6.1. We define the continued approximate solution $U : [0,T] \to S_h$ of problem (2.1) obtained by the midpoint-DG scheme in the following way: Let $m > 0$ and $s \in [0,\tau]$, we seek $U(t_{m-1}+s) \in S_h$ such that

$$\begin{aligned} \big(U(t_{m-1}+s) - U^{m-1}, w\big) + \frac{s\varepsilon}{2} A_h\big(U(t_{m-1}+s) + U^{m-1}, w\big) + s b_h\left(\frac{U(t_{m-1}+s) + U^{m-1}}{2}, w\right) \\ = s\ell_h(w)(t_{m-1}+s/2), \quad \forall w \in S_h. \end{aligned} \tag{6.2}$$

As in Definition 5.1, by setting $s := 0$, we obtain $U(t_{m-1}) = U^{m-1}$. By setting $s := \tau$, we obtain $U(t_m) = U^m$.

Similarly as for the BDF2 scheme we can prove existence, uniqueness and time-continuity of the continued midpoint-DG solution from Definition 6.1.

Lemma 6.2. *There exist constants $C_1, C_2 > 0$ independent of h, τ, t, ε, such that the following holds. Let $h \in (0, h_0)$ and $\tau \in [0, \tau_0)$, where $\tau_0 = \max\{C_1\varepsilon, C_2 h\}$. Then U, the continued solution from Definition 6.1 exists, is uniquely determined, $\|U(t)\|$ is uniformly bounded with respect to $t \in [0,T]$, $U(t_m) = U^m$ for all $m = 0, \ldots, r$ and $\|U(t)\|$ depends continuously on t.*

Proof.
(i) **Existence**: We denote the left- and right-hand side from (6.2)

$$B_s^m(v,w) = (v - U^{m-1}, w) + \frac{s\varepsilon}{2} A_h(v + U^{m-1}, w) + s b_h\left(\frac{v + U^{m-1}}{2}, w\right),$$
$$L_s^m(w) = s\ell_h(w)(t_{m-1} + s/2).$$

Then B_s^m is strongly monotone on S_h:

$$B_s^m(v, v-w) - B_s^m(w, v-w) \geq \|v-w\|^2 + \frac{s\varepsilon}{2}|\!|\!|v-w|\!|\!|^2 - Cs\|v-w\|\,|\!|\!|v-w|\!|\!|$$
$$\geq \left(1 - \frac{Cs}{h}\right)\|v-w\|^2 = M\|v-w\|^2$$

for sufficiently small s, τ with respect to h. On the other hand, we may estimate using Young's inequality as in (5.4) to obtain monotonicity for s, τ sufficiently small with respect to ε.

Now, we show that B_s^m is Lipschitz continuous on S_h:

$$B_s^m(v,w) - B_s^m(\bar{v},w) \leq \|v-w\|\,\|w\| + C\frac{s\varepsilon}{2}|\!|\!|v-\bar{v}|\!|\!|\,|\!|\!|w|\!|\!| + Cs\|v-\bar{v}\|\,|\!|\!|w|\!|\!|$$
$$\leq \left(1 + \frac{Cs\varepsilon}{h^2} + \frac{Cs}{h}\right)\|v-\bar{v}\|\,\|w\| = C\|v-\bar{v}\|\,\|w\|.$$

The right-hand side L_s^m is bounded, hence continuous, on S_h the nonlinear Lax-Milgram lemma gives us existence and uniqueness of the continued discrete solution and classical discrete solution, respectively.

(ii) **Continuity**: Continuity with respect to time can be proved in the same way as in the proof of Lemma 5.3. Again, we use the monotonicity of the form B_s^m and write

$$\begin{aligned} M\|U(t) - U(\bar{t})\|^2 &\leq B_t^m(U(t), U(t) - U(\bar{t})) - B_t^m(U(\bar{t}), U(t) - U(\bar{t})) \\ &= L_t^m(U(t) - U(\bar{t})) - L_{\bar{t}}^m(U(t) - U(\bar{t})) + B_{\bar{t}}^m(U(\bar{t}), U(t) - U(\bar{t})) - B_t^m(U(\bar{t}), U(t) - U(\bar{t})). \end{aligned}$$

Similarly as in BDF case we can estimate the terms on the second and third row and prove that they tend to zero as $|t - \bar{t}|$ tends to zero, therefore $\|U(t) - U(\bar{t})\|$ tends to zero as well. Analogically we can prove the continuity at $t_{m-1}+$. Since the exact solution u is continuous and since we have continuity on the closed interval $[0,T]$, we can see that the error $U(t) - u(t)$ is uniformly continuous. □

Lemma 6.3. *Let u satisfy regularity assumptions (6.1). Let $s \in (0, \tau]$. Then*

$$(u(t_{m-1} + s) - u^{m-1} - su'(t_{m-1} + s/2), w) \leq Cs\tau^2\|u\|_{W^{3,\infty}(L^2)}\|w\|$$

Proof. The proof is analogical to the proof of Lemma 5.4. We can formally rewrite

$$\begin{aligned} u(t_{m-1} + s) - u^{m-1} - su'(t_{m-1} + s/2) &= u(t_{m-1} + s) - u^{m-1} - su'(t_{m-1}) \\ &\quad - \frac{s^2}{2}u''(t_{m-1}) - su'(t_{m-1} + s/2) + su'(t_{m-1}) + \frac{s^2}{2}u''(t_{m-1}). \end{aligned}$$

Then it is easy to see that

$$\begin{aligned} \big(u(t_{m-1} + s) - u^{m-1} - su'(t_{m-1} + s/2), w\big) &= \int_{t_{m-1}}^{t_{m-1}+s}\int_{t_{m-1}}^{z_1}\int_{t_{m-1}}^{z_2}(u'''(z_3), w)\mathrm{d}z_3\mathrm{d}z_2\mathrm{d}z_1 \\ &\quad - s\int_{t_{m-1}}^{t_{m-1}+s/2}\int_{t_{m-1}}^{z_1}(u'''(z_2), w)\mathrm{d}z_2\mathrm{d}z_1 \\ &\leq \left(\frac{s^3}{6} + \frac{s^3}{8}\right)\|u\|_{W^{3,\infty}(L^2)}\|w\|. \end{aligned}$$

□

Lemma 6.4. *Let u satisfy regularity assumptions (6.1). Let $s \in (0,\tau]$. Then*

$$A_h\left(u(t_{m-1}+s/2) - \frac{u(t_{m-1}+s)+u^{m-1}}{2}, w\right) \le C\tau^2 \|u\|_{W^{2,\infty}(H^2)} \|\!|w|\!\| \tag{6.3}$$

$$b_h\big(u(t_{m-1}+s/2), w\big) - b_h\left(\frac{u(t_{m-1}+s)+u^{m-1}}{2}, w\right) \le C\tau^2 \|u\|_{W^{2,\infty}(H^2)} \|w\| \tag{6.4}$$

Proof. Let us denote $u_1 = u(t_{m-1}+s/2)$ and $u_2 = \frac{u(t_{m-1}+s)+u^{m-1}}{2}$. Moreover, it is possible to see that

$$\begin{aligned} u_1 - u_2 &= u(t_{m-1}+s/2) - \frac{u(t_{m-1}+s)+u^{m-1}}{2} \\ &= \frac{1}{2}u(t_{m-1}+s/2) - \frac{1}{2}u(t_{m-1}+s) + \frac{s}{2}u'(t_{m-1}+s/2) + \frac{1}{2}u(t_{m-1}+s/2) - \frac{1}{2}u^{m-1} - \frac{s}{2}u'(t_{m-1}+s/2) \\ &= -\frac{1}{2}\int_{t_{m-1}+s/2}^{t_{m-1}+s}\int_{t_{m-1}+s/2}^{z_1} u''(z_2)\mathrm{d}z_2\mathrm{d}z_1 - \frac{1}{2}\int_{t_{m-1}}^{t_{m-1}+s/2}\int_{z_1}^{t_{m-1}+s/2} u''(z_2)\mathrm{d}z_2\mathrm{d}z_1. \end{aligned} \tag{6.5}$$

Following the proof of ([8], Lem. 9) it can be shown

$$A_h(u_1-u_2, w) \le C(\|\!|u_1-u_2|\!\| + |u_1-u_2|_{H^2})\|\!|w|\!\|. \tag{6.6}$$

Since $u_1, u_2 \in H_0^1(\Omega)$, we can simplify (6.6) to

$$A_h(u_1-u_2, w) \le C\|u_1-u_2\|_{H^2}\|\!|w|\!\|.$$

Using (6.5) we get

$$A_h(u_1-u_2, w) \le C\frac{s^2}{2}\|u\|_{W^{2,\infty}(H^2)}\|\!|w|\!\|,$$

which implies (6.3). Since u_1 and u_2 are smooth enough, it implies

$$b_h(u_1, w) - b_h(u_2, w) = \int_\Omega \nabla\cdot(f(u_1)-f(u_2))w\mathrm{d}x \le \|\nabla\cdot(f(u_1)-f(u_2))\|\,\|w\|.$$

To prove (6.4) it is sufficient to estimate $\|\nabla\cdot(f(u_1)-f(u_2))\|$.

$$\begin{aligned} \|\nabla\cdot(f(u_1)-f(u_2))\| &\le \sum_{i=1}^d \left\| f_i'(u_1)\frac{\partial u_1}{\partial x_i} - f_i'(u_2)\frac{\partial u_2}{\partial x_i}\right\| \\ &\le \sum_{i=1}^d \left(\left\| f_i'(u_1)\left(\frac{\partial u_1}{\partial x_i} - \frac{\partial u_2}{\partial x_i}\right)\right\| + \left\|(f_i'(u_1)-f_i'(u_2))\frac{\partial u_2}{\partial x_i}\right\|\right) \\ &\le d\max_i \|f_i'(u_1)\|_{L^\infty}|u_1-u_2|_{H^1} + d\max_i\|f_i'(u_1)-f_i'(u_2)\|_{L^\infty}|u_2|_{H^1}. \end{aligned}$$

Then (6.4) is a consequence of (6.5) and $\|f_i'(u_1)-f_i'(u_2)\|_{L^\infty} \le C\|u_1-u_2\|_{L^\infty}$. □

Now, we shall derive the error estimate of the continued solution at arbitrary time $t \in [0,T]$ which immediately implies the error estimate for the original midpoint scheme (3.5).

Lemma 6.5. *Let $p > d/2$. Let $m > 0$ and $s \in (0,\tau]$. If $\|e(t)\| \le h^{1+d/2}$ for $t \in [0, t_{m-1}+s]$, then*

$$\sup_{t\in[0,t_{m-1}+s]} \|e(t)\|^2 \le C_T^2(h^{2p+1} + \varepsilon h^{2p} + \tau^4),$$

where the constant C_T is independent of h, τ, ε.

Proof. We set $y = t_{m-1} + s$. Multiplying (3.1) for $t = t_{m-1} + s/2$ by s, subtracting from (6.2) and adding several terms we get

$$\begin{aligned}(\xi(y) - \xi^{m-1}, w) + \frac{s\varepsilon}{2} A_h\big(\xi(y) + \xi^{m-1}), w\big) &\le \left(s\frac{\partial u}{\partial t}(t_{m-1} + s/2) - u(y) + u^{m-1}, w\right)\\ &+ s\left(b_h\big(u(t_{m-1} + s/2), w\big) - b_h\Big(\frac{u(y) + u^{m-1}}{2}, w\Big)\right) + \big(\eta(y) - \eta^{m-1}, w\big)\\ &+ s\left(b_h\Big(\frac{u(y) + u^{m-1}}{2}, w\Big) - b_h\Big(\frac{U(y) + U^{m-1}}{2}, w\Big)\right) - \frac{s\varepsilon}{2} A_h\big(\eta(y) + \eta^{m-1}), w\big)\\ &+ s\left(A_h(u(t_{m-1} + s/2), w) - A_h\Big(\frac{u(y) + u^{m-1}}{2}, w\Big)\right).\end{aligned}$$

Setting $w = \xi(y) + \xi^{m-1}$ and using Lemmas 4.1–4.3 and Lemma 6.4 to estimate the right-hand side, we get

$$\|\xi(y)\|^2 - \|\xi^{m-1}\|^2 \le Cs\left(1 + \frac{\|e(y) + e^{m-1}\|_\infty^2}{h^2}\right)(\varepsilon h^{2p} + h^{2p+1} + \tau^4 + \|\xi(y)\|^2 + \|\xi^{m-1}\|^2).$$

Using the assumptions we can get rid of the unpleasant term $\|e(s) + e^{m-1}\|_\infty^2/h^2$. Finally, by taking $s := \tau$ and $m = 1, \ldots$, we obtain a similar estimate for $\|\xi^m\|^2 - \|\xi^{m-1}\|^2$. By the discrete Gronwall lemma we can finish the proof. □

Theorem 6.6. *Let $p > 1 + d/2$. Let τ_0 be defined as in Lemma 6.2. Let $h \in (0, h_0)$ and $\tau_1 \in (0, \tau_0)$ be such that*

$$C_T^2(h^{2p+1} + \varepsilon h^{2p} + \tau^4) \le \frac{1}{4} h^{2+d}, \tag{6.7}$$

where C_T is the constant from Lemma 6.5 independent of h, τ, ε. Then the error of the midpoint–DG scheme satisfies

$$\sup_{t\in[0,T]} \|e(t)\|^2 \le C_T^2(h^{2p+1} + \varepsilon h^{2p} + \tau^4).$$

Proof. The proof is essentially identical to that of Theorem 5.8. We have the desired estimate for $t = 0$ and due to continuity and Lemma 6.5, we can extend its validity to time T by induction. □

Remark 6.7. Similarly as in Remark 6.7, the condition (6.7) can be essentially split into two parts: $p > 1 + d/2$ and $\tau = O(h^{1/2+d/4})$. The latter condition is weaker than for the backward Euler method, where we needed $\tau = O(h^{1+d/2})$.

7. Quadrature variant of time-DG

In this section, we will prove error estimates for the quadrature variant of the QT-DG. As in the previous sections, we will construct a suitable continuation of U from Definition 3.5. While for the BDF2 and midpoint schemes, the discrete solution is defined only in the nodes of the partition t_m and the continuation "fills in the gaps" between these points, the DG solution U is already inherently defined on the whole interval $(0, T)$. It is therefore a question how to define a continuation w.r.t. time for such an object. In our approach, we construct the continuation U_y with respect to an auxiliary parameter y. Then U_y will be a piecewise polynomial function defined on $(0, y)$ which will depend continuously on y in the $L^\infty(L^2)$-norm. Again, we will use an induction argument to pass with y from 0 to T. For our analysis we will need the following regularity

$$u \in W^{1,\infty}(H^{p+1}) \cap L^\infty(W^{1,\infty}) \cap W^{q+1,\infty}(H^1). \tag{7.1}$$

7.1. Construction of the continuation

Throughout this section, let $s \in (0,\tau]$ and $m \in \{1,\ldots,r\}$. We denote $y = t_{m-1} + s$, the continuation parameter and define $I_m(s) = (t_{m-1}, y)$. Let us generalize the quadrature Q_τ^m to Q_s^m:

$$\int_{I_m(s)} \Phi(t)\mathrm{d}t \approx Q_s^m[\Phi] = s\sum_{i=0}^{q} \omega_i \Phi(t_{m-1} + s\psi_i).$$

We define the space of piecewise polynomials up to degree p in space and up to degree q in time defined on I_m:

$$S_h^m = \Big\{v \in L^2(I_m; S_h) : v = \sum_{j=0}^{q} v_j t^j,\ v_j \in S_h\Big\}.$$

Definition 7.1. Let $y \in I_m \cup \{t_m\}$. We say that the function $U_y \in L^2(0,t_m;S_h)$ is a *continued approximate solution* of problem (2.1) obtained by the QT-DG scheme if $U_y|_{I_l} = U|_{I_l}$ for $l = 0,\ldots,m-1$, where U is the space-time DG solution from Definition 3.5 and $U_y|_{I_m} \in S_h^m$ satisfies

$$\int_{I_m(s)} (U_y', w) + \varepsilon A_h(U_y, w)\mathrm{d}t + Q_s^m[b_h(U_y,w)] + (\{U_y\}_{m-1}, w_+^{m-1}) = Q_s^m[\ell_h(w)] \quad \forall w \in S_h^m. \tag{7.2}$$

Remark 7.2. We note that by taking $s = \tau$, or equivalently $y = t_m$, we get $U_y|_{(0,t_m)} = U|_{(0,t_m)}$, *i.e.* we obtain the original space-time DG solution on $(0,t_m)$. Specifically, by taking $y = T$, we get $U_T = U$ on the whole interval $(0,T)$. We note also that relation (7.2) provides naturally the definition of U_y on $I_m(s)$. Since $U_y|_{I_m}$ is a polynomial with respect to time, U_y is uniquely defined on the remaining part of I_m and corresponds to the natural prolongation of $U_y|_{I_m(s)}$.

In order to prove existence, uniqueness and continuous dependence on y, we first need to establish monotonicity and Lipschitz continuity of the corresponding forms in (7.2). The same results can then be derived for (3.6) by taking $s := \tau$. Let us denote the left- and right-hand side of (7.2) by

$$\begin{aligned} B_s^m(v,w) &= \int_{I_m(s)} (v', w) + \varepsilon A_h(v,w)\mathrm{d}t + Q_s^m[b_h(v,w)] + (v_+^{m-1}, w_+^{m-1}),\\ L_s^m(w) &= Q_s^m[\ell_h(w)] + (U_-^{m-1}, w_+^{m-1}). \end{aligned}$$

Definition 7.3. We define the projection $P_s^m : C(\overline{I_m(s)}; L^2(\Omega)) \to S_h^m$ by

$$(P_s^m v)(t_{m-1} + s\psi_i) = v(t_{m-1} + s\psi_i), \quad \forall i = 0,\ldots,q. \tag{7.3}$$

Furthermore, for any function $v \in S_h^m$ we denote

$$\tilde{v}(t) = P_s^m\Big(\frac{s}{t - t_{m-1}} v(t)\Big). \tag{7.4}$$

We point out that the relevant factors $\frac{s}{t_{m-1}+s\psi_i - t_{m-1}} = \frac{1}{\psi_i} \geq 1$. We have the following approximation properties of P_s^m:

Lemma 7.4. *Let $u \in W^{q+1,\infty}(H^1)$. Then*

$$\begin{aligned} \sup_{I_m(s)} \|P_s^m u - u\| &\leq C s^{q+1} \sup_{I_m(s)} \|u^{(q+1)}\|,\\ \sup_{I_m(s)} \|\!|P_s^m u - u|\!\| &\leq C s^{q+1} \sup_{I_m(s)} \|\!|u^{(q+1)}|\!\|, \end{aligned}$$

where the constant C does not depend on s.

Proof. The proof is an analogy to (*e.g.* [4], Thm. 3.1.5) for Bochner spaces. The result is also derived in the Appendix of [20]. □

We shall use following technical lemmas.

Lemma 7.5. *For any $v \in S_h^m$ the following terms are equivalent with the equivalence constants depending only on q:*

$$\sup_{I_m(s)} \|v\|^2, \qquad \sup_{I_m(s)} \|\tilde{v}\|^2, \qquad \frac{1}{s}\int_{I_m(s)} \|\tilde{v}\|^2 \mathrm{d}t.$$

Proof. The proof follows immediately from the fact that S_h^m has finite dimension. □

Lemma 7.6. *Let $v \in S_h^m$ and $\tilde{v}$ defined by* (7.4)*. Then*

$$\int_{I_m(s)} (v', 2\tilde{v})\mathrm{d}t + (v_+^{m-1}, 2\tilde{v}_+^{m-1}) = \|v(y)\|^2 + \frac{1}{s}\int_{I_m(s)} \|\tilde{v}\|^2 \mathrm{d}t.$$

Proof. The proof can be made as a simple extension of ([1], Lem. 2.1), which describes the same result for scalar polynomials and on the unit time interval. □

Lemma 7.7. *Let $v \in S_h^m$ and $\tilde{v}$ be defined by* (7.4)*, then*

$$0 \le \int_{I_m(s)} A_h(v,v)\mathrm{d}t \le \int_{I_m(s)} A_h(v,\tilde{v})\mathrm{d}t.$$

Proof.

$$\begin{aligned} 0 \le \int_{I_m(s)} A_h(v,v)\mathrm{d}t &= Q_s^m[A_h(v,v)] = s\sum_{i=0}^{q} \omega_i A_h(v(t_{m-1}+s\psi_i), v(t_{m-1}+s\psi_i)) \\ &\le s\sum_{i=0}^{q} \omega_i \frac{1}{\psi_i} A_h(v(t_{m-1}+s\psi_i), v(t_{m-1}+s\psi_i)) = Q_s^m[A_h(v,\tilde{v})] = \int_{I_m(s)} A_h(v,\tilde{v})\mathrm{d}t, \end{aligned}$$

since $1/\psi_i \ge 1$. □

Now we are ready to prove fundamental properties of the forms B_s^m and L_s^m. We note that the mapping $v \to \tilde{v}$ is a bijection on S_h^m, therefore we can reformulate problem (7.2), *i.e.* $B_s^m(U_s, w) = L_s^m(w)$, for all $w \in S_h^m$ to the equivalent problem $B_s^m(U_s, \tilde{w}) = L_s^m(\tilde{w})$ for all $w \in S_h^m$. Hence for the purpose of proving existence and uniqueness of U_y, we can deal either with $B_s^m(.,.)$ or $B_s^m(.,\tilde{.})$ and similarly for L_s^m.

Lemma 7.8. *Let $s \le \tau \le C_1 h$, where C_1 is a suitable constant. Then the form $B_s^m(.,\tilde{.})$ is strongly monotone and Lipschitz continuous on S_h^m with respect to the $L^2(\Omega)$-norm, with the monotonicity and Lipschitz constants independent of s. Furthermore, L_s^m is bounded on this space, with norm uniformly bounded with respect to s but depending on $\|U_-^{m-1}\|$.*

Proof. To simplify the notation, all of the suprema in this proof are over the relevant interval $I_m(s)$.

(i) **Monotonicity of** B_s^m: Let $v, w \in S_h^m$, then

$$\begin{aligned}
B_s^m(v, \tilde{v} - \tilde{w}) - B_s^m(w, \tilde{v} - \tilde{w}) &= \int_{I_m(s)} (v' - w', \tilde{v} - \tilde{w}) + \varepsilon A_h(v - w, \tilde{v} - \tilde{w})\mathrm{d}t \\
&\quad + Q_s^m[b_h(v, \tilde{v} - \tilde{w}) - b_h(w, \tilde{v} - \tilde{w})] + (v_+^0 - w_+^0, \tilde{v}_+^0 - \tilde{w}_+^0) \\
&\geq \frac{1}{2}\|v(y) - w(y)\|^2 + \frac{1}{2s}\int_{I_m(s)} \|\tilde{v} - \tilde{w}\|^2\mathrm{d}t + \varepsilon \int_{I_m(s)} A_h(v - w, v - w)\mathrm{d}t \\
&\quad - Cs \sup \|v - w\| \sup |\!|\!|\tilde{v} - \tilde{w}|\!|\!| \\
&\geq c \sup \|v - w\|^2 - \frac{C}{h} s \sup \|v - w\| \sup \|\tilde{v} - \tilde{w}\| \\
&\geq \left(c - \frac{Cs}{h}\right) \sup \|v - w\|^2,
\end{aligned}$$

where the constant c comes from Lemma 7.5 and the generic constant C comes from Lemmas 4.3 and 7.5. If $s \leq \tau \leq C_1 h$ with a sufficiently small constant C_1, we obtain strong monotonicity with the monotonicity constant $M = c - \frac{C\tau}{h}$.

(ii) **Lipschitz continuity of** B_s^m: Let $v, \bar{v}, w \in S_h^m$. We estimate individual terms in B_s^m:

$$\begin{aligned}
\int_{I_m(s)} (v' - \bar{v}', w)\mathrm{d}t + (v_+^{m-1} - \bar{v}_+^{m-1}, w_+^{m-1}) &\leq s \sup \|v' - \bar{v}'\| \sup \|w\| + \sup \|v - \bar{v}\| \sup \|w\| \\
&\leq C \sup \|v - \bar{v}\| \sup \|w\|, \\
\varepsilon \int_{I_m(s)} A(v - \bar{v}, w)\mathrm{d}t \leq C\varepsilon \int_{I_m(s)} |\!|\!|v - \bar{v}|\!|\!| \, |\!|\!|w|\!|\!|\mathrm{d}t &\leq Ch^{-2} s\varepsilon \sup \|v - \bar{v}\| \sup \|w\|, \\
Q_s^m[b(v, w) - b(\bar{v}, w)] \leq CQ_s^m[\|v - \bar{v}\| \, |\!|\!|w|\!|\!|] &\leq Csh^{-1} \sup \|v - \bar{v}\| \sup \|w\|.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&B_s^m(v, w) - B_s^m(\bar{v}, w) \leq C \sup \|v - \bar{v}\| \sup \|w\|, \\
&B_s^m(v, \tilde{w}) - B_s^m(\bar{v}, \tilde{w}) \leq C \sup \|v - \bar{v}\| \sup \|\tilde{w}\| \leq C \sup \|v - \bar{v}\| \sup \|w\|.
\end{aligned}$$

Here the resulting constants C depend also on ε, h, s, however, for the sake of the existence and uniqueness proof, these may be considered as fixed quantities. Elsewhere, we can bound $s \leq \tau$ to obtain s-independence of the Lipschitz constant.

(iii) **Boundedness of** L_s^m:

$$\begin{aligned}
&L_s^m(v) = Q_s^m[\ell(v)] + (U_-^{m-1}, v_+^{m-1}) \leq s \sup \|g\| \sup \|v\| + \|U_-^{m-1}\| \sup \|v\| \leq C \sup \|v\|, \\
&L_s^m(\tilde{v}) \leq C \sup \|\tilde{v}\| \leq C \sup \|v\|.
\end{aligned}$$

The constant C in the resulting estimate depends also on $\|U_-^{m-1}\|$ and s, however by bounding $s \leq \tau$, we obtain s-independence of the boundedness constant. □

Existence and uniqueness of the continued solution U_y follows immediately from Lemma 7.8. We will also need uniform boundedness of $\|U_y\|$ and $\|U_y'\|$ with respect to $t \in [0, y]$. The resulting boundedness constants depend on ε and negative powers of h, however since the main goal is to prove continuous dependence of U_y on y, this is not a problem.

Lemma 7.9. *There exist constants $C_1, C_2 > 0$ independent of h, τ, t, ε, such that the following holds. Let $h \in (0, h_0)$ and $\tau \in [0, \tau_0)$, where $\tau_0 = \max\{C_1\varepsilon, C_2 h\}$. Then U_y, the continued solution from Definition 7.1 exists, is uniquely determined and $\|U_y(t)\|$, $\|U_y'(t)\|$ are uniformly bounded with respect to $t \in [0, y]$. Furthermore, for fixed $\|U_-^{m-1}\|$, the norms $\sup_{t \in I_m(s)} \|U_y(t)\|$, $\sup_{t \in I_m(s)} \|U_y'(t)\|$ are uniformly bounded with respect to $y \in I_m$.*

Proof. Remark 3.7 holds for U_y as well, therefore, we can prove unique existence and boundedness of U_y on each interval independently. From Lemma 7.8, we obtain existence and uniqueness of U_y.

(i) **Boundedness of** U_y: Due to Lemma 7.8,

$$M \sup_{I_m(s)} \|U_y\|^2 = M \sup_{I_m(s)} \|U_y - 0\|^2 \le B_s^m(U_y, \tilde{U}_y) - B_s^m(0, \tilde{U}_y) = B_s^m(U_y, \tilde{U}_y) = L_s^m(\tilde{U}_y) \le C \sup_{I_m(s)} \|U_y\|.$$

Due to Lemma 7.8, all the constants involved are independent of s, hence y.

(ii) **Boundedness of** U_y': setting $v(t) = (t - t_{m-1})U_y'(t) \in S_h^m$

$$\int_{I_m(s)} (t - t_{m-1})\|U_y'\|^2 + (t - t_{m-1})\varepsilon A_h(U_y, U_y')\mathrm{d}t + Q_s^m[(t - t_{m-1})b_h(U_y, U_y')] = Q_s^m[(t - t_{m-1})\ell_h(U_y')].$$

From this follows

$$\begin{aligned} cs \int_{I_m(s)} \|U_y'\|^2\mathrm{d}t &\le \int_{I_m(s)} (t - t_{m-1})\|U_y'\|^2\mathrm{d}t \\ &\le - \int_{I_m(s)} (t - t_{m-1})\varepsilon A_h(U_y, U_y')\mathrm{d}t - Q_s^m\left[(t - t_{m-1})b_h(U_y, U_y') - (t - t_{m-1})\ell_h(U_y')\right] \\ &\le s \int_{I_m(s)} C\varepsilon h^{-2}\|U_y\|\|U_y'\|\mathrm{d}t + sQ_s^m\left[(Ch^{-1}\|U_y\| + C)\|U_y'\|\right] \\ &= s \int_{I_m(s)} \|U_y'\|(C\varepsilon h^{-2}\|U_y\| + Ch^{-1}\|U_y\| + C)\mathrm{d}t \\ &\le \frac{cs}{2} \int_{I_m(s)} \|U_y'\|^2\mathrm{d}t + C(\varepsilon, h)s^2, \end{aligned}$$

where we have used Hölder's and Young's inequality in the last step. Since

$$s \sup_{I_m(s)} \|U_y'\|^2 \le C \int_{t_{m-1}}^{y} \|U_y'\|^2\mathrm{d}t,$$

we get the boundedness of $\|U_y'\|$. Moreover, after cancellation of the term s^2 from the resulting estimate, we obtain s-independence of the upper bound. □

Before we prove the main property of U_y, continuous dependence on y, we need one more technical lemma concerning the estimation of quadratures.

Lemma 7.10. *Let* $s, \bar{s} \in (0, \tau]$ *and* $m \in 1, \ldots, r$. *Let* $v, w \in S_h^m$. *Then* $|s - \bar{s}| \to 0$ *implies*

$$\begin{aligned} Q_s^m[b_h(v, w)] - Q_{\bar{s}}^m[b_h(v, w)] &\to 0, \\ Q_s^m[\ell_h(v)] - Q_{\bar{s}}^m[\ell_h(v)] &\to 0. \end{aligned}$$

Proof. Let us assume $|s - \bar{s}| \to 0$. In order to simplify the notation of quadrature points, we shall set $s_i := t_{m-1} + s\psi_i$ and $\bar{s}_i := t_{m-1} + \bar{s}\psi_i$. Then

$$\begin{aligned} Q_s^m[b_h(v, w)] - Q_{\bar{s}}^m[b_h(v, w)] &= \sum_{i=0}^{q} \left(s\omega_i b_h(v, w)|_{s_i} - \bar{s}\omega_i b_h(v, w)|_{\bar{s}_i}\right) \\ &= s \sum_{i=0}^{q} \omega_i \left(b_h(v, w)|_{s_i} - b_h(v, w)|_{\bar{s}_i}\right) + \sum_{i=0}^{q} (s - \bar{s})\, \omega_i b_h(v, w)|_{\bar{s}_i} \end{aligned}$$

and

$$Q_s^m[\ell_h(v)] - Q_{\bar{s}}^m[\ell_h(v)] = \sum_{i=0}^{q} \left(s\omega_i \ell_h(v)|_{s_i} - \bar{s}\omega_i \ell_h(v)|_{\bar{s}_i}\right) = s\sum_{i=0}^{q} \omega_i \left(\ell_h(v)|_{s_i} - \ell_h(v)|_{\bar{s}_i}\right) + \sum_{i=0}^{q} (s-\bar{s})\, \omega_i \ell_h(v)|_{\bar{s}_i}.$$

From continuity of $b_h(.,.)$ and v, w we get

$$b_h(v,w)|_{s_i} - b_h(v,w)|_{\bar{s}_i} \to 0$$

and since $\ell_h(v) = (g, v)$, where g is continuous with respect to time, we get

$$\ell_h(v)|_{s_i} - \ell_h(v)|_{\bar{s}_i} \to 0.$$

From boundedness of $b_h(v,w)|_{\bar{s}_i}$ and $\ell_h(v)|_{\bar{s}_i}$ we obtain $(s-\bar{s})\omega_i b_h(v,w)|_{\bar{s}_i} \to 0$ and $(s-\bar{s})\omega_i \ell_h(v,w)|_{\bar{s}_i} \to 0$. □

Lemma 7.11. *Let the assumptions of Lemma 7.9 hold. Then $U_{t_m} = U|_{(0,t_m)}$ for all $m = 0, \ldots, r$ and U_y depends continuously on the parameter y in the following sense:*

$$\sup_{(0,\min(y,\bar{y}))} \|U_y - U_{\bar{y}}\| \to 0, \text{ as } |y-\bar{y}| \to 0,$$
$$\sup_{(t_{m-1},y)} \|U_y - U_-^{m-1}\| \to 0, \text{ as } y \to t_{m-1}+ . \tag{7.5}$$

Proof. Let $y = t_{m-1} + s, \bar{y} = t_{m-1} + \bar{s}$ for some m. Without loss of generality, let $0 < s < \bar{s} \le \tau$. Since $U_y = U_{\bar{y}} = U$ on $(0, t_{m-1})$, it is sufficient to prove the first relation only on (t_{m-1}, y). Let us denote $w = U_y - U_{\bar{y}}$. Due to monotonicity of $B_s^m(.,\tilde{.})$ and Lemma 7.10, we have

$$M \sup_{(t_{m-1},y)} \|U_y - U_{\bar{y}}\|^2 \le B_s^m(U_y, \tilde{w}) - B_s^m(U_{\bar{y}}, \tilde{w}) = L_s^m(\tilde{w}) - L_{\bar{s}}^m(\tilde{w}) + B_{\bar{s}}^m(U_{\bar{y}}, \tilde{w}) - B_s^m(U_{\bar{y}}, \tilde{w})$$
$$= \int_{t_{m-1}+s}^{t_{m-1}+\bar{s}} (U'_{\bar{y}}, \tilde{w}) + \varepsilon A_h(U_{\bar{y}}, \tilde{w})\mathrm{d}t + Q_{\bar{s}}^m[b_h(U_{\bar{y}}, \tilde{w})] - Q_s^m[b_h(U_{\bar{y}}, \tilde{w})] - Q_{\bar{s}}^m[\ell_h(\tilde{w})] + Q_s^m[\ell_h(\tilde{w})].$$

Since the terms in the integral are bounded, the integral tends to zero as $|s-\bar{s}| \to 0$. According to Lemma 7.10 the quadrature terms tend to zero as well. From this it follows that $\sup_{(t_{m-1},y)} \|U_s - U_{\bar{s}}\| \to 0$ for $|s - \bar{s}| \to 0$.

It remains to prove the second formula in (7.5). Since U_y is continuous on (t_{m-1}, y), it is sufficient to prove $U_{y+}^{m-1} \to U_-^{m-1}$ as $y \to t_{m-1}+$, *i.e.* $s \to 0+$: Testing (7.2) with $w \equiv U_{y+}^{m-1} - U_-^{m-1}$, we get

$$\int_{t_{m-1}}^{y} (U'_y, U_{y+}^{m-1} - U_-^{m-1}) + \varepsilon A_h(U_y, U_{y+}^{m-1} - U_-^{m-1})\mathrm{d}t + Q_s^m[b(U_y, U_{y+}^{m-1} - U_-^{m-1})] + \|U_{y+}^{m-1} - U_-^{m-1}\|^2$$
$$= Q_s^m[\ell_h(U_{y+}^{m-1} - U_-^{m-1})].$$

Except for the last left-hand side term $\|U_{y+}^{m-1} - U_-^{m-1}\|^2$, all remaining terms tend to zero as $s \to 0+$, therefore $\|U_{y+}^{m-1} - U_-^{m-1}\|^2$ tends to zero as well. □

7.2. Error estimates

As the final step we shall derive the error estimate of the continued solution at arbitrary time $t \in [0, T]$ which immediately implies the error estimate for the classical method.

As usual, we shall split the error $e_y(t) = U_y(t) - u(t)$ into two parts $e_y(t) = \xi_y(t) + \eta_y(t)$, where we define:

$$\eta_y|_{I_i} = \begin{cases} \pi_\tau^i u|_{I_i} - u|_{I_i}, & i = 0, \ldots, m-1, \\ \pi_s^m u|_{I_m} - u|_{I_m}, & i = m, \end{cases}$$
$$\xi_y|_{I_i} = \begin{cases} U_y|_{I_i} - \pi_\tau^i u|_{I_i}, & i = 0, \ldots, m-1, \\ U_y|_{I_m} - \pi_s^m u|_{I_m}, & i = m, \end{cases}$$

where $\pi_s^i = P_s^i \Pi$. We have the following estimates for η_y and ξ_y:

Lemma 7.12. *Let u satisfy regularity assumptions (7.1). Then for all $v \in S_h^m$*

$$\sup_{I_m(s)} \|\eta_y\| \leq C(h^{p+1} + s^{q+1}), \tag{7.6}$$

$$Q_s^m[(\eta_y', v)] + (\{\eta_y\}_{m-1}, v_+^{m-1}) \leq sC(h^{p+1} + s^{q+1}) \sup_{I_m(s)} \|v\|. \tag{7.7}$$

Proof. The estimate (7.6) follows directly from Lemmas 4.1 and 7.4. The estimate (7.7) is proved in ([21], Lem. 4). □

Lemma 7.13. *Let u satisfy regularity assumptions (7.1). Then*

$$Q_s^m[b_h(u,\xi_y) - b_h(U_y,\xi_y)] \leq Cs\left(1 + \frac{\sup_{I_m(s)} \|U_y - u\|^2}{h^2}\right)(h^{2p+1} + \sup_{I_m(s)} \|\xi_y\|^2),$$

$$Q_s^m[b_h(u,\tilde{\xi}_y) - b_h(U_y,\tilde{\xi}_y)] \leq Cs\left(1 + \frac{\sup_{I_m(s)} \|U_y - u\|^2}{h^2}\right)(h^{2p+1} + \sup_{I_m(s)} \|\xi_y\|^2).$$

Proof. The proof is analogical for both of these inequalities, so we will prove only the second (more difficult) one.

$$\begin{aligned}
Q_s^m[b(u,\tilde{\xi}_y) - b(U_y,\tilde{\xi}_y)] &= s\sum_{i=0}^{q} \omega_i \left(b_h(u,\tilde{\xi}_y) - b_h(U_y,\tilde{\xi}_y)\right)|_{t=t_{m-1}+s\psi_i} \\
&= s\sum_{i=0}^{q} \omega_i \frac{1}{\psi_i}\big(b_h(u, U_y - \Pi_s^m u) - b_h(U_y, U_y - \Pi_s^m u)\big)|_{t=t_{m-1}+s\psi_i} \\
&\leq s\frac{1}{\psi_0} \sup_{I_m(s)} \big(b_h(u, U_u - \Pi_s^m u) - b_h(U_y, U_u - \Pi_s^m u)\big).
\end{aligned}$$

Now it is sufficient to apply Lemma 4.3. □

Now, we shall prove the analogy to Lemmas 5.7 and 6.5.

Lemma 7.14. *Let $p > d/2$. Let $s \in (0,\tau]$ and $y = t_{m-1} + s$. If $\|e_y(t)\| \leq h^{1+d/2}$ for $t \in [0,y]$, then*

$$\sup_{t\in[0,y]} \|e_y(t)\|^2 \leq C_T^2(h^{2p+1} + \varepsilon h^{2p} + \tau^{2q+2}),$$

where the constant C_T is independent of h, τ, ε.

Proof. Again, it is sufficient to estimate the error only on the last time interval $I_m(s)$, the previous ones are treated similarly. The error equation reads

$$\begin{aligned}
\int_{I_m(s)} (\xi_y', v) + \varepsilon A_h(\xi_y, v)\mathrm{d}t + (\{\xi_y\}_{m-1}, v_+^{m-1}) &= Q_s^m[\varepsilon A_h(\eta_y, v)] - Q_s^m[(\eta_y', v)] \\
&\quad - (\{\eta_y\}_{m-1}, v_+^{m-1}) + Q_s^m[b_h(u,v) - b_h(U_y,v)].
\end{aligned}$$

By setting $v = 2\xi_y$ we get

$$\begin{aligned}
&\|\xi_y(y)\|^2 - \|\xi_{y-}^{m-1}\|^2 + \|\{\xi_y\}_{m-1}\|^2 + 2\varepsilon \int_{I_m(s)} \|\xi_y\|^2\mathrm{d}t \\
&\quad\leq Cs\varepsilon(h^{2p} + s^{2q+2}) + \varepsilon \int_{I_m(s)} \|\xi_y\|^2\mathrm{d}t + sC(h^{p+1} + s^{q+1}) \sup_{I_m(s)} \|\xi_y\| \\
&\quad\quad + Cs\Big(1 + \frac{\sup_{I_m(s)} \|U_y - u\|_\infty^2}{h^2}\Big)(h^{2p+1} + \sup_{I_m(s)} \|\xi_y\|^2).
\end{aligned}$$

Therefore

$$\|\xi_y(y)\|^2 - \|\xi_{y-}^{m-1}\|^2 + \varepsilon \int_{I_m(s)} \|\!|\xi_y|\!\|^2 \mathrm{d}t \le Cs\Big(1 + \frac{\sup_{I_m(s)} \|U_y - u\|_\infty^2}{h^2}\Big)(h^{2p+1} + \varepsilon h^{2p} + s^{2q+2} + \sup_{I_m(s)} \|\xi_y\|^2). \tag{7.8}$$

With the aid of Lemmas 7.5–7.7 we get

$$\begin{aligned} c \sup_{I_m(s)} \|\xi_y\|^2 \le \frac{1}{s}\int_{I_m(s)} \|\xi_y\|^2 &\le \int_{I_m(s)} (\xi_y', 2\tilde{\xi}_y)\mathrm{d}t + (\xi_{y+}^{m-1}, 2\tilde{\xi}_{y+}^{m-1}) \\ &\le \int_{I_m(s)} (\xi_y', 2\tilde{\xi}_y) + 2\varepsilon A_h(\xi_y, \tilde{\xi}_y)\mathrm{d}t + (\xi_{y+}^{m-1}, 2\tilde{\xi}_{y+}^{m-1}). \end{aligned} \tag{7.9}$$

By setting $v = 2\tilde{\xi}_y$ in the error equation we get

$$\begin{aligned} &\int_{I_m(s)} (\xi_y', 2\tilde{\xi}_y) + 2\varepsilon A_h(\xi_y, \tilde{\xi}_y)\mathrm{d}t + (\xi_{y+}^{m-1}, 2\tilde{\xi}_{y+}^{m-1}) \\ &= Q_s^m[\varepsilon A_h(\eta_y, 2\tilde{\xi}_y)] - Q_s^m[(\eta_y', 2\tilde{\xi}_y)] - (\{\eta_y\}_{m-1}, 2\tilde{\xi}_{y+}^{m-1}) + (\xi_{y-}^{m-1}, 2\tilde{\xi}_{y+}^{m-1}) + Q_s^m[b(u, 2\tilde{\xi}_y) - b(U_y, 2\tilde{\xi}_y)] \\ &\le Cs\varepsilon(h^{2p} + s^{2q+2}) + (\xi_{y-}^{m-1}, 2\tilde{\xi}_{y+}^{m-1}) + \varepsilon\int_{I_m(s)} \|\!|\xi_y|\!\|^2\mathrm{d}t + sC(h^{p+1} + s^{q+1}) \sup_{I_m(s)} \|\xi_y\| \\ &\quad + Cs\Big(1 + \frac{\sup_{I_m(s)} \|U_y - u\|_\infty^2}{h^2}\Big)(h^{2p+1} + \sup_{I_m(s)} \|\xi_y\|^2) \\ &\le Cs\Big(1 + \frac{\sup_{I_m(s)} \|U_y - u\|_\infty^2}{h^2}\Big)(h^{2p+1} + \varepsilon h^{2p} + s^{2q+2} + \sup_{I_m(s)} \|\xi_y\|^2) + \frac{2C}{c}\|\xi_{y-}^{m-1}\|^2 + \frac{c}{4}\sup_{I_m(s)} \|\xi_y\|^2, \end{aligned} \tag{7.10}$$

where $\varepsilon\int_{I_m(s)} \|\!|\xi_y|\!\|^2\mathrm{d}t$ is estimated with the aid of (7.8).

Under the assumption $\|e(t)\| \le h^{1+d/2}$ inequality (7.8) can be simplified to

$$\|\xi_y(y)\|^2 - \|\xi_{y-}^{m-1}\|^2 \le Cs(h^{2p+1} + \varepsilon h^{2p} + s^{2q+2} + \sup_{I_m(s)} \|\xi_y\|^2) \tag{7.11}$$

and inequalities (7.9) and (7.10) give

$$\sup_{I_m(s)} \|\xi_y\|^2 \le \frac{C}{c}s(h^{2p+1} + \varepsilon h^{2p} + s^{2q+2} + \sup_{I_m(s)} \|\xi_y\|^2) + \frac{2C}{c^2}\|\xi_{y-}^{m-1}\|^2 + \frac{1}{4}\sup_{I_m(s)} \|\xi_y\|^2.$$

If $s \le \tau \le c/4C$ than the last inequality can be simplified to

$$\sup_{I_m(s)} \|\xi_y\|^2 \le h^{2p+1} + \varepsilon h^{2p} + s^{2q+2} + \frac{4C}{c^2}\|\xi_{y-}^{m-1}\|^2$$

Substituting this estimate into (7.11), we get

$$\|\xi(y)\|^2 - \|\xi_{y-}^{m-1}\|^2 \le Cs(h^{2p+1} + \varepsilon h^{2p} + s^{2q+2} + \|\xi_{y-}^{m-1}\|^2).$$

Similar estimates can be obtained on all previous time intervals. By application of the discrete Gronwall lemma we finish the proof. □

Theorem 7.15. *Let $p > 1 + d/2$. Let τ_0 be defined as in Lemma 7.9. Let $h \in (0, h_0)$ and $\tau_1 \in (0, \tau_0)$ be such that*

$$C_T^2(h^{2p+1} + \varepsilon h^{2p} + \tau^{2q+2}) \le \frac{1}{16}h^{2+d},$$

where C_T is the constant from Lemma 7.14 independent of h, τ, ε. Then the error of the QT-DG scheme satisfies

$$\sup_{t\in[0,T]} \|e(t)\|^2 \le C_T^2(h^{2p+1} + \varepsilon h^{2p} + \tau^{2q+2}).$$

Proof. Since the continuation $U_y(t)$ now depends on two variables, y and t, we proceed more carefully. We define the propositional function φ by

$$\varphi(y) \equiv \Big\{ \max_{t\in[0,y]} \|e_y(t)\|^2 \le C_T^2(h^{2p+1} + \varepsilon h^{2p} + \tau^{2q+2})\Big\}.$$

Due to the approximation of the initial condition, $\varphi(0)$ holds trivially. We want to prove $\varphi(T)$. We will proceed by continuous induction, *cf*. [17]. For this we need to prove that

$$\begin{aligned} &(A) \quad \forall y \in [0, T)\ \exists \delta(y) > 0 :\ \varphi(y) \text{ implies } \varphi(y + \delta),\ \forall \delta \in [0, \delta(y)] : y + \delta \in [0, T]\\ &(B) \quad \forall y_1, y_2 \in [0, T], y_1 < y_2 : \text{If } \varphi \text{ holds on } (y_1, y_2) \text{ then } \varphi(y_2) \text{ holds.} \end{aligned} \tag{7.12}$$

First we note, that due to the construction of U, U_y, it is sufficient to assume $y, y + \delta \in [t_{m-1}, t_m]$ and then proceed by induction with respect to $m = 1, \ldots, r$. Our main tools will be the continuity of U_y with respect to y, *cf*. Lemma 7.11, the uniform boundedness of $\|U_y'(t)\|$ with respect to t and y, *cf*. Lemma 7.9 and uniform continuity of u from $[t_{m-1}, t_m]$ to $L^2(\Omega)$. Specifically, if $y \in [t_{m-1}, t_m)$ there exists $\delta(y) > 0$ such that

$$\begin{aligned} \delta \in [0, \delta(y)], t \in [y, y + \delta] \implies \|u(y) - u(t)\| \le \frac{1}{4}h^{1+d/2},\\ \delta \in [0, \delta(y)] \implies \sup_{(t_{m-1},y)} \|U_{y+\delta} - U_y\| \le \frac{1}{4}h^{1+d/2}. \end{aligned}$$

Without loss of generality, $\delta(y)$ can be taken small enough so that $C\delta(y) \le \frac{1}{4}h^{1+d/2}$, where C is the uniform bound for $\|U_y'(t)\|$ from Lemma 7.9.

Induction step (A): Let us assume that $\varphi(y)$ holds. We want to prove that $\varphi(y+\delta)$ holds, where $\delta \in [0, \delta(y)]$. In other words, we want to estimate

$$\max_{t\in[0,y+\delta]} \|e_{y+\delta}(t)\| = \max\{\max_{t\in[0,y]} \|e_{y+\delta}(t)\|, \max_{t\in[y,y+\delta]} \|e_{y+\delta}(t)\|\}. \tag{7.13}$$

We estimate the first right-hand side term in (7.13) by

$$\begin{aligned} \max_{t\in[0,y]} \|e_{y+\delta}(t)\| &= \max_{t\in[0,y]} \|U_{y+\delta}(t) - u(t)\| \le \max_{t\in[0,y]} \|U_{y+\delta}(t) - U_y(t)\| + \max_{t\in[0,y]} \|U_y(t) - u(t)\|\\ &= \max_{t\in[t_{m-1},y]} \|U_{y+\delta}(t) - U_y(t)\| + \max_{t\in[0,y]} \|e_y(t)\| \le \frac{1}{4}h^{1+d/2} + C_T\sqrt{h^{2p+1} + \varepsilon h^{2p} + \tau^{2q+2}} \le \frac{1}{2}h^{1+d/2} \end{aligned} \tag{7.14}$$

by Lemma 7.11 and the induction assumption. As for the second right-hand side term in (7.13), we have

$$\begin{aligned} \max_{t\in[y,y+\delta]} \|e_{y+\delta}(t)\| &= \max_{t\in[y,y+\delta]} \|U_{y+\delta}(t) - u(t)\|\\ &\le \max_{t\in[y,y+\delta]} \|U_{y+\delta}(t) - U_{y+\delta}(y)\| + \|U_{y+\delta}(y) - U_y(y)\| + \|U_y(y) - u(y)\| + \max_{t\in[y,y+\delta]} \|u(y) - u(t)\|\\ &\le \delta \max_{t\in[t_{m-1},y]} \|U_{y+\delta}'(t)\| + \max_{t\in[0,y]} \|U_{y+\delta}(t) - U_y(t)\| + \max_{t\in[0,y]} \|e_y(t)\| + \frac{1}{4}h^{1+d/2}\\ &\le C\delta + \frac{1}{4}h^{1+d/2} + C_T\sqrt{h^{2p+1} + \varepsilon h^{2p} + \tau^{2q+2}} + \frac{1}{4}h^{1+d/2} \le h^{1+d/2}, \end{aligned} \tag{7.15}$$

due to Lemmas 7.9, 7.11 and the induction assumption. Collecting (7.13)–(7.15) gives us

$$\max_{t\in[0,y+\delta]} \|e_{y+\delta}(t)\| \le h^{1+d/2}. \tag{7.16}$$

Lemma 7.14 then gives us $\varphi(y+\delta)$.

Induction step (B)*:* We prove (B) in (7.12) by contradiction. Fix $y_1, y_2 \in [0,T]$. Assume that for all $y \in (y_1,y_2)$ the statement $\varphi(y)$ holds, but $\varphi(y_2)$ is false. In other words assume that

$$\max_{t\in[0,y]} \|e_y(t)\|^2 \le C_T^2(h^{2p+1}+\varepsilon h^{2p}+\tau^{2q+2}) \qquad \text{and} \qquad \max_{t\in[0,y_2]} \|e_{y_2}(t)\|^2 > C_T^2(h^{2p+1}+\varepsilon h^{2p}+\tau^{2q+2}). \tag{7.17}$$

Therefore, after taking the square root,

$$\max_{t\in[0,y_2]} \|e_{y_2}(t)\| - \max_{t\in[0,y]} \|e_y(t)\| \ge c_0 > 0, \qquad \text{for all } y \in (y_1,y_2), \tag{7.18}$$

where $c_0 > 0$ is an appropriate constant independent of $y \in (y_1,y_2)$.

We can estimate by the triangle inequality

$$\max_{t\in[y,y_2]} \|e_{y_2}(t)\| \le \|e_{y_2}(y)\| + \max_{t\in[y,y_2]} \|e_{y_2}(t)-e_{y_2}(y)\| \le \max_{t\in[0,y]} \|e_{y_2}(t)\| + C|y_2-y|,$$

since u is uniformly continuous and $U'_y(t)$ is uniformly bounded with respect to y,t. Therefore,

$$\max_{t\in[0,y_2]} \|e_{y_2}(t)\| \le \max\{\max_{t\in[0,y]} \|e_{y_2}(t)\|, \max_{t\in[y,y_2]} \|e_{y_2}(t)\|\} \le \max_{t\in[0,y]} \|e_{y_2}(t)\| + C|y_2-y|.$$

Hence, the left-hand side of (7.18) can be estimated as

$$\begin{aligned}
\max_{t\in[0,y_2]}\|e_{y_2}(t)\| - \max_{t\in[0,y]} \|e_y(t)\| &\le \max_{t\in[0,y]} \|e_{y_2}(t)\| + C|y_2-y| - \max_{t\in[0,y]} \|e_y(t)\| \\
&\le \max_{t\in[0,y]} \|U_{y_2}(t)-U_y(t)\| + \max_{t\in[0,y]} \|U_y(t)-u(t)\| + C|y_2-y| - \max_{t\in[0,y]} \|e_y(t)\| \\
&= \max_{t\in[0,y]} \|U_{y_2}(t)-U_y(t)\| + C|y_2-y| \longrightarrow 0, \qquad \text{as } y \to y_2,
\end{aligned} \tag{7.19}$$

which is a contradiction with (7.18), *i.e.* (7.17). Thus (B) is proved, which completes the proof. □

8. Conclusions

We have proved *a priori* error estimates for the discontinuous Galerkin method applied to a nonlinear time-dependent singularly perturbed, convection-diffusion problem. The BDF-2, midpoint and quadrature version of the space-time DG scheme were analyzed. The main contribution of the paper is that $L^\infty(L^2)$-estimates are derived that are uniform with respect to the diffusion parameter $\varepsilon \to 0$ and valid even in the purely convective case $\varepsilon = 0$. The paper extends the work [16], where similar estimates were derived for the space-semidiscretization and implicit Euler scheme as well as the paper [17], where similar estimates are obtained for the conforming finite element method. The basis of the technique is the idea of [24], where the analysis is carried out for an explicit Runge–Kutta scheme in time.

Similarly as in [16], the presented error analysis is based on construction of suitable continuations of the discrete solution with respect to time and performing, *via* induction. The resulting estimates are of the order $O(h^{p+1/2}+\varepsilon h^p+\tau^4)$ for the BDF-2 and midpoint schemes and $O(h^{p+1/2}+\varepsilon h^p+\tau^{q+1})$ for q-order quadrature time-DG. The estimates are derived under the CFL-like $\tau = O(h)$ condition guaranteeing the unique existence and continuity of the continuation. Furthermore, the estimates are derived under the order condition $p > 1+d/2$, or $p > (1+d)/2$ for $\varepsilon = 0$, where d is the spatial dimension of the problem.

Future work includes removing of the CFL and order conditions and extension to more difficult equations, *e.g.* nonlinear diffusion as in [15], derivation of optimal order $L^\infty(L^2)$-error estimates and analysis of other temporal discretizations, especially the space-time DG scheme without quadratures.

References

[1] G. Akrivis and C. Makridakis, Galerkin time-stepping methods for nonlinear parabolic equations. *ESAIM: M2AN* **38** (2004) 261–289.

[2] T. Barth and M. Ohlberger, Finite Volume Methods: Foundation and Analysis. Vol. 1 of *Encyclopedia of Computational Mechanics*. John Wiley & Sons, Chichester, New York, Brisbane (2004) 439–474.

[3] F. Bassi and S. Rebay, High-Order Accurate Discontinuous Finite Element Solution of the 2D Euler Equations. *J. Comput. Phys.* **138** (1997) 251–285.

[4] P.G. Ciarlet, *The finite element methods for elliptic problems*. North-Holland, Amsterdam (1978).

[5] B. Cockburn and C.-W. Shu, The Local Discontinuous Galerkin Method for Time-Dependent Convection-Diffusion Systems. *SIAM J. Numer. Anal.* **35** (1998) 2440–2463.

[6] V. Dolejší and M. Feistauer, *Discontinuous Galerkin Method: Analysis and Applications to Compressible Flow*. Springer (2015).

[7] V. Dolejší and M. Vlasák, Analysis of a BDF-DG scheme for nonlinear convection-diffusion problems. *Numer. Math.* **110** (2008) 405–447.

[8] V. Dolejší, M. Feistauer and V. Sobotíková, Analysis of the discontinuous Galerkin method for nonlinear convection-diffusion problems. *Comput. Methods Appl. Mech. Engrg.* **194** (2005) 2709–2733.

[9] V. Dolejší, M. Feistauer and J. Hozman, Analysis of semi-implicit DGFEM for nonlinear convection-diffusion problems. *Comput. Methods Appl. Mech. Engrg.* **196** (2007) 2813–2827.

[10] V. Dolejší, M. Feistauer, V. Kučera and V. Sobotíková, An optimal $L^\infty(L^2)$-error estimate for the discontinuous Galerkin approximation of a nonlinear non-stationary convection-diffusion problem. *IMA J. Numer. Anal.* **28** (2008) 496–521.

[11] M. Feistauer and V. Kučera, On a robust discontinuous Galerkin technique for the solution of compressible flow. *J. Comput. Phys.* **224** (2007) 208–221.

[12] M. Feistauer, J. Hájek and K. Švadlenka, Space-time discontinuous Galerkin method for solving nonstationary convection-diffusion-reaction problems. *Appl. Math.* **52** (2007) 197–233.

[13] S. Gianni, D. Schötzau and L. Zhu, An a posteriori error estimate for hp-adaptive DG methods for convection-diffusion problems on anisotropically refined meshes. *Comp. Math. Appl.* **67** (2014) 869–887.

[14] E. Hairer, S.P. Norsett and G. Wanner, *Solving ordinary differential equations I, Nonstiff problems*. Springer Verlag (2000).

[15] V. Kučera, Optimal $L^\infty(L^2)$-error Estimates for the DG Method Applied to Nonlinear Convection-Diffusion Problems with Nonlinear Diffusion. *Numer. Func. Anal. Opt.* **31** (2010) 285–312.

[16] V. Kučera, On diffusion-uniform error estimates for the DG method applied to singularly perturbed problems. *IMA J. Numer. Anal.* **32** (2014) 820–861.

[17] V. Kučera, Finite element error estimates for nonlinear convective problems. *J. Numer. Math.* **24** (2016) 143–165.

[18] S. Osher, Riemann solvers, the entropy condition, and difference approximations. *SIAM. J. Numer. Anal.* **21** (1984) 217–235.

[19] W.H. Reed and T.R. Hill, *Triangular mesh methods for the neutron transport equation*. Technical Report LA–UR–73–479, Los Alamos Scientific Laboratory (1973).

[20] M. Vlasák, Optimal spatial error estimates for DG time discretizations. *J. Numer. Math.* **21** (2013) 201–230.

[21] M. Vlasák and H.G. Roos, An optimal uniform *a priori* error estimate for an unsteady singularly perturbed problem. *Int. J. Numer. Anal. Model.* **11** (2014) 24–33.

[22] M. Vlasák, V. Dolejší and J. Hájek, *A priori* error estimates of an extrapolated space-time discontinuous Galerkin method for nonlinear convection-diffusion problems. *Numer. Methods Partial Differ. Eqs.* **27** (2011) 1456–1482.

[23] E. Zeidler, *Nonlinear Functional Analysis and Its Applications II/B: Nonlinear Monotone Operators*. Springer, Heidelberg (1986).

[24] Q. Zhang and C.W. Shu, Error estimates to smooth solutions of Runge-Kutta discontinuous Galerkin method for symmetrizable systems of conservation laws. *SIAM J. Numer. Anal.* **44** (2006) 1703–1720.