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CORRECTOR RESULTS FOR A PARABOLIC PROBLEM WITH A MEMORY EFFECT

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Abstract. The aim of this paper is to provide the correctors associated to the homogenization of a parabolic problem describing the heat transfer. The results here complete the earlier study in [Jose, *Rev. Roumaine Math. Pures Appl.* **54** (2009) 189–222] on the asymptotic behaviour of a problem in a domain with two components separated by an ε-periodic interface. The physical model established in [Carslaw and Jaeger, The Clarendon Press, Oxford (1947)] prescribes on the interface the condition that the flux of the temperature is proportional to the jump of the temperature field, by a factor of order ε^γ. We suppose that $-1 < \gamma \leq 1$. As far as the energies of the homogenized problems are concerned, we consider the cases $-1 < \gamma < 1$ and $\gamma = 1$ separately. To obtain the convergence of the energies, it is necessary to impose stronger assumptions on the data. As seen in [Jose, *Rev. Roumaine Math. Pures Appl.* **54** (2009) 189–222] and [Faella and Monsurrò, Topics on Mathematics for Smart Systems, World Sci. Publ., Hackensack, USA (2007) 107–121] (also in [Donato *et al.*, *J. Math. Pures Appl.* **87** (2007) 119–143]), the case $\gamma = 1$ is more interesting because of the presence of a memory effect in the homogenized problem.

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1. Introduction

This paper is devoted to the study of corrector results associated to the homogenization of the parabolic problem studied in [21]. In this work, the domain $\Omega \subset \mathbb{R}^n$ is given by $\Omega = \Omega_{1\varepsilon} \cup \overline{\Omega_{2\varepsilon}}$. By taking $Y = Y_1 \cup \overline{Y_2}$ to be the reference cell, $\Omega_{1\varepsilon}$ and $\Omega_{2\varepsilon}$ are respectively, the connected and disconnected union of ε-periodic translated sets of εY_1 and εY_2. On the other hand, $\Gamma^\varepsilon := \partial\Omega_{2\varepsilon}$ is the interface separating the two components

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with $\partial\Omega \cap \Gamma^\varepsilon = \emptyset$. For $T > 0$ and $-1 < \gamma \leq 1$, consider the following problem:

$$\begin{cases} {u_{1\varepsilon}}' - \operatorname{div}(A^\varepsilon \nabla u_{1\varepsilon}) = f_{1\varepsilon} + P_1^{\varepsilon *}(g) & \text{in } \Omega_{1\varepsilon} \times]0,T[\,, \\ {u_{2\varepsilon}}' - \operatorname{div}(A^\varepsilon \nabla u_{2\varepsilon}) = f_{2\varepsilon} & \text{in } \Omega_{2\varepsilon} \times]0,T[\,, \\ A^\varepsilon \nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -A^\varepsilon \nabla u_{2\varepsilon} \cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon \times]0,T[\,, \\ A^\varepsilon \nabla u_{1\varepsilon} \cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon (u_{1\varepsilon} - u_{2\varepsilon}) & \text{on } \Gamma^\varepsilon \times]0,T[\,, \\ u_{1\varepsilon} = 0 & \text{on } \partial\Omega \times]0,T[\,, \\ u_{1\varepsilon}(x,0) = U^0_{1\varepsilon} \ \text{ in } \Omega_{1\varepsilon}, \ u_{2\varepsilon}(x,0) = U^0_{2\varepsilon} \ \text{ in } \Omega_{2\varepsilon}, & \end{cases} \tag{1.1}$$

where $n_{i\varepsilon}$ is the unitary outward normal to $\Omega_{i\varepsilon}$ $(i = 1, 2)$, P_1^ε is a suitable extension operator and $P_1^{\varepsilon *}$ its adjoint. The coefficient A^ε is assumed to be independent of t, uniformly bounded in $L^\infty(\Omega)$ and satisfying the ellipticity condition given by (2.8)(i). Moreover, h^ε is an oscillating periodic function which is bounded in $L^\infty(\Gamma^\varepsilon)$. Meanwhile, the data $f_{i\varepsilon}$ and $U^0_{i\varepsilon}$ $(i = 1, 2)$, belongs to $L^2(0,T;\ L^2(\Omega))$ and $L^2(\Omega_{i\varepsilon})$ respectively.

This paper completes the investigation of the asymptotic behaviour of a parabolic problem earlier considered by Faella and Monsurrò [19] and Jose [21]. Our aim is to find corrector results in order to improve the weak approximations from [21]. The notion of corrector matrix, which was introduced by Tartar in [30,31], plays an important role in homogenization theory.

Let us recall the convergence results from [21]. If $\theta_i = \frac{|Y_i|}{|Y|}$ $(i = 1, 2)$ is the proportion of the material occupying $\Omega_{i\varepsilon}$ and

$$\begin{cases} (\widetilde{U^0_{1\varepsilon}}, \widetilde{U^0_{2\varepsilon}}) \rightharpoonup (\theta_1 U^0_1, \theta_2 U^0_2) & \text{weakly in } L^2(\Omega) \times L^2(\Omega), \\ (\widetilde{f_{1\varepsilon}}, \widetilde{f_{2\varepsilon}}) \rightharpoonup (\theta_1 f_1, \theta_2 f_2) & \text{weakly in } L^2(0,T;\ L^2(\Omega)) \times L^2(0,T;\ L^2(\Omega)), \end{cases} \tag{1.2}$$

where $\tilde{}$ denotes the zero extension to the whole of Ω, then for all $-1 < \gamma \leq 1$,

$$\begin{cases} (i)\ P_1^\varepsilon u_{1\varepsilon} \rightharpoonup u_1 & \text{weakly in } L^2(0,T;\ H^1_0(\Omega)), \\ (ii)\ \widetilde{u_{1\varepsilon}} \rightharpoonup \theta_1 u_1 & \text{weakly* in } L^\infty(0,T;\ L^2(\Omega)), \\ (iii)\ \widetilde{u_{2\varepsilon}} \rightharpoonup u_2 & \text{weakly* in } L^\infty(0,T;\ L^2(\Omega)), \\ (iv)\ \varepsilon^{\frac{\gamma}{2}} \|u_{1\varepsilon} - u_{2\varepsilon}\|_{L^2(0,T;\ L^2(\Gamma^\varepsilon))} < c. & \end{cases}$$

Furthermore,

$$\begin{cases} (i)\ A^\varepsilon \widetilde{\nabla u_{1\varepsilon}} \rightharpoonup A^0 \nabla u_1 & \text{weakly in } L^2(0,T;\ [L^2(\Omega)]^n), \\ (ii)\ A^\varepsilon \widetilde{\nabla u_{2\varepsilon}} \rightharpoonup 0 & \text{weakly in } L^2(0,T;\ [L^2(\Omega)]^n), \end{cases}$$

where A^0 is the homogenized matrix obtained by Cioranescu and Saint Jean Paulin in [9], for the Laplace problem in a perforated domain with a Neumann condition on the boundary of the holes.

The homogenized (limit) problems satisfied by the couple (u_1, u_2) are different for the two cases $-1 < \gamma < 1$ and $\gamma = 1$. We describe first the case $-1 < \gamma < 1$, where $u_2 = \theta_2 u_1$ and u_1 is the unique solution of the homogenized problem

$$\begin{cases} u_1' - \ \operatorname{div}\,(A^0 \nabla u_1) = \theta_1 f_1 + \theta_2 f_2 + g & \text{in } \Omega \times]0,T[\,, \\ u_1 = 0 & \text{on } \partial\Omega \times]0,T[\,, \\ u_1(0) = \theta_1 U^0_1 + \theta_2 U^0_2 & \text{in } \Omega. \end{cases} \tag{1.3}$$

For this case, the corrector result given in Theorem 3.4 and proved in Section 6 states the following convergences:

$$\begin{cases} (i)\ \widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}} \to u_1 & \text{in } \mathcal{C}^0([0,T]; L^2(\Omega)), \\ (ii)\ \lim\limits_{\varepsilon\to 0} \|\nabla u_{1\varepsilon} - C^\varepsilon \nabla u_1\|_{L^2(0,T;[L^1(\Omega_{1\varepsilon})]^n)} = 0, \\ (iii)\ \lim\limits_{\varepsilon\to 0} \|\nabla u_{2\varepsilon}\|_{L^2(0,T;[L^2(\Omega_{2\varepsilon})]^n)} = 0, \end{cases}$$

where $(u_{1\varepsilon}, u_{2\varepsilon})$ is the solution of problem (1.1) and C^ε is the corrector matrix associated with A^0.

To prove that result, stronger assumptions on the data than (1.2) are necessary, in order to establish the convergence of the energy of the ε-problem to that of the homogenized one. That was also the case in the homogenization of the wave and heat equations in a fixed domain Ω done by Bensoussan *et al.* in [2] and Brahim-Otsman *et al.* in [3] (see also [11]).

For the first case $-1 < \gamma < 1$, we make here the stronger assumptions that $f_{i\varepsilon} \in L^2(0,T;\ L^2(\Omega))$, $U^0_{i\varepsilon} \in L^2(\Omega_{i\varepsilon})$ $(i = 1,2)$, and satisfy

$$\begin{cases} (i)\ f_{i\varepsilon} \to f_i & \text{strongly in } L^2(0,T;\ L^2(\Omega)), \\ (ii)\ \widetilde{U^0_{1\varepsilon}} + \widetilde{U^0_{2\varepsilon}} \to U^0 & \text{strongly in } L^2(\Omega). \end{cases} \tag{1.4}$$

To describe the corrector results for the case $\gamma = 1$, we recall from [21] that (u_1, u_2) is the unique solution of the coupled system

$$\begin{cases} \theta_1 u_1' - \operatorname{div}(A^0 \nabla u_1) + c_h(\theta_2 u_1 - u_2) = \theta_1 f_1 + g & \text{in } \Omega \times]0,T[, \\ u_2' - c_h(\theta_2 u_1 - u_2) = \theta_2 f_2 & \text{in } \Omega \times]0,T[, \\ u_1 = 0 & \text{on } \partial\Omega \times]0,T[, \\ u_1(0) = U_1^0,\ u_2(0) = \theta_2 U_2^0 & \text{in } \Omega, \end{cases} \tag{1.5}$$

where $c_h = \dfrac{1}{|Y_2|}\displaystyle\int_\Gamma h(y)\, \mathrm{d}\sigma_y$. Solving the ODE in (1.5) and replacing u_2 in the PDE, shows that u_1 satisfies an equation of the form

$$\theta_1 u_1' - \operatorname{div}(A^0_\gamma \nabla u_1) + c_h \theta_2 u_1 - c_h^2 \theta_2 \int_0^t \mathcal{K}(t,s) u_1(s)\, \mathrm{d}s = F(x,t),$$

with $\mathcal{K}$ an exponential kernel, giving rise to a memory effect.

We now introduce the stronger assumptions on the data. We suppose that for $f_{i\varepsilon} \in L^2(0,T;\ L^2(\Omega))$ and $U^0_{i\varepsilon} \in L^2(\Omega_{i\varepsilon})$ $(i = 1,2)$, one has

$$\begin{cases} (i)\ f_{i\varepsilon} \to f_i \quad \text{strongly in } L^2(0,T;\ L^2(\Omega)), \\ (ii)\ \widetilde{U^0_{i\varepsilon}} \rightharpoonup \theta_i U^0_i \quad \text{weakly in } L^2(\Omega), \\ (iii)\ \|U^0_{1\varepsilon}\|^2_{L^2(\Omega_{1\varepsilon})} + \|U^0_{2\varepsilon}\|^2_{L^2(\Omega_{2\varepsilon})} \to \theta_1 \|U^0_1\|^2_{L^2(\Omega)} + \theta_2 \|U^0_2\|^2_{L^2(\Omega)}. \end{cases} \tag{1.6}$$

Then, assuming that Γ is of class C^2, the following corrector results for the case $\gamma = 1$ hold true:

$$\begin{cases} (i)\ \lim\limits_{\varepsilon\to 0} \|u_{1\varepsilon} - u_1\|_{\mathcal{C}^0(0,T;L^2(\Omega_{1\varepsilon}))} = 0, \\ (ii)\ \lim\limits_{\varepsilon\to 0} \|u_{2\varepsilon} - \theta_2^{-1} u_2\|_{\mathcal{C}^0(0,T;L^2(\Omega_{2\varepsilon}))} = 0, \\ (iii)\ \lim\limits_{\varepsilon\to 0} \|\nabla u_{1\varepsilon} - C^\varepsilon \nabla u_1\|_{L^2(0,T;[L^1(\Omega_{1\varepsilon})]^n)} = 0, \\ (iv)\ \lim\limits_{\varepsilon\to 0} \|\nabla u_{2\varepsilon}\|_{L^2(0,T;[L^2(\Omega_{2\varepsilon})]^n)} = 0. \end{cases}$$

As seen in Section 5, assumptions (1.4) and assumptions (1.6) are well adapted to the homogenized problems (1.3) and (1.5).

In both cases, the proof of the main results are based on a suitable upper semicontinuity-type inequality. Despite the fact that this approach is classical, we have a specific difficulty in the parabolic case because of the influence of the interface. Indeed, some compactness of the solution $(u_{1\varepsilon}, u_{2\varepsilon})$ in a space of type $\mathcal{C}^0([0,T];X)$ is needed. In the classical case of a fixed domain, such a compactness of the solution in $\mathcal{C}^0([0,T];L^2(\Omega))$ is straightforward. This is not true in perforated domains. In this case, a compactness result in $\mathcal{C}^0([0,T];H^{-1}(\Omega))$ was proved in [15], leading to a corrector result. The situation here is complicated by the fact that we have the couple of functions $u_{1\varepsilon}, u_{2\varepsilon}$. Nevertheless, we are able to prove that the sequence $\{\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}}\}$ is compact in $\mathcal{C}^0([0,T];H^{-1}(\Omega))$ (see Thm. 4.8). Moreover, for $\gamma = 1$, the sequences $\{\widetilde{u_{1\varepsilon}}\}$ and $\{\widetilde{u_{2\varepsilon}}\}$ are also separately compact in $\mathcal{C}^0([0,T];H^{-1}(\Omega))$. These compactness results play a crucial role when proving the corrector results.

For the case $\gamma = 1$ (see Step 2 of the proof of Prop. 6.8), we had to adapt to the parabolic case some technical lemmas for the elliptic and hyperbolic case [12,18]. In contrast to the hyperbolic case, the coefficient matrix A^ε is not necessarily symmetric in our case and this is a significant difference. Indeed, when proving the upper semicontinuity-type inequality there is an additional term (in both cases), which needs specific arguments (see Step 1.3 and Step 1 in Sects. 6.1 and 6.2). We refer to Remarks 6.6, 6.7 and 6.9 for more details on these technical points.

This paper is organized as follows. In Section 2, we recall the geometric and functional setting of the problem together with the homogenization results proved in [21]. In Section 3, the corrector results are stated as well as the necessary assumptions regarding the data. We also give a detailed description of these assumptions. In Section 4, we investigate the compactness of $\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}}$ in $\mathcal{C}^0([0,T];H^{-1}(\Omega))$ discussed above. Section 5 is devoted to the convergence of the energy of the ε-problems. Finally, in Section 6 we prove the corrector results stated in Section 3.

The homogenization of elliptic and hyperbolic problems in a domain with the same geometric and abstract framework as in the present paper, were already done by Monsurrò [26,27], Donato and Monsurrò [13], and Donato *et al.* [17,18]. For similar studies of problems with jump conditions in the elliptic case we refer to [1,20,23,24] and the references therein. Our results can be related to the case of parabolic problems in perforated domains that were studied by Donato and Nabil [15]. The homogenization of Neumann boundary problems in perforated domains were investigated by Cioranescu and Saint Jean Paulin in [9,10]. For associated correctors we refer to Donato *et al.* [16]. For the pioneer works on linear memory effects in the homogenization of parabolic problems, we refer to Mascarenhas in [25] and Tartar in [32]. For other homogenization of parabolic and hyperbolic problems for which memory effects occur, we also refer to the recent articles [28,29].

2. Preliminaries

We recall the geometric framework used for the homogenization of problem (1.1) in [21]. We consider an open bounded set Ω of $\mathbb{R}^n$ which is decomposed into the connected set $\Omega_{1\varepsilon}$ and the disconnected set $\Omega_{2\varepsilon}$, both of which are unions of ε^{-n} translated sets, with $\{\varepsilon\}$ a sequence of positive real numbers that converges to zero.

Let $Y =]0,\ell_1[\times ... \times]0,\ell_n[$ and let Y_1 and Y_2 be two nonempty open sets such that $Y = Y_1 \cup \overline{Y_2}$. We suppose that Y_1 is connected and Y_2 has a Lipschitz continuous boundary Γ.

For any $k \in \mathbb{Z}^n$, Y_i^k and Γ_k are the translated sets

$$Y_i^k := k_\ell + Y_i, \quad \Gamma_k := k_\ell + \Gamma \quad \text{where} \quad k_\ell = (k_1\ell_1, ..., k_n\ell_n) \quad \text{and} \quad i = 1,2.$$

For any given ε, set

$$K_\varepsilon := \{k \in \mathbb{Z}^n | \varepsilon Y_i^k \cap \Omega \neq \emptyset, \ i = 1,2\}.$$

We then define the two components of Ω and the interface respectively as follows:

$$\Omega_{i\varepsilon} := \Omega \cap \{\textstyle\bigcup_{k\in K_\varepsilon} \varepsilon Y_i^k\}, \quad i = 1,2 \quad \text{and} \quad \Gamma^\varepsilon = \partial\Omega_{2\varepsilon}.$$

Assume that

$$\partial\Omega \cap \left(\bigcup_{k\in\mathbb{Z}^n} (\varepsilon\Gamma_k) \right) = \emptyset, \tag{2.1}$$

and so $\partial\Omega \cap \Gamma^\varepsilon = \emptyset$.

Remark 2.1. The above geometric assumption is the one used in the homogenization of the parabolic problem (1.1). This gives a simpler presentation, as it was the case in the hyperbolic [17,18] and elliptic cases [12, 13,26,27]. Assumption (2.1) can be replaced by a different definition of $\Omega_{2\varepsilon}$ as the union of all the set εY_2^k such that $\overline{\varepsilon Y_2^k} \subset \Omega$. In such a case, all the previous results and those proved here are still true.

In the sequel, we will use the following notation:

- χ_ω the characteristic function of any open set $\omega \subset \mathbb{R}^n$;
- $m_\omega(v) = \frac{1}{|\omega|}\displaystyle\int_\omega v \,\mathrm{d}x$, the mean value of v over a measurable set ω;
- $\widetilde{v}$ the zero extension to $\mathbb{R}^n$ of any function v defined on $\Omega_{i\varepsilon}$ or Y_i for $i = 1, 2$.

Remark 2.2. To simplify notation, if a function v is defined on the whole of Ω, we still denote by v its restriction to $\Omega_{i\varepsilon}$ when no confusion arises. We will also use the fact that

$$\widetilde{v|_{\Omega_{i\varepsilon}}} = \chi_{\Omega_{i\varepsilon}} v, \qquad \text{for } i = 1, 2.$$

It is known that (for instance, see [8]),

$$\chi_{\Omega_{i\varepsilon}} \rightharpoonup \theta_i := \frac{|Y_i|}{|Y|} \ (i = 1, 2), \text{ weakly in } L^2(\Omega). \tag{2.2}$$

We consider the two spaces V^ε and H_γ^ε defined by

$$\begin{aligned} V^\varepsilon &:= \{v_1 \in H^1(\Omega_{1\varepsilon}) \mid v_1 = 0 \text{ on } \partial\Omega\}, \\ H_\gamma^\varepsilon &:= \{v = (v_1, v_2) \mid v_1 \in V^\varepsilon \text{ and } v_2 \in H^1(\Omega_{2\varepsilon})\}, \qquad \forall \gamma \in \mathbb{R}, \end{aligned} \tag{2.3}$$

which are Banach spaces respectively, for the norms

$$\|v_1\|_{V^\varepsilon} := \|\nabla v_1\|_{L^2(\Omega_{1\varepsilon})} \tag{2.4}$$

and

$$\|v\|_{H_\gamma^\varepsilon}^2 := \|\nabla v_1\|_{L^2(\Omega_{1\varepsilon})}^2 + \|\nabla v_2\|_{L^2(\Omega_{2\varepsilon})}^2 + \varepsilon^\gamma \|v_1 - v_2\|_{L^2(\Gamma^\varepsilon)}^2. \tag{2.5}$$

Remark 2.3. As already seen in [9,10], a uniform Poincaré inequality holds in V^ε, *i.e.*, there exists a constant $C > 0$ (independent of ε) such that for every ε

$$\|v\|_{L^2(\Omega_{1\varepsilon})} \le C\|\nabla v\|_{L^2(\Omega_{1\varepsilon})}, \quad \forall v \in V^\varepsilon.$$

On the other hand, observe that if $\gamma_1 \le \gamma_2$ then

$$\|v\|_{H_{\gamma_2}^\varepsilon}^2 \le \|v\|_{H_{\gamma_1}^\varepsilon}^2.$$

Hence in particular, for all $\gamma \le 1$ we have

$$\|v\|_{H_1^\varepsilon} \le \|v\|_{H_\gamma^\varepsilon}. \tag{2.6}$$

Let us recall the following result from [26,27] giving equivalence of norms.

Lemma 2.4 [26]**.** *There exist two positive constants C_1, C_2 (independent of ε) such that*

$$C_1\|v\|_{H_1^\varepsilon} \le \|v\|_{V^\varepsilon\times H^1(\Omega_{2\varepsilon})} \le C_2\|v\|_{H_1^\varepsilon}, \quad \forall v\in H_1^\varepsilon.$$

With this functional setting, we now can state our parabolic problem.

Suppose that

$$\begin{cases} g\in L^2(0,T;\ H^{-1}(\Omega)),\\ (U^0_{1\varepsilon},U^0_{2\varepsilon})\in L^2(\Omega_{1\varepsilon})\times L^2(\Omega_{2\varepsilon}),\\ (f_{1\varepsilon},f_{2\varepsilon})\in L^2(0,T;\ L^2(\Omega))\times L^2(0,T;\ L^2(\Omega)).\end{cases} \tag{2.7}$$

Furthermore, let A be a $n\times n$ matrix field in $(L^\infty(Y))^{n^2}$, Y-periodic and such that $\forall\lambda\in\mathbb{R}^n$ and a.e. in Y,

$$\begin{cases} (i)\ (A(x)\lambda,\lambda)\ge\alpha|\lambda|^2,\\ (ii)\ |A(x)\lambda|\le\beta\lambda,\end{cases} \tag{2.8}$$

where $\alpha,\beta\in\mathbb{R}$ with $0<\alpha<\beta$. For any $\varepsilon>0$, we set

$$A^\varepsilon(x) := A\left(\frac{x}{\varepsilon}\right)\cdot \tag{2.9}$$

We also suppose that h is a Y-periodic function satisfying

$$h\in L^\infty(\Gamma),\quad \exists h_0\in\mathbb{R}\quad \text{such that}\quad 0<h_0<h(y),\quad y\quad \text{a.e. in }\Gamma \tag{2.10}$$

and set

$$h^\varepsilon(x) := h\left(\frac{x}{\varepsilon}\right)\cdot \tag{2.11}$$

For $T>0$ and $-1<\gamma\le 1$, consider the following problem:

$$\begin{cases} {u_{1\varepsilon}}' - \operatorname{div}(A^\varepsilon\nabla u_{1\varepsilon}) = f_{1\varepsilon} + P_1^{\varepsilon *}(g) & \text{in } \Omega_{1\varepsilon}\times\,]0,T[\,,\\ {u_{2\varepsilon}}' - \operatorname{div}(A^\varepsilon\nabla u_{2\varepsilon}) = f_{2\varepsilon} & \text{in } \Omega_{2\varepsilon}\times\,]0,T[\,,\\ A^\varepsilon\nabla u_{1\varepsilon}\cdot n_{1\varepsilon} = -A^\varepsilon\nabla u_{2\varepsilon}\cdot n_{2\varepsilon} & \text{on } \Gamma^\varepsilon\times\,]0,T[\,,\\ A^\varepsilon\nabla u_{1\varepsilon}\cdot n_{1\varepsilon} = -\varepsilon^\gamma h^\varepsilon(u_{1\varepsilon}-u_{2\varepsilon}) & \text{on } \Gamma^\varepsilon\times\,]0,T[\,,\\ u_{1\varepsilon}=0 & \text{on } \partial\Omega\times\,]0,T[\,,\\ u_{1\varepsilon}(x,0)=U^0_{1\varepsilon}\ \text{ in }\Omega_{1\varepsilon},\quad u_{2\varepsilon}(x,0)=U^0_{2\varepsilon}\ \text{ in }\Omega_{2\varepsilon}, \end{cases} \tag{2.12}$$

where $n_{i\varepsilon}$ is the unitary outward normal to $\Omega_{i\varepsilon}$ $(i=1,2)$, P_1^ε is a suitable extension operator (see Lem. 4.5) and $P_1^{\varepsilon *}$ its adjoint.

By definition, for any $g\in L^2(0,T;\ H^{-1}(\Omega))$, $P_1^{\varepsilon *}g$ is given by

$$P_1^{\varepsilon *}g:\ v\in L^2(0,T;\ V^\varepsilon)\longmapsto \int_0^T \langle g, P_1^\varepsilon v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\mathrm{d}s. \tag{2.13}$$

Observe that

$$P_1^{\varepsilon *}\in\mathcal{L}(L^2(0,T;\ H^{-1}(\Omega)); L^2(0,T;\ (V^\varepsilon)')).$$

The variational formulation of problem (2.12) is

$$\begin{cases} \text{Find } u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon}) \text{ in } W^\varepsilon \text{ such that} \\ \displaystyle\int_0^T \langle u'_{1\varepsilon}, v_1\rangle_{(V^\varepsilon)', V^\varepsilon}\, \mathrm{d}t + \int_0^T \langle u'_{2\varepsilon}, v_2\rangle_{(H^1(\Omega_{2\varepsilon}))', H^1(\Omega_{2\varepsilon})}\, \mathrm{d}t + \int_0^T\int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon}\nabla v_1\, \mathrm{d}x\, \mathrm{d}t \\ \qquad\displaystyle + \int_0^T\int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon}\nabla v_2\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \varepsilon^\gamma \int_{\Gamma^\varepsilon} h^\varepsilon (u_{1\varepsilon} - u_{2\varepsilon})(v_1 - v_2)\, \mathrm{d}\sigma_x\, \mathrm{d}t \\ \qquad\displaystyle = \int_0^T\int_{\Omega_{1\varepsilon}} f_{1\varepsilon} v_1\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \langle g, P_1^\varepsilon v_1\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \int_0^T\int_{\Omega_{2\varepsilon}} f_{2\varepsilon} v_2\, \mathrm{d}x\, \mathrm{d}t \\ \text{for every } (v_1, v_2) \in L^2(0,T;\ V^\varepsilon) \times L^2(0,T;\ H^1(\Omega_{2\varepsilon})), \\ u_{1\varepsilon}(x,0) = U^0_{1\varepsilon} \text{ in } \Omega_{1\varepsilon} \ \text{ and } \ u_{2\varepsilon}(x,0) = U^0_{2\varepsilon} \text{ in } \Omega_{2\varepsilon}, \end{cases} \tag{2.14}$$

where

$$W^\varepsilon := \{v = (v_1, v_2) \in L^2(0,T;\ V^\varepsilon) \times L^2(0,T;\ H^1(\Omega_{2\varepsilon})) \ \text{ such that} \\ v' \in L^2(0,T;\ (V^\varepsilon)') \times L^2(0,T;\ (H^1(\Omega_{2\varepsilon}))')\},$$

equipped with the norm

$$\|v\|_{W^\varepsilon} = \|v_1\|_{L^2(0,T;\ V^\varepsilon)} + \|v_2\|_{L^2(0,T;\ H^1(\Omega_{2\varepsilon}))} + \|v_1'\|_{L^2(0,T;\ (V^\varepsilon)')} + \|v_2'\|_{L^2(0,T;\ (H^1(\Omega_{2\varepsilon}))')}.$$

In [21] (see also [19]), the limit behaviour as ε tends to zero of problem (2.14) has been described for $\gamma \le 1$. When $-1 < \gamma \le 1$, which is the case studied in this paper, two different homogenized (limit) problems were obtained and are given in Theorem 2.5 below. To do so, let $\widehat{w}_\lambda \in H^1(Y_1)$ for any $\lambda \in \mathbb{R}^n$, be the solution of the problem

$$\begin{cases} -\mathrm{div}\,(A\nabla \widehat{w}_\lambda) = 0 & \text{in } Y_1, \\ (A\nabla \widehat{w}_\lambda)\cdot n_1 = 0 & \text{in } \Gamma, \\ \widehat{w}_\lambda - \lambda\cdot y & Y\text{-periodic}, \\ \displaystyle\frac{1}{|Y_1|}\int_{Y_1} (\widehat{w}_\lambda - \lambda\cdot y)\, \mathrm{d}y = 0, \end{cases} \tag{2.15}$$

and A^0 the homogenized matrix given by

$$A^0\lambda := m_Y(A\nabla \widehat{w}_\lambda). \tag{2.16}$$

Theorem 2.5 [21]. *Let A^ε and h^ε be defined by (2.9) and (2.11) respectively. Let $-1 < \gamma \le 1$ and u_ε be the solution of problem (2.12). Moreover, suppose that*

$$\begin{cases} g \in L^2(0,T;\ H^{-1}(\Omega)), \\ (U^0_{1\varepsilon}, U^0_{2\varepsilon}) \in L^2(\Omega_{1\varepsilon}) \times L^2(\Omega_{2\varepsilon}), \\ (f_{1\varepsilon}, f_{2\varepsilon}) \in L^2(0,T;\ L^2(\Omega)) \times L^2(0,T;\ L^2(\Omega)) \end{cases} \tag{2.17}$$

and

$$\begin{cases} (\widetilde{U^0_{1\varepsilon}}, \widetilde{U^0_{2\varepsilon}}) \rightharpoonup (\theta_1 U^0_1, \theta_2 U^0_2) & \text{weakly in } L^2(\Omega) \times L^2(\Omega), \\ (\widetilde{f_{1\varepsilon}}, \widetilde{f_{2\varepsilon}}) \rightharpoonup (\theta_1 f_1, \theta_2 f_2) & \text{weakly in } L^2(0,T;\ L^2(\Omega)) \times L^2(0,T;\ L^2(\Omega)), \end{cases} \tag{2.18}$$

where θ_i $(i=1,2)$ *is given by* (2.2). *Then, there exists a suitable extension operator*

$$P_1^\varepsilon \in \mathcal{L}(L^2(0,T;V^\varepsilon);L^2(0,T;H_0^1(\Omega))) \cap \mathcal{L}(L^2(0,T;\ L^2(\Omega_{1\varepsilon}));L^2(0,T;\ L^2(\Omega)))$$

such that

$$\begin{cases} (i)\ P_1^\varepsilon u_{1\varepsilon} \rightharpoonup u_1 & \textit{weakly in } L^2(0,T;\ H_0^1(\Omega)),\\ (ii)\ \widetilde{u_{1\varepsilon}} \rightharpoonup \theta_1 u_1 & \textit{weakly}^* \textit{ in } L^\infty(0,T;\ L^2(\Omega)),\\ (iii)\ \widetilde{u_{2\varepsilon}} \rightharpoonup u_2 & \textit{weakly}^* \textit{ in } L^\infty(0,T;\ L^2(\Omega)),\\ (iv)\ \varepsilon^{\frac{\gamma}{2}}\|u_{1\varepsilon}-u_{2\varepsilon}\|_{L^2(0,T;\ L^2(\Gamma^\varepsilon))} < c, \end{cases} \tag{2.19}$$

where c *is a constant independent of* ε. *In addition,*

$$\begin{cases} (i)\ A^\varepsilon\widetilde{\nabla u_{1\varepsilon}} \rightharpoonup A^0\nabla u_1 & \textit{weakly in } L^2(0,T;\ [L^2(\Omega)]^n),\\ (ii)\ A^\varepsilon\widetilde{\nabla u_{2\varepsilon}} \rightharpoonup 0 & \textit{weakly in } L^2(0,T;\ [L^2(\Omega)]^n), \end{cases} \tag{2.20}$$

where A^0 *is given by* (2.16). *Moreover, the limit functions* u_1 *and* u_2 *are described as follows:*

- ***Case*** $-1<\gamma<1$. *We have* $u_2=\theta_2 u_1$, *where* θ_2 *is given by* (2.2) *and* $u_1\in\mathcal{C}^0([0,T];L^2(\Omega))\cap L^2(0,T;\ H_0^1(\Omega))$, *with* $u_1'\in L^2(0,T;H^{-1}(\Omega))$ *is the unique solution of the homogenized problem*

$$\begin{cases} u_1' - \operatorname{div}(A^0\nabla u_1) = \theta_1 f_1 + \theta_2 f_2 + g & \textit{in } \Omega\times\,]0,T[\,,\\ u_1 = 0 & \textit{on } \partial\Omega\times\,]0,T[\,,\\ u_1(0) = \theta_1 U_1^0 + \theta_2 U_2^0 & \textit{in } \Omega. \end{cases} \tag{2.21}$$

- ***Case*** $\gamma=1$. *The couple* $(u_1,u_2)\in\mathcal{C}^0([0,T];L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))\times\mathcal{C}^0([0,T];L^2(\Omega))$ *with* $(u_1',u_2')\in L^2(0,T;H^{-1}(\Omega))\times L^2(0,T;L^2(\Omega))$ *is the unique solution of the problem (a PDE coupled with an ODE)*

$$\begin{cases} \theta_1 u_1' - \operatorname{div}(A^0\nabla u_1) + c_h(\theta_2 u_1 - u_2) = \theta_1 f_1 + g & \textit{in } \Omega\times\,]0,T[\,,\\ u_2' - c_h(\theta_2 u_1 - u_2) = \theta_2 f_2 & \textit{in } \Omega\times\,]0,T[\,,\\ u_1 = 0 & \textit{on } \partial\Omega\times\,]0,T[\,,\\ u_1(0)=U_1^0,\ u_2(0)=\theta_2 U_2^0 & \textit{in } \Omega, \end{cases} \tag{2.22}$$

where $c_h = \dfrac{1}{|Y_2|}\displaystyle\int_\Gamma h(y)\,\mathrm{d}\sigma_y$.

3. Statement of the problem and main result

Let us first introduce the corrector matrix for the parabolic problem (2.12), which is the same as that obtained by Donato *et al.* in [16] for perforated domains.

Let $(e_j)_{j=1,\dots,n}$ be the canonical basis of $\mathbb{R}^n$. Set $\widehat{w}_j = \widehat{w}_{e_j}$, where $\widehat{w}_j \in H^1(Y_1)$ is the solution of problem (2.15) written for $\lambda=e_j,\ j=1,\dots,n$.

The corrector matrix $C^\varepsilon = \left(C_{ij}^\varepsilon\right)_{1\le i,j\le n}$ is defined by

$$\begin{cases} C_{ij}^\varepsilon(x) = \widetilde{C_{ij}}\left(\dfrac{x}{\varepsilon}\right) & \text{a.e. on } \Omega,\\ C_{ij}(y) := \dfrac{\partial\widehat{w}_j}{\partial y_i}(y), \quad i,j=1,\dots,n & \text{a.e. on } Y_1, \end{cases} \tag{3.1}$$

where $\widetilde{\ }$ denotes the zero extension to the whole of Y. Now, define $\widehat{w}_j^\varepsilon$ by

$$\widehat{w}_j^\varepsilon(x) := x_j - \varepsilon(Q_1(\widehat{\chi}_j)(x/\varepsilon)), \qquad \widehat{\chi}_j = y_j - \widehat{w}_j(y), \tag{3.2}$$

where Q_1 is a suitable extension operator introduced in [9].

It follows that if C^ε_j denotes the jth column, then

$$C^\varepsilon_j(x) = \nabla \widehat{w}^\varepsilon_j\left(\frac{x}{\varepsilon}\right), \quad j = 1, 2, ..., n \tag{3.3}$$

and for c independent of ε,

$$\|C^\varepsilon\|_{[L^2(\Omega_{1\varepsilon})]^{n^2}} \leq c. \tag{3.4}$$

Observe also that a change of scale in (2.15) gives

$$\int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla \widehat{w}^\varepsilon_j \nabla v \,\mathrm{d}x = 0, \qquad \forall v \in H^1_0(\Omega) \tag{3.5}$$

and the following convergences hold:

$$\begin{cases} (i)\ \widehat{w}^\varepsilon_j(x) \rightharpoonup x_j & \text{weakly in } H^1(\Omega),\\ (ii)\ \widehat{w}^\varepsilon_j(x) \to x_j & \text{strongly in } L^2(\Omega),\\ (iii)\ \nabla\widehat{w}^\varepsilon_j(x) \rightharpoonup e_j & \text{weakly in } [L^2(\Omega)]^n,\\ (iv)\ \chi_{\Omega_{1\varepsilon}} A^\varepsilon \nabla \widehat{w}^\varepsilon_j \rightharpoonup\ A^0 e_j & \text{weakly in } [L^2(\Omega)]^n. \end{cases} \tag{3.6}$$

We now introduce some assumptions on the data, stronger than (2.18), depending on γ. These assumptions, as already seen in [3,15], are necessary (see Sect. 5) to provide the convergence of the energy of problem (2.14) to that of the homogenized one. This convergence, observed in [3], plays an essential role in the proof of corrector results.

Concerning the data $f_{i\varepsilon}$ $(i = 1, 2)$, we suppose that for $-1 < \gamma \leq 1$, $f_{i\varepsilon}$ is the restriction of a function defined on the whole of Ω and

$$\begin{cases} f_{i\varepsilon} \in L^2(0,T;\ L^2(\Omega)), & i = 1, 2,\\ (f_{1\varepsilon}, f_{2\varepsilon}) \to (f_1, f_2) & \text{strongly in } L^2(0,T;\ L^2(\Omega)) \times L^2(0,T;\ L^2(\Omega)). \end{cases} \tag{3.7}$$

This will imply that (see also Rem. 2.2),

$$(\chi_{\Omega_{1\varepsilon}} f_{1\varepsilon}, \chi_{\Omega_{2\varepsilon}} f_{2\varepsilon}) \rightharpoonup (\theta_1 f_1, \theta_2 f_2) \quad \text{weakly in } L^2(0,T;\ L^2(\Omega)) \times L^2(0,T;\ L^2(\Omega)). \tag{3.8}$$

Let us now focus on the assumptions for the initial conditions.

– If $-1 < \gamma < 1$, we suppose that for some $U^0 \in L^2(\Omega)$,

$$\widetilde{U^0_{1\varepsilon}} + \widetilde{U^0_{2\varepsilon}} \to U^0 \quad \text{strongly in } L^2(\Omega). \tag{3.9}$$

The lemma below clarifies this assumption.

Lemma 3.1. *Let $U^0_{i\varepsilon} \in L^2(\Omega_{i\varepsilon})$ $(i = 1, 2)$ and $U^0 \in L^2(\Omega)$. Then (3.9) holds if and only if*

$$\begin{cases} (i)\ U^0_{i\varepsilon} \in L^2(\Omega_{i\varepsilon})\\ (ii)\ \widetilde{U^0_{i\varepsilon}} \rightharpoonup \theta_i U^0 \quad \textit{weakly in } L^2(\Omega),\\ (iii)\ \|U^0_{1\varepsilon}\|^2_{L^2(\Omega_{1\varepsilon})} + \|U^0_{2\varepsilon}\|^2_{L^2(\Omega_{2\varepsilon})} \to \|U^0\|^2_{L^2(\Omega)}. \end{cases} \tag{3.10}$$

Proof. Observe that from (3.10)(ii), we obtain

$$\widetilde{U^0_{1\varepsilon}} + \widetilde{U^0_{2\varepsilon}} \rightharpoonup \theta_1 U^0 + \theta_2 U^0 = U^0 \quad \text{weakly in } L^2(\Omega). \tag{3.11}$$

Now, since $\Omega_{1\varepsilon}$ and $\Omega_{2\varepsilon}$ are disjoint, from (3.10)(iii) we get

$$\int_\Omega (\widetilde{U^0_{1\varepsilon}} + \widetilde{U^0_{2\varepsilon}})^2 \,\mathrm{d}x = \int_\Omega (\widetilde{U^0_{1\varepsilon}})^2 \,\mathrm{d}x + \int_\Omega (\widetilde{U^0_{2\varepsilon}})^2 \,\mathrm{d}x \to \int_\Omega (U^0)^2 \,\mathrm{d}x, \tag{3.12}$$

which together with (3.11) gives (3.9).

Conversely, suppose that (3.9) is satisfied. For every $\varphi \in L^2(\Omega)$,

$$\int_\Omega \widetilde{U^0_{1\varepsilon}}\varphi \,\mathrm{d}x = \int_\Omega \chi_{\Omega_{1\varepsilon}} (\widetilde{U^0_{1\varepsilon}} + \widetilde{U^0_{2\varepsilon}})\varphi \,\mathrm{d}x \to \int_\Omega \theta_1 U^0 \varphi \,\mathrm{d}x.$$

Hence, $\widetilde{U^0_{1\varepsilon}} \rightharpoonup \theta_1 U^0$ weakly in $L^2(\Omega)$. Similarly, $\widetilde{U^0_{2\varepsilon}} \rightharpoonup \theta_2 U^0$ weakly in $L^2(\Omega)$. So we have (ii). Meanwhile, by the definition of L^2-norm and (3.12),

$$\|U^0_{1\varepsilon}\|^2_{L^2(\Omega_{1\varepsilon})} + \|U^0_{2\varepsilon}\|^2_{L^2(\Omega_{2\varepsilon})} = \|\widetilde{U^0_{1\varepsilon}} + \widetilde{U^0_{2\varepsilon}}\|^2_{L^2(\Omega)} \to \|U^0\|^2_{L^2(\Omega)}.$$

This shows (iii) and ends the proof. □

Remark 3.2. Observe that (3.10) holds, for instance, if $U^0_{i\varepsilon}$ is defined on the whole of Ω with $U^0_{i\varepsilon} \in L^2(\Omega)$ and there exists $U^0 \in L^2(\Omega)$ such that for $i = 1, 2$,

$$U^0_{i\varepsilon} \to U^0 \quad \text{strongly in } L^2(\Omega).$$

Indeed, from Remark 2.2 we have, $\widetilde{U^0_{i\varepsilon}|_{\Omega_{i\varepsilon}}} \rightharpoonup \theta_i U^0$ weakly in $L^2(\Omega)$. Also,

$$\begin{aligned}\int_{\Omega_{1\varepsilon}} (U^0_{1\varepsilon})^2 \,\mathrm{d}x + \int_{\Omega_{2\varepsilon}} (U^0_{2\varepsilon})^2 \,\mathrm{d}x &= \int_\Omega \chi_{\Omega_{1\varepsilon}} (U^0_{1\varepsilon})^2 \,\mathrm{d}x + \int_\Omega \chi_{\Omega_{2\varepsilon}} (U^0_{2\varepsilon})^2 \,\mathrm{d}x \\ &\to \int_\Omega \theta_1 (U^0)^2 + \int_\Omega \theta_2 (U^0)^2 \,\mathrm{d}x = \|U^0\|^2_{L^2(\Omega)}.\end{aligned}$$

Remark 3.3. Let $-1 < \gamma < 1$ and suppose that (3.7) (from which (3.8) follows) and (3.9) are satisfied. It is clear (see also Rem. 2.2) that Theorem 2.5 applies, with $U^0_1 = U^0_2 = U^0$. Hence, the initial conditions in problem (2.21) reads

$$u_1(0) = U^0.$$

Let us now state our first corrector results which will be proved in Section 6.

Theorem 3.4 (corrector results for the case $-1 < \gamma < 1$)**.** *Let A^ε and h^ε be defined by (2.9) and (2.11) respectively. Let u_ε be the solution of problem (2.14). Suppose that (3.7) and (3.9) hold. Then, we have the following convergences:*

$$\begin{cases} (i)\ \widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}} \to u_1 & \text{in } \mathcal{C}^0([0,T]; L^2(\Omega)), \\ (ii)\ \lim\limits_{\varepsilon\to 0} \|\nabla u_{1\varepsilon} - C^\varepsilon \nabla u_1\|_{L^2(0,T;[L^1(\Omega_{1\varepsilon})]^n)} = 0, \\ (iii)\ \lim\limits_{\varepsilon\to 0} \|\nabla u_{2\varepsilon}\|_{L^2(0,T;[L^2(\Omega_{2\varepsilon})]^n)} = 0, \end{cases} \tag{3.13}$$

where u_1 is the solution of the homogenized problem

$$\begin{cases} u_1' - \operatorname{div}(A^0 \nabla u_1) = \theta_1 f_1 + \theta_2 f_2 + g & \text{in } \Omega \times]0,T[, \\ u_1 = 0 & \text{on } \partial\Omega \times]0,T[, \\ u_1(0) = U^0 & \text{in } \Omega. \end{cases} \tag{3.14}$$

Remark 3.5. From the proof of Proposition 5.3, it can be seen that (3.7) and (3.9) gives the necessary conditions for the convergence of the energy of problem (2.14) to that of the homogenized one when $-1 < \gamma < 1$.

- Consider now the case $\gamma = 1$. We make the following assumptions on the initial conditions. For some $U_i^0 \in L^2(\Omega)$ $(i = 1, 2)$, suppose that

$$\begin{cases} (i)\ U_{i\varepsilon}^0 \in L^2(\Omega_{i\varepsilon}), \\ (ii)\ \widetilde{U_{i\varepsilon}^0} \rightharpoonup \theta_i U_i^0 \text{ weakly in } L^2(\Omega), \\ (iii)\ \|U_{1\varepsilon}^0\|^2_{L^2(\Omega_{1\varepsilon})} + \|U_{2\varepsilon}^0\|^2_{L^2(\Omega_{2\varepsilon})} \to \theta_1 \|U_1^0\|^2_{L^2(\Omega)} + \theta_2 \|U_2^0\|^2_{L^2(\Omega)}. \end{cases} \tag{3.15}$$

Remark 3.6. Assumption (3.15) holds, for instance, if $U_{i\varepsilon}^0$ is defined on the whole of Ω, with $U_{i\varepsilon}^0 \in L^2(\Omega)$ and if for some $U_i^0 \in L^2(\Omega)$, one has

$$U_{i\varepsilon}^0 \to U_i^0 \quad \text{strongly in } L^2(\Omega), \qquad \text{for } i = 1, 2.$$

Indeed, for every $\varphi \in L^2(\Omega)$,

$$\int_{\Omega_{i\varepsilon}} U_{i\varepsilon}^0 \varphi \,\mathrm{d}x = \int_\Omega \chi_{\Omega_{i\varepsilon}} U_{i\varepsilon}^0 \varphi \,\mathrm{d}x \to \int_\Omega \theta_i U_i^0 \varphi \,\mathrm{d}x, \qquad \text{for } i = 1, 2.$$

Hence, $\widetilde{U_{i\varepsilon}^0|_{\Omega_{i\varepsilon}}} \rightharpoonup \theta_i U_i^0$ weakly in $L^2(\Omega)$. On the other hand,

$$\begin{aligned} \|U_{1\varepsilon}^0\|^2_{L^2(\Omega_{1\varepsilon})} + \|U_{2\varepsilon}^0\|^2_{L^2(\Omega_{2\varepsilon})} &= \int_\Omega \chi_{\Omega_{1\varepsilon}} (U_{1\varepsilon}^0)^2 \,\mathrm{d}x + \int_\Omega \chi_{\Omega_{2\varepsilon}} (U_{2\varepsilon}^0)^2 \,\mathrm{d}x \\ &\to \int_\Omega \theta_1 (U_1^0)^2 \,\mathrm{d}x + \int_\Omega \theta_2 (U_2^0)^2 \,\mathrm{d}x = \theta_1 \|U_1^0\|^2_{L^2(\Omega)} + \theta_2 \|U_2^0\|^2_{L^2(\Omega)}. \end{aligned}$$

Remark 3.7. Observe also that if (3.15) holds with $U_1^0 = U_2^0 = U^0$ for some $U^0 \in L^2(\Omega)$, then

$$\widetilde{U_{1\varepsilon}^0} + \widetilde{U_{2\varepsilon}^0} \to U^0 \quad \text{strongly in } L^2(\Omega),$$

which is (3.9). This is because from (3.15)(ii) we obtain

$$\widetilde{U_{1\varepsilon}^0} + \widetilde{U_{2\varepsilon}^0} \rightharpoonup \theta_1 U^0 + \theta_2 U^0 = U^0 \quad \text{weakly in } L^2(\Omega).$$

Moreover, since the support of $\chi_{\Omega_{1\varepsilon}}$ and $\chi_{\Omega_{2\varepsilon}}$ are disjoint, we get the convergence of the norms from (iii), so that

$$\|\widetilde{U_{1\varepsilon}^0} + \widetilde{U_{2\varepsilon}^0}\|^2_{L^2(\Omega)} = \|U_{1\varepsilon}^0\|^2_{L^2(\Omega_{1\varepsilon})} + \|U_{2\varepsilon}^0\|^2_{L^2(\Omega_{2\varepsilon})} \to \|U^0\|^2_{L^2(\Omega)}.$$

Hence, (3.15) is a more general condition than (3.9).

Remark 3.8. Let $\gamma = 1$. Since (3.7) clearly implies (3.8), then under assumptions (3.7) and (3.15), Theorem 2.5 applies and the homogenized problem is still given by (2.22).

We give in the following theorem the corrector results for the second case, which is proved in Section 6.

Theorem 3.9 (corrector results for the case $\gamma = 1$)**.** *Let A^ε and h^ε be defined by* (2.9) *and* (2.11) *respectively, and assume that Γ is of class C^2. Let u_ε be the solution of problem* (2.14)*. Suppose that* (3.7) *and* (3.15) *hold.*

Then, we have the following convergences:

$$\begin{cases} (i) \ \lim\limits_{\varepsilon\to 0} \|u_{1\varepsilon} - u_1\|_{\mathcal{C}^0(0,T;L^2(\Omega_{1\varepsilon}))} = 0, \\ (ii) \ \lim\limits_{\varepsilon\to 0} \|u_{2\varepsilon} - \theta_2^{-1} u_2\|_{\mathcal{C}^0(0,T;L^2(\Omega_{2\varepsilon}))} = 0, \\ (iii) \ \lim\limits_{\varepsilon\to 0} \|\nabla u_{1\varepsilon} - C^\varepsilon \nabla u_1\|_{L^2(0,T;[L^1(\Omega_{1\varepsilon})]^n)} = 0, \\ (iv) \ \lim\limits_{\varepsilon\to 0} \|\nabla u_{2\varepsilon}\|_{L^2(0,T;[L^2(\Omega_{2\varepsilon})]^n)} = 0, \end{cases} \tag{3.16}$$

where (u_1, u_2) *is the solution of the homogenized problem* (2.22)*.*

Remark 3.10. It can be seen in Section 5 that (3.7) and (3.15) are exactly what we need for the convergence of the energy in the case $\gamma = 1$.

One of the main tools in proving the corrector results is a compactness result for $\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}}$ in $\mathcal{C}^0([0,T];H^{-1}(\Omega))$, which is shown in the next section (Thm. 4.8).

4. *A priori* estimates for $(u^\varepsilon)'$ and a compactness result

In this section, we first study the dual of H^ε_γ and complete the *a priori* estimates stated in [21], under the assumptions of Theorem 2.5. We begin with the following results concerning the space H^ε_γ and its dual.

Proposition 4.1. (i) *There exists a constant* $c > 0$ *such that*

$$\|v\|^2_{H^\varepsilon_\gamma} \le c(1+\varepsilon^{\gamma-1})\|v\|^2_{V^\varepsilon\times H^1(\Omega_{2\varepsilon})}, \quad \forall\gamma\in\mathbb{R}.$$

(ii) *If* $\gamma \le 1$*, there exist two positive constants* c_1, c_2 *(independent of* ε*) such that*

$$c_1\|v\|^2_{V^\varepsilon\times H^1(\Omega_{2\varepsilon})} \le \|v\|^2_{H^\varepsilon_\gamma} \le c_2(1+\varepsilon^{\gamma-1})\|v\|^2_{V^\varepsilon\times H^1(\Omega_{2\varepsilon})}.$$

Proof. (i) Observe that by the definition of the norms in H^ε_γ, V^ε and $H^1(\Omega_{2\varepsilon})$ and by Lemma 2.4, one has

$$\begin{aligned} \|v\|^2_{H^\varepsilon_\gamma} &= \|\nabla v_1\|^2_{L^2(\Omega_{1\varepsilon})} + \|\nabla v_2\|^2_{L^2(\Omega_{2\varepsilon})} + \varepsilon^\gamma \|v_1 - v_2\|^2_{L^2(\Gamma^\varepsilon)} \\ &= \|v_1\|^2_{V^\varepsilon} + \|\nabla v_2\|^2_{L^2(\Omega_{2\varepsilon})} + \varepsilon^\gamma\|v_1 - v_2\|^2_{L^2(\Gamma^\varepsilon)} \\ &\le \|v_1\|^2_{V^\varepsilon} + \|v_2\|^2_{H^1(\Omega_{2\varepsilon})} + \varepsilon^{\gamma-1}\varepsilon\|v_1 - v_2\|^2_{L^2(\Gamma^\varepsilon)} \\ &\le \|v_1\|^2_{V^\varepsilon} + \|v_2\|^2_{H^1(\Omega_{2\varepsilon})} + \varepsilon^{\gamma-1}\|v\|^2_{H^\varepsilon_1} \\ &\le c_2(1+\varepsilon^{\gamma-1})\|v\|^2_{V^\varepsilon\times H^1(\Omega_{2\varepsilon})}. \end{aligned}$$

(ii) The right-hand side inequality follows from (i). Now, if $\gamma \le 1$, then (2.6) implies that $\|v\|^2_{H^\varepsilon_1} \le \|v\|^2_{H^\varepsilon_\gamma}$. Together with Lemma 2.4, we have $c_1\|v\|^2_{V^\varepsilon\times H^1(\Omega_{2\varepsilon})} \le \|v\|^2_{H^\varepsilon_\gamma}$ with $c_1 = C_2^{-2}$. □

What can be said about the dual space $(H^\varepsilon_\gamma)'$ with respect to $(V^\varepsilon)'\times(H^1(\Omega_{2\varepsilon}))'$? An answer is provided by the next proposition, where we use the notation $x'(x)$ for the duality pairing between the dual space E' and a Banach space E.

Proposition 4.2. *For* $\gamma \le 1$*, we have*

$$v \in (V^\varepsilon)'\times(H^1(\Omega_{2\varepsilon}))' \qquad \textit{if and only if} \qquad v\in(H^\varepsilon_\gamma)'$$

and there exist positive constants k_1, k_2 *(independent of* ε*) such that*

$$k_1\|v\|^2_{(H^\varepsilon_\gamma)'} \le \|v\|^2_{(V^\varepsilon)'\times(H^1(\Omega_{2\varepsilon}))'} \le k_2(1+\varepsilon^{\gamma-1})\|v\|^2_{(H^\varepsilon_\gamma)'}.$$

Proof. Let $\gamma \le 1$ and suppose $v \in (V^\varepsilon)' \times (H^1(\Omega_{2\varepsilon}))'$. We have from Proposition 4.1(ii) that

$$\begin{aligned}|v(u)| &\le \|v\|_{(V^\varepsilon)' \times (H^1(\Omega_{2\varepsilon}))'} \|u\|_{V^\varepsilon \times H^1(\Omega_{2\varepsilon})} \\ &\le \frac{1}{\sqrt{c_1}} \|v\|_{(V^\varepsilon)' \times (H^1(\Omega_{2\varepsilon}))'} \|u\|_{H^\varepsilon_\gamma}.\end{aligned}$$

Hence,

$$\sup_{u \neq 0} \frac{|v(u)|}{\|u\|_{H^\varepsilon_\gamma}} \le \frac{1}{\sqrt{c_1}} \|v\|_{(V^\varepsilon)' \times (H^1(\Omega_{2\varepsilon}))'}.$$

Therefore, $v \in (H^\varepsilon_\gamma)'$ and we have the first inequality with $k_1 = c_1$.

Now, suppose that $v \in (H^\varepsilon_\gamma)'$. By using again Proposition 4.1, we get

$$\begin{aligned}|v(u)| &\le \|v\|_{(H^\varepsilon_\gamma)'} \|u\|_{H^\varepsilon_\gamma} \\ &\le \sqrt{c_2} \|v\|_{(H^\varepsilon_\gamma)'} (1 + \varepsilon^{\frac{\gamma - 1}{2}}) \|u\|_{V^\varepsilon \times H^1(\Omega_{2\varepsilon})}.\end{aligned}$$

This yields

$$\sup_{u \neq 0} \frac{|v(u)|}{\|u\|_{V^\varepsilon \times H^1(\Omega_{2\varepsilon})}} \le \sqrt{c_2} (1 + \varepsilon^{\frac{\gamma - 1}{2}}) \|v\|_{(H^\varepsilon_\gamma)'}.$$

Therefore, $v \in (V^\varepsilon)' \times (H^1(\Omega_{2\varepsilon}))'$ and we have the second inequality with $k_2 = c_2$. □

Remark 4.3. Proposition 4.2 implies that if $v = (v_1, v_2) \in (V^\varepsilon)' \times (H^1(\Omega_{2\varepsilon}))'$ and $u = (u_1, u_2) \in V^\varepsilon \times H^1(\Omega_{2\varepsilon})$, then $\langle v, u \rangle_{(H^\varepsilon_\gamma)', H^\varepsilon_\gamma} = \langle v_1, u_1 \rangle_{(V^\varepsilon)', V^\varepsilon} + \langle v_2, u_2 \rangle_{(H^1(\Omega_{2\varepsilon}))', H^1(\Omega_{2\varepsilon})}$.

Now, in order to show a compactness result for $\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}}$, we need to estimate first the quantity

$$\|(\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}})'\|_{L^2(0,T;\, H^{-1}(\Omega))}.$$

To this aim, we prove the following lemma.

Lemma 4.4. *Let*

$$v \in L^2(0, T;\ V^\varepsilon) (\textit{resp. } v \in L^2(0, T;\ H^1(\Omega_{2\varepsilon})))$$

with

$$v' \in L^2(0, T;\ (V^\varepsilon)') (\textit{resp. } v' \in L^2(0, T;\ (H^1(\Omega_{2\varepsilon}))')).$$

Then

$$(\widetilde{v})' \in L^2(0, T;\ H^{-1}(\Omega))$$

and for every $\psi \in L^2(0, T;\ H^1_0(\Omega))$,

$$\langle (\widetilde{v})', \psi \rangle_{L^2(0,T;\, H^{-1}(\Omega)), L^2(0,T;\, H^1_0(\Omega))} = \langle v', \psi|_{\Omega_{1\varepsilon}} \rangle_{L^2(0,T;\, (V^\varepsilon)'), L^2(0,T;\, V^\varepsilon)}$$

$$(\textit{resp. } \langle (\widetilde{v})', \psi \rangle_{L^2(0,T;\, H^{-1}(\Omega)), L^2(0,T;\, H^1_0(\Omega))} = \langle v', \psi|_{\Omega_{2\varepsilon}} \rangle_{L^2(0,T;\, (H^1(\Omega_{2\varepsilon}))'), L^2(0,T;\, H^1(\Omega_{2\varepsilon}))}).$$

Proof. Observe that for all $\psi \in \mathcal{D}((0, T) \times \Omega)$

$$\begin{aligned}\langle v', \psi|_{\Omega_{1\varepsilon}} \rangle_{L^2(0,T;\, (V^\varepsilon)'), L^2(0,T;\, V^\varepsilon)} &= \int_0^T \langle v', \psi|_{\Omega_{1\varepsilon}} \rangle_{(V^\varepsilon)', V^\varepsilon}\, \mathrm{d}t = -\int_0^T \int_{\Omega_{1\varepsilon}} v\psi'\, \mathrm{d}x\, \mathrm{d}t \\ &= -\int_0^T \int_\Omega \widetilde{v}\psi'\, \mathrm{d}x\, \mathrm{d}t = \langle (\widetilde{v})', \psi \rangle_{\mathcal{D}'((0,T)\times\Omega), \mathcal{D}((0,T)\times\Omega)}.\end{aligned}$$

Hence, by a density argument

$$
\begin{aligned}
|\langle (\widetilde{v})', \psi\rangle_{\mathcal{D}'((0,T)\times\Omega),\mathcal{D}((0,T)\times\Omega)}| &= |\langle v', \psi|_{\Omega_{1\varepsilon}}\rangle_{L^2(0,T;\,(V^\varepsilon)'),L^2(0,T;\,V^\varepsilon)}| \\
&\le \|v'\|_{L^2(0,T;\,(V^\varepsilon)')}\|\psi\|_{L^2(0,T;\,V^\varepsilon)} \le \|v'\|_{L^2(0,T;\,(V^\varepsilon)')}\|\psi\|_{L^2(0,T;\,H^1_0(\Omega))},
\end{aligned}
$$

for every $\psi \in L^2(0,T;\ H^1_0(\Omega))$. In a similar manner, if

$$
v \in L^2(0,T;\ H^1(\Omega_{2\varepsilon})) \text{ and } v' \in L^2(0,T;\ (H^1(\Omega_{2\varepsilon}))')
$$

then for every $\psi \in L^2(0,T;\ H^1_0(\Omega))$,

$$
|\langle (\widetilde{v})', \psi\rangle_{\mathcal{D}'((0,T)\times\Omega),\mathcal{D}((0,T)\times\Omega)}| \le \|v'\|_{L^2(0,T;\,(H^1(\Omega_{2\varepsilon}))')}\|\psi\|_{L^2(0,T;\,H^1_0(\Omega))}.
$$

Therefore, in both cases,

$$
(\widetilde{v})' \in L^2(0,T;\ H^{-1}(\Omega)). \qquad \square
$$

Let us also recall the following extension result proved by Cioranescu and Donato in [7] concerning the operator P_1^ε from (2.13).

Lemma 4.5 [7]**.** *There exists a linear continuous extension operator*

$$
P_1^\varepsilon \in \mathcal{L}(L^2(0,T;V^\varepsilon); L^2(0,T;H^1_0(\Omega))) \cap \mathcal{L}(L^2(0,T;\ L^2(\Omega_{1\varepsilon})); L^2(0,T;\ L^2(\Omega)))
$$

such that for some positive constant c independent of ε and for any $\varphi \in L^2(0,T;V^\varepsilon)$ with $\varphi' \in L^2(0,T;L^2(\Omega_{1\varepsilon}))$, we have

$$
\begin{cases}
P_1^\varepsilon\varphi = \varphi \quad \text{in } \Omega_{1\varepsilon}\times]0,T[\,, \\[2mm]
P_1^\varepsilon\varphi' = (P_1^\varepsilon\varphi)' \quad \text{in } \Omega\times]0,T[\,, \\[2mm]
\|P_1^\varepsilon\varphi\|_{L^2(0,T;\,L^2(\Omega))} \le c\|\varphi\|_{L^2(0,T;\,L^2(\Omega_{1\varepsilon}))}, \\[2mm]
\|P_1^\varepsilon\varphi'\|_{L^2(0,T;\,L^2(\Omega))} \le c\|\varphi'\|_{L^2(0,T;\,L^2(\Omega_{1\varepsilon}))}, \\[2mm]
\|P_1^\varepsilon\varphi(t)\|_{H^1_0(\Omega)} \le c\|\nabla\varphi(t)\|_{L^2(\Omega_{1\varepsilon})}, \quad \text{a.e. in }]0,T[\,, \\[2mm]
\|\nabla(P_1^\varepsilon\varphi)\|_{L^2(0,T;\,[L^2(\Omega)]^n)} \le c\|\nabla\varphi\|_{L^2(0,T;\,[L^2(\Omega_{1\varepsilon})]^n)}.
\end{cases}
$$

We will use in the sequel the following result (used in the proof of Cor. 2.8 of [21]), which is an adaptation to the time-dependent case of a lemma given in [4].

Lemma 4.6 [21]**.** *Suppose that (v_ε) and $(v_\varepsilon)'$ are bounded in $L^2(0,T;\ H^1_0(\Omega))$ and $L^2(0,T;\ L^2(\Omega))$ respectively, with $v_\varepsilon \to v$ strongly in $L^2(0,T;\ L^2(\Omega))$. Then,*

$$
P_1^\varepsilon(v_\varepsilon|_{\Omega_{1\varepsilon}}) \rightharpoonup v \qquad \text{weakly in } L^2(0,T;\ H^1_0(\Omega)).
$$

Consider now the solution $(u_{1\varepsilon}, u_{2\varepsilon})$ of problem (2.14). The next proposition deals with the norms $\|u'_{1\varepsilon}\|$ and $\|u'_{2\varepsilon}\|$ in spaces $L^2(0,T;\ (V^\varepsilon)')$ and $L^2(0,T;\ (H^1(\Omega_{2\varepsilon}))')$ respectively. In addition, it will provide *a priori* estimates for $\|(\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}})'\|_{L^2(0,T;\,H^{-1}(\Omega))}$.

Theorem 4.7. *Under the assumptions of Theorem 2.5, if $\gamma \le 1$, there exists positive constant c, independent of ε, such that*

$$\begin{cases} (i)\ \|u'_{1\varepsilon}\|_{L^2(0,T;\,(V^\varepsilon)')} \le c(1+\varepsilon^{\frac{\gamma-1}{2}}),\\ (ii)\ \|u'_{2\varepsilon}\|_{L^2(0,T;\,(H^1(\Omega_{2\varepsilon}))')} \le c(1+\varepsilon^{\frac{\gamma-1}{2}}),\\ (iii)\ \|(\widetilde{u_{1\varepsilon}}+\widetilde{u_{2\varepsilon}})'\|_{L^2(0,T;\,H^{-1}(\Omega))} \le c. \end{cases} \tag{4.1}$$

Proof. Using (2.8), the variational formulation (2.14) and the Hölder inequality, we deduce that

$$\begin{aligned}
&|\langle u'_{1\varepsilon}, v_1\rangle_{L^2(0,T;\,(V^\varepsilon)'),L^2(0,T;\,V^\varepsilon)} + \langle u'_{2\varepsilon}, v_2\rangle_{L^2(0,T;\,(H^1(\Omega_{2\varepsilon}))'),L^2(0,T;\,H^1(\Omega_{2\varepsilon}))}|\\
&= \bigg|\int_0^T\int_{\Omega_{1\varepsilon}} f_{1\varepsilon}v_1\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle g, P_1^\varepsilon v_1\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\mathrm{d}t + \int_0^T\int_{\Omega_{2\varepsilon}} f_{2\varepsilon}v_2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \int_0^T\int_{\Omega_{1\varepsilon}} A^\varepsilon\nabla u_{1\varepsilon}\nabla v_1\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_{\Omega_{2\varepsilon}} A^\varepsilon\nabla u_{2\varepsilon}\nabla v_2\,\mathrm{d}x\,\mathrm{d}t\\
&\quad - \varepsilon^\gamma\int_0^T\int_{\Gamma^\varepsilon} h^\varepsilon(u_{1\varepsilon}-u_{2\varepsilon})(v_1-v_2)\,\mathrm{d}\sigma_x\,\mathrm{d}t\bigg|\\
&\le \|\widetilde{f_{1\varepsilon}}\|_{L^2(0,T;\,L^2(\Omega))}\|v_1\|_{L^2(0,T;\,L^2(\Omega_{1\varepsilon}))} + \|g\|_{L^2(0,T;\,H^{-1}(\Omega))}\|P_1^\varepsilon v_1\|_{L^2(0,T;\,H_0^1(\Omega))}\\
&\quad + \|\widetilde{f_{2\varepsilon}}\|_{L^2(0,T;\,L^2(\Omega))}\|v_2\|_{L^2(0,T;\,L^2(\Omega_{2\varepsilon}))} + \beta\|\widetilde{\nabla u_{1\varepsilon}}\|_{L^2(0,T;\,L^2(\Omega))}\|\nabla v_1\|_{L^2(0,T;\,L^2(\Omega_{1\varepsilon}))}\\
&\quad + \beta\|\widetilde{\nabla u_{2\varepsilon}}\|_{L^2(0,T;\,L^2(\Omega))}\|\nabla v_2\|_{L^2(0,T;\,L^2(\Omega_{2\varepsilon}))}\\
&\quad + \|h^\varepsilon\|_{L^\infty(\Gamma^\varepsilon)}\varepsilon^{\frac{\gamma}{2}}\|u_{1\varepsilon}-u_{2\varepsilon}\|_{L^2(0,T;\,L^2(\Gamma^\varepsilon))}\varepsilon^{\frac{\gamma}{2}}\|v_1-v_2\|_{L^2(0,T;\,L^2(\Gamma^\varepsilon))}.
\end{aligned}$$

From (2.18), (2.19), Lemma 4.5, Remark 2.3 and Proposition 4.1(ii) it follows that

$$\begin{aligned}
&|\langle u'_{1\varepsilon}, v_1\rangle_{L^2(0,T;\,(V^\varepsilon)'),L^2(0,T;\,V^\varepsilon)} + \langle u'_{2\varepsilon}, v_2\rangle_{L^2(0,T;\,(H^1(\Omega_{2\varepsilon}))'),L^2(0,T;\,H^1(\Omega_{2\varepsilon}))}|\\
&\qquad \le c(\|v_1\|_{L^2(0,T;\,V^\varepsilon)} + \|v_2\|_{L^2(0,T;\,H^1(\Omega_{2\varepsilon}))} + \varepsilon^{\frac{\gamma}{2}}\|v_1-v_2\|_{L^2(0,T;\,L^2(\Gamma^\varepsilon))})\\
&\qquad \le c\|(v_1,v_2)\|_{L^2(0,T;\,H_\gamma^\varepsilon)}.
\end{aligned}\tag{4.2}$$

Therefore, together with Proposition 4.1(i), we have for every $v \in L^2(0,T;\ H_\gamma^\varepsilon)$

$$\begin{aligned}
&|\langle u'_{1\varepsilon}, v_1\rangle_{L^2(0,T;\,(V^\varepsilon)'),L^2(0,T;\,V^\varepsilon)} + \langle u'_{2\varepsilon}, v_2\rangle_{L^2(0,T;\,(H^1(\Omega_{2\varepsilon}))'),L^2(0,T;\,H^1(\Omega_{2\varepsilon}))}|\\
&\qquad \le c\|(v_1,v_2)\|_{L^2(0,T;\,H_\gamma^\varepsilon)}\\
&\qquad \le c(1+\varepsilon^{\frac{\gamma-1}{2}})\|(v_1,v_2)\|_{L^2(0,T;\,V^\varepsilon)\times L^2(0,T;\,H^1(\Omega_{2\varepsilon}))}.
\end{aligned}\tag{4.3}$$

Taking $v_2 = 0$ in (4.3), we get

$$\|u'_{1\varepsilon}\|_{L^2(0,T;\,(V^\varepsilon)')} = \sup\frac{|\langle u'_{1\varepsilon}, v_1\rangle_{L^2(0,T;\,(V^\varepsilon)'),L^2(0,T;\,V^\varepsilon)}|}{\|v_1\|_{L^2(0,T;\,V^\varepsilon)}} \le c(1+\varepsilon^{\frac{\gamma-1}{2}}).$$

Similarly, taking $v_1 = 0$ in (4.3), gives

$$\|u'_{2\varepsilon}\|_{L^2(0,T;\,(H^1(\Omega_{2\varepsilon}))')} \le c(1+\varepsilon^{\frac{\gamma-1}{2}}).$$

It remains to prove (iii). To do that, take $v_1 = v_2 = \psi$ in (4.2), where $\psi \in L^2(0,T;\ H_0^1(\Omega))$. Then, together with Lemma 4.4, one has

$$
\begin{aligned}
&|\langle (\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}})', \psi \rangle_{L^2(0,T;\ H^{-1}(\Omega)), L^2(0,T;\ H_0^1(\Omega))}| \\
&\qquad = |\langle u'_{1\varepsilon}, \psi|_{\Omega_{1\varepsilon}} \rangle_{L^2(0,T;\ (V^\varepsilon)'), L^2(0,T;\ V^\varepsilon)} + \langle u'_{2\varepsilon}, \psi|_{\Omega_{2\varepsilon}} \rangle_{L^2(0,T;\ (H^1(\Omega_{2\varepsilon}))'), L^2(0,T;\ H^1(\Omega_{2\varepsilon}))}| \\
&\qquad \leq c\|(\psi,\psi)\|_{L^2(0,T;\ H_\gamma^\varepsilon)} = c\|(\psi,\psi)\|_{L^2(0,T;\ V^\varepsilon)\times L^2(0,T;\ H^1(\Omega_{2\varepsilon}))} \leq 2c\|\psi\|_{L^2(0,T;\ H_0^1(\Omega))}.
\end{aligned}
$$

This concludes the proof. $\square$

We are now in the position to give the main result of this section stating the compactness of $\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}}$ in $\mathcal{C}^0([0,T]; H^{-1}(\Omega))$.

Theorem 4.8. *Let $\gamma \leq 1$. Then, under the assumptions of Theorem 2.5, the following strong convergence holds:*

$$
\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}} \to \theta_1 u_1 + u_2 \qquad \text{in} \quad \mathcal{C}^0([0,T]; H^{-1}(\Omega)). \tag{4.4}
$$

Moreover, if $\gamma = 1$, one has separately that

$$
\begin{cases}
\widetilde{u_{1\varepsilon}} \to \theta_1 u_1 & \text{in} \quad \mathcal{C}^0([0,T]; H^{-1}(\Omega)), \\
\widetilde{u_{2\varepsilon}} \to u_2 & \text{in} \quad \mathcal{C}^0([0,T]; H^{-1}(\Omega)).
\end{cases} \tag{4.5}
$$

Proof. As a consequence of (2.19)(ii and iii) and (4.1)(iii),

$$
\begin{cases}
\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}} \rightharpoonup \theta_1 u_1 + u_2 & \text{weakly* in } L^\infty(0,T;\ L^2(\Omega)), \\
(\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}})' & \text{is bounded in } L^2(0,T;\ H^{-1}(\Omega)).
\end{cases}
$$

Hence, from classical compactness results (see [22]),

$$
\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}} \text{ is relatively compact in } \mathcal{C}^0([0,T]; H^{-1}(\Omega)),
$$

whence, (4.4).

For the case $\gamma = 1$, we apply to each of $\widetilde{u_{1\varepsilon}}$ and $\widetilde{u_{2\varepsilon}}$ the same arguments as that of the previous case. Observe that in this case from (4.1)(i and ii), the norms $\|u'_{1\varepsilon}\|_{L^2(0,T;\ (V^\varepsilon)')}$ and $\|u'_{2\varepsilon}\|_{L^2(0,T;\ (H^1(\Omega_{2\varepsilon}))')}$ are bounded. Then, the result is a consequence of Theorem 2.5 and Lemma 3.3 of [15] (which still holds by replacing V^ε by $H^1(\Omega_{2\varepsilon})$). $\square$

Remark 4.9. We emphasize that for $\gamma < 1$, the above theorem does not provide separately uniform estimates for $\widetilde{u_{1\varepsilon}}$ and $\widetilde{u_{2\varepsilon}}$ so that in this case, we only have the compactness (4.4) for the sum $\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}}$. These results are sufficient for proving Propositions 6.5 and 6.8.

The relationship between the energies of problem (2.14) and those of limit problems (3.14) and (2.22) will play a crucial role when proving the corrector results. We discuss these in the following section.

5. Asymptotic behavior of the energy

In this section, we show that under the stronger assumptions on the data from Section 3, the energy of problem (2.14) converges in $\mathcal{C}^0([0,T])$ to that of the homogenized one. By definition, the energy d_ε associated to problem (2.14) is given by

$$
\begin{aligned}
d_\varepsilon(t) = {}& \frac{1}{2}\|u_{1\varepsilon}(t)\|^2_{L^2(\Omega_{1\varepsilon})} + \frac{1}{2}\|u_{2\varepsilon}(t)\|^2_{L^2(\Omega_{2\varepsilon})} + \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} \nabla u_{1\varepsilon}\, \mathrm{d}x\, \mathrm{d}s \\
&+ \int_0^t \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon} \nabla u_{2\varepsilon}\, \mathrm{d}x\, \mathrm{d}s + \varepsilon^\gamma \int_0^t \int_{\Gamma^\varepsilon} h^\varepsilon |u_{1\varepsilon} - u_{2\varepsilon}|^2\, \mathrm{d}\sigma_x\, \mathrm{d}s.
\end{aligned} \tag{5.1}
$$

Now, suppose $u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon})$ is the solution to problem (2.12). By taking $(u_{1\varepsilon}, u_{2\varepsilon})$ as test function in (2.14) and integrating by parts, we get

$$\begin{aligned}\frac{1}{2}\|u_{1\varepsilon}(t)\|^2_{L^2(\Omega_{1\varepsilon})} + \frac{1}{2}\|u_{2\varepsilon}(t)\|^2_{L^2(\Omega_{2\varepsilon})} + \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} \nabla u_{1\varepsilon} \, \mathrm{d}x \, \mathrm{d}s \\ + \int_0^t \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon} \nabla u_{2\varepsilon} \, \mathrm{d}x \, \mathrm{d}s + \varepsilon^\gamma \int_0^t \int_{\Gamma^\varepsilon} h^\varepsilon |u_{1\varepsilon} - u_{2\varepsilon}|^2 \, \mathrm{d}\sigma_x \, \mathrm{d}s \\ = \frac{1}{2}\|u_{1\varepsilon}(0)\|^2_{L^2(\Omega_{1\varepsilon})} + \frac{1}{2}\|u_{2\varepsilon}(0)\|^2_{L^2(\Omega_{2\varepsilon})} + \int_0^t \int_{\Omega_{1\varepsilon}} f_{1\varepsilon} u_{1\varepsilon} \, \mathrm{d}x \, \mathrm{d}s \\ + \int_0^t \langle g, P_1^\varepsilon u_{1\varepsilon} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, \mathrm{d}s + \int_0^t \int_{\Omega_{2\varepsilon}} f_{2\varepsilon} u_{2\varepsilon} \, \mathrm{d}x \, \mathrm{d}s.\end{aligned} \tag{5.2}$$

Hence, $d_\varepsilon(t)$ can be rewritten as

$$\begin{aligned}d_\varepsilon(t) = \frac{1}{2}\|U^0_{1\varepsilon}\|^2_{L^2(\Omega_{1\varepsilon})} + \frac{1}{2}\|U^0_{2\varepsilon}\|^2_{L^2(\Omega_{2\varepsilon})} + \int_0^t \int_{\Omega_{1\varepsilon}} f_{1\varepsilon} u_{1\varepsilon} \, \mathrm{d}x \, \mathrm{d}s \\ + \int_0^t \langle g, P_1^\varepsilon u_{1\varepsilon} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, \mathrm{d}s + \int_0^t \int_{\Omega_{2\varepsilon}} f_{2\varepsilon} u_{2\varepsilon} \, \mathrm{d}x \, \mathrm{d}s.\end{aligned} \tag{5.3}$$

We now show that $\{d_\varepsilon\}$ is relatively compact in $\mathcal{C}^0([0,T])$.

Proposition 5.1. *Under the hypotheses of Theorem 2.5, there exists a subsequence (still denoted by ε) and $d_0 \in \mathcal{C}^0([0,T])$ such that $d_\varepsilon \to d_0$ in $\mathcal{C}^0([0,T])$.*

Proof. From (5.3) and the Hölder inequality, we have

$$\begin{aligned}|d_\varepsilon(t)| \le \frac{1}{2}\|U^0_{1\varepsilon}\|^2_{L^2(\Omega_{1\varepsilon})} + \frac{1}{2}\|U^0_{2\varepsilon}\|^2_{L^2(\Omega_{2\varepsilon})} + \|\widetilde{f_{1\varepsilon}}\|_{L^2(0,T;\, L^2(\Omega))} \|\widetilde{u_{1\varepsilon}}\|_{L^2(0,T;\, L^2(\Omega))} \\ + \|\widetilde{f_{2\varepsilon}}\|_{L^2(0,T;\, L^2(\Omega))} \|\widetilde{u_{2\varepsilon}}\|_{L^2(0,T;\, L^2(\Omega))} + \|g\|_{L^2(0,T;\, H^{-1}(\Omega))} \|P_1^\varepsilon u_{1\varepsilon}\|_{L^2(0,T;\, H_0^1(\Omega))}\end{aligned}$$

so that in view of Theorem 2.5, $\{d_\varepsilon\}$ is bounded in $L^\infty(0,T)$.

Now, for any $t \in [0,T]$ and $h > 0$ small enough, one has

$$\begin{aligned}|d_\varepsilon(t+h) - d_\varepsilon(t)| \le h^{1/2} \|\widetilde{u_{1\varepsilon}}\|_{L^\infty(0,T;\, L^2(\Omega))} \|\widetilde{f_{1\varepsilon}}\|_{L^2(0,T; L^2(\Omega))} + h^{1/2} \|\widetilde{u_{2\varepsilon}}\|_{L^\infty(0,T;\, L^2(\Omega))} \|\widetilde{f_{2\varepsilon}}\|_{L^2(0,T; L^2(\Omega))} \\ + \|g\|_{L^2([t,t+h]; H^{-1}(\Omega))} \, \|P_1^\varepsilon u_{1\varepsilon}\|_{L^2(0,T; H_0^1(\Omega))}.\end{aligned}$$

Using again Theorem 2.5 and taking $h \to 0$, we get

$$|d_\varepsilon(t+h) - d_\varepsilon(t)| \le c(h^{1/2} + \|g\|_{L^2([t,t+h]; H^{-1}(\Omega))}) \to 0 \quad \text{uniformly in } \varepsilon.$$

Hence, by Ascoli-Arzela's Theorem, $d_\varepsilon \to d_0$ for some d_0 in $\mathcal{C}^0([0,T])$. $\square$

We introduce now the energy associated with the homogenized problem for the two cases.

- For the case $-1 < \gamma < 1$, the energy associated with the limit problem (3.14), denoted by d is defined by

$$d(t) = \frac{1}{2}\|u_1(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega A^0 \nabla u_1 \nabla u_1 \, \mathrm{d}x \, \mathrm{d}s. \tag{5.4}$$

We can rewrite this expression by multiplying the first equation of the homogenized problem (3.14) by u_1 and integrating by parts. Using the same argument as above for proving (5.2) and taking into account the initial conditions of (3.14), we get

$$\begin{aligned} d(t) &= \frac{1}{2}\|u_1(0)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega (\theta_1 f_1 + \theta_2 f_2) u_1 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \langle g, u_1\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} \,\mathrm{d}s \\ &= \frac{1}{2}\|U^0\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega (\theta_1 f_1 + \theta_2 f_2) u_1 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \langle g, u_1\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} \,\mathrm{d}s. \end{aligned} \tag{5.5}$$

- For $\gamma = 1$, the energy associated with the homogenized problem (2.22), denoted by d_1 is defined by

$$\begin{aligned} d_1(t) = {} & \frac{1}{2}\theta_1\|u_1(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\theta_2^{-1}\|u_2(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega A^0 \nabla u_1 \nabla u_1 \,\mathrm{d}x\,\mathrm{d}s \\ & + c_h \theta_2^{-1} \int_0^t \int_\Omega (\theta_2 u_1 - u_2)^2 \,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{5.6}$$

Proposition 5.2. *If $\gamma = 1$,*

$$\begin{aligned} d_1(t) = {} & \frac{1}{2}\theta_1\|U_1^0\|^2_{L^2(\Omega)} + \frac{1}{2}\theta_2\|U_2^0\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega \theta_1 f_1 u_1 \,\mathrm{d}x\,\mathrm{d}s \\ & + \int_0^t \langle g, u_1\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} \,\mathrm{d}s + \int_0^t \int_\Omega f_2 u_2 \,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{5.7}$$

Proof. Taking u_1 in the first equation and $\theta_2^{-1}u_2$ in the second equation of the homogenized problem (2.22) and integrating by parts, we have

$$\begin{aligned} & \int_0^t \langle \theta_1 u_1', u_1\rangle_{L^2(\Omega),L^2(\Omega)} \,\mathrm{d}s + \int_0^t \langle u_2', \theta_2^{-1} u_2\rangle_{L^2(\Omega),L^2(\Omega)} \,\mathrm{d}s + \int_0^t \int_\Omega A^0 \nabla u_1 \nabla u_1 \,\mathrm{d}x\,\mathrm{d}s \\ & \qquad + c_h \int_0^t \int_\Omega (\theta_2 u_1 - u_2) u_1 \,\mathrm{d}x\,\mathrm{d}s - c_h \int_0^t \int_\Omega (\theta_2 u_1 - u_2)\theta_2^{-1} u_2 \,\mathrm{d}x\,\mathrm{d}s \\ & \qquad\qquad = \int_0^t \int_\Omega \theta_1 f_1 u_1 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \langle g, u_1\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} \,\mathrm{d}s + \int_0^t \int_\Omega \theta_2 f_2 (\theta_2^{-1} u_2) \,\mathrm{d}x\,\mathrm{d}s. \end{aligned}$$

Now,

$$\int_0^t \langle \theta_1 u_1', u_1\rangle_{L^2(\Omega),L^2(\Omega)} \,\mathrm{d}s = \frac{1}{2}\theta_1\|u_1(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\theta_1\|u_1(0)\|^2_{L^2(\Omega)}.$$

Similarly,

$$\int_0^t \langle u_2', \theta_2^{-1} u_2\rangle_{L^2(\Omega),L^2(\Omega)} \,\mathrm{d}s = \frac{1}{2}\theta_2^{-1}\|u_2(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\theta_2^{-1}\|u_2(0)\|^2_{L^2(\Omega)}.$$

Also,

$$c_h \int_0^t \int_\Omega (\theta_2 u_1 - u_2) u_1 \,\mathrm{d}x\,\mathrm{d}s - c_h \int_0^t \int_\Omega (\theta_2 u_1 - u_2)\theta_2^{-1} u_2 \,\mathrm{d}x\,\mathrm{d}s = c_h \theta_2^{-1} \int_0^t \int_\Omega (\theta_2 u_1 - u_2)^2 \,\mathrm{d}x\,\mathrm{d}s.$$

Therefore,

$$d_1(t) = \frac{1}{2}\theta_1 \|u_1(0)\|^2_{L^2(\Omega)} + \frac{1}{2}\theta_2^{-1}\|u_2(0)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega \theta_1 f_1 u_1 \,\mathrm{d}x \,\mathrm{d}s$$
$$+ \int_0^t \langle g, u_1\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \,\mathrm{d}s + \int_0^t \int_\Omega f_2 u_2 \,\mathrm{d}x \,\mathrm{d}s.$$

Using the initial conditions in (2.22), we get (5.7) since

$$\frac{1}{2}\theta_2^{-1}\|\theta_2 U_2^0\|^2_{L^2(\Omega)} = \frac{1}{2}\theta_2 \|U_2^0\|^2_{L^2(\Omega)}. \qquad \square$$

In the following propositions, we prove that for both cases, the energy d_ε converges to the respective energy of the associated homogenized problem.

Proposition 5.3 (convergence of energy for $-1 < \gamma < 1$)**.** *Let A^ε and h^ε be defined by* (2.9) *and* (2.11) *respectively. Suppose that* (3.7) *and* (3.9) *hold. If $(u_{1\varepsilon}, u_{2\varepsilon})$ is the solution of problem* (2.14) *and u_1 is the solution of the homogenized problem* (3.14) *then*

$$d_\varepsilon \to d \qquad in \quad \mathcal{C}^0([0,T]),$$

where d_ε and d are given by (5.3) *and* (5.5) *respectively.*

Proof. As noted in Remark 3.3, we can apply the homogenization results stated in Theorem 2.5. From (3.7) and (2.19)(ii and iii), we obtain

$$\lim_{\varepsilon\to 0} \int_0^t \int_{\Omega_{1\varepsilon}} f_{1\varepsilon} u_{1\varepsilon} \,\mathrm{d}x \,\mathrm{d}s = \lim_{\varepsilon\to 0} \int_0^t \int_\Omega f_{1\varepsilon} \widetilde{u_{1\varepsilon}} \,\mathrm{d}x \,\mathrm{d}s = \int_0^t \int_\Omega f_1 \theta_1 u_1 \,\mathrm{d}x \,\mathrm{d}s \tag{5.8}$$

and

$$\lim_{\varepsilon\to 0} \int_0^t \int_{\Omega_{2\varepsilon}} f_{2\varepsilon} u_{2\varepsilon} \,\mathrm{d}x \,\mathrm{d}s = \lim_{\varepsilon\to 0} \int_0^t \int_\Omega f_{2\varepsilon} \widetilde{u_{2\varepsilon}} \,\mathrm{d}x \,\mathrm{d}s = \int_0^t \int_\Omega f_2 u_2 \,\mathrm{d}x \,\mathrm{d}s. \tag{5.9}$$

Also, from (2.19)(i),

$$\lim_{\varepsilon\to 0} \int_0^t \langle g, P_1^\varepsilon u_{1\varepsilon}|_{\Omega_{1\varepsilon}}\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \,\mathrm{d}s = \int_0^t \langle g, u_1\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \,\mathrm{d}s. \tag{5.10}$$

Clearly, from assumptions (3.9) and Lemma 3.1 we have

$$\frac{1}{2}\|U_{1\varepsilon}^0\|^2_{L^2(\Omega_{1\varepsilon})} + \frac{1}{2}\|U_{2\varepsilon}^0\|^2_{L^2(\Omega_{2\varepsilon})} \to \frac{1}{2}\|U^0\|^2_{L^2(\Omega)}.$$

Since $u_2 = \theta_2 u_1$, combining the above convergences and using Proposition 5.1, the conclusion follows. $\square$

Proposition 5.4 (convergence of energy for $\gamma = 1$)**.** *Let A^ε and h^ε be defined by* (2.9) *and* (2.11) *respectively. Suppose that* (3.7) *and* (3.15) *hold. If $(u_{1\varepsilon}, u_{2\varepsilon})$ is the solution of problem* (2.14) *and (u_1, u_2) is the solution of the homogenized problem* (2.22)*, then*

$$d_\varepsilon \to d_1 \qquad in\, \mathcal{C}^0([0,T]),$$

where d_ε and d_1 are given by (5.3) *and* (5.7) *respectively.*

Proof. Under assumptions (3.7) and (3.15), Theorem 2.5 applies (see also Rem. 3.8). By using (3.7) and (2.19)(i, ii and iii), we still have (5.8)–(5.10). Obviously, by (3.15)(iii),

$$\frac{1}{2}\|U^0_{1\varepsilon}\|^2_{L^2(\Omega_{1\varepsilon})} + \frac{1}{2}\|U^0_{2\varepsilon}\|^2_{L^2(\Omega_{2\varepsilon})} \to \frac{1}{2}\theta_1\|U^0_1\|^2_{L^2(\Omega)} + \frac{1}{2}\theta_2\|U^0_2\|^2_{L^2(\Omega)}.$$

Therefore, by Proposition 5.1 and the above results, $d_\varepsilon \to d_1$ in $\mathcal{C}^0([0,T])$. □

6. Proof of the corrector results

We prove in this section the corrector results stated in Theorems 3.4 and 3.9. The proofs are rather technical and are based on the results of Sections 4 and 5. We adapt to our cases techniques used by Donato in the elliptic case [12], Donato *et al.* in the hyperbolic case [18] and Donato and Nabil for the parabolic case in perforated domains [15].

We recall some technical lemmas, the first one being a classical density result.

Lemma 6.1. *Let $v \in L^2(0,T;\ H^1_0(\Omega)) \cap \mathcal{C}^0([0,T];L^2(\Omega))$. Then for any $\delta > 0$, there exists $\phi \in \mathcal{C}^\infty(0,T;\mathcal{D}(\Omega))$ such that*

$$\begin{cases} (i)\ \|v-\phi\|_{\mathcal{C}^0([0,T];L^2(\Omega))} \le \delta, \\ (ii)\ \|\nabla v - \nabla\phi\|_{L^2(0,T;\ L^2(\Omega))} \le \delta. \end{cases} \tag{6.1}$$

The next lemma proved by Donato and Nabil [15] overcomes the technical difficulty in passing to the limit in products with two weakly converging sequences when one of them is independent of t.

Lemma 6.2 [15]**.** *Let $(h_\varepsilon) \subset L^p(0,T;W^{1,q}_0(\Omega))$ and $(g_\varepsilon) \subset L^{q'}(\Omega)$ with $p,q \ge 1$ and $\frac{1}{q}+\frac{1}{q'} = 1$ be two sequences such that*

$$\begin{cases} h_\varepsilon \rightharpoonup h & \text{weakly in } L^p(0,T;W^{1,q}_0(\Omega)), \\ g_\varepsilon \rightharpoonup g & \text{weakly in } L^{q'}(\Omega). \end{cases}$$

Then

$$h_\varepsilon g_\varepsilon \rightharpoonup hg \qquad \text{weakly in } L^p(0,T;L^1(\Omega)).$$

We also state the following results, which are straightforward extensions to the time-dependent case of the results proved in [6,14] (see also [12]) respectively. The first one provides an inequality for weakly convergent sequences while the second one transforms integral on the boundary Γ^ε into volume integrals on $\Omega_{2\varepsilon}$.

Lemma 6.3 [14]**.** *Let $\mathcal{O}$ be an open set on $\mathbb{R}^n$ and, for every ε let $\mathcal{O}_\varepsilon$ be an open set such that $\mathcal{O}_\varepsilon \subset \mathcal{O}$. Suppose that $v_\varepsilon \subset L^2(0,T;L^2(\mathcal{O}_\varepsilon))$ and for some $v \in L^2(0,T;L^2(\mathcal{O}))$ the following convergences hold:*

$$\begin{cases} \chi_{\mathcal{O}_\varepsilon} \rightharpoonup \chi_0 & \text{weakly* in } L^\infty(\mathcal{O}), \\ \widetilde{v_\varepsilon} \rightharpoonup \chi_0 v & \text{weakly in } L^2(0,T;L^2(\mathcal{O})). \end{cases}$$

Then

$$\liminf_{\varepsilon\to 0} \int_0^T \int_{\mathcal{O}_\varepsilon} |v_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \ge \int_0^T \int_{\mathcal{O}} \chi_0 |v|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Lemma 6.4 [6,12]**.** *Suppose that Γ is of class C^2. Let g be a function in $L^\infty(\Gamma)$ and set $c_g = \dfrac{1}{|Y_2|}\displaystyle\int_\Gamma g(y)\,\mathrm{d}\sigma_y$. If for some positive constant c (independent of ε) one has*

$$\|v_\varepsilon\|_{L^2(0,T;W^{1,1}(\Omega_{2\varepsilon}))} \le c,$$

then

$$\liminf_{\varepsilon\to 0} \varepsilon \int_0^T \int_{\Gamma^\varepsilon} g(x/\varepsilon) v_\varepsilon(x,t) \, \mathrm{d}\sigma_x \, \mathrm{d}t = \liminf_{\varepsilon\to 0} c_g \int_0^T \int_{\Omega_{2\varepsilon}} v_\varepsilon(x,t) \, \mathrm{d}x \, \mathrm{d}t.$$

We are now in the position to prove the corrector results. We consider first the case $-1 < \gamma < 1$.

6.1. Corrector results for the case $-1 < \gamma < 1$

We prove first the following proposition which is needed in the proof of the corrector results.

Proposition 6.5. *Let Φ be in $\mathcal{C}^\infty(0,T;\mathcal{D}(\Omega))$ and set*

$$\eta_\varepsilon(t) = \frac{1}{2}\|\widetilde{u_{1\varepsilon}}(t) + \widetilde{u_{2\varepsilon}}(t) - \Phi(t)\|^2_{L^2(\Omega)} + \int_0^t \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon} \nabla u_{2\varepsilon} \, \mathrm{d}x \, \mathrm{d}s \\ + \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon (\nabla u_{1\varepsilon} - C^\varepsilon \nabla \Phi)(\nabla u_{1\varepsilon} - C^\varepsilon \nabla\Phi) \, \mathrm{d}x \, \mathrm{d}s. \tag{6.2}$$

Then, under the assumptions of Theorem 3.4,

$$\limsup_{\varepsilon\to 0} \|\eta_\varepsilon\|_{\mathcal{C}^0([0,T])} \le \|\eta\|_{\mathcal{C}^0([0,T])}, \tag{6.3}$$

where $\eta(t)$ is given by

$$\eta(t) = \frac{1}{2}\|u_1(t) - \Phi(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega A^0 (\nabla u_1 - \nabla\Phi)(\nabla u_1 - \nabla \Phi) \, \mathrm{d}x \, \mathrm{d}s. \tag{6.4}$$

Remark 6.6. The above proposition is a weaker result compared to what can be seen in the general literature (see for instance [3,15]), where for appropriate η_ε and η one has

$$\eta_\varepsilon \to \eta \quad \text{in } \mathcal{C}^0([0,T]).$$

Nevertheless, (6.3) is enough to prove our main theorem. A similar situation occurs in the same geometrical framework for the hyperbolic case [18].

Proof. We begin by decomposing (6.2) into three terms

$$\eta_\varepsilon(t) = \eta^1_\varepsilon(t) - \eta^2_\varepsilon(t) + \eta^3_\varepsilon(t), \tag{6.5}$$

where

$$\eta^1_\varepsilon(t) = \frac{1}{2}\|\Phi(t)\|^2_{L^2(\Omega)} + \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon C^\varepsilon \nabla\Phi C^\varepsilon \nabla \Phi \, \mathrm{d}x \, \mathrm{d}s,$$
$$\eta^2_\varepsilon(t) = \int_\Omega (\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}})\Phi \, \mathrm{d}x + \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon C^\varepsilon \nabla\Phi \nabla u_{1\varepsilon} \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} C^\varepsilon \nabla\Phi \, \mathrm{d}x \, \mathrm{d}s,$$
$$\eta^3_\varepsilon(t) = \frac{1}{2}\|\widetilde{u_{1\varepsilon}}(t) + \widetilde{u_{2\varepsilon}}(t)\|^2_{L^2(\Omega)} + \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} \nabla u_{1\varepsilon} \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon} \nabla u_{2\varepsilon} \, \mathrm{d}x \, \mathrm{d}s.$$

Step 1. Let us study first the term $\eta^2_\varepsilon(t)$ which is the most complicated. Our aim is to show that:

$$\lim_{\varepsilon\to 0} \eta^2_\varepsilon(t) = \int_\Omega u_1 \Phi \, \mathrm{d}x + \int_0^t \int_\Omega A^0 \nabla\Phi \nabla u_1 \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \int_\Omega A^0 \nabla u_1 \nabla \Phi \, \mathrm{d}x \, \mathrm{d}s \\ := \eta^2(t) \qquad \text{in } \mathcal{C}^0([0,T]). \tag{6.6}$$

To do that, we decompose it into three terms defined below, that will be treated separately

$$\eta_\varepsilon^2(t) = \kappa_\varepsilon^1(t) + \kappa_\varepsilon^2(t) + \kappa_\varepsilon^3(t), \tag{6.7}$$

where

$$\begin{aligned}
\kappa_\varepsilon^1(t) &= \int_\Omega (\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}})\Phi \,\mathrm{d}x,\\
\kappa_\varepsilon^2(t) &= \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon C^\varepsilon \nabla\Phi \nabla u_{1\varepsilon} \,\mathrm{d}x\,\mathrm{d}s,\\
\kappa_\varepsilon^3(t) &= \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} C^\varepsilon \nabla\Phi \,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

Step 1.1. For the first term $\kappa_\varepsilon^1(t)$, observe that

$$\max_{t\in[0,T]} \left| \int_\Omega [(\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}})\Phi - u_1\Phi] \mathrm{d}x \right| \le \|\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}} - u_1\|_{\mathcal{C}^0([0,T];H^{-1}(\Omega))} \|\Phi\|_{\mathcal{C}^0([0,T];H_0^1(\Omega))}.$$

Hence, from Theorem 4.8 (see (4.4)) and recalling that $u_2 = \theta_2 u_1$ for the case $-1 < \gamma < 1$, we have

$$\kappa_\varepsilon^1(t) \to \int_\Omega u_1 \Phi \,\mathrm{d}x \qquad \text{in } \mathcal{C}^0([0,T]). \tag{6.8}$$

Step 1.2. We consider now $\kappa_\varepsilon^2(t)$ and we follow the proof of Proposition 5.3 of [15]. Note that using (3.5) with $v = u_{1\varepsilon} \dfrac{\partial \Phi}{\partial x_i}$, we have

$$\kappa_\varepsilon^2(t) = \sum_{i=1}^n \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla \widehat{w}_i^\varepsilon \frac{\partial\Phi}{\partial x_i} \nabla u_{1\varepsilon} \,\mathrm{d}x\,\mathrm{d}s = -\sum_{i=1}^n \int_0^t \int_\Omega \chi_{\Omega_{1\varepsilon}} A^\varepsilon \nabla \widehat{w}_i^\varepsilon \nabla \left(\frac{\partial\Phi}{\partial x_i} \right) P_1^\varepsilon u_{1\varepsilon} \,\mathrm{d}x\,\mathrm{d}s.$$

By (2.19)(i), (3.6)(iv) and Lemma 6.2 with $h_\varepsilon = P_1^\varepsilon u_{1\varepsilon}$ and $g_\varepsilon = \chi_{\Omega_{1\varepsilon}} A^\varepsilon \nabla \widehat{w}_i^\varepsilon$, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0} \kappa_\varepsilon^2(t) &= -\sum_{i=1}^n \int_0^t \int_\Omega A^0 e_i \nabla \left(\frac{\partial\Phi}{\partial x_i} \right) u_1 \,\mathrm{d}x\,\mathrm{d}s\\
&= \int_0^t \int_\Omega A^0 \nabla\Phi \nabla u_1 \,\mathrm{d}x\,\mathrm{d}s \qquad \forall t \in [0,T].
\end{aligned} \tag{6.9}$$

Now, observe that by (2.19)(ii) and from the properties of $\Phi, A^\varepsilon, C^\varepsilon$ (given by (2.8)(ii) and (3.4)), it follows that $\kappa_\varepsilon^2(t)$ is bounded in $H^1(0,T)$.

Hence, by (6.9) and the compactness of the injection $H^1(0,T) \subset \mathcal{C}^0([0,T])$, we conclude that

$$\kappa_\varepsilon^2 \to \kappa^2 \qquad \text{in } \mathcal{C}^0([0,T]), \qquad \text{where} \quad \kappa^2(t) = \int_0^t \int_\Omega A^0 \nabla\Phi \nabla u_1 \,\mathrm{d}x\,\mathrm{d}s. \tag{6.10}$$

Step 1.3. In this step, we study the third and last term κ_ε^3, which is the most delicate. Our tasks are to identify the pointwise limit of $\kappa_\varepsilon^3(t)$ and to show its compactness in $\mathcal{C}^0([0,T])$, the former being the main difficulty. Let us show first that

$$\kappa_\varepsilon^3(t) \to \int_0^t \int_\Omega A^0 \nabla u_1 \nabla\Phi \,\mathrm{d}x\,\mathrm{d}s \qquad \forall t \in [0,T].$$

Observe that

$$\begin{aligned}\kappa_\varepsilon^3(t) &= \sum_{i=1}^n \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} \nabla \widehat{w}_i^\varepsilon \frac{\partial \Phi}{\partial x_i} \, \mathrm{d}x \, \mathrm{d}s \\ &= \sum_{i=1}^n \left\{ \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} \nabla \left(\widehat{w}_i^\varepsilon \frac{\partial \Phi}{\partial x_i} \right) \mathrm{d}x \, \mathrm{d}s - \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} \nabla \left(\frac{\partial \Phi}{\partial x_i} \right) \widehat{w}_i^\varepsilon \, \mathrm{d}x \, \mathrm{d}s \right\}.\end{aligned}$$

Using the variational formulation (2.14) with $\left(\widehat{w}_i^\varepsilon \frac{\partial \Phi}{\partial x_i}, x_i \frac{\partial \Phi}{\partial x_i} \right)$ as test function and integrating by parts gives

$$\begin{aligned}\kappa_\varepsilon^3(t) = \sum_{i=1}^n \Bigg\{ & \int_0^t \int_\Omega \widetilde{f_{1\varepsilon}} \widehat{w}_i^\varepsilon \frac{\partial \Phi}{\partial x_i} \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \int_\Omega \widetilde{f_{2\varepsilon}} x_i \frac{\partial \Phi}{\partial x_i} \, \mathrm{d}x \, \mathrm{d}s \\ & + \int_0^t \left\langle g, P_1^\varepsilon \left(\widehat{w}_i^\varepsilon \frac{\partial \Phi}{\partial x_i} \right) |_{\Omega_{1\varepsilon}} \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \mathrm{d}s - \int_0^t \int_\Omega A^\varepsilon \widetilde{\nabla u_{2\varepsilon}} \nabla \left(x_i \frac{\partial \Phi}{\partial x_i} \right) \mathrm{d}x \, \mathrm{d}s \\ & - \int_0^t \left\langle u_{1\varepsilon}', \widehat{w}_i^\varepsilon \frac{\partial \Phi}{\partial x_i} \right\rangle_{(V^\varepsilon)', V^\varepsilon} \mathrm{d}s - \int_0^t \left\langle u_{2\varepsilon}', x_i \frac{\partial \Phi}{\partial x_i} \right\rangle_{(H^1(\Omega_{2\varepsilon}))', H^1(\Omega_{2\varepsilon})} \mathrm{d}s \\ & - \varepsilon^\gamma \int_0^t \int_{\Gamma^\varepsilon} h^\varepsilon (u_{1\varepsilon} - u_{2\varepsilon}) \left(\widehat{w}_i^\varepsilon \frac{\partial \Phi}{\partial x_i} - x_i \frac{\partial \Phi}{\partial x_i} \right) \mathrm{d}\sigma_x \, \mathrm{d}s - \int_0^t \int_\Omega A^\varepsilon \widetilde{\nabla u_{1\varepsilon}} \nabla \left(\frac{\partial \Phi}{\partial x_i} \right) \widehat{w}_i^\varepsilon \, \mathrm{d}x \, \mathrm{d}s \Bigg\}.\end{aligned} \tag{6.11}$$

Now, we evaluate the limit of (6.11) term by term. By using (3.8), (3.6)(ii) and Lemma 4.6, we have

$$\begin{aligned}\sum_{i=1}^n \lim_{\varepsilon \to 0} \Bigg\{ \int_0^t \int_\Omega \widetilde{f_{1\varepsilon}} \widehat{w}_i^\varepsilon \frac{\partial \Phi}{\partial x_i} \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \int_\Omega \widetilde{f_{2\varepsilon}} x_i \frac{\partial \Phi}{\partial x_i} \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \left\langle g, P_1^\varepsilon (\widehat{w}_i^\varepsilon \frac{\partial \Phi}{\partial x_i}) |_{\Omega_{1\varepsilon}} \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \mathrm{d}s \Bigg\} \\ = \sum_{i=1}^n \left\{ \int_0^t \int_\Omega (\theta_1 f_1 + \theta_2 f_2) x_i \frac{\partial \Phi}{\partial x_i} \, \mathrm{d}x \, \mathrm{d}s + \int_0^t \left\langle g, x_i \frac{\partial \Phi}{\partial x_i} \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \mathrm{d}s \right\},\end{aligned} \tag{6.12}$$

for every $t \in [0, T]$. On the other hand, from (2.20)(ii),

$$\sum_{i=1}^n \lim_{\varepsilon \to 0} \int_0^t \int_\Omega A^\varepsilon \widetilde{\nabla u_{2\varepsilon}} \nabla \left(x_i \frac{\partial \Phi}{\partial x_i} \right) \mathrm{d}x \, \mathrm{d}s = 0. \tag{6.13}$$

Meanwhile, by (2.20)(i) and (3.6)(ii),

$$\sum_{i=1}^n \lim_{\varepsilon \to 0} \int_0^t \int_\Omega A^\varepsilon \widetilde{\nabla u_{1\varepsilon}} \nabla \left(\frac{\partial \Phi}{\partial x_i} \right) \widehat{w}_i^\varepsilon \, \mathrm{d}x \, \mathrm{d}s = \sum_{i=1}^n \int_0^t \int_\Omega A^0 \nabla u_1 \nabla \left(\frac{\partial \Phi}{\partial x_i} \right) x_i \, \mathrm{d}x \, \mathrm{d}s. \tag{6.14}$$

Now, for the boundary term, observe that by (2.19)(iv), (3.2) and a change of scales,

$$\begin{aligned}\sum_{i=1}^n \varepsilon^\gamma \int_0^t \int_{\Gamma^\varepsilon} h^\varepsilon (u_{1\varepsilon} - u_{2\varepsilon})(\widehat{w}_i^\varepsilon - x_i) \frac{\partial \Phi}{\partial x_i} \, \mathrm{d}\sigma_x \, \mathrm{d}s &\le \sum_{i=1}^n \varepsilon^\gamma \|h\|_{L^\infty(\Gamma)} \|u_{1\varepsilon} - u_{2\varepsilon}\|_{L^2(0,T;\, L^2(\Gamma^\varepsilon))} \|\varepsilon \widehat{\chi}_i^\varepsilon \left(\frac{x}{\varepsilon} \right) \|_{L^2(\Gamma^\varepsilon)} \\ &\le c \varepsilon^{\gamma+1} \varepsilon^{-\gamma/2} \varepsilon^{-1/2} = c \varepsilon^{(\gamma+1)/2} \to 0,\end{aligned} \tag{6.15}$$

as $\varepsilon \to 0$, since $\gamma > -1$ and $\|\widehat{\chi}_i^\varepsilon \left(\frac{x}{\varepsilon} \right) \|_{L^2(\Gamma^\varepsilon)} \le \varepsilon^{-1/2}$.

It remains to show that $\forall t \in [0,T]$,

$$\sum_{i=1}^{n} \lim_{\varepsilon\to 0}\left\{ \int_0^t \left\langle u'_{1\varepsilon}, \widehat{w}_i^\varepsilon \frac{\partial \Phi}{\partial x_i}\right\rangle_{(V^\varepsilon)',V^\varepsilon} \mathrm{d}s + \int_0^t \left\langle u'_{2\varepsilon}, x_i \frac{\partial \Phi}{\partial x_i}\right\rangle_{(H^1(\Omega_{2\varepsilon}))',H^1(\Omega_{2\varepsilon})} \mathrm{d}s\right\}$$
$$= \sum_{i=1}^{n} \int_0^t \left\langle \theta_1 u'_1 + u'_2, x_i \frac{\partial \Phi}{\partial x_i}\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \mathrm{d}s \tag{6.16}$$
$$= \sum_{i=1}^{n} \int_0^t \left\langle u'_1, x_i \frac{\partial \Phi}{\partial x_i}\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \mathrm{d}s.$$

To prove (6.16), we rewrite

$$\int_0^t \left\langle u'_{1\varepsilon}, \widehat{w}_i^\varepsilon \frac{\partial \Phi}{\partial x_i}\right\rangle_{(V^\varepsilon)',V^\varepsilon} \mathrm{d}s + \int_0^t \left\langle u'_{2\varepsilon}, x_i \frac{\partial \Phi}{\partial x_i}\right\rangle_{(H^1(\Omega_{2\varepsilon}))',H^1(\Omega_{2\varepsilon})} \mathrm{d}s$$
$$= \int_0^t \left\langle u'_{1\varepsilon}, \widehat{w}_i^\varepsilon \frac{\partial \Phi}{\partial x_i}\right\rangle_{(V^\varepsilon)',V^\varepsilon} \mathrm{d}s + \int_0^t \left\langle u'_{2\varepsilon}, \widehat{w}_i^\varepsilon \frac{\partial \Phi}{\partial x_i}\right\rangle_{(H^1(\Omega_{2\varepsilon}))',H^1(\Omega_{2\varepsilon})} \mathrm{d}s$$
$$+ \int_0^t \left\langle u'_{2\varepsilon}, (x_i - \widehat{w}_i^\varepsilon) \frac{\partial \Phi}{\partial x_i}\right\rangle_{(H^1(\Omega_{2\varepsilon}))',H^1(\Omega_{2\varepsilon})} \mathrm{d}s \tag{6.17}$$
$$= \int_0^t \left\langle (\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}})', \widehat{w}_i^\varepsilon \frac{\partial \Phi}{\partial x_i}\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \mathrm{d}s$$
$$+ \int_0^t \left\langle u'_{2\varepsilon}, (x_i - \widehat{w}_i^\varepsilon) \frac{\partial \Phi}{\partial x_i}\right\rangle_{(H^1(\Omega_{2\varepsilon}))',H^1(\Omega_{2\varepsilon})} \mathrm{d}s,$$

where we used Lemma 4.4 for the last equality.

Using (3.2), we have

$$\int_0^t \left\langle u'_{2\varepsilon}, (x_i - \widehat{w}_i^\varepsilon) \frac{\partial \Phi}{\partial x_i}\right\rangle_{(H^1(\Omega_{2\varepsilon}))',H^1(\Omega_{2\varepsilon})} \mathrm{d}s$$
$$= -\varepsilon \int_0^t \int_\Omega \widetilde{u_{2\varepsilon}} \widehat{\chi}_i^\varepsilon \left(\frac{x}{\varepsilon}\right) \frac{\partial \Phi'}{\partial x_i} \,\mathrm{d}x\, \mathrm{d}s + \varepsilon \int_\Omega \widetilde{u_{2\varepsilon}}(t) \widehat{\chi}_i^\varepsilon \left(\frac{x}{\varepsilon}\right) \frac{\partial \Phi}{\partial x_i}(t)\, \mathrm{d}x - \varepsilon \int_\Omega \widetilde{U_{2\varepsilon}^0} \widehat{\chi}_i^\varepsilon \left(\frac{x}{\varepsilon}\right) \frac{\partial \Phi}{\partial x_i}(0)\, \mathrm{d}x.$$

By (2.19)(iii), (3.10)(ii) of Lemma 3.1 and the Hölder inequality,

$$\sum_{i=1}^{n} \int_0^t \left\langle u'_{2\varepsilon}, (x_i - \widehat{w}_i^\varepsilon) \frac{\partial \Phi}{\partial x_i}\right\rangle_{(H^1(\Omega_{2\varepsilon}))',H^1(\Omega_{2\varepsilon})} \mathrm{d}s \le c\varepsilon \to 0 \qquad \text{as } \varepsilon \to 0. \tag{6.18}$$

For the first term of (6.17), note that for all $t \in [0,T]$,

$$\int_0^t \left\langle (\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}})', \widehat{w}_i^\varepsilon \frac{\partial \Phi}{\partial x_i}\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \mathrm{d}s$$
$$= -\int_0^t \int_\Omega (\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}}) \widehat{w}_i^\varepsilon \frac{\partial \Phi'}{\partial x_i} \,\mathrm{d}x\, \mathrm{d}s + \int_\Omega (\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}})(t) \widehat{w}_i^\varepsilon \frac{\partial \Phi}{\partial x_i}(t)\, \mathrm{d}x - \int_\Omega (\widetilde{U_{1\varepsilon}^0} + \widetilde{U_{2\varepsilon}^0}) \widehat{w}_i^\varepsilon \frac{\partial \Phi}{\partial x_i}(0)\, \mathrm{d}x,$$

where we pass to the limit term by term. Using the fact that $u_2 = \theta_2 u_1$ and by (2.19)(ii and iii) and (3.6)(ii), we get

$$\sum_{i=1}^{n} \lim_{\varepsilon\to 0} \int_0^t \int_\Omega (\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}})\widehat{w}_i^\varepsilon \frac{\partial \Phi'}{\partial x_i} \,\mathrm{d}x\,\mathrm{d}s = \sum_{i=1}^{n} \int_0^t \int_\Omega (\theta_1 u_1 + u_2) x_i \frac{\partial \Phi'}{\partial x_i} \,\mathrm{d}x\,\mathrm{d}s$$
$$= \sum_{i=1}^{n} \int_0^t \int_\Omega u_1 x_i \frac{\partial \Phi'}{\partial x_i} \,\mathrm{d}x\,\mathrm{d}s \qquad \forall t \in [0,T]. \tag{6.19}$$

On the other hand, by (3.10)(ii), (3.6)(ii) and in view of the initial conditions in (3.14),

$$\sum_{i=1}^{n} \lim_{\varepsilon\to 0} \int_\Omega (\widetilde{U^0_{1\varepsilon}} + \widetilde{U^0_{2\varepsilon}})\widehat{w}_i^\varepsilon \frac{\partial \Phi}{\partial x_i}(0) \,\mathrm{d}x = \sum_{i=1}^{n} \int_\Omega (\theta_1 U^0 + \theta_2 U^0) x_i \frac{\partial \Phi}{\partial x_i}(0) \,\mathrm{d}x$$
$$= \sum_{i=1}^{n} \int_\Omega u_1(0) x_i \frac{\partial \Phi}{\partial x_i}(0) \,\mathrm{d}x. \tag{6.20}$$

Lastly, the compactness result in Theorem 4.8 and (3.6)(ii and iii) imply that

$$\sum_{i=1}^{n} \lim_{\varepsilon\to 0} \int_\Omega (\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}})(t)\widehat{w}_i^\varepsilon \frac{\partial \Phi}{\partial x_i}(t) \,\mathrm{d}x = \sum_{i=1}^{n} \lim_{\varepsilon\to 0} \left\langle (\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}})(t), \widehat{w}_i^\varepsilon \frac{\partial \Phi}{\partial x_i}(t) \right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$$
$$= \sum_{i=1}^{n} \left\langle (\theta_1 u_1 + u_2)(t), x_i \frac{\partial \Phi}{\partial x_i}(t) \right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$$
$$= \sum_{i=1}^{n} \int_\Omega u_1(t) x_i \frac{\partial \Phi}{\partial x_i}(t) \,\mathrm{d}x \qquad \forall t \in [0,T]. \tag{6.21}$$

Combining (6.19), (6.20) and (6.21) together with (6.18), we get (6.16).

Using convergences (6.12)–(6.16) in (6.11) we get

$$\lim_{\varepsilon\to 0} \kappa_\varepsilon^3(t) = \sum_{i=1}^{n} \left\{ \int_0^t \int_\Omega (\theta_1 f_1 + \theta_2 f_2) x_i \frac{\partial \Phi}{\partial x_i} \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \left\langle g, x_i \frac{\partial \Phi}{\partial x_i} \right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \mathrm{d}s \right. \tag{6.22}$$
$$\left. - \int_0^t \left\langle u_1', x_i \frac{\partial \Phi}{\partial x_i} \right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \mathrm{d}s - \int_0^t \int_\Omega A^0 \nabla u_1 \nabla \left(\frac{\partial \Phi}{\partial x_i} \right) x_i \,\mathrm{d}x\,\mathrm{d}s \right\} \qquad \forall t \in [0,T].$$

On the other hand, observe that by taking $x_i \dfrac{\partial \Phi}{\partial x_i}$ as test function in the homogenized problem (3.14) and integrating by parts, we have

$$\sum_{i=1}^{n} \left\{ \int_0^t \left\langle u_1', x_i \frac{\partial \Phi}{\partial x_i} \right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \mathrm{d}s + \int_0^t \int_\Omega A^0 \nabla u_1 \nabla \left(x_i \frac{\partial \Phi}{\partial x_i} \right) \,\mathrm{d}x\,\mathrm{d}s \right\}$$
$$= \sum_{i=1}^{n} \left\{ \int_0^t \int_\Omega (\theta_1 f_1 + \theta_2 f_2) x_i \frac{\partial \Phi}{\partial x_i} \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \left\langle g, x_i \frac{\partial \Phi}{\partial x_i} \right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \mathrm{d}s \right\}. \tag{6.23}$$

Therefore, combining (6.22) and (6.23), we can deduce that

$$\begin{aligned}\lim_{\varepsilon\to 0}\kappa_\varepsilon^3(t) &= \sum_{i=1}^n \left\{ \int_0^t \int_\Omega A^0\nabla u_1 \nabla\left(x_i \frac{\partial\Phi}{\partial x_i}\right)\,\mathrm{d}x\,\mathrm{d}s - \int_0^t\int_\Omega A^0\nabla u_1\nabla\left(\frac{\partial\Phi}{\partial x_i}\right)x_i\,\mathrm{d}x\,\mathrm{d}s\right\}\\ &= \sum_{i=1}^n \int_0^t\int_\Omega A^0\nabla u_1\nabla x_i\left(\frac{\partial\Phi}{\partial x_i}\right)\,\mathrm{d}x\,\mathrm{d}s\\ &= \int_0^t\int_\Omega A^0\nabla u_1\nabla\Phi\,\mathrm{d}x\,\mathrm{d}s \qquad \forall t\in[0,T].\end{aligned}$$

The Ascoli-Arzela Theorem shows that the above convergence is actually in $\mathcal{C}^0([0,T])$. Indeed, by (2.20)(i), (3.4), our assumption on Φ and the Hölder inequality,

$$|\kappa_\varepsilon^3(t)| \le \|A^\varepsilon\widetilde{\nabla u_{1\varepsilon}}\|_{L^2(0,T;\,[L^2(\Omega)]^n)}\|C^\varepsilon\|_{[L^2(\Omega)]^{n^2}}\|\nabla\Phi\|_{L^\infty(0,T;\,[L^2(\Omega)]^n)} \le c,$$

where c is independent of t. Moreover, for any $h>0$ small enough,

$$\begin{aligned}|\kappa_\varepsilon^3(t+h)-\kappa_\varepsilon^3(t)| \le& \|A^\varepsilon\widetilde{\nabla u_{1\varepsilon}}\|_{L^2(0,T;\,[L^2(\Omega)]^n)}\|C^\varepsilon\|_{[L^2(\Omega)]^{n^2}}h^{1/2}\|\nabla\Phi\|_{L^\infty(0,T;\,[L^2(\Omega)]^n)}\\ &\le ch^{1/2}\to 0, \qquad \text{as } h\to 0,\ \text{uniformly in } \varepsilon.\end{aligned}$$

Hence,

$$\kappa_\varepsilon^3\to\kappa^3 \qquad \text{in } \mathcal{C}^0([0,T]), \qquad \text{where} \quad \kappa^3(t) = \int_0^t\int_\Omega A^0\nabla u_1\nabla\Phi\,\mathrm{d}x\,\mathrm{d}s. \tag{6.24}$$

Then, (6.6) follows from (6.7) and convergences (6.8), (6.10) and (6.24).

Step 2. Since

$$\|u_{1\varepsilon}(t)\|^2_{L^2(\Omega_{1\varepsilon})} + \|u_{2\varepsilon}(t)\|^2_{L^2(\Omega_{2\varepsilon})} = \|\widetilde{u_{1\varepsilon}}(t)+\widetilde{u_{2\varepsilon}}(t)\|^2_{L^2(\Omega)},$$

from (2.10) it follows that

$$\eta_\varepsilon^3(t)\le d_\varepsilon(t) \qquad \forall t\in[0,T],$$

where $d_\varepsilon(t)$ is the energy associated with problem (2.14) given by (5.1). This yields

$$0\le \eta_\varepsilon(t) = \eta_\varepsilon^3(t)+\eta_\varepsilon^1(t)-\eta_\varepsilon^2(t) \le d_\varepsilon(t)+\eta_\varepsilon^1(t)-\eta_\varepsilon^2(t), \qquad \forall t\in[0,T]. \tag{6.25}$$

On the other hand, it is known (see for instance, [16]) that

$$\eta_\varepsilon^1\to\eta^1 \qquad \text{in } \mathcal{C}^0([0,T]), \tag{6.26}$$

where

$$\eta^1(t) = \frac{1}{2}\|\Phi(t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega A^0\nabla\Phi\nabla\Phi\,\mathrm{d}x\,\mathrm{d}s.$$

Hence, by Proposition 5.3, (6.26) and (6.6), we have

$$d_\varepsilon+\eta_\varepsilon^1-\eta_\varepsilon^2\to d+\eta^1-\eta^2=\eta \qquad \text{in } \mathcal{C}^0([0,T]),$$

where $\eta(t)$ is given by (6.4) and d by (5.4). This, together with (6.25), implies that

$$\limsup_{\varepsilon\to 0}\|\eta_\varepsilon\|_{\mathcal{C}^0([0,T])} \le \lim_{\varepsilon\to 0}\|d_\varepsilon+\eta_\varepsilon^1-\eta_\varepsilon^2\|_{\mathcal{C}^0([0,T])} = \|\eta\|_{\mathcal{C}^0([0,T])},$$

which is (6.3) and the proof of Proposition 6.5 is complete. □

Remark 6.7. The non-symmetry of A^ε makes $\eta_\varepsilon^2(t)$ different from its counterpart studied in [18] for the hyperbolic case, giving rise to the third term $\kappa_\varepsilon^3(t)$. This term is complicated to treat as already seen in the study of the heat equation in perforated domains studied in [15]. Here, we are able to conclude thanks to Lemma 4.4 and the compactness result proved in Theorem 4.8.

Proof of Theorem 3.4. Let $\delta > 0$ be fixed and $\Phi \in \mathcal{C}^\infty(0,T;\mathcal{D}(\Omega))$ be the corresponding function satisfying Lemma 6.1 and associated to the solution u_1 of (3.14). First, observe that by (2.8)(ii), Lemma 6.1 and Proposition 6.5,

$$\begin{aligned}\limsup_{\varepsilon\to 0}\|\eta_\varepsilon\|_{\mathcal{C}^0([0,T])} &\leq \|\eta\|_{\mathcal{C}^0([0,T])}\\ &\leq \frac{1}{2}\|u_1-\Phi\|^2_{\mathcal{C}^0([0,T];L^2(\Omega))} + \int_0^T\int_\Omega A^0(\nabla u_1-\nabla\Phi)(\nabla u_1-\nabla\Phi)\,\mathrm{d}x\,\mathrm{d}t\\ &\leq \left(\frac{1}{2}+\|A^0\|_{[L^\infty(\Omega)]^{n^2}}\right)\delta^2,\end{aligned}\tag{6.27}$$

where $\eta_\varepsilon(t)$ and $\eta(t)$ are given by (6.2) and (6.4) respectively.

Now, using the triangle inequality and (6.1)(i), we obtain

$$\begin{aligned}\|\widetilde{u_{1\varepsilon}}+\widetilde{u_{2\varepsilon}}-u_1\|^2_{\mathcal{C}^0([0,T];L^2(\Omega))} &\leq 2(\|\widetilde{u_{1\varepsilon}}+\widetilde{u_{2\varepsilon}}-\Phi\|^2_{\mathcal{C}^0([0,T];L^2(\Omega))}+\|\Phi-u_1\|^2_{\mathcal{C}^0([0,T];L^2(\Omega))})\\ &\leq 2\|\widetilde{u_{1\varepsilon}}+\widetilde{u_{2\varepsilon}}-\Phi\|^2_{\mathcal{C}^0([0,T];L^2(\Omega))}+2\delta^2.\end{aligned}$$

Moreover, because of the ellipticity condition of A^ε given in (2.8)(i), one has

$$\frac{1}{2}\|\widetilde{u_{1\varepsilon}}(t)+\widetilde{u_{2\varepsilon}}(t)-\Phi(t)\|^2_{L^2(\Omega)} \leq \eta_\varepsilon(t).$$

Hence,

$$\limsup_{\varepsilon\to 0}\|\widetilde{u_{1\varepsilon}}+\widetilde{u_{2\varepsilon}}-u_1\|^2_{\mathcal{C}^0([0,T];L^2(\Omega))} \leq c_1(\limsup_{\varepsilon\to 0}\|\eta_\varepsilon\|_{\mathcal{C}^0([0,T])}+\delta^2).\tag{6.28}$$

On the other hand,

$$\nabla u_{1\varepsilon}-C^\varepsilon\nabla u_1 = (\nabla u_{1\varepsilon}-C^\varepsilon\nabla\Phi)+C^\varepsilon(\nabla\Phi-\nabla u_1).$$

By the triangle inequality, (3.4), (6.1)(ii) and the Hölder inequality,

$$\begin{aligned}&\int_0^T\|\nabla u_{1\varepsilon}(t)-C^\varepsilon\nabla u_1(t)\|^2_{[L^1(\Omega_{1\varepsilon})]^n}\,\mathrm{d}t+\int_0^T\|\nabla u_{2\varepsilon}(t)\|^2_{[L^2(\Omega_{2\varepsilon})]^n}\,\mathrm{d}t\\ &\qquad\leq 2\int_0^T\|\nabla u_{1\varepsilon}-C^\varepsilon\nabla\Phi\|^2_{[L^1(\Omega_{1\varepsilon})]^n}\,\mathrm{d}t+2\|C^\varepsilon\|_{[L^2(\Omega_{1\varepsilon})]^{n^2}}\int_0^T\|\nabla\Phi-\nabla u_1\|^2_{[L^2(\Omega_{1\varepsilon})]^n}\,\mathrm{d}t\\ &\qquad\quad+\int_0^T\|\nabla u_{2\varepsilon}(t)\|^2_{[L^2(\Omega_{2\varepsilon})]^n}\,\mathrm{d}t\\ &\qquad\leq c_2\int_0^T\|\nabla u_{1\varepsilon}-C^\varepsilon\nabla\Phi\|^2_{[L^2(\Omega_{1\varepsilon})]^n}\,\mathrm{d}t+c_3\delta^2+\int_0^T\|\nabla u_{2\varepsilon}(t)\|^2_{[L^2(\Omega_{2\varepsilon})]^n}\,\mathrm{d}t\\ &\qquad\leq c_4(\eta_\varepsilon(T)+\delta^2),\end{aligned}$$

where for the last inequality, we used the definition of η_ε given in (6.2) and the ellipticity condition of A^ε. Therefore,

$$\limsup_{\varepsilon\to 0}\left[\|\nabla u_{1\varepsilon}-C^\varepsilon\nabla u_1\|^2_{L^2(0,T;[L^1(\Omega_{1\varepsilon})]^n)}+\|\nabla u_{2\varepsilon}\|^2_{L^2(0,T;[L^2(\Omega_{2\varepsilon})]^n)}\right] \leq c_4(\limsup_{\varepsilon\to 0}\|\eta_\varepsilon\|_{\mathcal{C}^0([0,T])}+\delta^2).$$

Together with (6.27) and (6.28), we have

$$0 \leq \limsup_{\varepsilon \to 0} \left\{ \|\widetilde{u_{1\varepsilon}} + \widetilde{u_{2\varepsilon}} - u_1\|^2_{\mathcal{C}^0([0,T];L^2(\Omega))} + \|\nabla u_{1\varepsilon} - C^\varepsilon \nabla u_1\|^2_{L^2(0,T;[L^1(\Omega_{1\varepsilon})]^n)} + \|\nabla u_{2\varepsilon}\|^2_{L^2(0,T;[L^2(\Omega_{2\varepsilon})]^n)} \right\} \leq c\delta^2.$$

Since δ is arbitrary, the conclusion follows. □

6.2. Corrector results for the case $\gamma = 1$

The proof of Theorem 3.9 is based on the following proposition:

Proposition 6.8. *Let Φ and Ψ be in $\mathcal{C}^\infty(0,T;\mathcal{D}(\Omega))$ and set*

$$\beta_\varepsilon(t) = \frac{1}{2}\|u_{1\varepsilon}(t) - \Phi(t)\|^2_{L^2(\Omega_{1\varepsilon})} + \frac{1}{2}\|u_{2\varepsilon}(t) - \Psi(t)\|^2_{L^2(\Omega_{2\varepsilon})} + \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon(\nabla u_{1\varepsilon} - C^\varepsilon\nabla\Phi)(\nabla u_{1\varepsilon} - C^\varepsilon\nabla\Phi)\,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon}\nabla u_{2\varepsilon}\,\mathrm{d}x\,\mathrm{d}s. \tag{6.29}$$

Then, under the assumptions of Theorem 3.9, we have

$$\limsup_{\varepsilon\to 0} \|\beta_\varepsilon\|_{\mathcal{C}^0([0,T])} \leq \|\beta\|_{\mathcal{C}^0([0,T])} \tag{6.30}$$

where

$$\beta(t) = \frac{1}{2}\theta_1\|u_1(t) - \Phi(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\theta_2^{-1}\|u_2(t) - \theta_2\Psi(t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega A^0(\nabla u_1 - \nabla\Phi)(\nabla u_1 - \nabla\Phi)\,\mathrm{d}x\,\mathrm{d}s. \tag{6.31}$$

Remark 6.9. In this proposition as in Proposition 6.5, we only have an upper semi-continuity type inequality (see Rem. 6.6) which is likewise sufficient for the main theorem. The outline of the proof of the above proposition is the same as that of Proposition 6.5. However, the part β^3_ε (see (6.35) below for its definition) of the the energy studied in Step 2 of the proof, requires technical and specific arguments as already encountered in [18] for the corresponding hyperbolic case.

Proof. We closely follow the proof of Proposition 6.5. Only different points will be treated in a detailed manner. To begin with, we decompose β_ε into three terms

$$\beta_\varepsilon(t) = \beta^1_\varepsilon(t) - \beta^2_\varepsilon(t) + \beta^3_\varepsilon(t), \tag{6.32}$$

where

$$\beta^1_\varepsilon(t) = \frac{1}{2}\|\Phi(t)\|^2_{L^2(\Omega_{1\varepsilon})} + \frac{1}{2}\|\Psi(t)\|^2_{L^2(\Omega_{2\varepsilon})} + \int_0^t\int_{\Omega_{1\varepsilon}} A^\varepsilon C^\varepsilon\nabla\Phi C^\varepsilon\nabla\Phi\,\mathrm{d}x\,\mathrm{d}s, \tag{6.33}$$

$$\beta^2_\varepsilon(t) = \int_{\Omega_{1\varepsilon}} u_{1\varepsilon}\Phi\,\mathrm{d}x + \int_{\Omega_{2\varepsilon}} u_{2\varepsilon}\Psi\,\mathrm{d}x + \int_0^t\int_{\Omega_{1\varepsilon}} A^\varepsilon C^\varepsilon\nabla\Phi\nabla u_{1\varepsilon}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\Omega_{1\varepsilon}} A^\varepsilon\nabla u_{1\varepsilon}C^\varepsilon\nabla\Phi\,\mathrm{d}x\,\mathrm{d}s, \tag{6.34}$$

$$\beta^3_\varepsilon(t) = \frac{1}{2}\|u_{1\varepsilon}(t)\|^2_{L^2(\Omega_{1\varepsilon})} + \frac{1}{2}\|u_{2\varepsilon}(t)\|^2_{L^2(\Omega_{2\varepsilon})} + \int_0^t\int_{\Omega_{1\varepsilon}} A^\varepsilon\nabla u_{1\varepsilon}\nabla u_{1\varepsilon}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\Omega_{2\varepsilon}} A^\varepsilon\nabla u_{2\varepsilon}\nabla u_{2\varepsilon}\,\mathrm{d}x\,\mathrm{d}s. \tag{6.35}$$

We continue step by step in getting the limit of each term in (6.32).

Step 1. In this step, we study the limits of β^1_ε and β^2_ε. For the first term $\beta^1_\varepsilon(t)$, following the arguments of the proof of Proposition 5.3 of [15] (as in the proof of Proposition 6.5), it can be shown that

$$\lim_{\varepsilon\to 0}\beta^1_\varepsilon(t) = \frac{1}{2}\theta_1\|\Phi(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\theta_2\|\Psi(t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega A^0\nabla\Phi\nabla\Phi\,\mathrm{d}x\,\mathrm{d}s \qquad \text{in } \mathcal{C}^0([0,T]). \tag{6.36}$$

Now consider the second term $\beta^2_\varepsilon(t)$ which corresponds to the term $\eta^2_\varepsilon(t)$ of the previous case. We want to show that

$$\begin{aligned}\lim_{\varepsilon\to 0}\beta^2_\varepsilon(t) = &\int_\Omega(\theta_1 u_1\Phi + u_2\Psi)\,\mathrm{d}x + \int_0^t\int_\Omega A^0\nabla\Phi\nabla u_1\,\mathrm{d}x\,\mathrm{d}s\\ &+\int_0^t\int_\Omega A^0\nabla u_1\nabla\Phi\,\mathrm{d}x\,\mathrm{d}s \qquad \text{in } \mathcal{C}^0([0,T]).\end{aligned} \tag{6.37}$$

The main difference with the previous case lies on the first two terms of $\beta^2_\varepsilon(t)$ where we need to use convergence (4.5) instead of (4.4) of Theorem 4.8. Indeed, observe that

$$\begin{aligned}\max_{t\in[0,T]}\left|\int_{\Omega_{1\varepsilon}}(u_{1\varepsilon}\Phi-\theta_1u_1\Phi)\,\mathrm{d}x+\int_{\Omega_{2\varepsilon}}(u_{2\varepsilon}\Psi-u_2\Psi)\,\mathrm{d}x\right| &= \max_{t\in[0,T]}\left|\int_\Omega(\widetilde{u_{1\varepsilon}}-\theta_1u_1)\Phi\,\mathrm{d}x+\int_\Omega(\widetilde{u_{2\varepsilon}}-u_2)\Psi\,\mathrm{d}x\right|\\ &\le \|\widetilde{u_{1\varepsilon}}-\theta_1u_1\|_{\mathcal{C}^0([0,T];H^{-1}(\Omega))}\|\Phi\|_{\mathcal{C}^0([0,T];H^1_0(\Omega))}+\|\widetilde{u_{2\varepsilon}}-u_2\|_{\mathcal{C}^0([0,T];H^{-1}(\Omega))}\|\Psi\|_{\mathcal{C}^0([0,T];H^1_0(\Omega))}.\end{aligned}$$

Therefore, from (4.5)

$$\int_{\Omega_{1\varepsilon}}u_{1\varepsilon}\Phi\,\mathrm{d}x+\int_{\Omega_{2\varepsilon}}u_{2\varepsilon}\Psi\,\mathrm{d}x \to \int_\Omega(\theta_1u_1\Phi+u_2\Psi)\,\mathrm{d}x \qquad \text{in } \mathcal{C}^0([0,T]). \tag{6.38}$$

Moreover, the same arguments used to prove (6.10), give

$$\int_0^t\int_{\Omega_{1\varepsilon}}A^\varepsilon C^\varepsilon\nabla\Phi\nabla u_{1\varepsilon}\,\mathrm{d}x\,\mathrm{d}s \to \int_0^t\int_\Omega A^0\nabla\Phi\nabla u_1\,\mathrm{d}x\,\mathrm{d}s \qquad \text{in } \mathcal{C}^0([0,T]), \tag{6.39}$$

where u_1 is the solution of the homogenized problem (2.22).
Hence, to prove (6.37), it remains to show that

$$\int_0^t\int_{\Omega_{1\varepsilon}}A^\varepsilon\nabla u_{1\varepsilon}C^\varepsilon\nabla\Phi\,\mathrm{d}x\,\mathrm{d}s \to \int_0^t\int_\Omega A^0\nabla u_1\nabla\Phi\,\mathrm{d}x\,\mathrm{d}s \qquad \text{in } \mathcal{C}^0([0,T]). \tag{6.40}$$

The same arguments used to obtain (6.19) and (6.21), give now

$$\sum_{i=1}^n\lim_{\varepsilon\to 0}\int_0^t\int_\Omega(\widetilde{u_{1\varepsilon}}+\widetilde{u_{2\varepsilon}})\widehat{w}^\varepsilon_i\frac{\partial\Phi'}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}s = \sum_{i=1}^n\int_0^t\int_\Omega(\theta_1u_1+u_2)x_i\frac{\partial\Phi'}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}s \qquad \forall t\in[0,T] \tag{6.41}$$

and

$$\sum_{i=1}^n\lim_{\varepsilon\to 0}\int_\Omega(\widetilde{u_{1\varepsilon}}+\widetilde{u_{2\varepsilon}})(t)\widehat{w}^\varepsilon_i\frac{\partial\Phi}{\partial x_i}(t)\,\mathrm{d}x = \sum_{i=1}^n\int_\Omega(\theta_1u_1+u_2)(t)x_i\frac{\partial\Phi}{\partial x_i}(t)\,\mathrm{d}x \qquad \forall t\in[0,T]. \tag{6.42}$$

Comparing to limits (6.19) and (6.21), we observe here that we do not replace u_2 by $\theta_2 u_1$. Furthermore, by Theorem 4.8, (3.15)(ii), (3.6)(ii) and in view of the initial conditions in (2.22),

$$\begin{aligned} \lim_{\varepsilon\to 0}\int_\Omega (\widetilde{U^0_{1\varepsilon}}+\widetilde{U^0_{2\varepsilon}})\widehat{w}^\varepsilon_i \frac{\partial\Phi}{\partial x_i}(0)\,\mathrm{d}x &= \int_\Omega (\theta_1 U^0_1+\theta_2 U^0_2)x_i\frac{\partial\Phi}{\partial x_i}(0)\,\mathrm{d}x\\ &= \int_\Omega (\theta_1 u_1+u_2)(0)x_i\frac{\partial\Phi}{\partial x_i}(0)\,\mathrm{d}x. \end{aligned} \tag{6.43}$$

Hence, as we did to pass to the limit in (6.17),

$$\begin{aligned} \sum_{i=1}^n \lim_{\varepsilon\to 0}\left\{\int_0^t \left\langle u'_{1\varepsilon}, \widehat{w}^\varepsilon_i\frac{\partial\Phi}{\partial x_i}\right\rangle_{(V^\varepsilon)',V^\varepsilon}\mathrm{d}s+\int_0^t\left\langle u'_{2\varepsilon}, x_i\frac{\partial\Phi}{\partial x_i}\right\rangle_{(H^1(\Omega_{2\varepsilon}))',H^1(\Omega_{2\varepsilon})}\mathrm{d}s\right\}\\ =\sum_{i=1}^n\int_0^t\left\langle\theta_1 u'_1+u'_2, x_i\frac{\partial\Phi}{\partial x_i}\right\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\mathrm{d}s. \end{aligned} \tag{6.44}$$

Therefore, using (6.11)–(6.15) (which are still valid for $\gamma=1$) together with (6.44), yield

$$\begin{aligned} \lim_{\varepsilon\to 0}\int_0^t\int_{\Omega_{1\varepsilon}} A^\varepsilon\nabla u_{1\varepsilon}C^\varepsilon\nabla\Phi\,\mathrm{d}x\,\mathrm{d}s &= \sum_{i=1}^n\left\{\int_0^t\int_\Omega(\theta_1 f_1+\theta_2 f_2)x_i\frac{\partial\Phi}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\left\langle g, x_i\frac{\partial\Phi}{\partial x_i})\right\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\mathrm{d}s\right.\\ &\left.-\int_0^t\left\langle\theta_1 u'_1+u'_2, x_i\frac{\partial\Phi}{\partial x_i}\right\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\mathrm{d}s-\int_0^t\int_\Omega A^0\nabla u_1\nabla\left(\frac{\partial\Phi}{\partial x_i}\right)x_i\,\mathrm{d}x\,\mathrm{d}s\right\}. \end{aligned} \tag{6.45}$$

Now, taking $x_i\dfrac{\partial\Phi}{\partial x_i}$ as test function in the first and second equations of the homogenized problem (2.22) and integrating by parts, we obtain

$$\begin{aligned} \sum_{i=1}^n\left\{\int_0^t\left\langle\theta_1 u'_1+u'_2, x_i\frac{\partial\Phi}{\partial x_i}\right\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\mathrm{d}s+\int_0^t\int_\Omega A^0\nabla u_1\nabla\left(x_i\frac{\partial\Phi}{\partial x_i}\right)\,\mathrm{d}x\,\mathrm{d}s\right\}\\ =\sum_{i=1}^n\left\{\int_0^t\int_\Omega(\theta_1 f_1+\theta_2 f_2)x_i\frac{\partial\Phi}{\partial x_i}\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\left\langle g, x_i\frac{\partial\Phi}{\partial x_i}\right\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\mathrm{d}s\right\} \end{aligned}$$

which combined with (6.45) gives

$$\int_0^t\int_{\Omega_{1\varepsilon}} A^\varepsilon\nabla u_{1\varepsilon}C^\varepsilon\nabla\Phi\,\mathrm{d}x\,\mathrm{d}s\to\int_0^t\int_\Omega A^0\nabla u_1\nabla\Phi\,\mathrm{d}x\,\mathrm{d}s\qquad\forall t\in[0,T]. \tag{6.46}$$

To prove that this convergence takes place in $\mathcal{C}^0([0,T])$, one argues exactly as we did to show (6.24).

Step 2. It remains to study the term $\beta^3_\varepsilon(t)$ given in (6.35). Let us first show that

$$\{\beta^3_\varepsilon\}\text{ is compact in }\mathcal{C}^0([0,T]). \tag{6.47}$$

To do so, recall that the energy $d_{1\varepsilon}$ (see (5.1)) associated to (2.14) for $\gamma = 1$ is given by

$$d_{1\varepsilon}(t) = \frac{1}{2}\|u_{1\varepsilon}(t)\|^2_{L^2(\Omega_{1\varepsilon})} + \frac{1}{2}\|u_{2\varepsilon}(t)\|^2_{L^2(\Omega_{2\varepsilon})} + \int_0^t \int_{\Omega_{1\varepsilon}} A^\varepsilon \nabla u_{1\varepsilon} \nabla u_{1\varepsilon}\, \mathrm{d}x\, \mathrm{d}s + \int_0^t \int_{\Omega_{2\varepsilon}} A^\varepsilon \nabla u_{2\varepsilon} \nabla u_{2\varepsilon}\, \mathrm{d}x\, \mathrm{d}s + \varepsilon \int_0^t \int_{\Gamma^\varepsilon} h^\varepsilon |u_{1\varepsilon} - u_{2\varepsilon}|^2\, \mathrm{d}\sigma_x\, \mathrm{d}s.$$

Clearly, from (6.35) we have

$$\beta^3_\varepsilon(t) = d_{1\varepsilon}(t) - \varepsilon \int_0^t \int_{\Gamma^\varepsilon} h^\varepsilon |u_{1\varepsilon} - u_{2\varepsilon}|^2\, \mathrm{d}\sigma_x\, \mathrm{d}s \qquad \forall t \in [0,T]. \tag{6.48}$$

From Proposition 5.4, we know that

$$d_{1\varepsilon} \to d_1 \qquad \text{in } \mathcal{C}^0([0,T]). \tag{6.49}$$

Hence, it is enough to show that $\alpha_\varepsilon(t) = \varepsilon \int_0^t \int_{\Gamma^\varepsilon} h^\varepsilon |u_{1\varepsilon} - u_{2\varepsilon}|^2\, \mathrm{d}\sigma_x\, \mathrm{d}s$ is also compact in $\mathcal{C}^0([0,T])$. Now, recalling (2.19)(iv) and (2.11), it follows that $\alpha_\varepsilon(t)$ is bounded in $H^1(0,T)$. Since $H^1(0,T) \subset \mathcal{C}^0([0,T])$ is a compact injection, we get (6.47).

Let us now prove that

$$\begin{aligned}\limsup_{\varepsilon\to 0} \beta^3_\varepsilon(t) &\le \frac{1}{2}\theta_1 \|u_1(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\theta_2^{-1}\|u_2(t)\|^2_{L^2(\Omega)} + \int_0^t \int_\Omega A^0 \nabla u_1 \nabla u_1\, \mathrm{d}x\, \mathrm{d}s \\ &= d_1 - c_h \theta_2^{-1} \int_0^t \int_\Omega (\theta_2 u_1 - u_2)^2\, \mathrm{d}x\, \mathrm{d}s \qquad \forall t \in [0,T],\end{aligned} \tag{6.50}$$

where $d_1(t)$ is given by (5.6), the energy associated with the homogenized problem (2.22).

Due to (6.49), it suffices to show that

$$\liminf_{\varepsilon\to 0} \varepsilon \int_0^t \int_{\Gamma^\varepsilon} h^\varepsilon |u_{1\varepsilon} - u_{2\varepsilon}|^2\, \mathrm{d}\sigma_x\, \mathrm{d}s \ge c_h \theta_2^{-1} \int_0^t \int_\Omega (\theta_2 u_1 - u_2)^2\, \mathrm{d}x\, \mathrm{d}s \qquad \forall t \in [0,T]. \tag{6.51}$$

To do that, we adapt to the parabolic case the same ideas as in [12,18]. Since in this case Γ is assumed of class C^2, we can apply Lemma 6.4 with $g = h$ and $v_\varepsilon = (P_1^\varepsilon u_{1\varepsilon} - u_{2\varepsilon})^2$, to get

$$\begin{aligned}\liminf_{\varepsilon\to 0} \varepsilon \int_0^t \int_{\Gamma^\varepsilon} h^\varepsilon |u_{1\varepsilon} - u_{2\varepsilon}|^2\, \mathrm{d}\sigma_x\, \mathrm{d}s &= \liminf_{\varepsilon\to 0} \varepsilon \int_0^t \int_{\Gamma^\varepsilon} h^\varepsilon |P_1^\varepsilon u_{1\varepsilon} - u_{2\varepsilon}|^2\, \mathrm{d}\sigma_x\, \mathrm{d}s \\ &= \liminf_{\varepsilon\to 0} c_h \int_0^t \int_{\Omega_{2\varepsilon}} |P_1^\varepsilon u_{1\varepsilon} - u_{2\varepsilon}|^2\, \mathrm{d}x\, \mathrm{d}s \qquad \forall t \in [0,T],\end{aligned} \tag{6.52}$$

where $c_h = \frac{1}{|Y_2|}\int_\Gamma h(y)\, \mathrm{d}\sigma_y$. Now, by (2.2), (2.19)(i and iii) and Lemma 6.2, we obtain

$$\begin{aligned}\widetilde{P_1^\varepsilon u_{1\varepsilon}|_{\Omega_{2\varepsilon}}} - \widetilde{u_{2\varepsilon}} &= \chi_{\Omega_{2\varepsilon}} P_1^\varepsilon u_{1\varepsilon} - \widetilde{u_{2\varepsilon}} \\ &\rightharpoonup \theta_2 u_1 - u_2 = \theta_2(u_1 - \theta_2^{-1} u_2) \quad \text{weakly in } L^2(0,T;\, L^2(\Omega)).\end{aligned}$$

Using Lemma 6.3 with $\mathcal{O}_\varepsilon = \Omega_{2\varepsilon}$, $\chi_0 = \theta_2$, $v_\varepsilon = P_1^\varepsilon u_{1\varepsilon}|_{\Omega_{2\varepsilon}} - u_{2\varepsilon}$ and $v = u_1 - \theta_2^{-1} u_2$, we have

$$\liminf_{\varepsilon\to 0} c_h \int_0^t \int_{\Omega_{2\varepsilon}} (P_1^\varepsilon u_{1\varepsilon} - u_{2\varepsilon})^2 \,\mathrm{d}\sigma_x \,\mathrm{d}s \geq c_h \int_0^t \int_\Omega \theta_2 (u_1 - \theta_2^{-1} u_2)^2 \,\mathrm{d}x \,\mathrm{d}s$$
$$= c_h \theta_2^{-1} \int_0^t \int_\Omega (\theta_2 u_1 - u_2)^2 \,\mathrm{d}x \,\mathrm{d}s \qquad \forall t \in [0,T].$$

Together with (6.52), this gives (6.51) and so (6.50) follows.

Step 3 (conclusion of the proof). From (6.31)–(6.35), together with (6.36), (6.37), (6.47) and (6.50), we deduce that

$$\{\beta_\varepsilon\} \text{ compact in } \mathcal{C}^0([0,T]) \quad \text{and} \quad 0 \leq \limsup_{\varepsilon\to 0} \beta_\varepsilon(t) \leq \beta(t), \quad \forall t \in [0,T].$$

This gives us the desired result (6.30). $\square$

Proof of Theorem 3.9. The proof of Theorem 3.9 will be patterned accordingly to the proof of Theorem 3.4. Let $\delta > 0$ be fixed and $\Phi \in \mathcal{C}^\infty(0,T;\mathcal{D}(\Omega))$ be the corresponding function satisfying Lemma 6.1 and associated to the solution u_1 of (2.22). Moreover, let $\Psi \in \mathcal{C}^\infty(0,T;\mathcal{D}(\Omega))$ such that

$$\|u_2 - \theta_2 \Psi\|_{\mathcal{C}^0([0,T];L^2(\Omega))} \leq \delta.$$

Note that by (2.8)(ii), Lemma 6.1 and Proposition 6.8,

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \|\beta_\varepsilon\|_{\mathcal{C}^0([0,T])} &\leq \|\beta\|_{\mathcal{C}^0([0,T])} \\
&\leq \frac{1}{2}\theta_1 \|u_1 - \Phi\|^2_{\mathcal{C}^0([0,T];L^2(\Omega))} + \frac{1}{2}\theta_2^{-1} \|u_2 - \theta_2 \Psi\|^2_{\mathcal{C}^0([0,T];L^2(\Omega))} \\
&\quad + \int_0^T \int_\Omega A^0 (\nabla u_1 - \nabla \Phi)(\nabla u_1 - \nabla \Phi) \,\mathrm{d}x \,\mathrm{d}t \\
&\leq \left(\frac{1}{2}\theta_1 + \frac{1}{2}\theta_2^{-1} + \|A^0\|_{[L^\infty(\Omega)]^{n^2}} \right) \delta^2,
\end{aligned} \tag{6.53}$$

where $\beta_\varepsilon(t)$ and $\beta(t)$ are given by (6.29) and (6.31) respectively.

Now, using the triangle inequality and (6.1)(i), we get

$$\begin{aligned}
&\|u_{1\varepsilon} - u_1\|^2_{\mathcal{C}^0(0,T;L^2(\Omega_{1\varepsilon}))} + \|u_{2\varepsilon} - \theta_2^{-1} u_2\|^2_{\mathcal{C}^0(0,T;L^2(\Omega_{2\varepsilon}))} \\
&\quad \leq 2 (\|u_{1\varepsilon} - \Phi\|^2_{\mathcal{C}^0(0,T;L^2(\Omega_{1\varepsilon}))} + \|\Phi - u_1\|^2_{\mathcal{C}^0(0,T;L^2(\Omega_{1\varepsilon}))} \\
&\qquad + \|u_{2\varepsilon} - \Psi\|^2_{\mathcal{C}^0(0,T;L^2(\Omega_{2\varepsilon}))} + \|\Psi - \theta_2^{-1} u_2\|^2_{\mathcal{C}^0(0,T;L^2(\Omega_{2\varepsilon}))}) \\
&\quad \leq 2 (\|u_{1\varepsilon} - \Phi\|^2_{\mathcal{C}^0(0,T;L^2(\Omega_{1\varepsilon}))} + \|u_{2\varepsilon} - \Psi\|^2_{\mathcal{C}^0(0,T;L^2(\Omega_{2\varepsilon}))}) + c_1 \delta^2.
\end{aligned}$$

Furthermore, by the ellipticity of A^ε

$$\frac{1}{2} \|u_{1\varepsilon}(t) - \Phi(t)\|^2_{L^2(\Omega_{1\varepsilon})} + \frac{1}{2} \|u_{2\varepsilon}(t) - \Psi(t)\|^2_{L^2(\Omega_{2\varepsilon})} \leq \beta_\varepsilon(t),$$

so that

$$\limsup_{\varepsilon\to 0} \{\|u_{1\varepsilon} - u_1\|^2_{\mathcal{C}^0(0,T;L^2(\Omega_{1\varepsilon}))} + \|u_{2\varepsilon} - \theta_2^{-1} u_2\|^2_{\mathcal{C}^0(0,T;L^2(\Omega_{2\varepsilon}))}\} \leq k_1 (\limsup_{\varepsilon\to 0} \|\beta_\varepsilon\|_{\mathcal{C}^0([0,T])} + \delta^2) \leq c\delta^2. \tag{6.54}$$

In a similar manner as in the proof of Theorem 3.4,

$$\limsup_{\varepsilon\to 0}\Big[\|\nabla u_{1\varepsilon}-C^{\varepsilon}\nabla u_1\|^2_{L^2(0,T;[L^1(\Omega_{1\varepsilon})]^n)}+\|\nabla u_{2\varepsilon}\|^2_{L^2(0,T;[L^2(\Omega_{2\varepsilon})]^n)}\Big]\le k_2(\limsup_{\varepsilon\to 0}\|\beta_\varepsilon\|_{\mathcal{C}^0([0,T])}+\delta^2)\le c\delta^2.$$

Hence, together with (6.53) and (6.54), we have

$$\begin{aligned}0\le \limsup_{\varepsilon\to 0}\big\{&\|u_{1\varepsilon}-u_1\|^2_{\mathcal{C}^0(0,T;L^2(\Omega_{1\varepsilon}))}+\|u_{2\varepsilon}-\theta_2^{-1}u_2\|^2_{\mathcal{C}^0(0,T;L^2(\Omega_{2\varepsilon}))}\\ &+\|\nabla u_{1\varepsilon}-C^{\varepsilon}\nabla u_1\|^2_{L^2(0,T;[L^1(\Omega_{1\varepsilon})]^n)}+\|\nabla u_{2\varepsilon}\|^2_{L^2(0,T;[L^2(\Omega_{2\varepsilon})]^n)}\big\}\le 2c\delta^2.\end{aligned}$$

Since δ is arbitrary, the conclusion follows. □

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