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ON VARIETIES OF LITERALLY IDEMPOTENT LANGUAGES *

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Abstract. A language $L \subseteq A^*$ is literally idempotent in case that $ua^2v \in L$ if and only if $uav \in L$, for each $u, v \in A^*$, $a \in A$. Varieties of literally idempotent languages result naturally by taking all literally idempotent languages in a classical (positive) variety or by considering a certain closure operator on classes of languages. We initiate the systematic study of such varieties. Various classes of literally idempotent languages can be characterized using syntactic methods. A starting example is the class of all finite unions of $B_1^* B_2^* \ldots B_k^*$ where $B_1, \ldots, B_k$ are subsets of a given alphabet A.

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1. Introduction

Papers by Straubing [11] on $\mathbb{C}$-varieties and Ésik *et al.* [4,5] on literal varieties of languages enable us to consider new significant classes of languages. Due to the result by Kunc [7] we also have an equational logic for those classes.

(Positive) varieties of languages corresponding to pseudovarieties of (ordered) idempotent semigroups/monoids are not very important from the point of language theory. This is far from being the case for languages corresponding to pseudovarieties of literally idempotent homomorphisms.

Most of our classes result by considering intersections of well-known classical (positive) varieties with literally idempotent languages. Our new classes nicely fit into the table in Section 8 by Pin [10]. We characterize languages from certain

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classes of languages in various ways. More precisely, we describe the languages which are literally idempotent and which belong to the level 1/2, level 1, level 3/2 of the so-called Straubing-Thérien hierarchy, respectively. We also consider other interesting classes of languages, *e.g.* languages which are finite unions of the languages of the form $B_0^* B_1^* \dots B_k^*$, where k is a non-negative integer and $B_0, \dots, B_k$ are subsets of a given alphabet.

Notice that the motivation for studying literally idempotent languages also comes from the linear temporal logic. The formulas of LTL without the "next" operator determine literally idempotent languages. We give a logical characterization of languages from one of our classes.

The paper is organized as follows. In Section 2 we recall known results and techniques related to syntactic methods. Section 3 presents several new classes of languages. In Section 4 we introduce literally idempotent languages and their basic properties. Section 5 contains results concerning intersections of literally idempotent languages with some well-known classes (level 1/2, 1, 3/2, right-trivial languages, finite languages). The last section collects remarks dealing with the relationship to the linear temporal logic.

2. Preliminaries

Valuable treatments on syntactic methods in language theory are books by Almeida [1], Pin [9] and his chapter [10].

Let $\mathcal{M}$ (resp. $\mathcal{O}$) be the class of all surjective homomorphisms from free monoids over non-empty finite sets onto finite (ordered) monoids. A class $\mathcal{V} \subseteq \mathcal{M}$ is a *literal pseudovariety* if it is closed with respect to the homomorphic images, literal substructures and products of finite families (see Ésik *et al.* [4,5] or Straubing [11] for a more general notion of a $\mathbb{C}$-pseudovariety). More precisely, such class $\mathcal{V}$ satisfies the following:

- (H) for each $(\varphi : A^* \to M) \in \mathcal{V}$ and a surjective monoid homomorphism $\sigma : M \to N$, we have $\sigma\varphi \in \mathcal{V}$;
- (S) for each $(\varphi : A^* \to M) \in \mathcal{V}$ and for each $f : B^* \to A^*$ with $f(B) \subseteq A$, we have $(\varphi f : B^* \to (\varphi f)(B^*)) \in \mathcal{V}$;
- (P) each mapping of A^* onto one-element monoid is in V, and for each $(\varphi : A^* \to M)$, $(\psi : A^* \to N) \in \mathcal{V}$; the natural homomorphism of A^* onto A^* factorized by the intersections of kernels of φ and ψ is in $\mathcal{V}$.

Similarly, we define the literal pseudovarieties in the ordered case.

Let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let I_n, for $n \in \mathbb{N}$, be the set of all n-ary implicit operations for the class of finite monoids – see *e.g.* [1]. We write $\pi^M : M^n \to M$ for the realization of $\pi \in I_n$ on a finite monoid M. A pseudoidentity $\pi = \rho$, where $\pi, \rho \in I_n$, is *literally* satisfied in $(\varphi : A^* \to M) \in \mathcal{M}$ if

$$(\forall\, a_1, \dots, a_n \in A)\ \pi^M(\varphi(a_1), \dots, \varphi(a_n)) = \rho^M(\varphi(a_1), \dots, \varphi(a_n)).$$

We write $\varphi \models_{\mathscr{L}} \pi = \rho$ in this case.

Similarly, a pseudoinequality $\pi \leq \rho$, where $\pi, \rho \in I_n$, is *literally* satisfied in $(\varphi : A^* \to (M, \leq)) \in \mathcal{O}$ if

$$(\forall\ a_1, \ldots, a_n \in A)\ \pi^M(\varphi(a_1), \ldots, \varphi(a_n)) \leq \rho^M(\varphi(a_1), \ldots, \varphi(a_n)).$$

We write $\varphi \models_{\mathscr{L}} \pi \leq \rho$ in this case.

Usually we fix an alphabet $\Sigma = \{x_1, \ldots, x_n\}$ of variables and we identify a word $u = x_{i_1} \ldots x_{i_k} \in \Sigma^*$ with the implicit operation given by $u^M(a_1, \ldots, a_n) = a_{i_1} \ldots a_{i_k}$, where M is a finite monoid and $a_1, \ldots, a_n \in M$. Examples of implicit operations which are not of this form are u^ω, for $u \in \Sigma^+$. We define

$$((x_{i_1} \ldots x_{i_k})^\omega)^M(a_1, \ldots, a_n) = a^\omega,$$

where $a = a_{i_1} \ldots a_{i_k}$ and a^ω is the unique idempotent in the set $\{a, a^2, a^3, \ldots\}$.

A special case of the main result of Kunc [7] follows.

Result 2.1 (Kunc)**.** The literal pseudovarieties of homomorphisms onto finite monoids are exactly the subclasses of $\mathcal{M}$ defined by the literal satisfaction of sets of pseudoidentities.

One can expect an analogous result in the ordered case – we do not need it here, we only support it by examples.

By a *quotient* of $L \subseteq A^*$ we mean any set $u^{-1}Lv^{-1} = \{\, w \in A^* \mid uwv \in L \,\}$ where $u, v \in A^*$.

A *class* of (recognizable) languages is an operator $\mathscr{V}$ assigning to each non-empty finite set A a set $\mathscr{V}(A)$ of recognizable languages over the alphabet A.

Such a class is a *positive variety* if

(0) for each A, we have $\emptyset$, $A^* \in \mathscr{V}(A)$;
(i) each $\mathscr{V}(A)$ is closed with respect to finite unions, finite intersections and quotients; and
(ii) for each non-empty finite sets A and B and a homomorphism $f : B^* \to A^*$, $K \in \mathscr{V}(A)$ implies $f^{-1}(K) \in \mathscr{V}(B)$.

Adding the condition

(iii) each $\mathscr{V}(A)$ is closed with respect to complements,

we get a *boolean* variety.

A modification of (ii) to

(ii') for each non-empty finite sets A and B and a homomorphism $f : B^* \to A^*$ with $f(B) \subseteq A$, $K \in \mathscr{V}(A)$ implies $f^{-1}(K) \in \mathscr{V}(B)$

leads to the notions of *literal* positive/boolean variety of languages.

For $L \in \mathcal{V}(A)$, we put $L^{\mathsf{c}} = A^* \setminus L$. Moreover, $\mathcal{V}^{\mathsf{c}}(A) = \{\, L^{\mathsf{c}} \mid L \in \mathcal{V}(A) \,\}$ for each A.

Let $L \subseteq A^*$ be a recognizable language. Recall that the *syntactic congruence* $\sim_L$ on A^* is defined by

$$u \sim_L v \text{ if and only if } (\forall\ p, q \in A^*)\ (puq \in L \ \Leftrightarrow\ pvq \in L).$$

Further, the structure $\mathsf{O}(L) = A^*/\sim_L$ is called the *syntactic monoid* of L and the mapping $\varphi_L : A^* \to \mathsf{O}(L),\ u \mapsto u\sim_L$ is the *syntactic homomorphism*.

Moreover, $\mathsf{O}(L)$ is implicitly ordered by

$$u\sim_L \le\ v\sim_L \ \text{ if and only if } (\forall\ p,q \in A^*)\ (pvq \in L\ \Rightarrow\ puq \in L).$$

We speak about the *ordered syntactic monoid* and the *ordered syntactic homomorphism*.

Often we will use the pseudoinequality $x \le 1$. Notice that $(\mathsf{O}(L),\cdot,\le)$ satisfies this pseudoinequality if and only if

$$(\forall\ u,v,w \in A^*)\ (uw \in L\ \Rightarrow\ uvw \in L).$$

Similarly, the homomorphism φ_L satisfies $x \le 1$ literally if and only if

$$(\forall\ u,v \in A^*,\ a \in A)\ (uv \in L\ \Rightarrow\ uav \in L).$$

In this case the both variants of the satisfiability coincide. This is far from being true for the pseudoidentity $x^2 = x$.

For a class $\mathscr{V}$ of languages, let

$$\mathsf{M}(\mathscr{V}) =\ \langle\ \{\ \varphi_L : A^* \to \mathsf{O}(L) \mid A \text{ non-empty finite alphabet},\ L \in \mathscr{V}(A)\ \}\ \rangle$$

be the literal pseudovariety generated by the syntactic homomorphisms of members of $\mathscr{V}$, and conversely, for $\mathcal{V} \subseteq \mathcal{M}$,

$$\mathcal{V} \mapsto \mathsf{L}(\mathcal{V}),\ \text{ where } (\mathsf{L}(\mathcal{V}))(A) = \{\, L \subseteq A^* \mid \varphi_L \in \mathcal{V}\,\} \text{ for each } A.$$

Result 2.2 (Ésik and Larsen [5], Straubing [11])**.** The operators M and L are mutually inverse bijections between the classes of literal boolean varieties of languages and literal pseudovarieties of homomorphisms onto finite monoids.

Similarly as in Result 2.1 one can expect an ordered version of Result 2.2 – we do not need it here, we only support it by examples.

We recall certain classical (positive) varieties of languages – see [9,10].

(i) The languages of the level 1/2 over A are exactly finite unions of languages of the form

$$A^*a_1A^*a_2\ldots a_kA^*,\ k \in \mathbb{N}_0,\ a_1,\ldots,a_k \in A. \qquad (1/2)$$

We denote this positive variety of languages by $\mathscr{V}_{1/2}$ and it is known that $L \in \mathscr{V}_{1/2}(A)$ if and only if the ordered syntactic monoid of the language L satisfies the pseudoinequality $x \le 1$.

(ii) The languages of the level 1 over A are exactly boolean combinations of languages of the form (1/2). We denote this variety of languages by $\mathscr{V}_1$ and it is known that $L \in \mathscr{V}_1(A)$ if and only if the syntactic monoid of the language L is $\mathcal{J}$-trivial, *i.e.* it satisfies the pseudoidentities $x^\omega = x^{\omega+1}$ and $(xy)^\omega = (yx)^\omega$.

(iii) The languages of the level 3/2 over A are exactly finite unions of

$$B_0^*a_1B_1^*a_2\dots a_kB_k^*,\ k\in\mathbb{N}_0,\ a_1,\dots,a_k\in A,\ B_0,\dots,B_k\subseteq A. \qquad (3/2)$$

We denote this positive variety of languages by $\mathscr{V}_{3/2}$ and it is known that $L\in\mathscr{V}_{3/2}(A)$ if and only if the ordered syntactic monoid of the language L satisfies the pseudoinequalities $x^\omega yx^\omega\le x^\omega$ for every $x,y\in\Sigma^*$ such that $\mathsf{c}(x)=\mathsf{c}(y)$ ($\mathsf{c}(x)$ is the set of all variables occurring in x).

(iv) We denote by $\mathscr{R}$ the positive variety of languages which can be written as (disjoint) finite unions of languages of the form $B_0^*a_1B_1^*a_2\dots a_kB_k^*$, where

$$k\in\mathbb{N}_0,\ a_1,\dots,a_k\in A,\ B_0,\dots,B_k\subseteq A,\ B_0\not\ni a_1,\dots,B_{k-1}\not\ni a_k. \qquad (R)$$

The language L belongs to $\mathscr{R}$ if and only if its syntactic monoid is $\mathcal{R}$-trivial, *i.e.* it satisfies the pseudoidentity $(xy)^\omega x=(xy)^\omega$.

Finally, we consider two well-known classes of +-languages together with the corresponding pseudovarieties of semigroups.

(v) The class of all finite languages generates the positive variety of languages consisting of the finite languages and the full languages. This variety corresponds to the pseudovariety of ordered nilpotent semigroups with 0 being the greatest element. Such semigroups are characterized by the following pseudoinequalities $x^\omega y=x^\omega=yx^\omega,\ y\le x^\omega$.

(vi) The boolean variety of languages generated by the class of all finite languages is the class consisting of all finite and cofinite languages. This class corresponds to nilpotent semigroups.

3. New natural classes of languages

In this paper we deal mainly with the following classes of languages (we will see in the next sections that they are literally idempotent). Observe the similarities with the classes of languages from Section 2.

(i) Finite unions of languages

$$A^*a_1A^*a_2\dots a_kA^*,\ k\in\mathbb{N}_0,\ a_1,\dots,a_k\in A,\ a_1\ne a_2\ne\dots\ne a_k. \qquad (\mathscr{L}\ 1/2)$$

(ii) Finite unions of languages

$$B_1^*B_2^*\dots B_k^*,\ k\in\mathbb{N}_0,\ B_1,\dots,B_k\subseteq A. \qquad (\mathscr{L}\ 1/2\ c)$$

(iii) Boolean combinations of languages of the form ($\mathscr{L}$ 1/2).

(iv) Finite unions of languages of the form

$$B_0^*a_1B_1^*a_2\dots a_kB_k^*,\ k\in\mathbb{N}_0,\ a_1,\dots,a_k\in A,\ B_0,\dots,B_k\subseteq A,$$
$$a_1\ne a_2\ne\dots\ne a_k,\ a_1\in B_1,\dots,a_k\in B_k. \qquad (\mathscr{L}\ 3/2)$$

(v) Finite unions of languages of the form

$$B_0^* a_1 B_1^* a_2 \dots a_k B_k^*,\ k \in \mathbb{N}_0,\ a_1, \dots, a_k \in A,\ B_0, \dots, B_k \subseteq A,$$

$$a_1 \neq a_2 \neq \cdots \neq a_k,\ B_0 \not\ni a_1 \in B_1 \not\ni a_2 \in \cdots \not\ni a_k \in B_k. \qquad (\mathscr{L}\ R)$$

(vi) Finite unions of languages of the form

$$B_0^* a_1 B_1^* a_2 \dots a_k B_k^*,\ k \in \mathbb{N}_0,\ a_1, \dots, a_k \in A,\ B_0, \dots, B_k \subseteq A$$

$$a_1 \neq a_2 \neq \cdots \neq a_k,\ a_1 \in B_0 \cap B_1, \dots, a_k \in B_{k-1} \cap B_k. \qquad (\mathscr{L}\ E)$$

(vii) The class of all finite languages generates the literal positive variety of languages, denoted by $\mathscr{N}_f$, consisting of the finite languages and the full languages. This variety corresponds to the variety of homomorphisms onto ordered monoids which result from nilpotent semigroups satisfying the pseudoinequality $x \leq 0$ with units incomparable with other elements adjoined. This means $L \in \mathscr{N}_f(A)$ if and only if

$$\varphi_L \models_{\mathscr{L}} u^\omega x = u^\omega,\ xu^\omega = u^\omega,\ x \leq u^\omega,\ \text{for any } u \in \Sigma^+,\ x \in \Sigma.$$

(viii) The literal boolean variety of languages generated by the class of finite languages is the class $\mathscr{N}$ consisting of all finite and cofinite languages. This class corresponds to homomorphisms onto nilpotent semigroups with the extra unit elements adjoined.

4. Literally idempotent languages

A recognizable language L over a finite non-empty alphabet A is *literally idempotent* if its syntactic homomorphism $\varphi_L : A^* \to \mathsf{O}(L)$ satisfies the pseudoidentity $x^2 = x$ literally, which means

$$(\forall\ a \in A)\ a^2 \sim_L a$$

or equivalently

$$(\forall\ u, v \in A^*,\ a \in A)\ (uav \in L\ \Leftrightarrow\ ua^2v \in L). \qquad (*)$$

We denote the class of all such languages by $\mathscr{L}$.

We can introduce a string rewriting system which is given by rules $pa^2q \to paq$ for each $a \in A$, $p, q \in A^*$. Let $\to^*$ be the reflexive-transitive closure of the relation $\to$. We say that a word $u \in A^*$ is the *normal form* of a word w if it satisfies the properties

$$w \to^* u \quad \text{and} \quad (\,u \to^* v \text{ implies } u = v\,).$$

It is easy to see that this system is confluent and terminating. Consequently, for any word $w \in A^*$, there exists the unique normal form $\overrightarrow{w} \in A^*$ of the word w.

We will denote by $\sim$ the equivalence relation on A^* generated by the relation $\rightarrow$. In fact, this equivalence relation is a congruence of the monoid A^*.

In what follows we are interested in literal positive/boolean varieties consisting of literally idempotent languages. These varieties can be induced by classical varieties in two natural ways. At first, for a class of languages $\mathscr{C}$, we can consider the class of languages from $\mathscr{C}$ which are also literally idempotent languages, *i.e.* the intersection $\mathscr{C} \cap \mathscr{L}$ of the classes $\mathscr{C}$ and $\mathscr{L}$. The second possibility is to consider the following (closure) operator on languages. For any language $L \subseteq A^*$, we define

$$\overline{L} = \{ w \in A^* \mid (\exists\, u \in L)\ u \sim w \} \text{ which is } \{ w \in A^* \mid (\exists\, u \in L)\ \overrightarrow{u} = \overrightarrow{w} \}.$$

Lemma 4.1. *For $K, L \subseteq A^*$, we have:*

(i) *$\overline{L}$ is recognizable whenever L is recognizable;*
(ii) *$\overline{K \cup L} = \overline{K} \cup \overline{L}$;*
(iii) *$\overline{K \cap L} \subseteq \overline{K} \cap \overline{L}$.*

Proof. (i) Considering the regular substitution $\varphi : A^* \to A^*$ defined by the rule $\varphi(a) = a^+$, for each $a \in A$, we can write $\overline{L} = \varphi(\varphi^{-1}(L))$. Then we can apply Theorem 4.4 from [13] saying that the family of recognizable languages is closed under regular substitutions and inverse regular substitutions.

(ii) and (iii) are trivial observations. □

The following is obvious.

Lemma 4.2. *For a recognizable $L \subseteq A^*$, the following statements are equivalent:*

(i) *L is literally idempotent;*
(ii) *$\overline{L} = L$;*
(iii) *$\sim \subseteq \sim_L$;*
(iv) *L is a (disjoint) union (not necessarily finite!) of the languages of the form*

$$a_1^+ a_2^+ \dots a_k^+,\ k \in \mathbb{N}_0,\ a_1, \dots, a_k \in A,\ a_1 \neq a_2 \neq \dots \neq a_k.$$

For a class of languages $\mathscr{C}$, we can consider the class of literally idempotent languages $\overline{\mathscr{C}}$ where

$$\overline{\mathscr{C}}(A) = \{ \overline{L} \mid L \in \mathscr{C}(A) \}$$

for each A. Clearly, the following holds.

Lemma 4.3. *Let $\mathscr{C}$ be a class of languages. Then:*

(i) *The class $\overline{\mathscr{C}}$ is closed under union whenever $\mathscr{C}$ is closed under union.*
(ii) *$\mathscr{C} \cap \mathscr{L} \subseteq \overline{\mathscr{C}}$.* □

5. Varieties of literally idempotent languages

Our main results consist in syntactic characterizations of languages from Section 3, their relationship to the Straubing-Thérien hierarchy – see Propositions 5.4, 5.6 and 5.9 together with the following result – see Propositions 5.4–5.10.

Theorem 5.1. *For each* $\mathscr{V} \in \{\mathscr{V}_{1/2}, (\mathscr{V}_{1/2})^{\mathsf{c}}, \mathscr{R}, \mathscr{V}_1, \mathscr{V}_{3/2}\}$ *we have* $\mathscr{V} \cap \mathscr{L} = \overline{\mathscr{V}}$.

The key parts of our proofs consist in showing that $\overline{\mathscr{V}} \subseteq \mathscr{V}$. The reasonings are quite different – we have not found any general method. Moreover, the following examples show that $\overline{\mathscr{V}}$ needs not be a (positive) literal variety if $\mathscr{V}$ is a (positive) literal variety.

Example 5.2. We consider the class $\mathscr{N}_f$. Now $\mathscr{N}_f \cap \mathscr{L}$ consists of full languages, the empty language and the unit language, *i.e.* $(\mathscr{N}_f \cap \mathscr{L})(A) = \{\emptyset, \{\epsilon\}, A^*\}$. It is an easy observation that this literal variety is given by the literal pseudoidentities $x = y,\ x^2 = x$ and the pseudoinequality $1 \leq x$.

On the other hand, $\emptyset, \{\epsilon\}, A^* \in \overline{\mathscr{N}_f}(A)$ and a language $L \notin \{\emptyset, A^*\}$ over A belongs to $\overline{\mathscr{N}_f}(A)$ if and only if L is a finite union of languages of the form $a_1^+ a_2^+ \dots a_k^+$ where $a_1, \dots, a_k \in A,\ a_1 \neq a_2 \neq \dots \neq a_k$ (for $k = 0$ we mean the language $\{\epsilon\}$). This implies that $\overline{\mathscr{N}_f}$ is not a literal positive variety of languages, because $\overline{\mathscr{N}_f}$ is not closed under inverse literal homomorphic images. Indeed, for $A = \{a\},\ B = \{b, c\},\ f : b, c \mapsto a,\ L = a^+$ we have $f^{-1}(L) = \{b, c\}^+$.

If we consider the literal positive variety of languages generated by $\overline{\mathscr{N}_f}$ then it is easy to see that L belongs to $< \overline{\mathscr{N}_f} >_{plv} (A)$ if and only if L is a finite union of languages of the form $B_1^+ B_2^+ \dots B_k^+$ where $\{B_1, B_2, \dots, B_k\}$ is a partition of a subset of the alphabet A (*i.e.* different B_i's are disjoint) and $B_1 \neq B_2 \neq \dots \neq B_k$. For example, if $A = \{a, b, c\}$ then $\{a, c\}^+ \cup \{a, b\}^+ \{c\}^+ \{a, b\}^+ \in < \overline{\mathscr{N}_f} >_{plv} (A)$ but $\{a, c\}^+ \{b\}^+ \{a\}^+ \notin < \overline{\mathscr{N}_f} >_{plv} (A)$.

In a subsequent paper the first author will show that this positive variety is given by the literal satisfaction of the following pseudoidentities and pseudoinequalities

$$x^2 = x,\ u^\omega v x = u^\omega v y,\ x v u^\omega = y v u^\omega,\ x \leq u^\omega,$$

$$\text{for all } u, v \in \Sigma^+ \text{ such that } x, y \in \mathsf{c}(u),\ x, y \in \Sigma.$$

Example 5.3. We can also consider the variety $\mathscr{N}$. We have

$$(\mathscr{N} \cap \mathscr{L})(A) = \{\emptyset, \{\epsilon\}, A^+, A^*\}.$$

Moreover, if the language L over A is cofinite then $\overline{L} \in \{A^+, A^*\}$. For this reason $\overline{\mathscr{N}}(A) = \overline{\mathscr{N}_f}(A) \cup \{A^+\}$ and again it is not a literal variety.

Now, we will study the new classes from Section 3. We start with the variety $\mathscr{V}_{1/2}$. For a word $u = a_1a_2 \dots a_k,\ a_1, \dots, a_k \in A$, we denote

$$L_u = A^*a_1A^*a_2 \dots a_kA^*$$

the set of all words which contain the word u as a subword.

Proposition 5.4. *For a language L over A, the following are equivalent:*

(i) *L is a finite union of languages of the form ($\mathscr{L}$ 1/2);*
(ii) *$L \in (\mathscr{V}_{1/2} \cap \mathscr{L})(A)$;*
(iii) *the syntactic homomorphism $\varphi_L : A^* \to \mathsf{O}(L)$ of the language L satisfies the pseudoinequalities $x \le 1$ and $x^2 = x$ literally;*
(iv) *$L \in \overline{\mathscr{V}_{1/2}}(A)$.*

Proof. "(i) $\Rightarrow$ (ii)". It follows from the fact that each language L from (i) satisfies the condition $(*)$.

"(ii) $\Leftrightarrow$ (iii)" is clear because φ_L satisfies the pseudoinequality $x \le 1$ literally if and only if $\mathsf{O}(L)$ satisfies this pseudoinequality in the classical sense.

"(ii) $\Rightarrow$ (iv)" follows from Lemma 4.3 (ii).

"(iv) $\Rightarrow$ (i)". If $L \in \overline{\mathscr{V}_{1/2}}(A)$ then, by Lemma 4.1 (ii), L is a finite union of languages of the form $\overline{L_u}$. We prove that $\overline{L_u}$ is of the form ($\mathscr{L}$ 1/2).

First, we claim that $\overline{L_u} = \overline{L_{\overrightarrow{u}}}$. The inclusion $L_u \subseteq L_{\overrightarrow{u}}$ is trivial and $\overline{L_u} \subseteq \overline{L_{\overrightarrow{u}}}$ follows. Assuming that $w \in \overline{L_{\overrightarrow{u}}}$ then there is a word $s \in L_{\overrightarrow{u}}$ such that $w \sim s$. We define the word $s_{|u|}$ in such a way, that we replace any letter a in s by $a^{|u|}$, where $|u|$ is the length of the word u. Because s contains the word $\overrightarrow{u}$ as a subword, we can see that $s_{|u|}$ contains the word u. Hence $w \sim s_{|u|} \in L_u$ and we can conclude that $w \in \overline{L_u}$.

We proved that $\overline{L_u} = \overline{L_{\overrightarrow{u}}}$ and because $L_{\overrightarrow{u}}$ is of the form ($\mathscr{L}$ 1/2), *i.e.* it is literally idempotent as we proved in "(i) $\Rightarrow$ (ii)" at the beginning, we have $\overline{L_{\overrightarrow{u}}} = L_{\overrightarrow{u}}$ which implies that $\overline{L_u}$ is of the form ($\mathscr{L}$ 1/2). □

We prove now a similar proposition for the class $(\mathscr{V}_{1/2})^{\mathsf{c}}$. At first, we formulate the following technical lemma which describes the basic properties of languages of the form $(L_u)^{\mathsf{c}}$.

Lemma 5.5. *Let $u, u_1, \dots, u_n, w \in A^*,\ n \in \mathbb{N},\ u = a_1 \dots a_k,\ a_1, \dots, a_k \in A$. Then:*

(i) *$w \in \overline{(L_u)^{\mathsf{c}}}$ if and only if $\overrightarrow{w} \in (L_u)^{\mathsf{c}}$.*
(ii) $\overline{(L_u)^{\mathsf{c}}} = (A \setminus \{a_1\})^*\ a_1^*\ (A \setminus \{a_2\})^*\ a_2^*\ \dots\ a_{k-1}^*\ (A \setminus \{a_k\})^*$.
(iii) $\overline{(L_{u_1})^{\mathsf{c}} \cap \dots \cap (L_{u_n})^{\mathsf{c}}} = \overline{(L_{u_1})^{\mathsf{c}}} \cap \dots \cap \overline{(L_{u_n})^{\mathsf{c}}}$.

Proof. (i). The implication "$\Leftarrow$" is trivial.

"$\Rightarrow$" If $w \in \overline{(L_u)^{\mathsf{c}}}$ then there is a word $v \in (L_u)^{\mathsf{c}}$ such that $w \sim v$. This means that v does not contain the word u as a subword. Hence $\overrightarrow{v}$ does not contain the word u as a subword too, *i.e.* $\overrightarrow{w} = \overrightarrow{v} \in (L_u)^{\mathsf{c}}$.

(ii). We denote $K = (A \setminus \{a_1\})^*\ a_1^*\ (A \setminus \{a_2\})^*\ a_2^*\ \dots\ a_{k-1}^*\ (A \setminus \{a_k\})^*$ and we will prove that $\overline{(L_u)^{\mathsf{c}}} = K$.

"$\subseteq$". If $w \in \overline{(L_u)^{\mathsf{c}}}$ then $\overrightarrow{w} \in (L_u)^{\mathsf{c}}$ by (i). If we read $\overrightarrow{w}$ from left to right and look for the first occurrence of a_1 (if it exists) and then look for the first occurrence of a_2 after this first occurrence of a_1 (if it exists) and so on, we obtain the following factorization of $\overrightarrow{w}$:

$$\overrightarrow{w} = w_1 a_1 w_2 a_2 \dots a_l w_{l+1}, \text{ where } l < k,\ w_i \in (A \setminus \{a_i\})^*.$$

Hence $\overrightarrow{w} \in K$ and because $\overline{K} = K$ we have $w \in K$.

"$\supseteq$". Let $w \in K$. Then $w = w_1 a_1^{\alpha_1} w_2 a_2^{\alpha_2} \dots a_{k-1}^{\alpha_{k-1}} w_k$ where $w_i \in (A \setminus \{a_i\})^*$ and $\alpha_i \in \mathbb{N}_0$ for $i = 1, \dots, k-1$. Hence $\overrightarrow{w}$ is a subword of $\overrightarrow{w_1} a_1 \overrightarrow{w_2} a_2 \dots a_{k-1} \overrightarrow{w_k}$ and one can check by induction with respect to i that the word $\overrightarrow{w_1} a_1 \overrightarrow{w_2} a_2 \dots a_{i-1} \overrightarrow{w_i}$ does not contain the word $a_1 \dots a_i$ as a subword. This implies that $\overrightarrow{w} \in (L_u)^{\mathsf{c}}$ and $w \in \overline{(L_u)^{\mathsf{c}}}$ follows by (i).

(iii). The inclusion "$\subseteq$" is a trivial consequence of Lemma 4.1 (iii) and the inclusion "$\supseteq$" is a consequence of (i). Indeed, $w \in \overline{(L_{u_1})^{\mathsf{c}}} \cap \dots \cap \overline{(L_{u_n})^{\mathsf{c}}}$ implies $w \in \overline{(L_{u_i})^{\mathsf{c}}}$ and $\overrightarrow{w} \in (L_{u_i})^{\mathsf{c}}$, for $i = 1, \dots, n$, follows. Hence $\overrightarrow{w} \in (L_{u_1})^{\mathsf{c}} \cap \dots \cap (L_{u_n})^{\mathsf{c}}$ and consequently $w \in \overline{(L_{u_1})^{\mathsf{c}} \cap \dots \cap (L_{u_n})^{\mathsf{c}}}$. ☐

Proposition 5.6. *For a language L over A, the following are equivalent:*

(i) *L is a finite union of the languages of the form ($\mathscr{L}$ 1/2 c).*
(ii) $L \in ((\mathscr{V}_{1/2})^{\mathsf{c}} \cap \mathscr{L})(A)$.
(iii) *The syntactic homomorphism $\varphi_L : A^* \to \mathsf{O}(L)$ satisfies the pseudoinequalities $x^2 = x$ and $1 \le x$ literally.*
(iv) $L \in \overline{(\mathscr{V}_{1/2})^{\mathsf{c}}}(A)$.
(v) $L \in \overleftarrow{(\mathscr{V}_{1/2})^{\mathsf{c}}}(A)$.
(vi) *L is a finite intersection of the languages of the form ($\mathscr{L}$ 1/2 c).*

Proof. "(i) $\Rightarrow$ (iii)". Again the condition $(*)$ is satisfied, and moreover, for all $u, v \in A^*,\ a \in A,\ uav \in L \ \Rightarrow\ uv \in L$.

As in the previous proof we have that (iii) is equivalent to (ii) and by Lemma 4.3 we have that (ii) implies (iv).

"(iv) $\Rightarrow$ (vi)". Let $L \in \overline{(\mathscr{V}_{1/2})^{\mathsf{c}}}(A)$. Then $L = \overline{R}$, where $R \in (\mathscr{V}_{1/2})^{\mathsf{c}}(A)$. So, R^{c} is a finite union of the languages of the form $A^* a_1 A^* a_2 \dots a_k A^*$, $k \in \mathbb{N}_0$, $a_1, \dots, a_k \in A$. This means that R is a finite intersection of the languages $(L_u)^{\mathsf{c}}$.

The language $L = \overline{R}$ is an intersection of languages of the form $\overline{(L_u)^{\mathsf{c}}}$ by (iii) in Lemma 5.5. Moreover, any of these languages is of the form ($\mathscr{L}$ 1/2 c) by (ii) of the same lemma.

"(vi) $\Rightarrow$ (i)". Let $K = B_1^* B_2^* \dots B_k^*$ and $L = C_1^* C_2^* \dots C_l^*$, where $k, l \in \mathbb{N}_0$ and $B_1, \dots, B_k, C_1, \dots, C_l \subseteq A$, be two languages of the form ($\mathscr{L}$ 1/2 c). We prove that $K \cap L$ is a union of the languages of this form by induction with respect to $k + l$.

If one of k, l is equal to 0, then the corresponding language is $\{\epsilon\}$ and the statement is obvious. If $k = 1$ then

$$B_1^* \cap C_1^* C_2^* \dots C_l^* = (B_1 \cap C_1)^* (B_1 \cap C_2)^* \dots (B_1 \cap C_l)^*$$

and analogically for $l = 1$.

For $k, l > 1$ we put $K' = B_2^* \dots B_k^*$ and $L' = C_2^* \dots C_l^*$. Then

$$K \cap L = (B_1 \cap C_1)^* (K' \cap L) \cup (B_1 \cap C_1)^* (K \cap L')$$

and we can use the induction assumption for $K' \cap L$ and $K \cap L'$ and the distributivity law.

So, we proved that the conditions (i)–(iv) and (vi) are equivalent. The condition (v) is equivalent to those by Proposition 5.4 (iii). □

Proposition 5.7. *For a language L over A, the following are equivalent:*

(i) *L is a finite union of languages of the form ($\mathscr{L}$ R).*
(ii) *$L \in (\mathscr{R} \cap \mathscr{L})(A)$.*
(iii) *$L \in \overline{\mathscr{R}}(A)$.*

Proof. "(i) $\Rightarrow$ (ii)" is similar to the previous proofs.

"(ii) $\Rightarrow$ (iii)". Again by Lemma 4.3.

"(iii) $\Rightarrow$ (i)". If $L \in \overline{\mathscr{R}}(A)$, then $L = \overline{R}$, where $R \in \mathscr{R}(A)$. So, R is a finite union of the languages of the form (R). If we apply Lemma 4.1 we see that L is a finite union of the languages of the form $K = \overline{B_0^* a_1 B_1^* a_2 \dots a_k B_k^*}$, $B_0 \not\ni a_1, \dots, B_{k-1} \not\ni a_k$. We show that each such language K can be written as a finite union of languages of the form ($\mathscr{L}$ R). We prove that such K with the set of "bad" indices $\{\, i \mid a_i = a_{i+1}$ or $a_i \notin B_i \,\}$ can be written as a union of languages of the same form, but with the set of bad indices of a less cardinality. Indeed, let i be such that $a_i = a_{i+1}$ or $a_i \notin B_i$. First, assume that $a_i = a_{i+1}$. Then $a_i \notin B_i$. If $B_i = \emptyset$ we can simply remove $B_i^* a_{i+1}$ from the expression of the language K. Otherwise we write the language K as a union of certain languages $L(c)$ for $c \in B_i \cup \{a_i\}$ as follows. The language $L(a_i)$ comes from our expression if we exchange the string $a_i B_i^* a_{i+1}$ with a_i, *i.e.*

$$L(a_i) = \overline{B_0^* a_1 B_1^* a_2 \dots a_{i-1} B_{i-1}^* a_i B_{i+1}^* \dots a_k B_k^*}.$$

This language consists of words from K which do not use letters from B_i. For $c \in B_i$ the language $L(c)$ comes from our expression if we exchange the part $a_i B_i^* a_{i+1}$ with $a_i a_i^* c B_i^* a_{i+1}$, *i.e.*

$$L(c) = \overline{B_0^* a_1 B_1^* a_2 \dots a_{i-1} B_{i-1}^* a_i a_i^* c B_i^* a_{i+1} B_{i+1}^* \dots a_k B_k^*}.$$

The language $L(c)$ consists of words from K which use letters from B_i and the first such letter is c.

In the second case we have $a_i \neq a_{i+1}$ and $a_i \notin B_i$ and we can apply a similar construction. More precisely, we have $a_i \notin B_i$, $a_{i+1} \notin B_i$ and we write the language K as the union of the languages $L(c)$ for $c \in B_i$ given above and of the following language $K(a_i)$. The language $K(a_i)$ comes from the expression of K if we exchange B_i^* with a_i^*, *i.e.*

$$K(a_i) = \overline{B_0^* a_1 B_1^* a_2 \dots a_{i-1} B_{i-1}^* a_i a_i^* a_{i+1} B_{i+1}^* \dots a_k B_k^*}.$$

The language $K(a_i)$ consists of words from K which do not use letters from B_i. □

For a class $\mathscr{V}$ of languages we put:

$$\mathscr{V}^{\mathsf{d}}(A) = \{ L^{\mathsf{d}} \mid L \in \mathscr{V}(A) \}$$

– the class *dual* to $\mathscr{V}$, where

$$L^{\mathsf{d}} = \{ a_k \dots a_1 \mid a_1 \dots a_k \in L,\, a_1, \dots, a_k \in A \};$$

– the language dual to L.

Corollary 5.8. $\mathscr{V}_1 \cap \mathscr{L} = \overline{\mathscr{V}_1}$.

Proof. By Proposition 5.7 we have $\mathscr{R} \cap \mathscr{L} = \overline{\mathscr{R}}$ which has the dual version $\mathscr{R}^{\mathsf{d}} \cap \mathscr{L} = \overline{\mathscr{R}^{\mathsf{d}}}$. It is well-known that $\mathscr{R} \cap \mathscr{R}^{\mathsf{d}} = \mathscr{V}_1$.

The inclusion $\mathscr{V}_1 \cap \mathscr{L} \subseteq \overline{\mathscr{V}_1}$ follows from Lemma 4.3.

If $L \in \overline{\mathscr{V}_1}(A)$ then $L = \overline{K}$ where $K \in \mathscr{V}_1(A)$. Hence $K \in \mathscr{R}(A) \cap \mathscr{R}^{\mathsf{d}}(A)$ and we obtain $L = \overline{K} \in \overline{\mathscr{R}}(A) \cap \overline{\mathscr{R}^{\mathsf{d}}}(A)$. Now we use the previous proposition to get $L \in (\mathscr{R} \cap \mathscr{L})(A)$ and $L \in (\mathscr{R}^{\mathsf{d}} \cap \mathscr{L})(A)$. Hence $L \in (\mathscr{R} \cap \mathscr{R}^{\mathsf{d}} \cap \mathscr{L})(A) = (\mathscr{V}_1 \cap \mathscr{L})(A)$. □

Proposition 5.9. *For a language L over A, the following are equivalent:*

(i) *L is a boolean combination of languages of the form ($\mathscr{L}$ 1/2).*
(ii) *L is a boolean combination of languages of the form ($\mathscr{L}$ 1/2 c).*
(iii) *$L \in (\mathscr{V}_1 \cap \mathscr{L})(A)$.*
(iv) *The syntactic homomorphism $\varphi_L : A^* \to \mathsf{O}(L)$ of the language L satisfies the pseudoidentity $x^2 = x$ literally and $\mathsf{O}(L)$ is $\mathcal{J}$-trivial.*
(v) *$L \in \overline{\mathscr{V}_1}(A)$.*

Proof. The conditions (i) and (ii) are equivalent by Proposition 5.6. The equivalence of conditions (iii) and (iv) follows from the characterization of varieties $\mathscr{V}_1$ and $\mathscr{L}$. The equivalence of conditions (iii) and (v) is contained in Corollary 5.8. The implication (i) $\Rightarrow$ (iii) holds as $\mathscr{V}_1 \cap \mathscr{L}$ is closed under boolean operations. It remains to show the implication (iii) $\Rightarrow$ (i).

Let $L \in (\mathscr{V}_1 \cap \mathscr{L})(A)$. Then L is a literally idempotent and it is a boolean combination of the languages of the form

$$A^* a_1 A^* a_2 \dots a_k A^*,\ k \in \mathbb{N}_0,\ a_1, \dots, a_k \in A,$$

i.e. L is a finite union of the languages of the form

$$L_{u_1} \cap \cdots \cap L_{u_g} \cap (L_{v_1})^{\mathsf{c}} \cap \cdots \cap (L_{v_h})^{\mathsf{c}}.$$

We will show that L can be written as a boolean combination of the languages of the form ($\mathscr{L}$ 1/2). In fact, we will follow the original proof of a characterization of the class $\mathscr{V}_1$, where a piecewise testable language is decomposed into a boolean combination of the languages L_u – compare our proof with that of Simon's theorem in [9].

Because our literally idempotent language L is fully given by the words in normal form contained in it, we will concentrate on such words. We denote by r the maximal length of words which occur in the mentioned description of L as a boolean combination of languages of the form L_u. Now for any word $w \in L$ in normal form we consider two following lists of words in normal forms:

$s_1, \ldots, s_p$ are all words in normal form of the length at most $2r$ which are subwords of w;

$t_1, \ldots, t_q$ are all words in normal form of the length at most $2r$ which are not subwords of w.

We consider the language

$$N_w = L_{s_1} \cap \cdots \cap L_{s_p} \cap (L_{t_1})^{\mathsf{c}} \cap \cdots \cap (L_{t_q})^{\mathsf{c}}.$$

In this way we define finitely many languages (for all w's we have only finitely many s's and t's). We see that N_w is a boolean combination of languages of the form ($\mathscr{L}$ 1/2), in particular, N_w is literally idempotent.

We will show that

$$L = \bigcup_{w \in L} N_w.$$

Recall that the languages on both sides of this equality are literally idempotent.

"$\subseteq$". Let $x \in L$. Then $w = \overrightarrow{x} \in L$ and $x \in N_w$.

"$\supseteq$". Let $x \in N_w$ where $w \in L$ is in normal form. Assume that

$$w \in K = L_{u_1} \cap \cdots \cap L_{u_g} \cap (L_{v_1})^{\mathsf{c}} \cap \cdots \cap (L_{v_h})^{\mathsf{c}} \subseteq L.$$

Because N_w is literally idempotent we have $\overrightarrow{x} \in N_w$. We show that $\overrightarrow{x} \in K$, which implies $x \in L$. So, it is enough to prove that $\overrightarrow{x} \in L_{u_i}$ and $\overrightarrow{x} \notin L_{v_j}$.

If we take an arbitrary $u \in \{u_1, \ldots, u_g\}$ then u is contained in w. Because w is in normal form and the length of u is $\leq r$, we can find a word s which has the following properties:

i) u is a subword of the word s;
ii) s is a subword of w;
iii) s is a word in normal form; and
iv) s has the length at most $2r$.

The construction of s is straightforward: we look on w and mark the subword u, then for each two consecutive occurrences of the same letter in u, say aa, we mark some different letter, say $b \neq a$, in w between these two a's. Such a letter exists because w is in normal form. If we separate all marked "double-letters" from u in w, then we obtain a word s with required properties.

Hence u is a subword of some s_i which is a subword of $\overrightarrow{x}$ and $\overrightarrow{x} \in L_u$ follows.

Now we take an arbitrary $v \in \{v_1, \ldots, v_h\}$ and assume for a moment that $\overrightarrow{x} \in L_v$. This means that $\overrightarrow{x}$ contains v as a subword. Again, we can find a subword s of the word $\overrightarrow{x}$ such that s is in normal form, contains v as a subword and it is of length at most $2r$. Because $\overrightarrow{x} \in N_w$ we know that $s \in \{s_1, \ldots, s_p\}$. Hence v is a subword of s, which is a subword of w. This implies that $w \in L_v$ and it is a contradiction with $w \in K$. □

One can prove the following result in the similar way as Proposition 5.9.

Proposition 5.10. *For a language L over A, the following are equivalent:*

(i) *L is a finite union of languages of the form ($\mathscr{L}$ 3/2).*
(ii) *$L \in (\mathscr{V}_{3/2} \cap \mathscr{L})(A)$.*
(iii) *$L \in \overline{\mathscr{V}_{3/2}}(A)$.*

Example 5.11. We can consider similar variety of all languages which are finite unions of languages of the form ($\mathscr{L}$ E). It is clear that this class is a literal positive variety contained in $\overline{\mathscr{V}_{3/2}}$. The inclusion is proper as we have an example of the language $a^*b^+ = a^*bb^* \in \overline{\mathscr{V}_{3/2}}(\{a,b\})$ which can not be written in the form ($\mathscr{L}$ E).

6. Linear temporal logic without the operator "next"

In this section we mention a connection between the Linear Temporal Logic (LTL) and the concept of the literal idempotency. The expressive power of certain variants of the temporal logic were successfully characterized applying algebraic methods; in particular, the concept of the syntactic monoid was used in [2,3,12]. In the center of our interest is the expressive power of LTL formulas which do not use the "next" operator.

First, we recall basic definitions. *A formula of linear temporal logic without next operator* (LTLWN) over a finite set A of letters is built from the elements of the alphabet A and the logical constant T (true) using the boolean connectives $\neg$ and $\vee$ and the temporal logic operator U (until).

Let $w \in A^*$ be a word over A. The length of w is denoted by $|w|$. For any $1 \leq i \leq n = |w|$ we denote by $w(i)$ the ith letter of w and for any $0 \leq i \leq n = |w|$ we denote by w_i the suffix of w starting after the ith position, *i.e.* $w_i = w(i+1) \ldots w(n)$, in particular $w_0 = w$ and w_n is the empty word.

The validity of the formula φ of LTLWN on $w \in A^*$ is defined as follows:

$$
\begin{aligned}
& w \models \mathsf{T} \\
& w \models a \;\Leftrightarrow\; w(1) = a \\
& w \models \neg\varphi \;\Leftrightarrow\; w \not\models \varphi \\
& w \models \varphi_1 \vee \varphi_2 \;\Leftrightarrow\; w \models \varphi_1 \vee w \models \varphi_2 \\
& w \models \varphi_1 \mathsf{U}\, \varphi_2 \;\Leftrightarrow\; (\exists\, i \in \{0, \dots, |w|\})\ (w_i \models \varphi_2 \wedge \forall\, 0 \le j < i : w_j \models \varphi_1).
\end{aligned}
$$

Every formula φ defines the language $L_\varphi = \{\, w \in A^* \mid w \models \varphi \,\}$.

Traditionally, in linear temporal logic we consider also operator "next" and it is well-known that language is definable by linear logic formula if and only if it is star-free, *i.e.* it has aperiodic syntactic monoid.

It was proved that languages definable by LTLWN are exactly those which are stutter-invariant (see [8]). This observation can be rephrased to our terminology.

Result 6.1 (Peled and Wilke [8])**.** The class of languages definable by LTLWN is a literal boolean variety of languages corresponding to the pseudovariety of homomorphisms which are aperiodic and literally idempotent. *I.e.* $L \in$ LTLWN if and only if

$$\varphi_L \models_{\mathscr{L}} u^\omega = u^\omega u,\ x^2 = x, \text{ for any } u \in \Sigma^+,\ x \in \Sigma.$$

We expect that analogical results could be established for many significant subclasses of LTLWN and LTL, *e.g.* subclasses which correspond to certain fragments of linear temporal logic. Concrete examples could be hierarchies studied in literature, *e.g.* in [6,12].

In this paper we just show an example of such subclass. In fact, we will consider formulas of a special form which correspond to certain literal variety of languages.

For a non-empty subset $B = \{b_1, \dots, b_k\} \subseteq A$ we consider the formula $\varphi_B = b_1 \vee b_2 \vee \dots \vee b_k$ which defines the language BA^*. We can also put $\varphi_\epsilon = \neg\varphi_A$, a formula getting the language $\{\epsilon\}$.

We say that the formula φ is *easy* if it is of the form

$$\psi = \varphi_{B_1} \mathsf{U}\, (\varphi_{B_2} \mathsf{U}\, (\dots(\varphi_{B_n} \mathsf{U}\, \varphi_\epsilon))\dots).$$

The language is *easy* if it is definable by boolean combinations of easy formulas.

Proposition 6.1. *Let L be a language over A. Then L is easy if and only if $L \in \overline{\mathscr{V}_1}(A)$.*

Proof. It is easy to see that an easy formula $\psi = \varphi_{B_1} \mathsf{U}\, (\varphi_{B_2} \mathsf{U}\, (\dots(\varphi_{B_n} \mathsf{U}\, \varphi_\epsilon))\dots)$ define the language $B_1^* B_2^* \dots B_n^*$. Hence the statement follows from Proposition 5.9. □

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