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EXISTENCE OF SOLUTIONS TO AN ELASTO-VISCOPLASTIC MODEL WITH KINEMATIC HARDENING AND r-LAPLACIAN FRACTURE APPROXIMATION

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Abstract. This paper deals with an existence theorem for a model describing an elasto-viscoplastic evolution of a 2D material with linear kinematic hardening and fracture where the Griffith fracture energy is regularized using a r-Laplacian.

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1. Introduction

The goal of this paper is a mathematical study of a model that takes into consideration three dissipative terms: plastic flow, fracture and viscoplastic dissipation. The study of a such model is motivated by the modeling of the Earth crust considered as an elasto-visco-plastic solid in which cracks are allowed to propagate. This hypothesis is qualitatevely supported by analogue 2D-experiments of Peltzer and Tapponnier [20] that show faults propagation in a layer of plasticine. Unfortunately, the rheology of plasticine is not well known.

For those reasons, we proposed a class of models that combine several dissipation phenomena: anelastic deformation, fracture and viscous dissipation [3, 16] and studied numerically if such models can reproduce (at least partially) Peltzer and Tapponnier experiment. In particular, the numerical experiments have shown that a model combining plasticity with kinematic hardening and regularized fracture permits to express simultaneously the dissipation phenomena as observed in plasticine. Kinematic hardening allows the translation of the yield surface and thus the elastic energy can increase after plastification, so that plastification does not prevent the appearance of cracks.

For this reason, we study from mathematical point of view a model in $\mathbb{R}^2$ of elasto-plastic material with kinematic hardening and regularized fracture that may account for the behaviour observed in the plasticine experiments. For the plastic behaviour, we use a similar visco-plastic approximation as in [10, 11, 21].

To model the fracture, we use the approximate models to the variational fracture model proposed by Francfort and Marigo [14]. In our model, we only consider fracture *via* a diffuse interface model using Ambrosio−Tortorelli functional. In other words, the geometry of possible cracks is captured by a function v with values between 0

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and 1, $v = 1$ in the healthy parts that do not contain cracks. A convenience of a such model is the fact, that it can be studied numerically, see [4–7] for the numerical studies in elastic case and [3, 16] for the numerical studies of the elasto-plastic models with fracture in the case of traction and plasticine experiments.

An existence result for a quasi-static evolution of the elastic model with the Ambrosio−Tortorelli functional was proposed by Giacomini [15], and by Larsen, Ortner, Suli [18] for an elasto-dynamic evolution with regularized fracture.

Particularly, in this paper, we prove an existence result for a continuous elasto-viscoplastic model with kinematic hardening and regularized fracture using a r-Laplacian [2, 13] fracture approximation in $\mathbb{R}^2$ with $r > 2$, but our results extend to any dimension $n > 2$ such that $r > n$. We can also prove an existence result for a 2D visco-elasto-plastic model with regularized fracture in the case $r = 2$ that could reproduce plasticine experiments (see [17]). A model coupling perfect plasticity and brittle fracture was studied in [9] and an other model coupling plasticity with damage in [8].

The unknowns of our model are u a displacement field, e an elastic strain, p a plastic strain, v a phase field variable representing fracture. In our case, we will consider a modified Ambrosio−Tortorelli functional, for all $(e, v) \in L^2(\Omega, \mathbb{M}^{2\times 2}_{\text{sym}}) \times W^{1,r}(\Omega, \mathbb{R})$,

$$\mathcal{E}_\varepsilon(e, v) := \frac{1}{2}\int_\Omega \left(v^2 + \eta\right) Ae : e \,\mathrm{d}x + \int_\Omega \frac{\varepsilon^{r-1}}{r}|\nabla v|^r \mathrm{d}x + \int_\Omega \frac{\alpha}{r'\varepsilon}|1 - v|^r \,\mathrm{d}x,$$

where $\alpha > 0$ is a some regularization constant and $r' = r/(r - 1)$ with $r > 2$. The advantage of r-Laplacian approach is the gain of compactness on the variables v, and then (u, e, p), of sequences of approximate solutions.

Babadjian, Francfort, Mora [1] studied an evolution elasto-visco-plastic model and proved that the approximate semi-discrete time solutions $(e_h)_h$, $(p_h)_h$ are Cauchy sequences in $L^\infty(0, T_f, L^2(\Omega, \mathbb{M}^{2\times 2}_{\text{sym}}))$. This result allows passage to the limit in the discrete plastic-flow rule and proving an existence result for the continuous elasto-visco-plastic model. The presence of v in our model requires the control of some additional terms (see Lem. 3.7). In our model, to pass to the limit in the discrete plastic flow rule and in discrete fracture propagation condition, we prove particularly that for fixed $t \in (0, T_f]$, $e^+_h(t)$, $p_h(t)$, $\dot{p}_h(t)$, the piecewise constant and affine interpolants defined in Section 3.2, are Cauchy sequences in $L^2(\Omega, \mathbb{M}^{2\times 2}_{\text{sym}})$. This compactness result is proved using Helly's selection principle [19].

The paper is organised as follows. After a short introduction, Section 2 is devoted to the definitions, mathematical and mechanical settings. This is followed by the model description. In Section 3, we prove the existence of solutions for discrete variational problem. Then, we study the convergence of these approximate evolutions as the time step $h \to 0$. Finally, the main result of the paper is an existence theorem for elasto-viscoplastic model with kinematic linear hardening and fracture. There exists at least one evolution (u, v, e, p) satisfying Theorem 2.1.

2. Description of the model

2.1. Preliminaries and mathematical setting

Throughout the paper, Ω is a bounded connected open set in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ where $\partial\Omega_D$, $\partial\Omega_N$ are disjoint relatively open sets in $\partial\Omega$. Given $T_f > 0$, we denote by $L^p((0, T_f), X)$, $W^{k,p}((0, T_f), X)$, the Lebesgue and Sobolev spaces involving time [see [12] p. 285], where X is a Banach space. We note for $1 \le p \le \infty$ the L^p-norm by $\| \,.\, \|_p$. The set of symmetric 2×2 matrices is denoted by $\mathbb{M}^{2\times 2}_{\text{sym}}$. For $\xi, \zeta \in \mathbb{M}^{2\times 2}_{\text{sym}}$ we define the scalar product between matrices $\zeta : \xi := \sum_{ij} \zeta_{ij}\xi_{ij}$, and the associated matrix norm by $|\xi| := \sqrt{\xi : \xi}$. Let A be the fourth order tensor of Lamé coefficients and B a suitable symmetric-fourth order tensor. We assume that for some constants $0 < \alpha_1 \le \alpha_2 < \infty$, they satisfy the ellipticity conditions

$$\forall\, e \in \mathbb{M}^{2\times 2}_{\text{sym}}, \quad \alpha_1|e|^2 \le Ae : e \le \alpha_2|e|^2 \quad \text{and} \quad \alpha_1|e|^2 \le Be : e \le \alpha_2|e|^2.$$

We recall that the mechanical unknowns of our model are the displacement field $u : \Omega \times [0, T_f] \to \mathbb{R}^2$, the elastic strain $e : \Omega \times [0, T_f] \to \mathbb{M}^{2\times 2}_{\text{sym}}$, the plastic strain $p : \Omega \times [0, T_f] \to \mathbb{M}^{2\times 2}_{\text{sym}}$. We assume u and ∇u remain small. So that the relation between the deformation tensor E and the displacement field is given by

$$Eu := \frac{1}{2}(\nabla u + \nabla u^T).$$

We also assume that Eu decomposes as an elastic part and a plastic part

$$Eu = e + p.$$

For $w \in H^1(0, T_f, H^1(\Omega, \mathbb{R}^2))$, which represents an applied boundary displacement, we define for $t \in [0, T_f]$ the set of kinematically admissible fields by

$$\begin{aligned} A_{adm}(w(t)) := \{(u, e, p) \in H^1(\Omega, \mathbb{R}^2) \times L^2(\Omega, \mathbb{M}^{2\times 2}_{\text{sym}}) \times L^2(\Omega, \mathbb{M}^{2\times 2}_{\text{sym}}) \ : \\ Eu = e + p \quad \text{a.e. in } \Omega,\ u = w(t) \quad \text{a.e. on } \partial\Omega_D\}. \end{aligned}$$

For a fixed constant $\tau > 0$, we define $K := \{q \in \mathbb{M}^{2\times 2}_{\text{sym}};\ |q| \le \tau\}$ and $H : \mathbb{M}^{2\times 2}_{\text{sym}} \to [0, \infty]$ the support function of K by

$$H(p) := \sup_{\theta \in K} \ \theta : p = \tau |p|.$$

For $\eta > 0$, the elastic energy is defined as

$$\begin{aligned} &\mathcal{E}_{el} \ : \ L^2(\Omega, \mathbb{M}^{2\times 2}_{\text{sym}}) \times W^{1,r}(\Omega, \mathbb{R}) \to \mathbb{R} \\ &(e, v) \longmapsto \mathcal{E}_{el}(e, v) = \frac{1}{2} \int_\Omega \left(v^2 + \eta\right) Ae : e \, \mathrm{d}x. \end{aligned}$$

In the following, we will define an evolution as a limit of time discretizations with a step h: p and p_0 represent the plastic deformation at 2 consecutive time steps, so that $\dfrac{p - p_0}{h} \sim \dot{p}$. The plastic dissipated energy is defined, by

$$\begin{aligned} &\mathcal{E}_p \ : \ L^2(\Omega, \mathbb{M}^{2\times 2}_{\text{sym}}) \times L^2(\Omega, \mathbb{M}^{2\times 2}_{\text{sym}}) \to \mathbb{R} \\ &(p, p_0) \longmapsto \mathcal{E}_p(p, p_0) = \int_\Omega H(p - p_0) \, \mathrm{d}x, \end{aligned}$$

and the hardening energy by

$$\begin{aligned} &\mathcal{E}_{KH} \ : \ L^2(\Omega, \mathbb{M}^{2\times 2}_{\text{sym}}) \to \mathbb{R} \\ &p \longmapsto \mathcal{E}_{KH}(p) = \frac{1}{2} \int_\Omega Bp : p \, \mathrm{d}x. \end{aligned}$$

Given $\beta > 0$, the viscoplastic energy is defined by

$$\begin{aligned} &\mathcal{E}_{vp} \ : \ L^2(\Omega, \mathbb{M}^{2\times 2}_{\text{sym}}) \times L^2(\Omega, \mathbb{M}^{2\times 2}_{\text{sym}}) \to \mathbb{R} \\ &(p, p_0) \longmapsto \mathcal{E}_{vp}(p, p_0) = \frac{\beta}{2h} \int_\Omega (p - p_0) : (p - p_0) \, \mathrm{d}x. \end{aligned}$$

For $r > 2$, ans $\varepsilon > 0$, we define the phase-field surface energy

$$\begin{aligned} &\mathcal{E}^r_S \ : \ W^{1,r}(\Omega, \mathbb{R}) \to \mathbb{R} \\ &\quad v \quad \longmapsto \mathcal{E}^r_S(v) = \int_\Omega \frac{\varepsilon^{r-1}}{r} |\nabla v|^r \mathrm{d}x + \int_\Omega \frac{\alpha}{r'\varepsilon} |1 - v|^r \, \mathrm{d}x \end{aligned}$$

where $r' := \dfrac{r}{r-1}$ and $\alpha := \left(\dfrac{r}{2}\right)^{r'}$. In the next section we describe the evolution of the proposed model.

2.2. The evolution in elasto-viscoplastic model with linear kinematic hardening and fracture

Consider $w \in H^1(0, T_f, H^1(\Omega, \mathbb{R}^2))$. We define the evolution of the model by a seeking functions

$$(u, v, e, p) : \Omega \times [0, T_f] \longrightarrow \mathbb{R}^2 \times \mathbb{R} \times \mathbb{M}^{2\times 2}_{\text{sym}} \times \mathbb{M}^{2\times 2}_{\text{sym}}$$

that satisfy the following conditions:

- (A1) Initial condition: $(u(0), v(0), e(0), p(0)) = (u_0, v_0, e_0, p_0)$ with $(u_0, e_0, p_0) \in A_{adm}(w(0))$, and $v_0 \in W^{1,r}(\Omega)$ with $v_0 = 1$ on $\partial\Omega_D$ and $0 \leq v_0 \leq 1$ a.e. in Ω, such that $-\operatorname{div} \sigma_0 = 0$ a.e. in Ω where $\sigma_0 := (v_0^2 + \eta)Ae_0$ and $\sigma_0.\boldsymbol{n} = 0$ on $\partial\Omega_N$. $\boldsymbol{n}$ is outward normal to $\partial\Omega$.
- (A2) Irreversibility condition: $0 \leq v(t) \leq v(s) \leq 1$ in Ω for every $0 \leq s \leq t \leq T_f$.
- (A3) Kinematic compatibility: for every $t \in [0, T_f]$,

$$(u(t), e(t), p(t)) \in A_{adm}(w(t)).$$

- (A4) Equilibrium condition: for $t \in [0, T_f]$,

$$\begin{cases} -\operatorname{div}(\sigma(t)) = 0, & \text{a.e. in} \quad \Omega, \\ \sigma(t).\boldsymbol{n} = 0, & \text{a.e. on} \quad \partial\Omega_N, \\ (u(t), v(t)) = (w(t), 1), & \text{a.e on} \quad \partial\Omega_D, \end{cases}$$

  where $\sigma(t) = (v(t)^2 + \eta)Ae(t)$.
- (A5) Plastic flow rule: for a.e. $t \in [0, T_f]$,

$$\sigma(t) - Bp(t) - \beta\dot{p}(t) \in \partial H(\dot{p}(t)) \quad \text{for a.e.} \quad x \in \Omega.$$

- (A6) Crack propagation condition: for $t \in [0, T_f]$,

$$\mathcal{E}_{el}(e(t), v(t)) + \mathcal{E}^r_S(v(t)) = \inf_{v=1 \text{ on } \partial\Omega_D, v \leq v(t),\, v(t) \in W^{1,r}(\Omega)} \mathcal{E}_{el}(e(t), v) + \mathcal{E}^r_S(v).$$

  The condition $v = 1$ on $\partial\Omega_D$ means that the Dirichlet part of boundary do not crack.

The main result of the paper is the following existence result.

Theorem 2.1. *There exists at least one evolution*

$$\begin{cases} u \in L^\infty(0, T_f, H^1(\Omega, \mathbb{R}^2)), \\ v \in L^\infty(0, T_f, W^{1,r}(\Omega, \mathbb{R})), \\ e \in L^\infty(0, T_f, L^2(\Omega, \mathbb{M}^{2\times 2}_{sym})), \\ p \in W^{1,\infty}(0, T_f, L^2(\Omega, \mathbb{M}^{2\times 2}_{sym})), \end{cases}$$

that satisfies (A1)−(A6).

Remark 2.2. We present our result for a simplified model of plasticity, in which one assumes that the yield set depends on the whole stress tensor, and is a ball. More physical models only assume that K is a closed convex set with non-empty interior, and constraints only the deviatoric part of the stress. Nevertheless, our arguments can easily be extended to this general case. The only noticeable changes concern the regularity of the plastic strain p when K is not bounded. In that case, one obtains estimates of p in $H^1(L^2)$ instead of the $W^{1,\infty}(L^2)$ estimates shown here. However, this does not fundamentally affect our arguments (see the Rems. 3.4, 3.9, 3.11 and 3.13).

3. Proof of the existence theorem

3.1. Time discretization

The proof of Theorem 2.1 is based on a time discretization. It the whole paper $C > 0$ denotes a generic constant which is independent of the discretization parameters. Let us consider a partition of the time interval $[0, T_f]$ into N_f sub-intervals of equal length h:

$$0 = t_h^0 < t_h^1 < \ldots < t_h^n < \ldots < t_h^{N_f} = T_f, \quad \text{with} \quad h = \frac{T_f}{N_f} = t_h^n - t_h^{n-1} \to 0.$$

Let $v_h^0 = v_0$, $u_h^0 = u_0$, $e_h^0 = e_0$, $p_h^0 = p_0$. We suppose that v_0 satisfies the crack propagation condtion (A6). For $n = 0, \ldots, N_f$, we set $w_h^n := w(t_h^n)$. We also define the total energy

$$\begin{aligned}
\mathcal{E}_{\text{total}}(z,\phi,\xi,q) =& \frac{1}{2}\int_\Omega (\phi^2+\eta)A\xi : \xi \,\mathrm{d}x + \frac{1}{2}\int_\Omega Bq : q \,\mathrm{d}x \\
&+ \frac{1}{2h}\beta \parallel q - p_h^{n-1} \parallel_2^2 + \tau \int_\Omega |q - p_h^{n-1}| \,\mathrm{d}x \\
&+ \int_\Omega \frac{\varepsilon^{r-1}}{r} |\nabla \phi|^r \mathrm{d}x + \int_\Omega \frac{\alpha}{r'\varepsilon} |1-\phi|^r \,\mathrm{d}x \\
=& \mathcal{E}_{el}(\phi,\xi) + \mathcal{E}_{KH}(q) + \mathcal{E}_{vp}(q, p_h^{n-1}) + \mathcal{E}_p(q, p_h^{n-1}) + \mathcal{E}_S^r(\phi).
\end{aligned}$$

Proposition 3.1. *Given $(u_h^{n-1}, v_h^{n-1}, e_h^{n-1}, p_h^{n-1})$ that satisfy $(u_h^{n-1}, e_h^{n-1}, p_h^{n-1}) \in A_{adm}(w_h^{n-1})$, $v_h^{n-1} \in W^{1,r}(\Omega)$, $0 \le v_h^{n-1} \le 1$, $v_h^{n-1} = 1$ on $\partial\Omega_D$. There exist a minimizer $(u_h^n, v_h^n, e_h^n, p_h^n)$ to the variational problem*

$$\min_{(z,\xi,q)\in A_{adm}(w_h^n),\, \phi\in W^{1,r}(\Omega),\, \phi\le v_h^{n-1},\, \phi=1 \text{ on } \partial\Omega_D} \mathcal{E}_{\text{total}}(z,\phi,\xi,q). \tag{3.1}$$

Proof. Since $(w_h^n, v_h^{n-1}, Ew_h^n, 0)$ is admissible for (3.1), we have that

$$m := \inf_{(z,\xi,q)\in A_{adm}(w_h^n),\, \phi\in W^{1,r}(\Omega),\, \phi\le v_h^{n-1},\, \phi=1 \text{ on } \partial\Omega_D} \mathcal{E}_{\text{total}}(z,\phi,\xi,q) < \infty$$

Let (u_k, v_k, e_k, p_k) be a minimizing sequence. It follows from the Poincaré inequality and the Korn inequalities that

$$\parallel u_k \parallel_{H^1} + \parallel v_k \parallel_{W^{1,r}} + \parallel e_k \parallel_{L^2} + \parallel p_k \parallel_{L^2} \le C_{n,h}.$$

Therefore can be extracted a subsequence (u_k, v_k, e_k, p_k) such that

$$\begin{aligned}
u_k &\rightharpoonup u \quad \text{in} \quad H^1(\Omega, \mathbb{R}^2), \\
v_k &\rightharpoonup v \quad \text{in} \quad W^{1,r}(\Omega, \mathbb{R}), \\
e_k &\rightharpoonup e \quad \text{in} \quad L^2(\Omega, \mathbb{M}^{2\times 2}_{\text{sym}}), \\
p_k &\rightharpoonup p \quad \text{in} \quad L^2(\Omega, \mathbb{M}^{2\times 2}_{\text{sym}}).
\end{aligned}$$

It follows that $(u, e, p) \in A_{adm}(w_h^n)$ and since $r > 2$, $v_k \to v$ in $C^0(\overline{\Omega})$ by the Sobolev imbedding theorem. As $v_k \le v_h^{n-1}$ and $v_k = 1$ on $\partial\Omega_D$ for all k, we have $v \le v_h^{n-1}$ and $v = 1$ on $\partial\Omega_D$. Furthermore, $v_k e_k \rightharpoonup ve$ weakly in $L^2(\Omega)$. By lower semicontinuity,

$$\int_\Omega v^2 Ae : e \,\mathrm{d}x = \int_\Omega Ave : ve \,\mathrm{d}x \le \liminf_{k\to\infty} \int_\Omega Av_k e_k : v_k e_k \,\mathrm{d}x,$$

and

$$\int_\Omega (v^2+\eta)Ae : e \,\mathrm{d}x \le \liminf_{k\to\infty} \int_\Omega (v_k^2+\eta)Ae_k : e_k \,\mathrm{d}x.$$

The other terms of $\mathcal{E}_{\text{total}}$ are weakly lower semicontinuous with respect to the weak topology $H^1(\Omega,\mathbb{R}^2)\times W^{1,r}(\Omega)\times L^2(\Omega,\mathbb{M}^{2\times 2}_{\text{sym}})\times L^2(\Omega,\mathbb{M}^{2\times 2}_{\text{sym}})$ and thus

$$\begin{aligned} m \le \mathcal{E}_{\text{total}}(u,v,e,p) &\le \liminf_{k\to\infty} \mathcal{E}_{\text{total}}(u_k,v_k,e_k,p_k) \\ &= \lim_{k\to\infty} \mathcal{E}_{\text{total}}(u_k,v_k,e_k,p_k) = m, \end{aligned}$$

so that (u,v,e,p) is indeed a minimizer. □

We now define $(u_h^n,v_h^n,e_h^n,p_h^n)$ as one solution of (3.1) and we derive the Euler-Lagrange equation satisfied by this solution. We define for all $n\geqslant 1$,

$$\delta p_h^n := \frac{p_h^n - p_h^{n-1}}{h}.$$

Proposition 3.2. *For $1\le n\le N_f$, let $(u_h^n,v_h^n,e_h^n,p_h^n)$ be a solution of (3.1) and let*

$$\sigma_h^n := a_h^n A e_h^n$$

with $a_h^n := (v_h^n)^2+\eta$. Then we have:

$$\begin{cases} -\operatorname{div}(\sigma_h^n) = 0, & \text{a.e. } in \quad \Omega, \\ \sigma_h^n.\boldsymbol{n} = 0, & \text{a.e. } on \quad \partial\Omega_N, \\ \sigma_h^n - Bp_h^n - \beta\delta p_h^n \in \partial H(p_h^n - p_h^{n-1}), & \text{a.e. } in \quad \Omega. \end{cases}$$

Furthermore,

$$v_h^n = \underset{\phi\in W^{1,r}(\Omega),\, \phi\le v_h^{n-1},\, \phi=1 \text{ on } \partial\Omega_D}{\operatorname{argmin}} \left\{ \mathcal{E}_{el}(e_h^n,\phi) + \int_\Omega \frac{\varepsilon^{r-1}}{r}|\nabla\phi|^r + \frac{\alpha}{r'\varepsilon}|1-\phi|^r \,\mathrm{d}x \right\}. \tag{3.2}$$

Proof. Let $(z,\xi,q)\in A_{adm}(0)$, so that $(u_h^n+sz,e_h^n+s\xi,p_h^n+sq)\in A_{adm}(w_h^n)$ is an admissible triplet for every $0<s<1$. We have

$$\mathcal{E}_{\text{total}}(u_h^n,v_h^n,e_h^n,p_h^n) \le \mathcal{E}_{\text{total}}(u_h^n+sz,v_h^n,e_h^n+s\xi,p_h^n+sq),$$

and thus

$$\begin{aligned} 0 \le s\int_\Omega a_h^n A e_h^n : \xi \,\mathrm{d}x + s\int_\Omega Bp_h^n : q\,\mathrm{d}x + s\int_\Omega \beta\frac{p_h^n-p_h^{n-1}}{h} : q\,\mathrm{d}x \\ +\tau\int_\Omega |p_h^n+sq-p_h^{n-1}| - |p_h^n-p_h^{n-1}|\mathrm{d}x + o(s). \end{aligned}$$

Let $\Psi(s) := \tau\int_\Omega |p_h^n+sq-p_h^{n-1}|\mathrm{d}x$. Using the convexity of Ψ we have $\Psi(s)-\Psi(0)\le s(\Psi(1)-\Psi(0))$. Dividing this inequality by s and letting s tend to zero implies that for all $(z,\xi,q)\in A_{adm}(0)$,

$$\int_\Omega a_h^n A e_h^n : \xi\,\mathrm{d}x + \int_\Omega Bp_h^n : q\,\mathrm{d}x + \int_\Omega \beta\frac{p_h^n-p_h^{n-1}}{h} : q\,\mathrm{d}x + \tau\int_\Omega |p_h^n-p_h^{n-1}+q| - |p_h^n-p_h^{n-1}|\,\mathrm{d}x \geqslant 0. \tag{3.3}$$

Testing (3.3) with $\pm(\phi,E(\phi),0)$ for any $\phi\in C_c^\infty(\Omega,\mathbb{R}^2)$, we obtain

$$\int_\Omega \sigma_h^n : E(\phi)\,\mathrm{d}x = 0 \tag{3.4}$$

and from which we deduce that $-\mathrm{div}(\sigma_h^n) = 0$ a.e. in Ω. Furher, picking $\phi \in C^\infty(\overline{\Omega}, \mathbb{R}^2)$, with $\phi = 0$ on $\partial\Omega_D$ in $\pm(\phi, E\phi, 0)$ as a test function for (3.3) and integrating (3.4) by parts, we also obtain that $\sigma_h^n.\boldsymbol{n} = 0$ a.e. on $\partial\Omega_N$. Testing (3.3) with $(0, -q + p_h^n - p_h^{n-1}, q - p_h^n + p_h^{n-1})$ for any $q \in L^2(\Omega, \mathbb{M}^{2\times 2}_{\mathrm{sym}})$, we have

$$\tau \int_\Omega |q| \mathrm{d}x \geq \tau \int_\Omega |p_h^n - p_h^{n-1}| \mathrm{d}x + \int_\Omega \left(a_h^n A e_h^n - B p_h^n - \beta \frac{p_h^n - p_h^{n-1}}{h} \right) : (q - (p_h^n - p_h^{n-1}))\, \mathrm{d}x. \tag{3.5}$$

By a standard localization argument, it follows that

$$\tau|q| \geq \tau|p_h^n - p_h^{n-1}| + (a_h^n A e_h^n - B p_h^n - \beta\delta p_h^n) : (q - (p_h^n - p_h^{n-1})) \quad \text{for all } q \in \mathbb{M}^{2\times 2}_{\mathrm{sym}}, \quad \text{a.e.}\ in\ \Omega$$

which by definition of the subdifferential implies that

$$a_h^n A e_h^n - B p_h^n - \beta\delta p_h^n \in \partial H(p_h^n - p_h^{n-1}) \quad \text{a.e.} \quad in \quad \Omega. \tag{3.6}$$

We also have

$$\mathcal{E}_{\mathrm{total}}(u_h^n, v_h^n, e_h^n, p_h^n) \leq \mathcal{E}_{\mathrm{total}}(u_h^n, \varphi, e_h^n, p_h^n).$$

for every $\varphi \in W^{1,r}(\Omega)$, $\varphi \leq v_h^{n-1}$ and $\varphi = 1$ on $\partial\Omega_D$, which implies

$$v_h^n = \underset{\varphi \in W^{1,r}(\Omega),\, \varphi \leq v_h^{n-1},\, \varphi = 1\, on\, \partial\Omega_D}{\mathrm{argmin}} \left\{ \mathcal{E}_{el}(e_h^n, \varphi) + \int_\Omega \frac{\varepsilon^{r-1}}{r} |\nabla\varphi|^r + \frac{\alpha}{r'\varepsilon} |1 - \varphi|^r \, \mathrm{d}x \right\}. \qquad \square$$

Remark that by a truncation argument, we have $v_h^n \geqslant 0$ in Ω.

3.2. *A priori* estimates

We define piecewise affine interpolants of the sequences $(u_h^n)_{n=0}^{N_f}$, $(v_h^n)_{n=0}^{N_f}$, $(e_h^n)_{n=0}^{N_f}$, $(p_h^n)_{n=0}^{N_f}$ as follows:

$$\begin{aligned}
u_h(t) &= u_h^n + (t - t_h^n)\delta u_h^n, \quad \text{for} \quad t \in [t_h^{n-1}, t_h^n], \quad n = 1, \ldots, N_f, \\
v_h(t) &= v_h^n + (t - t_h^n)\delta v_h^n, \quad \text{for} \quad t \in [t_h^{n-1}, t_h^n], \quad n = 1, \ldots, N_f, \\
e_h(t) &= e_h^n + (t - t_h^n)\delta e_h^n, \quad \text{for} \quad t \in [t_h^{n-1}, t_h^n], \quad n = 1, \ldots, N_f, \\
p_h(t) &= p_h^n + (t - t_h^n)\delta p_h^n, \quad \text{for} \quad t \in [t_h^{n-1}, t_h^n], \quad n = 1, \ldots, N_f.
\end{aligned}$$

Remark that $u_h(0) = u_0$, $v_h(0) = v_0$, $e_h(0) = e_0$, $p_h(0) = p_0$. We also define piecewise constant interpolants

$$\begin{aligned}
u_h^+(t) &= u_h^n, \quad \text{for} \quad t \in (t_h^{n-1}, t_h^n], \quad n = 1, \ldots, N_f, \\
v_h^+(t) &= v_h^n, \quad \text{for} \quad t \in (t_h^{n-1}, t_h^n], \quad n = 1, \ldots, N_f, \\
a_h^+(t) &= a_h^n, \quad \text{for} \quad t \in (t_h^{n-1}, t_h^n], \quad n = 1, \ldots, N_f, \\
e_h^+(t) &= e_h^n, \quad \text{for} \quad t \in (t_h^{n-1}, t_h^n], \quad n = 1, \ldots, N_f, \\
p_h^+(t) &= p_h^n, \quad \text{for} \quad t \in (t_h^{n-1}, t_h^n], \quad n = 1, \ldots, N_f, \\
w_h^+(t) &= w_h^n, \quad \text{for} \quad t \in (t_h^{n-1}, t_h^n], \quad n = 1, \ldots, N_f,
\end{aligned}$$

with $u_h^+(0) = u_0$, $v_h^+(0) = v_0$, $a_h^+(0) := v_0^2 + \eta$, $e_h^+(0) = e_0$, $p_h^+(0) = p_0$, $w_h^+(0) = w(0)$. We also set

$$\sigma_h^+(t) = (v_h^+(t)^2 + \eta) A e_h^+(t) \quad \text{for} \quad t \in (0, T_f],$$

with $\sigma_h^+(0) = \sigma_0$.

Proposition 3.3. *There exists a constant $C > 0$ independent of h, n such that*

$$\sup_{[0,T_f]} \|u_h^+(t)\|_{H^1} \leq C,\ \sup_{[0,T_f]} \|v_h^+(t)\|_{W^{1,r}} \leq C,\ \sup_{[0,T_f]} \|p_h^+(t)\|_2 \leq C,$$
$$\sup_{[0,T_f]} \|e_h^+(t)\|_2 \leq C,\ \sup_{(0,T_f)} \|\dot{p}_h(t)\|_2 \leq C.$$

Proof. Firstly, we observe that $(u_h^{n-1} + w_h^n - w_h^{n-1}, v_h^{n-1}, e_h^{n-1} + Ew_h^n - Ew_h^{n-1}, p_h^{n-1})$ is admissible for the minimisation problem (3.1), and

$$\mathcal{E}_{\text{total}}(u_h^n, v_h^n, e_h^n, p_h^n) \leq \mathcal{E}_{\text{total}}(u_h^{n-1} + w_h^n - w_h^{n-1}, v_h^{n-1}, e_h^{n-1} + Ew_h^n - Ew_h^{n-1}, p_h^{n-1}).$$

So that

$$\begin{aligned}
&\mathcal{E}_{el}(e_h^n, v_h^n) + \mathcal{E}_S^r(v_h^n) + \frac{1}{2}\int_\Omega Bp_h^n : p_h^n \,\mathrm{d}x + \frac{\beta}{2h} \parallel p_h^n - p_h^{n-1} \parallel_2^2 + \tau \int_\Omega |p_h^n - p_h^{n-1}| \,\mathrm{d}x \\
\leq\, &\mathcal{E}_{el}(e_h^{n-1} + Ew_h^n - Ew_h^{n-1}, v_h^{n-1}) + \mathcal{E}_S^r(v_h^{n-1}) + \frac{1}{2}\int_\Omega Bp_h^{n-1} : p_h^{n-1} \,\mathrm{d}x \\
=\, &\mathcal{E}_{el}(e_h^{n-1}, v_h^{n-1}) + \mathcal{E}_{el}(Ew_h^n - Ew_h^{n-1}, v_h^{n-1}) \\
&+ \int_\Omega a_h^{n-1} Ae_h^{n-1} : (Ew_h^n - Ew_h^{n-1}) \,\mathrm{d}x + \mathcal{E}_S^r(v_h^{n-1}) + \frac{1}{2}\int_\Omega Bp_h^{n-1} : p_h^{n-1} \,\mathrm{d}x. \qquad (3.7)
\end{aligned}$$

Since Ew is absolutely continuous in time with values in $L^2(\Omega; \mathbb{M}_{sym}^{2\times 2})$,

$$Ew_h^n - Ew_h^{n-1} = \int_{t_h^{n-1}}^{t_h^n} E\dot{w}(s) \,\mathrm{d}s.$$

We now estimate,

$$\begin{aligned}
&\mathcal{E}_{el}(Ew_h^n - Ew_h^{n-1}, v_h^{n-1}) \leq \mathcal{E}_{el}(Ew_h^n - Ew_h^{n-1}, 1) \\
&\leq \frac{\alpha_2}{2}(1+\eta) \left\| \int_{t_h^{n-1}}^{t_h^n} E\dot{w}(s) \,\mathrm{d}s \right\|_2^2 \leq \frac{\alpha_2}{2}(1+\eta) \left(\int_{t_h^{n-1}}^{t_h^n} \|E\dot{w}(s)\|_2 \,\mathrm{d}s \right)^2 \\
&\leq \frac{\alpha_2}{2}(1+\eta) f(h) \int_{t_h^{n-1}}^{t_h^n} \|E\dot{w}(s)\|_2 \,\mathrm{d}s, \qquad (3.8)
\end{aligned}$$

where

$$f(h) := \max_{k\in\{1,N_f\}} \int_{t_h^{k-1}}^{t_h^k} \|E\dot{w}(s)\|_2 \,\mathrm{d}s \to 0 \quad \text{as}\, h \to 0, \qquad (3.9)$$

and

$$\begin{aligned}
\int_\Omega a_h^{n-1} Ae_h^{n-1} : \left(\int_{t_h^{n-1}}^{t_h^n} E\dot{w}(s) \,\mathrm{d}s \right) \mathrm{d}x &\leq (1+\eta) \parallel Ae_h^{n-1} \parallel_2 \int_{t_h^{n-1}}^{t_h^n} \parallel E\dot{w}(s) \parallel_2 \,\mathrm{d}s \\
&\leq (1+\eta) 2\alpha_2 \sup_{\{0,..,N_f\}} \parallel e_h^n \parallel_2 \int_{t_h^{n-1}}^{t_h^n} \parallel E\dot{w}(s) \parallel_2 \,\mathrm{d}s. \qquad (3.10)
\end{aligned}$$

Thanks to (3.7)–(3.10) we obtain

$$\begin{aligned} &\mathcal{E}_{el}(e_h^n, v_h^n) + \mathcal{E}_S^r(v_h^n) + \frac{1}{2}\int_\Omega Bp_h^n : p_h^n \,\mathrm{d}x + \frac{\beta}{2h} \parallel p_h^n - p_h^{n-1} \parallel_2^2 + \tau \int_\Omega |p_h^n - p_h^{n-1}| \,\mathrm{d}x \\ \leq &\mathcal{E}_{el}(e_h^{n-1}, v_h^{n-1}) + \mathcal{E}_S^r(v_h^{n-1}) + \frac{1}{2}\int_\Omega Bp_h^{n-1} : p_h^{n-1} \,\mathrm{d}x \\ &+ Cf(h)\int_{t_h^{n-1}}^{t_h^n} \|E\dot{w}(s)\|_2 \,\mathrm{d}s + (1+\eta)2\alpha_2 \sup_{\{0,..,N_f\}} \parallel e_h^n \parallel_2 \int_{t_h^{n-1}}^{t_h^n} \parallel E\dot{w}(s) \parallel_2 \,\mathrm{d}s. \end{aligned} \tag{3.11}$$

Summing the inequalities (3.11) for $1 \leq n \leq N \leq N_f$ we obtain

$$\begin{aligned} &\mathcal{E}_{el}(e_h^N, v_h^N) + \mathcal{E}_S^r(v_h^N) + \frac{1}{2}\int_\Omega Bp_h^N : p_h^N \,\mathrm{d}x \\ &+ \frac{\beta}{2}\sum_{n=1}^N h \left|\left|\frac{p_h^n - p_h^{n-1}}{h}\right|\right|_2^2 + \tau h \sum_{n=1}^N \int_\Omega \left|\frac{p_h^n - p_h^{n-1}}{h}\right| \mathrm{d}x \\ \leq &\mathcal{E}_{el}(e_0, v_0) + \mathcal{E}_S^r(v_0) + \frac{1}{2}\int_\Omega Bp_0 : p_0 \,\mathrm{d}x \\ &+ Cf(h)\int_0^{t_h^N} \|E\dot{w}(s)\|_2 \,\mathrm{d}s + (1+\eta)2\alpha_2 \sup_{\{0,..,N_f\}} \parallel e_h^n \parallel_2 \int_0^{t_h^N} \parallel E\dot{w}(s) \parallel_2 \,\mathrm{d}s. \end{aligned} \tag{3.12}$$

From the last inequality, and from the coercivity and boundedness of the tensor A we deduce that

$$\begin{aligned} \sup_{\{0,\ldots,N_f\}} \parallel e_h^n \parallel_2^2 \leq C \parallel e_0 \parallel_2^2 + \mathcal{E}_S^r(v_0) + \int_\Omega Bp_0 : p_0 \,\mathrm{d}x \\ + C \sup_{\{0,..,N_f\}} \parallel e_h^n \parallel_2 \int_0^{T_f} \parallel E\dot{w}(s) \parallel_2 \,\mathrm{d}s + Cf(h). \end{aligned}$$

This last estimate, the coercivity and boundedness of the tensor B and (3.12) leads to

$$\sup_{[0,T_f]} \left\{ \|u_h^+(t)\|_{H^1}, \, \|v_h^+(t)\|_{W^{1,r}}, \, \|p_h^+(t)\|_2, \, \|e_h^+(t)\|_2 \right\} \leq C.$$

Furthermore, from the discrete plastic flow rule (3.6), we deduce that

$$|a_h^n A e_h^n - Bp_h^n - \beta\delta p_h^n| \leq \tau \quad \text{a.e. } in\ \Omega,$$

and consenquently,

$$\sup_{(0,T_f]} \|\dot{p}_h(t)\|_2 \leq C. \qquad \square$$

Remark 3.4. In the general case, when K is a convex, closed set of $\mathbb{M}^{2\times 2}_{\mathrm{Sym}}$ with non empty interior, the bound of $\dot{p}_h$ is only in $L^2(L^2)$ thanks to the bound obtained in formula (3.12).

3.3. Compactness results

Our aim is to study the limit of the discrete plastic flow rule, and of the discrete variational problem for v_h^n. To this end, we show the strong compactness on the sequence of stresses $(\sigma_h^+)_h$, and the sequences of elastic and plastic strains $(e_h^+)_h$, $(p_h)_h$.

Let $M_f \geqslant 2$ with $M_f \neq N_f$ and consider an other partition of the time interval $[0, T_f]$ into M_f sub-intervals of equals length $l = \dfrac{T_f}{M_f} = t_l^m - t_l^{m-1} \to 0$:

$$0 = t_l^0 < t_l^1 < \ldots < t_l^{m-1} < t_l^m < \ldots < T_f.$$

In the same way we define all interpolant functions with indexes l and m.

Lemma 3.5. *For all $t \in (0, T_f]$ we have*

$$\beta \parallel \dot{p}_h(t) - \dot{p}_l(t) \parallel_2 \leq \parallel (\sigma_h^+(t) - Bp_h^+(t)) - (\sigma_l^+(t) - Bp_l^+(t)) \parallel_2 .$$

Proof. By the homogeneity of degree 1 of H, we have

$$\sigma_h^n - Bp_h^n - \beta\delta p_h^n \in \partial H(\delta p_h^n) \qquad \text{a.e.} \quad \text{in } \Omega. \tag{3.13}$$

We obtain for $m = 1, \ldots, M_f$,

$$\sigma_l^m - Bp_l^m - \beta\delta p_l^m \in \partial H(\delta p_l^m). \tag{3.14}$$

By a standard result of convex analysis we have

$$\langle(\sigma_l^m - Bp_l^m - \beta\delta p_l^m) - (\sigma_h^n - Bp_h^n - \beta\delta p_h^n), \delta p_h^n - \delta p_l^m\rangle \leq 0. \tag{3.15}$$

We deduce from (3.15) and the Cauchy–Schwarz inequality

$$\begin{aligned} \beta \parallel \delta p_h^n - \delta p_l^m \parallel_2^2 &\leq \langle(\sigma_h^n - Bp_h^n) - (\sigma_l^m - Bp_l^m), \delta p_h^n - \delta p_l^m\rangle \\ &\leq \parallel (\sigma_h^n - Bp_h^n) - (\sigma_l^m - Bp_l^m) \parallel_2 \parallel \delta p_h^n - \delta p_l^m \parallel_2, \end{aligned} \tag{3.16}$$

to obtain

$$\beta \parallel \delta p_h^n - \delta p_l^m \parallel_2 \leq \parallel (\sigma_h^n - Bp_h^n) - (\sigma_l^m - Bp_l^m) \parallel_2,$$

or in other words, for all $t \in (0, T_f]$

$$\beta \parallel \dot{p}_h(t) - \dot{p}_l(t) \parallel_2 \leq \parallel (\sigma_h^+(t) - Bp_h^+(t)) - (\sigma_l^+(t) - Bp_l^+(t)) \parallel_2 . \qquad \square$$

The proof of the next proposition is similar to the proof of Lemma 4.1 in [15] or of the Lemma 4.9 in [2].

Proposition 3.6. *There exists a subsequence (not relabeled) $h \to 0$ and a function $v : [0, T_f] \to W^{1,r}(\Omega)$ such that $v_h^+(t) \rightharpoonup v(t)$ weakly in $W^{1,r}(\Omega)$ for every $t \in [0, T_f]$. Furthermore, we have $v(0) = v_0$, $0 \leq v(s) \leq v(t) \leq 1$ for every $0 \leq t \leq s \leq T_f$ and*

$$v \in L^\infty(0, T_f, W^{1,r}(\Omega)). \tag{3.17}$$

Proof. By definition $v_h^+ : [0, T_f] \to L^1(\Omega)$ is monotone non-increasing, for every $t \in [0, T_f]$. By a generalized version of Helly's selection principle (see [19]), there exists a subsequence (not relabeled) $h \to 0$ and a map $v : [0, T_f] \to L^1(\Omega)$ such that $v_h^+(t) \rightharpoonup v(t)$ weakly in $L^1(\Omega)$ for every $t \in [0, T_f]$. By Proposition 3.3, for every $t \in [0, T_f]$, up to a subsequence, $v_{h_n}^+(t) \rightharpoonup w$ weakly in $W^{1,r}(\Omega)$ and so weakly in $L^1(\Omega)$. As $v_h^+(t) \rightharpoonup v(t)$ weakly in $L^1(\Omega)$ we deduce that $w = v(t)$, $v(t) \in W^{1,r}(\Omega)$ and the whole sequence $v_h^+(t) \rightharpoonup v(t)$ weakly in $W^{1,r}(\Omega)$, since the limit of $v_{h_n}^+(t)$ does not depend of the subsequence. Consenquently, by the Sobolev imbedding theorem, $v_h^+(t) \to v(t)$ strongly in $C^0(\overline{\Omega})$ for every $t \in [0, T_f]$. Since $v_h^+(t) = 1$ on $\partial\Omega_D$, $0 \leq v_h^+(t) \leq 1$ in Ω for all $t \in [0, T_f]$ and $0 \leq v_h^+(s) \leq v_h^+(t) \leq 1$ for every $0 \leq t \leq s \leq T_f$, we obtain $v(t) = 1$ on $\partial\Omega_D$, $0 \leq v(t) \leq 1$ in Ω for all $t \in [0, T_f]$ and $0 \leq v(s) \leq v(t) \leq 1$ for every $0 \leq t \leq s \leq T_f$. By lower semicontinuity, we have

$$\sup_{[0,T_f]} \parallel v(t) \parallel_{W^{1,r}(\Omega)} \leq C. \tag{3.18}$$

$\square$

In the following results, we only consider the subsequence given by the Proposition 3.6.

Lemma 3.7. *Define*

$$Y_{h,l}(t) := \| e_l^+(t)(v_l^+(t)^2 - v_h^+(t)^2) \|_2,$$
$$Q_{h,l}(t) := \int_0^t Y_{h,l}(s)\,\mathrm{d}s.$$

Then for all $t \in [0, T_f]$, $Y_{h,l}(t) \to 0$, $Q_{h,l}(t) \to 0$ as $h, l \to 0$.

Proof. Let $t \in [0, T_f]$. By the Proposition 3.6, $v_h^+(t) \rightharpoonup v(t)$ weakly in $W^{1,r}(\Omega)$. By the Sobolev imbedding theorem, $v_h^+(t) \to v(t)$ strongly in $C^0(\overline{\Omega})$:

$$\lim_{h\to 0}\left(\sup_{x\in\overline{\Omega}} |v_h^+(t) - v(t)|\right) = 0 \tag{3.19}$$

which implies $(v_h^+(t))_h$ is a Cauchy sequence in $C^0(\overline{\Omega})$:

$$\lim_{h,l\to 0}\left(\sup_{x\in\overline{\Omega}} |v_h^+(t) - v_l^+(t)|\right) = 0. \tag{3.20}$$

Since $v_h^+(t) \leq 1$ and $\| e_l^+(t) \|_2 \leq C$,

$$\begin{aligned}
Y_{h,l}(t)^2 &= \int_\Omega (v_l^+(t)^2 - v_h^+(t)^2)^2 e_l^+(t) : e_l^+(t)\,\mathrm{d}x \\
&\leq \sup_{x\in\overline{\Omega}} |v_h^+(t) - v_l^+(t)| \int_\Omega |v_h^+(t) - v_l^+(t)||(v_h^+(t) + v_l^+(t))|^2 e_l^+(t) : e_l^+(t)\,\mathrm{d}x \\
&\leq C \sup_{x\in\overline{\Omega}} |v_h^+(t) - v_l^+(t)|
\end{aligned}$$

with $C > 0$ independent of h and l. By (3.20) $Y_{h,l}(t) \to 0$ as $h, l \to 0$. By the Lebesgue Dominated Convergence Theorem, it follows that $Q_{h,l}(t) \to 0$ as $h, l \to 0$. □

Lemma 3.8. *For all $t \in [0, T_f]$ we have*

$$\begin{aligned}
\| p_h^+(t) - p_l^+(t) \|_2 \leq C &\left(\int_0^t \| a_h^+(s)(e_h^+(s) - e_l^+(s)) \|_2 \,\mathrm{d}s + Q_{h,l}(t)\right) \\
&+ C\int_0^t \| p_h^+(s) - p_l^+(s) \|_2 \,\mathrm{d}s + C(h + l)
\end{aligned} \tag{3.21}$$

with $C > 0$, independent of h and l.

Proof. We have $p_h^n - p_h^{n-1} = h\delta p_h^n$. Summation for $n = 1$ to N gives

$$p_h^N - p_0 = \sum_{n=1}^N \int_{t_h^{n-1}}^{t_h^n} \delta p_h^n\,\mathrm{d}s. \tag{3.22}$$

Let $t \in (t_h^{N-1}, t_h^N]$, then

$$p_h^+(t) - p_0 = \int_0^t \dot{p}_h(s)\,\mathrm{d}s + R_h(t) \quad \text{with} \quad R_h(t) = \int_t^{t_h^N} \delta p_h^N\,\mathrm{d}s,$$
$$\text{and } \| R_h(t) \|_2 \leq \int_t^{t_h^n} \| \delta p_h^N \|_2 \,\mathrm{d}s \leq Ch. \tag{3.23}$$

In the same way we have for $t \in (t_l^{M-1}, t_l^M]$,

$$p_l^+(t) - p_0 = \int_0^t \dot p_l(s)\,\mathrm{d}s + R_l(t) \quad \text{with} \quad R_l(t) = \int_t^{t_l^M} \delta p_l^M\,\mathrm{d}s,$$
$$\text{and} \parallel R_l(t) \parallel_2 \leq \int_t^{t_l^m} \parallel \delta p_l^M \parallel_2 \,\mathrm{d}s \leq Cl. \tag{3.24}$$

Let $t \in (0, T_f]$, and $m, n \geqslant 1$ such that $t \in (t_l^{m-1}, t_l^m] \cap (t_h^{n-1}, t_h^n]$. Then

$$p_h^+(t) - p_l^+(t) = \int_0^t \dot p_h(s) - \dot p_l(s)\,\mathrm{d}s + R_h(t) - R_l(t) \tag{3.25}$$

and by the Lemma 3.5 we deduce that

$$\begin{aligned} \parallel p_h^+(t) - p_l^+(t) \parallel_2 &\leq \int_0^t \parallel \dot p_h(s) - \dot p_l(s) \parallel_2 \,\mathrm{d}s + C(h+l) \\ &\leq C \int_0^t \parallel (\sigma_h^+(t) - Bp_h^+(t)) - (\sigma_l^+(t) - Bp_l^+(t)) \parallel_2 \,\mathrm{d}s + C(h+l). \end{aligned} \tag{3.26}$$

Further,

$$\begin{aligned} \parallel (\sigma_h^+(t) - Bp_h^+(t)) - (\sigma_l^+(t) - Bp_l^+(t)) \parallel_2 \leq & \parallel \sigma_h^+(t) - \sigma_l^+(t) \parallel_2 + C \parallel p_h^+(t) - p_l^+(t) \parallel_2 \\ \leq & C \parallel a_h^+(t)(e_h^+(t) - e_l^+(t)) \parallel_2 + C \parallel e_l^+(t)(a_l^+(t) - a_h^+(t)) \parallel_2 \\ & + C \parallel p_h^+(t) - p_l^+(t) \parallel_2 . \end{aligned} \tag{3.27}$$

From (3.26) and (3.27) we obtain

$$\begin{aligned} \parallel p_h^+(t) - p_l^+(t) \parallel_2 \leq C & \int_0^t \parallel a_h^+(s)(e_h^+(s) - e_l^+(s)) \parallel_2 \,\mathrm{d}s \\ & + C \int_0^t \parallel e_l^+(s)(a_l^+(s) - a_h^+(s)) \parallel_2 \,\mathrm{d}s \\ & + C \int_0^t \parallel p_h^+(s) - p_l^+(s) \parallel_2 \,\mathrm{d}s + C(h+l). \end{aligned} \tag{3.28}$$

$\square$

Remark 3.9. In the general case, when K is a convex, closed set of $\mathbb{M}^{2\times 2}_{\mathrm{Sym}}$ with non empty interior, we can replace the term $C(h+l)$ in Lemma 3.8 by $C(\sqrt{h}+\sqrt{l})$ since in formula (3.23), on can apply the Cauchy–Schwarz inequality and use the bound of $\dot p_h$ in $L^2(L^2)$ (see Rem. 3.4) to get that $\parallel R_h(t) \parallel_2 \leq C\sqrt{h}$.

Proposition 3.10. *For all $t \in [0, T_f]$, $(u_h^+(t), e_h^+(t), p_h^+(t))$ is a Cauchy sequence in $H^1(\Omega, \mathbb{R}^2) \times L^2(\Omega, \mathbb{M}^{2\times 2}_{sym}) \times L^2(\Omega, \mathbb{M}^{2\times 2}_{sym})$.*

Proof. Let $t \in (0, T_f]$. Since $a_h^+(t) \geqslant \eta$ and

$$\begin{aligned} a_h^+(t)(e_h^+(t) - e_l^+(t)) &= a_h^+(t)e_h^+(t) - a_l^+(t)e_l^+(t) + e_l^+(t)(a_l^+(t) - a_h^+(t)) \\ &= \sigma_h^+(t) - \sigma_l^+(t) + e_l^+(t)(a_l^+(t) - a_h^+(t)), \end{aligned} \tag{3.29}$$

we estimate the difference $e_h^+(t) - e_l^+(t)$ as follows:

$$
\begin{aligned}
\eta\alpha_A \parallel e_h^+(t) - e_l^+(t) \parallel_2^2 &\leq \eta \int_\Omega A(e_h^+(t) - e_l^+(t)) : (e_h^+(t) - e_l^+(t))\,\mathrm{d}x \\
&\leq \int_\Omega a_h^+(t) A(e_h^+(t) - e_l^+(t)) : (e_h^+(t) - e_l^+(t))\,\mathrm{d}x \\
&= \int_\Omega (\sigma_h^+(t) - \sigma_l^+(t)) : (e_h^+(t) - e_l^+(t))\,\mathrm{d}x \\
&\quad + \int_\Omega (a_l^+(t) - a_h^+(t)) A e_l^+(t) : (e_h^+(t) - e_l^+(t))\,\mathrm{d}x.
\end{aligned}
\tag{3.30}
$$

Applying the compatibility condition

$$
E(u_h^+(t) - u_l^+(t)) = e_h^+(t) - e_l^+(t) + p_h^+(t) - p_l^+(t), \tag{3.31}
$$

which leads to

$$
\begin{aligned}
\eta\alpha_A \parallel e_h^+(t) - e_l^+(t) \parallel_2^2 \leq\ & \int_\Omega (\sigma_h^+(t) - \sigma_l^+(t)) : E(u_h^+(t) - u_l^+(t))\,\mathrm{d}x \\
& - \int_\Omega (\sigma_h^+(t) - \sigma_l^+(t)) : (p_h^+(t) - p_l^+(t))\,\mathrm{d}x \\
& + \int_\Omega A e_l^+(t)(a_l^+(t) - a_h^+(t)) : (e_h^+(t) - e_l^+(t))\,\mathrm{d}x \\
:=\ & I_1 - I_2 + I_3.
\end{aligned}
$$

Since $\mathrm{div}\sigma_h^+(t) = \mathrm{div}\sigma_l^+(t) = 0$ a.e in Ω, $u_h^+(t) - u_l^+(t) = w_h^+(t) - w_l^+(t)$ a.e. on $\partial\Omega_D$, and $\sigma_h^+(t).\boldsymbol{n} = \sigma_l^+(t).\boldsymbol{n} = 0$ a.e. on $\partial\Omega_N$, we have

$$
I_1 = \int_\Omega (\sigma_h^+(t) - \sigma_l^+(t)) : E(w_h^+(t) - w_l^+(t))\,\mathrm{d}x,
$$

and we estimate thanks to Proposition 3.3,

$$
\begin{aligned}
|I_1| &\leq \parallel \sigma_h^+(t) - \sigma_l^+(t) \parallel_2 \parallel E(w_h^+(t) - w_l^+(t)) \parallel_2 \\
&\leq C \parallel E(w_h^+(t) - w_l^+(t)) \parallel_2 .
\end{aligned}
$$

Since $Ew \in H^1(0, T_f, L^2(\Omega, \mathbb{M}^{2\times 2}_{\mathrm{sym}}))$, it is Hölder continuous with value in $L^2(\Omega, \mathbb{M}^{2\times 2}_{\mathrm{sym}})$. $(Ew_h^+)_h$ is a Cauchy sequence in $L^\infty(0, T_f; L^2(\Omega, \mathbb{M}^{2\times 2}_{\mathrm{sym}}))$, thus $\parallel E(w_h^+(t) - w_l^+(t)) \parallel_2 \leq \delta_{h,l}$ with $\delta_{h,l} \to 0$ as $h, l \to 0$. Further, we have

$$
\begin{aligned}
I_2 = &\int_\Omega a_h^+(t) A(e_h^+(t) - e_l^+(t)) : (p_h^+(t) - p_l^+(t))\,\mathrm{d}x \\
&- \int_\Omega A e_l^+(t)(a_l^+(t) - a_h^+(t)) : (p_h^+(t) - p_l^+(t))\,\mathrm{d}x.
\end{aligned}
$$

Thus, we get using Proposition 3.3 and Lemma 3.8

$$\begin{aligned}
\eta\alpha_A \parallel e_h^+(t) - e_l^+(t) \parallel_2^2 \leq & \parallel \sigma_h^+(t) - \sigma_l^+(t) \parallel_2 \parallel E(w_h^+(t) - w_l^+(t)) \parallel_2 \\
& + C \parallel e_h^+(t) - e_l^+(t) \parallel_2 \parallel p_h^+(t) - p_l^+(t) \parallel_2 \\
& + C \parallel e_l^+(t)(a_l^+(t) - a_h^+(t)) \parallel_2 \parallel p_h^+(t) - p_l^+(t) \parallel_2 \\
& + C \parallel e_l^+(t)(a_l^+(t) - a_h^+(t)) \parallel_2 \parallel e_h^+(t) - e_l^+(t) \parallel_2 \\
\leq & \; C \parallel E(w_h^+(t) - w_l^+(t)) \parallel_2 \qquad (3.32) \\
& + C Y_{h,l}(t)(\parallel e_h^+(t) - e_l^+(t) \parallel_2 + \parallel p_h^+(t) - p_l^+(t) \parallel_2) \\
& + C \parallel e_h^+(t) - e_l^+(t) \parallel_2 \int_0^t \parallel (e_h^+(s) - e_l^+(s)) \parallel_2 \, \mathrm{d}s \\
& + C \parallel e_h^+(t) - e_l^+(t) \parallel_2 \int_0^t \parallel p_h^+(s) - p_l^+(s) \parallel_2 \, \mathrm{d}s \\
& + C \parallel e_h^+(t) - e_l^+(t) \parallel_2 (Q_{h,l}(t) + (h+l)). \qquad (3.33)
\end{aligned}$$

On the other hand, using the Lemma 3.8 again leads to the estimate

$$\begin{aligned}
\parallel p_h^+(t) - p_l^+(t) \parallel_2^2 \leq C \parallel p_h^+(t) - p_l^+(t) \parallel_2 \Bigg(& \int_0^t \parallel (e_h^+(s) - e_l^+(s)) \parallel_2 \, \mathrm{d}s \\
& + Q_{h,l}(t) + (h+l) + \int_0^t \parallel p_h^+(s) - p_l^+(s) \parallel_2 \, \mathrm{d}s \Bigg). \qquad (3.34)
\end{aligned}$$

Set

$$X_{h,l}(t) = \parallel p_h^+(t) - p_l^+(t) \parallel_2 + \parallel e_h^+(t) - e_l^+(t) \parallel_2 .$$

Adding (3.33) and (3.34) yields

$$\begin{aligned}
\eta\alpha_A \parallel e_h^+(t) - e_l^+(t) \parallel_2^2 + \parallel p_h^+(t) - p_l^+(t) \parallel_2^2 \leq & \; C X_{h,l}(t) \left(\int_0^t X_{h,l}(s) \, \mathrm{d}s + Y_{h,l}(t) + Q_{h,l}(t) + h + l \right) \\
& + C \parallel E(w_h^+(t) - w_l^+(t)) \parallel_2 .
\end{aligned}$$

The Cauchy inequality $(a+b)^2 \leq 2a^2 + 2b^2$ leads to

$$\begin{aligned}
X_{h,l}(t)^2 \leq & \; C X_{h,l}(t) \left(\int_0^t X_{h,l}(s) \, \mathrm{d}s + Y_{h,l}(t) + Q_{h,l}(t) + h + l \right) \qquad (3.35) \\
& + C \parallel E(w_h^+(t) - w_l^+(t)) \parallel_2,
\end{aligned}$$

from which we deduce that

$$\begin{aligned}
\parallel p_h^+(t) - p_l^+(t) \parallel_2 + \parallel e_h^+(t) - e_l^+(t) \parallel_2 \leq & \; C \int_0^t \parallel (e_h^+(s) - e_l^+(s)) \parallel_2 + \parallel p_h^+(s) - p_l^+(s) \parallel_2 \, \mathrm{d}s \\
& + C \left(Y_{h,l}(t) + Q_{h,l}(t) + h + l + \sqrt{\frac{2}{C} \parallel E(w_h^+(t) - w_l^+(t)) \parallel_2} \right),
\end{aligned}$$

for some constant $C > 0$ independent on h, l, t. Applying the Gronwall's inequality leads to

$$\begin{aligned}
& \parallel p_h^+(t) - p_l^+(t) \parallel_2 + \parallel e_h^+(t) - e_l^+(t) \parallel_2 \\
& \leq C \left(Y_{h,l}(t) + Q_{h,l}(t) + h + l + \sqrt{\frac{2}{C} \parallel E(w_h^+(t) - w_l^+(t)) \parallel_2} \right) \\
& \quad + C^2 e^{CT_f} \int_0^t \left(Y_{h,l}(s) + Q_{h,l}(s) + h + l + \sqrt{\frac{2}{C} \parallel E(w_h^+(s) - w_l^+(s)) \parallel_2} \right) \mathrm{d}s. \qquad (3.36)
\end{aligned}$$

Since $Y_{h,l}$ and $Q_{h,l}$ tend to 0 for all $s \in [0,T_f]$ (Lem. 3.7), and since these functions are uniformly bounded on $[0,T_f]$, from (3.36) and the Lebesgue's dominated convergence theorem we deduce that $X_{h,l}(t) \longrightarrow 0$ as $h,l \to 0$. Finally we conclude that for fixed $t \in [0,T_f]$, $(u_h^+(t), e_h^+(t), p_h^+(t))_h$ is a Cauchy sequence in $H^1(\Omega,\mathbb{R}^2) \times L^2(\Omega,\mathbb{M}^{2\times 2}_{\text{sym}}) \times L^2(\Omega,\mathbb{M}^{2\times 2}_{\text{sym}})$. $\square$

Remark 3.11. In the general case, when K is convex, closed set of $\mathbb{M}^{2\times 2}_{\text{Sym}}$ with non empty interior, in the proof of Proposition 3.10, we can replace everywhere the term $(h+l)$ by $(\sqrt{h}+\sqrt{l})$.

Proposition 3.12. *There exists a function $t \to (u(t), e(t), p(t))$, such that for all $t \in [0,T_f]$ the next results hold:*

$$\begin{aligned}
&(u(t), e(t), p(t)) \in A_{adm}(w(t)),\\
&u_h^+(t) \to u(t) \quad \textit{strongly in}\, H^1(\Omega,\mathbb{R}^2),\\
&e_h^+(t) \to e(t) \quad \textit{strongly in}\, L^2(\Omega,\mathbb{M}^{2\times 2}_{sym}),\\
&p_h^+(t) \to p(t) \quad \textit{strongly in}\, L^2(\Omega,\mathbb{M}^{2\times 2}_{sym}),\\
&p_h(t) \to p(t) \quad \textit{strongly in}\, L^2(\Omega,\mathbb{M}^{2\times 2}_{sym}).
\end{aligned}$$

Furthermore, for a.e. $t \in [0,T_f]$

$$\dot{p}_h(t) \to \dot{p}(t) \quad \textit{strongly in}\, L^2(\Omega,\mathbb{M}^{2\times 2}_{sym}),$$

and

$$\begin{aligned}
&u \in L^\infty(0,T_f, H^1(\Omega,\mathbb{R}^2)),\\
&e \in L^\infty(0,T_f, L^2(\Omega,\mathbb{M}^{2\times 2}_{sym})),\\
&p \in W^{1,\infty}(0,T_f, L^2(\Omega,\mathbb{M}^{2\times 2}_{sym})).
\end{aligned}$$

Proof. Let $t \in [0,T_f]$. By Proposition 3.10 there exist $u(t) \in H^1(\Omega,\mathbb{R}^2)$, $e(t) \in L^2(\Omega,\mathbb{M}^{2\times 2}_{\text{sym}})$, and $p(t) \in L^2(\Omega,\mathbb{M}^{2\times 2}_{\text{sym}})$ such that for all $t \in [0,T_f]$ the next convergence results hold:

$$u_h^+(t) \to u(t) \quad \text{strongly in}\, H^1(\Omega,\mathbb{R}^2), \tag{3.37}$$
$$e_h^+(t) \to e(t) \quad \text{strongly in}\, L^2(\Omega,\mathbb{M}^{2\times 2}_{\text{sym}}), \tag{3.38}$$
$$p_h^+(t) \to p(t) \quad \text{strongly in}\, L^2(\Omega,\mathbb{M}^{2\times 2}_{\text{sym}}). \tag{3.39}$$

By the compatibility condition we have for all $t \in [0,T_f]$,

$$Eu_h^+(t) = e_h^+(t) + p_h^+(t) \quad \text{and} \quad u_h^+(t) = w_h^+(t) \text{ on } \partial\Omega_D \quad \text{a.e.} \quad \text{in} \quad \Omega.$$

The convergence results (3.37)–(3.39) imply that

$$(u(t), e(t), p(t)) \in A_{adm}(w(t)), \quad \text{for all } t \in [0,T_f].$$

On the other hand, for all $t \in (0,T_f]$,

$$\| p_h(t) - p_h^+(t) \|_2 \leq h \| \dot{p}_h(t) \|_2 . \tag{3.40}$$

Since $\dot{p}_h(t)$ is uniformly bounded in $L^2(\Omega)$, $p_h(0) = p_h^+(0) = p_0$, we deduce from (3.39) and (3.40) that for all $t \in [0,T_f]$,

$$p_h(t) \to p(t) \quad \text{strongly in}\, L^2(\Omega,\mathbb{M}^{2\times 2}_{\text{sym}}). \tag{3.41}$$

From Lemma 3.5 and (3.27), we deduce that

$$\| \dot{p}_h(t) - \dot{p}_l(t) \|_2 \leq \| a_h^+(t)(e_h^+(t) - e_l^+(t)) \|_2 + Y_{h,l}(t) + C \| p_h^+(t) - p_l^+(t) \|_2 . \tag{3.42}$$

Combining this last inequality, Lemma 3.7 and Proposition 3.10 we have that for all $t \in (0, T_f]$, $(\dot{p}_h(t))_h$ is a Cauchy sequence in $L^2(\Omega, \mathbb{M}^{2\times 2}_{\mathrm{sym}})$. As a consequence, there is a function $\zeta(t) \in L^2(\Omega, \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ such that

$$\dot{p}_h(t) \to \zeta(t) \quad \text{strongly in } L^2(\Omega, \mathbb{M}^{2\times 2}_{\mathrm{sym}}). \tag{3.43}$$

Due to the *a priori* estimate of Proposition 3.3

$$\sup_{(0,T_f]} \| \dot{p}_h(t) \|_{L^2} \leq C. \tag{3.44}$$

Thanks to the previous convergence result (3.43) we have

$$\sup_{(0,T_f]} \| \zeta(t) \|_{L^2} \leq C, \quad \text{and} \quad \zeta \in L^\infty(0, T_f, L^2(\Omega, \mathbb{M}^{2\times 2}_{\mathrm{sym}})). \tag{3.45}$$

From Proposition 3.3 we also deduce that

$$\|p_h\|_{W^{1,\infty}(0,T_f,L^2(\Omega,\mathbb{M}^{2\times 2}_{\mathrm{sym}}))} \leq C, \tag{3.46}$$

so that up to a subsequence, there exists $\hat{p} \in W^{1,\infty}(0, T_f, L^2(\Omega, \mathbb{M}^{2\times 2}_{\mathrm{sym}}))$ such that

$$p_h, \dot{p}_h \rightharpoonup \hat{p}, \dot{\hat{p}} \quad \text{weakly* in} \quad L^\infty(0, T_f, L^2(\Omega, \mathbb{M}^{2\times 2}_{\mathrm{sym}})). \tag{3.47}$$

Then, by the Arzelà–Ascoli theorem $p_h(t) \rightharpoonup \hat{p}(t)$ weakly in $L^2(\Omega, \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ for all $t \in [0, T_f]$. It follows from (3.41) that for all $t \in [0, T_f]$, $p(t) = \hat{p}(t)$, and

$$p \in W^{1,\infty}(0, T_f; L^2(\Omega, \mathbb{M}^{2\times 2}_{\mathrm{sym}})). \tag{3.48}$$

Since $\dot{p_h}(t) \to \zeta(t)$ strongly in $L^2(\Omega, \mathbb{M}^{2\times 2}_{\mathrm{sym}})$ for all $t \in (0, T_f]$, by the Lebesgue dominated convergence theorem and Proposition 3.3 we deduce that

$$\dot{p}_h \rightharpoonup \zeta \quad \text{weakly* in} \quad L^\infty(0, T_f, L^2(\Omega, \mathbb{M}^{2\times 2}_{\mathrm{sym}})). \tag{3.49}$$

The convergence results (3.47), (3.48) and (3.49) lead to $\dot{p} = \zeta$ a.e. in $[0, T_f] \times \Omega$ which implies due to (3.43) that for a.e. $t \in [0, T_f]$

$$\dot{p}_h(t) \to \dot{p}(t) \quad \text{strongly in } L^2(\Omega, \mathbb{M}^{2\times 2}_{\mathrm{sym}}). \tag{3.50}$$

Furthermore, by the *a priori* estimates of Proposition 3.3 we have

$$\sup_{[0,T_f]} \| u_h^+(t) \|_{H^1} \leq C, \ \sup_{[0,T_f]} \| e_h^+(t) \|_{L^2} \leq C,$$

for some constant $C > 0$ independent on h. Thanks to the convergence (3.37)–(3.39),

$$\sup_{[0,T_f]} \| u(t) \|_{H^1} \leq C, \ \sup_{[0,T_f]} \| e(t) \|_{L^2} \leq C.$$

We conclude that

$$u \in L^\infty(0, T_f, H^1(\Omega, \mathbb{R}^2)), e \in L^\infty(0, T_f, L^2(\Omega, \mathbb{M}^{2\times 2}_{\mathrm{sym}})), p \in W^{1,\infty}(0, T_f, L^2(\Omega, \mathbb{M}^{2\times 2}_{\mathrm{sym}})). \qquad \square$$

Remark 3.13. In the case, when K is a convex, closed set of $\mathbb{M}^{2\times 2}_{\mathrm{Sym}}$ with non empty interior, we can proceed as follows: we can show that $\| p_h(t) - p_h^+(t) \|_{L^2(L^2)} \leq h \| \dot{p}_h(t) \|_{L^2(L^2)}$ (which replaces formula (3.40)). Thanks to (3.39), using the *a priori* bounds and the Lebesgue dominated convergence theorem, we can show that $p_h^+ \to p$ strongly in $L^2(L^2)$ (which replaces formula (3.41)) and so obtain that $p_h \to p$ strongly in $L^2(L^2)$. Using (3.42), we can show that $\dot{p_h}$ is a Cauchy sequence in $L^2(L^2)$. Since p_h is uniformly bounded in $H^1(L^2)$, we obtain using the Arzelà−Ascoli theorem that for almost every $t \in [0, T_f]$, $\dot{p_h}(t)$ converges to $\dot{p}(t)$ strongly in L^2. Finally, we obtain $p \in H^1(L^2)$ instead of $p \in W^{1,\infty}(L^2)$.

3.4. The proof of Theorem 2.1

Let $t \in (0, T_f]$. The convergence result (3.38) and Proposition 3.6 imply that

$$\sigma_h^+(t) \to \sigma(t) \quad \text{strongly in } L^2(\Omega, \mathbb{M}^{2\times 2}_{\text{sym}}), \tag{3.51}$$

with $\sigma(t) = (v^2(t) + \eta)Ae(t)$. Since $-\operatorname{div} \sigma_h^+(t) = 0$ a.e. in Ω,

$$-\operatorname{div} \sigma(t) = 0 \text{ a.e. in } \Omega.$$

We rewrite the discrete plastic flow rule as follows:

$$\begin{aligned} \tau \int_\Omega |q| \mathrm{d}x \geq \tau \int_\Omega |\dot{p}_h(t)| \mathrm{d}x& \\ + \int_\Omega \left(\sigma_h^+(t) - Bp_h^+(t) - \beta \dot{p}_h(t)\right) &: (q - \dot{p}_h(t))\, \mathrm{d}x. \end{aligned} \tag{3.52}$$

By the convergence results (3.39), (3.50), (3.51) we obtain for a.e $t \in [0, T_f]$

$$\begin{aligned} \tau \int_\Omega |q| \mathrm{d}x \geq \tau \int_\Omega |\dot{p}(t)| \mathrm{d}x& \\ + \int_\Omega (\sigma(t) - Bp(t) - \beta \dot{p}(t)) &: (q - \dot{p}(t))\, \mathrm{d}x, \end{aligned} \tag{3.53}$$

which implies

$$\sigma(t) - Bp(t) - \beta \dot{p}(t) \in \partial H(\dot{p}(t)) \quad \text{for a.e.} \quad x \in \Omega.$$

We now pass to the limit in the crack propoagation condition. A similar treatement was used in [2, 15]. We rewrite the problem (3.2) as follows: for every $\varphi \in W^{1,r}(\Omega)$, $\varphi \leq v_h^{n-1}$, $\varphi = 1$ on $\partial\Omega_D$ we have

$$\mathcal{E}_{el}(e_h^+(t), v_h^+(t)) + \mathcal{E}_S^r(v_h^+(t)) \leq \mathcal{E}_{el}(e_h^+(t), \varphi) + \mathcal{E}_S^r(\varphi). \tag{3.54}$$

Let $v \in W^{1,r}(\Omega)$, $v = 1$ on $\partial\Omega_D$, with $v \leq v(t)$ in Ω. We define

$$v_h^*(t) := \min(v, v_h^+(t)).$$

By definition $v_h^*(t) \in W^{1,r}(\Omega)$ and $v_h^*(t) \leq v_h^+(t) \leq v_h^{n-1}$ and $v_h^*(t) = 1$ on $\partial\Omega_D$, so that $v_h^*(t)$ is an admissible test function for the problem (3.54). We obtain

$$\begin{aligned} \frac{1}{2} \int_\Omega \left(v_h^+(t)^2 + \eta\right) Ae_h^+(t) : e_h^+(t)\, \mathrm{d}x + \int_\Omega \frac{\varepsilon^{r-1}}{r} |\nabla v_h^+(t)|^r \mathrm{d}x + \int_\Omega \frac{\alpha}{r'\varepsilon} \left(1 - v_h^+(t)\right)^r \, \mathrm{d}x& \\ \leq \frac{1}{2} \int_\Omega \left(v_h^*(t)^2 + \eta\right) Ae_h^+(t) : e_h^+(t)\, \mathrm{d}x& \\ + \int_\Omega \frac{\varepsilon^{r-1}}{r} |\nabla v_h^*(t)|^r \mathrm{d}x + \int_\Omega \frac{\alpha}{r'\varepsilon} \left(1 - v_h^*(t)\right)^r \, \mathrm{d}x&. \end{aligned} \tag{3.55}$$

Set $A_h := \{x \in \Omega;\, v(x) \leq v_h^+(t, x)\}$. As $v_h^+(t) \rightharpoonup v(t)$ weakly in $W^{1,r}(\Omega)$; $1_{A_h} \to 1$, and $1_{A_h^c} \to 0$ pointwise in Ω. As a consequence, by the Lebesque Dominated Convergence Theorem we get

$$\int_\Omega 1_{A_h^c}(x)\, \mathrm{d}x \to 0. \tag{3.56}$$

We now prove that $1_{A_h}\nabla v_h^+(t) \rightharpoonup \nabla v(t)$ weakly in $L^r(\Omega)$. Let $q \in L^{r/(r-1)}(\Omega)$. Since $v_h^+(t) \rightharpoonup v(t)$ weakly in $W^{1,r}(\Omega)$, we have

$$\int_\Omega \nabla v_h^+(t) q \,\mathrm{d}x = \int_{A_h} \nabla v_h^+(t) q \,\mathrm{d}x + \int_{A_h^c} \nabla v_h^+(t) q \,\mathrm{d}x \to \int_\Omega \nabla v(t) q \,\mathrm{d}x. \tag{3.57}$$

By the Lebesque dominated convergence

$$\int_{A_h^c} \nabla v_h^+(t) q \,\mathrm{d}x = \int_\Omega 1_{A_h^c} \nabla v_h^+(t) q \,\mathrm{d}x \to 0,$$

which, using (3.57) yields

$$\int_\Omega 1_{A_h} \nabla v_h^+(t) q \,\mathrm{d}x = \int_{A_h} \nabla v_h^+(t) q \,\mathrm{d}x \to \int_\Omega \nabla v(t) q \,\mathrm{d}x.$$

By lower semicontinuity,

$$\liminf_{h\to 0} \int_{A_h} |\nabla v_h^+(t)|^r \,\mathrm{d}x = \liminf_{h\to 0} \int_\Omega |1_{A_h} \nabla v_h^+(t)|^r \,\mathrm{d}x \geqslant \int_\Omega |\nabla v(t)|^r \,\mathrm{d}x. \tag{3.58}$$

Using the same arguments, we also prove that $v_h^*(t) \rightharpoonup v$ weakly in $W^{1,r}(\Omega)$. The Sobolev imbedding theorem implies that, $v_h^*(t) \to v$ strongly in $C^0(\overline{\Omega})$, using Proposition 3.6 and (3.38) we show that as $h \to 0$,

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \left(v_h^+(t)^2 + \eta\right) Ae_h^+(t) : e_h^+(t) \,\mathrm{d}x \to \frac{1}{2}\int_\Omega \left(v(t)^2 + \eta\right) Ae(t) : e(t) \,\mathrm{d}x,\\
&\frac{1}{2}\int_\Omega \left(v_h^*(t)^2 + \eta\right) Ae_h^+(t) : e_h^+(t) \,\mathrm{d}x \to \frac{1}{2}\int_\Omega \left(v^2 + \eta\right) Ae(t) : e(t) \,\mathrm{d}x,\\
&\int_\Omega \frac{\alpha}{r'\varepsilon} \left(1 - v_h^+(t)\right)^r \,\mathrm{d}x \to \int_\Omega \frac{\alpha}{r'\varepsilon} \left(1 - v(t)\right)^r \,\mathrm{d}x,\\
&\int_\Omega \frac{\alpha}{r'\varepsilon} \left(1 - v_h^*(t)\right)^r \,\mathrm{d}x \to \int_\Omega \frac{\alpha}{r'\varepsilon} \left(1 - v\right)^r \,\mathrm{d}x,\\
&\int_{A_h} \frac{\varepsilon^{r-1}}{r} |\nabla v|^r \mathrm{d}x \to \int_\Omega \frac{\varepsilon^{r-1}}{r} |\nabla v|^r \,\mathrm{d}x.
\end{aligned} \tag{3.59}$$

The definition of $v_h^*(t)$ gives

$$\int_\Omega \frac{\varepsilon^{r-1}}{r} |\nabla v_h^*(t)|^r \,\mathrm{d}x = \int_{A_h} \frac{\varepsilon^{r-1}}{r} |\nabla v|^r \,\mathrm{d}x + \int_{A_h^c} \frac{\varepsilon^{r-1}}{r} |\nabla v_h^+(t)|^r \,\mathrm{d}x.$$

From (3.55), we obtain

$$\begin{aligned}
\frac{1}{2}\int_\Omega \left(v_h^+(t)^2 + \eta\right) Ae_h^+(t) : e_h^+(t) \,\mathrm{d}x + \int_{A_h} \frac{\varepsilon^{r-1}}{r} |\nabla v_h^+(t)|^r \mathrm{d}x + \int_\Omega \frac{\alpha}{r'\varepsilon} \left(1 - v_h^+(t)\right)^r \,\mathrm{d}x&\\
\leq \frac{1}{2}\int_\Omega \left(v_h^*(t)^2 + \eta\right) Ae_h^+(t) : e_h^+(t) \,\mathrm{d}x&\\
+ \int_{A_h} \frac{\varepsilon^{r-1}}{r} |\nabla v|^r \mathrm{d}x + \int_\Omega \frac{\alpha}{r'\varepsilon} \left(1 - v_h^*(t)\right)^r \,\mathrm{d}x.&
\end{aligned} \tag{3.60}$$

The previous convergence results (3.58), (3.59), and the last inequality yield to

$$\begin{aligned}\frac{1}{2}\int_{\Omega}\left(v(t)^2+\eta\right)Ae(t):e(t)\,\mathrm{d}x+\int_{\Omega}\frac{\varepsilon^{r-1}}{r}|\nabla v(t)|^r\mathrm{d}x+\int_{\Omega}\frac{\alpha}{r'\varepsilon}\left(1-v(t)\right)^r\,\mathrm{d}x\\ \leq\frac{1}{2}\int_{\Omega}\left(v^2+\eta\right)Ae(t):e(t)\,\mathrm{d}x\\ +\int_{\Omega}\frac{\varepsilon^{r-1}}{r}|\nabla v|^r\mathrm{d}x+\int_{\Omega}\frac{\alpha}{r'\varepsilon}\left(1-v\right)^r\,\mathrm{d}x.\end{aligned}\tag{3.61}$$

for all $v\in W^{1,r}(\Omega)$, $v=1$ on $\partial\Omega_D$, with $v\leq v(t)$ in Ω, which completes the proof. □

4. Conclusion

In this paper, we studied an elasto-viscoplastic continuous evolution with kinematic hardening and fracture. We proved an existence result of an evolution to the proposed model via a study of a discrete time evolutions obtained resolving incremental variational problems.

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