ROBUST OPERATOR ESTIMATES AND THE APPLICATION TO SUBSTRUCTURING METHODS FOR FIRST-ORDER SYSTEMS

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Abstract. We discuss a family of discontinuous Petrov–Galerkin (DPG) schemes for quite general partial differential operators. The starting point of our analysis is the DPG method introduced by [Demkowicz *et al.*, *SIAM J. Numer. Anal.* **49** (2011) 1788–1809; Zitelli *et al.*, *J. Comput. Phys.* **230** (2011) 2406–2432]. This discretization results in a sparse positive definite linear algebraic system which can be obtained from a saddle point problem by an element-wise Schur complement reduction applied to the test space. Here, we show that the abstract framework of saddle point problems and domain decomposition techniques provide stability and *a priori* estimates. To obtain efficient numerical algorithms, we use a second Schur complement reduction applied to the trial space. This restricts the degrees of freedom to the skeleton. We construct a preconditioner for the skeleton problem, and the efficiency of the discretization and the solution method is demonstrated by numerical examples.

1. INTRODUCTION

Petrov–Galerkin schemes for the numerical solution of partial differential equations are among the most flexible and powerful discretization techniques. In particular, these methods allow for different test and ansatz spaces in the variational problem formulation. This additional flexibility possibly increases the robustness and stability in case of parameter dependent systems. Uniform stability can be granted for a large class of partial differential equations including, *e.g.*, the nearly incompressible linear elasticity case. However, the choice of suitable finite dimensional test and trial spaces is far from trivial in the conforming setting since uniform inf-sup conditions have to be satisfied. Recently introduced discontinuous Petrov–Galerkin schemes [15,16,18,34] are of special interest and provide attractive alternatives to standard conforming schemes, see, also *e.g.*, [9,13,17].

Here we focus on this newly introduced abstract family of discretizations and provide robust *a priori* estimates and define iterative solvers. More precisely, we show that a broad class of linear partial differential equations can be efficiently approximated by methods which allow for a local reduction to positive definite Schur complement problems. Schur complement reductions are extensively investigated for finite element methods of least-squares type [4,5] and, recently, for least-square problems on element level resulting in discontinuous Petrov–Galerkin

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methods [15,16]. In combination with the reduction to the interface [10,25,27] efficient iterative solvers can be designed. Note that there is also a close link to the FOSLS scheme [1,2,11,22], to the Trefftz method (see [21] for an overview), and to a least-squares approach in negative norms for first-order systems [7].

The starting point is a linear first-order partial differential system. This is obtained either directly from balance equations and constitutive laws, *e.g.*, in the case of Darcy's law, or alternatively we rewrite a second order equation as first order system by introducing auxiliary unknowns. To discretize this system, we use element-wise domain decomposition techniques involving traces, fluxes and suitable lifting operators. Once the traces on the element boundaries are known, the interior values can be reconstructed approximately by solving local low dimensional problems.

Before we consider the abstract setting, we sketch the ideas for a simple model problem in Section 2. The abstract setting for the DPG method fits perfectly well into the standard saddle point framework. In order to obtain optimal *a priori* bounds and to explain the concept of Petrov–Galerkin discretizations with optimal test spaces and the connection to Schur complements, we briefly review saddle point problems in Section 3. Then in Section 4, we verify the stability for several examples including elasticity, the Stokes system, the Helmholtz problem, and Maxwell's equation. In Section 5, it is shown that the stability of the skeleton reduction only depends on the stability constant of the first-order system. The discretization of the skeleton reduction is analyzed in Section 6 within the abstract framework. Finally in Section 7, we conclude with some numerical examples demonstrating the robustness of the method.

2. A scalar model problem

We consider the second order elliptic partial differential equation

$$-\Delta p + b \cdot \nabla p + ap = f \quad \text{in } \Omega, \qquad p = 0 \quad \text{on } \partial\Omega \tag{2.1}$$

with $f \in \mathrm{L}_2(\Omega)$. The domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is assumed to be bounded and Lipschitz. Here $b \in \mathrm{H}^1(\Omega, \mathbb{R}^d)$ is the solution of an incompressible flow problem with $\nabla \cdot b = 0$, and $a \in \mathrm{L}_2(\Omega)$ is a non-negative function. The regularity assumptions on a and b will guarantee that our weak formulation is well-defined. To define a first-order system, we introduce a flux variable $\sigma = bp - \nabla p$ and rewrite (2.1) as $ap + \nabla \cdot \sigma = f$. Then the model equation reads as: find $(\sigma^{\mathrm{sol}}, p^{\mathrm{sol}}) \in U = \mathrm{H}(\mathrm{div}, \Omega) \times \mathrm{H}_0^1(\Omega)$ such $L(\sigma^{\mathrm{sol}}, p^{\mathrm{sol}}) = (0, f)$, with

$$L(\sigma, p) = (\sigma - bp + \nabla p, ap + \nabla \cdot \sigma). \tag{2.2}$$

2.1. Local substructuring

Following the ideas of non-overlapping domain decomposition techniques, the discretization scheme is based on a disjoint partitioning $\bar{\Omega} = \bigcup_{\tau \in \mathcal{T}} \bar{\tau}$ into open subdomains $\tau \subset \Omega$. Let $\Gamma_{\mathcal{T}} = \bigcup_{\tau \in \mathcal{T}} \partial\tau$ be the skeleton of this decomposition. Then Schur complement techniques allow to eliminate the inner subdomain degrees of freedom and to reduce the system to trace values at the skeleton.

On each subdomain τ, we define the local spaces $V_\tau = \mathrm{H}(\mathrm{div}, \tau) \times \mathrm{H}^1(\tau)$, on the boundaries $\partial\tau$ the local trace spaces $\hat{V}_\tau = \mathrm{H}^{-1/2}(\partial\tau) \times \mathrm{H}^{1/2}(\partial\tau)$, and the local trace operators $\gamma_\tau \colon V_\tau \longrightarrow \hat{V}_\tau$ with $\gamma_\tau(\sigma, p) = (\sigma|_{\partial\tau} \cdot n_\tau, p|_{\partial\tau})$, where n_τ is the outer normal vector on $\partial\tau$. We define the adjoint trace operator $\gamma_\tau^{\mathrm{ad}}(\eta, q) = (q|_{\partial\tau}, \eta|_{\partial\tau} \cdot n_\tau)$ and the adjoint differential operator

$$L^{\mathrm{ad}}(\eta, q) = (\eta - \nabla q, aq - b \cdot \eta - \nabla \cdot \eta). \tag{2.3}$$

Then integration by parts yields for all test functions $(\eta, q) \in V_\tau$

$$\begin{aligned}
\big(L(\sigma, p), (\eta, q)\big)_\tau &= \big(\sigma - bp + \nabla p, \eta\big)_\tau + \big(ap + \nabla \cdot \sigma, q\big)_\tau \\
&= (\sigma, \eta - \nabla q)_\tau + (p, aq - b \cdot \eta - \nabla \cdot \eta)_\tau + \big\langle \sigma|_{\partial\tau} \cdot n_\tau, q|_{\partial\tau} \big\rangle + \big\langle \eta|_{\partial\tau} \cdot n_\tau, p|_{\partial\tau} \big\rangle \\
&= \big((\sigma, p), L^{\mathrm{ad}}(\eta, q)\big)_\tau + \big\langle \gamma_\tau(\sigma, p), \gamma_\tau^{\mathrm{ad}}(\eta, q) \big\rangle,
\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes a vectorial duality pairing of $H^{-1/2}(\partial\tau)$ and $\mathrm{H}^{1/2}(\partial\tau)$. Combining the local traces defines for $(\sigma, p) \in \mathrm{H}(\mathrm{div}, \Omega) \times \mathrm{H}^1(\Omega)$ the skeleton values

$$\gamma_\Gamma(\sigma, p) = \big(\gamma_\tau(\sigma, p)\big)_{\tau \in \mathcal{T}} \in \hat{V} = \prod_{\tau \in \mathcal{T}} \hat{V}_\tau.$$

This gives for the solution $(\sigma^{\mathrm{sol}}, p^{\mathrm{sol}}) \in U$ of the system $L(\sigma^{\mathrm{sol}}, p^{\mathrm{sol}}) = (0, f)$ and its trace $\gamma_\Gamma(\sigma^{\mathrm{sol}}, p^{\mathrm{sol}}) \in \hat{V}$ the variational equality

$$\big((\sigma^{\mathrm{sol}}, p^{\mathrm{sol}}), L^{\mathrm{ad}}(\eta, q)\big)_\tau + \big\langle \gamma_\tau \big(\sigma^{\mathrm{sol}}, p^{\mathrm{sol}}\big), \gamma_\tau^{\mathrm{ad}}(\eta, q)\big\rangle = (f, q)_\tau, \quad (\eta, q) \in V_\tau. \tag{2.4}$$

Restricting the test space $V = \prod_{\tau \in \mathcal{T}} V_\tau$ to the kernel of the adjoint operator

$$\mathcal{N}(L^{\mathrm{ad}}) = \big\{(\eta, q) \in V \colon L^{\mathrm{ad}}(\eta, q) = 0\big\},$$

we observe that the skeleton values $\gamma_\Gamma(\sigma^{\mathrm{sol}}, p^{\mathrm{sol}})$ satisfy

$$\sum_\tau \big\langle \gamma_\tau(\sigma^{\mathrm{sol}}, p^{\mathrm{sol}}), \gamma_\tau^{\mathrm{ad}}(\eta, q)\big\rangle = (f, q)_\Omega, \quad (\eta, q) \in \mathcal{N}\big(L^{\mathrm{ad}}\big). \tag{2.5}$$

Knowing $\gamma_\Gamma(\sigma^{\mathrm{sol}}, p^{\mathrm{sol}})$, we can recover $(\sigma^{\mathrm{sol}}, p^{\mathrm{sol}})$ by solving independently for each $\tau \in \mathcal{T}$ a local problem. Now the main idea is to define suitable discrete approximation spaces for $\gamma_\Gamma(U)$ and $\mathcal{N}(L^{\mathrm{ad}})$ such that the discrete version of (2.5) is well-defined. We point out that the spaces $\gamma_\Gamma(U)$ and $\mathcal{N}(L^{\mathrm{ad}})$ have a completely different structure. The kernel $\mathcal{N}(L^{\mathrm{ad}})$ is purely subdomain-wise defined. Thus $\mathcal{N}(L^{\mathrm{ad}})$ can be written as a product space on the subdomains, whereas each $(\sigma, p) \in U$ with a non-trivial trace $\gamma_\Gamma(\sigma, p)$ has at least two subdomains as support. Thus a discretization of $\gamma_\Gamma(U)$ and $\mathcal{N}(L^{\mathrm{ad}})$ will naturally lead to a Petrov–Galerkin scheme.

2.2. A Petrov–Galerkin discretization

We approximate the trace of the solution $\hat{u}^{\mathrm{sol}} = \gamma_\Gamma(\sigma^{\mathrm{sol}}, p^{\mathrm{sol}})$ of (2.5) in a given discrete space $\hat{U}_h \subset \gamma_\Gamma(U)$. A natural choice of $\hat{U}_h$ is to take the trace of the fluxes of mixed finite elements, such as, $e.g.$, Raviart–Thomas or BDM elements, see, $e.g.$, [8], for the first component, and for the second component the traces of standard conforming finite elements. The Petrov–Galerkin solution $\hat{u}_h^{\mathrm{sol}} = (\hat{u}_{\tau,h}^{\mathrm{sol}}) \in \hat{U}_h$ of (2.5) is defined by

$$\sum_\tau \big\langle \hat{u}_{\tau,h}^{\mathrm{sol}}, \gamma_\tau^{\mathrm{ad}} v_h\big\rangle = \big((0, f), v_h\big)_\Omega, \qquad v_h \in \mathcal{N}_h. \tag{2.6}$$

To obtain a well-defined system, the choice of $\mathcal{N}_h$ is crucial. Is the dimension of $\mathcal{N}_h$ too small compared to the dimension of $\hat{U}_h$, (2.6) has no unique solution. Is the dimension too large, then there is possibly no solution. To guarantee a unique solution of (2.6), the construction of $\mathcal{N}_h$ is done in two steps. Firstly, we choose finite element spaces $H_{\tau,h} \subset \mathrm{L}_2(\tau, \mathbb{R}^d \times \mathbb{R})$ and $V_{\tau,h} \subset V_\tau$ such that the dimension of the discrete kernel $\mathcal{N}(L_h^{\mathrm{ad}}) = \prod_\tau \mathcal{N}_\tau(L_h^{\mathrm{ad}})$ with

$$\mathcal{N}_\tau(L_h^{\mathrm{ad}}) = \big\{v_{\tau,h} \in V_{\tau,h} \colon (L^{\mathrm{ad}} v_{h,\tau}, u_{h,\tau})_\tau = 0 \text{ for } u_{h,\tau} \in H_{\tau,h}\big\}$$

is larger or equal than the dimension of $\hat{U}_h$. Secondly, we select a positive definite, continuous and symmetric bilinear form $a_\tau(\cdot, \cdot)$ on $V_{\tau,h} \times V_{\tau,h}$ and define for $\hat{u}_h \in \hat{U}_h$ the solution $N_\tau(\hat{u}_h) \in \mathcal{N}_\tau(L_h^{\mathrm{ad}})$ by

$$a_\tau(N_\tau(\hat{u}_h), v) = \big\langle \hat{u}_{\tau,h}, \gamma_\tau^{\mathrm{ad}} v\big\rangle, \quad v \in \mathcal{N}_\tau(L_h^{\mathrm{ad}}). \tag{2.7}$$

We point out that $N_\tau(\hat{u}_h)$ is, by construction, well-defined. It can also be obtained by the solution of the following local saddle point problem: find $(N_\tau(\hat{u}_h), M_\tau(\hat{u}_h)) \in V_{\tau,h} \times H_{\tau,h}$ such that

$$\begin{aligned} a_\tau(N_\tau(\hat{u}_h), v) + \big(M_\tau(\hat{u}_h), L^{\mathrm{ad}} v\big)_\tau &= \big\langle \hat{u}_{\tau,h}, \gamma_\tau^{\mathrm{ad}} v\big\rangle, & v &\in V_{\tau,h}, \\ \big(L^{\mathrm{ad}} N_\tau(\hat{u}_h), w\big)_\tau &= 0, & w &\in H_{\tau,h}. \end{aligned} \tag{2.8}$$

If the pairing $(V_{\tau,h}, H_{\tau,h})$ satisfy a suitable inf-sup condition, a unique solution exists of (2.8). Otherwise the first solution component is unique but not the second one. Associated with N_τ is the global operator N_h which maps $\hat{U}_h$ into $\mathcal{N}(L_h^{\mathrm{ad}})$ by $N_h(\hat{u}_h)|_\tau = N_\tau(\hat{u}_h)$. Now we set $\mathcal{N}_h = N_h(U_h)$. Obviously, this choice guarantees the existence of a solution of (2.6). Uniqueness is obtained if $\dim \mathcal{N}_h = \dim U_h$ or equivalently if the operator N_h is injective. By construction, we find $\dim \mathcal{N}_h \leq \dim U_h$ and if $\dim \mathcal{N}_h = \dim U_h$, then (2.6) is equivalent to the symmetric positive definite variational problem: find $\hat{u}_h^{\mathrm{sol}} \in \hat{U}_h$ such that

$$\sum_\tau a_\tau \left(N_h\left(\hat{u}_h^{\mathrm{sol}}\right), N_h(\hat{u}_h)\right) = \left((0, f), N_h(\hat{u}_h)\right)_\Omega, \quad \hat{u}_h \in \hat{U}_h. \tag{2.9}$$

The variational problem (2.9) is not very well suited for a direct numerical realization since it requires the knowledge of the action of the operator N_h on each element of $\hat{U}_h$. Alternatively, combining (2.7) and (2.8), we can obtain the solution of (2.5) by the solving the saddle point formulation: find $(v_h^{\mathrm{sol}}, u_h^{\mathrm{sol}}, \hat{u}_h^{\mathrm{sol}}) \in \prod_\tau V_{\tau,h} \times \prod_\tau H_{\tau,h} \times \hat{U}_h$ such that

$$\begin{aligned}
a(v_h^{\mathrm{sol}}, v) + \left(u_h^{\mathrm{sol}}, L^{\mathrm{ad}}v\right)_\Omega - \left\langle \hat{u}_h^{\mathrm{sol}}, \gamma_\Gamma^{\mathrm{ad}} v\right\rangle &= 0, & v &\in \prod_\tau V_{\tau,h}, \\
(L^{\mathrm{ad}}v_h^{\mathrm{sol}}, w)_\Omega &= 0, & w &\in \prod_\tau H_{\tau,h}, \\
\left\langle \hat{u}, \gamma_\Gamma^{\mathrm{ad}} v_h^{\mathrm{sol}}\right\rangle &= \left((0, f), N_h(\hat{u})\right)_\Omega, & \hat{u} &\in \hat{U}_h,
\end{aligned} \tag{2.10}$$

where $a(v, w) = \sum_\tau a_\tau(v|_\tau, w|_\tau)$, $v, w \in \prod_\tau V_{\tau,h}$, and $\left\langle \cdot, \gamma_\Gamma^{\mathrm{ad}} \cdot \right\rangle$ and $(L^{\mathrm{ad}}\cdot, \cdot)_\Omega$ have to be understood in the broken subdomain-wise sense. By construction, (2.10) has a solution for all possible choices of $V_{\tau,h}$, $H_{\tau,h}$ and $\hat{U}_h$. However uniqueness is not granted. Under the assumption $\dim \mathcal{N}_h = \dim U_h$, uniqueness of $\hat{u}_h$ is given.

3. Saddle point estimates in operator dependent norms

It is shown in ([30], Thm. 3) for the Stokes problem and in [9] for Friedrichs' systems that an *a priori* estimate for the variational problem (2.4) can be derived from a single stability constant, and in ([20], Thm. 2.1) it is proved that the upper bound for the error of the discrete DPG method relies only on a local inf-sup constant. Optimal estimates for Petrov–Galerkin discretizations using Kato's identity on idempotent operators are derived in [33], and optimal estimates for saddle point problems are recently presented in [26]. These estimates depend on the considered norms, and thus smaller constants can be obtained by selecting proper norms. It is obvious that in the well-posed elliptic case with a symmetric bilinear form, the ellipticity and continuity constants are one if the norm is defined as energy norm. Here, we show that the concept of energy norms can be extended to the saddle point setting.

Let X and Y be Hilbert spaces, and let $A \in \mathcal{L}(X, X')$ be a positive and self-adjoint operator, so that $\|x\|_A = \sqrt{\langle Ax, x\rangle}$ is a norm in X. Moreover, let $B \in \mathcal{L}(Y, X')$ be an injective operator and assume that the Schur complement $S = B'A^{-1}B$ defines a norm in Y by

$$\|y\|_S = \sqrt{\langle Sy, y\rangle} = \sup_{x \in X} \frac{|\langle By, x\rangle|}{\|x\|_A}, \qquad y \in Y. \tag{3.1}$$

It is easy to see that if B is continuous and the system is inf-sup stable, then the Schur complement norm is equivalent to the Hilbert space norm in Y. We now study in $W = X \times Y$ the saddle point operator

$$K := \begin{pmatrix} A & B \\ B' & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_X & 0 \\ B'A^{-1} & \mathrm{id}_Y \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & -S \end{pmatrix} \begin{pmatrix} \mathrm{id}_X & A^{-1}B \\ 0 & \mathrm{id}_Y \end{pmatrix}.$$

This block decomposition motivates us to define the self-adjoint and positive definite operator

$$\widetilde{K} := \begin{pmatrix} \mathrm{id}_X & 0 \\ B'A^{-1} & \mathrm{id}_Y \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} \mathrm{id}_X & A^{-1}B \\ 0 & \mathrm{id}_Y \end{pmatrix} = \begin{pmatrix} A & B \\ B' & 2S \end{pmatrix}$$

and the norm $\|(x,y)\|_{\widetilde{K}}^2 = \langle \widetilde{K}(x,y), (x,y)\rangle$ in W. Observing the identity $K'\widetilde{K}^{-1}K = \widetilde{K}$, we obtain

$$\|K(x,y)\|_{\widetilde{K}^{-1}}^2 = \langle K(x,y), \widetilde{K}^{-1}K(x,y)\rangle = \langle \widetilde{K}(x,y), (x,y)\rangle = \|(x,y)\|_{\widetilde{K}}^2$$

for all $(x,y) \in W$, i.e., $\|K\|_{\mathcal{L}(W,W')} = \|K^{-1}\|_{\mathcal{L}(W',W)} = 1$ with respect to the operator dependent norm in W.

Let $X_h \subset X$ and $Y_h \subset Y$ be a family of finite dimensional subspaces, and let $A_h \in \mathcal{L}(X_h, X_h')$ and $B_h \in \mathcal{L}(Y_h, X_h')$ be the Galerkin approximations of A and B, respectively. The corresponding discrete Schur complement is given by $S_h = B_h'A_h^{-1}B_h$. We assume uniform inf-sup stability of the discrete saddle point problem, i.e., there exists a $1 \geq \beta_0 > 0$ independent of h such that

$$\sup_{x_h \in X_h} \frac{|\langle B_h y_h, x_h\rangle|}{\|x_h\|_A} \geq \beta_0 \sup_{x \in X} \frac{|\langle B y_h, x\rangle|}{\|x\|_A}, \qquad y_h \in Y_h.$$

This implies that the discrete operator $K_h(x_h, y_h) = (A_h x_h + B_h y_h, B_h' x_h)$ is invertible, and we have the spectral bounds

$$\beta_0^2 \langle S y_h, y_h\rangle \leq \langle S_h y_h, y_h\rangle \leq \langle S y_h, y_h\rangle, \qquad y_h \in Y_h. \tag{3.2}$$

The following lemma shows that K_h^{-1} inherits its continuity constant from the inf-sup constant.

Lemma 3.1. $\|K_h^{-1}\|_{\mathcal{L}(W',W)} \leq 2\beta_0^{-2} - 1$.

Proof. Define $\widetilde{K}_h(x_h, y_h) = (A_h x_h + B_h y_h, B_h' x_h + 2S_h y_h)$ in $W_h = X_h \times Y_h$. As above, we find $\|K_h\|_{\mathcal{L}(W_h, W_h')} = \|K_h^{-1}\|_{\mathcal{L}(W_h', W_h)} = 1$ with respect to the discrete norm $\|(x_h, y_h)\|_{\widetilde{K}_h}^2 = \langle \widetilde{K}_h(x_h, y_h), (x_h, y_h)\rangle$ in W_h. Next, we consider the natural embedding $E_h \in \mathcal{L}(W_h, W)$ with $E_h(x_h, y_h) = (x_h, y_h)$. From (3.2) and

$$\|(x_h, y_h)\|_{\widetilde{K}}^2 = \|x_h + A^{-1}B y_h\|_A^2 + \langle S y_h, y_h\rangle = \|x_h + A_h^{-1}B_h y_h\|_A^2 + \langle(2S - S_h)y_h, y_h\rangle$$
$$\leq \|x_h + A_h^{-1}B_h y_h\|_A^2 + (2\beta_0^{-2} - 1)\langle S_h y_h, y_h\rangle \leq (2\beta_0^{-2} - 1)\|(x_h, y_h)\|_{\widetilde{K}_h}^2$$

we obtain the bound $\|E_h\|_{\mathcal{L}(W_h, W)} \leq \sqrt{2\beta_0^{-2} - 1}$, and the assertion follows from

$$\|K_h^{-1}\|_{\mathcal{L}(W',W)} \leq \|E_h\|_{\mathcal{L}(W_h,W)}\|K_h^{-1}\|_{\mathcal{L}(W_h',W_h)}\|E_h'\|_{\mathcal{L}(W',W_h')} = \|E_h\|_{\mathcal{L}(W_h,W)}^2 \leq 2\beta_0^{-2} - 1. \qquad \square$$

For given $(x^{\mathrm{sol}}, y^{\mathrm{sol}}) \in W$, let $(x_h^{\mathrm{sol}}, y_h^{\mathrm{sol}}) \in W_h$ be the Galerkin approximation, i.e., $(x_h^{\mathrm{sol}}, y_h^{\mathrm{sol}}) \in W_h$ is the unique solution of

$$\langle K_h(x_h^{\mathrm{sol}}, y_h^{\mathrm{sol}}), (x_h, y_h)\rangle = \langle K(x^{\mathrm{sol}}, y^{\mathrm{sol}}), (x_h, y_h)\rangle, \quad (x_h, y_h) \in W_h.$$

Then we obtain the following upper bound for the discretization error in terms of the best approximation error.

Theorem 3.2. $\|(x^{\mathrm{sol}}, y^{\mathrm{sol}}) - (x_h^{\mathrm{sol}}, y_h^{\mathrm{sol}})\|_{\widetilde{K}} \leq (2\beta_0^{-2} - 1) \inf_{w_h \in W_h} \|(x^{\mathrm{sol}}, y^{\mathrm{sol}}) - w_h\|_{\widetilde{K}}$.

Proof. We use the observation in [33] on the Hilbert space norm of idempotents. The Galerkin property yields $(K_h^{-1}K)(K_h^{-1}K) = K_h^{-1}K_h K_h^{-1}K = K_h^{-1}K$, so that $K_h^{-1}K$ is non-trivial and idempotent. Then, Kato's identity, see, e.g., [31], reads $\|\operatorname{id}_W - K_h^{-1}K\|_{\mathcal{L}(W,W)} = \|K_h^{-1}K\|_{\mathcal{L}(W,W)}$. In terms of Lemma 3.1, the Galerkin orthogonality for all $(x_h, y_h) \in W_h$ gives

$$\begin{aligned}
\|(x^{\mathrm{sol}}, y^{\mathrm{sol}}) - (x_h^{\mathrm{sol}}, y_h^{\mathrm{sol}})\|_{\widetilde{K}} &= \|(\operatorname{id}_W - K_h^{-1}K)(x^{\mathrm{sol}}, y^{\mathrm{sol}})\|_{\widetilde{K}} \\
&= \|(\operatorname{id}_W - K_h^{-1}K)((x^{\mathrm{sol}}, y^{\mathrm{sol}}) - (x_h, y_h))\|_{\widetilde{K}} \\
&\leq \|K\|_{\mathcal{L}(W,W')}\|K_h^{-1}\|_{\mathcal{L}(W',W)}\|(x^{\mathrm{sol}}, y^{\mathrm{sol}}) - (x_h, y_h)\|_{\widetilde{K}} \\
&\leq (2\beta_0^{-2} - 1)\|(x^{\mathrm{sol}}, y^{\mathrm{sol}}) - (x_h, y_h)\|_{\widetilde{K}}. \qquad \square
\end{aligned}$$

As we have already seen for the model problem, a well-posed Petrov–Galerkin formulation can be related to a saddle point formulation. Here we state the reverse. The saddle point approach also provides a Petrov–Galerkin discretization. Therefore, we consider a right-hand side $f \in B(Y) \subset X'$, and let $y^{\mathrm{sol}} = S^{-1}B'A^{-1}f \in Y$ be the unique solution of the equation $By^{\mathrm{sol}} = f$. Then, we observe that the approximation $y_h^{\mathrm{sol}} = S_h^{-1}B_h'A_h^{-1}f$ is the unique Petrov–Galerkin solution of the variational problem: find $y_h^{\mathrm{sol}} \in Y_h$ such that

$$\langle B_h y_h^{\mathrm{sol}}, x_h \rangle = \langle f, x_h \rangle, \qquad x_h \in X_h^{\mathrm{red}} := A_h^{-1}B_h(Y_h) \subset X_h.$$

The operator $S_h^{-1}B_h'A_h^{-1}B$ is idempotent, and we have

$$\|B\|_{\mathcal{L}(Y,X')} = \|A_h^{-1}\|_{\mathcal{L}(X',X_h)} = 1, \qquad \|B_h'\|_{\mathcal{L}(X_h,Y')} = \|S_h^{-1}\|_{\mathcal{L}(Y',Y)} \leq \beta_0^{-1}.$$

Following the lines of the proof of Theorem 3.2 (see also [33], Thm. 2.1), we obtain

$$\|y^{\mathrm{sol}} - y_h^{\mathrm{sol}}\|_S \leq \beta_0^{-1} \inf_{y_h \in Y_h} \|y^{\mathrm{sol}} - y_h\|_S, \tag{3.3}$$

provided that we can show that $\|S_h^{-1}B_h'\|_{\mathcal{L}(X_h,Y)} \leq \beta_0^{-1}$. To do so, we start with the straightforward estimate

$$\|S_h^{-1}B_h'x_h\|_{S_h}^2 = \langle x_h, B_h S_h^{-1}B_h'x_h \rangle \leq \|x_h\|_A \langle A_h^{-1}B_h S_h^{-1}B_h'x_h, B_h S_h^{-1}B_h'x_h \rangle^{\frac{1}{2}}$$
$$= \|x_h\|_A \langle x_h, B_h S_h^{-1}B_h'A_h^{-1}B_h S_h^{-1}B_h'x_h \rangle^{\frac{1}{2}} = \|x_h\|_A \langle x_h, B_h S_h^{-1}B_h'x_h \rangle^{\frac{1}{2}} = \|x_h\|_A \|S_h^{-1}B_h'x_h\|_{S_h}.$$

Using now the upper spectral bound for S, we find $\|S_h^{-1}B_h'x_h\|_S \leq \beta_0^{-1}\|S_h^{-1}B_h'x_h\|_{S_h} \leq \beta_0^{-1}\|x_h\|_A$.

Remark 3.3. Additionally, the identity

$$\sup_{x_h \in X_h^{\mathrm{red}}} \frac{|\langle B_h y_h, x_h \rangle|^2}{\|x_h\|_A^2} = \sup_{\tilde{y}_h \in Y_h} \frac{|\langle B_h y_h, A_h^{-1}B_h\tilde{y}_h \rangle|^2}{\|A_h^{-1}B_h\tilde{y}_h\|_A^2} = \sup_{\tilde{y}_h \in Y_h} \frac{\langle S_h y_h, \tilde{y}_h \rangle^2}{\langle S_h\tilde{y}_h, \tilde{y}_h \rangle} = \|y_h\|_{S_h}^2$$

implies $\inf_{y_h \in Y_h} \sup_{x_h \in X_h^{\mathrm{red}}} \frac{|\langle y_h, B_h'x_h \rangle|}{\|x_h\|_A \|y_h\|_S} \geq \beta_0$ for the reduced space $X_h^{\mathrm{red}} \subset X_h$ (see also [20], Thm. 2.1).

Before we focus on stability estimates in the next section, we relate the Schur complement norm (3.1) with an inf-sup constant. We assume that $\gamma > 0$ exists such that

$$\sup_{y \in Y} \frac{|\langle By, x \rangle|}{\|y\|_Y} \geq \gamma \|x\|_X, \qquad x \in X.$$

Then, since B is injective, we obtain by duality $\|y\|_S = \sup_{x \in X} \frac{|\langle By, x \rangle|}{\|x\|_A} \geq \gamma \|y\|_Y$ for $y \in Y$ (cf. [6], Lem. 4.4.2).

4. Stability estimates for first-order systems

For representative first-order differential operators L we specify the stability constant $C_L < \infty$.

4.1. An abstract framework

Let H be a Hilbert space with norm $\|\cdot\|_H$, and let $U \subset H$ be a dense subspace. We assume that $L\colon U \longrightarrow H$ is a closed linear operator, *i.e.*, the graph of the operator $\{(u, Lu)\colon u \in U\}$ is closed in $H \times H$, the domain U is a Hilbert space in the graph norm $\|u\|_U = \sqrt{\|u\|_H^2 + \|Lu\|_H^2}$, and the operator L has closed range. Moreover we assume that L is injective and surjective, and that its inverse is bounded.

In our examples, this setting is obtained as follows: for a dense subset $\mathcal{D}(L) \subset H$, we verify the existence of a stability constant $C_L < \infty$ satisfying

$$\|u\|_H \leq C_L\|Lu\|_H, \qquad u \in \mathcal{D}(L). \tag{4.1}$$

We then define U as the closure of $\mathcal{D}(L)$ with respect to the graph norm $\|u\|_U^2 = \|u\|_H^2 + \|Lu\|_H^2$ and L can be extended to U. The stability estimate (4.1) holds also for $u \in U$, and the range $L(U) \subset H$ is closed. Thus the choice $H = L(U)$ guarantees that we are in the above abstract setting.

4.2. Examples of second order partial differential equations and first order systems

We recall that $d = 2, 3$ is the spatial dimension and that $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain. Let $\partial\Omega = \partial\Omega_\mathrm{D} \cup \partial\Omega_\mathrm{N}$ be a decomposition of the boundary into Dirichlet and Neumann part, and let n be the outer unit normal on $\partial\Omega$. For simplicity, we consider, without loss of generality, only homogeneous boundary conditions. The standard L_2-norm on Ω is denoted by $\|\cdot\|_\Omega$. In our first examples, we start with a second order partial differential equation and transform it to a first-order system. In all cases, the Hilbert space H is a subset of $\mathrm{L}_2(\Omega, \mathbb{R}^M)$.

A scalar elliptic model problem. We reconsider the model problem (2.1) with Dirichlet boundary $\partial\Omega_\mathrm{D} = \partial\Omega$ and $\mathcal{D}(L) = \mathrm{H}(\mathrm{div}, \Omega) \times \mathrm{H}_0^1(\Omega)$. Let $C_\mathrm{P} > 0$ be the Poincaré constant with $\|p\|_\Omega \le C_\mathrm{P} \|\nabla p\|_\Omega$ for $p \in \mathrm{H}_0^1(\Omega)$. Furthermore, the continuous Sobolev embedding $\mathrm{H}^1(\Omega) \subset \mathrm{L}_4(\Omega)$ yields that there exists a $C_b > 0$ such that

$$\|bp\|_\Omega \le \|p\|_{\mathrm{L}_4(\Omega)}\|b\|_{\mathrm{L}_4(\Omega)} \le C_b\|\nabla p\|_\Omega, \qquad p \in \mathrm{H}_0^1(\Omega). \tag{4.2}$$

The first-order operator L is given in (2.2).

Lemma 4.1. *The stability constants of the scalar elliptic model problem can be defined as*

$$C_L = \sqrt{2 + \left(4 + C_P^2 + 4C_b^2\right)\left(1 + C_\mathrm{P}^2\right)}.$$

Proof. To verify (4.1), we firstly provide an upper bound of $\|\nabla p\|_\Omega$ in terms of $\|L(\sigma, p)\|_\Omega$. Due to the homogeneous Dirichlet boundary condition and the assumption $\nabla \cdot b = 0$ on b, we have $(p, b \cdot \nabla p)_\Omega = -(p, b \cdot \nabla p)_\Omega + (p^2, b \cdot n)_{\partial\Omega} = -(p, b \cdot \nabla p)_\Omega$, which gives $\left(p, b \cdot \nabla p\right)_\Omega = 0$. Then, we get

$$\begin{aligned}
\|\nabla p\|_\Omega^2 &\le \|\nabla p\|_\Omega^2 + (ap, p)_\Omega = \|\nabla p\|_\Omega^2 - \left(p, b \cdot \nabla p\right)_\Omega + (ap, p)_\Omega \\
&= \left(\sigma - bp + \nabla p, \nabla p\right)_\Omega + \left(\nabla \cdot \sigma, p\right)_\Omega + (ap, p)_\Omega = \left(L(\sigma, p), (\nabla p, p)\right)_\Omega \\
&\le \|L(\sigma, p)\|_\Omega\|(\nabla p, p)\|_\Omega \le \sqrt{1 + C_\mathrm{P}^2}\,\|L(\sigma, p)\|_\Omega\|\nabla p\|_\Omega.
\end{aligned}$$

Secondly, we bound $\|\sigma\|_\Omega$ in terms of $\|L(\sigma, p)\|_\Omega$ and $\|\nabla p\|_\Omega$ by

$$\|\sigma\|_\Omega^2 = (\sigma - bp + \nabla p, \sigma)_\Omega + (bp - \nabla p, \sigma)_\Omega \le \left(\|L(\sigma, p)\|_\Omega + \|bp - \nabla p\|_\Omega\right)\|\sigma\|_\Omega$$

which yields $\|\sigma\|_\Omega \le \|L(\sigma, p)\|_\Omega + \|bp - \nabla p\|_\Omega$. This gives the estimate

$$\begin{aligned}
\|\sigma\|_\Omega^2 + \|p\|_\Omega^2 &\le 2\|L(\sigma, p)\|_\Omega^2 + 2\|bp - \nabla p\|_\Omega^2 + \|p\|_\Omega^2 = 2\|L(\sigma, p)\|_\Omega^2 + 4\|bp\|^2 + 4\|\nabla p\|_\Omega^2 + \|p\|_\Omega^2 \\
&\le 2\|L(\sigma, p)\|_\Omega^2 + (4C_b^2 + 4 + C_\mathrm{P}^2)\|\nabla p\|_\Omega^2 \le \left(2 + (4C_b^2 + 4 + C_\mathrm{P}^2)(1 + C_\mathrm{P}^2)\right)\|L(\sigma, p)\|_\Omega^2. \qquad \square
\end{aligned}$$

We note that the stability constant does not depend on the norm of a but depends on b. Stable estimates for dominating convection are considered in [12, 18] with suitable weighted norms in order to compensate strong boundary layers. Stability for the pure transport problem is discussed in Theorem 2.4 from [13].

The Helmholtz problem. As second example, we consider the Helmholtz equation

$$-\Delta p - \omega^2 p = \mathrm{i}\omega g \quad \text{in } \Omega, \qquad \nabla p \cdot n + \mathrm{i}\omega p = 0 \quad \text{on } \partial\Omega, \tag{4.3}$$

where $\omega > 0$ is a given positive wave number. Introducing $\sigma = -1/(\mathrm{i}\omega)\nabla p$, we obtain $L(\sigma, p) = (0, g)$ with

$$L(\sigma, p) = (\mathrm{i}\omega\sigma + \nabla p, \mathrm{i}\omega p + \nabla \cdot \sigma) \tag{4.4}$$

defined on $\mathcal{D}(L) = \{(\sigma, p) \in \mathrm{H}(\mathrm{div}, \Omega) \times \mathrm{H}^1(\Omega) : \sigma \cdot n - p = 0 \text{ on } \partial\Omega\} \subset H$ with $H = \mathrm{L}_2(\Omega, \mathbb{C}^d \times \mathbb{C})$.

Lemma 4.2. *Assume that a constant $\alpha > 0$ exists such that $x \cdot n(x) \geq \alpha$ for a.a. $x \in \partial\Omega$. Then, the stability constant for the Helmholtz problem can be defined as*

$$C_L = 3R + \frac{R^2}{\alpha} + \frac{d-1}{\omega^2},$$

where d is the space dimension and $|x| < R$ for $x \in \partial\Omega$.

We note that such a constant α exists, *e.g.*, for star-shaped domains with the origin placed in Ω.

Proof. Following the ideas in [28] Proposition 8.1.4, [16, 19], Section 4.3, the stability estimate can be shown in three steps. Firstly, we provide upper bounds for $\|p\|_{\partial\Omega}$ and $\|\sigma\|_\Omega$. Secondly we consider smooth curl-free vector fields σ and bound $\|p\|_\Omega$. Finally a Helmholtz decomposition is applied.

Using the boundary conditions for $(\sigma, p) \in \mathcal{D}(L)$, we get $\|p\|_{\partial\Omega}^2 = (\sigma \cdot n, p)_{\partial\Omega} = (\nabla p, \sigma)_\Omega + (p, \nabla \cdot \sigma)_\Omega$. The definition of $L(\sigma, p)$ now yields

$$\|p\|_{\partial\Omega}^2 = \text{Re}\left(\mathrm{i}\omega \, \|(\sigma, p)\|_\Omega^2 + \left(\nabla p, \sigma\right)_\Omega + \left(\nabla \cdot \sigma, p\right)_\Omega\right) \tag{4.5a}$$
$$= \text{Re}\left(\left(\mathrm{i}\omega\sigma + \nabla p, \sigma\right)_\Omega + \left(\mathrm{i}\omega p + \nabla \cdot \sigma, p\right)_\Omega\right) = \text{Re}\left(L(\sigma, p), (\sigma, p)\right)_\Omega \leq \|L(\sigma, p)\|_\Omega \|(\sigma, p)\|_\Omega$$

$$\|\sigma\|_\Omega^2 = \|p\|_\Omega^2 + \frac{1}{\omega^2}\text{Re}\left(L(\sigma, p), (\mathrm{i}\omega\sigma, -\mathrm{i}\omega p)\right)_\Omega \leq \|p\|_\Omega^2 + \frac{1}{\omega^2}\|L(\sigma, p)\|_\Omega \|(\sigma, p)\|_\Omega. \tag{4.5b}$$

In the second step, we restrict ourselves to $(\sigma, p) \in \mathcal{D}(L)$ such that σ is smooth enough and $\nabla \times \sigma = 0$. A crucial role within this step plays the following equalities which can be easily established by a straightforward computation

$$\nabla \cdot (|\sigma|^2 x) = \sigma \cdot \overline{\nabla(\sigma \cdot x)} + \overline{\sigma} \cdot \nabla(\sigma \cdot x) + (d-2)|\sigma|^2, \tag{4.6a}$$
$$\nabla \cdot (|p|^2 x) = p \overline{\nabla p \cdot x} + \nabla p \cdot x \, \overline{p} + d \, |p|^2. \tag{4.6b}$$

In terms of (4.6), we can now express boundary terms by volume terms and find by integration by parts

$$\left(|\sigma|^2, x \cdot n\right)_{\partial\Omega} = 2\text{Re}\left(\sigma, \nabla(x \cdot \sigma)\right)_\Omega + (d-2)(\sigma, \sigma)_\Omega, \tag{4.7a}$$
$$\left(|p|^2, x \cdot n\right)_{\partial\Omega} = 2\text{Re}\left(p, x \cdot \nabla p\right)_\Omega + d \, (p, p)_\Omega. \tag{4.7b}$$

A reformulation of $\text{Re}\left(L(\sigma, p), (xp, x \cdot \sigma)\right)_\Omega$ in terms of (4.7) and the use of the boundary condition give

$$2\text{Re}\left(L(\sigma, p), (xp, x \cdot \sigma)\right)_\Omega = 2\text{Re}\left(\nabla \cdot \sigma, x \cdot \sigma\right)_\Omega + 2\text{Re}\left(p, x \cdot \nabla p\right)_\Omega$$
$$= 2\text{Re}\left(n \cdot \sigma, x \cdot \sigma\right)_{\partial\Omega} - 2\text{Re}\left(\sigma, \nabla(x \cdot \sigma)\right)_\Omega + 2\text{Re}\left(p, x \cdot \nabla p\right)_\Omega$$
$$= 2\text{Re}\left(p, x \cdot \sigma\right)_{\partial\Omega} - \left(|\sigma|^2, x \cdot n\right)_{\partial\Omega} + (d-2)(\sigma, \sigma)_\Omega + \left(|p|^2, x \cdot n\right)_{\partial\Omega} - d \, (p, p)_\Omega$$

or equivalently

$$d\|p\|_\Omega^2 + \left(|\sigma|^2, x \cdot n\right)_{\partial\Omega} = \left(|p|^2, x \cdot n\right)_{\partial\Omega} + 2\text{Re}\left(p, x \cdot \sigma\right)_{\partial\Omega} + (d-2)(\sigma, \sigma)_\Omega - 2\text{Re}\left(L(\sigma, p), (xp, x \cdot \sigma)\right)_\Omega.$$

Using the assumptions on Ω there exits $R < \infty$ with $|x| < R$ for $x \in \partial\Omega$ and thus $\alpha \leq x \cdot n \leq R$ for almost all $x \in \partial\Omega$. This observation guarantees that we can provide an upper bound for $\|p\|_\Omega^2$ by using Young's inequality

$$d\|p\|_\Omega^2 + \alpha \|\sigma\|_{\partial\Omega}^2 \leq R \|p\|_{\partial\Omega}^2 + 2R \|p\|_{\partial\Omega}\|\sigma\|_{\partial\Omega} + (d-2)\|\sigma\|_\Omega^2 + 2R \|L(\sigma, p)\|_\Omega \|(\sigma, p)\|_\Omega,$$
$$d\|p\|_\Omega^2 \leq \left(R + \frac{R^2}{\alpha}\right) \|p\|_{\partial\Omega}^2 + 2R \|L(\sigma, p)\|_\Omega \|(\sigma, p)\|_\Omega + (d-2)\|\sigma\|_\Omega^2.$$

Now the upper bounds (4.5a) and (4.5b) can be used, and we get

$$d\|p\|_\Omega^2 \leq \left(R + \frac{R^2}{\alpha} + 2R + \frac{d-2}{\omega^2} \right) \|L(\sigma,p)\|_\Omega \|(\sigma,p)\|_\Omega + (d-2)\|p\|_\Omega^2,$$

$$\|p\|_\Omega^2 \leq \frac{1}{2} \left(3R + \frac{R^2}{\alpha} + \frac{d-2}{\omega^2} \right) \|L(\sigma,p)\|_\Omega \|(\sigma,p)\|_\Omega.$$

Combining the results of the first two steps, we obtain for all $(\sigma,p) \in \mathcal{D}(L)$ with $\sigma = \nabla\psi$ the stability bound

$$\|(\sigma,p)\|_\Omega \leq \left(3R + \frac{R^2}{\alpha} + \frac{d-1}{\omega^2} \right) \|L(\sigma,p)\|_\Omega = C_L \|L(\sigma,p)\|_\Omega. \tag{4.8}$$

A density argument yields that the above estimate hold also for all $(\sigma,p) \in \mathcal{D}(L)$ with $\nabla \times \sigma = 0$.

Finally in the third step, we have to show that the above stability estimate holds true for all $(\sigma,p) \in \mathcal{D}(L)$. To do so, we use a Helmholtz decomposition. Introducing

$$H_0 = \left\{ \sigma \in \mathrm{L}_2(\Omega,\mathbb{C}^d) \colon (\sigma, \nabla \times v)_\Omega = 0 \text{ for all } v \in \mathrm{H}_0(\mathrm{curl},\Omega) \right\},$$

$(\sigma,p) \in \mathcal{D}(L)$ has a unique orthogonal decomposition into $(\sigma_0,p) + (\sigma_1,0)$ with $\sigma_0 \in H_0$ and $(\sigma_1, \nabla q)_\Omega = 0$ for all $q \in H^1(\Omega)$. Then the definition of L trivially gives $L(\sigma_1,0) = (\mathrm{i}\omega\sigma_1,0)$. Moreover the operator L preserves the orthogonality between (σ_0,p) and $(\sigma_1,0)$. These observations guarantee now in terms of (4.8)

$$\|(\sigma,p)\|_\Omega^2 = \|(\sigma_0,p)\|_\Omega^2 + \|(\sigma_1,0)\|_\Omega^2 \leq C_L^2 \|L(\sigma_0,p)\|_\Omega^2 + \frac{1}{\omega^2} \|L(\sigma_1,p)\|_\Omega^2$$

$$\leq C_L^2 \left(\|L(\sigma_0,p)\|_\Omega^2 + \|L(\sigma_1,p)\|_\Omega^2 \right) = C_L^2 \|L(\sigma,p)\|_\Omega^2. \qquad \square$$

Remark 4.3. We note that the stability constant C_L is uniformly bounded for large wave numbers.

Linear elasticity. We consider the linear elasticity system

$$-\mathrm{div}(\mathcal{C}\varepsilon(u)) = f \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega_\mathrm{D} \text{ and } \mathcal{C}\varepsilon(u)n = 0 \text{ on } \partial\Omega_\mathrm{N}, \tag{4.9}$$

where $\mathcal{C}$ is the positive definite elasticity tensor in $\mathbb{R}^{d \times d}_\mathrm{sym}$, $\varepsilon(u) = \mathrm{sym}(\mathrm{D}u)$ is the linearized strain tensor, $f \in \mathrm{L}_2(\Omega,\mathbb{R}^d)$ is a prescribed volume load, and u is the displacement. We assume that $\partial\Omega_\mathrm{D}$ has a positive measure in $d-1$. Introducing the stress $\sigma = \mathcal{C}\varepsilon(u)$, we can rewrite (4.9) as a first-order system $L(\sigma,u) = (0,-f)$ with

$$L(\sigma,u) = (\sigma - \mathcal{C}\varepsilon(u), \mathrm{div}(\sigma)) \tag{4.10}$$

and domain $\mathcal{D}(L) \subset H = \mathrm{L}_2(\Omega,\mathbb{R}^{d \times d}_\mathrm{sym} \times \mathbb{R}^d)$ given by

$$\mathcal{D}(L) = \left\{ (\sigma,u) \in \mathrm{H}(\mathrm{div},\Omega)^d \times \mathrm{H}^1(\Omega)^d \colon \sigma^\top = \sigma, \ \sigma n = 0 \text{ on } \partial\Omega_\mathrm{N} \text{ and } u = 0 \text{ on } \partial\Omega_\mathrm{D} \right\}.$$

On H we use the norm $\|(\sigma,u)\|_H^2 = (\mathcal{C}^{-1}\sigma,\sigma)_\Omega + \|u\|_\Omega^2$ and recall that the stability constant has to be specified with respect to this norm.

Lemma 4.4. *The stability constant of the elasticity problem can be defined as*

$$C_L = \sqrt{1 + 2C_K^2|\mathcal{C}^{-1}|(1 + C_K^2|\mathcal{C}^{-1}|)}.$$

Proof. Firstly we provide a bound for $\|u\|_\Omega$ by $\|L(\sigma, u)\|_H$. By assumption the Dirichlet boundary part is non-trivial, and thus no rigid body motion is in $\mathcal{D}(L)$ and Korn's inequality shows, *i.e.*, for all $u \in \mathrm{H}^1(\Omega)^d$ with $u = 0$ on $\partial\Omega_\mathrm{D}$ there exists a constant $C_\mathrm{K} > 0$ only depending on Ω and $\partial\Omega_\mathrm{D}$ such that

$$\|u\|_\Omega^2 \leq C_\mathrm{K}^2 \|\varepsilon(u)\|_\Omega^2 \leq C_\mathrm{K}^2 |\mathcal{C}^{-1}| \big(\mathcal{C}\varepsilon(u), \varepsilon(u)\big)_\Omega.$$

Korn's inequality show that to bound $\|u\|_\Omega^2$, we have to bound $\big(\mathcal{C}\varepsilon(u), \varepsilon(u)\big)_\Omega$. For $(\sigma, u) \in \mathcal{D}(L)$ we have, due to the boundary conditions,

$$\big(\mathcal{C}\varepsilon(u), \varepsilon(u)\big)_\Omega = -\big(\sigma - \mathcal{C}\varepsilon(u), \varepsilon(u)\big)_\Omega + \big(\sigma, \varepsilon(u)\big)_\Omega = -\big(\sigma - \mathcal{C}\varepsilon(u), \varepsilon(u)\big)_\Omega - \big(\operatorname{div}(\sigma), u\big)_\Omega$$

$$\leq \|L(\sigma, u)\|_H \|(\mathcal{C}\varepsilon(u), u)\|_H \leq \|L(\sigma, u)\|_H (1 + C_\mathrm{K}^2 |\mathcal{C}^{-1}|)^{\frac{1}{2}} \big(\mathcal{C}\varepsilon(u), \varepsilon(u)\big)_\Omega^{\frac{1}{2}}$$

and thus $\|u\|_\Omega^2 \leq C_\mathrm{K}^2 |\mathcal{C}^{-1}| (1 + C_\mathrm{K}^2 |\mathcal{C}^{-1}|) \|L(\sigma, u)\|_H^2$.

Secondly we bound $(\mathcal{C}^{-1}\sigma, \sigma)_\Omega$ by $\|L(\sigma, u)\|_H$ and $\|u\|_\Omega$. Integration by parts and the homogeneous boundary conditions guarantee

$$(\mathcal{C}^{-1}\sigma, \sigma)_\Omega = (\sigma - \mathcal{C}\varepsilon(u), \mathcal{C}^{-1}\sigma)_\Omega + (\varepsilon(u), \sigma)_\Omega = (\sigma - \mathcal{C}\varepsilon(u), \mathcal{C}^{-1}\sigma)_\Omega - (\operatorname{div}(\sigma), u)_\Omega$$

$$\leq \|L(\sigma, u)\|_H \|(\sigma, u)\|_H \leq \frac{1}{2}\|L(\sigma, u)\|_H + \frac{1}{2}(\mathcal{C}^{-1}\sigma, \sigma)_\Omega + \frac{1}{2}\|u\|_\Omega^2.$$

Combining these two bounds, we obtain the assertion by

$$\|(\sigma, u)\|_H^2 \leq \|L(\sigma, u)\|_H^2 + 2\|u\|_\Omega^2 \leq \big(1 + 2C_\mathrm{K}^2 |\mathcal{C}^{-1}| (1 + C_\mathrm{K}^2 |\mathcal{C}^{-1}|)\big) \|L(\sigma, u)\|_H^2. \qquad \square$$

Remark 4.5. Since $|\mathcal{C}^{-1}|$ is bounded in the incompressible limit, the estimate is robust for nearly incompressible materials.

The Stokes system. We consider the stationary linear Stokes system which can be regarded as the incompressible limit of the elasticity system

$$\begin{aligned} -\Delta u + \nabla p &= f, \\ \operatorname{div}(u) &= 0 \end{aligned} \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{4.11}$$

Introducing σ by $\sigma = \varepsilon(u) - pI$, we can rewrite (4.11) as a first-order system $L(\sigma, u, p) = (0, -f, 0)$ with

$$L(\sigma, u, p) = \big(\sigma + pI - \varepsilon(u), \operatorname{div}(\sigma), \operatorname{div}(u)\big) \tag{4.12}$$

defined on $\mathcal{D}(L) = \big\{(\sigma, u, p) \in \mathrm{H}(\operatorname{div}, \Omega)^d \times \mathrm{H}^1(\Omega)^d \times \mathrm{L}_2(\Omega) \colon \sigma^\top = \sigma, \ u = 0 \text{ on } \partial\Omega, \ \int_\Omega p \, dx = 0\big\}$.

Lemma 4.6. *The stability constant of the Stokes problem can be defined as*

$$C_L = \sqrt{C_1^2 \left(C_\mathrm{K}^2 + \sqrt{3}/c_p^2\right) + \left(1 + \left(\frac{\sqrt{3d}}{c_p} + 1\right)C_1\right)^2}$$

with $C_1 = \sqrt{1 + d + C_\mathrm{K}^2}$ and the inf-sup constant c_p.

Proof. We proceed in three steps and consider the terms $\|\varepsilon(u)\|_\Omega$, $\|p\|_\Omega$ and $\|\sigma\|_\Omega$ separately. For $(\sigma, u, p) \in \mathcal{D}(L)$ Korn's inequality can be applied and thus, we get

$$\begin{aligned} \|\varepsilon(u)\|_\Omega^2 &= \big(\varepsilon(u) - \sigma - pI, \varepsilon(u)\big)_\Omega + \big(\sigma, \varepsilon(u)\big)_\Omega + \big(pI, \varepsilon(u)\big)_\Omega \\ &= \big(\varepsilon(u) - \sigma - pI, \varepsilon(u)\big)_\Omega - \big(\operatorname{div}(\sigma), u\big)_\Omega + \big(p, \operatorname{div}(u)\big)_\Omega \\ &\leq \|L(\sigma, u, p)\|_\Omega \|(\varepsilon(u), u, \operatorname{div}(u))\|_\Omega \leq \sqrt{1 + C_\mathrm{K}^2 + d} \, \|L(\sigma, u, p)\|_\Omega \|\varepsilon(u)\|_\Omega. \end{aligned}$$

To bound $\|p\|_\Omega$, we use an inf-sup estimate, which yields

$$c_p\|p\|_\Omega \le \sup_{(0,v,0)\in\mathcal{D}(L)} \frac{\big(p,\mathrm{div}(v)\big)_\Omega}{\|\varepsilon(v)\|_\Omega} \le \sup_{(0,v,0)\in\mathcal{D}(L)} \frac{\big(pI+\sigma,\varepsilon(v)\big)_\Omega + \big(\mathrm{div}(\sigma),v\big)_\Omega}{\|\varepsilon(v)\|_\Omega}$$

$$\le \|pI+\sigma - \varepsilon(u)\|_\Omega + \|\varepsilon(u)\|_\Omega + C_\mathrm{K}\|\mathrm{div}(\sigma)\|_\Omega \le C_1\sqrt{3}\,\|L(\sigma,u,p)\|_\Omega.$$

Finally in terms of the two previous upper estimates, we can bound

$$(\sigma,\sigma)_\Omega = (\sigma - pI - \varepsilon(u),\sigma)_\Omega + (pI,\sigma)_\Omega + (\varepsilon(u),\sigma)_\Omega$$

$$\le \Big(\|L(\sigma,u,p)\|_\Omega + \sqrt{d}\|p\|_\Omega + \|\varepsilon(u)\|_\Omega\Big)\|\sigma\|_\Omega \le \left(1 + \left(\frac{\sqrt{3d}}{c_p}+1\right)C_1\right)\|L(\sigma,u,p)\|_\Omega\|\sigma\|_\Omega$$

which yields $C_L^2 = C_1^2(C_\mathrm{K}^2 + \sqrt{3}/c_p^2) + (1+(\frac{\sqrt{3d}}{c_p}+1)C_1)^2$. $\qquad\square$

Maxwell's equations. For linear materials, electro-magnetic waves are determined by the first-order system for the magnetic $\mathcal{H}$ and the electric $\mathcal{E}$ field

$$L(\mathcal{H},\mathcal{E}) = (\partial_t\mathcal{H} + \mu^{-1}\nabla\times\mathcal{E},\ \partial_t\mathcal{E} - \varepsilon^{-1}\nabla\times\mathcal{H})$$

Here the permeability μ and the permittivity ε are positive constants. The boundary is disjointly decomposed in $\partial\Omega_\mathcal{E}$ and $\partial\Omega_\mathcal{H}$. In Ω, we define

$$\mathcal{D}(L) = \big\{(\mathcal{H},\mathcal{E} \in \mathrm{L}_2((0,T),\mathrm{H}(\mathrm{curl},\Omega)^2)\cap\mathrm{H}^1((0,T),\mathrm{L}_2(\Omega)^6)\colon\ \mathrm{div}(\mu\mathcal{H}) = \mathrm{div}(\varepsilon\mathcal{E}) = 0 \text{ in } \Omega\times(0,T),$$
$$\mathcal{H}(0) = \mathcal{E}(0) = 0,\text{ in } \Omega,\quad \mathcal{H}\times n = 0 \text{ on } \partial\Omega_\mathcal{H}\times(0,T),\ \mathcal{E}\times n = 0 \text{ on } \partial\Omega_\mathcal{E}\times(0,T)\big\}.$$

By H we denote the closure of $\mathcal{D}(L)$ in $\mathrm{L}_2((0,T)\times\Omega,\mathbb{R}^3\times\mathbb{R}^3)$ associated with the inner product

$$\big((\mathcal{H},\mathcal{E}),(\tilde{\mathcal{H}},\tilde{\mathcal{E}})\big)_H = \int_0^T \Big(\big(\mu\mathcal{H}(t),\tilde{\mathcal{H}}(t)\big)_\Omega + \big(\varepsilon\mathcal{E}(t),\tilde{\mathcal{E}}(t)\big)_\Omega\Big)\,\mathrm{d}t.$$

Lemma 4.7. *The stability constants of Maxwell's equations can be defined as*

$$C_L = 2T.$$

Proof. Due to the boundary conditions in $\mathcal{D}(L)$, we have $(\nabla\times\mathcal{E}(s),\mathcal{H}(s))_\Omega - (\nabla\times\mathcal{H}(s),\mathcal{E}(s))_\Omega = 0$. Then, a straightforward use of the definition of the norm and of the initial condition shows for $(\mathcal{H},\mathcal{E})\in\mathcal{D}(L)$

$$\|(\mathcal{H},\mathcal{E})\|_H^2 = \int_0^T\int_0^t \partial_t\Big(\big(\mu\mathcal{H}(s),\mathcal{H}(s)\big)_\Omega + \big(\varepsilon\mathcal{E}(s),\mathcal{E}(s)\big)_\Omega\Big)\,\mathrm{d}s\,\mathrm{d}t$$

$$= 2\int_0^T\int_0^t \Big(\big(\mu\partial_t\mathcal{H}(s),\mathcal{H}(s)\big)_\Omega + \big(\varepsilon\partial_t\mathcal{E}(s),\mathcal{E}(s)\big)_\Omega\Big)\,\mathrm{d}s\,\mathrm{d}t$$

$$= 2\int_0^T\int_0^t \Big(\big(L(\mathcal{H},\mathcal{E})(s),(\mu\mathcal{H},\varepsilon\mathcal{E})\big)_\Omega - \big(\nabla\times\mathcal{E}(s),\mathcal{H}(s)\big)_\Omega + \big(\nabla\times\mathcal{H}(s),\mathcal{E}(s)\big)_\Omega\Big)\,\mathrm{d}s\,\mathrm{d}t$$

$$= 2\int_0^T\int_0^t \Big(\big(L(\mathcal{H},\mathcal{E})(s),(\mu\mathcal{H},\varepsilon\mathcal{E})\big)_\Omega\,\mathrm{d}s\,\mathrm{d}t \le 2T\,\|L(\mathcal{H},\mathcal{E})\|_H\|(\mathcal{H},\mathcal{E})\|_H. \qquad\square$$

The monochromatic Maxwell problem. Now we consider special monochromatic solutions of Maxwell's equation of the form $(\mathcal{H},\mathcal{E})(x,t) = \exp(\mathrm{i}\omega t)(h(x),e(x))$ for given frequency ω; this results in

$$L(h,e) = \big(\mathrm{i}\omega h + \mu^{-1}\nabla\times e,\ \mathrm{i}\omega e - \varepsilon^{-1}\nabla\times h\big).$$

In analogy to the Helmholtz case, we show stability on

$$\mathcal{D}(L) = \big\{ (h,e) \in \mathrm{H}(\mathrm{curl},\Omega)^2 \colon (\mu h, \nabla\psi)_\Omega = (\varepsilon e, \nabla\psi)_\Omega = 0 \text{ for all } \psi \in \mathrm{C}_0^\infty(\Omega),$$
$$\langle n \times h, \phi \rangle = \langle n \times e, n \times \phi \rangle \text{ for all } \phi \in \mathrm{C}^\infty(\partial\Omega)^3 \big\}.$$

Smooth functions with $\nabla\cdot(\varepsilon e) = \nabla\cdot(\mu h) = 0$ in Ω and Robin-type boundary condition $n \times h - (n \times e) \times n = 0$ on $\partial\Omega$ are dense in $\mathcal{D}(L)$. We set H as the closure of $\mathcal{D}(L)$ in $\mathrm{L}_2(\Omega, \mathbb{C}^3 \times \mathbb{C}^3)$ with $\|(h,e)\|_H^2 = (h, \mu h)_\Omega + (e, \varepsilon e)_\Omega$. Following the Helmholtz case, we consider domains with $x\cdot n(x) \geq \alpha > 0$ for a.a. $x \in \partial\Omega$ and $|x| \leq R$ for $x \in \Omega$.

Lemma 4.8. *The stability constants of the monochromatic Maxwell equation can be defined as*

$$C_L = 2\sqrt{\varepsilon\mu}R + \frac{(\mu+\varepsilon)R^2}{\alpha}.$$

Proof. For the proof, we use the techniques in ([23] Thm. 3.3, [29] Thm. 5.4.5). We start with an upper bound for $\|n \times e\|_{\partial\Omega}$. For all smooth functions in $(h,e) \in \mathcal{D}(L)$, we find

$$\big(L(h,e),(h,e)\big)_H = \big(\mathrm{i}\omega\mu h + \nabla \times e, h\big)_\Omega + \big(\mathrm{i}\omega\varepsilon e - \nabla \times h, e\big)_\Omega = \mathrm{i}\omega \,\|(h,e)\|_H^2 + \big(\nabla \times e, h\big)_\Omega - \big(\nabla \times h, e\big)_\Omega.$$

Integration by parts then yields in terms of the boundary condition for the real part

$$\mathrm{Re}\big(L(h,e),(h,e)\big)_H = \mathrm{Re}\Big(\big(\nabla \times e, h\big)_\Omega - \big(\nabla \times h, e\big)_\Omega\Big) = -\mathrm{Re}\big(n \times h, e\big)_{\partial\Omega} = -\|n \times e\|_{\partial\Omega}^2.$$

This gives $\|n \times e\|_{\partial\Omega}^2 \leq \|L(h,e)\|_H \|(h,e)\|_H$.

The rest of the proof is based on the two following identities which can be obtained by a straightforward computation

$$(x \times \bar{h}) \cdot (\nabla \times h) + (x \times h) \cdot (\nabla \times \bar{h}) + h \cdot \bar{h} = \nabla \cdot ((h \cdot \bar{h})x) - \nabla \cdot ((x \cdot h)\bar{h}) \tag{4.13a}$$
$$+ (\nabla \cdot \bar{h})(x \cdot h) - \nabla \cdot ((x \cdot \bar{h})h) + (\nabla \cdot h)(x \cdot h),$$
$$(x \times h) \cdot (n \times \bar{h}) = x \cdot n |h|^2 - (x \cdot h)(n \cdot \bar{h}) \tag{4.13b}$$

see, *e.g.*, Lemma 5.3.1 from [29], for (4.13b). Using $\nabla \cdot h = 0$, we get in terms of the Gauss theorem

$$2\mathrm{Re}\big(\nabla \times h, x \times h\big)_\Omega + \big(h, h\big)_\Omega = \big(n \cdot x, |h|^2\big)_{\partial\Omega} - 2\mathrm{Re}\big(n \cdot h, x \cdot h\big)_{\partial\Omega} = 2\mathrm{Re}\big(n \times h, x \times h\big)_{\partial\Omega} - \big(n \cdot x, |h|^2\big)_{\partial\Omega}.$$

Finally by means of $h \cdot (x \times \bar{e}) = (h \times x) \cdot \bar{e} = -(x \times h) \cdot \bar{e}$, we can establish the following equality

$$2\mathrm{Re}\big(L(h,e),(x \times \varepsilon e, -x \times \mu h)\big)_H$$
$$= 2\mathrm{Re}\Big(\big(\mathrm{i}\omega\mu h, x \times \varepsilon e\big)_\Omega + \big(\nabla \times e, x \times \varepsilon e\big)_\Omega - \big(\mathrm{i}\omega\varepsilon e, x \times \mu h\big)_\Omega + \big(\nabla \times h, x \times \mu h\big)_\Omega\Big)$$
$$= 2\mu\mathrm{Re}\big(\nabla \times h, x \times h\big)_\Omega + 2\varepsilon\mathrm{Re}\big(\nabla \times e, x \times e\big)_\Omega$$
$$= 2\mu\mathrm{Re}\big(n \times h, x \times h\big)_{\partial\Omega} - \mu\big(n \cdot x, |h|^2\big)_{\partial\Omega} - \big(h, \mu h\big)_\Omega + 2\varepsilon\mathrm{Re}\big(n \times e, x \times e\big)_{\partial\Omega} - \varepsilon\big(n \cdot x, |e|^2\big)_{\partial\Omega} - \big(e, \varepsilon e\big)_\Omega.$$

Now we can use the definition of the boundary conditions and get

$$\big(h, \mu h\big)_\Omega + \big(e, \varepsilon e\big)_\Omega + \mu\big(n \cdot x, |h|^2\big)_{\partial\Omega} + \varepsilon\big(n \cdot x, |e|^2\big)_{\partial\Omega}$$
$$= -2\mathrm{Re}\big(L(h,e),(x \times \varepsilon e, -x \times \mu h)\big)_H + 2\mu\mathrm{Re}\big(n \times e, n \times (x \times h)\big)_{\partial\Omega} + 2\varepsilon\mathrm{Re}\big(n \times e, x \times e\big)_{\partial\Omega}.$$

Applying Young's inequality results in the estimate

$$\|(h,e)\|_H^2 + \alpha\mu\|h\|_{\partial\Omega}^2 + \alpha\varepsilon\|e\|_{\partial\Omega}^2$$
$$\leq 2\|L(h,e)\|_H \|(x \times \varepsilon e, -x \times \mu h)\|_H + 2\mu\|n \times e\|_{\partial\Omega}\|n \times (x \times h)\|_{\partial\Omega} + 2\varepsilon\|n \times e\|_{\partial\Omega}\|x \times e\|_{\partial\Omega}$$
$$\leq 2\sqrt{\varepsilon\mu}R\|L(h,e)\|_H \|(h,e)\|_H + 2\mu R\|n \times e\|_{\partial\Omega}\|h\|_{\partial\Omega} + 2\varepsilon R\|n \times e\|_{\partial\Omega}\|e\|_{\partial\Omega}$$
$$\leq 2\sqrt{\varepsilon\mu}R\|L(h,e)\|_H \|(h,e)\|_H + \frac{\mu R^2}{\alpha}\|n \times e\|_{\partial\Omega}^2 + \alpha\mu\|h\|_{\partial\Omega}^2 + \frac{\varepsilon R^2}{\alpha}\|n \times e\|_{\partial\Omega}^2 + \alpha\varepsilon\|e\|_{\partial\Omega}^2.$$

Observing that the $L_2(\partial\Omega)$-terms cancel, we obtain by means of the upper bound for $\|n \times e\|^2_{\partial\Omega}$

$$\|(h,e)\|^2_H \leq 2\sqrt{\varepsilon\mu}R\,\|L(h,e)\|_H\|(h,e)\|_H + \frac{(\mu+\varepsilon)R^2}{\alpha}\,\|L(h,e)\|_H\|(h,e)\|_H.$$

This is shown for all sufficiently smooth functions satisfying (4.13) and tangential trace in L_2; a density argument now yields the stability constant C_L in $\mathcal{D}(L)$. $\qquad\square$

Remark 4.9. We note that C_L does not depend on the wave number.

5. The continuous substructuring method

In this section, we provide a Schur complement formulation of the problem. It gives rise to a symmetric positive definite formulation and is thus of special interest for efficient numerical solvers. To do so, we start with the introduction of a suitable partitioning and trace spaces, formulate the weak problem and apply a local static condensation.

We assume that we are in the abstract setting of Subsection 4.1. Then, for all $f \in H$ a unique solution $u^{\mathrm{sol}} \in U$ of $Lu^{\mathrm{sol}} = f$ exists such that $\|u^{\mathrm{sol}}\|_U \leq \sqrt{C_L^2 + 1}\|f\|_H$. We recall, that we use the graph norm $\|u\|^2_U = \|u\|^2_H + \|Lu\|^2_H$ for $u \in U$. Moreover, let $U^{\mathrm{ad}} = \{v \in H : f \in H \text{ exists such that } (v, Lu)_H = (f,u)_H \text{ for all } u \in U\}$ be the adjoint space, and the Hilbert adjoint L^{ad} is given by $(v, Lu)_H = (L^{\mathrm{ad}}v, u)_H$ for $v \in U^{\mathrm{ad}}$.

Partitioning and product space. For a given disjoint partitioning $\Omega \setminus \Gamma_{\mathcal{T}} = \bigcup_{\tau \in \mathcal{T}} \tau$, we define the local spaces $H_\tau = \{u|_\tau : u \in H\} \subset L_2(\tau)^M$ for all $\tau \in \mathcal{T}$, and we assume that the restrictions $L|_\tau$ and $L^{\mathrm{ad}}|_\tau$ of L and its adjoint L^{ad}, respectively, are defined in the extended space $V_\tau \supset \{u|_\tau : u \in U + U^{\mathrm{ad}}\}$. We assume that V_τ is a Hilbert space with the graph norm $\|v\|^2_{V_\tau} = \|v\|^2_{H_\tau} + \|L^{\mathrm{ad}}v\|^2_{H_\tau}$ of the adjoint operator. We will use V_τ as test space. In general, V_τ will not include boundary conditions. Associated with the local spaces is the product space $V = \prod_\tau V_\tau$ equipped with the norm $\|v\|^2_V = \sum_\tau \|v_\tau\|^2_{V_\tau}$.

Traces and trace norms. On τ, we assume that trace spaces $\hat{V}_\tau$ and surjective trace mappings $\gamma_\tau \colon V_\tau \longrightarrow \hat{V}_\tau$ exist such that $\hat{V}_\tau$ is identified with the quotient spaces $V_\tau/\mathcal{N}(\gamma_\tau)$, and we use the corresponding trace norm

$$\|\hat{v}_\tau\|_{\hat{V}_\tau} = \inf_{\gamma_\tau v_\tau = \hat{v}_\tau} \|v_\tau\|_{V_\tau}.$$

Associated with the trace mapping γ_τ is the adjoint trace $\gamma_\tau^{\mathrm{ad}} \colon V_\tau \longrightarrow \hat{V}_\tau'$ defined by

$$(Lv_\tau, \tilde{v}_\tau)_{H_\tau} = (v_\tau, L^{\mathrm{ad}}\tilde{v}_\tau)_{H_\tau} + \langle \gamma_\tau v_\tau, \gamma_\tau^{\mathrm{ad}}\tilde{v}_\tau \rangle, \qquad v_\tau, \tilde{v}_\tau \in V_\tau.$$

The embedding $U \subset V$ allows to define $\hat{U} = \gamma_\Gamma(U)$, where $\gamma_\Gamma \colon V \longrightarrow \prod_\tau \hat{V}_\tau$ is the product trace mapping. We observe for the kernel of γ_Γ that $\mathcal{N}(\gamma_\Gamma) \subset U$. The trace space $\hat{U}$ will be identified with the quotient space $\hat{U} = U/\mathcal{N}(\gamma_\Gamma)$ associated with the norm

$$\|\hat{u}\|_{\hat{U}} = \inf_{\gamma_\Gamma u = \hat{u}} \|u\|_U.$$

For $u \in U$ and $\hat{u} = \gamma_\Gamma u$ we set $\hat{u}_\tau = \gamma_\tau u$. Then, we have for $\hat{v}_\tau \in V_\tau$ and $\hat{u} = (\hat{u}_\tau) \in \hat{U}$ the estimates $\|\gamma_\tau v_\tau\|_{\hat{V}_\tau} \leq \|v_\tau\|_{V_\tau}$ and

$$\left|\sum_\tau \langle \hat{u}_\tau, \gamma_\tau^{\mathrm{ad}} v_\tau \rangle\right| = \inf_{\gamma_\Gamma u = \hat{u}} \left|\sum_\tau \langle \gamma_\tau u, \gamma_\tau^{\mathrm{ad}} v_\tau \rangle\right| = \inf_{\gamma_\Gamma u = \hat{u}} \left|\sum_\tau (Lu, v_\tau)_{H_\tau} - (u, L^{\mathrm{ad}}v_\tau)_{H_\tau}\right|$$

$$\leq \inf_{\gamma_\Gamma u = \hat{u}} \sum_\tau \left(\|Lu\|_{H_\tau}\|v_\tau\|_{H_\tau} + \|u\|_{H_\tau}\|L^{\mathrm{ad}}v_\tau\|_{H_\tau}\right)$$

$$\leq \inf_{\gamma_\Gamma u = \hat{u}} \left(\sum_\tau \|Lu\|^2_{H_\tau} + \|u\|^2_{H_\tau}\right)^{\frac{1}{2}} \left(\sum_\tau \|v_\tau\|^2_{H_\tau} + \|L^{\mathrm{ad}}v_\tau\|^2_{H_\tau}\right)^{\frac{1}{2}}$$

$$\leq \inf_{\gamma_\Gamma u = \hat{u}} \|u\|_U\|v\|_V = \|\hat{u}\|_{\hat{U}}\|v\|_V. \tag{5.1}$$

A weak formulation. For given $f \in H$, we set $F_\tau \in V_\tau'$ and $F \in V'$ by $\langle F_\tau, v \rangle = (f, v)_{H_\tau}$ and $\langle F, v \rangle = (f, v)_H$, respectively. In a next step, we define the local bilinear forms $B_\tau \in \mathcal{L}(H_\tau, V_\tau')$ and $\hat{B}_\tau \in \mathcal{L}(\hat{U}, V_\tau')$ by

$$\langle B_\tau u, v \rangle = \left(u, L^{\mathrm{ad}} v\right)_{H_\tau}, \quad \langle \hat{B}_\tau \hat{u}, v \rangle = \langle \hat{u}_\tau, \gamma_\tau^{\mathrm{ad}} v \rangle,$$

and observe that $\left(Lu, v\right)_{H_\tau} = \langle B_\tau u, v \rangle + \langle \hat{B}_\tau \gamma_\tau u, v \rangle$. For the global operator $B \in \mathcal{L}(H \times \hat{U}, V')$ given by $\langle B(u, \hat{u}), v \rangle = \sum_\tau \langle B_\tau u_\tau + \hat{B}_\tau \hat{u}, v_\tau \rangle$, we find that the solution $u^{\mathrm{sol}} \in U$ of $Lu^{\mathrm{sol}} = f$ and its trace $\hat{u}^{\mathrm{sol}} = \gamma_\Gamma u^{\mathrm{sol}}$ satisfy

$$B\left(u^{\mathrm{sol}}, \hat{u}^{\mathrm{sol}}\right) = F. \tag{5.2}$$

This observation motivates our interest to focus on the properties of $B(\cdot, \cdot)$. The following lemma shows that (5.2) has a unique solution and thus is equivalent to our original problem.

Lemma 5.1. *The operator $B \in \mathcal{L}(H \times \hat{U}, V')$ is bounded by*

$$\left|\langle B(u, \hat{u}), v \rangle\right| \leq C_B \|(u, \hat{u})\|_{H \times \hat{U}} \|v\|_V, \qquad (u, \hat{u}) \in H \times \hat{U}$$

with $C_B = \sqrt{2}$, and for given $f \in H$ a unique solution of (5.2) exists.

Proof. Using (5.1) we obtain for $(u, \hat{u}) \in H \times \hat{U}$

$$\left|\sum_\tau \langle B_\tau u_\tau, v_\tau \rangle\right| \leq \sum_\tau \|u_\tau\|_{H_\tau} \|L^{\mathrm{ad}} v_\tau\|_{H_\tau} \leq \|u\|_H \|v\|_V,$$
$$\left|\sum_\tau \langle \hat{B}_\tau \hat{u}, v_\tau \rangle\right| = \left|\sum_\tau \langle \hat{u}, \gamma_\tau^{\mathrm{ad}} v_\tau \rangle\right| \leq \|\hat{u}\|_{\hat{U}} \|v\|_V,$$

and thus $\left|\langle B(u, \hat{u}), v \rangle\right| \leq \left(\|u\|_H + \|\hat{u}\|_{\hat{U}}\right)\|v\|_V \leq \sqrt{2}\|(u, \hat{u})\|_{H \times \hat{U}}\|v\|_V$.

The solution $u^{\mathrm{sol}} \in U$ of $Lu^{\mathrm{sol}} = f$ is unique, and we have $B(u^{\mathrm{sol}}, \gamma_\Gamma u^{\mathrm{sol}}) = F$ by construction. Thus, it is sufficient to show that B is injective. Let $(u, \hat{u}) \in U \times \hat{U}$ be a solution of the homogeneous problem $B(u, \hat{u}) = 0$. In the first step, we test with $v \in \mathcal{N}(\gamma_\Gamma) \subset U$ which yields

$$0 = \langle B(u, \hat{u}), v \rangle = (u, L^{\mathrm{ad}} v)_H = (Lu, v)_H.$$

Since $\mathcal{N}(\gamma_\Gamma)$ is dense in H, this implies $Lu = 0$ and since L is injective, we obtain $u = 0$. Then, we have $0 = \langle B(0, \hat{u}), v \rangle = \sum_\tau \langle \hat{B}_\tau \hat{u}, v_\tau \rangle$ for all $v \in V$, with implies $\hat{u} = 0$, since $\hat{B}_\tau$ is injective. This shows that B is injective on $U \times \hat{U}$, and since U is dense in H and B is continuous, B is also injective on $H \times \hat{U}$. $\qquad \square$

The Schur complement. For the solution of the equation $B(u^{\mathrm{sol}}, \hat{u}^{\mathrm{sol}}) = F$ we now construct a Schur complement problem. To do so, we introduce self-adjoint and positive operators $A_\tau \in \mathcal{L}(V_\tau, V_\tau')$ and $A \in \mathcal{L}(V, V')$ by

$$\langle A_\tau v_\tau, \tilde{v}_\tau \rangle = \left(v_\tau, \tilde{v}_\tau\right)_{H_\tau} + \left(L^{\mathrm{ad}} v_\tau, L^{\mathrm{ad}} \tilde{v}_\tau\right)_{H_\tau}, \qquad \langle A v, \tilde{v} \rangle = \sum_\tau \langle A_\tau v_\tau, \tilde{v}_\tau \rangle,$$

and the Schur complement $S = B' A^{-1} B \in \mathcal{L}(H \times \hat{U}, H' \times \hat{U}')$. This gives

$$S(u^{\mathrm{sol}}, \hat{u}^{\mathrm{sol}}) = B' A^{-1} F.$$

We now show that this problem is well-posed using the abstract theory of Section 3 with $Y = H \times \hat{U}$.

Lemma 5.2. *We have for all $(u, \hat{u}) \in H \times \hat{U}$*

$$c_S \|(u, \hat{u})\|_{H \times \hat{U}}^2 \leq \langle S(u, \hat{u}), (u, \hat{u}) \rangle \leq C_S \|(u, \hat{u})\|_{H \times \hat{U}}^2$$

with $c_S = (4C_L^2 + 2)^{-1}$ and $C_S = 2$.

Proof. We start with the proof of the upper bound. We have $\langle Av, v \rangle = \|v\|_V^2$, $\langle F, A^{-1}F \rangle = \|F\|_{V'}^2$, and

$$\langle S(u, \hat{u}), (u, \hat{u}) \rangle = \langle B(u, \hat{u}), A^{-1}B(u, \hat{u}) \rangle = \|B(u, \hat{u})\|_{V'}^2 = \sup_{v \in V} \frac{|\langle B(u, \hat{u}), v \rangle|^2}{\|v\|_V^2}. \tag{5.3}$$

Thus, Lemma 5.1 yields the upper bound $C_S = C_B^2 = 2$.

To show the lower bound, we use that for all $v \in V \subset H$ a function $u_v \in U$ with $Lu_v = v$ exists. Using the definition of the norm in U and $\hat{U}$, we get

$$\|(u_v, \gamma_\Gamma u_v)\|_{H \times \hat{U}}^2 = \|u_v\|_H^2 + \|\gamma_\Gamma u_v\|_{\hat{U}}^2 \leq \|u_v\|_H^2 + \|u_v\|_U^2 = 2\|u_v\|_H^2 + \|Lu_v\|_H^2$$
$$\leq (2C_L^2 + 1)\|Lu_v\|_H^2 = (2C_L^2 + 1)\|v\|_H^2.$$

This observation and $\langle B(u_v, \gamma_\Gamma u_v), v \rangle = (Lu_v, v)_H = \|v\|_H^2$ yield

$$\sup_{(u, \hat{u}) \in H \times \hat{U}} \frac{|\langle B(u, \hat{u}), v \rangle|}{\|(u, \hat{u})\|_{H \times \hat{U}}} \geq \frac{\langle B(u_v, \gamma_\Gamma u_v), v \rangle}{\|(u_v, \gamma_\Gamma u_v)\|_{H \times \hat{U}}} \geq \frac{1}{\sqrt{2C_L^2 + 1}} \|v\|_H.$$

Moreover testing with $(u, \hat{u}) = (L^{\mathrm{ad}}v, 0)$, a second lower bound is obtained

$$\sup_{(u, \hat{u}) \in H \times \hat{U}} \frac{|\langle B(u, \hat{u}), v \rangle|}{\|(u, \hat{u})\|_{H \times \hat{U}}} \geq \frac{\langle B(L^{\mathrm{ad}}v, 0), v \rangle}{\|L^{\mathrm{ad}}v\|_H} = \|L^{\mathrm{ad}}v\|_H.$$

Combining these two estimates gives

$$\sup_{(u, \hat{u}) \in H \times \hat{U}} \frac{|\langle B(u, \hat{u}), v \rangle|}{\|(u, \hat{u})\|_{H \times \hat{U}}} \geq \max \left\{ \frac{\|v\|_H}{\sqrt{2C_L^2 + 1}}, \|L^{\mathrm{ad}}v\|_H \right\} \geq \frac{\sqrt{\|v\|_H^2 + \|L^{\mathrm{ad}}v\|_H^2}}{\sqrt{2}\sqrt{2C_L^2 + 1}} = \frac{1}{\sqrt{4C_L^2 + 2}} \|v\|_V.$$

Recalling that B is injective, the lower bound is now obtained by duality, see, *e.g.*, [6], Lemma 4.4.2. $\square$

The skeleton reduction. Finally we eliminate u^{sol} and obtain a positive definite system for the trace $\hat{u}^{\mathrm{sol}}$. We point out that u^{sol} can be recovered locally from $\hat{u}^{\mathrm{sol}}$. This is an attractive feature for the discrete setting.

Inserting $\hat{u} = 0$ gives for the local Schur complement $S_\tau = B_\tau' A_\tau^{-1} B_\tau$ the estimate $c_S \|u_\tau\|_{H_\tau}^2 \leq \langle S_\tau u_\tau, u_\tau \rangle$. Thus, we can construct the local projection $P_\tau = \mathrm{id}_{V_\tau} - A_\tau^{-1} B_\tau S_\tau^{-1} B_\tau'$ on $\mathcal{N}(B_\tau')$ with $P_\tau A_\tau^{-1} B_\tau = 0$. Moreover, we have $A_\tau P_\tau = P_\tau' A_\tau$, *i.e.*, P_τ is an orthogonal projection in V_τ.

We use the projection to eliminate the local solutions $u_\tau^{\mathrm{sol}} \in H_\tau$ and thus for the reduction to the skeleton values $\hat{U}$. We obtain from $B_\tau u^{\mathrm{sol}} + \hat{B}_\tau \hat{u}^{\mathrm{sol}} = F_\tau$ the local equation $P_\tau A_\tau^{-1} \hat{B}_\tau \hat{u}^{\mathrm{sol}} = P_\tau A_\tau^{-1} F_\tau$. Setting on the skeleton $\hat{S} = \sum_\tau \hat{B}_\tau' P_\tau A_\tau^{-1} \hat{B}_\tau$ and $\hat{F} = \sum_\tau \hat{B}_\tau' P_\tau A_\tau^{-1} F_\tau$, the global equation for u^{sol} can be rewritten as

$$\hat{S}\hat{u}^{\mathrm{sol}} = \hat{F}. \tag{5.4}$$

Lemma 5.3. *We have* $c_S \|\hat{u}\|_{\hat{U}}^2 \leq \langle \hat{S}\hat{u}, \hat{u} \rangle \leq \|\hat{u}\|_{\hat{U}}^2$ *for* $\hat{u} \in \hat{U}$.

Proof. For given $\hat{u} \in \hat{U}$ and the special choice $u_\tau = -S_\tau^{-1} B_\tau' A_\tau^{-1} \hat{B}_\tau \hat{u}$, we have

$$B_\tau u_\tau + \hat{B}_\tau \hat{u} = (-B_\tau S_\tau^{-1} B_\tau' A_\tau^{-1} + \mathrm{id}_{V_\tau'})\hat{B}_\tau \hat{u} = P_\tau' \hat{B}_\tau \hat{u}.$$

Using $P_\tau A_\tau^{-1} P_\tau' = P_\tau A_\tau^{-1}$, we obtain for $u = (u_\tau)$

$$\langle S(u, \hat{u}), (u, \hat{u}) \rangle = \sum_\tau \langle B_\tau u_\tau + \hat{B}_\tau \hat{u}, A_\tau^{-1}(B_\tau u_\tau + \hat{B}_\tau \hat{u}) \rangle = \sum_\tau \langle P_\tau' \hat{B}_\tau \hat{u}, A_\tau^{-1} P_\tau' \hat{B}_\tau \hat{u} \rangle = \langle \hat{S}\hat{u}, \hat{u} \rangle.$$

Then, the lower bound $c_S \|\hat{u}\|_{\hat{U}}^2 \leq c_S \|(u, \hat{u})\|_{H \times \hat{U}}^2 \leq \langle \hat{S}\hat{u}, \hat{u} \rangle$ follows from Lemma 5.2, and the upper bound follows from (5.1) by

$$
\begin{aligned}
\langle \hat{S}\hat{u}, \hat{u} \rangle &= \sum_\tau \langle \hat{B}_\tau \hat{u}, P_\tau A_\tau^{-1} \hat{B}_\tau \hat{u} \rangle \leq \sum_\tau \|\hat{B}_\tau \hat{u}\|_{V_\tau'} \|P_\tau A_\tau^{-1} \hat{B}_\tau \hat{u}\|_{V_\tau} \\
&\leq \sum_\tau \|\hat{B}_\tau \hat{u}\|_{V_\tau'}^2 \leq \sum_\tau \inf_{\|v_\tau\|_{V_\tau} = 1} \langle \hat{u}_\tau, \gamma_\tau^{\mathrm{ad}} v_\tau \rangle^2 \leq \|\hat{u}\|_{\hat{U}}^2 . \qquad \square
\end{aligned}
$$

This shows that $\hat{S}$ is positive and the equation (5.4) has a unique solution $\hat{u}^{\mathrm{sol}}$. Then, the local solutions $u_\tau^{\mathrm{sol}} = S_\tau^{-1} B_\tau' A_\tau^{-1} (F_\tau - \hat{B}_\tau \hat{u}^{\mathrm{sol}})$ are determined by the skeleton solution and the right-hand side.

6. The discrete substructuring method

In this section, we firstly proceed exactly as in the continuous setting and then comment on inf-sup stability and preconditioning. Since the Schur complement equation for the continuous problem corresponds to a saddle point problem, we need a uniform inf-sup stable discretization. Let $H_{\tau,h} \subset H_\tau$ be local finite element spaces, and let $\hat{U}_h \subset \hat{U}$ be the trace space of a conforming finite element space. Then, we assume that discrete spaces $V_{\tau,h} \subset V_\tau$ and $\beta_0 \in (0, 1)$ exists satisfying for all $(u_{\tau,h}, \hat{u}_h) \in H_{\tau,h} \times \hat{U}_h$ and $\tau \in \mathcal{T}$

$$
\sup_{v_{\tau,h} \in V_{\tau,h}} \frac{|\langle B_\tau u_{\tau,h} + \hat{B}_\tau \hat{u}_h, v_{\tau,h} \rangle|}{\|v_{\tau,h}\|_{V_\tau}} \geq \beta_0 \sup_{v_\tau \in V_\tau} \frac{|\langle B_\tau u_{\tau,h} + \hat{B}_\tau \hat{u}_h, v_\tau \rangle|}{\|v_\tau\|_{V_\tau}} .
$$

Let $B_{\tau,h}$, $\hat{B}_{\tau,h}$, $A_{\tau,h}$, and $F_{\tau,h}$ be the Galerkin approximations of B_τ, $\hat{B}_\tau$, A_τ, and F_τ, i.e., for $u_{\tau,h} \in H_{\tau,h}$, $\hat{u}_{\tau,h} \in \hat{U}_h$, and $v_{\tau,h}, \tilde{v}_{\tau,h} \in V_{\tau,h}$ we have

$$
\begin{aligned}
\langle B_{\tau,h} u_{\tau,h}, v_{\tau,h} \rangle &= \langle B_\tau u_{\tau,h}, v_{\tau,h} \rangle, & \langle \hat{B}_{\tau,h} \hat{u}_{\tau,h}, v_{\tau,h} \rangle &= \langle \hat{B}_\tau \hat{u}_{\tau,h}, v_{\tau,h} \rangle, \\
\langle A_{\tau,h} v_{\tau,h}, \tilde{v}_{\tau,h} \rangle &= \langle A_\tau v_{\tau,h}, \tilde{v}_{\tau,h} \rangle, & \langle F_{\tau,h}, v_{\tau,h} \rangle &= \langle F_\tau, v_{\tau,h} \rangle.
\end{aligned}
$$

As in the continuous setting, we define the spaces $H_h = \prod_\tau H_{\tau,h}$ and $V_h = \prod_\tau V_{\tau,h}$. The Galerkin approximations of B, A, and F are denoted by $B_h \in \mathcal{L}(H_h \times \hat{U}_h, V_h)$, $A_h \in \mathcal{L}(V_h, V_h')$, and $F_h \in V_h'$. Since A_h is symmetric and positive, the discrete approximation $S_h = B_h' A_h^{-1} B_h \in \mathcal{L}(H_h \times \hat{U}_h, H_h' \times \hat{U}_h')$ of S is well-defined.

A direct approximation of the equation $B(u^{\mathrm{sol}}, \hat{u}^{\mathrm{sol}}) = F$ requires that the discrete solution space $H_h \times \hat{U}_h$ and the test space have the same dimension. This can be achieved by selecting the test space $A_h^{-1} B_h (H_h \times \hat{U}_h) \subset V_h$, or equivalently, by the solution of the discrete Schur complement problem given by $S_h(u_h^{\mathrm{sol}}, \hat{u}_h^{\mathrm{sol}}) = B_h' A_h^{-1} F_h$.

We now show that the local inf-sup stability guarantees that S_h is uniformly positive in $Y_h = H_h \times \hat{U}_h$, so that a unique solution of the Schur complement problem exists.

Lemma 6.1. *We have for all $(u_h, \hat{u}_h) \in H_h \times \hat{U}_h$*

$$
c_S \beta_0^2 \|(u_h, \hat{u}_h)\|_{H \times \hat{U}}^2 \leq \langle S_h(u_h, \hat{u}_h), (u_h, \hat{u}_h) \rangle \leq C_S \|(u_h, \hat{u}_h)\|_{H \times \hat{U}}^2 .
$$

Proof. The upper bound follows from the continuity of the operator B. More precisely, we get

$$
\sup_{v_h \in V_h} \frac{|\langle B_h(u_h, \hat{u}_h), v_h \rangle|}{\|v_h\|_V} \leq \sup_{v \in V} \frac{|\langle B(u_h, \hat{u}_h), v \rangle|}{\|v\|_V} \leq C_B \|(u, \hat{u})\|_{H \times \hat{U}} .
$$

By construction of $V_{\tau,h}$, we have for all $(u_h, \hat{u}_h) \in H_h \times \hat{U}_h$

$$
\begin{aligned}
\langle S_h(u_h, \hat{u}_h), (u_h, \hat{u}_h) \rangle &= \sup_{v_h \in V_h} \frac{|\langle B_h(u_h, \hat{u}_h), v_h \rangle|^2}{\|v_h\|_V^2} = \sum_\tau \sup_{v_{\tau,h} \in V_{\tau,h}} \frac{|\langle B_{\tau,h} u_{\tau,h} + \hat{B}_{\tau,h} \hat{u}_h, v_{\tau,h} \rangle|^2}{\|v_{\tau,h}\|_{V_\tau}^2} \\
&\geq \beta_0^2 \sum_\tau \sup_{v_\tau \in V_\tau} \frac{|\langle B_\tau u_{\tau,h} + B_\tau \hat{u}_h, v_\tau \rangle|^2}{\|v_\tau\|_{V_\tau}^2} = \beta_0^2 \langle S(u_h, \hat{u}_h), (u_h, \hat{u}_h) \rangle,
\end{aligned}
$$

so that the lower bound is obtained from Lemma 5.2. $\qquad \square$

Lemma 6.1 allows us to formulate an *a priori* bound for $(u^{\text{sol}}, \hat{u}^{\text{sol}}) - (u_h^{\text{sol}}, \hat{u}_h^{\text{sol}})$.

Theorem 6.2. *We have the* a priori *bound*

$$\| \left(u^{\text{sol}}, \hat{u}^{\text{sol}}\right) - \left(u_h^{\text{sol}}, \hat{u}_h^{\text{sol}}\right) \|_{H \times \hat{U}} \leq \frac{\sqrt{C_S}}{\beta_0 \sqrt{c_S}} \inf_{(u_h, \hat{u}_h) \in H_h \times \hat{U}_h} \| \left(u^{\text{sol}}, \hat{u}^{\text{sol}}\right) - (u_h, \hat{u}_h) \|_{H \times \hat{U}}.$$

Proof. Combining (3.3) with Lemmas 5.2 and 6.1, see also Theorem 2.1 from [20], gives the *a priori* bound. □

Remark 6.3. We note that Theorem 6.2 can be regarded as an extension of Céa's lemma to the more general setting of first order systems. More precisely, the stability and the inf-sup constants of first order systems play the same role as the continuity and coercivity constants for Céa's lemma, respectively. For parameter dependent problems, the ratio $\sqrt{C_S}/(\sqrt{c_S}\beta_0) = 2\sqrt{C_L^2 + 1}/\beta_0$ has to be uniformly bounded to obtain robust estimates with respect to the parameter space.

The skeleton reduction. For the elimination of the inner degrees of freedom, we proceed as in the continuous case. From Lemma 6.1 we obtain for the discrete local Schur complement $S_{\tau,h} = B'_{\tau,h} A_{\tau,h}^{-1} B_{\tau,h}$ the estimate $c_S \beta_0^2 \|u_{\tau,h}\|_{H_\tau}^2 \leq \langle S_{\tau,h} u_{\tau,h}, u_{\tau,h}\rangle$. Thus, the local operators $S_{\tau,h}$ can be inverted, and we can define the local projections $P_{\tau,h} = \text{id}_{V_{\tau,h}} - A_{\tau,h}^{-1} B_{\tau,h} S_{\tau,h}^{-1} B'_{\tau,h}$ on $\mathcal{N}(B'_{\tau,h})$ and

$$\hat{S}_h = \sum_\tau \hat{B}'_{\tau,h} P_{\tau,h} A_{\tau,h}^{-1} \hat{B}_{\tau,h}, \qquad \hat{F}_h = \sum_\tau \hat{B}'_{\tau,h} P_{\tau,h} A_{\tau,h}^{-1} F_{\tau,h}.$$

Analogously to Lemma 5.3, we obtain spectral bounds

$$c_S \beta_0^2 \|\hat{u}_h\|_{\hat{U}}^2 \leq \langle \hat{S}_h \hat{u}_h, \hat{u}_h\rangle \leq \|\hat{u}_h\|_{\hat{U}}^2, \qquad \hat{u}_h \in \hat{U}_h. \tag{6.1}$$

This shows that $\hat{S}_h$ is positive, and $\hat{u}_h^{\text{sol}} \in \hat{U}_h$ is the unique solution of

$$\hat{S}_h \hat{u}_h^{\text{sol}} = \hat{F}_h. \tag{6.2}$$

Then, local computations give $u_{\tau,h}^{\text{sol}} = S_{\tau,h}^{-1} B'_{\tau,h} A_{\tau,h}^{-1}(F_{\tau,h} - \hat{B}_{\tau,h}\hat{u}_h^{\text{sol}})$ for all τ.

Inf-sup stability. So far we have been assuming the existence of a uniform discrete inf-sup constant. Here, we derive a simple criterion and show that inf-sup stability can be achieved if V_h is large enough.

We start with the explicit evaluation of the supremum. For given $(u_{\tau,h}, \hat{u}_h) \in H_{\tau,h} \times \hat{U}_h$, we define locally $F_\tau = B_\tau u_{\tau,h} + \hat{B}_\tau \hat{u}_h \in V'_\tau$ and $F_{\tau,h} = B_{\tau,h} u_{\tau,h} + \hat{B}_{\tau,h}\hat{u}_h \in V'_{\tau,h}$, and we observe

$$\|A_{\tau,h}^{-1} F_{\tau,h}\|_{V_\tau} = \sqrt{\langle A_{\tau,h} A_{\tau,h}^{-1} F_{\tau,h}, A_{\tau,h}^{-1} F_{\tau,h}\rangle} = \sqrt{\langle F_{\tau,h}, A_{\tau,h}^{-1} F_{\tau,h}\rangle}$$

$$= \sup_{v_{\tau,h} \in V_{\tau,h}} \frac{|\langle B_{\tau,h} u_{\tau,h} + \hat{B}_{\tau,h}\hat{u}_h, v_{\tau,h}\rangle|}{\|v_{\tau,h}\|_{V_\tau}} = \sup_{v_{\tau,h} \in V_{\tau,h}} \frac{|\langle B_\tau u_{\tau,h} + \hat{B}_\tau \hat{u}_h, v_{\tau,h}\rangle|}{\|v_{\tau,h}\|_{V_\tau}}$$

$$\leq \sup_{v_\tau \in V_\tau} \frac{|\langle B_\tau u_{\tau,h} + B_\tau \hat{u}_h, v_\tau\rangle|}{\|v_\tau\|_{V_\tau}} = \sqrt{\langle F_\tau, A_\tau^{-1} F_\tau\rangle} = \|A_\tau^{-1} F_\tau\|_{V_\tau}.$$

Thus, uniform inf-sup stability is obtained if we can show $\|A_{\tau,h}^{-1} F_{\tau,h}\|_{V_\tau} \geq \beta_0 \|A_\tau^{-1} F_\tau\|_{V_\tau}$ for some $\beta_0 \in (0, 1)$ independent of the mesh size. For a more detailed discussion, we define the spaces $W_\tau = A_\tau^{-1}(B_\tau H_{\tau,h} + \hat{B}_\tau \hat{U}_h)$ and $W_{\tau,h} = A_{\tau,h}^{-1}(B_{\tau,h} H_{\tau,h} + \hat{B}_{\tau,h}\hat{U}_h)$. Note that $\dim(W_\tau) \leq \dim(H_{\tau,h}) + \dim(\hat{U}_h)$ in our applications. An obvious requirement for inf-sup stability is $\dim(W_{\tau,h}) = \dim(W_\tau)$. Since we have $\dim(W_{\tau,h}) \leq \dim(V_{\tau,h})$, this implies that $V_{\tau,h}$ has to be large enough.

Now we discuss two cases. In the first case, we assume that W_τ can be determined explicitly. Then, we can compute $\beta_{0,\tau} > 0$ for every τ by a simple eigenvalue analysis in $\mathbb{R}^{\dim(W_\tau)}$. For an affine family of subdomains τ

this reduces to an eigenvalue analysis on a reference element. In this case, a suitable Fortin mapping can explicitly be constructed, see [20] for the Poisson problem and for linear elasticity.

For the more general case, we assume that $V_{\tau,h,p} \subset V_\tau$, $p \in \mathcal{P}$, is a dense family of discrete subspaces, so that the orthogonal projections $P_{\tau,h,p} \in \mathcal{L}(W_\tau, V_{\tau,h,p})$ satisfy $\lim_{p \in \mathcal{P}} \|P_{\tau,h,p} - \mathrm{id}_{W_\tau}\|_{\mathcal{L}(W_\tau, V_\tau)} = 0$. Recalling that W_τ is finite dimensional, for each $\beta_0 \in (0,1)$ there exists a $p \in \mathcal{P}$ such that

$$\inf_{v_{\tau,h,p} \in V_{\tau,h,p}} \|v_{\tau,h,p} - w_\tau\|_{V_\tau} \leq (1 - \beta_0)\|w_\tau\|_{V_\tau}, \qquad w_\tau \in W_\tau.$$

For given $(u_{\tau,h}, \hat{u}_h) \in H_{\tau,h} \times \hat{U}_h$, we define $w_\tau = A_\tau^{-1}(B_\tau u_{\tau,h} + \hat{B}_\tau \hat{u}_h) \in W_\tau$ and the approximation $w_{\tau,h,p} = A_{\tau,h,p}^{-1}(B_{\tau,h,p} u_{\tau,h} + \hat{B}_{\tau,h,p} \hat{u}_h) \in W_{\tau,h,p}$. Then we observe

$$\|w_{\tau,h,p} - w_\tau\|_{V_\tau} = \inf_{v_{\tau,h,p} \in V_{\tau,h,p}} \|v_{\tau,h,p} - w_\tau\|_{V_\tau}.$$

This yields

$$\sup_{v_{\tau,h,p} \in V_{\tau,h,p}} \frac{|\langle B_\tau u_{\tau,h} + \hat{B}_\tau \hat{u}_h, v_{\tau,h,p}\rangle|}{\|v_{\tau,h,p}\|_V} = \|w_{\tau,h,p}\|_{V_\tau} \geq \|w_\tau\|_{V_\tau} - \|w_{\tau,h,p} - w_\tau\|_{V_\tau}$$

$$\geq \beta_0 \|w_\tau\|_{V_\tau} = \beta_0 \sup_{v \in V} \frac{|\langle B_\tau u_{\tau,h} + B_\tau \hat{u}_h, v\rangle|}{\|v\|_V}.$$

In some applications it may be required to choose $p \in \mathcal{P}$ depending of τ adaptively which can be achieved by a local variant of the greedy algorithm in [14].

Preconditioning. Although the discrete problem (6.2) is a well-defined symmetric positive definite system, the condition number depends on the mesh size. Thus, for an efficient solver a good preconditioner is required. To our knowledge, up to now for discontinuous Petrov–Galerkin methods only a one-level additive Schwarz preconditioner is discussed, see [3].

Here, we show that an efficient preconditioner for $\hat{S}_h$ can be constructed from an efficient preconditioner in $V_h \cap U$. For this purpose, assume that a subspace $U_h \subset U \cap V_h$ with trace space $\gamma_\Gamma(U_h) = \hat{U}_h$, a self-adjoint positive operator $A_h^U \in \mathcal{L}(U_h, U_h')$, and a self-adjoint preconditioner $G_h \in \mathcal{L}(U_h', U_h)$ exists.

Let $\gamma_{\Gamma,h} \in \mathcal{L}(U_h, \hat{U}_h)$ be the Galerkin approximation of $\gamma_\Gamma|_{U_h}$ and define the corresponding skeleton preconditioner $\hat{G}_h = \gamma_{\Gamma,h} G_h \gamma_{\Gamma,h}' \in \mathcal{L}(\hat{U}_h', \hat{U}_h)$ for $\hat{S}_h$. The following theorem shows that the skeleton preconditioner inherits its properties from the self-adjoint preconditioner, the inf-sup constant and suitable norm equivalences.

Theorem 6.4. *Assume that $0 < c_A \leq C_A$ and $0 < c_G \leq C_G$ exists such that*

$$c_A \langle A_h^U u_h, u_h\rangle \leq \|\gamma_\Gamma u_h\|_{\hat{U}}^2 \leq C_A \langle A_h^U u_h, u_h\rangle, \tag{6.3}$$

$$c_G \langle A_h^U u_h, u_h\rangle \leq \langle A_h^U u_h, G_h A_h^U u_h\rangle \leq C_G \langle A_h^U u_h, u_h\rangle \tag{6.4}$$

holds for $u_h \in U_h$. Then, with $c_{\hat{G}} = c_G c_A c_S \beta_0^2$ and $C_{\hat{G}} = C_G C_A$, we have

$$c_{\hat{G}} \langle \hat{S}_h \hat{u}_h, \hat{u}_h\rangle \leq \langle \hat{S}_h \hat{u}_h, \hat{G}_h \hat{S}_h \hat{u}_h\rangle \leq C_{\hat{G}} \langle \hat{S}_h \hat{u}_h, \hat{u}_h\rangle, \qquad \hat{u}_h \in \hat{U}_h.$$

Proof. We proceed in two steps. Firstly, we bound $\langle \hat{S}_h \hat{u}_h, \hat{G}_h \hat{S}_h \hat{u}_h\rangle$ in terms of (6.4). Secondly, we show the spectral equivalence of $\hat{S}_h$ and $\hat{S}_h \gamma_{\Gamma,h} A_h^{-1} \gamma_{\Gamma,h}' \hat{S}_h$. We have for $\hat{u}_h \in \hat{U}_h$

$$\langle \hat{S}_h \hat{u}_h, \hat{G}_h \hat{S}_h \hat{u}_h\rangle = \langle A_h^U (A_h^U)^{-1} \gamma_{\Gamma,h}' \hat{S}_h \hat{u}_h, G_h A_h^U (A_h^U)^{-1} \gamma_{\Gamma,h}' \hat{S}_h \hat{u}_h\rangle$$

and thus, using (6.4),

$$c_G \langle \gamma_{\Gamma,h}' \hat{S}_h \hat{u}_h, (A_h^U)^{-1} \gamma_{\Gamma,h}' \hat{S}_h \hat{u}_h\rangle \leq \langle \hat{S}_h \hat{u}_h, \hat{G}_h \hat{S}_h \hat{u}_h\rangle \leq C_G \langle \gamma_{\Gamma,h}' \hat{S}_h \hat{u}_h, (A_h^U)^{-1} \gamma_{\Gamma,h}' \hat{S}_h \hat{u}_h\rangle.$$

Introducing $\hat{A}_h^U = \left(\gamma_{\Gamma,h}(A_h^U)^{-1}\gamma'_{\Gamma,h}\right)^{-1} \in \mathcal{L}(\hat{U}, \hat{U}')$, this equivalence reads as

$$c_G \, \langle \hat{S}_h \hat{u}_h, (\hat{A}_h^U)^{-1} \hat{S}_h \hat{u}_h \rangle \leq \langle \hat{S}_h \hat{u}_h, \hat{G}_h \hat{S}_h \hat{u}_h \rangle \leq C_G \, \langle \hat{S}_h \hat{u}_h, (\hat{A}_h^U)^{-1} \hat{S}_h \hat{u}_h \rangle. \tag{6.5}$$

Thus it remains to show that $\hat{S}_h(\hat{A}_h^U)^{-1}\hat{S}_h$ is spectrally equivalent to $\hat{S}_h$.

For given $\hat{u}_h \in \hat{U}_h$ define $u_h = (A_h^U)^{-1}\gamma'_{\Gamma,h}\hat{A}_h^U \hat{u}_h$ so that $\gamma_{\Gamma,h} u_h = \hat{u}_h$ and $\langle A_h^U u_h, u_h \rangle = \langle \hat{A}_h^U \hat{u}_h, \hat{u}_h \rangle$. Inserting these equalities in (6.3) yields

$$c_A \, \langle \hat{A}_h^U \hat{u}_h, \hat{u}_h \rangle \leq \|\hat{u}_h\|_{\hat{U}}^2 \leq C_A \, \langle \hat{A}_h^U \hat{u}_h, \hat{u}_h \rangle, \qquad \hat{u}_h \in \hat{U}_h. \tag{6.6}$$

Together with the spectral bound (6.1), we obtain

$$c_A c_S \beta_0^2 \, \langle \hat{A}_h^U \hat{u}_h, \hat{u}_h \rangle \leq \langle \hat{S}_h \hat{u}_h, \hat{u}_h \rangle \leq C_A \, \langle \hat{A}_h^U \hat{u}_h, \hat{u}_h \rangle, \qquad \hat{u}_h \in \hat{U}_h$$

or equivalently

$$c_A c_S \beta_0^2 \, \langle \hat{F}_h, \hat{S}_h^{-1} \hat{F}_h \rangle \leq \langle \hat{F}_h, (\hat{A}_h^U)^{-1} \hat{F}_h \rangle \leq C_A \, \langle \hat{F}_h, \hat{S}_h^{-1} \hat{F}_h \rangle, \qquad \hat{F}_h \in \hat{U}'_h.$$

The choice $\hat{F}_h = \hat{S}_h \hat{u}_h$ now gives

$$c_A c_S \beta_0^2 \, \langle \hat{S}_h \hat{u}_h, \hat{u}_h \rangle \leq \langle \hat{S}_h \hat{u}_h, (\hat{A}_h^U)^{-1} \hat{S}_h \hat{u}_h \rangle \leq C_A \, \langle \hat{S}_h \hat{u}_h, \hat{u}_h \rangle$$

and in combination with (6.5) the assertion is verified. □

We shortly comment on a suitable construction of U_h satisfying (6.3). Let $\hat{P}_\Gamma \in \mathcal{L}(U, \mathcal{N}(\gamma_\Gamma))$ be the orthogonal projection to $\mathcal{N}(\gamma_\Gamma)$. Then, we have

$$\|\gamma_\Gamma u_h\|_{\hat{U}} = \inf_{v \in \mathcal{N}(\gamma_\Gamma)} \|u_h - v\|_U = \|u_h - \hat{P}_\Gamma u_h\|_U.$$

This shows that choosing $U_h = \left(\mathrm{id}_{U_h} - \hat{P}_{\Gamma,h}\right)(U \cap V_h)$ with the orthogonal projection $\hat{P}_{\Gamma,h} \in \mathcal{L}(U \cap V_h, \mathcal{N}(\gamma_{\Gamma,h}))$ gives (6.3) with $c_A \in (0,1)$ close to 1 and $C_A = 1$, if V_h is fine enough and if the operator A_h^U is given by $\langle A_h^U u_h, \tilde{u}_h \rangle = (u_h, \tilde{u}_h)_H + (Lu_h, L\tilde{u}_h)_H$.

Error control. The condition $\hat{U}_h \subset \gamma_\Gamma(V_h)$ allows for a local reconstruction of the discrete skeleton solution $\hat{u}_h^{\mathrm{sol}} \in \hat{U}_h$ in U by determining $v_h^{\mathrm{sol}} \in V_h$ such that $\gamma_\Gamma v_h^{\mathrm{sol}} = \hat{u}_h^{\mathrm{sol}}$. Then, we have $v_h^{\mathrm{sol}} \in U \cap V_h$ and the error bound

$$\|\hat{u}^{\mathrm{sol}} - \hat{u}_h^{\mathrm{sol}}\|_{\hat{U}} \leq \|u^{\mathrm{sol}} - v_h^{\mathrm{sol}}\|_U \leq \sqrt{1 + C_L^2} \, \|L(u^{\mathrm{sol}} - v_h^{\mathrm{sol}})\|_H = \sqrt{1 + C_L^2} \, \|f - Lv_h^{\mathrm{sol}}\|_H. \tag{6.7}$$

In the special case $\hat{U}_h = \gamma_\Gamma(V_h)$ and $\mathcal{N}(\gamma_\Gamma) = \{0\}$ the reconstruction is unique. Then, the error bound is minimal for the result of the standard least squares method in $U_h = U \cap V_h$, i.e., by computing a minimizer of $R(v_h) = \frac{1}{2}\|f - Lv_h\|_H^2$.

Better results can be expected if $\mathcal{N}(\gamma_\Gamma) \neq \{0\}$, where the lifting $v_{\tau,h}^{\mathrm{sol}}$ with $\gamma_\Gamma v_h^{\mathrm{sol}} = \hat{u}_h^{\mathrm{sol}}$ is not unique, i.e.,

$$v_{\tau,h}^{\mathrm{sol}} \in A_{\tau,h}^{-1}\gamma'_{\tau,h}\left(\gamma_{\tau,h}A_{\tau,h}^{-1}\gamma'_{\tau,h}\right)^{-1}\hat{u}_h^{\mathrm{sol}} + \mathcal{N}(\gamma_\tau).$$

Uniqueness can be achieved by the additional condition $(Lv_{\tau,h}^{\mathrm{sol}} - f, v_h)_{H_\tau} = 0$ for $v_h \in \mathcal{N}(\gamma_\tau)$.
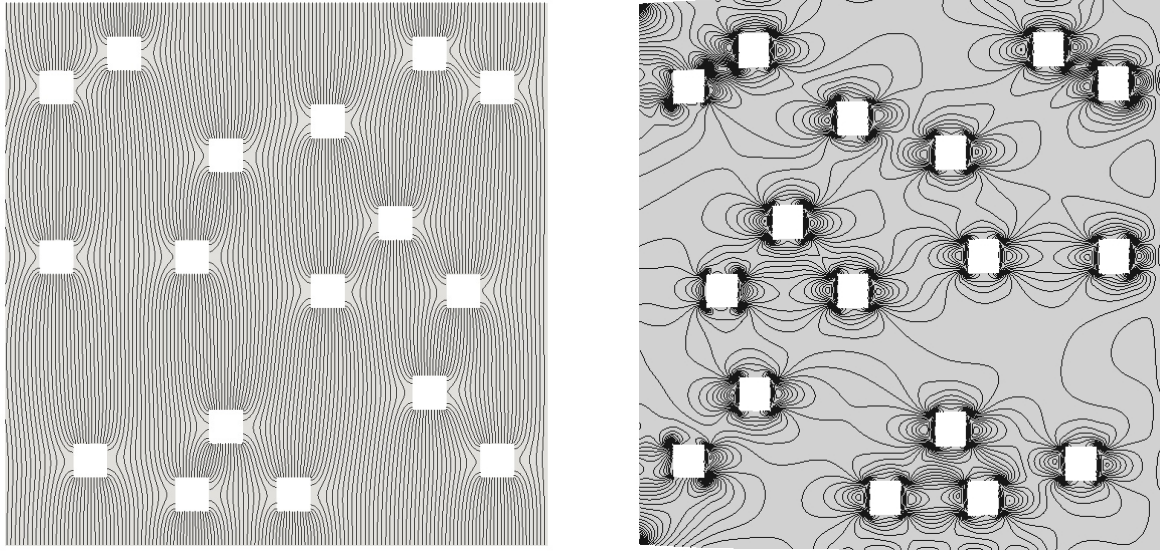
FIGURE 1. Isolines of p for a diffusion problem with homogeneous Dirichlet boundary values on the top boundary ($x_2 = 1$), Neumann boundary values $\sigma \cdot n = 1$ on the bottom boundary ($x_2 = 0$), and homogeneous Neumann boundary values else (left), and isolines of $|\sigma|$ for nearly incompressible linear elasticity with homogeneous Dirichlet boundary values on the bottom boundary ($x_2 = 0$), a pressure load for $x_2 = 1$, and homogeneous Neumann boundary values else (right).

7. NUMERICAL EXAMPLES

As a proof of concept, we consider two simple examples. In both cases we test the convergence of the discretization by computing the error bound (6.7), and we test the convergence of the skeleton preconditioner $\hat{G}_h$.

The first example is a diffusion problem $-\Delta p = 0$ in $\Omega \subset (0,1)^2$ with impenetrable obstacles in the domain, see Figure 1 for the geometry and boundary conditions. Let $U_h = \mathrm{RT}_{0,\partial\Omega_{\mathrm{N}}}(\mathcal{T}_h) \times \mathrm{S}_{1,\partial\Omega_{\mathrm{D}}}(\mathcal{T}_h)$ be the lowest order finite element space, $\hat{U}_h = \gamma_\Gamma(U_h)$ its trace space, $H_K = \mathbb{P}_0(K)^2 \times \mathbb{P}_0(K)$, and $V_K = \mathbb{P}_1(K)^2 \times \mathbb{P}_1(K)$. For preconditioning, we choose

$$\langle A_h^U\big((\sigma_h, p_h),(\tilde{\sigma}_h, \tilde{p}_h)\big) = (\sigma_h, \tilde{\sigma}_h)_\Omega + (\nabla \cdot \sigma_h, \nabla \cdot \tilde{\sigma}_h)_\Omega + (p_h, \tilde{p}_h)_\Omega + (\nabla p_h, \nabla \tilde{p}_h)_\Omega, \qquad (\sigma_h, p_h),(\tilde{\sigma}_h, \tilde{p}_h) \in U_h.$$

In the first component, G_h is a H(div, Ω) multigrid preconditioner with hybrid smoothing [24], and in the second component we use a parallel damped block Jacobi smoothing and local Gauß–Seidel smoothing on each processor [32]. As stopping criteria for the parallel Krylov iteration with V(2,2)-multigrid preconditioning, we use a residual reduction by the factor 10^{-12}. The results in the upper part of Table 1 show asymptotically the expected convergence rate $O(h^{\frac{2}{3}})$ for the lowest order approximation and mesh independent convergence for the multigrid preconditioner.

In our second example, we test the robustness of the discrete scheme for nearly incompressible 2D linear elasticity $-\operatorname{div}\mathcal{C}\varepsilon(u) = 0$ with constant pressure load $\sigma n = -p_0 n$ on $\partial\Omega_{\mathrm{N}} \subset \partial\Omega$, see Figure 1. Here, we use $H_K = \mathbb{P}_0(K)^{2\times2}_{\mathrm{sym}} \times \mathbb{P}_0(K)^2$ and $V_K = \mathbb{P}_1(K)^{2\times2}_{\mathrm{sym}} \times \mathbb{P}_1(K)^2$, and a V(4,4)-multigrid cycle is applied. The results in the lower part of Table 1 show robustness of the approximation and for the multigrid preconditioner in the incompressible limit.

The transfer to more challenging problems requires the proper choice of A_h^U in $U_h = V_h \cap U$, an efficient preconditioner G_h, and for the error control a suitable reconstruction operator.

TABLE 1. Numerical results for the diffusion problem and for linear elasticity. The reconstruction in (6.7) is determined by the evaluation of $\hat{u}^{\mathrm{sol}}$ at the nodal points of U_h, and the error is estimated by $\rho_h = \|(\sigma + \nabla p, \nabla \cdot \sigma - f)\|_\Omega$ for the diffusion problem and by $\rho_h = \|(\mathcal{C}^{-1}\sigma - \mathrm{D}u, \mathrm{div}\,\sigma + f)\|_\Omega$ for elasticity.

	number of triangles	1 912	7 648	30 592	122 368	489 472	1 957 888	7 831 552
diffusion	$\dim(\hat{U}_h)$	4 072	15 808	62 224	246 832	983 152	3 924 208	15 679 984
	ρ_h	0.24044	0.15507	0.09852	0.06227	0.03928	0.02475	0.01560
	$\log_2 \frac{\rho_{2h}}{\rho_h}$		0.633	0.654	0.662	0.665	0.666	0.666
	multigrid steps		19	20	20	21	21	21
elasticity	$\dim(\hat{U}_h)$	8 144	31 616	124 448	493 664	1 966 304	7 848 416	
$\nu = 0.49$	ρ_h	0.010224	0.006978	0.004593	0.003028	0.002010	0.0013434	
	$\log_2 \frac{\rho_{2h}}{\rho_h}$		0.551	0.60333	0.601	0.591	0.582	
	multigrid steps		32	32	33	34	36	
$\nu = 0.499$	ρ_h	0.010170	0.006941	0.004569	0.003014	0.002002	0.001338	
	$\log_2 \frac{\rho_{2h}}{\rho_h}$		0.550	0.603	0.600	0.590	0.580	
	multigrid steps		32	33	33	34	36	
$\nu = 0.4999$	ρ_h	0.010164	0.006938	0.004567	0.003012	0.002001	0.001338	
	$\log_2 \frac{\rho_{2h}}{\rho_h}$		0.550	0.603	0.600	0.590	0.580	
	multigrid steps		33	33	33	34	36	

Concluding remarks. Robust operator estimates and, as a consequence, *a priori* estimates in operator depending norms are essential for the design of robust discretization methods and robust numerical solvers. Here we have shown that for representative problem classes suitable stability estimates can be provided leading to convergence estimates of discontinuous Petrov−Galerkin methods. The efficiency of the preconditioning is demonstrated for diffusion and elasticity. Further numerical results for the Poisson problem [15], for the transport equation [13], for convection diffusion problems [12, 18], for the Helmholtz problem [16], and for the Stokes problem [30] underline the flexibility and efficiency of discontinuous Petrov–Galerkin methods. Nevertheless, to exploit the full potential of a Petrov–Galerkin approach, further research is required to identify the optimal balance of local approximations and skeleton discretizations.

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