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PICTURE CODES

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Abstract. We introduce doubly-ranked (DR) monoids in order to study picture codes. We show that a DR-monoid is free iff it is pictorially stable. This allows us to associate with a set C of pictures a picture code $B(C)$ which is the basis of the least DR-monoid including C. A weak version of the defect theorem for pictures is established. A characterization of picture codes through picture series is also given.

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1. Introduction

During the last decade, many people dealt with the task to investigate how the word language environment is transferred into that of picture languages (see [5, 7–10, 12]). Participating in this effort we study the 2-dimensional analog of a code. A picture code is a set C of pictures with the property that any picture p can be constructed by the elements of C at most in one way.

To build up our theory we introduce the notion of the doubly-ranked (DR) monoid which is a very convenient tool to describe phenomena in the theory of picture languages. They play the role corresponding to that of ordinary monoids in word language theory. A DR-monoid is a doubly ranked set $M = (M_{m,n})$ equipped with a horizontal and a vertical ranked multiplication

$$ⓗ : M_{m,n} \times M_{m,n'} \to M_{m,n+n'} \quad ⓥ : M_{m,n} \times M_{m',n} \to M_{m+m',n} \quad m, m', n, n' \in \mathbb{N}$$

satisfying certain natural compatibility axioms.

For a given doubly-ranked alphabet $X = (X_{m,n})$, the set $pict(X)$ of all pictures from it together with horizontal and vertical picture concatenation is the free DR-monoid over X (Sect. 2).

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A picture code over the alphabet $X = (X_{m,n})$ is a subset $C \subseteq pict(X)$ such that the canonically induced morphism of DR-monoids $pict(C) \to pict(X)$ is injective.

In Section 3 we display a characterization of picture codes through picture series. The $\circ$-star operator we use is an extension to picture series of the generalized Kleene star of Simplot. Precisely we show that $L \subseteq pict(X)$ is a code if and only if

$$ch(L^\circ) = ch(L)^\circ$$

where $ch(L)$ is the characteristic series of L.

The main result of section 4 is that any DR-submonoid M of $pict(X)$ admits a minimum set of generators $C(M)$. M is free over $C(M)$ if and only if $C(M)$ is a picture code.

A basic well-known necessary and sufficient condition for a submonoid N of A^* (A an ordinary alphabet) to be free is to satisfy next stability condition

$$r, rs, st, t \in N \Rightarrow s \in N \ \text{ for all } r, s, t \in A^* \ \ (\textit{cf.}\ [3]).$$

Trying to transfer the above result in the framework of pictures we should notice that neither horizontal stability (HS) nor vertical stability (VS) nor both guarantee that a DR-submonoid of $pict(X)$ is of the form $pict(Y)$, for some DR-alphabet Y. We must incorporate to (HS)+(VS) the extra condition that we call circular stability (CS) which has as follows:

(CS) $\quad r\textcircled{h}s, s\textcircled{v}t, u\textcircled{h}t, r\textcircled{v}u \in M \Rightarrow r, s, t, u \in M$ for all $r, s, t, u \in pict(X)$ of suitable rank.

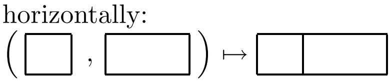
It turns out that the intersection of any family of free DR-submonoids of pict(X) is again free and this permits us to associate with any set C of pictures a picture code $B(C)$ which is the basis of the least free DR-monoid including C.

A weak version of the Defect Theorem (*cf.* [3]) holds for pictures: for any finite set C, the set $B(C)$ is also finite.

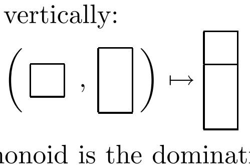
2. Doubly-ranked monoids

Traditionally, pictures are two dimensional words, that is rectangular arrays of symbols called *pixels* (*cf.* [13,14]). They can be concatenated in two different ways

- horizontally:



- vertically:



As monoid is the dominating algebraic structure in word language theory, the corresponding notion for pictures is that of doubly-ranked monoid.

Precisely, a *doubly ranked semi-group* (DR semi-group for short) is a doubly ranked set $M = (M_{m,n})$ endowed with two operations (simulating the above ones)

$$ⓗ : M_{m,n} \times M_{m,n'} \to M_{m,n+n'} \text{ (horizontal multiplication)}$$

$$ⓥ : M_{m,n} \times M_{m',n} \to M_{m+m',n} \text{ (vertical multiplication)}$$

$(m, m', n, n' \in \mathbb{N})$ which are associative, *i.e.*

$$aⓗ(bⓗc) = (aⓗb)ⓗc$$

$$aⓥ(bⓥc) = (aⓥb)ⓥc$$

and compatible to each other, *i.e.*

$$(aⓗa')ⓥ(bⓗb') = (aⓥb)ⓗ(a'ⓥb')$$

for all a, a', b, b' of suitable rank.

A DR semi-group $M = (M_{m,n})$ whose operations ⓗ and ⓥ are *unitary*, that is there are two sequences $E = (e_m)$ and $F = (f_n)$ with $e_m \in M_{m,0}$, $f_n \in M_{0,n}$ $(m, n \in \mathbb{N})$ such that

$$e_mⓗa = a = aⓗe_m \ , \ f_nⓥa = a = aⓥf_n \ , \ e_0 = f_0 \ ,$$
$$e_mⓥe_n = e_{m+n} \ , \ f_mⓗf_n = f_{m,n}$$

is called a *DR-monoid*. E and F are called respectively the *horizontal* and *vertical* units of M. Submonoids and morphisms of such structures are defined in a natural way.

The transpose M^T of the DR-monoid $M = (M_{m,n})$ is given by

$$M^T_{m,n} = M_{n,m}.$$

The horizontal (resp. vertical) operation of M^T is the vertical (resp. horizontal) operation of M. Thus to any statement concerning DR-monoids, a dual statement can be obtained by interchanging the roles of horizontal and vertical operations.

We shall see that the set of all pictures over a certain alphabet is organized into a DR-monoid which actually is the free DR-monoid generated by that alphabet. The alphabets we deal with have pixels of arbitrary rank and thus our notion of picture is more general than the one proposed in the literature (*cf.* [5, 7–12, 15]). Let $X = (X_{m,n})$ be a DR-alphabet. We first define the sets $P_{m,n}(X)$ $(m, n \in \mathbb{N})$ inductively as follows:

- $X_{m,n} \subseteq P_{m,n}(X)$;
- $e_m \in P_{m,0}(X)$, $f_n \in P_{0,n}(X)$ where $e_m, f_n (m, n \in \mathbb{N})$ are two sequences of specified symbols not belonging to $X (e_0 = f_0)$;
- if $a \in P_{m,n}(X), b \in P_{m,n'}(X), c \in P_{m',n}(X)$, then the schemes

$$ab \in P_{m,n+n'}(X) \ , \ \begin{pmatrix} a \\ c \end{pmatrix} \in P_{m+m',n}(X);$$

– the sets $P_{m,n}(X)$, $m,n \in \mathbb{N}$ are exclusively constructed by using the above three items.

Now the set $pict(X) = (pict_{m,n}(X))$ of all pictures from X is obtained by dividing the set $\bigcup\limits_{m,n} P_{m,n}(X)$ by the equivalence generated by the relations

$$a(a'a'') \sim (aa')a''$$

$$\begin{pmatrix} b \\ \begin{pmatrix} b' \\ b'' \end{pmatrix} \end{pmatrix} \sim \begin{pmatrix} \begin{pmatrix} b \\ b' \end{pmatrix} \\ b'' \end{pmatrix}$$

$$ae_m \sim a \sim e_m a$$

$$\begin{pmatrix} e_m \\ e_n \end{pmatrix} \sim e_{m+n}$$

$$\begin{pmatrix} f_n \\ b \end{pmatrix} \sim b \sim \begin{pmatrix} b \\ f_n \end{pmatrix}$$

$$f_m f_n \sim f_{m+n}$$

$$\begin{pmatrix} aa' \\ bb' \end{pmatrix} \sim \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix}$$

for all a, a', b, b' of suitable rank.

Convention. Taking into account vertical associativity, we may omit inner parentheses in the same column, for instance

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ \begin{pmatrix} a_3 \\ a_4 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a_1 \\ \begin{pmatrix} a_2 \\ a_3 \end{pmatrix} \\ a_4 \end{pmatrix} = \ldots$$

It is often convenient to represent in figures the element aa' by $\boxed{a \mid a'}$ and the element $\begin{pmatrix} b \\ b' \end{pmatrix}$ by $\boxed{\begin{array}{c} b \\ \hline b' \end{array}}$ respectively.

Proposition 1. *$pict(X)$ is the free DR-monoid generated by X, i.e. each function of DR-sets $F : X \to M$ ($M = (M_{m,n})$ a DR-monoid) can be uniquely extended into a morphism of DR-monoids $\widetilde{F} : pict(X) \to M$ defined by the following inductive clauses:*

– $\widetilde{F}(x) = F(x)$, *for all* $x \in X$;
– $\widetilde{F}(aa') = \widetilde{F}(a)\,ⓗ\,\widetilde{F}(a')$;
– $\widetilde{F}\begin{pmatrix} b \\ b' \end{pmatrix} = \widetilde{F}(b)\,ⓥ\,\widetilde{F}(b')$;

for all $a, a', b, b' \in pict(X)$ of suitable rank.

In the case our alphabet $X = (X_{m,n})$ is a monadic DR-alphabet, that is $X_{1,1} = \Sigma$ and $X_{m,n} = \emptyset$ for all $m \neq 1$ or $n \neq 1$ each element of $pict_{m,n}(\Sigma)$ can be depicted as

$$a = \boxed{\begin{matrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{matrix}} \quad , \ a_{ij} \in \Sigma.$$

For every DR-monoid $M = (M_{m,n})$, each function $F : \Sigma \to M_{1,1}$ is uniquely extended into a morphism of DR-monoids $\widetilde{F} : pict(\Sigma) \to M$ whose value at the above picture is

$$\begin{aligned} \widetilde{F}(a) &= (F(a_{11}) ⓗ \dots ⓗ F(a_{1n})) ⓥ \dots ⓥ (F(a_{m1}) ⓗ \dots ⓗ F(a_{mn})) \\ &= (F(a_{11}) ⓥ \dots ⓥ F(a_{m1})) ⓗ \dots ⓗ (F(a_{1n}) ⓥ \dots ⓥ F(a_{mn})). \end{aligned}$$

3. Picture codes

A 2-dimensional code notion related to DR-monoids is introduced in the present section.

Using proposition 1 we can explicitly describe the elements of the DR-submonoid generated by a set. More precisely, let $M = (M_{m,n})$ be a DR-monoid and $C = (C_{m,n})$ be a subset of M : $C_{m,n} \subseteq M_{m,n}$ for all m, n. Further, let us denote by C° the least DR-submonoid of M which includes C. We introduce the auxiliary DR-alphabet $X(C)$ such that $X(C)_{m,n}$ is a copy of $C_{m,n}$, that is there are bijections

$$F(C)_{m,n} : X(C)_{m,n} \xrightarrow{\sim} C_{m,n} \quad m, n \text{ non-negatives.}$$

Proposition 2. *It holds $C^\circ = \widetilde{F}(C)(pict(X(C)))$ where $\widetilde{F}(C)$ is the unique extension of $F(C)$ according to proposition 1.*

Proof. Left to the reader. □

Remark. It should be pointed out that the operator $\circ$-star above is just the generalized Kleene star of Simplot (*cf.* [12])

The present framework enables us to speak of codes in a quite natural way.

Thus, $C \subseteq pict(X)$ is a *picture code* whenever the canonical morphism of DR-monoids induced by the function

$$F(C) : X(C) \to pict(X)$$

is injective.

Manifestly, C can not contain any element of the units E, F.

Example 1. Let $X = \{a, b, c\}$ with $rank(a) = (1,1)$, $rank(b) = (1,2)$, $rank(c) = (2,1)$. Then the set $C = \left\{ \begin{pmatrix} aa \\ b \end{pmatrix}, \begin{pmatrix} a \\ a \end{pmatrix} c \begin{pmatrix} a \\ a \end{pmatrix}, b \right\}$ is a code.

Example 2. For fixed $m, n \geqslant 0$ the set $pict_{m,n}(X)$ of all pictures with rank (m,n) is a picture code.

Example 3. Consider the alphabet $X = \{a\}$, $rank(a) = (1,1)$ and let $C \subseteq pict(X)$ be an arbitrary code.

If $Card(C) > 1$, then there exist two distinct elements $p, q \in C$ with $rank(p) = (m,n)$, $rank(q) = (m', n')$ and the picture

$$\text{m-times}\left\{ \overbrace{\begin{pmatrix} q & \dots & q \\ \vdots & & \vdots \\ q & \dots & q \end{pmatrix}}^{n-times} = \overbrace{\begin{pmatrix} p & \dots & p \\ \vdots & & \vdots \\ p & \dots & p \end{pmatrix}}^{n'-times} \right\} m'\text{-times} \qquad \in C^{\circ}$$

has two distinct factorizations. However this is not true since C is a code. Therefore C is a singleton.

Remark.

1. Aigrain and Beauquier introduced a nice notion of code for patterns of the plane of integers, called polyominoes (*cf.* [1]).

   Beauquier and Nivat proved that for a given finite set $\mathcal{C}$ of such objects it is undecidable whether or not it is a code. This undecidability result also holds for dominoes (*cf.* [2]). In the case of pictures, the problem remains open.
2. In the free binoid generated by an ordinary alphabet Σ, two notions of code are considered according to the concatenation operation used. For these pseudo-2-dimensional codes a Sardinas-Patterson algorithm is established (*cf.* [6]).

Now we point out how our code notion is related with series on pictures. We need some additional notation.

The *valuation morphism* $val_M : pict(M) \to M$ associated with a DR-monoid M, is the unique extension of the identity function $id : M \to M$.

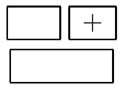
For instance, if $m, m' \in pict_{1,2}(M)$ and $m'' \in pict_{2,1}(M)$, then val_M sends the picture

m	m''
m'	

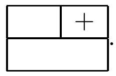
of $pict_{2,3}(M)$ to the element $(m \textcircled{v} m') \textcircled{h} m''$ of $M_{2,3}$.

In particular for $M = pict(X)$ we have the morphism of DR-monoids $val_X : pict(pict(X)) \to pict(X)$.

Given $p \in \mathit{pict}(X)$, any picture $\mathbf{p} \in \mathit{val}_X^{-1}(p)$ is called a *partition* of p. For instance



is a partition of the picture



Given a partition $\mathbf{p}$ of a picture $p \in \mathit{pict}(X)$ we say that $r \in \mathit{pict}(X)$ *belongs* to $\mathbf{p}$ if r is a piece of $\mathbf{p}$.

Now given a commutative semiring K, mappings of the form

$$S : \mathit{pict}(X) \to K$$

are called *picture series*. The value of S at p is denoted by (S, p) and is referred to as the *coefficient* of S at p (*cf.* [4]).

Given a series $S : \mathit{pict}(X) \to K$ and a partition $\mathbf{p}$ of a picture p, we set

$$(S, \mathbf{p}) = \prod_{r \in \mathbf{p}} (S, r).$$

Then we define the $\circ$-star of S by the formula

$$(S^\circ, p) = \sum_{\mathbf{p} \in \mathit{val}_X^{-1}(p)} (S, \mathbf{p}), \qquad\qquad p \in \mathit{pict}(X).$$

Thus in the case of a picture language $L \subseteq \mathit{pict}(X)$,

$$\mathit{ch}(L)^\circ : \mathit{pict}(X) \to \mathbb{N}$$

sends every picture p to the number of its partitions with pieces in L, where $\mathit{ch}(L)$ is the characteristic series of L.

The equality

$$\mathit{ch}(L)^\circ = \mathit{ch}(L^\circ)$$

means that each picture p has at most one partition into pieces of L. Therefore:

Proposition 3. *$L \subseteq \mathit{pict}(X)$ is a picture code if and only if it holds*

$$\mathit{ch}(L)^\circ = \mathit{ch}(L^\circ).$$

4. Minimum generation of DR-monoids of pictures

Proposition 4. *Consider, a DR-submonoid M of $\mathit{pict}(X)$ and let*

$$\overline{M} = (M - E) - F.$$

Then M has a minimum, with respect to inclusion, set of generators

$$C(M) = \overline{M} - (\overline{M} ⓗ \overline{M} \cup \overline{M} ⓥ \overline{M}).$$

Proof. First, we show that $C(M)$ generates M.

Obviously, $C(M)^\circ \subseteq M$.

To establish the opposite inclusion, let $p \in \overline{M}_{m,n}$. If $p \notin (\overline{M} ⓗ \overline{M})_{m,n} \cup (\overline{M} ⓥ \overline{M})_{m,n}$, then $p \in C(M)_{m,n} \subseteq (C(M)^\circ)_{m,n}$. Otherwise, either $p \in (\overline{M} ⓗ \overline{M})_{m,n}$ or $p \in (\overline{M} ⓥ \overline{M})_{m,n}$. In the first case, $p = p_1 p_2$ with $rank(p_i) = (m, n_i)$, $n_i < n (i = 1, 2)$, $n_1 + n_2 = n$. Using double induction on $rank(p)$, $p_i \in (C^\circ)_{m,n_i} (i = 1, 2)$ and so $p \in (C(M)^\circ)_{m,n}$. The other case is treated analogously.

Now, let D be a set of generators of M. We may suppose that D does not contain any identity element, *i.e.* $(D - E) - F = D$. Let $p \in C(M)_{m,n}$. Since $M = D^\circ$, either $p \in D_{m,n}$ or

$$p = p_1 p_2 \quad \text{or} \quad p = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \tag{1}$$

with $p_i \in (D^\circ)_{m,n_i}$, $n_1 + n_2 = n$, $q_i \in (D^\circ)_{m_i,n}$, $m_1 + m_2 = m$, $i = 1, 2$.

Case (1) is excluded, because $p \notin \overline{M} ⓗ \overline{M} \cup \overline{M} ⓥ \overline{M}$. Thus, $p \in D_{m,n}$ for all m, n, that is $C(M) \subseteq D$. □

Proposition 5. *The minimum set of generators of a free DR-submonoid M of $pict(X)$, is a picture code.*

Conversely for any picture code $C \subseteq pict(X)$, C° is a free DR-submonoid of $pict(X)$ and its minimum set of generators is again C.

Proof. Since M is a free DR-submonoid of $pict(X)$, there is an isomorphism $\alpha : pict(B) \to M$ for some DR-alphabet B. Thus $C(M) = \alpha(B)$ is a code.

In addition, $M = \alpha(pict(B)) = (\alpha(B))^\circ = C(M)^\circ$ and thus $C(M)$ generates M.

Moreover, by applying α to

$$B = \overline{pict(B)} - (\overline{pict(B)} ⓗ \overline{pict(B)} \cup \overline{pict(B)} ⓥ \overline{pict(B)})$$

we obtain

$$\alpha(B) = \alpha(\overline{pict(B)}) - (\alpha(\overline{pict(B)}) ⓗ \alpha(\overline{pict(B)}) \cup \alpha(\overline{pict(B)}) ⓥ \alpha(\overline{pict(B)}))$$

which implies

$$C(M) = \overline{C^\circ} - (\overline{C^\circ} ⓗ \overline{C^\circ} \cup \overline{C^\circ} ⓥ \overline{C^\circ})$$

and so

$$C(M) = \overline{M} - (\overline{M} ⓗ \overline{M} \cup \overline{M} ⓥ \overline{M}).$$

Conversely, let $C \subseteq pict(X)$ be a code. Then there is a doubly ranked alphabet B and a bijection $\alpha : B \to C$ which is extended into an isomorphism of DR-monoids $\alpha' : pict(B) \to C^\circ$. Thus C° is free.

Furthermore, from the equality

$$B = \overline{pict(B)} - (\overline{pict(B)} ⓗ \overline{pict(B)} \cup \overline{pict(B)} ⓥ \overline{pict(B)})$$

we get

$$\alpha(B) = \overline{\alpha'(pict(B))} - (\overline{\alpha'(pict(B))} ⓗ \overline{\alpha'(pict(B))} \cup \overline{\alpha'(pict(B))} ⓥ \overline{\alpha'(pict(B))}).$$

It turns out that

$$C = \overline{C^\circ} - (\overline{C^\circ} ⓗ \overline{C^\circ} \cup \overline{C^\circ} ⓥ \overline{C^\circ})$$

and by virtue of proposition 1, C is the minimum set of generators of C°. $\square$

We need the following definition. Let M be a DR-monoid; if $C(M)$ is a picture code, then we say that $C(M)$ is the *basis* of M.

Next remarkable result is a consequence of the previous proposition.

Corollary 1. Given picture codes $C_1, C_2 \subseteq pict(X)$, it holds

$$C_1^\circ = C_2^\circ \text{ implies } C_1 = C_2.$$

Let X be a DR-alphabet. A picture code C is *maximal* over X whenever C is not properly contained in any other picture code over X, that is if

$$C \subseteq C' \ , \ C' \text{ code} \Rightarrow C = C'.$$

Let M be a free DR-submonoid of $pict(X)$, $M \neq pict(X)$. Then M is *maximal* if it is not properly contained in any other free DR-submonoid excepted $pict(X)$.

Proposition 6. *If M is a maximal free DR-submonoid of $pict(X)$, then its basis $C(M)$ is a maximal picture code.*

Proof. Let $C_1 \subseteq pict(X)$ be a code with $C(M) \subset C_1$. Then $C(M)^\circ \subseteq C_1^\circ$ and $C(M)^\circ \neq C_1^\circ$ (by corollary 1). But M is maximal and thus $C_1^\circ = pict(X)$.

Let $p \in X - C(M)$. The set $C_2 = C(M) \cup pp$ is a code and $M \subset C_2^\circ \subset pict(X)$ since $pp \notin M$ and $p \notin C_2^\circ$. This contradicts the maximality of M. $\square$

5. Stable DR-monoids

The next fundamental characterization of monoid freeness is established in [3]: given an ordinary alphabet A, a submonoid N of A^* is free if-f it is stable, that is

$$r, rs, st, t \in N \Rightarrow s \in N \qquad \text{for all } \ r, s, t \in A^*.$$

In pictures, we have the possibility to generalize the above stability condition in two directions: horizontally and vertically. Precisely, we say that a DR-submonoid M of $pict(X)$ is *horizontally stable* (HS) whenever for all $a \in pict_{m,n_1}(X)$, $b \in pict_{m,n_2}(X)$, $c \in pict_{m,n_3}(X)$ it holds

$$a \in M_{m,n_1} \ , \ a ⓗ b \in M_{m,n_1+n_2} \ , \ b ⓗ c \in M_{m,n_2+n_3} \ , \ c \in M_{m,n_3} \Rightarrow b \in M_{m,n_2}.$$

M is said to be *vertically stable* (VS) whenever its transpose M^T is horizontally stable, that is for all $a \in pict_{m_1,n}(X),\ b \in pict_{m_2,n}(X),\ c \in pict_{m_3,n}(X)$ it holds

$$a \in M_{m_1,n}\ ,\ a \circledv b \in M_{m_1+m_2,n}\ ,\ b \circledv c \in M_{m_2+m_3,n}\ ,\ c \in M_{m_3,n} \Rightarrow b \in M_{m_2,n}.$$

Example 4. A DR-submonoid M of *pict*(X) fulfilling both (HS) and(VS) may not be free. Take the monadic alphabet $X = \{a, b, c, d, e, f\}$,

$$C = \left\{ ab, ef, c, d, \begin{pmatrix} a \\ d \end{pmatrix}, \begin{pmatrix} b \\ e \end{pmatrix}, \begin{pmatrix} c \\ f \end{pmatrix} \right\}$$

and $M = C^\circ$.

M is (HS)+(VS) but fails to be free since the picture

$$\begin{pmatrix} abc \\ def \end{pmatrix}$$

has two distinct decompositions in elements of C.

Therefore, in order to characterize freeness in the setup of pictures, we need an additional condition.

A DR-submonoid M of *pict*(X) is termed *circularly stable* (CS) whenever for all $r \in pict_{m_1,n_1}(X)\ ,\ s \in pict_{m_1,n_2}(X)\ ,\ t \in pict_{m_2,n_2}(X)\ ,\ u \in pict_{m_2,n_1}(X)$

$$\begin{array}{|c|c|} \hline r & s \\ \hline u & t \\ \hline \end{array} \qquad \text{(Fig. 1)}$$

it holds

$$r \circledh s \in M_{m_1,n_1+n_2}\ ,\ s \circledv t \in M_{m_1+m_2,n_2}\ ,\ u \circledh t \in M_{m_2,n_1+n_2}\ ,\ r \circledv u \in M_{m_1+m_2,n_1}$$

implies

$$r \in M_{m_1,n_1}\ ,\ s \in M_{m_1,n_2}\ ,\ t \in M_{m_2,n_2}\ ,\ u \in M_{m_2,n_1}.$$

Now we state

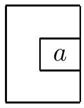
Theorem 1. *A DR-submonoid M of pict(X) is free if and only if it is horizontally, vertically and circularly stable.*

Proof. Assume that M is free, that is $M = C^\circ$ with $C \subseteq pict(X)$ a picture code and let r, s, t, u of suitable rank so that $rs\ ,\ \begin{pmatrix} s \\ t \end{pmatrix},\ ut\ ,\ \begin{pmatrix} r \\ u \end{pmatrix} \in C^\circ$. If for instance r does not belong to C°, then in the unique factorization $\mathbf{p}$ of rs in C°, there is at least a piece x such that

$$x = ab$$

with $a \in pict(X) - C^\circ$ lying on the eastern border of r and $b \in \overline{pict(X)}$ lying on the western border of s.

Similarly a (or $a' \in pict(X) - C^\circ$ which lies on the south-eastern border of a) must lie on the western part of a piece $y \in C$

$y =$ 

in the unique factorization $\mathbf{p}'$ of $\begin{pmatrix} r \\ u \end{pmatrix}$ in C°. Therefore x and y belong in two distinct factorizations of $\begin{pmatrix} rs \\ ut \end{pmatrix}$ in C° which is impossible. Thus $M = C^\circ$ has the circular property.

Now, if a, b, c are pictures of suitable rank such that a , $\begin{pmatrix} a \\ b \end{pmatrix}$, $\begin{pmatrix} b \\ c \end{pmatrix}$, $c \in M$ and b does not belong to M, then

$$\begin{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ c \end{pmatrix} \;,\; \begin{pmatrix} a \\ \begin{pmatrix} b \\ c \end{pmatrix} \end{pmatrix}$$

would be two different factorizations of the same picture, contradiction. Hence M is v-stable.

Horizontal stability is established analogously.

Conversely, assume that M is hv- and circularly stable. Furthermore suppose that the minimum set of generators $C(M)$ is not a code and let p be a picture of minimal rank which admits two different decompositions (D_1) and (D_2) in $C(M)^\circ$. The following alternatives may arise.

`Case 1.` $p = p_1 p_2 = q_1 q_2$.

`Subcase 1a.` If $rank(p_1) = rank(q_1)$ and p_1, q_1 have the same decomposition in $C(M)^\circ$, then the picture $p_2 = q_2$ has smaller rank than that of p and admits two different decompositions in $C(M)^\circ$, which contradicts the choice of p. By the same argument, we exclude the case the picture $p_1 = q_1$ has two different decompositions.

`Subcase 1b.` $rank(p_1) > rank(q_1)$. Then there exists a picture $s \in pict(X)$ such that $p_1 = q_1 s$ and $s p_2 = q_2$. As M is h-stable, it comes $s \in M = C(M)^\circ$. This implies that the decompositions (D_1) and (D_2) are identical, contradiction.

`Case 2.` $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$. Dual to Case 1.

`Case 3.` $p = p_1 p_2 = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$.

$$p_1 \left\{ \overbrace{\underbrace{\begin{array}{|c|c|} \hline \mathrm{r} & \mathrm{s} \\ \hline \mathrm{u} & \mathrm{t} \\ \hline \end{array}}_{q_2}}^{q_1} \right\} p_2$$

Since by hypothesis, M has the circular property $r, s, t, u \in C(M)^\circ$ and therefore the factorizations $(D_1), (D_2)$ coincide, contradiction.

We conclude that $C(M)$ is a picture code, *i.e.* M is free. $\square$

Example 4 (continue)**.** We observe that M is not CS since

$$abc \in M_{1,3}\ ,\ def \in M_{1,3}\ ,\ \begin{pmatrix} a \\ d \end{pmatrix} \in M_{2,1}\ ,\ \begin{pmatrix} bc \\ ef \end{pmatrix} \in M_{2,2}$$

while $a \notin M_{1,1}$.

Thus it is not free.

Example 5. Although prefix (suffix) word subsets constitute a wide class of codes, in the 2-dimensional case this fails to be true with respect to horizontal (vertical) concatenation.

For instance given $a \in pict(X)$ the set $C = \left\{ aa\ , \begin{pmatrix} a \\ a \end{pmatrix} \right\}$ is not a code as it can be easily seen.

6. Free hull of a picture language

The results of the previous section enable us to study more in depth picture codes.

First, from Theorem 1 we immediately deduce that:

Proposition 7. *The intersection of any family of free DR-submonoids of* $pict(X)$ *is again free.*

Proof. It is omitted. $\square$

This allows us to speak of the *free hull* of a picture language $C \subseteq pict(X)$; it is the intersection of all free DR-submonoids of $pict(X)$ including C, *i.e.* it is the smallest free DR-submonoid of $pict(X)$ including C.

Next lemma is useful.

Lemma 1. Let $r, s, t, u \in pict_{m,n}(X)$ and assume that B° is the free hull of the set $C = \left\{ rs, ut, \begin{pmatrix} r \\ u \end{pmatrix}, \begin{pmatrix} s \\ t \end{pmatrix} \right\}$. Then $B = \{r, s, t, u\}$.

Proof. It holds $C \subseteq \{r, s, t, u\}^\circ$ and by the definition of the free hull,

$$B(C)^\circ \subseteq \{r, s, t, u\}^\circ.$$

On the other hand, since $rs, ut, \begin{pmatrix} r \\ u \end{pmatrix}, \begin{pmatrix} s \\ t \end{pmatrix} \in B(C)^\circ$ which has the circular property, we obtain $r, s, t, u \in B(C)^\circ$. Therefore $\{r, s, t, u\}^\circ \subseteq B(C)^\circ$.

We conclude that the free hull of C is $\{r, s, t, u\}^\circ$. $\square$

In the sequel, we are interested in the cardinality of the basis $B(C)$ of the free hull $B(C)^\circ$ of a set $C \subseteq pict(X)$. The well known Defect Theorem for words, states that $B(C)$ is a finite set and even $CardB(C) \leqslant CardC$, the equality holding exactly when C is a code (*cf.* [3]).

Example 6. As we have seen (Lem. 1) the basis of the free hull of the set $C = \{rs, \begin{pmatrix} s \\ t \end{pmatrix}, ut, \begin{pmatrix} r \\ u \end{pmatrix}\}$ is $B(C) = \{r, s, t, u\}$. Although $CardC = CardB(C)$, C is not a picture code.

Example 7. Consider the set

$$C = \left\{ abc, def, \begin{pmatrix} a \\ d \end{pmatrix}, \begin{pmatrix} b \\ e \end{pmatrix}, \begin{pmatrix} c \\ f \end{pmatrix} \right\}$$

where all the symbols a, b, c, d, e, f have rank $(1, 1)$. The basis of its free hull is $B(C) = \{a, b, c, d, e, f\}$ and thus $CardB(C) > CardC$.

Proposition 8. *Let $C = \{c_1, ..., c_\zeta\}$ be a subset of $pict(X)$ whose free hull is $B(C)^\circ$ and $rank(c_i) = (m_i, n_i)$, $1 \leqslant i \leqslant \zeta$. Then*

$$CardB(C) \leqslant \sum_{i=1}^{\zeta} \frac{m_i(m_i + 1)}{2} \cdot \frac{n_i(n_i + 1)}{2}.$$

Proof. For all $i = 1, ..., \zeta$ in the picture $c_i \in B(C)^\circ$ appear at most $(m_i+1-\kappa_i)(n_i+1-\lambda_i)$ distinct elements of $B(C)$ with rank (κ_i, λ_i) , $1 \leqslant \kappa_i \leqslant m_i$, $1 \leqslant \lambda_i \leqslant n_i$. Therefore, in c_i appear at most

$$\begin{aligned} \sum_{\kappa_i=1}^{m_i} \sum_{\lambda_i=1}^{n_i} (m_i + 1 - \kappa_i)(n_i + 1 - \lambda_i) &= \sum_{\kappa_i=1}^{m_i} (m_i + 1 - \kappa_i) \sum_{\lambda_i=1}^{n_i} (n_i + 1 - \lambda_i) \\ &= \frac{m_i \cdot (m_i + 1)}{2} \cdot \frac{n_i \cdot (n_i + 1)}{2} \end{aligned}$$

distinct elements of $B(C)$ and so

$$CardB(C) \leqslant \sum_{i=1}^{\zeta} \frac{m_i(m_i + 1)}{2} \cdot \frac{n_i(n_i + 1)}{2}. \qquad \square$$

Remark. If all pictures of C have rank $(1, 1)$, then $B(C) = C$ and in this case we have $CardB(C) = CardC$.

Next proposition gives us more information about how the basis of the free hull is located.

Proposition 9. *Assume that $C \subseteq pict(X)$ and $C_p = C - \{p\}$ for some $p \in C$. If $B(C)^\circ$ is the free hull of C and $B(C_p)^\circ$ is the free hull of C_p, then $B(C)^\circ$ is also the free hull of $B(C_p) \cup \{p\}$.*

Proof. Assume that D° is the free hull of $B(C_p) \cup \{p\}$. From

$$C = C_p \cup \{p\} \subseteq B(C_p)^\circ \cup \{p\} \subseteq (B(C_p) \cup \{p\})^\circ \subseteq D^\circ$$

we get $B(C)^\circ \subseteq D^\circ$ because D° is free and $B(C)^\circ$ is the free hull of C.

Conversely, from $C_p \subseteq C \subseteq B(C)^\circ$ and the fact that $B(C_p)^\circ$ is the free hull of C_p we deduce that $B(C_p)^\circ \subseteq B(C)^\circ$. Hence, $B(C_p) \cup \{p\} \subseteq B(C_p)^\circ \cup \{p\} \subseteq B(C)^\circ$ and so $D^\circ \subseteq B(C)^\circ$ as wanted. $\square$

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