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APPROXIMATION BY HARMONIC POLYNOMIALS IN STAR-SHAPED DOMAINS AND EXPONENTIAL CONVERGENCE OF TREFFTZ *hp*-dGFEM *

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Abstract. We study the approximation of harmonic functions by means of harmonic polynomials in two-dimensional, bounded, star-shaped domains. Assuming that the functions possess analytic extensions to a δ-neighbourhood of the domain, we prove exponential convergence of the approximation error with respect to the degree of the approximating harmonic polynomial. All the constants appearing in the bounds are explicit and depend only on the shape-regularity of the domain and on δ. We apply the obtained estimates to show exponential convergence with rate $O(\exp(-b\sqrt{N}))$, N being the number of degrees of freedom and $b > 0$, of a *hp*-dGFEM discretisation of the Laplace equation based on piecewise harmonic polynomials. This result is an improvement over the classical rate $O(\exp(-b\sqrt[3]{N}))$, and is due to the use of *harmonic polynomial spaces*, as opposed to complete polynomial spaces.

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1. Introduction

We fix a domain that meets the following requirements, see Figure 1.

Assumption 1.1. The domain $D \subset \mathbb{C}$ is open and satisfies[5]

(i) $\operatorname{diam}(D) = 1$,
(ii) there exists $0 < \rho \leq 1/2$ such that $B_\rho \subseteq D$,
(iii) there exists $0 < \rho_0 < \rho$ such that D is star-shaped with respect to B_{ρ_0}, *i.e.*, $\forall w \in D$ and $\forall v \in B_{\rho_0}$, the straight segment with endpoints w and v is contained in $\overline{D}$.

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[5]We write $B_r(w_0) := \{w \in \mathbb{C} : \ |w - w_0| < r\}$ and $B_r := B_r(0)$.

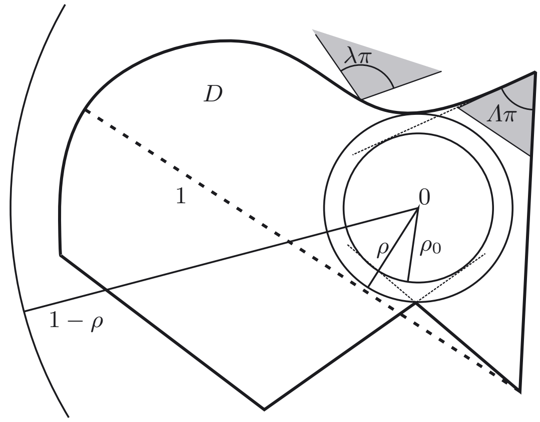

FIGURE 1. Geometry of domain D, see Assumption 1.1.

In this article we investigate the best approximation on D of a function $f : D \to \mathbb{C}$ by means of (complex variable) polynomials. We obtain *exponential convergence* in the polynomial degree provided that f is holomorphic in an open neighbourhood of D. Our main approximation result from Section 4.2 reads as follows[6].

Theorem 1.2. *Fix $0 < \delta \leq 1/2$ and define the inflated domain $D_\delta = \{w \in \mathbb{C} : \ d(w, D) < \delta\}$. There exist $C, b > 0$ only depending on ρ, ρ_0 and δ such that, for any function f which is holomorphic and bounded in D_δ, there is a sequence of polynomials $\{q_p\}_{p\geq 1}$ of degree at most p such that*

$$\|f - q_p\|_{L^\infty(D)} \leq C\,\mathrm{e}^{-bp}\,\|f\|_{L^\infty(D_\delta)}\,.$$

In Section 4, the values of C and b will be made *fully explicit* in terms of δ and the geometry of D (Thm. 4.7), and we will prove similar results for the derivatives of f (Cor. 4.9).

Our considerations follow the pioneering work of Melenk in [27], Chapter II and [28], refining and completing his arguments. The linchpin is Hermite's representation formula for the error of polynomial interpolation of holomorphic functions in complex domains, see Section 4.2. It is applied using, as integration contours, the level lines of the holomorphic mapping $\varphi_D : \mathbb{C}\backslash\overline{B}_1 \to \mathbb{C}\backslash\overline{D}$ provided by the Riemann mapping theorem. Thus we need rather precise information about the position of these level lines, and this information is gleaned in Section 3 by means of fairly intricate estimates. A result similar to Theorem 1.2 was stated in [27], Theorem 2.2.10, the novelty of the present contribution lies in the *explicit expressions* for the constants C and b in terms of the parameters δ, ρ and ρ_0 only. The importance of having explicit dependence of the constants on the geometry, in the context of Trefftz methods, is described in Remark 5.6.

Our work was motivated by the desire to obtain convergence estimates for the hp-version of Trefftz-type discontinuous Galerkin finite element methods (dGFEM) for second-order scalar elliptic boundary value problems. For the Laplace equation $\Delta u = 0$, these methods rely on harmonic polynomials for the local approximation on the mesh cells. Thus, with D standing for a mesh cell (after the identification of $\mathbb{R}^2$ with $\mathbb{C}$ and, possibly, a similarity transformation), estimates like that of Theorem 1.2 become instrumental for showing exponential convergence of the discretisation error in terms of the dimensions of the trial spaces. This will be outlined in Section 5, in the case of (straight) triangular and quadrilateral meshes, building on the hp-dGFEM convergence theory of [40]. On geometrically graded meshes, this scheme features faster exponential convergence than standard methods: the energy norm of the error decays as $\exp(-b\sqrt{N})$, N being the number of degrees of freedom and $b > 0$, as opposed to standard schemes which achieve only $\exp(-b\sqrt[3]{N})$. A dGFEM based on harmonic polynomials was already introduced in [24,25], only the convergence under mesh refinement was discussed there.

[6]Here and in the following, we denote the distance between a point $w \in \mathbb{C}$ and a set $\mathcal{D} \subset \mathbb{C}$ and the distance between two sets $\mathcal{D}_1, \mathcal{D}_2 \subset \mathbb{C}$ by $d(w, \mathcal{D}) := \inf_{w' \in \mathcal{D}} |w - w'|$ and $d(\mathcal{D}_1, \mathcal{D}_2) := \inf_{w_1 \in \mathcal{D}_1, w_2 \in \mathcal{D}_2} |w_1 - w_2|$.

The results of the present paper can be extended to general second-order elliptic equations by means of the so-called Vekua theory [31,36]. This technique provides continuous bijections between spaces of harmonic functions and spaces of solutions to the considered elliptic equation. In particular, the case of Helmholtz boundary value problems is relevant as several Trefftz-type numerical schemes have recently been proposed for their efficient approximation at medium and high wave numbers, see [8,13,14,19,20,23,29,32], the references therein, and the review in [30], Section 1.2. In this case, Vekua's theory allows to translate approximation estimates for harmonic polynomials into similar bounds for circular waves and can be related to approximation results for plane waves. This is pursued in recent work [21].

We close this introduction with some remarks on the geometry of the domain D in our approximation results. We refer to Figure 1 for an illustration of the notation in the following statements. By Assumption 1.1, D is bounded, simply connected, $0 \in D$ and $D \subseteq B_{1-\rho}$. Moreover, D satisfies the following *uniform cone conditions*: there exist $H_0 > 0$, and $\Lambda, \lambda \in (0, 1]$ satisfying

$$\min\{\Lambda, \lambda\} \geq \frac{2}{\pi} \arcsin \frac{\rho_0}{1-\rho}, \tag{1.1}$$

such that, for any $w \in \partial D$,

a) there exists a cone[7] with vertex w, opening angle $\Lambda\pi$ and height H_0 contained in $\overline{D}$,
b) there exists an infinite cone with vertex w and opening angle $\lambda\pi$ contained in $\mathbb{C} \setminus D$.

The proof can be found in Theorem A.1 of [22] Appendix A. The uniform cone conditions imply that D is Lipschitz (see, *e.g.*, [16], Thm. 1.2.2.2).

Remark 1.3. If D is convex, we could choose $\rho_0 = \rho$. However, in order to avoid the discussion of special cases, we will always assume $\rho_0 < \rho$, obviously with no loss of generality.

We also notice that, in the convex case, the exterior cone condition holds with $\lambda = 1$ (the cone is a half plane through w that does not intersect D), while for the interior cone condition one always has $\Lambda < 1$.

Remark 1.4. We chose to consider star-shaped domains since, as mentioned above, the main application we have in mind involves the use of the Vekua operators, which are defined only under this assumption, see [31]. On the other hand, all the proofs in the present article would be hugely simplified if we assumed convex domains instead (*e.g.* compare the proof of bound (3.2) in the convex case in Section 3.2 and in the star-shaped one in the appendix), and it could be argued that convex elements suffice for applications to finite elements. However, we believe that the abstract approximation results we state may be of much wider interest than for Galerkin schemes only. Moreover, if a boundary value problem is to be solved in a piecewise-smooth, non-convex domain, it is not possible to partition it into convex elements. Finally, we envisage the use of the bounds proved here in the context of finite elements for more general elliptic PDEs, where elements might be analytically mapped to transform the original PDE into the Laplace equation, in this case convexity would not be preserved.

2. Exterior conformal mappings

Let $\mathcal{D} \subset \mathbb{C}$ be a non-empty, simply connected "generic" domain that is either compact or open and bounded. Set $\mathcal{D}^c := (\mathbb{C} \cup \{\infty\}) \setminus \overline{\mathcal{D}}$ and $B_1^c := (\mathbb{C} \cup \{\infty\}) \setminus \overline{B}_1$. Owing to the Riemann mapping theorem, there exists a unique one-to-one conformal mapping $\varphi_{\mathcal{D}} : B_1^c \to \mathcal{D}^c$ such that $\varphi_{\mathcal{D}}(\infty) = \infty$ and $\varphi'_{\mathcal{D}}(\infty)$ is real and positive. The proof can be found in [17], Corollary 5.10c (where "regions" are non-empty, open, connected sets as defined in [17], Sect. 3.2) or in [26], Volume III, Theorems 1.2 and 1.3, after using the inversion across ∂B_1. If $\partial\mathcal{D}$ is a Jordan curve, then $\varphi_{\mathcal{D}}$ can be extended to a homeomorphism from $\overline{B_1^c} \to \overline{\mathcal{D}^c}$, *i.e.*, it is bijective and continuous, with continuous inverse also on the boundary (see [33], Sect. 17.20 or [17], Thm. 5.10e).

[7]Following [27], Proposition 2.1.6, we call "cone" an isosceles triangle, "infinite cone" the sector of the plane delimited by two half lines with common origin, and "opening angle" the angle adjacent to the two sides with equal length of a cone, or to the two half lines of a infinite cone.

For every $h > 0$, we define the *level line of* $\varphi_{\mathcal{D}}$ by

$$L_h[\overline{\mathcal{D}}] := \varphi_{\mathcal{D}}(\partial B_{1+h}). \tag{2.1}$$

Since $\varphi_{\mathcal{D}}$ is a homeomorphism, the level line $L_h[\overline{\mathcal{D}}]$ partitions $\mathbb{C}$ into two connected components and we denote by $\operatorname{Int} L_h[\overline{\mathcal{D}}]$ the closure of the bounded connected component. Whenever $\mathcal{D} = D$, which satisfies Assumption 1.1, we set for brevity $\varphi := \varphi_D$ and $L_h := L_h[\overline{D}]$.

In Section 16.5.II of [18] (Eqs. (16.5−7), p. 374) and in [37] (Sect. 4.4, p. 74)[8], the value $\varphi'_{\mathcal{D}}(\infty)$ (which is real and positive by definition of $\varphi_{\mathcal{D}}$) is identified as the classic analytic *capacity* of $\mathcal{D}$.

If $\mathcal{D}_1 \subset \mathcal{D}_2$, then $\varphi'_{\mathcal{D}_1}(\infty) \leq \varphi'_{\mathcal{D}_2}(\infty)$. Besides, Theorem 16.6j of [18] asserts that $\varphi'_{B_R(w)}(\infty) = R$ and, thus, for the domain D,

$$\rho < \varphi'_D(\infty) < 1 - \rho. \tag{2.2}$$

Let P be a bounded *polygon* with counterclockwise ordered vertices $\{w_k\}_{k=1}^N$ and corresponding internal angles $\{\alpha_k \pi\}_{k=1}^N$. Then, using conformal inversion across ∂B_1 and ([11], Eq. (4.6)), the conformal mapping φ_P is given by the Schwartz–Christoffel formula

$$\varphi_P(z) = A + C_{SC} \int^{1/z} \zeta^{-2} \prod_{k=1}^{N} \left(1 - \frac{\zeta}{z_k}\right)^{1-\alpha_k} \mathrm{d}\zeta \qquad |z| > 1, \tag{2.3}$$

where $z_k = \varphi_P^{-1}(w_k)$, $|z_k| = 1$. We have $\sum_{k=1}^N \alpha_k = N - 2$ (or $\sum_{k=1}^N (1 - \alpha_k) = 2$), see also ([26], Vol. III, Eq. (9.10), p. 331). The constant $A \in \mathbb{C}$ depends on translations of P and on the initial point in the integration, the constant C_{SC} is related to rotations/dilations and from [11], page 53, we have

$$|C_{SC}| = \varphi'_P(\infty). \tag{2.4}$$

The complex derivative of the Schwarz–Christoffel mapping can easily be computed as

$$\varphi'_P(z) = -C_{SC} \prod_{k=1}^{N} \left(1 - \frac{1}{z z_k}\right)^{1-\alpha_k} = -C_{SC} \frac{1}{z^2} \prod_{k=1}^{N} (z - \overline{z}_k)^{1-\alpha_k}, \tag{2.5}$$

where in the last step we have used $|z_k| = 1$ and $\sum_k (1 - \alpha_k) = 2$. When z approaches one of the z_k's, then $\varphi'_P(z)$ tends either to 0 or to ∞, depending on the sign of $1 - \alpha_k$.

Next, we recall the estimates of ([27], Lem. 2.1.3), applied to our domain D.

Lemma 2.1. *Let φ be the conformal mapping from B_1^c onto D^c. Then, for $0 < h' < h$*

$$d(L_h, L_{h'}) \geq \frac{\rho^2}{8\varphi'(\infty)} \frac{h'}{(1+h)^3}(h - h'), \qquad |\varphi'(z)| \leq \frac{\varphi'(\infty)|z|}{|z| - 1} \qquad |z| > 1.$$

Proof. We refer to [27], Appendix A.2, for the second bound, which is based on the "area formula" of ([26], Vol. III, Thm. 1.4), while here we report the proof of the first bound given in [27], Appendix A.2, taking into account our assumptions on D.

[8]Notice that, in both these references, the inverse conformal map $\varphi_{\mathcal{D}}^{-1}$ is used.

Fix $0 < h' < h$, $L_h, L_{h'}$ are compact, thus we can choose $1 < |z_1| = 1 + h' < |z_2| = 1 + h$ such that $d(L_h, L_{h'}) = |\varphi(z_2) - \varphi(z_1)|$. Then,

$$\begin{aligned}
h - h' &\leq |z_2 - z_1| = |\varphi^{-1}(\varphi(z_2)) - \varphi^{-1}(\varphi(z_1))| \\
&= \left| \int_{\varphi(z_1)}^{\varphi(z_2)} (\varphi^{-1})'(w)\, \mathrm{d}w \right| \leq |\varphi(z_2) - \varphi(z_1)| \sup_{1+h' \leq |\varphi^{-1}(w)| \leq 1+h} \left|(\varphi^{-1})'(w)\right| \\
&= |\varphi(z_2) - \varphi(z_1)| \sup_{1+h' \leq |z| \leq 1+h} \left| \frac{1}{\varphi'(z)} \right| \\
&\overset{[27], \text{ page } 165}{\leq} |\varphi(z_2) - \varphi(z_1)|\, \varphi'(\infty) \sup_{1+h' \leq |z| \leq 1+h} \frac{(1+|z|)^3}{(|z|-1)|\varphi(z)|^2} \\
&\leq d(L_h, L_{h'}) \varphi'(\infty) \frac{(2+h)^3}{h'\rho^2} \leq d(L_h, L_{h'}) \varphi'(\infty) \frac{8(1+h)^3}{h'\rho^2},
\end{aligned}$$

which gives the result. The bound we used from ([27], p. 165), is a consequence of the "distortion theorem", see [26], Volume III, Theorems 1.7 and 1.9. □

The following result is a direct consequence of Schwarz's Lemma [17], Theorem 5.10b, *i.e.*, of the fact that every holomorphic function $f : B_1 \to B_1$ satisfies $|f(z)| \leq |z|\ \forall z \in B_1$, applied to the function $z \mapsto 1/(\varphi_{\mathcal{D}_1}^{-1}(\varphi_{\mathcal{D}_2}(1/z)))$.

Lemma 2.2. *Let $\mathcal{D}_1 \subset \mathcal{D}_2$ be two bounded, simply connected, Lipschitz domains. Then,* $\operatorname{Int} L_h[\mathcal{D}_1] \subset \operatorname{Int} L_h[\mathcal{D}_2]$ *for all $h > 0$.*

3. Distance estimates for level lines of φ_D

We need precise quantitative information of how far the level lines L_h move away from ∂D as h increases. It is provided by the following key result.

Theorem 3.1. *Let L_h be the h-level line of the conformal mapping of D. Define $0 < \xi \leq 1$ as*

$$\xi := \begin{cases} \dfrac{2}{\pi} \arcsin \dfrac{\rho_0}{1-\rho} & \text{if } D \text{ is non convex,} \\ 1 & \text{if } D \text{ is convex.} \end{cases}$$

Then, provided that $0 < h \leq 1$, we have

$$\forall w \in \partial D,\ \forall w_h \in L_h, \qquad |w - w_h| \geq C_I h^2, \tag{3.1}$$

$$\forall w_h \in L_h\ \ \exists w \in \partial D: \qquad |w - w_h| \leq C_E h^{\xi}, \tag{3.2}$$

where we have set $C_I := \dfrac{\rho}{4}$ and $C_E := \dfrac{27}{\xi}$.

Remark 3.2. In the case of a convex polygonal domain D, (3.2) holds with $C_E = 9$ instead of 27 and, for more general convex domains, C_E can be improved up to $9 + c_0$, with any $c_0 > 0$, see Section 3.2 below.

Remark 3.3. The bounds (3.1) and (3.2) can be rewritten as

$$d(L_h, \partial D) \geq C_I h^2, \quad d(w_h, \partial D) \leq C_E h^{\xi} \qquad \forall w_h \in L_h.$$

A result in the spirit of Theorem 3.1 is proved in [27], Proposition 2.1.6. There, an upper bound for $d(w, L_h)$ with $w \in \partial D$ is given, which is different from our estimate (3.2). Another difference is that the exponents of h in the bounds (3.1) and (3.2), as well as the constants C_I and C_E, are specified under our assumptions on D.

The proofs of the two bounds (3.1) and (3.2) are given in the following Sections 3.1 and 3.2, respectively. On first reading these may be skipped.

3.1. Proof of the lower bound (3.1)

Lemma 3.4. *Let $S \subset \mathbb{C}$ be the segment $[-\rho, \rho]$, $\rho > 0$, on the real axis. Then $d(\rho, L_h[S]) = \frac{\rho h^2}{2(1+h)}$ for all $h > 0$.*

Proof. For any $\rho > 0$, the Joukowski map ([17], Sect. 5.1, p. 294), $J(z) = \frac{\rho}{2}\left(z + \frac{1}{z}\right)$ is the conformal mapping that maps B_1^c in the exterior of the segment S, with $J(\partial B_1) = S$, $J(\infty) = \infty$ and $J'(\infty) = \rho/2$. It level lines are ellipses whose foci are the endpoints of S. For every $h > 0$,

$$
\begin{aligned}
d(\rho, L_h[S]) &= \min_{z \in \partial B_{1+h}} |\rho - J(z)| = \min_{\theta \in [-\pi,\pi]} \left| \rho - \frac{\rho}{2}\left((1+h)\mathrm{e}^{\mathrm{i}\theta} + \frac{1}{(1+h)\mathrm{e}^{\mathrm{i}\theta}}\right)\right| \\
&= \frac{\rho}{2(1+h)} \min_{\theta \in [-\pi,\pi]} \left| \frac{2(1+h)\mathrm{e}^{\mathrm{i}\theta} - (1+h)^2\mathrm{e}^{2\mathrm{i}\theta} - 1}{\mathrm{e}^{\mathrm{i}\theta}} \right| = \frac{\rho}{2(1+h)} \min_{\theta \in [-\pi,\pi]} \left|(1+h)\mathrm{e}^{\mathrm{i}\theta} - 1\right|^2,
\end{aligned}
$$

the minimum is h^2 and it is achieved for $\theta = 0$, and the proof is complete. □

Proof of (3.1). The proof proceeds along the lines of [27], Proposition 2.1.6. Since D is star-shaped with respect to the origin and $B_\rho \subseteq D$, then for any $w \in \partial D$, there exists a (closed) straight segment S_w with one endpoint at w and length 2ρ such that $S_w \subset \overline{D}$. By Lemmas 3.4 and 2.2, we have

$$
\frac{\rho h^2}{2(1+h)} = d(w, L_h[S_w]) \leq d(w, L_h) \qquad \forall w \in \partial D,
$$

which implies (3.1) with $C_I = \rho/4$, since $h \leq 1$. □

Remark 3.5. In Proposition 2.1.6 of [27] a bound similar to (3.1) was established with a better power of h, *i.e.*, $2 - \Lambda$ instead of 2. This was proved by comparing the level lines of D with those of a triangle, instead of comparing with those of a segment. We were not able to prove this result with a fully explicit constant C_I. On the other hand, exponent 2 is sufficient to establish exponential convergence for the approximations of holomorphic functions by complex polynomials.

3.2. Proof of the upper bound (3.2) for convex domains

In this section we consider the case of convex D, which already reveals the key ideas with moderate technical complexity. For the much more intricate case of general D with non convex boundary, we refer to Appendix A.

Proof of (3.2) *for convex domains.* We consider first the case when D is a convex polygon (with straight sides) with vertices $\{w_k\}_{k=1}^N$ and corresponding internal angles $\{\alpha_k \pi\}_{k=1}^N$. Set $z_k = \varphi^{-1}(w_k) \in \partial B_1$, $k = 1, \ldots, N$.

Fix $w_h \in L_h$ and set $z_h = \varphi^{-1}(w_h) \in \partial B_{1+h}$. Thus $z_h = (1+h)\mathrm{e}^{\mathrm{i}\theta}$, for some $\theta \in [-\pi, \pi]$. Define $z = \mathrm{e}^{\mathrm{i}\theta}$, $w = \varphi(z)$, and denote by S the straight segment of length h connecting z and z_h. From (2.4) and (2.5) we have

$$
|w_h - w| = |\varphi(z_h) - \varphi(z)| \leq \int_S |\varphi'(y)| \,\mathrm{d}y \leq \varphi'(\infty) \int_S \frac{1}{|y|^2} \prod_{k=1}^N |y - \overline{z}_k|^{1-\alpha_k} \,\mathrm{d}y.
$$

For any $y \in S$, we have $|y - \overline{z}_k| \leq 2 + h$ and, due to the convexity of D, $1 - \alpha_k \geq 0$, $k = 1, \ldots, N$. Then, recalling that $\sum_{k=1}^N (1 - \alpha_k) = 2$, we arrive at

$$
\prod_{k=1}^N |y - \overline{z}_k|^{1-\alpha_k} \leq (2+h)^{\sum_{k=1}^N (1-\alpha_k)} = (2+h)^2.
$$

Notice that this bound is independent of the number N of the vertices of P. Using $|y| \geq 1$ and (2.2), since $h \leq 1$, we obtain

$$
|w_h - w| \leq (1-\rho)(2+h)^2 \int_S 1 \,\mathrm{d}y \leq (2+h)^2 h \leq 9h.
$$

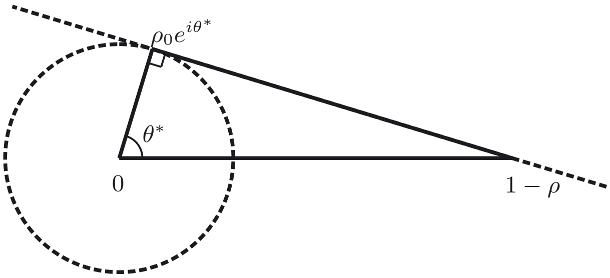

FIGURE 2. The extremal case in the proof of Lemma 4.1 and the angle θ^*.

If a convex D has more general shape, we exploit the fact that, for any fixed $\varepsilon > 0$, we can find a convex polygon P_ε containing D such that, for all $w \in \partial P_\varepsilon$, $d(w, \partial D) < \varepsilon$ ([38], Thm. 3.1.6). For ε small enough, $P_\varepsilon \subset B_1$, thus $\varphi'_{P_\varepsilon}(\infty) \leq 1$.

Fix $w_h \in L_h = L_h[\overline{D}]$. Let P_ε be an approximating polygon as before, with $\varepsilon \leq \frac{1}{2} d(w_h, \partial D)$. Then, $w_h \in L_{h'}[\overline{P}_\varepsilon]$ with $h' \leq h$, as a consequence of Lemma 2.2. Let $z_{h'} = \varphi_{P_\varepsilon}^{-1}(w_h) = (1+h')\mathrm{e}^{\mathrm{i}\theta}$, and define $z = \mathrm{e}^{\mathrm{i}\theta}$. Then,

$$\begin{aligned} d(w_h, \partial D) &\leq d\big(w_h, \varphi_{P_\varepsilon}(z)\big) + d\big(\varphi_{P_\varepsilon}(z), \partial D\big) = |\varphi_{P_\varepsilon}(z_{h'}) - \varphi_{P_\varepsilon}(z)| + d\big(\varphi_{P_\varepsilon}(z), \partial D\big) \\ &\leq (2+h')^2 h' + \varepsilon \leq (2+h)^2 h + \frac{1}{2} d\big(w_h, \partial D\big), \end{aligned}$$

which implies $d(w_h, \partial D) \leq 2(2+h)^2 h \leq 18h$. □

4. Interpolation estimates

In this section, we prove error estimates for the approximation of holomorphic functions by means of polynomials. We first state some auxiliary results.

4.1. Auxiliary results

We define the "polar parametrisation" $\Psi : \mathbb{C} \to \mathbb{C}$ such that

$$\Psi(B_1) = D, \qquad \Psi(r\mathrm{e}^{\mathrm{i}\theta}) = \psi(\theta) r\mathrm{e}^{\mathrm{i}\theta}, \qquad \psi : [-\pi, \pi) \to [\rho, 1-\rho].$$

Lemma 4.1. *The function $\psi : [-\pi, \pi) \to [\rho, 1-\rho]$ is Lipschitz continuous with constant L_ψ satisfying*

$$L_\psi := \sup_{\theta \in [-\pi, \pi]} \psi'(\theta) \leq \frac{(1-\rho)^2}{\rho_0}.$$

Proof. Assumption 1.1 guarantees that D is a Lipschitz domain, therefore by Rademacher's theorem (see [12], Sect. 3.1.2), ψ is differentiable almost everywhere and, for almost every point of ∂D, there exists a tangent line. Because of the star-shapedness requirement, no tangent line to ∂D can intersect the open ball B_{ρ_0}.

Therefore the steepest (in polar coordinates) possible tangent line at a point $\psi(\theta)$ is tangent to ∂B_{ρ_0}. Since the angular derivative of a straight line is larger for points with larger moduli, we can bound $\psi'(\theta)$ with the angular derivative at $\theta = 0$ of one of the two straight lines through $1-\rho$ that are tangent to B_{ρ_0}.

This line has polar representation $r(\theta) = \rho_0 / \cos(\theta^* - \theta)$, where $\theta^* = \arccos \frac{\rho_0}{1-\rho}$ (*i.e.*, θ^* is the angle at 0 of the rectangular triangle of vertices 0, $1-\rho$ and the tangent point of the line to ∂B_{ρ_0}, see Fig. 2). Its "polar slope" in $\theta = 0$ is given by

$$|r'(\theta)|_{\theta=0} = \left[\frac{\rho_0 |\sin(\theta^* - \theta)|}{\cos^2(\theta^* - \theta)}\right]_{\theta=0} = \frac{\rho_0 |\sin\theta^*|}{\cos^2\theta^*} = \frac{(1-\rho)^2}{\rho_0}\left|\sin\arccos\frac{\rho_0}{1-\rho}\right| \leq \frac{(1-\rho)^2}{\rho_0}.$$

Then $|\psi'(0)| \leq \frac{(1-\rho)^2}{\rho_0}$ and the proof is complete. □

The inverse of Ψ is given by $\Psi^{-1}(r\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{\psi(\theta)} r\mathrm{e}^{\mathrm{i}\theta}$ or, in Cartesian coordinates (after identifying $\mathbb{C}$ with $\mathbb{R}^2$),

$$\Psi^{-1}(r\cos\theta, r\sin\theta) = \left(\frac{r}{\psi(\theta)}\cos\theta, \frac{r}{\psi(\theta)}\sin\theta\right) =: (F_1, F_2). \tag{4.1}$$

Moreover, Ψ^{-1} is Lipschitz continuous, and an estimate for its Lipschitz constant is given in the next Lemma.

Lemma 4.2. *The function $\Psi^{-1} : \mathbb{C} \to \mathbb{C}$ is Lipschitz continuous with constant $L_{\Psi^{-1}}$ satisfying*

$$L_{\Psi^{-1}} := \sup_{w,v\in\mathbb{C},\, w\neq v} \frac{|w-v|}{|\Psi(w)-\Psi(v)|} \leq \frac{2(2\rho+L_\psi)}{\rho^2},$$

with L_ψ as in Lemma 4.1.

Proof. Let $D\Psi^{-1}$ be the Jacobian of Ψ^{-1}. Considering the representation (4.1) of Ψ^{-1}, we have

$$\frac{\partial F_1}{\partial x} = \frac{x}{r}\frac{\cos\theta}{\psi(\theta)} - \frac{y}{r}\frac{-\sin\theta\psi(\theta)+\cos\theta\psi'(\theta)}{(\psi(\theta))^2}, \qquad \frac{\partial F_2}{\partial x} = \frac{x}{r}\frac{\sin\theta}{\psi(\theta)} - \frac{y}{r}\frac{\cos\theta\psi(\theta)+\sin\theta\psi'(\theta)}{(\psi(\theta))^2},$$
$$\frac{\partial F_1}{\partial y} = \frac{y}{r}\frac{\cos\theta}{\psi(\theta)} + \frac{x}{r}\frac{-\sin\theta\psi(\theta)+\cos\theta\psi'(\theta)}{(\psi(\theta))^2}, \qquad \frac{\partial F_2}{\partial y} = \frac{y}{r}\frac{\sin\theta}{\psi(\theta)} + \frac{x}{r}\frac{\cos\theta\psi(\theta)+\sin\theta\psi'(\theta)}{(\psi(\theta))^2}.$$

Since $|x|, |y| \leq r$ and $0 < \rho < |\psi(\theta)| < 1$, we can bound $\left\|D\Psi^{-1}\right\|_{L^\infty(\mathbb{C})}$ (in the matrix ∞-norm) as

$$\left\|D\Psi^{-1}\right\|_{L^\infty(\mathbb{C})} \leq 2\left(\frac{1}{\rho}+\frac{1}{\rho}+\frac{L_\psi}{\rho^2}\right) = \frac{2(2\rho+L_\psi)}{\rho^2}.$$

Since $L_{\Psi^{-1}} = \left\|D\Psi^{-1}\right\|_{L^\infty(\mathbb{C})}$ the proof is complete. $\square$

Lemma 4.3. *For every positive h, the following bound holds:*

$$\left|\mathrm{e}^{\mathrm{i}\theta} - (1+h)\right|^2 \geq \left(\frac{2}{\pi}\right)^2 (\theta^2+h^2) =: C_B^2(\theta^2+h^2) \qquad \forall\theta\in[-\pi,\pi].$$

Proof. Using $1-\cos\theta \geq \frac{2}{\pi^2}\theta^2$ for any $\theta\in[-\pi,\pi]$, we have

$$\left|\mathrm{e}^{\mathrm{i}\theta}-(1+h)\right|^2 = (1+h-\cos\theta)^2 + (\sin\theta)^2 = h^2 + 2(1-\cos\theta)(h+1) \geq \frac{4}{\pi^2}(\theta^2+h^2). \qquad \square$$

Now, we provide a refined version of [27], Lemma 2.1.8.

Lemma 4.4. *If $0 < h \leq C_I$ is such that $L_h \subset B_{1+\rho}$ and $w_0 \in L_h$, then*

$$\int_{\partial D} \frac{1}{|w-w_0|}\,\mathrm{d}w \leq C_D\,|\log h|, \qquad \textit{where} \qquad C_D = 4\pi\sqrt{2}L_\psi L_{\Psi^{-1}},$$

with L_ψ and $L_{\Psi^{-1}}$ as in Lemmas 4.1 and 4.2, respectively.

Proof. Fix $w_0 \in L_h$, and assume, with no loss of generality, that w_0 is on the positive real axis. Define $d := w_0 - \psi(0)$ and notice that $d(w_0, \partial D) \leq d \leq 1$.

Setting $w(\theta) := \Psi(\mathrm{e}^{\mathrm{i}\theta}) = \psi(\theta)\mathrm{e}^{\mathrm{i}\theta} \in \partial D$, using Lemmas 4.2, and 4.3 and $\psi(\theta) < 1$, we obtain, for all $\theta\in[-\pi,\pi]$,

$$\begin{aligned} |w(\theta)-w_0|^2 &\geq L_{\Psi^{-1}}^{-2}\left|\Psi^{-1}(w(\theta)) - \Psi^{-1}(w_0)\right|^2 = L_{\Psi^{-1}}^{-2}\left|\mathrm{e}^{\mathrm{i}\theta} - w_0/\psi(0)\right|^2 \\ &\geq L_{\Psi^{-1}}^{-2}C_B^2\left[\theta^2 + \left(\frac{w_0}{\psi(0)}-1\right)^2\right] = L_{\Psi^{-1}}^{-2}C_B^2\left[\theta^2+\left(\frac{w_0-\psi(0)}{\psi(0)}\right)^2\right] \\ &> L_{\Psi^{-1}}^{-2}C_B^2\left[\theta^2+(w_0-\psi(0))^2\right] = \frac{4}{\pi^2}L_{\Psi^{-1}}^{-2}(\theta^2+d^2) =: L_D^2(\theta^2+d^2). \end{aligned} \tag{4.2}$$

Then,

$$\int_{\partial D} \frac{1}{|w-w_0|}\,\mathrm{d}w = \int_{-\pi}^{\pi} \frac{1}{|w(\theta)-w_0|}|w'(\theta)|\,\mathrm{d}\theta \overset{\text{Lem. 4.1}}{\leq} L_\psi \int_{-\pi}^{\pi} \frac{1}{|w(\theta)-w_0|}\,\mathrm{d}\theta$$
$$\overset{(4.2)}{\leq} L_\psi L_D^{-1} \int_{-\pi}^{\pi} \frac{1}{\sqrt{\theta^2+d^2}}\,\mathrm{d}\theta \leq 2\sqrt{2} L_\psi L_D^{-1} \int_0^{\pi} \frac{1}{\theta+d}\,\mathrm{d}\theta \leq 2\sqrt{2} L_\psi L_D^{-1}\big(\log(\pi+d)-\log d\big).$$

Since

$$h \leq C_I = \frac{\rho}{4} \leq \frac{1}{8} < \frac{1}{\pi+1} \leq \frac{1}{\pi+d} < 1,$$

we have $\log(\pi+d) \leq |\log h|$ and

$$\begin{aligned}
\int_{\partial D} \frac{1}{|w-w_0|}\,\mathrm{d}w \quad &\leq \quad 2\sqrt{2} L_\psi L_D^{-1}(|\log h| + |\log d|) \\
&\overset{d \geq d(w_0,\partial D)}{\leq} 2\sqrt{2} L_\psi L_D^{-1}(|\log h| + |\log d(w_0,\partial D)|) \\
&\overset{w_0\in L_h,\,(3.1)}{\leq} 2\sqrt{2} L_\psi L_D^{-1}\big(|\log C_I| + 3|\log h|\big) \\
&\overset{h\leq C_I<1}{\leq} 8\sqrt{2} L_\psi L_D^{-1}|\log h| = \underbrace{4\pi\sqrt{2} L_\psi L_{\Psi^{-1}}}_{=:C_D}|\log h|,
\end{aligned}$$

where we can use (3.1), because $h \leq C_I < 1$. $\square$

Remark 4.5. Using Lemma 4.2, Lemma 4.1 and $\rho_0 < \rho \leq 1/2$, we have the bound

$$\begin{aligned}
C_D = 4\pi\sqrt{2} L_\psi L_{\Psi^{-1}} &\leq 4\pi\sqrt{2} L_\psi \frac{2(2\rho+L_\psi)}{\rho^2} \leq 8\pi\sqrt{2}\frac{(1-\rho)^2(2\rho\rho_0+(1-\rho)^2)}{\rho_0^2\rho^2} \\
&\leq \frac{8\pi\sqrt{2}}{\rho_0^2\rho^2}[(1-\rho)^2(2\rho^2+(1-\rho)^2)] \leq \frac{8\pi\sqrt{2}}{\rho_0^2\rho^2}[(1-\rho)^2(1+\rho^2)] \leq \frac{8\pi\sqrt{2}}{\rho_0^2\rho^2},
\end{aligned}$$

since $(1-\rho)^2(1+\rho^2) = 1-2\rho+2\rho^2-2\rho^3+\rho^4 < 1$.

Define the sequence of complex polynomials $\{\omega_p\}_{p\in\mathbb{N}}$ with

$$\omega_p(w) := \prod_{k=0}^{p-1}\left(w - \varphi\left(\mathrm{e}^{2\pi \mathrm{i}k/p}\right)\right),$$

where φ is the exterior conformal mapping of D.

Lemma 4.6 ([27], Lem. 2.2.9)**.** *Under the same hypothesis on h as in Lemma 4.4 we find*

$$h^{C_D}|\varphi'(\infty)|^p(1+h)^p \leq |\omega_p(w)| \leq h^{-C_D}|\varphi'(\infty)|^p(1+h)^p \quad \forall w \in L_h,\ \forall p \in \mathbb{N},$$

where C_D is the constant in Lemma 4.4.

Proof. We refer to the proof of [27], Lemma 2.2.9. The constant at the exponents of h is equal to C_D and the threshold on h is the one needed by Lemma 4.4. $\square$

4.2. Main interpolation estimates

As in Theorem 1.2, for $\delta > 0$, define the inflated domain

$$D_\delta := \{w \in \mathbb{C} : \; d(w, D) < \delta\}. \tag{4.3}$$

Assume $\ell > 0$, then Theorem 3.1 guarantees that, if $0 < h < \frac{1}{\ell}\big(\frac{\delta}{C_E}\big)^{1/\xi}$, then $L_{\ell h} \subset D_\delta$.

Our main approximation results is a refinement of [27], Theorem 2.2.10.

Theorem 4.7. *Fix $0 < \delta \le 1/2$. Provided that*

$$0 < h < h^*(\delta) := \min\left\{ \frac{1}{3}\left(\frac{\delta}{C_E}\right)^{1/\xi}, \; \frac{\rho}{4} \right\}, \tag{4.4}$$

there exist $C_{\text{appr}} > 0$ and $\alpha > 0$ depending only on D through ρ and ρ_0, such that, for any f holomorphic in D_δ, there is a sequence of polynomials $\{q_p\}_{p\ge 1}$ of degree at most p such that

$$\|f - q_p\|_{L^\infty(\operatorname{Int} L_h)} \le C_{\text{appr}} h^{-\alpha}(1+h)^{-p} \, \|f\|_{L^\infty(\operatorname{Int} L_{3h})},$$

where

$$C_{\text{appr}} \le \frac{20(1-\rho)^2}{3\rho^2} \le \frac{7}{\rho^2}, \qquad \alpha \le 3 + \frac{16\sqrt{2}\pi}{\rho_0^2\rho^2} \le \frac{72}{\rho_0^2\rho^2}.$$

Remark 4.8. Compared to ([27], Thm. 2.2.10), this estimate features fully explicit bounds in terms of shape parameters of D. Moreover, no complete proof of Theorem 2.2.10 was given in [27], *cf.* Remark 3.3.

Proof of Theorem 4.7. We choose q_p as the polynomial of degree p which interpolates f at the $p+1$ points $\varphi(\mathrm{e}^{2\pi i k/(p+1)})$, $k = 0, \ldots, p$. Since $L_{3h} \subset D_\delta$, using the Hermite interpolation error formula (see [27], p. 17 or [9], Thm. 3.6.1), we have

$$\|f - q_p\|_{L^\infty(\operatorname{Int} L_h)} = \sup_{w \in \operatorname{Int} L_h} \left| \frac{1}{2\pi i} \int_{L_{3h}} \frac{\omega_p(w) f(t)}{\omega_p(t)(t-w)} \,\mathrm{d}t \right| \le \frac{\operatorname{length}(L_{3h}) \; \sup_{w\in \operatorname{Int} L_h} |\omega_p(w)| \;\; \|f\|_{L^\infty(\operatorname{Int} L_{3h})}}{2\pi \; \inf_{t\in \operatorname{Int} L_{3h}} |\omega_p(t)| \;\; d(L_h, L_{3h})}.$$

Since φ is a curve parametrisation $\varphi : \partial B_{1+3h} \to L_{3h}$, it satisfies $\operatorname{length}(L_{3h}) \le 2\pi(1+3h)\sup_{|z|=1+3h} |\varphi'(z)|$. This, together with the lower bound of $d(L_h, L_{3h})$ and the upper bound of $|\varphi'(z)|$ given in Lemma 2.1, and the bounds in Lemma 4.6, gives

$$\begin{aligned}
\|f - q_p\|_{L^\infty(\operatorname{Int} L_h)} &\le \frac{8(1+3h)^5 \varphi'(\infty)^2}{6h^3\rho^2} (3h^2)^{-C_D} \left(\frac{1+h}{1+3h}\right)^p \|f\|_{L^\infty(\operatorname{Int} L_{3h})} \\
&\le \frac{4\varphi'(\infty)^2}{3^{1+C_D}\rho^2} h^{-3-2C_D} \left(\frac{1+h}{1+3h}\right)^p (1+3h)^5 \, \|f\|_{L^\infty(\operatorname{Int} L_{3h})} \\
&\le \frac{20(1-\rho)^2}{3\rho^2} h^{-3-2C_D} \left(\frac{1}{1+h}\right)^p \|f\|_{L^\infty(\operatorname{Int} L_{3h})},
\end{aligned}$$

where in the last step we have used $3^{1+C_D} > 3$, $|\varphi'(\infty)| < 1-\rho$, $\frac{1+h}{1+3h} \le \frac{1}{1+h}$, and $(1+3h)^5 < 5$, since $h \le \rho/4 \Rightarrow h < 1/8$. The use of Lemma 4.6 (and thus of Lem. 4.4) is legitimate due to the hypothesis imposed on h and δ. The result of the theorem follows from the bound of C_D derived in Remark 4.5. □

Obviously, Theorem 1.2 from the Introduction is an immediate consequence of Theorem 4.7: given $0 < h < h^*$, just define $C := C_{\text{appr}}(h^*(\delta))^{-\alpha}$ and $b := \log(1 + h^*(\delta))$.

The polynomials q_p defined in the proof of Theorem 4.7 as the complex interpolants of f in special points, *simultaneously* approximate the first p derivatives of f (denoted $f^{(j)}$, $j = 1, \ldots, p$), as established by the following corollary.

Corollary 4.9. *Under the assumptions of Theorem 4.7, for any*[9] $j \in \mathbb{N}$, $j \le p$, *we have*

$$\left\| f^{(j)} - q_p^{(j)} \right\|_{L^\infty(D)} \le C_{\mathrm{appr}} \frac{j!}{(C_I h^2)^j} h^{-\alpha}(1+h)^{-p} \|f\|_{L^\infty(\mathrm{Int}\, L_{3h})}.$$

Proof. We use Cauchy's inequalities ([26], Vol. I, Thm. 14.7) for the interpolation error $f^{(j)} - q_p^{(j)}$ to obtain a sharp bound on the complex derivatives of holomorphic functions:

$$\left\| f^{(j)} - q_p^{(j)} \right\|_{L^\infty(D)} \le \frac{j!}{d(L_h, \partial D)^j} \|f - q_p\|_{L^\infty(\mathrm{Int}\, L_h)},$$

the assertion of the corollary follows from the bound (3.1) and from Theorem 4.7. □

As a consequence of the previous results, we can gauge the approximation of real-valued harmonic functions by harmonic polynomials. To this purpose, setting $z = x + iy$ we identify $S \subseteq \mathbb{C}$ and $\{(x, y) \in \mathbb{R}^2 | \; z = x + iy \in S\}$ and now regard $f : D_\delta \to \mathbb{C}$ as a *real analytic* function of two real variables $f = f(x, y)$. We also adopt this perspective for the polynomials q_p, which have been defined in the proof of Theorem 4.7 as the complex interpolants of f in special points.

The statement of the following results makes use of the (standard) $W^{j,\infty}(S)$-seminorms, $j \in \mathbb{N}$, and of the weighted Sobolev $W^{1,\infty}(S)$-norm, for sufficiently smooth functions, and $S \subseteq D_\delta$:

$$|u|_{W^{j,\infty}(S)} := \sup_{\boldsymbol{\beta} \in \mathbb{N}_0^2,\ |\boldsymbol{\beta}| = j} \left\| D^{\boldsymbol{\beta}} u \right\|_{L^\infty(S)}, \qquad \|u\|_{W^{1,\infty}(S)} := \|u\|_{L^\infty(S)} + \mathrm{diam}(D_\delta) \|\nabla u\|_{L^\infty(S)}.$$

Theorem 4.10. *Fix* $0 < \delta \le 1/2$, *and let* h *satisfy* (4.4). *For any real, harmonic function* u *in the inflated domain* D_δ *defined in* (4.3), *there is a sequence of harmonic polynomials* $\{Q_p\}_{p \ge 1}$ *of degree at most* p *such that*

$$\begin{aligned}
\|u - Q_p\|_{L^\infty(D)} &\le C_{\mathrm{appr}} h^{-\alpha}(1+h)^{-p} \|u\|_{W^{1,\infty}(\mathrm{Int}\, L_{3h})},\\
|u - Q_p|_{W^{j,\infty}(D)} &\le C_{\mathrm{appr}} \left(\frac{2j}{C_I h^2}\right)^j h^{-\alpha}(1+h)^{-p} \|u\|_{W^{1,\infty}(\mathrm{Int}\, L_{3h})},\\
\|u - Q_p\|_{L^2(D)} &\le \sqrt{|D|}\, C_{\mathrm{appr}} h^{-\alpha}(1+h)^{-p} \|u\|_{W^{1,\infty}(\mathrm{Int}\, L_{3h})},\\
|u - Q_p|_{H^j(D)} &\le \sqrt{|D|(j+1)}\, C_{\mathrm{appr}} \left(\frac{2j}{C_I h^2}\right)^j h^{-\alpha}(1+h)^{-p} \|u\|_{W^{1,\infty}(\mathrm{Int}\, L_{3h})}
\end{aligned}$$

for all $j \in \mathbb{N}$, $j \le p$, *where* $|D| < 1$ *is the Lebesgue measure of* D, *and the constants* C_{appr} *and* α *are the same as in Theorem 4.7.*

Proof. For any real, harmonic function u on a simply-connected domain $\mathcal{D} \ni (x_0, y_0)$, there exists a unique holomorphic function f on $\mathcal{D}$, with $f(x_0 + \mathrm{i}y_0) \in \mathbb{R}$, such that $u(x, y) = \mathrm{Re}\, f(x + \mathrm{i}y)$ ([26], Vol. II, Thm. 5.2). More precisely, $f(z) = u(x, y) + iv(x, y)$, with $z = x + iy$ and v a real, harmonic function satisfying the Cauchy–Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, and $v(x_0, y_0) = 0$. If $\mathcal{D}$ is star-shaped with respect to (x_0, y_0), and $\|u\|_{L^\infty(\mathcal{D})}$, $\|\nabla u\|_{L^\infty(\mathcal{D})}$ are bounded, it holds $\|f\|_{L^\infty(\mathcal{D})} \le \|u\|_{L^\infty(\mathcal{D})} + \mathrm{diam}(\mathcal{D}) \|\nabla u\|_{L^\infty(\mathcal{D})}$. Moreover, if f is a holomorphic function, then $u(\mathrm{Re}\, z, \mathrm{Im}\, z) = \mathrm{Re}\, f(z)$ is harmonic, thus, the real part of any complex polynomial is a harmonic polynomial. Obviously, $\|u\|_{L^\infty(\mathcal{D})} \le \|f\|_{L^\infty(\mathcal{D})}$ holds true.

With these considerations, defining $Q_p := \mathrm{Re}\, q_p$, with q_p as in Theorem 4.7, the desired bound in L^∞-norm is direct consequence of Theorem 4.7. Notice that $\|u\|_{L^\infty(\mathrm{Int}\, L_{3h})}$ and $\|\nabla u\|_{L^\infty(\mathrm{Int}\, L_{3h})}$ are bounded (and thus $\|u\|_{W^{1,\infty}(\mathrm{Int}\, L_{3h})} < +\infty$) because, by (4.4), the (closed) set $\mathrm{Int}\, L_{3h}$ is contained in D_δ, the (open) domain of analyticity of u.

[9]We use the following standard notation: $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

For the bounds in $W^{j,\infty}$-norms, the inclusion $D \subset L_h$, the interior estimates for the derivatives of harmonic functions in [15], Theorem 2.10, and the bound (3.1) give

$$|u - Q_p|_{W^{j,\infty}(D)} = \sup_{\boldsymbol{\beta}\in\mathbb{N}_0^2,\ |\boldsymbol{\beta}|=j} \left\| D^{\boldsymbol{\beta}}(u - Q_p) \right\|_{L^\infty(D)}$$
$$\leq \left(\frac{2j}{d(L_h, \Omega)}\right)^j \|u - Q_p\|_{L^\infty(\mathrm{Int}\, L_h)} \leq \left(\frac{2j}{C_I h^2}\right)^j \|u - Q_p\|_{L^\infty(\mathrm{Int}\, L_h)},$$

again, Theorem 4.7 allows to conclude. Finally, the bounds in integral norms follow from

$$|u - Q_p|^2_{H^j(D)} := \sum_{\boldsymbol{\beta}\in\mathbb{N}_0^2,\ |\boldsymbol{\beta}|=j} \int_D \left| D^{\boldsymbol{\beta}}\big(u(x) - Q_p(x)\big)\right|^2 \, \mathrm{d}x \leq |D|(j+1)\ |u - Q_p|^2_{W^{j,\infty}(D)}$$

and the previous inequalities. □

From Theorem 1.2, with the same considerations as in the proof of Theorem 4.10, we obtain the following result.

Corollary 4.11. *Fix $0 < \delta \leq 1/2$ and $j \in \mathbb{N}_0$. There exist $C > 0$ and $b > 0$,* depending only on ρ, ρ_0, δ and *j, such that, for any real-valued, harmonic function u which is bounded along with its first-order derivatives in the inflated domain D_δ defined in (4.3), there is a sequence of harmonic polynomials $\{Q_p\}_p$ of degree at most p such that*

$$|u - Q_p|_{W^{j,\infty}(D)} \leq C\, \mathrm{e}^{-bp}\, \|u\|_{W^{1,\infty}(D_\delta)}, \qquad |u - Q_p|_{H^j(D)} \leq C\, \mathrm{e}^{-bp}\, \|u\|_{W^{1,\infty}(D_\delta)}.$$

Remark 4.12. The boundedness of f, u and ∇u in Theorem 1.2 and Corollary 4.11 is assumed only in order to write estimates with L^∞-norms in the whole D_δ on the right-hand side. Actually, the estimates hold true also with $\|f\|_{L^\infty(\mathrm{Int}\, L_{3h})}$ and $\|u\|_{W^{1,\infty}(\mathrm{Int}\, L_{3h})}$, respectively, on the right-hand side, for any $0 < h < h^*$, with no need of assuming boundedness of f, u and ∇u in D_δ.

The constants C and b in Theorem 1.2 and Corollary 4.11 depend on δ only through $h^*(\delta)$ defined in (4.4).

Remark 4.13. The interpolating polynomials q_p (and Q_p) in Theorems 1.2, and 4.7 and Corollary 4.9 (Thm. 4.10 and Cor. 4.11, respectively) interpolate exactly the function f (u, respectively) in at least $p+1$ points lying on the boundary of D. The exact location of the points depend on the conformal map φ_D. This fact follows from the definition of q_p given in the proof of Theorem 4.7 and the relations $u = \mathrm{Re}\, f$ and $Q_p = \mathrm{Re}\, q_p$.

5. Application: exponential convergence of Trefftz *hp*-dGFEM

In this section, we outline how to apply the estimates of Corollary 4.11 and prove exponential convergence of a *Trefftz hp-dGFEM* for the mixed Laplace boundary value problem (BVP), *i.e.* a FEM with discontinuous, piecewise harmonic, polynomial basis functions on a geometrically graded mesh. We establish exponential convergence with rate $O(\exp(-b\sqrt{N}))$, for some $b > 0$, in terms of the overall number N of degrees of freedom. This result is an improvement over the classical rate $O(\exp(-b\sqrt[3]{N}))$ shown for inhomogeneous problems in [2,4]. This improvement is due to the use of harmonic polynomials, instead of complete polynomials, in the trial spaces. Indeed, as it was already observed *e.g.* in [28], page 38, the $(2p+1)$-dimensional space of harmonic polynomials of degree at most p enjoys the same approximation properties (when approximating harmonic functions) as the space of continuous polynomials of the same degree, which has higher dimension $\frac{1}{2}(p+1)(p+2)$. This is the reason of the better asymptotic properties (both in h and in p) of Trefftz methods compared to classic schemes, and is reflected in the *hp*-analysis performed here.

Since we rely on the *hp*-dGFEM theory from [39], we restrict ourselves to the case of (straight) polygonal domains and meshes comprising (straight) triangles or parallelograms. The extension to curvilinear domains and mesh elements would require to develop, for such elements, several tools as polynomial *hp*-inverse estimates, scaling estimates of Sobolev seminorms, and approximation estimates for linear and bilinear polynomials near corners. This goes beyond the scope of this paper.

5.1. The Laplace BVP

Without further explanation, we use the notation for the weighted Sobolev spaces ($H^{m,l}_{\underline{\beta}}(\Omega)$) and the countably normed spaces ($\mathcal{B}^{\ell}_{\underline{\beta}}(\Omega)$ and $\mathcal{C}^{\ell}_{\underline{\beta}}(\Omega)$) from [2], Section 2, along with the analyticity and analytic continuation results given in [2–5].

Let $\Omega \subset \mathbb{R}^2$ be a bounded, Lipschitz polygon with corners $\mathbf{c}_\nu$, $1 \le \nu \le n_a$, whose boundary is partitioned into a Dirichlet and a Neumann boundary $\Gamma^{[0]}$ and $\Gamma^{[1]}$, respectively, such that the interiors of $\Gamma^{[0]}$ and $\Gamma^{[1]}$ do not overlap and $\overline{\Gamma}^{[0]} \cup \overline{\Gamma}^{[1]} = \partial\Omega$. Moreover, we assume that $\Gamma^{[0]}$ has positive 1-dimensional measure. Consider the following (well-posed) boundary value problem: given $g^{[i]}$, $i = 0, 1$, find $u \in H^1(\Omega)$ such that

$$\Delta u = 0 \qquad \text{in } \Omega, \tag{5.1a}$$

$$\gamma_0 u\big|_{\Gamma^{[0]}} = g^{[0]} \qquad \text{on } \Gamma^{[0]}, \quad \gamma_1 u\big|_{\Gamma^{[1]}} = g^{[1]} \qquad \text{on } \Gamma^{[1]}. \tag{5.1b}$$

Here, γ_0 and γ_1 denote trace and normal derivative operators, respectively.

There exists a weight vector $\underline{\beta} \in (0,1)^{n_a}$ such that, if $g^{[i]} \in \mathcal{B}^{\frac{3}{2}-i}_{\underline{\beta}}(\Gamma^{[i]})$, $i = 0, 1$, problem (5.1) admits a unique solution u which belongs to $\mathcal{C}^2_{\underline{\beta}}(\Omega)$, [2], Theorem 3.5. Moreover, as in [2], page 841, it can be proved that there exist two constants $C_u > 0$ and $d_u \ge 1$ such that

$$|(D^\alpha u)(\mathbf{x}_0)| \le C_u \left(\frac{d_u}{\Phi(\mathbf{x}_0)}\right)^k k! \qquad \forall \mathbf{x}_0 \in \Omega,\ \alpha \in \mathbb{N}_0^2,\ |\alpha| = k \ge 1, \tag{5.2}$$

where $\Phi(\mathbf{x}_0) := \prod_{\nu=1}^{n_a} \min\{1, |\mathbf{x}_0 - \mathbf{c}_\nu|\}$, thus u admits a real analytic continuation to the set

$$\mathcal{N}(u) := \bigcup_{\mathbf{x}_0 \in \overline{\Omega} \setminus \bigcup_{\nu=1}^{n_a} \mathbf{c}_\nu} \left\{ \mathbf{x} \in \mathbb{R}^2 :\ |\mathbf{x} - \mathbf{x}_0| < \frac{\Phi(\mathbf{x}_0)}{2 d_u} \right\} \subset \mathbb{R}^2. \tag{5.3}$$

5.2. Trefftz *hp*-dGFEM

We now formulate the *hp*-dGFEM discretisation of the BVP (5.1) on *geometric mesh families* $\mathfrak{M}_\sigma = \{\mathcal{T}^\ell_\sigma\}_{\ell=1}^\infty$ in Ω, with increasing number ℓ of layers and geometric grading factor $0 < \sigma < 1$.

5.2.1. Geometric meshes

Given $\ell \in \mathbb{N}$, the mesh $\mathcal{T}^\ell_\sigma$ is a partition of the domain Ω into open triangles or parallelograms Ω^ℓ_{ij} (such that $\overline{\Omega} = \bigcup_{i,j} \overline{\Omega}^\ell_{ij}$ and $\Omega^\ell_{ij} \cap \Omega^\ell_{i'j'} = \emptyset$ if $(i,j) \ne (i',j')$). The elements are grouped in *layers*, denoted by $\mathcal{L}^\ell_{\sigma,i}$, $1 \le i \le \ell$, such that

$$\mathcal{T}^\ell_\sigma = \bigcup_{i=1}^{\ell} \mathcal{L}^\ell_{\sigma,i} = \left\{ \Omega^\ell_{ij} :\ 1 \le i \le \ell,\ 1 \le j \le \widehat{J}(i) \right\},$$

where $\widehat{J}(i) \ge 1$ is the number of the elements in ith layer $\mathcal{L}^\ell_{\sigma,i}$. Given an element $\Omega^\ell_{ij} \in \mathcal{T}^\ell_\sigma$, the index i denotes the layer Ω^ℓ_{ij} belongs to, and j identifies it among the $\widehat{J}(i)$ elements belonging to the ith layer. We say that $\mathcal{T}^\ell_\sigma$ is a geometric mesh if it belongs to a family $\mathfrak{M}_\sigma = \{\mathcal{T}^\ell_\sigma\}_{\ell=1}^\infty$ that satisfies the assumptions (GM1)–(GM4) listed below.

For every element Ω^ℓ_{ij}, we define the following parameters: $h^\ell_{ij} := \mathrm{diam}(\Omega^\ell_{ij})$, ρ^ℓ_{ij} and $\mathbf{x}^\ell_{ij}$ the radius and the centre, respectively, of the largest ball inscribed in Ω^ℓ_{ij}, and $r^\ell_{ij} := \min_{1 \le \nu \le n_a} d(\mathbf{c}_\nu, \Omega^\ell_{ij})$ its distance from the nearest corner of Ω.

Assumption 5.1. The family $\mathfrak{M}_\sigma = \{\mathcal{T}_\sigma^\ell\}_{\ell=1}^\infty$ satisfies the following conditions.

(GM1) The elements are *uniformly shape-regular* triangles/parallelograms: $\exists\ 0 < \kappa_1 \le 1/2$, independent of σ, ℓ, i and j, such that, for all $\mathcal{T}_\sigma^\ell \in \mathfrak{M}_\sigma$ and $\Omega_{ij}^\ell \in \mathcal{T}_\sigma^\ell$,

$$\rho_{ij}^\ell \ge \kappa_1\, h_{ij}^\ell > 0.$$

(GM2) The distance r_{ij}^ℓ between an element Ω_{ij}^ℓ and the nearest corner of Ω depends geometrically on its layer index i: $\exists\ 0 < \kappa_{2-} \le \kappa_{2+} < \infty$, independent of σ, ℓ, i and j, such that for all $\mathcal{T}_\sigma^\ell \in \mathfrak{M}_\sigma$ and $\Omega_{ij}^\ell \in \mathcal{T}_\sigma^\ell$, with $1 \le i < \ell$,

$$\kappa_{2-}\, \sigma^i \le r_{ij}^\ell \le \kappa_{2+}\, \sigma^i.$$

The ℓth layer is the set of the elements abutting at domain corners (*i.e.*, $r_{ij}^\ell = 0 \Leftrightarrow i = \ell$).

(GM3) The size of an element Ω_{ij}^ℓ depends geometrically on its layer index i: $\exists\ 0 < \kappa_{3-} \le \kappa_{3+} < \infty$, independent of σ, ℓ, i and j, such that for all $\mathcal{T}_\sigma^\ell \in \mathfrak{M}_\sigma$ and $\Omega_{ij}^\ell \in \mathcal{T}_\sigma^\ell$,

$$\kappa_{3-}\, \sigma^i \le h_{ij}^\ell \le \kappa_{3+}\, \sigma^i.$$

(GM4) For $\ell \ge 2$, $\mathcal{T}_\sigma^\ell$ is obtained from $\mathcal{T}_\sigma^{\ell-1}$ by only refining the elements in the layer $\mathcal{L}_{\sigma,\ell-1}^{\ell-1}$ adjacent to the domain corners, forming two new layers $\mathcal{L}_{\sigma,\ell-1}^\ell$ and $\mathcal{L}_{\sigma,\ell}^\ell$. Equivalently, the elements of $\mathcal{L}_{\sigma,i}^\ell$ are uniquely defined for all $\ell \ge i+1$:

$$\mathcal{L}_{\sigma,i}^\ell = \mathcal{L}_{\sigma,i}^{\ell'} \quad \forall i \in \{1, 2, \dots, \min(\ell, \ell') - 1\}; \qquad \mathcal{L}_{\sigma,\ell}^\ell = \bigcup_{i=\ell}^{\ell'} \mathcal{L}_{\sigma,i}^{\ell'} \quad \forall \ell' > \ell \ge 1. \tag{5.4}$$

Note that (GM2) and (GM3) imply that the diameter of an element Ω_{ij}^ℓ is proportional to its distance from the domain corners:

$$\frac{\kappa_{3-}}{\kappa_{2+}}\, r_{ij}^\ell \le h_{ij}^\ell \le \frac{\kappa_{3+}}{\kappa_{2-}}\, r_{ij}^\ell \qquad 1 \le i < \ell,\ 1 \le j \le \widehat{J}(\ell). \tag{5.5}$$

Using (GM1) and (GM3), we can control the area $|\Omega_{ij}^\ell|$ of each element: for all $\Omega_{ij}^\ell \in \mathcal{T}_\sigma^\ell$, $\ell \in \mathbb{N}$,

$$(h_{ij}^\ell)^2 \ge |\Omega_{ij}^\ell| \ge \left| B_{\rho_{ij}^\ell}(\mathbf{x}_{ij}^\ell) \right| = \pi(\rho_{ij}^\ell)^2 \ge \pi(\kappa_1 h_{ij}^\ell)^2 \ge \pi(\kappa_1\kappa_{3-})^2\sigma^{2i}.$$

Moreover, (GM2) and (GM3) imply

$$\bigcup_{j=1}^{\widehat{J}(i)} \Omega_{ij}^\ell \subseteq \bigcup_{\nu=1}^{n_a} \left(B_{\max\limits_{1\le j\le \widehat{J}(i)} (r_{ij}^\ell + h_{ij}^\ell)}(\mathbf{c}_\nu) \right) \subseteq \bigcup_{\nu=1}^{n_a} \left(B_{(\kappa_{2+}+\kappa_{3+})\sigma^i}(\mathbf{c}_\nu) \right) \qquad 1 \le i \le \ell,$$

from which $\left| \bigcup_{j=1}^{\widehat{J}(i)} \Omega_{ij}^\ell \right| \le n_a \pi (\kappa_{2+} + \kappa_{3+})^2 \sigma^{2i}$. Therefore, the number of elements per layer is uniformly bounded in i:

$$\widehat{J}(i) \le \frac{\left| \bigcup_{j=1}^{\widehat{J}(i)} \Omega_{ij}^\ell \right|}{\min_{1\le j\le \widehat{J}(i)} |\Omega_{ij}^\ell|} \le n_a \Big(\frac{\kappa_{2+} + \kappa_{3+}}{\kappa_1 \kappa_{3-}} \Big)^2 =: J^*(\mathfrak{M}_\sigma) \qquad 1 \le i \le \ell,\ \ell \in \mathbb{N}. \tag{5.6}$$

5.2.2. hp-subspaces on $\mathfrak{M}_\sigma$

For a positive integer p, let $\mathbb{P}_p(D)$ be the space of bivariate real polynomials of degree at most p on a domain $D \subset \mathbb{R}^2$. Define the spaces $\mathcal{S}^p(\mathcal{T}_\sigma^\ell)$ of *discontinuous*, piecewise polynomial functions of degree at most p on $\mathcal{T}_\sigma^\ell$:

$$\mathcal{S}^p(\mathcal{T}_\sigma^\ell) := \left\{ v \in L^2(\Omega) : \, v\big|_{\Omega_{ij}^\ell} \in \mathbb{P}_p(\Omega_{ij}^\ell) \text{ for every } \Omega_{ij}^\ell \in \mathcal{T}_\sigma^\ell \right\}, \tag{5.7}$$

and its subspace of discontinuous, piecewise harmonic polynomials (*i.e.*, the Trefftz subspace):

$$\mathcal{S}^{p,\Delta}(\mathcal{T}_\sigma^\ell) := \left\{ v \in \mathcal{S}^p(\mathcal{T}_\sigma^\ell) : \, \Delta\big(v|_{\Omega_{ij}^\ell}\big) = 0 \text{ for every } \Omega_{ij}^\ell \in \mathcal{T}_\sigma^\ell \right\}. \tag{5.8}$$

For the sake of simplicity, we confine ourselves to the case where the same polynomial degree is used in every element of the mesh; the results below can be extended to more sophisticated degree distributions. For example, in the elements adjacent to the domain corners, the use of linear polynomials on triangles and bilinear polynomials on parallelograms preserves the exponential convergence. Thus, if quadrilateral elements are used near the domain corners, we will choose $p \geq 2$ in order to include bilinear functions in the trial space. Polynomial degrees linearly decreasing with the layer index will also give the same convergence.

Lemma 5.2. *If the family $\mathfrak{M}_\sigma$ satisfies Assumption 5.1, for all $p, \ell \geq 1$, we have*

$$\dim\big(\mathcal{S}^p(\mathcal{T}_\sigma^\ell)\big) \leq J^*(\mathfrak{M}_\sigma)\,\frac{(p+1)(p+2)\ell}{2} \;=\; O(p^2\ell), \qquad \dim\big(\mathcal{S}^{p,\Delta}(\mathcal{T}_\sigma^\ell)\big) \leq J^*(\mathfrak{M}_\sigma)\,(2p+1)\ell \;=\; O(p\ell),$$

where $J^*(\mathfrak{M}_\sigma)$ is defined in (5.6) and is independent of ℓ and p.

Proof. The mesh $\mathcal{T}_\sigma^\ell$ has at most $J^*(\mathfrak{M}_\sigma)$ elements in each layer $\mathcal{L}_{\sigma,i}^\ell$, $1 \leq i \leq \ell$, thus at most $\ell J^*(\mathfrak{M}_\sigma)$ elements in total. Thus, $\dim(\mathbb{P}_p(\Omega_{ij}^\ell)) = (p+1)(p+2)/2$ and $\dim(\mathbb{P}_p(\Omega_{ij}^\ell) \cap \{v : \Delta v = 0\}) = 2p+1$ imply the assertion. □

5.2.3. hp-dGFEM

We consider both the *symmetric interior penalty* (SIP) and the *non symmetric interior penalty* (NIP) methods introduced, respectively, in [6,10], and in [7,34] (see [1] for a survey of interior penalty and other dGFEM for elliptic problems).

For a given mesh $\mathcal{T}_\sigma^\ell \in \mathfrak{M}_\sigma$ on Ω, let $V_p(\mathcal{T}_\sigma^\ell)$ be either of the subspaces defined in (5.7) and (5.8). For simplicity, we denote here by K a generic element of $\mathcal{T}_\sigma^\ell$, instead of using the more detailed notation Ω_{ij}^ℓ. Let $\mathcal{E}_{\text{int}}$ be the set the interior edges of $\mathcal{T}_\sigma^\ell$, *i.e.*, the intersections between two elements of $\mathcal{T}_\sigma^\ell$ that have positive 1-dimensional measure; moreover, let $\mathcal{E}_D$ be the set of the edges of $\mathcal{T}_\sigma^\ell$ lying on $\Gamma^{[0]}$, and set $\mathcal{E}_{\text{int},D} := \mathcal{E}_{\text{int}} \cup \mathcal{E}_D$. For a piecewise smooth function v, we define jumps and averages across the edges $e \in \mathcal{E}_{\text{int},D}$:

$$\begin{aligned} \{\!\{v\}\!\}|_e &:= \frac{v|_K + v|_{K'}}{2}, \qquad [\![v]\!]|_e := v|_K \mathbf{n}_K + v|_{K'}\mathbf{n}_{K'} \qquad e \in \overline{K} \cap \overline{K}', \quad K, K' \in \mathcal{T}_\sigma^\ell, \\ \{\!\{v\}\!\}|_e &:= [\![v]\!]|_e := v|_e \qquad e \subseteq \Gamma^{[0]}, \end{aligned}$$

where $\mathbf{n}_K$ is the outgoing unit normal on ∂K. We set $h_K := \operatorname{diam}(K)$ and define the meshwidth function $h : \mathcal{E}_{\text{int},D} \to \mathbb{R}$ as $h(\mathbf{x}) := \min\{h_K : \mathbf{x} \in K \in \mathcal{T}_\sigma^\ell\}$.

For $\theta \in \{1, -1\}$ and $v, w \in V_p(\mathcal{T}_\sigma^\ell)$, define the two bilinear forms and linear functionals

$$\begin{aligned} B_{\mathcal{T}}^\theta(v,w) &:= \sum_{K \in \mathcal{T}_\sigma^\ell} \int_K \nabla v \cdot \nabla w \,\mathrm{d}\mathbf{x} + \sum_{e \in \mathcal{E}_{\text{int},D}} \int_e \Big(-\{\!\{\nabla v\}\!\} \cdot [\![w]\!] + \theta [\![v]\!] \cdot \{\!\{\nabla w\}\!\} + \mathtt{a}[\![v]\!] \cdot [\![w]\!] \Big)\,\mathrm{d}s, \\ L_{\mathcal{T}}^\theta(w) &:= \int_{\Gamma^{[1]}} g^{[1]} w \,\mathrm{d}s + \theta \int_{\Gamma^{[0]}} g^{[0]}\, \gamma^{[1]} w \,\mathrm{d}s + \int_{\Gamma^{[0]}} \mathtt{a}\, g^{[0]}\, w \,\mathrm{d}s. \end{aligned}$$

Here, $\mathtt{a}$ is the discontinuity stabilisation function given by $\mathtt{a}(\mathbf{x}) := \alpha\, p^2/h(\mathbf{x})$, where $\alpha > 0$ is a parameter independent of h and p. Fixing $\theta \in \{1, -1\}$, the hp-dGFEM reads: find $u_p^\theta \in V_p(\mathcal{T}_\sigma^\ell)$ such that

$$B_{\mathcal{T}}^\theta(u_p^\theta, v_p) = L_{\mathcal{T}}^\theta(v_p) \quad \forall v_p \in V_p(\mathcal{T}_\sigma^\ell). \tag{5.9}$$

The method defined in (5.9) is SIP, for $\theta = -1$, and NIP, for $\theta = 1$.

Integrating by parts the volume term in $B_{\mathcal{T}}^\theta$, and using the fact that the discrete functions are harmonic, one would obtain an "ultra weak formulation", containing only skeleton and boundary integrals. For Helmholtz BVPs, a similar approach has been first adopted in [8]; the corresponding hp-version is analysed in [21].

We recall the following result from [39], where the mesh-dependent norm $\|\cdot\|_{dG}$ is defined by

$$\|w\|_{dG}^2 := \sum_{K\in\mathcal{T}_\sigma^\ell} \|\nabla w\|_{L^2(K)}^2 + \sum_{e\in\mathcal{E}_{\mathrm{int},D}} \left\|\sqrt{\mathtt{a}}\, [\![w]\!]\right\|_{L^2(e)}^2 \qquad w \in V_p(\mathcal{T}_\sigma^\ell).$$

Proposition 5.3 ([39], Thm. 2.3.7, Cor. 2.4.2)**.** *Let $\underline{\beta} \in (0,1)^{n_a}$ be such that the analytical solution u to (5.1) belongs to $\mathcal{C}_{\underline{\beta}}^2(\Omega)$. If either $\theta = 1$ and α is positive, or $\theta = -1$ and α is sufficiently large, then the hp-dGFEM (5.9) admits a unique solution.*

Moreover, let $\pi_{\mathcal{T}} : H_{\underline{\beta}}^{2,2}(\Omega) \to V_p(\mathcal{T}_\sigma^\ell)$ be an arbitrary operator such that, for every element $K \in \mathcal{T}_\sigma^\ell$, there exist at least two zeros of $\eta := u - \pi_{\mathcal{T}} u$ in $\overline{K}$. For $\theta = \pm 1$ (with sufficiently large α, if $\theta = -1$), it holds

$$\|u - u_p^\theta\|_{dG}^2 \le C\, p^2 \left\{ \sum_{K\in\mathcal{T}_\sigma^\ell} |\eta|_{H^1(K)}^2 + \sum_{K\in\mathcal{T}_\sigma^\ell\setminus\mathcal{K}_\sigma^\ell} h_K^2\, |\eta|_{H^2(K)}^2 + \sum_{K\in\mathcal{K}_\sigma^\ell} h_K^{2(1-\beta_{[K]})}\, |\eta|_{H_{\underline{\beta}}^{2,2}(K)}^2 \right\} \tag{5.10}$$

where $C > 0$ is independent of σ, ℓ and p. Here, $\mathcal{K}_\sigma^\ell := \mathcal{L}_{\sigma,\ell}^\ell \subseteq \mathcal{T}_\sigma^\ell$ designates the set of elements abutting at domain corners and, for any $K \in \mathcal{K}_\sigma^\ell$, $\beta_{[K]} := \sup\{\beta_\nu : \mathbf{c}_\nu \in \partial K\}$.

5.3. Exponential convergence of *hp*-dGFEM

We apply the approximation estimates proved in Section 4.2 to establish exponential convergence of the hp-dGFEM scheme. We begin with the following lemma, which puts in relation the domain of analyticity of u and the geometric mesh family $\mathfrak{M}_\sigma$.

Lemma 5.4. *Let $\mathfrak{M}_\sigma$ be a family of geometric meshes $\mathcal{T}_\sigma^\ell$ on Ω satisfying Assumption 5.1, and let u be the solution of the BVP (5.1) on Ω. Then, there exists $\delta_* > 0$ depending on u (only through d_u in (5.2)), σ and $\mathfrak{M}_\sigma$, such that u is analytic in $\Omega_{ij}^\ell + B_{\delta_* h_{ij}^\ell} = \{\mathbf{x} \in \mathbb{R}^2 : d(\mathbf{x}, \Omega_{ij}^\ell) < \delta_* h_{ij}^\ell\}$ for all $\Omega_{ij}^\ell \in \mathcal{T}_\sigma^\ell \setminus \mathcal{K}_\sigma^\ell$, $\mathcal{T}_\sigma^\ell \in \mathfrak{M}_\sigma$.*

Proof. We define the domain parameter

$$E_\Omega := \min\left\{1, \min_{1\le\nu\ne\nu'\le n_a} \frac{|\mathbf{c}_\nu - \mathbf{c}_{\nu'}|}{2}\right\},$$

which depends only on the position of the corners of Ω, and consider an arbitrary element $\Omega_{ij}^\ell \in \mathcal{T}_\sigma^\ell \setminus \mathcal{K}_\sigma^\ell$.

First, we consider the case $\Omega_{ij}^\ell \subseteq B_{E_\Omega}(\mathbf{c}_{\nu'})$ for some $\nu' \in \{1, \ldots, n_a\}$. Fix $\mathbf{x} \in \Omega_{ij}^\ell$; by the triangular inequality $|\mathbf{x} - \mathbf{c}_\nu| \ge |\mathbf{c}_\nu - \mathbf{c}_{\nu'}| - |\mathbf{x} - \mathbf{c}_{\nu'}| \ge E_\Omega$, for all $\nu \ne \nu'$. The definition of Φ and the bound (5.5) give

$$\Phi(\mathbf{x}) = \prod_{\nu=1}^{n_a} \max\{1, |\mathbf{x} - \mathbf{c}_\nu|\} \ge |\mathbf{x} - \mathbf{c}_{\nu'}|\, E_\Omega^{n_a-1} \ge r_{ij}^\ell E_\Omega^{n_a-1} \ge h_{ij}^\ell \frac{\kappa_{2-}}{\kappa_{3+}} E_\Omega^{n_a-1} \quad \forall \mathbf{x} \in \Omega_{ij}^\ell.$$

This, together with the definition of the domain of analyticity $\mathcal{N}(u)$ in (5.3) and of the parameter d_u in (5.2), implies that

$$\frac{d\big(\Omega_{ij}^\ell, \partial\mathcal{N}(u)\big)}{h_{ij}^\ell} \ge \frac{\inf_{\mathbf{x}\in\Omega_{ij}^\ell} \Phi(\mathbf{x})}{2d_u\, h_{ij}^\ell} \ge \frac{\kappa_{2-}\, E_\Omega^{n_a-1}}{2d_u\, \kappa_{3+}} =: \delta_1.$$

Now consider the case when $\Omega_{ij}^{\ell} \not\subseteq B_{E_\Omega}(\mathbf{c}_\nu)$ for any $\nu \in \{1,\dots,n_a\}$. Fix $\mathbf{x} \in \Omega_{ij}^{\ell}$ such that $|\mathbf{x}-\mathbf{c}_\nu| \geq E_\Omega$ for every $\nu \in \{1,\dots,n_a\}$. Thus, by (GM2) and (GM3),

$$E_\Omega \leq \inf_{1\leq\nu\leq n_a} |\mathbf{x}-\mathbf{c}_\nu| \leq r_{ij}^{\ell} + h_{ij}^{\ell} \leq (\kappa_{2+}+\kappa_{3+})\sigma^{\mathrm{i}} \quad \Rightarrow \quad i \leq \left\lceil \frac{\log \frac{E_\Omega}{\kappa_{2+}+\kappa_{3+}}}{\log \sigma} \right\rceil =: i^*,$$

i.e., Ω_{ij}^{ℓ} belongs to one of the first i^* layers. The elements in first i^* layers are uniquely defined in all the meshes with at least i^*+1 layers, see (GM4). Thus we can define

$$\delta_2 := \min_{\substack{i,j,\ell\in\mathbb{N}\\ i\leq i^*,\, i<\ell,\, 1\leq j\leq \widehat{J}(i)}} \frac{d\big(\Omega_{ij}^{\ell}, \partial\mathcal{N}(u)\big)}{h_{ij}^{\ell}},$$

which is positive since is the minimum of a finite number of positive values, although ℓ can take any value in $\mathbb{N}$.

Therefore, if $\delta_* := \min\{\delta_1,\delta_2\}$, for any element $\Omega_{ij}^{\ell} \in \mathcal{T}_\sigma^{\ell} \setminus \mathcal{K}_\sigma^{\ell}$, for any $\mathcal{T}_\sigma^{\ell} \in \mathfrak{M}_\sigma$, the solution u is analytic in $\Omega_{ij}^{\ell} + B_{\delta_* h_{ij}^{\ell}}$. Note that δ_* depends on u through d_u, on σ through i^*, but is independent of i, j and ℓ. □

Theorem 5.5. *Consider the solution $u \in \mathcal{C}^2_{\underline{\beta}}(\Omega)$ of the Laplace mixed BVP (5.1) and its approximation $u_p^\theta \in V_p(\mathcal{T}_\sigma^{\ell}) := \mathcal{S}^{p,\Delta}(\mathcal{T}_\sigma^{\ell})$ computed with the Trefftz hp-dGFEM (5.9) (with $\alpha > 0$, if $\theta = 1$, or α sufficiently large, if $\theta = -1$) on a family $\mathfrak{M}_\sigma$ of geometric meshes $\mathcal{T}_\sigma^{\ell}$ satisfying Assumption 5.1. Assume uniform polynomial degree $p = \ell$ and define $N := \dim(V_p(\mathcal{T}_\sigma^{\ell}))$. Then, u_p^θ converges exponentially to u: there exist $b, C > 0$ (depending on u, Ω, σ and $\mathfrak{M}_\sigma$, but independent of $p = \ell$) such that*

$$\|u - u_p^\theta\|_{dG} \leq C \exp(-b\sqrt{N}).$$

Proof. Since $N = O(p\ell)$ by Lemma 5.2 and $p = \ell$, we have to prove $\left\|u - u_p^\theta\right\|_{dG} \leq C\mathrm{e}^{-b\ell}$. Thanks to Proposition 5.3, we only need to define an operator $\pi_{\mathcal{T}} : H^{2,2}_{\underline{\beta}}(\Omega) \to V_p(\mathcal{T}_\sigma^{\ell})$ with suitable approximation and interpolation properties. We treat separately the elements $\Omega_{\ell j}^{\ell}$ adjacent to a domain corner ($\Omega_{\ell j}^{\ell} \in \mathcal{K}_\sigma^{\ell}$) and the remaining ones (Ω_{ij}^{ℓ} with $1 \leq i \leq \ell - 1$).

In the elements $\Omega_{\ell j}^{\ell} \in \mathcal{K}_\sigma^{\ell}$, we define $\pi_{\mathcal{T}}(u)$ as the (piecewise) linear or bilinear interpolant of u at the vertices of $\Omega_{\ell j}^{\ell}$, if $\Omega_{\ell j}^{\ell}$ is a triangle or a parallelogram, respectively. Then, $\pi_{\mathcal{T}}(u)$ is obviously harmonic. Using ([35], Lem. 4.16, Lem. 4.25) see also ([39], Lem. 2.5.2), and taking into account (GM3) with $i = \ell$, the contribution of the elements $\Omega_{\ell j}^{\ell} \in \mathcal{K}_\sigma^{\ell}$ to the right-hand side of (5.10) has exponential order of convergence in $p = \ell$ (for some $b \geq (1 - \max_\nu \beta_\nu)(-\log\sigma)$).

Consider now the elements $\Omega_{ij}^{\ell} \in \mathcal{T}_\sigma^{\ell} \setminus \mathcal{K}_\sigma^{\ell}$. For any $\Omega_{ij}^{\ell} \in \mathcal{T}_\sigma^{\ell} \setminus \mathcal{K}_\sigma^{\ell}$, due to Lemma 5.4, the solution u is analytic in $\Omega_{ij}^{\ell} + B_{\delta_* h_{ij}^{\ell}}$, for some δ_* independent of i, j and ℓ. Define the corresponding scaled element $D := \widehat{\Omega}_{ij}^{\ell} := \{\widehat{\mathbf{x}} := (\mathbf{x} - \mathbf{x}_{ij}^{\ell})/h_{ij}^{\ell} \in \mathbb{R}^2 : \mathbf{x} \in \Omega_{ij}^{\ell}\}$ and the scaled solution $\widehat{u}(\widehat{\mathbf{x}}) := u(h_{ij}^{\ell}\widehat{\mathbf{x}} + \mathbf{x}_{ij}^{\ell})$. The scaled element satisfies Assumption 1.1 with $\rho = \rho_{ij}^{\ell}/h_{ij}^{\ell} \geq \kappa_1$ and for any $0 < \rho_0 < \rho$, due to (GM1) and the convexity of Ω_{ij}^{ℓ}. The domain of analyticity of $\widehat{u}$ is dilated in the same way, therefore the hypothesis of Corollary 4.11 are verified with $\delta = \delta_*$. Thus, there exists a harmonic polynomial $\widehat{Q}_p$ of degree at most p such that

$$\left|\widehat{u} - \widehat{Q}_p\right|_{H^m(\widehat{\Omega}_{ij}^{\ell})} \leq C\mathrm{e}^{-bp} \qquad m = 0, 1, 2,$$

for some constants C and $b > 0$ depending only on κ_1, δ_* (which, again, depends on $\mathfrak{M}_\sigma$, σ and u, through d_u) and $\|\widehat{u}\|_{W^{1,\infty}(\widehat{\mathcal{N}}(u))}$ (which, again, depends only on u and Ω). We scale $\widehat{Q}_p$ back to Ω_{ij}^{ℓ} and define the local interpolant as

$$\big(\pi_{\mathcal{T}} u(\mathbf{x})\big)|_{\Omega_{ij}^{\ell}} := \widehat{Q}_p\big((\mathbf{x} - \mathbf{x}_{ij}^{\ell})/h_{ij}^{\ell}\big).$$

Remark 4.13 guarantees that the interpolation is exact in at least $p+1$ points on the boundary of Ω_{ij}^{ℓ}. From the usual scaling of Sobolev seminorms $|\cdot|_{H^k(\Omega_{ij}^{\ell})} \le C(h_{ij}^{\ell})^{1-k}\,|\cdot|_{H^k(\widehat{\Omega}_{ij}^{\ell})}$, we obtain

$$\sum_{1\le i\le \ell-1,\,1\le j\le \widehat{J}(i)} \left(|\eta|^2_{H^1(\Omega_{ij}^{\ell})} + (h_{ij}^{\ell})^2\,|\eta|^2_{H^2(\Omega_{ij}^{\ell})} \right) \le C\ell \mathrm{e}^{-b\ell},$$

with C and b depending only on u, σ, Ω and $\mathfrak{M}_\sigma$. Here we used the fact that the number of elements in $\mathcal{T}_\sigma^\ell$ is $O(\ell)$, as proved in Lemma 5.2.

The assertion is then obtained by combining the last bound with the one previously obtained for the elements incident to the corners, using $\ell = O(\sqrt{N})$, and noting that $\pi_{\mathcal{T}}(u)$ interpolates u at least in two points per element, thus Proposition 5.3 applies, and the *hp*-dGFEM error is bounded by the approximation error. □

Remark 5.6. In standard FEM convergence analysis, approximation estimates are derived only for *few reference elements*, which are then mapped to the "physical" mesh elements. For Trefftz schemes this is usually not possible: spaces made of harmonic functions (or harmonic polynomials) are not invariant under general affine mappings but only under similarity transformations, thus estimates that are *uniform for every element shape* must be proven, up to scaling and isometry only. This is one of the reasons for deriving the approximation estimates of Section 4.2; however, they hold in much more generality than what we used in the *hp*-dGFEM analysis (*i.e.*, for star-shaped elements instead of triangles and parallelograms). The explicit dependence on the geometry, only through ρ and ρ_0, shows that these bounds are uniform for all the elements of a shape-regular family of meshes. The obstruction to extending the results of Theorem 5.5 to more general (*e.g.*, curvilinear) geometries is not due to the new approximation estimates, but only to the limitations of the existing theory on quasi-optimality of dGFEM solutions.

Appendix A. Proof of the upper bound (3.2) for non convex domains

We consider first the case of polygonal domains (with straight sides) in Section A.1, then we extend the result to more general curvilinear domains in Section A.2. We recall that we are assuming $0 < h \le 1$.

A.1. Polygonal domains

Denote by $\{\alpha_k^C\pi\}_{k=1}^{n_C}$ and $\{\alpha_k^{NC}\pi\}_{k=1}^{n_{NC}}$ the convex and non convex internal angles, respectively, of D, by $\{w_k^C\}_{k=1}^{n_C}$ and $\{w_k^{NC}\}_{k=1}^{n_{NC}}$ the corresponding vertices and set

$$z_k^C = \varphi^{-1}(w_k^C) \qquad \text{for } k=1,\dots,n_C, \qquad\qquad z_k^{NC} = \varphi^{-1}(w_k^{NC}) \qquad \text{for } k=1,\dots,n_{NC}.$$

The following relations hold (see the left plot of Fig. 3 for the geometrical meaning of the parameters):

$$\begin{aligned} &0 \le \alpha_k^C \le 1, && 0 \le \beta_k^C := 1-\alpha_k^C \le 1 && k=1,\dots,n_C,\\ &1 < \alpha_k^{NC} \le 2, && -1 \le \beta_k^{NC} := 1-\alpha_k^{NC} < 0 && k=1,\dots,n_{NC},\\ &\sum_{k=1}^{n_C}\beta_k^C + \sum_{k=1}^{n_{NC}}\beta_k^{NC} = 2. \end{aligned}$$

Recalling the definition $\xi = \frac{2}{\pi}\arcsin\frac{\rho_0}{1-\rho}$ for non convex D, from Assumption 1.1 and ([22], Rem. A.1), we have

$$\alpha_k^C \ge \xi, \qquad \beta_k^C \le 1-\xi \qquad k=1,\dots,n_C, \qquad\qquad 2-\alpha_k^{NC} \ge \xi, \qquad \beta_k^{NC} \ge \xi - 1 \qquad k=1,\dots,n_{NC}. \tag{A.1}$$

One of the crucial ideas of this proof is the fact that the sum of the β_k's corresponding to an arbitrary set V of *consecutive* vertices of a polygon $P \subset B_{1-\rho}$, which is star-shaped with respect to B_{ρ_0}, satisfies the inequalities

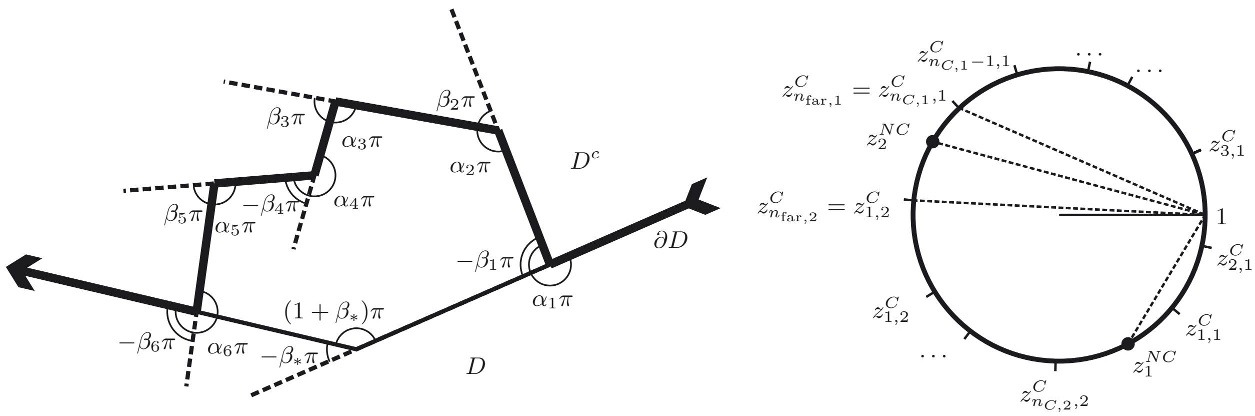

FIGURE 3. *Left plot*: the geometrical meaning of the parameters α_k's and β_k's. The α_k's are all positive, while the β_k's are positive only on convex corners: $\beta_1, \beta_4, \beta_6 < 0 < \beta_2, \beta_3, \beta_5$. The angle between the first and the last segment can be computed by summing over the β_k's, *i.e.*, $\beta_* := \sum_{k=1}^{6} \beta_k$. In this example β_* is negative since the corresponding internal angle is non convex. *Right plot*: the location of the pre-vertices z_k's in case *ii)* with two non consecutive non convex corners. The four dashed segments have lengths $\max\left\{\left|1-\overline{z}_1^{NC}\right|;\ \left|1-\overline{z}^C_{n_{\mathrm{far},1}}\right|\right\} \le \min\left\{\left|1-\overline{z}_2^{NC}\right|;\ \left|1-\overline{z}^C_{n_{\mathrm{far},2}}\right|\right\}$.

$\xi - 1 \le \sum_{k\in V} \beta_k \le 1-\xi$. It will be therefore necessary to take into account the ordering of the vertices along the polygon.

As in Section 3.2, fix $w_h \in L_h$ and set $z_h = \varphi^{-1}(w_h) \in \partial B_{1+h}$; thus $z_h = (1+h)\mathrm{e}^{\mathrm{i}\theta}$, for some $\theta \in [-\pi, \pi]$. Define $z = \mathrm{e}^{\mathrm{i}\theta}$, $w = \varphi(z)$, and denote by S the (straight) segment of length h connecting z and z_h. From (2.5) and (2.4) we have

$$|w_h - w| = |\varphi(z_h) - \varphi(z)| \le \int_S |\varphi'(y)|\,\mathrm{d}y \le \varphi'(\infty)\int_S \frac{1}{|y|^2}\prod_{k=1}^{n_C}\left|y-\overline{z}_k^C\right|^{1-\alpha_k^C}\prod_{k=1}^{n_{NC}}\left|y-\overline{z}_k^{NC}\right|^{1-\alpha_k^{NC}}\,\mathrm{d}y$$
$$\le \int_S \prod_{k=1}^{n_C}\left|y-\overline{z}_k^C\right|^{\beta_k^C}\prod_{k=1}^{n_{NC}}\left|y-\overline{z}_k^{NC}\right|^{\beta_k^{NC}}\,\mathrm{d}y =: T, \tag{A.2}$$

since $\varphi'(\infty) < 1$ and $|y| \ge 1$. Finally, for any $y \in S$,

$$\left|y-\overline{z}_k^C\right| \le 2+h \qquad k = 1,\dots,n_C, \qquad\qquad \left|y-\overline{z}_k^{NC}\right| \le 2+h \qquad k=1,\dots,n_{NC}.$$

With no loss of generality, we consider $\theta = 0$, *i.e.*, $z = 1$, $z_h = 1+h$ and S lies in the positive real axis.

We consider separately four situations.

i) *D has only one non convex angle.* In this case, the term T in (A.2) can be bounded by

$$T \le (2+h)^{\sum_k \beta_k^C}\int_S \left|y-\overline{z}_1^{NC}\right|^{\beta_1^{NC}}\,\mathrm{d}y \le 27\int_S\left|y-\overline{z}_1^{NC}\right|^{\beta_1^{NC}}\,\mathrm{d}y,$$

since $h \le 1$ and $\sum_k \beta_k^C \le 3$, due to $\sum_k \beta_k^C + \beta_1^{NC} = 2$ and $\beta_1^{NC} \ge -1$.
Since $\beta_1^{NC} < 0$ and $\left|y-\overline{z}_1^{NC}\right| \ge |y-1|$ for all $y \in S$, we have

$$T \le 27\int_S |y-1|^{\beta_1^{NC}}\,\mathrm{d}y = 27\int_0^h s^{\beta_1^{NC}}\,\mathrm{d}s = 27\frac{h^{\beta_1^{NC}+1}}{\beta_1^{NC}+1} \le \frac{27h^\xi}{\xi},$$

because $\beta_1^{NC} > -1$, $h \le 1$ and $\beta_1^{NC}+1 \ge \xi$.

ii) *D has only two non convex angles, and these angles are non consecutive.* Assume $\left|1-\overline{z}_1^{NC}\right| \leq \left|1-\overline{z}_2^{NC}\right|$. The points $\overline{z}_1^{NC}$ and $\overline{z}_2^{NC}$ separate the points in $\{\overline{z}_k^C\}_{k=1}^{n_C}$ into two blocks, $\{\overline{z}_{j,1}^C\}_{j=1}^{n_{C,1}}$ and $\{\overline{z}_{j,2}^C\}_{j=1}^{n_{C,2}}$. We set

$$n_{\mathrm{far},1} = \underset{j=1,\dots,n_{C,1}}{\arg\max} \left|1-\overline{z}_{j,1}^C\right|, \qquad n_{\mathrm{far},2} = \underset{j=1,\dots,n_{C,2}}{\arg\max} \left|1-\overline{z}_{j,2}^C\right|,$$

and assume

$$\left|1-\overline{z}_{n_{\mathrm{far},1}}^C\right| \leq \left|1-\overline{z}_{n_{\mathrm{far},2}}^C\right|;$$

consequently, as can be inferred from the right plot in Figure 4,

$$\left|1-\overline{z}_{n_{\mathrm{far},1}}^C\right| \leq \left|1-\overline{z}_2^{NC}\right|. \tag{A.3}$$

We have

$$\begin{aligned}
T &= \int_S \left|y-\overline{z}_1^{NC}\right|^{\beta_1^{NC}} \left|y-\overline{z}_2^{NC}\right|^{\beta_2^{NC}} \prod_{j=1}^{n_{C,1}} \left|y-\overline{z}_{j,1}^C\right|^{\beta_{j,1}^C} \prod_{j=1}^{n_{C,2}} \left|y-\overline{z}_{j,2}^C\right|^{\beta_{j,2}^C} \,\mathrm{d}y \\
&\leq \int_S \left|y-\overline{z}_1^{NC}\right|^{\beta_1^{NC}} \left|y-\overline{z}_2^{NC}\right|^{\beta_2^{NC}} \left|y-\overline{z}_{n_{\mathrm{far},1}}^C\right|^{\sum_j \beta_{j,1}^C} \left|y-\overline{z}_{n_{\mathrm{far},2}}^C\right|^{\sum_j \beta_{j,2}^C} \,\mathrm{d}y \\
&\overset{(A.3),\, \beta_{j,1}^C \geq 0}{\leq} \int_S \left|y-\overline{z}_1^{NC}\right|^{\beta_1^{NC}} \left|y-\overline{z}_2^{NC}\right|^{\beta_2^{NC}+\sum_j \beta_{j,1}^C} \left|y-\overline{z}_{n_{\mathrm{far},2}}^C\right|^{\sum_j \beta_{j,2}^C} \,\mathrm{d}y.
\end{aligned}$$

a) If $\beta_2^{NC} + \sum_j \beta_{j,1}^C \geq 0$,

$$T \leq (2+h)^{2-\beta_1^{NC}} \int_S \left|y-\overline{z}_1^{NC}\right|^{\beta_1^{NC}} \,\mathrm{d}y \leq 27 \frac{h^{\beta_1^{NC}+1}}{\beta_1^{NC}+1} \leq \frac{27h^\xi}{\xi}.$$

b) If $\beta_2^{NC} + \sum_j \beta_{j,1}^C < 0$, we write

$$T \leq (2+h)^{\sum_j \beta_{j,2}^C} \int_S \left|y-\overline{z}_1^{NC}\right|^{\beta_1^{NC}+\beta_2^{NC}+\sum_j \beta_{j,1}^C} \,\mathrm{d}y.$$

If we prove that

$$\beta^* := \beta_1^{NC} + \beta_2^{NC} + \sum_j \beta_{j,1}^C \geq \xi - 1, \tag{A.4}$$

then $\sum_j \beta_{j,2}^C = 2 - \beta^* < 3$, from which

$$T \leq 27 \int_S \left|y-\overline{z}_1^{NC}\right|^{\beta^*} \,\mathrm{d}y \leq \frac{27h^\xi}{\xi}. \tag{A.5}$$

In order to conclude, we only need to prove (A.4).

Consider the counterclockwise oriented part of ∂D formed by the consecutive (oriented) sides s_i, $i = 1,\dots,m := n_{C,1}+3$, abutting w_1^{NC}, $w_{j,1}^C$, $j=1,\dots,n_{C,1}$, and w_2^{NC}. Let ℓ_i be the oriented line containing s_i, $i=1,\dots,m$. Since D is star-shaped with respect to B_{ρ_0}, then B_{ρ_0} lies in the intersection of the half planes lying on the left of the ℓ_i's.

Let K be the infinite cone obtained by intersecting the right half planes generated by ℓ_1 and ℓ_m. Its opening is $(1+\beta^*)\pi < \pi$, with $\beta^* < 0$ (*cf.* the left plots of Figs. 3 and 4).

Define $D' := D \setminus \overline{K}$; D' only has one non convex angle of internal amplitude $(1-\beta^*)\pi$. The ball B_{ρ_0} lies on the left side of every edge of D', thus this domain is star-shaped with respect to B_{ρ_0} and $D' \subset B_\rho$, by the bounds (A.1) we have $1+\beta^* \geq \xi$ (*cf.* the left plot of Fig. 4). Therefore, $\beta^* \geq \xi - 1 > -1$, which concludes the argument.

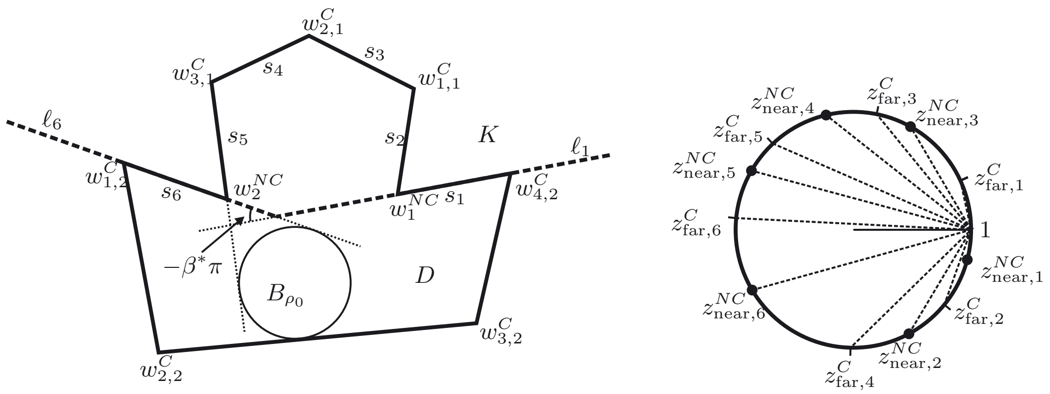

FIGURE 4. *Left plot*: an example of a domain D with two non consecutive, non convex corners (w_1^{NC} and w_1^{NC}). In this case $n_{C,1} = 3$, $n_{C,2} = 4$ and $m = 6$. The cone K is delimited by the dashed lines ℓ_1 and ℓ_6. The ball B_{ρ_0} lies on the left of every edge when ∂D is run counterclockwise. The set $D' = D \setminus \overline{K}$ is star-shaped with respect to B_{ρ_0}. *Right plot*: the pre-vertices $z_k^\cdot$ satisfy the ordering relation (A.7). Notice that $z^{NC}_{\mathrm{near},1}$ and $z^C_{\mathrm{far},n}$ (in the picture $n = 6$) do not enter the relation. Therefore it is not relevant which one between $z^C_{\mathrm{far},1}$ and $z^{NC}_{\mathrm{near},1}$ is closest to 1. The number of pre-vertices lying in the upper and in the lower half of the complex plane does not affect the ordering of the distances.

iii) *D has only two non convex angles, and these angles are consecutive.* We have

$$T \le (2+h)^{\sum_j \beta_j^C} \int_S \left|y - \overline{z}_1^{NC}\right|^{\beta_1^{NC}+\beta_2^{NC}} \,\mathrm{d}y,$$

assuming again $\left|1 - \overline{z}_1^{NC}\right| \le \left|1 - \overline{z}_2^{NC}\right|$. If we prove that

$$\beta^* := \beta_1^{NC} + \beta_2^{NC} \ge \xi - 1 > -1, \tag{A.6}$$

then $\sum_j \beta_j^C = 2 - \beta^* < 3$, from which we get again (A.5).

For the proof of (A.6), consider the part of ∂D formed by the $m = 3$ consecutive sides abutting w_1^{NC} and w_2^{NC}, the rest of the proof is identical to that of (A.4).

iv) *D has more than two non convex angles.* We generalise the argument of step (ii). Assume that we have n blocks of consecutive convex angles, alternated by n blocks of consecutive convex angles. With a similar notation as before, we can write

$$T \le \int_S \prod_{i=1}^{n} \left[\prod_{j=1}^{n_{NC,i}} \left|y - \overline{z}_{j,i}^{NC}\right|^{\beta_{j,i}^{NC}} \prod_{j=1}^{n_{C,i}} \left|y - \overline{z}_{j,i}^{C}\right|^{\beta_{j,i}^{C}} \right] \mathrm{d}y.$$

Setting, for $i = 1, \ldots, n$,

$$n_{\mathrm{far},i} = \operatorname*{arg\,max}_{j=1,\ldots,n_{C,i}} \left|1 - \overline{z}_{j,i}^{C}\right|, \qquad n_{\mathrm{near},i} = \operatorname*{arg\,min}_{j=1,\ldots,n_{NC,i}} \left|1 - \overline{z}_{j,i}^{NC}\right|,$$

we can bound T as

$$T \le \int_S \prod_{i=1}^{n} \left[\left|y - \overline{z}_{\mathrm{near},i}^{NC}\right|^{\sum_j \beta_{j,i}^{NC}} \left|y - \overline{z}_{\mathrm{far},i}^{C}\right|^{\sum_j \beta_{j,i}^{C}} \right] \mathrm{d}y =: \int_S P(y)\,\mathrm{d}y.$$

We order the blocks in such a way that

$$\left|1-\overline{z}^{NC}_{\mathrm{near},i}\right| \le \left|1-\overline{z}^{NC}_{\mathrm{near},i+1}\right| \qquad \text{and} \qquad \left|1-\overline{z}^{C}_{\mathrm{far},i}\right| \le \left|1-\overline{z}^{C}_{\mathrm{far},i+1}\right| \qquad i=1,\ldots,n-1,$$

consequently (see the left plot in Fig. 4),

$$\left|1-\overline{z}^{C}_{\mathrm{far},i}\right| \le \left|1-\overline{z}^{NC}_{\mathrm{near},i+1}\right| \qquad i=1,\ldots,n-1. \tag{A.7}$$

Thus, we have

$$P(y) \le \quad \left|y-\overline{z}^{NC}_{\mathrm{near},1}\right|^{\sum_j \beta^{NC}_{j,i}} \left[\prod_{i=1}^{n-1} \left|y-\overline{z}^{NC}_{\mathrm{near},i+1}\right|^{\sum_j \beta^{C}_{j,i}+\sum_j \beta^{NC}_{j,i+1}}\right] (2+h)^{\sum_j \beta^{C}_{j,n}}.$$

We consider the term with index $n-1$ in the product and look at its exponent $(\sum_j \beta^{C}_{j,n-1}+\sum_j \beta^{NC}_{j,n})$,

a) if it is ≥ 0, we combine the term with index $n-1$ with the following term (the last one) and obtain

$$P(y) \le \left|y-\overline{z}^{NC}_{\mathrm{near},1}\right|^{\sum_j \beta^{NC}_{j,1}} \left[\prod_{i=1}^{n-2} \left|y-\overline{z}^{NC}_{\mathrm{near},i+1}\right|^{\sum_j \beta^{C}_{j,i}+\sum_j \beta^{NC}_{j,i+1}}\right] (2+h)^{\sum_j \beta^{C}_{j,n-1}+\sum_j \beta^{NC}_{j,n}+\sum_j \beta^{C}_{j,n}},$$

b) if it is < 0, we combine the term with index $n-1$ with the previous term (the one with index $n-2$) and obtain

$$\begin{aligned} P(y) \le{}& \left|y-\overline{z}^{NC}_{\mathrm{near},1}\right|^{\sum_j \beta^{NC}_{j,1}} \left[\prod_{i=1}^{n-3} \left|y-\overline{z}^{NC}_{\mathrm{near},i+1}\right|^{\sum_j \beta^{C}_{j,i}+\sum_j \beta^{NC}_{j,i+1}}\right] \\ &\times \left|y-\overline{z}^{NC}_{\mathrm{near},n-1}\right|^{\sum_j \beta^{C}_{j,n-2}+\sum_j \beta^{NC}_{j,n-1}+\sum_j \beta^{C}_{j,n-1}+\sum_j \beta^{NC}_{j,n}} (2+h)^{\sum_j \beta^{C}_{j,n}}. \end{aligned}$$

Then, we proceed backward, considering the term of with index $i = n-2$ and, depending on whether its exponent is ≥ 0 or < 0, we combine it either with the following term or with the previous term the way we did before, and so on, until the term $i = 1$. We end up with three factors in the upper bound of $P(y)$: the first one is $\left|y-\overline{z}^{NC}_{\mathrm{near},1}\right|^{\sum_j \beta^{NC}_{j,1}}$, the third one is $(2+h)^B$, with $B > 0$, and the second one is $\left|y-\overline{z}^{NC}_{\mathrm{near},2}\right|^{A}$. If $A \ge 0$, we conclude as in step *ii)*, case *a)*, while if $A < 0$, in order to conclude as in step *ii)*, case *b)*, we need to prove that

$$\beta^* := \sum_j \beta^{NC}_{j,1} + A \ge \xi - 1 > -1. \tag{A.8}$$

Since the blocks of (convex an non convex) angles corresponding to the β's entering the expression $\beta^{NC}_{j,1}+A$ are consecutive, the proof of (A.8) can be carried out as the proof of (A.4).

The proof in the polygonal case is complete.

A.2. Domains with non-polygonal boundaries

We begin with the following trigonometric lemma.

Lemma A.7. *Let $0 < R_1 < R_2 < +\infty$, and fix two distinct straight lines ℓ_1 and ℓ_2 that are tangent to B_{R_1} in the two points y_1 and y_2 ($y_1 \ne y_2$), respectively. We denote by x_1 the intersection between the circle ∂B_{R_2} and the line ℓ_1 such that, in a counterclockwise orientation, x_1, y_1 and 0 appear with this ordering as vertices of a triangle. Symmetrically, we denote by x_2 the intersection between the circle ∂B_{R_2} and the line ℓ_2 such that, in a clockwise orientation, x_2, y_2 and 0 appear with this ordering. We denote by C_η the infinite convex sector with opening $\eta\pi$, $0 \le \eta \le 1$, defined by the two half lines generating at the origin and passing through x_1 and x_2, respectively (see the left plot in Fig. 5).*

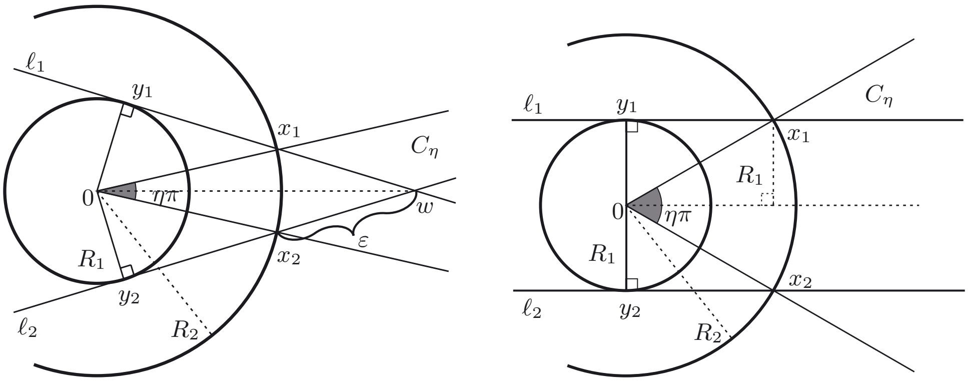

FIGURE 5. *Left plot:* the geometric configuration in Lemma A.7. *Right plot*: the limit case $\eta = \frac{2}{\pi}\arcsin\frac{R_1}{R_2}$ in the proof.

If $\eta < \frac{2}{\pi}\arcsin\frac{R_1}{R_2}$, then ℓ_1 and ℓ_2 intersect at a point w that lies in the interior of C_η. Moreover, if $|w| > R_2$ and if we define $\varepsilon := |w - x_1| = |w - x_2| > 0$, η is related to ε, R_1 and R_2 by the following formula:

$$0 < \eta(\varepsilon, R_1, R_2) = \frac{2}{\pi}\arccos\frac{R_2^2 + \varepsilon\sqrt{R_2^2 - R_1^2}}{R_2\sqrt{\varepsilon^2 + R_2^2 + 2\varepsilon\sqrt{R_2^2 - R_1^2}}} < \frac{2}{\pi}\arcsin\frac{R_1}{R_2}\cdot \tag{A.9}$$

For $\varepsilon > 0$, the function $\varepsilon \mapsto \eta(\varepsilon, R_1, R_2)$ is continuous and strictly increasing. For $R_2 > R_1$, the function $R_2 \mapsto \eta(\varepsilon, R_1, R_2)$ is continuous and strictly decreasing.

Proof. We consider the limit case $\eta = \frac{2}{\pi}\arcsin\frac{R_1}{R_2} < 1$. Then, $R_2 \sin\frac{\eta\pi}{2} = R_1$ and, as depicted in the right plot of Figure 5, the lines ℓ_1 and ℓ_2 are parallel to each other. Therefore, whenever η is smaller than this threshold value, ℓ_1 and ℓ_2 will intersect on the central half line of C_η.

We apply Pitagoras' theorem twice: to the triangle of vertices x_1, y_1 and 0, yielding $|x_1 - y_1|^2 = R_2^2 - R_1^2$, and then to the triangle of vertices w, y_1 and 0, leading to

$$|w|^2 = R_1^2 + \left(\varepsilon + \sqrt{R_2^2 - R_1^2}\right)^2 = \varepsilon^2 + R_2^2 + 2\varepsilon\sqrt{R_2^2 - R_1^2}.$$

From the law of cosines applied to the triangle of vertices w, x_1 and 0, we obtain $2|w|R_2\cos\frac{\eta\pi}{2} = |w|^2 + R_2^2 - \varepsilon^2$ from which the identity in the assertion follows.

The monotonicity in dependence of ε and R_2 can be verified by computing the derivative of the expression in (A.9). The last inequality in the assertion follows from

$$\eta(\varepsilon, R_1, R_2) < \lim_{\varepsilon\to\infty}\eta(\varepsilon, R_1, R_2) = \frac{2}{\pi}\arccos\sqrt{1 - \frac{R_1^2}{R_2^2}} = \frac{2}{\pi}\arcsin\frac{R_1}{R_2}$$

which uses the monotonicity of η as a function of ε, and the identity $\sin\arccos\sqrt{1 - t^2} = |t|$. □

We can now complete the proof of the bound (3.2) in the general case. In order to do that, we will construct a polygon $P_\varepsilon \supset D$, star-shaped with respect to B_{ρ_0}. The maximal distance $\sup_{w\in\partial P_\varepsilon} d(w, \partial D)$ will be made arbitrarily small, and the parameter ξ (defined in Thm. 3.1) relative to P_ε will converge to the one relative to D. Then, invoking the result of Section A.1 completes the proof in the case of non-polygonal domains.

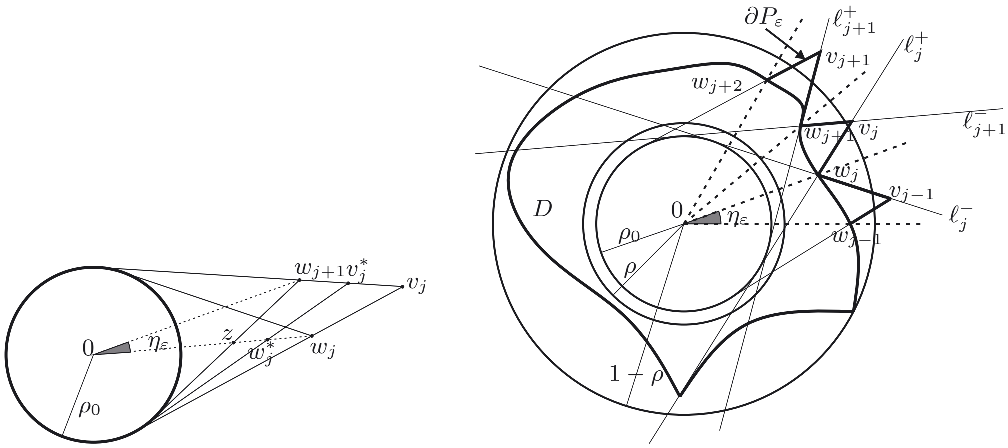

FIGURE 6. *Left plot*: the comparison of the cases $|w_j| = |w_{j+1}|$ and $|w_j^*| < |w_{j+1}|$ for a fixed w_{j+1}. In the second case, the constructed point v_j^* is closer to w_{j+1} than in the first case, namely, $|v_j^* - w_{j+1}| < |v_j - w_{j+1}|$. *Right plot*: the construction of the star-shaped polygon P_ε enclosing the non-polygonal, non-convex domain D.

Consider a domain D satisfying Assumption 1.1. Fix $\varepsilon > 0$. Define an integer $N \in \mathbb{N}$ such that $\eta_\varepsilon := \frac{2}{N} \leq \eta(\varepsilon, \rho_0, 1-\rho)$, where $\eta(\cdot,\cdot,\cdot)$ was defined in formula (A.9). We select the points $w_j \in \partial D$, $j = 1, \ldots, N$, that have complex argument (namely, angular polar coordinate) equal to $\theta_j \pi := j\eta_\varepsilon \pi$ for $j = 1, \ldots, N$. (In this proof we assume that all the indices j are taken modulus N.)

Let ℓ_j^+ and ℓ_j^- be the two tangent (straight) lines to B_{ρ_0} passing through w_j and such that, sitting in w_j and looking at B_{ρ_0}, ℓ_j^+ is on the left and ℓ_j^- is on the right (notice that the two lines do not coincide, since $\rho_0 < \rho \leq |w_j|$), see the right plot in Figure 6.

Consider the two lines ℓ_j^+ and ℓ_{j+1}^-. If $|w_j| = |w_{j+1}|$, then they satisfy the assumptions of Lemma A.7 (since $R_2 \mapsto \frac{2}{\pi} \arcsin \frac{\rho_0}{R_2}$ is monotonically decreasing, the definition of η_ε made above guarantees the needed bound for any value of $\rho_0 \leq R_2 = |w_j| \leq 1-\rho$). Thus they intersect at a point v_j such that $|v_j| \geq |w_j| = |w_{j+1}|$ and whose complex argument satisfies $\theta_j \pi \leq \arg(v_j) \leq \theta_{j+1}\pi$. Moreover, $|v_j - w_j| = |v_j - w_{j+1}| \leq \varepsilon$, due to the monotonicity of the map $\eta(\varepsilon, \rho_0, |w_j|) \mapsto \varepsilon$.

On the other hand, if $|w_j| < |w_{j+1}|$ (the opposite case is analogous), then v_j lies closer to w_{j+1} than in the previous case (see the left plot in Fig. 6), therefore, in all the situations, we have

$$d(v_j, \partial D) \leq \min\{|v_j - w_j|, |v_j - w_{j+1}|\} \leq \varepsilon. \tag{A.10}$$

Notice that, given $|w_{j+1}|$, ρ_0 and η_ε, due to the star-shapedness assumption, $|w_j|$ can not be arbitrarily small, namely it can not trespass the point denoted with z in the left plot in Figure 6.

Notice that every domain which is star-shaped with respect to B_{ρ_0} and such that its boundary contains the point w_j (*e.g.*, the domain D satisfies these requests) can not cross the segments $[v_{j-1}, w_j]$ and $[w_j, v_j]$.

Now we define the polygon P_ε with $2N$ sides whose vertices are $w_1, v_1, w_2, \ldots, w_N, v_N$. Every edge of P_ε is part of either ℓ_j^+ or ℓ_j^-. The polygon P_ε satisfies the following conditions:

(i) P_ε is star-shaped with respect to B_{ρ_0}, since the continuation of each of its edges is tangent to B_{ρ_0} and (in a counterclockwise orientation of ∂P_ε) leaves B_{ρ_0} on its left,

(ii) $D \subseteq P_\varepsilon$, as it contains *every* domain D' star-shaped with respect to B_{ρ_0} and satisfying $\{w_j\}_{j=1}^N \subset \partial D'$,

(iii) for every $w \in \partial P_\varepsilon$, $d(w, \partial D) \leq \varepsilon$, in fact, since $\{w_j\}_{j=1}^N \subset \partial D$, the maximum distance from ∂D is achieved in one of the vertices v_j and this is controlled by the bound (A.10).

Then we can conclude as in the convex case. Fix $w_h \in L_h = L_h[\overline{D}]$. Choose $0 < \varepsilon < \rho$ and define the polygon P_ε as above (so that $P_\varepsilon \subset B_1$ and $\varphi'_{P_\varepsilon}(\infty) \leq 1$). Then, $w_h \in L_{h'}[\overline{P_\varepsilon}]$ with $h' \leq h$, as a consequence of Lemma 2.2. Let $z_{h'} := \varphi^{-1}_{P_\varepsilon}(w_h) = (1+h')\mathrm{e}^{\mathrm{i}\theta}$, and define $z := \mathrm{e}^{\mathrm{i}\theta}$, $\xi_\varepsilon := \frac{2}{\pi}\arcsin\frac{\rho_0}{1-\rho+\varepsilon} < \xi$. Then, from Section A.1,

$$d(w_h, \partial D) \leq d\big(w_h, \varphi_{P_\varepsilon}(z)\big) + d\big(\varphi_{P_\varepsilon}(z), \partial D\big) = |\varphi_{P_\varepsilon}(z_{h'}) - \varphi_{P_\varepsilon}(z)| + d\big(\varphi_{P_\varepsilon}(z), \partial D\big) \leq 27\xi_\varepsilon^{-1}h^{\xi_\varepsilon} + \varepsilon.$$

Since this is true for every $0 < \varepsilon < \rho$, by taking the limit for $\varepsilon \to 0$, we get $d(w_h, \partial D) \leq \frac{27}{\xi}h^\xi$ for all $w_h \in L_h$.

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