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# ON CRITICAL EXPONENTS IN FIXED POINTS OF $k$-UNIFORM BINARY MORPHISMS

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**Abstract.** Let $\mathbf{w}$ be an infinite fixed point of a binary $k$-uniform morphism $f$, and let $E(\mathbf{w})$ be the critical exponent of $\mathbf{w}$. We give necessary and sufficient conditions for $E(\mathbf{w})$ to be bounded, and an explicit formula to compute it when it is. In particular, we show that $E(\mathbf{w})$ is always rational. We also sketch an extension of our method to non-uniform morphisms over general alphabets.

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## 1. Introduction

Let $\mathbf{w}$ be a right-infinite word over a finite alphabet $\Sigma$. The *critical exponent* of $\mathbf{w}$, denoted by $E(\mathbf{w})$, is the supremum of the set of exponents $r \in \mathbb{Q}_{\geq 1}$, such that $\mathbf{w}$ contains an $r$-power (see Sect. 2 for the definition of fractional powers). Given an infinite word $\mathbf{w}$, a natural question is to determine its critical exponent.

The first critical exponent to be computed was probably that of the Thue-Morse word, $\mathbf{t}$ [2,22]. This word, defined as the fixed point beginning with 0 of the Thue-Morse morphism $\mu(0) = 01$, $\mu(1) = 10$, was proved by Thue in 1912 to be *overlap-free*, that is, to contain no subword of the form $axaxa$, where $a \in \{0, 1\}$ and $x \in \{0, 1\}^*$. In other words, $\mathbf{t}$ is *$r$-power-free* for all $r > 2$; and since it contains 2-powers (squares), by our definition $E(\mathbf{t}) = 2$. Another famous word for which the critical exponent has been computed is the Fibonacci word $\mathbf{f}$, defined as the fixed point of the Fibonacci morphism $f(0) = 01$, $f(1) = 0$. In 1992, Mignosi and Pirillo [19] showed that $E(\mathbf{f}) = 2 + \varphi$, where $\varphi = (1 + \sqrt{5})/2$ is the golden mean. This gives an example of an irrational critical exponent. Another result we mention here is the critical exponent of the ternary Arshon sequence, generated

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by iterating two alternating 3-uniform ternary morphisms: in 2001, Klepinin and Sukhanov [12] showed it to be $\frac{7}{4}$.

In a more general setting, critical exponents have been studied mainly with relation to Sturmian words (for the definition, properties and structure of Sturmian words, see *e.g.* [16] Chap. 2). In 1989, Mignosi [18] proved that for a Sturmian word $\mathbf{s}$, $E(\mathbf{s}) < \infty$ if and only if the continued fraction expansion of the slope of $\mathbf{s}$ has bounded partial quotients; an alternative proof was given in 1999 by Berstel [3]. In 2000, Vandeth [23] gave an explicit formula for $E(\mathbf{s})$, where $\mathbf{s}$ is a Sturmian word which is a fixed point of a morphism, in terms of the continued fraction expansion of the slope of $\mathbf{s}$. In particular, $E(\mathbf{s})$ is algebraic quadratic. Alternative proofs for the results of Mignosi and Vandeth, with some generalizations, were given in 2000 by Carpi and de Luca [5], and in 2001 by Justin and Pirillo [11]. Carpi and de Luca also showed that $2 + \varphi$ is the minimal critical exponent for any Sturmian word. In 2002, Damanik and Lenz [7] gave a formula for critical exponents of general Sturmian words, again in terms of the continued fraction expansion of the slope. An alternative proof for this result was given in 2003 by Cao and Wen [4].

In this work we consider infinite words generated by iterating a morphism, also known as *pure morphic sequences* or *D0L-words*. For such words, most of the research has focused on deciding whether a given word has bounded critical exponent; see [6,8,13,20,21]. We study words generated by binary $k$-uniform morphisms. Let $f$ be a binary $k$-uniform morphism, and let $\mathbf{w}$ be a fixed point of $f$. After some preliminary definitions and results, given in Section 2, we analyze in Section 3 the structure of powers occurring in $\mathbf{w}$. We show that when $E(\mathbf{w})$ is bounded, sufficiently large powers must have a power block divisible by $k$, and must be produced by a simple iterative process, which we describe. Based on this analysis, we give necessary and sufficient conditions for $E(\mathbf{w})$ to be bounded, and an explicit formula to compute it when it is. In particular, we show that if $E(\mathbf{w}) < \infty$ then it is rational. We also show that, given a rational number $0 < r < 1$, we can construct a binary $k$-uniform morphism $f$ such $E(f^{\omega}(0)) = n + r$ for some positive integer $n$.

The results presented in this paper where generalized in [15] to non-uniform morphisms over an arbitrary finite alphabet. The main result, sketched in Section 4, is that if $f$ is uniform, then $E(\mathbf{w})$ is rational when bounded; and if $f$ is non-erasing, then $E(\mathbf{w})$ lies in the field extension $\mathbb{Q}[\lambda_1, \ldots, \lambda_\ell]$, where $\lambda_1, \ldots, \lambda_\ell$ are the eigenvalues of the incidence matrix of $f$. In particular, it is algebraic of degree at most $|\Sigma|$. In [15] we also gave (under certain conditions) an algorithm for computing $E(\mathbf{w})$.

Though the results of [15] are much more general, two points distinguish the results of this paper: first, we give a simple formula for computing $E(\mathbf{w})$, based on only 4 iterations of $f$; for the general case, there are no such formula and bound. Secondly, the proof is based on very elementary tools, while the general results relies on heavy combinatorial and algebraic machinery.

This paper is an extended and corrected version of [14].

## 2. Preliminaries

We use $\mathbb{Z}_{\geq r}$ (and similarly $\mathbb{Q}_{\geq r}, \mathbb{R}_{\geq r}$) to denote the integers (similarly rational or real numbers) greater than or equal to $r$.

Let $\Sigma$ be a finite alphabet. We use the notation $\Sigma^*$, $\Sigma^+$ and $\Sigma^\omega$ to denote the sets of finite words, non-empty finite words, and right-infinite words over $\Sigma$, respectively. We use $\varepsilon$ to denote the empty word. Infinite words are usually denoted by bold letters. For a finite word $w \in \Sigma^*$, $|w|$ is the length of $w$, and $|w|_a$ is the number of occurrences in $w$ of the letter $a \in \Sigma$. For both finite and infinite words, $w_i$ is the letter at position $i$, starting from zero; *e.g.*, $w = w_0 w_1 \cdots w_n$, $\mathbf{w} = w_0 w_1 w_2 \cdots$, where $w_i \in \Sigma$. A word $u \in \Sigma^*$ is a *subword* of a word $w \in \Sigma^* \cup \Sigma^\omega$ if $w = xuy$ for some words $x \in \Sigma^*$ and $y \in \Sigma^* \cup \Sigma^\omega$. If $x = \varepsilon$ (resp. $y = \varepsilon$) then $u$ is a *prefix* (resp. a *suffix*) of $w$. A prefix (resp. suffix) $u$ of $w$ is *proper* if $u \neq w$. The set of subwords of a word $w$ is denoted by $S(w)$.

Let $\mathbf{w}$ be an infinite word. An *occurrence* of a subword within $\mathbf{w}$ is a triple $(z, i, j)$, where $z \in S(\mathbf{w})$, $0 \leq i \leq j$, and $w_i \cdots w_j = z$. In other words, $z$ occurs in $\mathbf{w}$ at positions $i, \ldots, j$. For convenience, we usually omit the indices, and refer to an occurrence $(z, i, j)$ as $z = w_i \cdots w_j$. The set of all occurrences of subwords within $\mathbf{w}$ is denoted by $OC(\mathbf{w})$.

We use the notation $\lhd$ to denote the relation of subword or *suboccurrence*: if $x$ and $y$ are words, $x$ finite and $y$ finite or infinite, then $x \lhd y$ stands for "$x$ is a subword of $y$"; if $(x, i, j)$ and $(y, i', j')$ are occurrences within a word $\mathbf{w}$, then $x \lhd y$ means that $i' \leq i$ and $j' \geq j$.

Let $z = a_0 \cdots a_{n-1} \in \Sigma^+$, $a_i \in \Sigma$. A positive integer $q \leq |z|$ is a *period* of $z$ if $a_{i+q} = a_i$ for $i = 0, \cdots, n-1-q$. An infinite word $\mathbf{z} = a_0 a_1 \cdots \in \Sigma^\omega$ has a period $q \in \mathbb{Z}_{\geq 1}$ if $a_{i+q} = a_i$ for all $i \geq 0$; in this case, $\mathbf{z}$ is *periodic*, and we write $\mathbf{z} = x^\omega$, where $x = a_0 \cdots a_{q-1}$. If $\mathbf{z}$ has a periodic suffix, we say it is *ultimately periodic*.

A *fractional power* is a word of the form $z = x^n y$, where $n \in \mathbb{Z}_{\geq 1}$, $x \in \Sigma^+$, and $y$ is a proper prefix of $x$. Equivalently, $z$ has a $|x|$-period and $|y| = |z| \mod |x|$. If $|z| = p$ and $|x| = q$, we say that $z$ is a *$p/q$-power*, or $z = x^{p/q}$. Since $q$ stands for both the fraction's denominator and the period, we use non-reduced fractions to denote fractional powers: for example, 10101 is a $\frac{5}{2}$-power (as well as a $\frac{5}{4}$-power), while 1010101010 is a $\frac{10}{4}$-power (as well as a $\frac{10}{2}$-power). The word $x$ is referred to as the *power block*.

Let $\alpha$ be a real number. We say that a word $w$ (finite or infinite) is *$\alpha$-power-free* if no subword of it is an $r$-power for any rational $r \geq \alpha$; otherwise, $w$ *contains an $\alpha$-power*. The *critical exponent* of an infinite word $\mathbf{w}$ is defined by

$$E(\mathbf{w}) = \sup\{r \in \mathbb{Q}_{\geq 1} : \mathbf{w} \text{ contains an } r\text{-power}\}. \tag{1}$$

By this definition, $\mathbf{w}$ contains $\alpha$-powers for all $1 \leq \alpha < E(\mathbf{w})$, but no $\alpha$-powers for $\alpha > E(\mathbf{w})$; it may or may not contain $E(\mathbf{w})$-powers.

A morphism $f : \Sigma^* \to \Gamma^*$ is *$k$-uniform* if $|f(a)| = k$ for all $a \in \Sigma$, where $k$ is a positive integer. A morphism $f : \Sigma^* \to \Sigma^*$ is *prolongable* on a letter $a \in \Sigma$ if $f(a) = ax$ for some $x \in \Sigma^+$, and furthermore $f^n(x) \neq \varepsilon$ for all $n \geq 0$. If this is the case, then $f^n(a)$ is a proper prefix of $f^{n+1}(a)$ for all $n \geq 0$, and by applying $f$ successively we get an infinite *fixed point* of $f$,

$$f^\omega(a) = \lim_{n\to\infty} f^n(a) = axf(x)f^2(x)f^3(x)\cdots$$

Moreover, for a uniform morphism $f \neq \mathrm{Id}$, $\mathbf{w}$ is a fixed point of $f$ if and only if $\mathbf{w} = f^\omega(a)$ for some $a \in \Sigma$ on which $f$ is prolongable. Such fixed points are called *pure morphic sequences* or *D0L words*. In this work we consider powers in fixed points of uniform morphisms defined over a binary alphabet $\Sigma = \{0, 1\}$, therefore we assume that $f$ is prolongable on 0.

We now give three key definitions.

**Definition 1.** Let $z \in \Sigma^+$ be a $p/q$-power. We say that $z$ is *reducible* if it contains a $p'/q'$-power, such that $p'/q' > p/q$, or $p'/q' = p/q$ and $q' < q$. If $p'/q' > p/q$ then $z$ is *strictly reducible*.

**Example 1.** The $\frac{8}{4}$-power 1011 1011 is strictly reducible, since it contains the $\frac{3}{1}$-power 111. The $\frac{6}{3}$-power 101 101 is reducible since it contains the $\frac{2}{1}$-power 11. The word 1111 is strictly reducible as a $\frac{4}{2}$-power and irreducible as a $\frac{4}{1}$-power.

**Definition 2.** Let $\mathbf{w} \in \Sigma^\omega$, and let $z = w_i \cdots w_j \in OC(\mathbf{w})$ be a $p/q$-power. We say that $z$ is *left stretchable* (resp. *right stretchable*) if $w_{i-1} \cdots w_j$ (resp. $w_i \cdots w_{j+1}$) is a $(p+1)/q$-power. If $z$ is neither left nor right stretchable we say it is an *unstretchable power*.

**Example 2.** Let $f : \{0,1\}^* \to \{0,1\}^*$ be the morphism defined by $f(0) = 01$ and $f(1) = 00$, and let $\mathbf{w} = f^\omega(0) = 0100\ 01\ 01\ 01\ 0\ 0\cdots$. Then the $\frac{6}{2}$-power $w_4 \cdots w_9 = 010101$ is right-stretchable to the $\frac{7}{2}$-power $w_4 \cdots w_{10} = 0101010$.

Since $E(\mathbf{w})$ is an upper bound, it is enough to consider irreducible, unstretchable powers when computing it. Therefore, over a binary alphabet, we can assume that $p/q \geq 2$: since any binary word of length 4 or more contains a square, a $p/q$-power over $\{0, 1\}$ with $1 < p/q < 2$ is always reducible, save for the $\frac{3}{2}$-power 101 (010).

**Definition 3.** Let $f$ be a binary $k$-uniform morphism. The *left stretch* of $f$, denoted by $\sigma_f$, is the longest word $\sigma \in \{0,1\}^*$ satisfying $f(0) = x\sigma$ and $f(1) = y\sigma$ for some $x, y \in \{0,1\}^*$. Similarly, the *right stretch* of $f$, denoted by $\rho_f$, is the longest word $\rho \in \{0,1\}^*$ satisfying $f(0) = \rho x$, $f(1) = \rho y$ for some $x, y \in \{0,1\}^*$. The *stretch size* of $f$ is the combined length $\lambda_f = |\rho_f| + |\sigma_f|$.

**Example 3.** The morphism defined by $f(0) = 0110$ and $f(1) = 1010$ satisfies $\sigma_f = 10$, $\rho_f = \varepsilon$, and $\lambda_f = 2$.

We can now state our main theorem:

**Theorem 1.** *Let $f$ be a binary $k$-uniform morphism prolongable on 0, and let $\mathbf{w} = f^\omega(0)$. Then:*

(1) $E(\mathbf{w}) = \infty$ *if and only if at least one of the following holds:*
  (a) $f(0) = f(1)$;

(b) $f(0) = 0^k$;
(c) $f(1) = 1^k$;
(d) $k = 2m + 1$, $f(0) = (01)^m 0$, $f(1) = (10)^m 1$.

(2) *Suppose* $E(\mathbf{w}) < \infty$. *Let* $\mathcal{E}$ *be the set of exponents* $r = p/q$, *such that* $q < k$ *and* $f^4(0)$ *contains an* $r$*-power. Then*

$$E(\mathbf{w}) = \max_{p/q \in \mathcal{E}} \left\{ \frac{p(k-1) + \lambda_f}{q(k-1)} \right\}.$$

*In particular,* $E(\mathbf{w})$, *when bounded, is always rational. The bound* $E(\mathbf{w})$ *is attained if and only if* $\lambda_f = 0$.

Here is an example of an application of Theorem 1:

**Example 4.** The Thue-Morse word is overlap-free.

*Proof.* The Thue-Morse morphism $\mu$ satisfies $\lambda_\mu = 0$; and since the largest power in $\mu^4(0)$ is a square, we get that $E(\mu^\omega(0)) = 2$, and the bound is attained. □

The three following theorems are fundamental results in the area of combinatorics on words. They constitute the main tools we use in this paper. Theorem 2 can be found in [16], Theorem 8.1.4; Theorems 3 and 4 can be found in [1], Theorems 1.5.2 and 1.5.3. In this setting, $\Sigma$ is any finite alphabet.

**Theorem 2** (Fine and Wilf [9])**.** *Let* $w$ *be a word having periods* $p$ *and* $q$, *with* $p \le q$, *and suppose that* $|w| \ge p + q - \gcd(p, q)$. *Then* $w$ *also has period* $\gcd(p, q)$.

**Theorem 3** (Lyndon and Schützenberger [17])**.** *Let* $y \in \Sigma^*$ *and* $x, z \in \Sigma^+$. *Then* $xy = yz$ *if and only if there exist* $u, v \in \Sigma^*$ *and an integer* $e \ge 0$ *such that* $x = uv$, $z = vu$, *and* $y = (uv)^e u$.

**Theorem 4** (Lyndon and Schützenberger [17])**.** *Let* $x, y \in \Sigma^+$. *Then the following three conditions are equivalent:*

(1) $xy = yx$;
(2) *There exist integers* $i, j > 0$ *such that* $x^i = y^j$;
(3) *There exist* $z \in \Sigma^+$ *and integers* $k, \ell > 0$ *such that* $x = z^k$ *and* $y = z^\ell$.

We end this section by stating a theorem that will be useful later; the proof can be found in, *e.g.*, [1] Theorem 10.9.5. A word $\mathbf{w} \in \Sigma^\omega$ is *recurrent* if every finite subword of $\mathbf{w}$ occurs infinitely often. It is *uniformly recurrent* if for each finite subword $x$ of $\mathbf{w}$ there exists an integer $m$, such that every subword of $\mathbf{w}$ of length $m$ contains $x$. A morphism $h : \Sigma^* \to \Sigma^*$ is *primitive* if there exists an integer $n$ such that $b$ occurs in $h^n(a)$ for all letters $a, b \in \Sigma$.

**Theorem 5.** *Let* $h : \Sigma^* \to \Sigma^*$ *be a primitive morphism, prolongable on* $a$. *Then* $h^\omega(a)$ *is uniformly recurrent.*

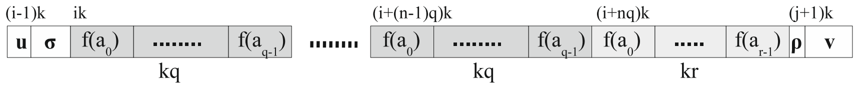


FIGURE 1. Applying $f$ to $w_{i-1} \cdots w_{j+1}$.

## 3. Power structure

For the rest of this section, $\Sigma = \{0, 1\}$.

**Lemma 6.** *Let $f$ be a binary $k$-uniform morphism prolongable on $0$, and let $\mathbf{w} = f^{\omega}(0)$. Let $\sigma$, $\rho$ and $\lambda$ be the left stretch, right stretch, and stretch size of $f$, respectively. Suppose $z = w_i \cdots w_j \in OC(\mathbf{w})$ is a $p/q$-power. Then*

$$E(\mathbf{w}) \geq \frac{p(k-1)+\lambda}{q(k-1)}.$$

*Proof.* If $f(0) = f(1)$ or $f(0) = 0^k$ or $f(1) = 1^k$, then it is easy to see that $E(\mathbf{w}) = \infty$. Otherwise, $f$ is primitive, thus $\mathbf{w}$ is recurrent by Theorem 5, and we can assume that $i > 0$. Let $p = nq + r$, where $n, r \in \mathbb{N}$ and $r < q$, and let $z = x^n y$, where $x = a_0 \cdots a_{q-1}$ and $y = a_0 \cdots a_{r-1}$. Let $f(w_{i-1}) = u\sigma$ and $f(w_{j+1}) = \rho v$ for some $u, v \in \Sigma^*$. Applying $f$ to $w_{i-1} \cdots w_{j+1}$, we get a subword of $\mathbf{w}$ which is a fractional power with period $kq$, as illustrated in Figure 1.

Since $\sigma$ is a common suffix of $f(0)$ and $f(1)$, it is a suffix of $f(a_{q-1})$ as well; similarly, $\rho$ is a prefix of $f(a_r)$. Therefore, we can stretch the $kq$-period of $f(z)$ by $\sigma$ to the left and $\rho$ to the right. We get that $z' = \sigma f(z) \rho$ is a $(kp+\lambda)/kq$-power.

The process of applying $f$ and stretching the resulting power can be repeated infinitely. Successive applications of $f$ give a sequence of powers, $\{p_m/q_m\}_{m \geq 0}$, which satisfy $p_0 = p$, $q_0 = q$, and for $m > 0$, $p_m = kp_{m-1} + \lambda$, and $q_m = kq_{m-1}$. Let $\pi : OC(\mathbf{w}) \times \mathbb{Q} \to OC(\mathbf{w}) \times \mathbb{Q}$ be the map defined by

$$\pi\left(z, \frac{p}{q}\right) = \left(\sigma f(z) \rho, \frac{kp+\lambda}{kq}\right). \tag{2}$$

We use $\pi(z)$ and $\pi(p/q)$ to denote the first and second component, respectively. Iterating $\pi$ on $p/q$, we get

$$\pi^m\left(\frac{p}{q}\right) = \frac{k^m p + \lambda \sum_{i=0}^{m-1} k^i}{k^m q} = \frac{k^m p + \lambda \frac{k^m - 1}{k-1}}{k^m q} \xrightarrow[m \to \infty]{} \frac{p(k-1)+\lambda}{q(k-1)}. \qquad \square$$

Our goal is to show that the $\pi$ map defined in (2) is what generates $E(\mathbf{w})$, and that it is enough to apply it to powers that appear in $f^4(0)$. Though the details are a bit tedious, the proof idea is very simple:

(1) every $p/q$-power in $\mathbf{w}$ that satisfies $q \equiv 0 \pmod{k}$ is an image under the $\pi$ map;

(2) every $p/q$-power in $\mathbf{w}$ that satisfies $q \not\equiv 0 \pmod{k}$ and $q > k$ is reducible to a $p'/q'$-power that satisfies $q' \leq k$;

(3) all the distinct $p/q$-powers in $\mathbf{w}$ that satisfy $q < k$ must appear in $f^4(0)$.

We start with a few lemmas which describe power behavior in a more general setting, namely, in an infinite word $\mathbf{v} = h(\mathbf{u})$, where $h$ is a $k$-uniform binary morphism and $\mathbf{u} \in \Sigma^\omega$ is an arbitrary infinite word.

**Definition 4.** Let $h$ be a binary $k$-uniform morphism, and let $\mathbf{v} = h(\mathbf{u})$ for some $\mathbf{u} \in \Sigma^\omega$. We refer to the decomposition of $\mathbf{v}$ into images of $h$ as decomposition into *$k$-blocks*. Let $\alpha = v_i \cdots v_j \in OC(\mathbf{v})$. The *outer closure* and *inner closure* of $\alpha$, denoted by $\hat{\alpha}$, $\check{\alpha}$, respectively, are defined as follows:

$$\hat{\alpha} = v_{\hat{\imath}} \cdots v_{\hat{\jmath}}, \quad \hat{\imath} = \left\lfloor \frac{i}{k} \right\rfloor k, \quad \hat{\jmath} = \left\lceil \frac{j+1}{k} \right\rceil k - 1;$$

$$\check{\alpha} = v_{\check{\imath}} \cdots v_{\check{\jmath}}, \quad \check{\imath} = \left\lceil \frac{i}{k} \right\rceil k, \quad \check{\jmath} = \left\lfloor \frac{j+1}{k} \right\rfloor k - 1.$$

Thus $\hat{\alpha} \in OC(\mathbf{v})$ consists of the minimal number of $k$-blocks that contain $\alpha$; similarly, $\check{\alpha} \in OC(\mathbf{v})$ consists of the maximal number of $k$-blocks that are contained in $\alpha$. By this definition, both $\hat{\alpha}$ and $\check{\alpha}$ have inverse images under $h$, denoted by $h^{-1}(\hat{\alpha})$ and $h^{-1}(\check{\alpha})$, respectively. Note that $\check{\alpha}$ may be empty.

**Lemma 7.** *Let $h$ be an injective binary $k$-uniform morphism, let $\mathbf{v} = h(\mathbf{u})$ for some $\mathbf{u} \in \Sigma^\omega$, and let $\alpha = v_i \cdots v_j \in OC(\mathbf{v})$ be an unstretchable $p/q$-power. Suppose $q \equiv 0 \pmod{k}$. Then $\alpha$ is an image under the $\pi$ map defined in (2).*

*Proof.* Let $q = mk$, $m \geq 1$. By definition, $\check{\imath} - i, j - \check{\jmath} \leq k - 1$. Therefore,

$$|\check{\alpha}| = |\alpha| - (\check{\imath} - i) - (j - \check{\jmath}) \geq |\alpha| - (2k - 2).$$

Since $|\alpha| = p$ and $p/q \geq 2$, we get that $|\check{\alpha}| \geq 2q - (2k-2) = (2m-2)k + 2$. If $m \geq 2$, this implies that $|\check{\alpha}| > q$; if $m = 1$, this implies that $|\check{\alpha}| \geq 2$, and since $|\check{\alpha}| \equiv 0 \pmod{k}$, necessarily $|\check{\alpha}| \geq k = q$. Thus $|\check{\alpha}| \geq q$, and since $\check{\alpha} \lhd \alpha$, we get that $\check{\alpha}$ has a $q$-period. It follows that $\check{\alpha}$ is a $p'/q$-power, where $p' = nk$ for some $n \geq 1$. Let $\alpha' = h^{-1}(\check{\alpha})$. Since $h$ is injective, necessarily $\alpha'$ is an $n/m$-power.

Now apply $h$ to $\alpha'$. By the proof of Lemma 6, the $mk$-period of $h(\alpha')$ can be stretched by at least $|\sigma|$ to the left and $|\rho|$ to the right, to create a power with exponent $(kn + \lambda)/km = (p + \lambda)/q$. On the other hand, $h(\alpha') = \check{\alpha}$, and by inner closure definition, the $q$-period of $\check{\alpha}$ can be stretched by at least $\check{\imath} - i$ to the left and at least $j - \check{\jmath}$ to the right to create the $p/q$-power $\alpha$. But, since by assumption $\alpha$ is unstretchable, it cannot be stretched more than this, *i.e.*, it can be stretched by exactly $\check{\imath} - i$ to the left and $j - \check{\jmath}$ to the right. Therefore, $\sigma = w_i \cdots w_{\check{\imath}-1}$, $\rho = w_{\check{\jmath}+1} \cdots w_j$, and

$$\alpha = \sigma\check{\alpha}\rho = \sigma h(\alpha')\rho = \pi(\alpha'). \qquad \square$$

**Lemma 8.** *Let $h$ be a binary $k$-uniform morphism, let $\mathbf{v} = h(\mathbf{u})$ for some $\mathbf{u} \in \Sigma^\omega$, and let $\alpha \in OC(\mathbf{v})$ be a $p/q$-power with $p/q \geq 2$. Let $Q$ be the power block. Suppose*

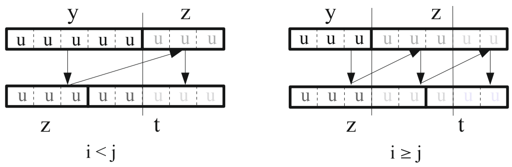


FIGURE 2. $yz = zt$, $y = u^j$, $|z| = i|u|$.

*$q > 2k$ and $q \not\equiv 0 \pmod k$. Then either $h(0) = h(1)$, or $\alpha$ is reducible to a $p'/q'$-power, which satisfies $q' \leq k$. If the second case holds, we get one of the following:*

(1) *$Q = u^c$ for some integer $c \geq 4$ and $u \in \Sigma^+$ satisfying $|u| < k$;*
(2) *$p/q < 5/2$.*

*Proof.* We start with three propositions that will be useful for the proof. All are extensions of the theorems of Lyndon and Schützenberger (Thms. 3, 4) to systems of word equations.

**Proposition 9.** *Let $x, y, z, t \in \Sigma^+$, and suppose the following equalities hold:*

(1) *$xy = yx$;*
(2) *$yz = zt$ (equivalently, $tz = zy$);*
(3) *$|z| = |x|$.*

*Then $x = z$ and $y = t$.*

*Proof.* By Theorem 4, $xy = yx$ if and only if there exists $u \in \Sigma^+$ and integers $i, j > 0$ such that $x = u^i$ and $y = u^j$. Therefore, it is enough to prove the following: $u^j z = zt$ and $|z| = i|u|$ imply $z = u^i$ and $t = u^j$. We consider two cases:

**$i < j$:** in this case, $|z| = |x| < |y|$, and since $yz = zt$, $z$ is a prefix (equivalently suffix) of $y$. Since $y = u^j$ and $|z| = i|u|$, necessarily $z = u^i = x$. This gives us $u^{j+i} = u^i t$, which implies $t = u^j = y$.

**$i \geq j$:** here we use induction on $|z|$. If $|z| = 1$, then either $|z| < |y|$, which implies $i < j$, or $|y| = 1$, which implies $x = y = z = t \in \Sigma$. Let $|z| > 1$. Since $|z| \geq |y|$, we get that $y = u^j$ is a prefix (suffix) of $z$. Let $z = u^j z'$ ($z = z' u^j$). Then $u^j z' = z' t$ ($z' u^j = t z'$) and $|z'| = |u^i| - |u^j| = (i - j)|u|$. Therefore, by the induction hypothesis, $z' = u^{i-j}$, and so $z = u^i$ and $t = u^j$. $\square$

Figure 2 illustrates Proposition 9.

**Proposition 10.** *Let $x, y, z, t \in \Sigma^+$, and suppose the following equalities hold:*

(1) *$xy = yz$;*
(2) *$yx = zt$ (equivalently, $tx = zy$).*

*Then $x = z$ and $y = t$.*

*Proof.* We prove this proposition for $yx = zt$. The proof for $tx = zy$ is identical.

By Theorem 3, $xy = yz$ if and only if there exist $u, v \in \Sigma^*$ and an integer $e \geq 0$ such that $x = uv$, $z = vu$, and $y = (uv)^e u$. If $e > 0$, then $y$ has $uv$ as a prefix, and so $yx = zt$ implies $uv = vu$, *i.e.*, $x = z$. The conditions translate to $xy = yx = xt$, thus $y = t$.

Suppose $e = 0$. Then $yx = zt$ translates to $uuv = vut$. By Theorem 3, there exist $\alpha, \beta \in \Sigma^*$ and a $c \geq 0$ such that $uu = \alpha\beta$, $ut = \beta\alpha$, and $v = (\alpha\beta)^c\alpha$. Therefore, $uuv = (\alpha\beta)^{c+1}\alpha = (uu)^{c+1}\alpha = u^{2c+2}\alpha$, where $\alpha$ is a prefix of $uu$. We get that $uuv$ has a $|u|$ period, and therefore $vut$ has a $|u|$ period. Since $|t| = |u|$ (as implied by $|uuv| = |vut|$), we get that $t = u = y$.

The equality $yx = zt$ now translates to $uuv = vuu$. By Theorem 4, there exists $w \in \Sigma^+$ and $i, j > 0$ such that $uu = w^i$ and $v = w^j$. If $i > 1$, then $uu$ has both periods $|u|$ and $|w|$ and $|uu| > |u| + |w| - 1$, so by Theorem 2, it has a $g = \gcd(|u|, |w|)$ period; thus there exists $w' \in \Sigma^+$ such that $u = w'^{|u|/g}$, $w = w'^{|w|/g}$, and $v = w'^{j|w|/g}$. If $i = 1$, then $uu = w$, and $v = u^{2j}$. In any case, $u$ and $v$ are integral powers of the same word, which means $uv = vu$, and $x = z$. $\square$

**Proposition 11.** *Let $x, y, z, t \in \Sigma^+$, and suppose the following equalities hold:*

(1) $xy = yz$;
(2) $zt = tx$;

*Then there exists $u \in \Sigma^+$, $v \in \Sigma^*$, and integers $i \geq 1$, $j, m \geq 0$, such that $x = (uv)^i$, $z = (vu)^i$, $y = (uv)^j u$, $t = (vu)^m v$. If in addition $|y| = |t|$, then either $v = \varepsilon$ and $m = j + 1$ ($y = t = u^m$), or $|u| = |v|$ and $m = j$.*

*Proof.* From the first equation, we get by Theorem 3 that $x = rs$, $z = sr$, and $y = (rs)^e r$ for some $r, s \in \Sigma^*$, and an integer $e \geq 0$. Plugging into the second equation, we get $srt = trs$, therefore $srtr = trsr$, and so by Theorem 4, $sr = w^i$ and $tr = w^{i'}$ for some $u \in \Sigma^+$ and $i, i' \geq 1$. This implies $s = w^a v$, $r = uw^b$, and $t = w^m v$, where $w = vu$, $0 \leq |v| < w$, $i = a+b+1$, and $i' = m+b+1$. Altogether we get:

$$
\begin{aligned}
x &= rs = u(vu)^b(vu)^a v = (uv)^{a+b+1} = (uv)^i;\\
z &= sr = (vu)^i;\\
y &= (rs)^e r = (uv)^{ie} u(vu)^b = (uv)^{ie+b} u;\\
t &= (vu)^m v.
\end{aligned}
$$

If $v \neq \varepsilon$, the lemma assertion holds for $j = ie+b$; if $v = \varepsilon$, it holds for $j = ie+b+1$.

Now suppose that $|y| = |t|$. If $v = \varepsilon$, necessarily $y = t = u^m$. Otherwise, the equation $|(uv)^j u| = |(vu)^m v|$ implies $j = m$ and $|u| = |v|$. $\square$

We now go back to the proof of Lemma 8. Let $\alpha = Q^{p/q} \in OC(\mathbf{v})$ be a $p/q$-power, where $p/q \geq 2$ and $q > 2k$, $q \not\equiv 0 \pmod{k}$. Since $\mathbf{v}$ is an image under a $k$-uniform morphism, it can be decomposed into $k$-blocks, which are images of either 0 or 1. Assume the decomposition of $\alpha$ into $k$-blocks starts from its first character; we will show at the end of the proof that this assumption causes no loss of generality. Since $q \not\equiv 0 \pmod{k}$, the last $k$-block of the first $q$-block extends into the second $q$-block; since the first and second $q$-blocks are identical, we get overlaps of $k$-blocks in the second $q$-block. An example is given in Figure 3.

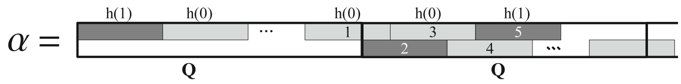


FIGURE 3. Overlaps of $k$-blocks. The bold rectangles represent the power blocks, the light grey ones stand for $h(0)$, the dark grey ones for $h(1)$.

The bold rectangles denote the power's $q$-blocks; the light grey and dark grey rectangles stand for $h(0), h(1)$, respectively; the top line of $h$ rectangles stands for the $k$-decomposition of $\alpha$; and the bottom line shows the repetition of the $q$-block.

The fact that $q > 2k$ implies that there are at least 5 $k$-blocks involved in the overlap. We shall now analyze the different overlap cases. For case notation, we order the $k$-blocks by their starting index (the numbers 1-5 in Fig. 3), and denote the case by the resulting 5-letter binary word; in the Figure 3 case, it is 01001. By symmetry arguments, it is enough to consider words that start with 0, therefore we need to consider 16 overlap combinations. Figure 4 shows these overlaps. Each overlap induces a partition on the $k$-blocks involved, denoted by dashed lines. We mark the $k$-block parts by the letters $x, y, z, t$ in the following manner: we start by marking the leftmost part by $x$, and then mark by $x$ all the parts we know are identical to it. We then mark the leftmost unmarked part by $y$, and so on. Since the $k$-decomposition starts from the first letter of $\alpha$, we have $|x| = q \mod k$, *i.e.*, $|x| > 0$.

We begin with combinations that imply $h(0) = h(1)$ straightforwardly.

**00010, 00100, 01000, 01110:** $h(0) = xy = yx = h(1)$.

**00011:** $h(0) = xy = yx$, $h(1) = yz = zt$. By Proposition 9, $x = z$ and $y = t$, *i.e.*, $h(0) = h(1)$.

**01001:** $h(0) = xy = yz$, $h(1) = yx = zt$. By Proposition 10, $x = z$ and $y = t$.

**00111:** $h(0) = xy = yz$, $h(1) = zt = tz$. By Proposition 9 (set $x \leftrightarrow t$, $y \leftrightarrow z$), $x = z$ and $y = t$.

**01101:** $h(0) = xy = tz$, $h(1) = yz = zt$. By Proposition 10 (set $x \to t$, $y \to x$, $z \to y$, $t \to z$), $x = z$ and $y = t$.

For the rest of the combinations, we need to consider possible continuations of the $q$-block. As mentioned above, $q = mk + |x|$ for some $m \geq 2$. If $m \geq 3$, the $q$-block continues with another $k$-block on the bottom row; the top row continues with the $k$-decomposition regardless of $m$. Let $K_t$ be the next $k$-block of the $k$-decomposition, and let $K_b$ be the next $k$-block in the bottom row if $m \geq 3$. Note that by the first block, $Q$ must end with an $x$.

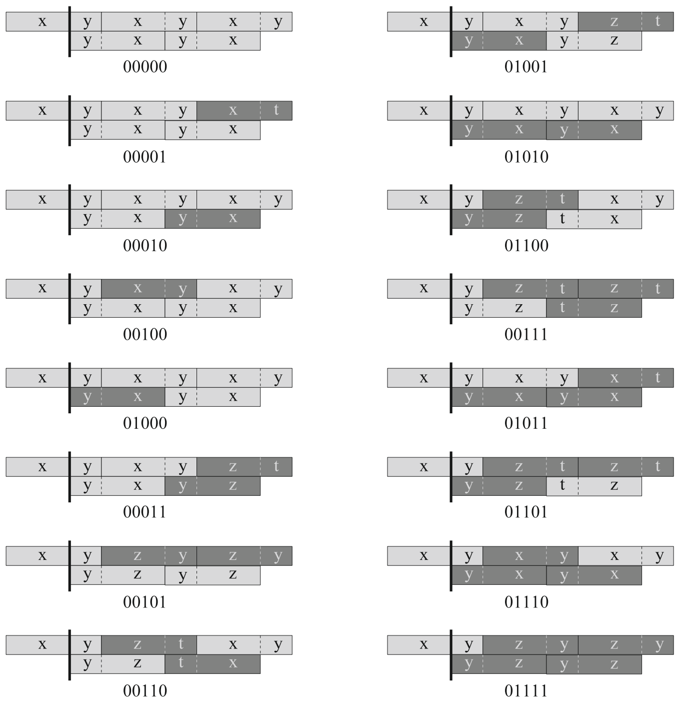


FIGURE 4. Possible overlaps of $k$-blocks.

**00000:** $h(0) = xy = yx$, therefore, by Theorem 4, $x = u^i$ and $y = u^j$ for some $u \in \Sigma^+$, $i, j \in \mathbb{Z}_{\geq 1}$. If $m = 2$, or $Q$ is continued with $h(0)$ blocks all the way, then $Q = (yx)^m x = u^{m(i+j)+i} = u^c$, $c \geq 5$; we get that the $p/q$-power contains at least a $5\,\lfloor p/q \rfloor$-power, *i.e.*, it is reducible with $q' = |u| < k$. Otherwise, if $m > 2$ and the continuation of $k$-blocks is not strictly by $h(0)$ blocks, then at some point an $h(1)$ block is introduced, and the behavior is similar to one of 00001, 00010, or 00011. The latter two imply $h(0) = h(1)$, as was shown above; the first one is discussed next.

**00001:** $h(0) = xy = yx$, $h(1) = xt$. By Theorem 4, $x = u^i$ and $y = u^j$ for some $u \in \Sigma^+$ and $i, j \in \mathbb{Z}_{\geq 1}$. Thus $|t| = |y| = j|u|$.

Suppose $m \geq 3$. If $K_b = h(0)$, we get $t = y$, and so $h(1) = xy = h(0)$; if $K_b = h(1)$, we get $h(1) = xt = tz$, where $|t| = |y|$, therefore by Proposition 9 (set $x \leftrightarrow y$ and $z \leftrightarrow t$) $y = t$ and $x = z$, and again $h(1) = xy = h(0)$. Therefore we can assume that $m = 2$. Thus $Q = yxyxx = u^{3i+2j} = u^c$, $c \geq 5$, *i.e.*, $\alpha$ is reducible with $q' = |u| < k$.

**00101:** $h(0) = xy = yz$, $h(1) = zy$. If we switch the roles of $h(0)$ and $h(1)$, we get a mirror image of case 01011, which is proved below to be reducible.

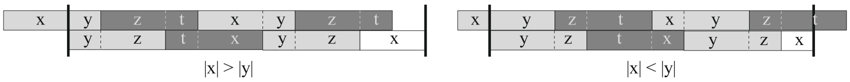


FIGURE 5. Overlap 00110, $Q = (h(0)h(1))^{(m-1)/2}h(0)x$.

**00110:** $h(0) = xy = yz$, $h(1) = zt = tx$. By Proposition 11, either $x = z = u^i$ and $y = t = u^j$, or there exist $u, v \in \Sigma^+$ and integers $i \geq 1$, $j \geq 0$, such that $|u| = |v|$, $x = (uv)^i$, $z = (vu)^i$, $y = (uv)^j u$, and $t = (vu)^j v$. In the first case, $h(0) = h(1) = u^\ell$, where $\ell = i + j$; in the second, $h(0) = (uv)^\ell u$ and $h(1) = (vu)^\ell v$, thus $h(0)h(1) = (uv)^{2\ell+1}$.

Suppose $m \geq 3$. If $K_b = h(1)$, we get $y = t$, and by Proposition 10, $h(0) = h(1)$. If $K_b = h(0)$ and $K_t = h(0)$, we get $x = z$, and again $h(0) = h(1)$. The only way to continue the $q$-block without forcing $h(0) = h(1)$ is to have $h(0)h(1)h(0)h(1)\ldots$ in the bottom row and $yh(1)h(0)h(1)h(0)\ldots$ in the top row. Thus we can assume that $Q$ has the form $(h(0)h(1))^{m/2}x$ ($m$ even) or $(h(0)h(1))^{(m-1)/2}h(0)x$ ($m$ odd); here $m \geq 2$.

If $m$ is odd, then either $x$ is a prefix of $t$ or $t$ is a prefix of $x$ (Fig. 5); since $x$ begins with $u$, $t$ begins with $v$ and $|u| = |v|$, we get $u = v$ and $h(0) = h(1)$. If $m$ is even, then $Q = (h(0)h(1))^{m/2}x = ((uv)^{2\ell+1})^{m/2}(uv)^i = (uv)^c$, $c \geq 4$, thus $\alpha$ is reducible with $q' = |uv| = |x|/i < k$.

**01010:** $h(0) = xy$, $h(1) = yx$. Suppose $m \geq 3$. The 4 possible continuations for $K_bK_t$ are 00, 01, 11, 10. The first two, when combined with the last 3 $k$-blocks (010) yield the combinations 01000, 01001, which were shown above to imply $h(0) = h(1)$. The 11 continuation yields the combination 01011, which will be dealt with next. The 10 continuation yields the original configuration again. Therefore, we can assume that the $q$-block continues with $h(1)$, and $Q = h(1)^m x$.

Suppose $p/q \geq 5/2$. The first two blocks have the form:

$$QQ = (yx)^m x (yx)^m x = (yx)^m (xy)^m xx = h(1)^m h(0)^m xx.$$

We get that either $xx$ is a prefix of a $k$-block, or *vice versa*; and since the $k$ blocks are $xy$ and $yx$, either $x$ is a prefix of $y$ or $y$ is a prefix of $x$. If $x = y$, then $h(0) = h(1)$. If $x = yy_1$, then $h(0) = yy_1y$, $h(1) = yyy_1$, and $xx = yy_1yy_1 = h(0)y_1$; we get that $y_1$ begins at the beginning of a $k$-block, therefore $y_1$ is a prefix of $y$ or $y$ is a prefix of $y_1$. If $y = xx_1$, then $h(0) = xxx_1$, $h(1) = xx_1x$, and the continuation of $xx$ implies again that $x$ is a prefix of $x_1$ or *vice versa*. By induction, we can continue to split the $k$-block into shorter and shorter substrings, until finally we must get equality. We conclude that for $p/q \geq 5/2$, $h(0) = h(1)$.

If $p/q < 5/2$ then obviously $\alpha$ is reducible, since it contains the cube $h(0)^3$.

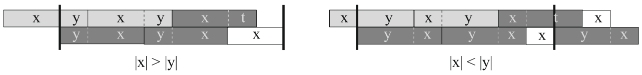


FIGURE 6. Overlap 01011, $Q = h(1)h(1)x$.

**01100:** $h(0) = xy = tx$, $h(1) = yz = zt$. This case is symmetric to the 00110 case. Use Proposition 11 to show that either $h(0) = h(1)$ or $\alpha$ is reducible.

**01011:** $h(0) = xy$, $h(1) = yx = xt$. Suppose $m \geq 3$. Similar arguments to the ones used in previous cases show that any continuation of the $q$-block implies $h(0) = h(1)$. Therefore, we can assume $m = 2$, $Q = h(1)h(1)x$. This means that either $x$ is a prefix of $t$ or $t$ is a prefix of $x$ (Fig. 6).

By Theorem 3, there exist $u, v \in \Sigma^*$ and $e \geq 0$ such that $y = uv$, $t = vu$, $x = (uv)^e u$. If $|x| > |y|$ then $e > 0$, and since $t$ is a prefix of $x$ we get $uv = vu = w^j$ for some $j \geq 1$, which implies $h(0) = h(1)$. If $|x| < |y|$, then $x = u$, *i.e.*, it is a prefix of $y$. Since the $q$-block ends with $xx$ and starts with $y$, $\alpha$ contains the cube $xxx$.

Suppose $p/q \geq 5/2$. Since $x$ is a prefix of both $h(0)$, $h(1)$, the third $q$-block will give us the equality $tx = xy$, which together with $xt = yx$ implies $h(0) = h(1)$. Thus $h(0) \neq h(1)$ implies $p/q < 5/2$, and $\alpha$ is reducible with $q' = |x| < k$.

**01111:** $h(0) = xy$, $h(1) = yz = zy$. By Theorem 4 $z = u^i$, $y = u^j$, therefore $|x| = |z| = i|u|$.

Suppose $m \geq 3$. If $K_t = h(0)$, then for $K_b = h(1)$ we get $x = z$, and so $h(0) = h(1)$; for $K_b = h(0)$ we get case 00011 with 0 and 1 flipped, and again $h(0) = h(1)$. Therefore we can assume $K_t = h(1)$. If $K_b = h(0)$ we then get $h(0) = yz = h(1)$. The only continuation that does not immediately force $h(0) = h(1)$ is $K_b = K_t = h(1)$. Thus we can assume $Q = h(1)^m x = u^{m(i+j)}x$, $m \geq 2$. But then, like in the 00110 case, $x$ is a prefix of $y$ or a prefix of $yz$, and since $|x| = i|u|$, we get $x = u^i = z$ and $h(0) = h(1)$.

To finish the proof of the lemma, we need to justify our assumption that the $k$-decomposition starts from the first letter of $\alpha$. Recall that $\alpha = v_i \cdots v_j \in OC(\mathbf{v})$. Suppose $i \not\equiv 0 \pmod{k}$. Let $\breve{\alpha} = v_{\breve{\imath}} \cdots v_{\breve{\jmath}}$ be the inner closure of $\alpha$, and let $\beta = v_{\breve{\imath}} \cdots v_j$. Let $c = \breve{\imath} - i$. Then $\beta \in OC(\mathbf{v})$ is a $(p-c)/q$-power, $\beta \lhd \alpha$, and the $k$-decomposition of $\beta$ starts from the first letter. Also, by the definition of $\breve{\alpha}$ we have $c \leq k-1$, and so $p/q - (p-c)/q \leq (k-1)/q < \frac{1}{2}$.

Let $Q'$ be the power block of $\beta$. Suppose $p/q \geq 5/2$. If $p/q \geq 3$, then $(p-c)/q \geq 5/2$; by the analysis above, either $h(0) = h(1)$, or $Q' = u^d$ for some $u \in \Sigma^*$ and $d \geq 4$. If $5/2 \leq p/q < 3$, then $(p-c)/q \geq 2$, and by the analysis above $\beta$ contains at least a cube. In both cases, $\alpha$ is reducible. We can therefore assume that $p/q < 5/2$.

Let $p = 2q + r$, where $0 \leq r < q$, and let $q = mk + s$, where $0 \leq s < k$. By the theorem's conditions, $m \geq 2$ and $s \geq 1$. If $r \geq c$ then $\beta$ is at least a square,

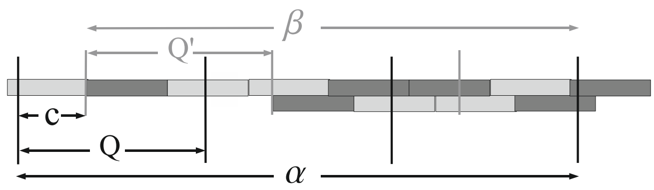


FIGURE 7. $n \geq 3$.

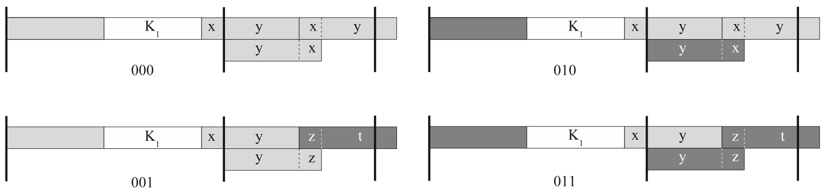


FIGURE 8. Overlaps of $k$-blocks, $q = 2k + s$, $p = 2q + r$.

and by the analysis above $\alpha$ is reducible to a cube. We can therefore assume that $r < c$. We can also assume that $r > 0$, since a square of length more than 4 is always reducible.

Suppose that $m \geq 3$. Then

$$p - q - c = q + r - c \geq 3k + 2 - (k - 1) = 2k + 3.$$

Therefore, the bottom row of $k$-blocks contains more than 2 blocks, and altogether we have at least 5 blocks involved in the overlap. By the analysis above, $\beta$ (and therefore $\alpha$) is reducible to a cube. We can therefore assume that $m = 2$, *i.e.*, $q = 2k + s$, and $p - q - c = 2k + s + r - c$. If $s + r \geq c$ then again we get that $\beta$ is reducible to a cube, and so we can assume that $2 \leq s + r < c$. Since $c < k$ we also get that $2k + s + r - c > k + s + r$. To summarize the setting, we have:

- $2 < p/q < 5/2$;
- $q = 2k + s, p = 2q + r$;
- $1 \leq r, s$; $r + s < c \leq k - 1$;
- $k + s + r < p - q - c < 2k$.

Again we use overlap analysis. This time we have 4 combinations to consider (Fig. 8). We will use the following notation:

- $K_1$ - the white $k$-block in Figure 8;
- $K_2$ - the $k$-block preceding $\beta$.

Since $|x| = s < c$, the $|x|$ letters to the right of $\beta$ belong to $\alpha$ and must equal $x$. Therefore $K_2 = wx$ for some $w \in \Sigma^+$.

**000:** $h(0) = xy = yx$. By Theorem 4, $xyx = u^\ell$ for some $\ell \geq 3$, and $\alpha$ is reducible.

**001:** $h(0) = xy = yz$, $h(1) = zt$. By Theorem 3, $x = uv$, $z = vu$, and $y = (uv)^e u$, for some $u, v \in \Sigma^*$ and $e \geq 0$. If $e > 0$, then $xyz$ contains the cube $(uv)^3$. Suppose $e = 0$. If $K_1 = h(0)$, then $\alpha$ contains the cube $h(0)^3$, thus we can assume $K_1 = h(1)$. Since $x$ is a suffix of $K_2$, we get that $\alpha$ contains the suboccurrence $xh(0)h(1) = xxyzt = uvuvuvut$, which again contains the cube $(uv)^3$.

**010:** $h(0) = xy$, $h(1) = yx$. Here, we do not show that $\alpha$ is reducible *per se*; rather, we show that $\mathbf{v}$ contains a cube with power block of length at most $k$.

Consider possible choices for $K_1$ and $K_2$. If $K_1 = h(0)$, we get the cube $h(0)^3$. Assume $K_1 = h(1)$. If $K_2 = h(1)$, we get the cube $h(1)^3$. Otherwise, if $K_2 = h(0)$, we get that $h(0) = wx = xy$, and so $w = uv$, $x = (uv)^e u$, $y = vu$. If $e > 0$, then $wxy$ (which must occur in $\mathbf{v}$, being a prefix of $h(0)h(1)$) contains the cube $(uv)^3$; otherwise, $h(1)h(0) = yxxy = vuuuvu$, containing the cube $u^3$.

**011:** $h(0) = xy$, $h(1) = yz = zt$. By Theorem 3, $y = uv$, $t = vu$, and $z = (uv)^e u$, for some $u, v \in \Sigma^*$ and $e \geq 0$. If $e > 0$, then $yzt$ contains the cube $(uv)^3$. Suppose $z = u$. If $K_2 = h(1)$, then both $h(1) = yz = wx$, and since $|z| = |x|$ we get that $x = z = u$. Thus $h(0) = uuv$, $h(1) = uvu$, and $h(1)h(0)$ contains the cube $uuu$. If $K_2 = h(0)$, then $h(0) = wx = xy$, therefore $w = ab$, $x = (ab)^e a$, and $y = ba$ for some $a, b \in \Sigma^*$ and $e \geq 0$. Since $|x| = |u| < |uv| = |y|$, necessarily $e = 0$, and $x = a$. We get:

$$\begin{array}{lllllllll} w & = & ab, & x & = & a, & y & = & ba; \\ y & = & uv, & z & = & u, & t & = & vu; \\ |a| & = & |u|, & |b| & = & |v|. & & & \end{array}$$

Assume $|b| \leq |u|$. Then $y = ba = uv$ implies that $u = bd$ and $a = dv$ for some $d \in \Sigma^*$, and so

$$h(0)h(1) = xy\ zt = aba\ uvu = dvbdv\ bdvbd,$$

containing the cube $(dvb)^3$. Now assume $|b| > |u|$. Then $b = ud$ and $v = da$ for some $d \in \Sigma^*$, and

$$h(0)h(1) = auda\ udau = (aud)^{(6|u|+2|d|)/(2|u|+|d|)}.$$

If $|d| \leq 2|u|$, then $(6|u| + 2|d|)/(2|u| + |d|) \geq 5/2$, making $\alpha$ reducible. Suppose $|d| > 2|u|$. Since $t = dau$, and $p - q - c > k + s + r$, either $a$ or $u$ must be a prefix of $d$, depending on whether $K_1 = h(0)$ or $K_1 = h(1)$ (recall that $|a| = |u| = s$, thus the first $s$ letters of $t$ are still part of $\beta$). If $K_1 = h(1)$, then $u$ is a prefix of $d$, and $h(1)h(1)$ contains the cube $uuu$. If $K_1 = h(0)$, then $a$ is a prefix of $d$, and $h(1)h(0)$ contains the subword $auaua$, which is a $5s/2s$-power. In both cases, $\alpha$ is reducible. $\square$

The overlap analysis we used in the previous lemma, though a bit long, was straightforward enough. This was due to the fact that when $q > 2k$, the power is sufficiently long to imply easy constraints. Things are a little more subtle when dealing with short powers, especially when $k < q < 2k$ and $\lfloor p/q \rfloor = 2$. While powers with $q > 2k$ are always strictly reducible, and in most cases to a power with a much bigger exponent, this is not necessarily the case for small powers. Consider the following example of a 10-uniform morphism:

$$\begin{aligned} h(0) &= 0110010110 \\ h(1) &= 1001011010 \\ h(101) &= 1001011010\,0110010110\,1001011010. \end{aligned}$$

The prefix of length 24 of $h(101)$ is a square with block size 12, which contains no overlaps; thus it is not strictly reducible, though it does contain squares of block size smaller than $k$. This example demonstrates that we need to be more careful when analyzing powers with $k < q < 2k$.

**Definition 5.** Let $h$ be a binary $k$-uniform morphism, and let $\mathbf{v} = v_0v_1v_2\cdots = h(\mathbf{u})$ for some $\mathbf{u} \in \Sigma^\omega$. Let $0 < i < k$. The *$k$-partition of the $i$-shift* of $\mathbf{v}$, denoted by $T_{i,k}(\mathbf{v})$ is the partition of $v_iv_{i+1}v_{i+2}\cdots$ into $k$-blocks.

Let $\alpha = v_i\cdots v_j \in OC(\mathbf{v})$ be a $p/q$-power, and suppose $i \not\equiv 0 \pmod k$. Consider $\alpha$ as an occurrence of $T_{i,k}(\mathbf{v})$. In general, $T_{i,k}(\mathbf{v})$ can have up to 4 different $k$-blocks. However, if there are only two composing $\alpha$, we can do the overlap analysis of $\alpha$ using the blocks of $T_{i,k}(\mathbf{v})$. Clearly, if $\alpha$ is reducible with respect to $T_{i,k}(\mathbf{v})$ then it is also reducible with respect to $\mathbf{v}$. If the $k$-blocks of $T_{i,k}(\mathbf{v})$ are proved to be equal it does not necessarily imply that $h(0) = h(1)$; however, it would still imply that $\alpha$ is reducible.

**Lemma 12.** *Let $h$ be a binary $k$-uniform morphism, let $\mathbf{v} = h(\mathbf{u})$ for some $\mathbf{u} \in \Sigma^\omega$, and let $\alpha = v_i\cdots v_j \in OC(\mathbf{v})$ be a $p/q$-power with $p/q \geq 2$. Let $Q$ be the power block. Suppose $k < q < 2k$. Then either $h(0) = h(1)$, or $\alpha$ is reducible to a $p'/q'$-power, which satisfies $q' \leq k$.*

*Proof.* Let $q = k+s$, $1 \leq s \leq k-1$, and let $p = nq+r = nk+ns+r$, $0 \leq r \leq q-1$. Let $\beta = v_{\breve{\imath}}\cdots v_j$, $c = \breve{\imath} - i$, and let $Q'$ be the power block of $\beta$. If $p - q - c \geq 2k$, then there are at least 5 $k$-blocks involved in the overlap of $\beta$. Using the same analysis applied in the previous lemma, we can show that either $h(0) = h(1)$, or $Q' = u^\ell$ for some $\ell \geq 4$ and $u \in \Sigma^+$ satisfying $|u| \leq k$. The two cases where this assertion does not hold are 01010 and 01011, but these cases cannot hold for $k < q < 2k$ (see Fig. 9). Since $\beta$ is a $(p-c)/q$-power and $c/q \leq (k-1)/(k+1) < 1$, we get that $\alpha$ is also reducible.

Suppose $p - q - c = (n-1)k + (n-1)s + r - c < 2k$. Then we get:

$$(n-1)k + n \leq (n-1)k + (n-1)s + r < 2k + c \leq 3k - 1.$$

Therefore, $n \leq (4k-2)/(k+1) < 4$.

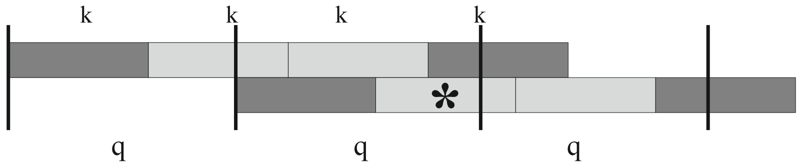


FIGURE 9. Overlaps of $k$-blocks, $k < q < 2k$. The $k$-block marked by '*' cannot be an $h(1)$ block, or the period will be violated.

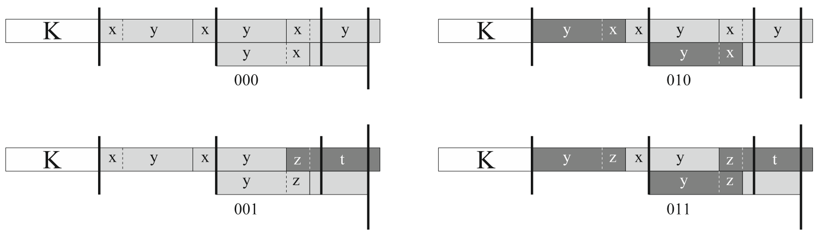


FIGURE 10. Overlaps of $k$-blocks, $k < q < 2k$, $n = 3$.

Suppose $n = 3$ and $p - q - c = 2k + 2s + r - c < 2k$. Then $2s + r < c \leq k - 1$, *i.e.*, $s < k/2$. Also, $p - q - c \geq k + 2s + r + 1$, and so there are 3 $k$-blocks involved in the overlap. The cases are illustrated in Figure 10. We denote by $K$ the $k$-block preceding $\beta$ (the white block in Fig. 10). Note that in all cases $|x| < |y|$, since $|x| = s < k/2$, and $|x| + |y| = k$.

**000:** $h(0) = xy = yx$. By Theorem 4, $h(0)h(0) = u^\ell$ for some $\ell \geq 4$, and $\alpha$ is reducible.

**001:** $h(0) = xy = yz$, $h(1) = zt$. Since $|x| < |y|$, we get that $x = uv$, $z = vu$, and $y = (uv)^e u$ for some $u, v \in \Sigma^*$ and $e \geq 1$. If $e > 2$, then $xyz$ contains the 4-power $(uv)^4$, making $\alpha$ reducible. If $e = 1$, then $|t| = |y| = |uvu| = s + |u| < 2s$, and so all of the $h(1)$ block is included in $\beta$, implying that $t = y$. Consider $K$: since $|x| = s < c$, the last $|x|$ letters of $K$ belong to $\alpha$, and must equal $x$. Since both $h(0)$ and $h(1)$ end with $y$, and $|y| > |x|$, necessarily $x$ is a suffix of $y$. Therefore, the word $yxyz$ contains the word $xxyz = (uv)^4 u$. Again, $\alpha$ is reducible.

**010:** $h(0) = xy$, $h(1) = yx$. Also, the last block on the second row implies that $x$ is a prefix of $y$. Let $y'$ be the suffix of $y$ of length $s$. Consider $K$: as in the previous case, $x$ is a suffix of $K$. If $K = h(0) = xy$, then $y' = x$. If $K = h(1) = yx$, then, since $c > 2s$, the first power block of $\beta$ must end with $y'x$, and again $y' = x$. We get that $x$ is both a prefix and a suffix of $y$, and so $yxxy$ contains $x^4$, making $\alpha$ reducible.

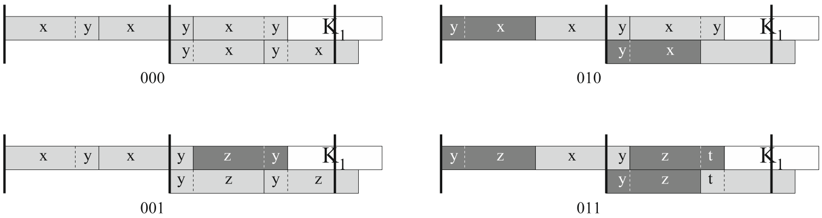


FIGURE 11. Overlaps of $k$-blocks, $k < q < 2k$, $n = 2$.

**011:** $h(0) = xy$, $h(1) = yz = zt$. Since $|z| < |y|$, we get that $y = uv$, $z = u$, and $t = vu$ for some $u, v \in \Sigma^*$. Suppose that $K = h(1)$. Then $x$ is a suffix of $h(1)$ therefore $x = z = u$. Also, the last block on the second row implies that $x$ is a prefix of $t = vu$. Therefore, $zxyz = uuuvu$ contains $u^4$.

Now assume that $K = h(0)$. Then $\alpha \lhd h(0)h(1)h(0)h(1)$, and contains at most two different $k$-blocks when considered as a subword of $T_{i,k}(\mathbf{v})$: the block resulting from $h(0)h(1)$, and the block resulting from $h(1)h(0)$. Using the $T_{i,k}(\mathbf{v})$ decomposition, we get 5 blocks involved in the overlap of $\alpha$, making it reducible.
Finally, suppose that $n = 2$. The cases are illustrated in Figure 11. We denote by $K_1$ the $k$-block that continues the $k$-decomposition on the first row, and by $K_2$ the $k$-block preceding it. Figure 11 depicts powers that start at an index $i \equiv 0 \pmod{k}$, but we also consider the case where $\alpha$ begins at some index $i$, and $p - c - q \geq k$.

**000:** $\alpha$ contains $h(0)^3$, and must be reducible.

**001:** $h(0) = xy = yz$, and so $x = uv$, $z = vu$, and $y = (uv)^e u$ for some $u, v \in \Sigma^*$ and $e \geq 0$. If $|x| < |y|$ (as illustrated in Fig. 10), then $e \geq 1$, and $xyz$ contains the cube $(uv)^3$. Otherwise, $|x| < |y|$ and $h(1) = zy$ (as illustrated in Fig. 11). We get that $h(0) = uvu$, $h(1) = vuu$, and $h(1)h(0)$ contains the cube $u^3$.

If $\alpha$ starts at some index $i \not\equiv 0 \pmod{k}$, consider $K_2$: if $K_2 = h(0)$, we get the cube $h(0)^3$. If $K_2 = h(1)$, assume $p - c - q \geq k$, *i.e.*, the whole of the $z$ part of the $h(1)$ block is included in $\alpha$. Then $h(1) = vut$ for some $t$, and by comparing $K_2$ to $x$, we get that $t = v$. Therefore $h(0) = uvu$, $h(1) = vuv$, and $h(0)h(1) = (uv)^3$.

**010:** $h(0) = xy$, $h(1) = yx$. Also, the last $k$-block in the second row implies that either $x$ is a prefix of $y$ or *vice versa*. If $x = ya$ for some $a \in \Sigma^*$, then $h(0) = yay$, $h(1) = yya$, and $h(0)h(1)$ contains the cube $y^3$; if $y = xb$ for some $b \in \Sigma^*$, then $h(0) = xxb$, $h(1) = xbx$, and $h(1)h(0)$ contains the cube $x^3$. In any case, $\alpha$ is reducible.

If $\alpha$ starts at some index $i \not\equiv 0 \pmod{k}$ and $p - c - q \geq k$, we still get $h(0) = xy$ and $h(1) = yx$. If $K_2 = h(0)$, then $y$ is a suffix of $x$ or *vice versa*, and again we get a cube. If $K_2 = h(1)$, then either $\alpha$ is the square $(xyx)^2$ (in which case it must be reducible), or again either $x$ is a prefix (suffix) of $y$ or *vice versa*.


FIGURE 12. Overlaps 011, $|z| < |y|$.

**011:** $h(1) = yz = zt$, therefore $y = uv$, $t = vu$, and $z = (uv)^e u$. If $|z| > |y|$, then $e > 0$, and $\alpha$ contains the cube $uvuvuv$. Assume $|z| < |y|$, *i.e.*, $z = u$ and $|x| = |z| = |u|$. We get the picture illustrated in Figure 12a. Here we assume $|u| < |v|$; assuming $|u| > |v|$ leads to similar results, while $|u| = |v|$ implies $QQ = (uv)^4$. Using the fact that $x$ is a prefix of $v$, we get the picture illustrated in Figure 12b.

Suppose $r > 0$. Then $u$ and $w$ share a prefix of size $\min(r, |u|, |w|)$. If $r$ is long enough, then either $u$ is a prefix of $w$, or $u$ is a prefix of $wu$. In both cases, we get that $\alpha$ contains the $5|u|/2|u|$-power $uxuxu$. In order for $\alpha$ to be irreducible, we must have $r > \frac{1}{2}q = \frac{1}{2}(4|u|+|w|)$, but then there is enough information in the third power block to imply a cube. Therefore, we must assume that $\min(r, |u|, |w|) = r$. Let $u = da$ and $w = db$, where $|d| = r$. Then $uxuxw = daxdaxdb$, containing the power $(dax)^{2+|d|/|dax|} = (dax)^{2+r/2|x|}$. Since $\alpha$ is a $(2 + r/q)$-power and $2|x| < k < q$, we get that $\alpha$ is reducible. Therefore we must have $r = 0$. But then $\alpha$ is a square, and must be reducible.

If $\alpha$ starts at some index $i \not\equiv 0 \pmod{k}$, consider $K_2$: if $K_2 = h(0)$, then $\alpha \lhd h(0)h(1)h(0)h(1)$, and contains at most two different $k$-blocks when considered as a subword of $T_{i,k}(\mathbf{v})$. We can apply the overlap analysis starting from the first character of $\alpha$, and get that it is reducible. If $K_2 = h(1)$, then $x = u$, and $\alpha$ contains the 4-power $x^4$. Again, $\alpha$ is reducible.

The only cases not yet covered are when $p - q - c < k$. Recall that $\alpha = v_i \cdots v_j$, $q = k + s$, $p = 2k + 2s + r$, and $c = k - (i \mod k)$. By assumption, $p - q - c = k + s + r - c < k$, *i.e.*, $s + r < k - i$ (here we consider $i$ modulo $k$). Therefore, $k > s + i$.

If $p \leq 3k$, then there can be at most two different $k$-blocks in the $T_{i,k}(\mathbf{v})$ decomposition of $\alpha$, and we can apply the analysis from the first character of $\alpha$. Otherwise, $2k + 2s + r > 3k$. Now,

$$j \mod k = i+2s+r \mod k = (i+s)+(s+r) \mod k < k+s+r \mod k = s+r.$$

Consider now $\alpha^R$ (that is, the reverse of $\alpha$), with the $k$-decomposition of $h(0)^R$ and $h(1)^R$. Let $c^R = j - \hat{j}$, where $\hat{j}$ is the rightmost index of the inner closure of $\alpha$. Then $c^R = j \mod k < s + r$, and so $p - q - c^R \geq k$, and we can show that $\alpha$ is reducible.

This concludes the proof of the lemma. $\square$

**Corollary 13.** *Let $f$ be a binary $k$-uniform morphism prolongable on 0, and let $\mathbf{w} = f^{\omega}(0)$. Suppose that $E(\mathbf{w}) < \infty$, and let $\mathcal{E}'$ be the set of exponents $r = p/q$, such that $q < k$ and $\mathbf{w}$ contains an $r$-power. Then*

$$E(\mathbf{w}) = \max_{p/q \in \mathcal{E}'} \left\{ \frac{p(k-1) + \lambda_f}{q(k-1)} \right\}. \tag{3}$$

*Proof.* By Lemmata 7, 8, 12, if $z \in OC(\mathbf{w})$ is an irreducible, unstretchable $p/q$-power and $p/q \notin \mathcal{E}'$, then $z$ is an image under the $\pi$ map; thus the exponent of every such power is an element of a sequence of the form $\{\pi^i(r)\}_{i=0}^{\infty}$, where $r \in \mathcal{E}'$. The set $\mathcal{E}'$ is finite, since $E(\mathbf{w}) < \infty$, thus there are only finitely many such sequences. By Lemma 6, the limit of each such sequence is given by the expression in (3); and since each of these sequences increases towards its limit, the critical exponent is the maximum of those limits. □

**Lemma 14.** *Let $h$ be a binary $k$-uniform morphism, let $\mathbf{v} = h(\mathbf{u})$ for some $\mathbf{u} \in \Sigma^{\omega}$, let $\alpha = v_i \cdots v_j \in OC(\mathbf{v})$ be a $p/q$-power with $p/q \geq 2$, and let $Q$ be the power block. Suppose $q < k$. Then at least one of the following holds:*

(1) $q|k$;
(2) $q \nmid k$ *and* $q|2k$;
(3) $\alpha$ *is reducible*;
(4) $p < 4k - 1$.

*Proof.* Let $\check{\alpha}$ be the inner closure of $\alpha$. Suppose $p = |\alpha| \geq 4k - 1$. Then $|\check{\alpha}| \geq 3k$, and so $\alpha$ contains an occurrence of the form $h(a_1 a_2 a_3)$ for some $a_1, a_2, a_3 \in \Sigma$. There are two cases: either $a_1 a_2 a_3$ contains a square, or $a_1 a_2 a_3$ is a 3/2-power.

Suppose $a_1 a_2 a_3$ contains a square, and assume w.l.o.g. it is 00. Then $h(0)h(0)$ is a suboccurrence of $\alpha$ that has both $k$ and $q$ periods. Since $2k > k + q$, by Theorem 2 $h(0)h(0)$ has a $g = \gcd(k, q)$ period. Since $q = |Q| < |h(0)| = k$, there must be an occurrence of $Q$ within $h(0)h(0)$, thus $Q$ has a $g$ period as well. We get that $Q = w^{q/g}$ for some $w \in \Sigma^*$ satisfying $|w| = g$, and $\alpha = w^{p/g}$. This implies that either $q|k$, or $\alpha$ is reducible: if $q \nmid k$, then $g < q$, and $p/g > p/q$.

Now suppose that $a_1 a_2 a_3 = 010$. Then $h(0)h(1)h(0)$ has both $q$ and $2k$ periods, thus either $q|2k$ or $\alpha$ is reducible. □

**Corollary 15.** *If $q < k$ and $\alpha$ is irreducible, then at least one of the following holds:*

(1) $h(0) = h(1)$;
(2) $h^{-1}(\hat{\alpha}) = ac^{\ell}b$, *where* $a, b \in \{0, 1, \varepsilon\}$, $c \in \{0, 1\}$, *and* $\ell \geq 0$;
(3) $h^{-1}(\hat{\alpha}) = ax^{\ell}b$, *where* $a, b \in \{0, 1, \varepsilon\}$, $x \in \{01, 10\}$, *and* $\ell \geq 0$;
(4) $|h^{-1}(\hat{\alpha})| \leq 5$.

*Proof.* By Lemma 14, either $q|k$, $q|2k$, or $p < 4k - 1$. Suppose $q|k$. Let $k = mq$, and let $\check{Q}$ denote the $q$ block of $\check{\alpha}$. Let $\ell = |\check{\alpha}|/k$. Then $\check{\alpha} = \check{Q}^{m\ell} = (h(a))^{\ell}$ for some $a \in \Sigma$. If $h(0) \neq h(1)$ this means that $h^{-1}(\check{\alpha}) = a^{\ell}$ and $h^{-1}(\hat{\alpha}) = ba^{\ell}c$, where $b, c \in \{0, 1, \varepsilon\}$.

If $q \nmid k$ and $q|2k$, we get similarly that $\check{\alpha} = \check{Q}^{m\lfloor \ell/2 \rfloor} = (h(x))^{\lfloor \ell/2 \rfloor}$, where $x \in \{01, 10\}$ and $\ell \geq 0$. Suppose $q \nmid 2k$. Then $p = |\alpha| < 4k - 1$, thus $|\hat{\alpha}| \leq 5k$, and $|h^{-1}(\hat{\alpha})| \leq 5$. □

**Corollary 16.** *Let $f$ be a $k$-uniform binary morphism prolongable on $0$, and let $\mathbf{w} = f^{\omega}(0)$. Then $E(\mathbf{w}) = \infty$ if and only if at least one of the following holds: $f(0) = f(1)$, $f(0) = 0^k$, $f(1) = 1^k$, or $f(0) = (01)^m 0$ and $f(1) = (10)^m 1$, where $k = 2m + 1$.*

*Proof.* It is easy to see that any of the 4 conditions implies $E(\mathbf{w}) = \infty$. For the converse, suppose $f(0) \neq f(1)$, and $\mathbf{w}$ contains unbounded powers. Then by Lemmata 7, 8, 12, 14 and Corollary 15, $\mathbf{w}$ must contain unbounded powers of the form $0^m$, $1^m$, or $(01)^m$. If it contains unbounded $0^m$ powers, then $f(a) = 0^k$ for some $a \in \Sigma$. Suppose $f(1) = 0^k$. Then $\mathbf{w}$ must contain unbounded $1^m$ powers as well, and so necessarily $f(0) = 1^k$, a contradiction: $f$ is prolongable on 0. Thus $\mathbf{w}$ contains unbounded $0^m$ powers if and only if $f(0) = 0^k$, and similarly it contains unbounded $1^m$ powers if and only if $f(1) = 1^k$. Finally, it is easy to see using similar inverse image arguments that $\mathbf{w}$ contains unbounded $(01)^m$ powers if and only if the last condition holds. □

**Note.** Another proof of Corollary 16 can be found in [13].

To complete the proof of Theorem 1, it remains to show that in order to compute $E(\mathbf{w})$, it is enough to consider $f^4(0)$. We do this by showing that any subword of $\mathbf{w}$ of the form $ab$, $a^\ell$, or $(a\bar{a})^\ell$, where $\ell$ is a positive integer, $a, b \in \Sigma$ and $\bar{a} = 1 - a$, must occur in $f^2(0)$ or $f^3(0)$. We then apply Corollary 15. The details are given below.

For the rest of this section $f : \Sigma^* \to \Sigma^*$ is a $k$-uniform morphism prolongable on 0, and $\mathbf{w} = f^{\omega}(0)$.

**Lemma 17.** *Let $a, b \in \Sigma$, and suppose $ab \lhd \mathbf{w}$. If $ab \in \{01, 10, 11\}$, then $ab \lhd f^2(0)$; if $ab = 00$, then either $00 \lhd f^2(0)$, or $00 \lhd f^3(0)$ and $000 \ntriangleleft \mathbf{w}$.*

*Proof.* The assertion clearly holds for $f(0) = 0^k$, thus we can assume $1 \lhd f(0)$. Suppose $ab \lhd \mathbf{w}$. Then either $ab \lhd f(c)$ for some $c \in \Sigma$, or $ab \lhd f(a'b')$ for some $a', b' \in \Sigma$; the first case implies that $ab \lhd f^2(0)$, since both $0, 1 \lhd f(0)$.

For $k = 2$, it is easy to check that the assertion holds. Assume $k \geq 3$. Then $f(0)$ contains at least two distinct pairs $ab$. If it contains four, we are done. Assume it contains exactly two. Then necessarily $f(0) \in \{0^{k-1}1, 01^{k-1}\}$. If the first case holds, then $f^2(0) = 0^{k-1}10^{k-1}1 \cdots 0^{k-1}1f(1)$, where $k - 1 \geq 2$; *i.e.*, it contains the pairs $00, 01, 10$. Assume $ab = 11 \ntriangleleft f^2(0)$. then necessarily $f(1) = 0y$, where $11 \ntriangleleft y$; but then $11 \ntriangleleft \mathbf{w}$, a contradiction. Thus the assertion holds for $f(0) = 0^{k-1}1$.

Assume now that $f(0) = 01^{k-1}$. Then $f^2(0) = 01^{k-1}f(1) \cdots f(1)$, *i.e.*, it contains the pairs $01, 11$. Suppose $ab = 10 \ntriangleleft f^2(0)$. Then necessarily $f(1) = 1^k$; but then $w = 01^{\omega}$, and $10 \ntriangleleft \mathbf{w}$, a contradiction. Now suppose $ab = 00 \ntriangleleft f^2(0)$. Then $00$ is not a subword of either $f(0), f(1), f(01)$, or $f(11)$. If $00 \ntriangleleft f(10)$, the only remaining option is $00 \not\triangleleft f(00)$; but then $00 \not\triangleleft f^3(0)$, and by induction, $00 \not\triangleleft f^n(0)$ for all $n$, a contradiction. Thus $00 \not\triangleleft f(10)$, and $00 \triangleleft f^3(0)$. The assertion $000 \not\triangleleft \mathbf{w}$ follows from the fact that $00 \not\triangleleft f(0), f(1)$.

Finally, assume $f(0)$ contains three distinct pairs. Let $ab \not\triangleleft f(0)$, and suppose $ab \not\triangleleft f^2(0)$. Then $ab$ is not a subword of either $f(0), f(1)$, or $f(a'b')$, where $a'b' \neq ab$. The only option is $ab \triangleleft f(ab)$, but then again $ab \not\triangleleft f^n(0)$ for all $n$, a contradiction. This completes the proof of the lemma. $\square$

**Lemma 18.** *Let $\ell$ be the maximal integer such that $0^\ell \triangleleft \mathbf{w}$ ($\ell = \infty$ if such integer does not exist). Then*

| | | | | |
|---|---|---|---|---|
| 1. | $f(0) = 0^k$ | $\Rightarrow$ | $\ell = \infty$; | |
| 2. | $f(0) \neq 0^k, f(1) \neq 0^k$ | $\Rightarrow$ | $\ell \leq 2k-2$ | *and* $0^\ell \triangleleft f^3(0)$; |
| 3. | $f(0) \neq 0^k, f(1) = 0^k$ | $\Rightarrow$ | $\ell \leq k^2 - k + 1$ | *and* $0^\ell \triangleleft f^3(0)$. |

*The bounds on $\ell$ and on the first occurrence of $0^\ell$ are tight. If $1 < \ell < \infty$, then $0^\ell$ occurs as a non-prefix subword of $f^3(0)$.*

*Proof.* Let $z = w_i \cdots w_j = 0^\ell \triangleleft \mathbf{w}$. Let $\hat{z}$ be the outer closure of $z$, and let $z' = f^{-1}(\hat{z})$. Observe that for any $m \in \mathbb{Z}_{\geq 0}$, if $\ell \geq 2k - 1 + mk$ then the $k$-decomposition of $\hat{z}$ contains at least $m+1$ consecutive 0-blocks (blocks of the form $0^k$).

Clearly, $f(0) = 0^k$ implies $\ell = \infty$. Assume $f(0) \neq 0^k$, $f(1) \neq 0^k$. Suppose $\ell \geq 2k-1$. Then the $k$-decomposition of $\hat{z}$ must contain a 0-block, a contradiction. Thus $\ell \leq 2k-2$, and $|z'| \leq 2$. By Lemma 17, $z' \triangleleft f^2(0)$, or $z' = 00$ and $z' \triangleleft f^3(0)$. The first case implies $0^\ell \triangleleft f^3(0)$; the second case implies that $\ell = 2$, and so again $0^\ell \triangleleft f^3(0)$. The bounds are tight: for the Thue-Morse morphism $\mu$, we get $\ell = 2k - 2 = 2$, and the first occurrence of $0^2$ is in $f^3(0)$.

To see that $0^\ell$ occurs in $f^3(0)$ as a non-prefix subword when $\ell > 1$, consider 4 possible values of $z'$, namely $\{0, 1, 0a, 1a\}$, where $a \in \Sigma$.

- If $z' = 1$, then, since $1 \triangleleft f(0)$ as a non-prefix, $0^\ell \triangleleft f^2(0)$ as a non-prefix.
- If $z = 0$, then either $0 \triangleleft f^2(0)$ as a non-prefix, or $f(0) = 01^{k-1}$ and $f(1) = 1^k$. In the first case, $0^\ell \triangleleft f^3(0)$ as a non-prefix; in the second case, $\mathbf{w} = 01^\omega$ and $\ell = 1$.
- If $z' = 1a$ for some $a \in \Sigma$, then it must occur as a non-prefix subword of $f^2(0)$, thus $0^\ell \triangleleft f^3(0)$ as a non-prefix.
- If $z = 0a$, and it is a prefix of $\mathbf{w}$, then $z \triangleleft f(0)$, and again, $z \triangleleft f^3(0)$ as a non-prefix unless $f(0) = 01^{k-1}$ and $f(1) = 1^k$.

Now assume $f(1) = 0^k$. Then $1^k \not\triangleleft \mathbf{w}$, and for any $m < k$, we get that $1^m \triangleleft \mathbf{w}$ if and only if $1^m \triangleleft f(0)$. Suppose $\ell \geq 2k - 1$ (at least one 0-block in the $k$-decomposition of $\hat{z}$). Then $z' = a1^m b$, where $a, b \in \{0, \varepsilon\}$, $1 \leq m \leq k-1$, and $1^m \triangleleft f(0)$. Therefore, $|f(0)|_0 \leq k - m$. Let $x, y$ be the longest prefix and suffix of $f(0)$, respectively, that do not contain 1. Then $|x| + |y| \leq k - m$. Since $z' \triangleleft 01^m0$, we get that $z \triangleleft yf(1)^m x = 0^{|x|+|y|+km}$; *i.e.*, $\ell \leq k - m + km = k + (k-1)m \leq k + (k-1)^2 = k^2 - k + 1$. Moreover, since $1^m \triangleleft f(0)$, we get that $z' \lhd f^2(0)$, thus $0^\ell \lhd f^3(0)$. The bounds are tight: let $f(0) = 01$, $f(1) = 00$. Then $\ell = k^2 - k + 1 = 3$, and the first occurrence of $0^3$ is in $f^3(0)$.

To see that $0^\ell$ occurs as a non-prefix subword of $f^3(0)$, observe that $0^k$ is not a prefix of $\mathbf{w}$ (since $1 \lhd f(0)$); on the other hand, $\ell \geq k + 1$, since $f(1) = 0^k$. Therefore $0^\ell$ must occur as a non-prefix subword of $f^3(0)$. $\square$

**Lemma 19.** *Let $\ell$ be the maximal integer such that $1^\ell \lhd \mathbf{w}$ ($\ell = \infty$ if such an integer does not exist). Then*

1. $f(1) = 1^k \quad \Rightarrow \quad \ell = \infty$;
2. $f(1) \neq 1^k \quad \Rightarrow \quad \ell \leq 2k - 2$ *and* $1^\ell \lhd f^3(0)$ *as a non-prefix subword.*

*The bounds on $\ell$ and on the first occurrence of $1^\ell$ are tight.*

*Proof.* If $f(1) = 1^k$, then $1^{k^n} \lhd f^n(0)$ for all $n$, and so $\ell = \infty$. Otherwise, if $0 \lhd f(1)$, then both $f(0) \neq 1^k$ and $f(1) \neq 1^k$, and the proof is similar to the proof of Lemma 18. The non-prefix statement is trivial, since $\mathbf{w}$ begins with 0. For tightness of the bound on $\ell$, observe that for a morphism of the form $0 \to 01^{k-1}$, $1 \to 1^{k-1}0$, we get $\ell = 2k-2$; for tightness of the bound on the first occurrence of $0^\ell$, observe that for a morphism of the form $0 \to 0^{k-1}1$, $1 \to 101^{k-2}$ we get that $\ell = k - 1$, and the first occurrence of $1^\ell$ is in $f^3(0)$. $\square$

**Lemma 20.** *Let $\ell$ be the maximal integer such that $(a\bar{a})^\ell \lhd \mathbf{w}$, where $a \in \Sigma$ and $\bar{a} = 1 - a$ ($\ell = \infty$ if such an integer does not exist). Assume $f(0) \neq f(1)$. Then either $\ell = \infty$ and $\mathbf{w} = (01)^\omega$, or*

1. *$k$ is even* $\Rightarrow$ $\ell \leq k - 1 + (k^2 + k)/2$ *and* $(a\bar{a})^\ell \lhd f^4(0)$;
2. *$k$ is odd* $\Rightarrow$ $\ell \leq k - 1$ *and* $(a\bar{a})^\ell \lhd f^3(0)$.

*The bounds on $\ell$ and on the first occurrence of $(a\bar{a})^\ell$ are tight. If $1 < \ell < \infty$, then $(a\bar{a})^\ell$ occurs as a non-prefix subword of $f^4(0)$.*

*Proof.* Let $z = w_i \cdots w_j = (a\bar{a})^\ell \in \mathbf{w}$. Let $\hat{z}$ be the outer closure of $z$, and let $z' = f^{-1}(\hat{z})$. Observe that for any $m \in \mathbb{Z}_{\geq 0}$, if $\ell > k - 1 + mk/2$ then the $k$-decomposition of $z$ contains at least $m + 1$ $k$-blocks. For even $k$, these blocks have the form $(b\bar{b})^{k/2}$, $b \in \Sigma$; for odd $k$, they alternate between $(b\bar{b})^{(k-1)/2}b$ and $(\bar{b}b)^{(k-1)/2}\bar{b}$.

Suppose $k$ is even. If $\ell > k - 1$, then $f(a) = (b\bar{b})^{k/2}$ for some $a \in \Sigma$. Let $m$ be the maximal integer satisfying $\ell > k - 1 + mk/2$. Since $f(0) \neq f(1)$, and the $m + 1$ $k$-blocks of $z$ are all the same, $z' = ca^{m+1}c'$ for $a, c, c' \in \Sigma$. Suppose $m > k$. Then $a^{k+2} \lhd \mathbf{w}$, and $f(a) = (b\bar{b})^{k/2}$. It is easy to see that the only way this situation is possible is if $f(\bar{a}) = a^k$. Since by assumption $f(0) = 0x$ where $1 \lhd x$, this implies that $f(0) = (01)^{k/2}$ and $f(1) = 0^k$. But in this case, it is easy to check that $a^{k+2} \not\lhd \mathbf{w}$, a contradiction. Thus $m \leq k$, and $\ell \leq k - 1 + (k+1)k/2$.

For tightness of the bound on $\ell$, observe that for the $f$ just defined, $0^{k+1} \lhd \mathbf{w}$, thus $(01)^{k(k+1)/2} \lhd \mathbf{w}$. The bound on the first occurrence of $z$ follows from the first occurrence bounds given in Lemmata 17, 18, 19. From these lemmata, we also get that $z$ occurs as a non-prefix. For tightness of this bound, observe that for $0 \to 010101, 1 \to 000110$, we get $\ell = 12$, and the first occurrence of $(01)^{12}$ is in $f^4(0)$.

Suppose that $k$ is odd. Then for $m \geq 1$, the $k$-decomposition contains both the blocks $(01)^{(k-1)/2}0$ and $(10)^{(k-1)/2}1$. This implies that $f(0) = (01)^{(k-1)/2}0$, $f(1) = (10)^{(k-1)/2}1$, and $\mathbf{w} = (01)^{\omega}$. Suppose $k - 1 < \ell \leq k - 1 + k/2$. Then $|z| \geq 2k$, *i.e.*, $z = xf(b)y = x(a\bar{a})^{k/2}ay$, where $b \in \Sigma$, $|xy| \geq k$, and $x, y$ has the following form: $x = \bar{a}(a\bar{a})^i$, $y = (\bar{a}a)^j$, $i + j \geq k/2$; or $x = (a\bar{a})^i$, $y = (\bar{a}a)^j\bar{a}$, $i + j \geq k/2$. Since $f(b)$ begins and ends with $a$, this implies that $x$ is a suffix of $f(\bar{b})$, and $y$ is a prefix of $f(\bar{b})$; and since $|xy| \geq k$, this implies that $f(\bar{b}) = (\bar{a}a)^{k/2}\bar{a}$. Again, we get $\mathbf{w} = (01)^{\omega}$. Thus $\mathbf{w} \neq (01)^{\omega}$ implies $\ell \leq k - 1$. Moreover, $z'$ must have the form $a\bar{a}$ or $a\bar{a}a$. If $z' = a\bar{a}$, then by Lemma 17, $z' \lhd f^2(0)$, and unless $\ell = 1$ (*i.e.*, $\mathbf{w} = 01^{\omega}$), it must occur as a non-prefix; if $z' = a\bar{a}a$, it is easy to show, by similar arguments, that $z' \lhd f^2(0)$ as a non-prefix. Thus $z \lhd f^3(0)$ as a non-prefix. For tightness of the bound on $\ell$, consider $f(0) = 010$, $f(1) = 111$. For tightness on the bound on the first occurrence of $z$, consider $f(0) = 01110$, $f(1) = 10101$. □

**Lemma 21.** *Let $k$ be even, and suppose there exist $n \geq 1$ and $x, y \in \Sigma^+$ such that $f(0) = (xy)^n x$ and $f(1) = (yx)^n y$. Let $\ell$ be the maximal integer such that $(a\bar{a})^{\ell} \lhd \mathbf{w}$, where $a \in \Sigma$ and $\bar{a} = 1 - a$. Assume $f(0) \neq f(1)$. Then $\ell \leq k - 1$ and $(a\bar{a})^{\ell} \lhd f^3(0)$ as a non-prefix.*

*Proof.* Let $z = w_i \cdots w_j = (a\bar{a})^{\ell} \in \mathbf{w}$. Let $\hat{z}$ be the outer closure of $z$, and let $z' = f^{-1}(\hat{z})$. Observe that the conditions imply $|x| = |y|$ and $|x|$ even. If there is at least one $k$-block in the $k$-decomposition of $z'$, then $f(a) = (b\bar{b})^{k/2}$ for some $a, b \in \Sigma$. This implies $x = y = (b\bar{b})^t$ for some $t \geq 1$, *i.e.*, $f(0) = f(1)$, a contradiction. Therefore $\ell \leq k - 1$ and $|z'| \leq 2$. By Lemma 17, $(a\bar{a})^{\ell} \lhd f^3(0)$ as a non-prefix. □

**Corollary 22.** *Let $z \lhd \mathbf{w}$ be an irreducible $p/q$-power satisfying $q < k$. Suppose $f(0) \neq f(1)$, $f(0) \neq 0^k$, $f(1) \neq 1^k$ and $\mathbf{w} \neq (01)^{\omega}$. Then $z \lhd f^4(0)$ as a non-prefix and $\lfloor p/q \rfloor \in O(k^3)$.*

*Proof.* Suppose $q < k$. By Corollary 15, either $f^{-1}(\hat{z}) = ab^{\ell}c$, $f^{-1}(\hat{z}) = a(b\bar{b})^{\ell}c$, or $|f^{-1}(\hat{z})| \leq 5$; here $\ell \geq 0$ is an integer and $a, b, c \in \{0, 1, \varepsilon\}$.

- If $f^{-1}(\hat{z}) = ab^{\ell}c$, then by Lemmata 18, 19, $\ell \leq k^2 - k + 1$ and $z \lhd f^4(0)$ as a non-prefix.
- If $f^{-1}(\hat{z}) = a(b\bar{b})^{\ell}c$ and $k$ is odd, then by Lemma 20, $\ell \leq k - 1$ and $z \lhd f^4(0)$ as a non-prefix.
- If $f^{-1}(\hat{z}) = a(b\bar{b})^{\ell}c$ and $k$ is even, then $f(b\bar{b}) = u^{2k/q}$ for some $u \in \Sigma^+$, where $|u| \nmid k$. This is possible only if there exist $n \geq 1$ and $x, y \in \Sigma^+$, such that $f(b) = (xy)^n x$ and $f(\bar{b}) = (yx)^n y$; but then by Lemma 21, $\ell \leq k - 1$ and $z \lhd f^4(0)$ as a non-prefix.
- If neither of the above cases hold, then $|f^{-1}(\hat{z})| \leq 5$, and $q \nmid 2k$, thus $q \geq 3$. Since $q < k$, we therefore get either $q = 3$ and $k = 4$, or $k \geq 5$. In both cases, a subword $x \lhd \mathbf{w}$ of length 5 satisfies $|f^{-1}(\hat{x})| \leq 2$, therefore by

Lemma 17, $f^{-1}(\hat{z}) \lhd f^3(0)$ as a non-prefix. Again we get that $z \lhd f^4(0)$ as a non-prefix. □

**Corollary 23.** *Suppose $E(\mathbf{w}) < \infty$. Let $\mathcal{E}$ be the set of exponents $r = p/q$, such that $q < k$ and $f^4(0)$ contains an $r$-power. Then*

$$E(\mathbf{w}) = \max_{p/q \in \mathcal{E}} \left\{ \frac{p(k-1) + \lambda_f}{q(k-1)} \right\}. \tag{4}$$

*The bound is attained if and only if $\lambda_f = 0$.*

*Proof.* Equation (4) is an immediate result of Corollaries 13, 22. The second assertion follows directly from the definition of $\pi$. □

Corollary 23 completes the proof of Theorem 1. We end this section with a couple of examples.

**Example 5.** As implied by the tightness assertions of Corollary 22, the prefix $f^4(0)$ is best possible. Consider the morphism $0 \to 010101$, $1 \to 000110$. In this example, $E(\mathbf{w}) = 12\frac{3}{5}$, and the first occurrence of a 12-power is in $f^4(0)$.

**Example 6.** Let $r, s$ be natural numbers satisfying $0 < r \leq s$. Let $f$ be the following binary $(s+1)$-uniform morphism:

$$f: \begin{array}{ccl} 0 & \to & 01^s; \\ 1 & \to & 01^{r-1}0^{s-r+1}. \end{array}$$

Then $f$ is an $(s+1)$-uniform morphism, satisfying $\rho_f = 01^{r-1}$, $\sigma_f = \varepsilon$, and $\lambda_f = r$. Let $\mathbf{w} = f^\omega(0)$. Then $1^s$ is a subword of $f^1(0)$; also, $0^{s(s+1)+1}$ is a subword of $f^3(0)$ if $r = 1$. Set $z = 1^s$ for $r > 1$ and $z = 0^{s(s+1)+1}$ for $r = 1$. It is easy to check that by applying $\pi$ to $z$ we get the maximal number in the set $\left\{ \frac{p(k-1)+\lambda_f}{q(k-1)} : p/q \in \mathcal{E} \right\}$; thus

$$\begin{aligned} r > 1 \quad &\Rightarrow \quad E(\mathbf{w}) = \frac{s \cdot s + r}{1 \cdot s} = s + \frac{r}{s}; \\ r = 1 \quad &\Rightarrow \quad E(\mathbf{w}) = s(s+1) + 1 + \frac{r}{s}. \end{aligned}$$

**Corollary 24.** *For any rational number $0 < t < 1$ there exist a binary $k$-uniform morphism $f$, such that $E(\mathbf{w}) = n + t$ for some $n \in \mathbb{Z}_{\geq 2}$.*

## 4. Generalizing the results

The definitions of $\rho, \sigma, \pi$ (Def. 3, Eq. (2)) can be generalized to arbitrary morphisms over finite alphabets. Let $\Sigma = \Sigma_t = \{0, \ldots, t-1\}$, let $f : \Sigma^* \to \Sigma^*$ be a morphism prolongable on 0, and let $\mathbf{w} = f^\omega(0)$. For a word $u \in \Sigma^*$, let $[u]$ be the *Parikh vector* of $u$, *i.e.*, $[u] = (|u|_0, \ldots, |u|_{t-1})^T$, where $|u|_i$ is the number of occurrences of the letter $i$ in $u$. Let $F$ be the *incidence matrix associated with* $f$, *i.e.*, $F_{i,j} = |f(j)|_i$, $0 \le i, j < t$. It is easy to check that for all $u \in \Sigma^*$ we have $[f(u)] = F[u]$.

Let $z = w_i \cdots w_j \lhd \mathbf{w}$ be a $p/q$-power, $z = x^{p/q}$, and let $Q, P$ be the Parikh vectors of $x, z$ respectively. In order to keep track of the components $|x|_i, |z|_i$, we introduce the notation "$z$ is a $P/Q$-power", where $P/Q$ stands for $\sum_{i=0}^{t-1} |z|_i / \sum_{i=0}^{t-1} |x|_i$. Under this notation, $f^m(z)$ is an $F^m P/F^m Q$-power; this power may be stretchable.

Assume $E(\mathbf{w}) < \infty$, and let $z = w_i \cdots w_j \lhd \mathbf{w}$ be an unstretchable $P/Q$-power. Define

$$\pi\left(z, \frac{P}{Q}\right) = \left(\sigma f(z)\rho, \frac{FP + \Lambda}{FQ}\right), \tag{5}$$

where $\sigma, \rho \in \Sigma^*$ are the words that stretch the $FQ$ period of $f(z)$ on the left and on the right, respectively, to an unstretchable power, and $\Lambda = [\sigma\rho]$. We call $\Lambda$ the *stretch vector* of $(f(z), FQ)$. Note that $\pi(z, P/Q)$ depends on the context of $z$, in particular on the letters $w_{i-1}, w_{j+1}$. Iterating $\pi$ on the initial power $z$, we get a sequence of stretch vectors, $\{\Lambda_m\}_{m\ge 0}$, and we have:

$$\pi^m\left(\frac{P}{Q}\right) = \frac{F^m P + \sum_{i=0}^{m-1} F^{m-1-i}\Lambda_i}{F^m Q}. \tag{6}$$

We call the sequence $\{\pi^m(P/Q)\}_{m\ge 0}$ a $\pi$-*sequence*. If the sequence $\{\Lambda_m\}_{m\ge 0}$ is ultimately periodic, it can be shown that in the uniform case, the $\pi$-sequence converges to a rational number, and in the non-uniform case, its $\limsup$ is a rational expression of the eigenvalues of $F$. In particular, it is algebraic of degree at most $t$.

To prove that $E(\mathbf{w})$ is algebraic, we need to show that the sequence of stretch vectors is indeed ultimately periodic for every choice of initial power $z$; we also need to show that $E(\mathbf{w})$ is generated by the $\pi$ map, *i.e.*, we need to rule out arbitrary powers. The overlap analysis method we used in the binary uniform case is too tedious for uniform morphisms over larger alphabets, and will not work at all for non-uniform morphisms. Instead, we use *circularity* arguments.

Basically, a fixed point $f^\omega(0)$ is circular if every sufficiently long subword of it has an unambiguous decomposition into images under $f$, save maybe for a prefix and a suffix of bounded length. The notion was introduced by Mignosi and Séébold in [20], where they showed that bounded critical exponent implies circularity in fixed points of morphisms over a finite alphabet; see also [6,21]. In [15], we use circularity arguments to show that when $f$ is a non-erasing morphism and $E(\mathbf{w}) < \infty$, the following holds:

(1) the sequence of stretch vectors is ultimately periodic for every choice of an initial power $z$;
(2) every unstretchable power with a sufficiently long power block belongs to some $\pi$-sequence;
(3) there are only finitely many distinct $\pi$-sequences occurring in $\mathbf{w}$.

Thus we have proved that if $\mathbf{w}$ is a fixed point of a non-erasing morphism, then either $E(\mathbf{w}) = \infty$, or $E(\mathbf{w})$ is algebraic of degree at most $t$, where $t$ is the alphabet size. Under certain conditions, our method also gives an algorithm for computing $E(\mathbf{w})$, which essentially reduces the problem to computing the Jordan decomposition of the incidence matrix.

It yet remains to extend the results to erasing morphisms; nevertheless, in light of the observations above, the following conjecture seems reasonable:

**Conjecture 25.** Let $f$ be a (possibly erasing) morphism over a finite alphabet $\Sigma$, and let $\mathbf{w}$ be an infinite fixed point of $f$. Assume $E(\mathbf{w}) < \infty$. Then

(1) if $f$ is uniform, then $E(\mathbf{w})$ is rational;

(2) if $f$ is non-uniform, then $E(\mathbf{w}) \in \mathbb{Q}[s_1, \ldots, s_\ell]$, where $s_1, \ldots, s_\ell$ are the eigenvalues of the incidence matrix of $f$. In particular, $E(\mathbf{w})$ is algebraic of degree at most $|\Sigma|$.

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