# PACKING OF $(0,1)$-MATRICES 

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#### Abstract

The Matrix Packing Down problem asks to find a row permutation of a given $(0,1)$-matrix in such a way that the total sum of the first non-zero column indexes is maximized. We study the computational complexity of this problem. We prove that the Matrix Packing Down problem is NP-complete even when restricted to zero trace symmetric $(0,1)$-matrices or to $(0,1)$-matrices with at most two 1 's per column. Also, as intermediate results, we introduce several new simple graph layout problems which are proved to be NP-complete.


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## 1. Introduction

Reordering of $(0,1)$-matrix is a very common task in computer science. For example, to minimize the amount of computation and storage for parallel sparse factorization, sparse matrices have to be reordered prior to factorization [18]. Another example is concerned with the minimization of the profile of a matrix [15]. The profile of a symmetric matrix is the number of elements in the envelope (the set of index pairs that lies between the first non-zero element in each row and the diagonal) plus the number of elements on the diagonal. The $L U$ factorization of a symmetric, positive definite matrix can be performed within the space given by the profile.

In a contribution to the tribute to Professor P. Erdös, H.S. Wilf asked about the complexity of finding permutations of the row and of the columns of a given square matrix such that after carrying out these permutations the obtained matrix is triangular [24]. Addressing this problem is important for job scheduling with precedence constraints, a well-known problem in theoretical computer science. H.S. Wilf concluded that the problem falls in difficulty between a known easy case

[^0]and a known hard case. In fact, the general problem of transforming a square $(0,1)$-matrix into a triangular matrix by permutations of the rows and columns is NP-hard as proved by B. DasGupta et al. [4] ${ }^{1}$. In the present paper, we consider the related but less restrictive problem of computing the maximum packing down of a given $(0,1)$-matrix by row permutation. Informally, the packing down of a $(0,1)$-matrix $A$ is computed by summing the first non-zero row indexes. In other words, we are asked to find a row permutation of $A$ in such a way that the sum of the first 0 's of each column is maximized. It is easily seen that if a given ( 0,1 )-matrix $A$ of order $n$ can be transformed, by independent permutations of its rows and then its columns, into a triangular matrix, then there exists a permutation matrix $P$ such that the packing down of $P A$ is greater or equal to $\frac{1}{2} n(n+1)$. The converse is false, as is shown by considering the following $(0,1)$-matrix:
\[

A=\left[$$
\begin{array}{llll}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1
\end{array}
$$\right]
\]

Unfortunately, as will be detailed below, we prove that the Matrix Packing Down problem is NP-complete even for sparse ( 0,1 )-matrices. More precisely, it is shown that this problem is NP-complete even when restricted to $(0,1)$-matrices with at most two 1's per column.

Another interest in that problem lies in its closed relationships with graph layout problems. Graph layout problems are a particular class of combinatorial optimization problems whose goal is to find a linear layout of an input graph in such a way that a certain objective function is optimized, see [6] for a recent survey. Because of their practical importance, there exist a lot of results with layout problems. For example, the Bandwidth problem corresponds to that of minimizing the bandwidth of a symmetric matrix by simultaneous row and column permutation [20,22], the Optimal Linear Arrangement problem asks to find a layout $\phi$ of a graph $G=(V, E)$ in such a way that $\sum_{\{u, v\} \in E}|\phi(u)-\phi(v)|$ is minimized $[8,10]$, the CuTwidth problem is concerned with the number of edges passing over a vertex when all vertices are aligned on a horizontal line [5,12] and the Vertex SEPARATION problem asks for a layout minimizing the maximum cut (number of vertices in the first partition connected to the second one) [13, 21]. Particular instances of the Matrix Packing Down problem may be easily rephrased in terms of graph layout problems. Indeed, restricted to zero trace symmetric $(0,1)$ matrices, the Matrix Packing Down is nothing but the following graph layout

[^1]problem: find a linear layout $\phi$ in such a way that $\sum_{u \in V} \min \{\phi(v) \mid v \in N(u)\}$ is maximized where $N(u)$ is the neighborhood of $u$, i.e., the set of vertices adjacent to $u$. This problem is proved to be NP-complete in the present paper by showing that the Matrix Packing Down problem remains NP-complete for zero trace symmetric $(0,1)$-matrices. We will prove more, namely that this graph layout problem is NP-complete even when restricted to split graphs, bipartite graphs and co-bipartite graphs, and hence to chordal graphs, comparability graphs and co-comparability graphs. Furthermore, as intermediate results, we introduce several new simple graph layout problems: (1) find a linear arrangement $\phi$ of the vertices of a graph $G=(V, E)$ such that $\sum_{\{u, v\} \in E} \max \{\phi(u), \phi(v)\}$ is maximized and (2) find a linear arrangement $\phi$ of the vertices of a graph $G=(V, E)$ such that $\sum_{\{u, v\} \in E} \min \{\phi(u), \phi(v)\}$ is maximized. All of these problems are hard; the first one is proved to be NP-complete even when restricted to planar graphs with maximum degree bounded by three.

This paper is organized as follows: we review the related notations and definitions used in this paper and introduce formally our problem in Section 2; in Section 3, it is shown that the Matrix Packing Down problem is NP-complete even when restricted to $(0,1)$-matrices with at most two 1 's per column; Section 4 deals with the case of zero trace symmetric $(0,1)$-matrices.

## 2. PRELIMINARIES

### 2.1. Notations and definitions

Let us resume in this subsection the principal notations and definitions. For any positive integer $n$, we will use $\llbracket n \rrbracket$ to denote the set $\{1,2, \ldots, n\}$. Let $A=\left[a_{i, j}\right]$, $1 \leq i \leq m, 1 \leq j \leq n$, be a matrix of $m$ rows and $n$ columns. We say that $A$ is of size $m$ by $n$. In the case that $m=n$ then the matrix is square of order $m$. A line of $A$ is either a row or a column of $A$. We will be concerned with matrices whose entries consist exclusively of the integers 0 and 1 . Such matrices are referred to as $(0,1)$-matrices. We always designate a zero matrix of size $m$ by $n$ by $0_{m, n}$ or, if no ambiguity arises, simply by 0 , and a matrix of size $m$ by $n$ with every entry equal to 1 by $J_{m, n}$. The identity matrix of order $n$ is denoted $I_{n}$, and $K_{n}$ stands for $J_{n}-I_{n}$. A permutation matrix is a square matrix which has exactly one entry 1 in each row and column and all other entries 0 . We will denote by $\Pi_{n}$ the set of permutation matrices of order $n$.

A graph $G$ consists of a finite set $V=\left\{u_{1}, u_{2}, \ldots\right\}$ of elements called vertices together with a prescribed set $E$ of undirected pair of distinct vertices of $V$. The number $n$ of elements in $V$ is called the order of the graph. An element $e=\{u, v\}$ of $E$ is called an edge with endpoints $u$ and $v$, and $e$ is incident with both $u$ and $v$. Two distinct vertices are adjacent if there exists an edge $e=\{u, v\} \in E$. The neighbor of $u \in V$ is the set $N(u)=\{v \in V \mid \exists\{u, v\} \in E\}$. The complement of graph $G=(V, E)$ is the graph $\bar{G}=(V, \bar{E})$ where $\bar{E}$ is the set of all pairs of distinct vertices which are not edges in $G$. Let $V=\left\{u_{1}, u_{2}, \ldots, u_{n}\right\}$ and $E=$
$\left\{e_{1}, e_{2}, \ldots, e_{m}\right\}$. The vertex-edge incidence matrix [7] of $G$ is the ( 0,1 )-matrix $A=\left[a_{i, j}\right]$ of size $n$ by $m$ where

$$
a_{i, j}= \begin{cases}1 & \text { when edge } e_{j} \text { is incident with vertex } u_{i}, \text { i.e., } u_{i} \in e_{j} \\ 0 & \text { otherwise. }\end{cases}
$$

A graph $G=(V, E)$ is bipartite if its vertices can be decomposed into two disjoint sets such that no two vertices within the same set are adjacent. Clearly, bipartite graphs form a subclass of comparability graphs (for a deeper discussion of comparability graphs we refer the reader to [14]). The graph $G$ is a co-bipartite graph if it is the complement of a bipartite graph. An undirected graph is called chordal if every cycle of length greater than three possesses a chord, that is, an edge joining two non consecutive vertices of the cycle $[1,2,11,14,19]$. A split graph is a chordal graph with a chordal complement [14, 16]; this terminology arises because a graph $G$ is a split graph if there is a partition $V=K \cup I$ where $K$ is a clique and $I$ is an independent set, i.e., $G$ can be split into a clique and an independent set. A vertex cover $V^{\prime}$ for $G$ is a set of vertices in $G$ such that every edge in $E$ has at least one endpoint in $V^{\prime}$. The Vertex Cover problem is to construct for a given graph a vertex cover of the minimum number of vertices. A layout of a graph $G=(V, E)$ of order $n$ is a bijective function $\phi: V \rightarrow \llbracket n \rrbracket$. We denote by $\Phi(G)$ the set of all layouts of the graph $G$. A layout is also called a linear arrangement $[10,23]$ or a linear labeling [17] of the vertices of the graph. For a recent account of the theory of graph layout problems, we refer the reader to [6].

### 2.2. The Matrix Packing Down problem

We introduce formally in this subsection the Matrix Packing Down problem and the related Matrix Packing Right problem. Let $A=\left[a_{i, j}\right]$ be a $(0,1)$-matrix of size $m$ by $n$. We will denote by $\mathrm{R}_{A}: \llbracket m \rrbracket \rightarrow \llbracket n+1 \rrbracket$ and $\mathrm{C}_{A}: \llbracket n \rrbracket \rightarrow \llbracket m+1 \rrbracket$ the mappings defined as follows:

$$
\begin{aligned}
& \forall i, 1 \leq i \leq m, \quad \mathrm{R}_{A}(i)= \begin{cases}j & \text { if } a_{i, j}=1 \text { and } a_{i, k}=0 \text { for all } 1 \leq k<j \\
n+1 & \text { if } a_{i, j}=0 \text { for all } 1 \leq j \leq n\end{cases} \\
& \forall j, 1 \leq j \leq n, \quad \mathrm{C}_{A}(j)= \begin{cases}i & \text { if } a_{i, j}=1 \text { and } a_{k, j}=0 \text { for all } 1 \leq k<i \\
m+1 & \text { if } a_{i, j}=0 \text { for all } 1 \leq i \leq m\end{cases}
\end{aligned}
$$

In other words, $\mathrm{R}_{A}(i)$ (resp. $\mathrm{C}_{A}(j)$ ) is the least column index $j$ (resp. the least row index $i$ ) such that $a_{i, j}=1$. Observe that $\mathrm{R}_{A}(i)=n+1\left(\right.$ resp. $\left.\mathrm{C}_{A}(j)=m+1\right)$ if row $i$ (resp. column $j$ ) contains only 0 's.

For the purpose of packing, convenient forms of matrices are needed. A ( 0,1 )-matrix $A$ of size $m$ by $n$ is in row standard form if $\mathrm{R}_{A}(i) \geq \mathrm{R}_{A}(j)$ for all $1 \leq i \leq j \leq m$. The matrix $A$ is in column standard form if $C_{A}(i) \geq C_{A}(j)$ for all
$1 \leq i \leq j \leq n$. The packing down of $A$, denoted by $\operatorname{pd}(A)$, is defined by

$$
\operatorname{pd}(A)=\sum_{1 \leq j \leq n} \mathrm{C}_{A}(j)
$$

Likewise, the packing right of $A$, denoted by $\operatorname{pr}(A)$, is defined by

$$
\operatorname{pr}(A)=\sum_{1 \leq i \leq m} \mathrm{R}_{A}(i)
$$

In others words, the packing down of $A$ is concerned with summing the first nonzero row indexes and the packing right of $A$ consists in summing the first non-zero column indexes.

Example 2.1. Let $A_{1}, A_{2}$ and $A_{3}$ be the ( 0,1 )-matrices defined by

$$
A_{1}=\left[\begin{array}{llll}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0
\end{array}\right] \quad A_{2}=\left[\begin{array}{llll}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0
\end{array}\right] \quad A_{3}=\left[\begin{array}{llll}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0
\end{array}\right]
$$

Then, $A_{1}$ is in row standard form, $A_{2}$ is in column standard form and $A_{3}$ is in row-column standard form. Furthermore, $\operatorname{pd}\left(A_{3}\right)=4+4+2+1=11$ and $\operatorname{pr}\left(A_{3}\right)=4+3+3+1=11$.

Here are some elementary properties of these concepts.
Property 1. Let $A$ be a $(0,1)$-matrix of size $m$ by $n$. Then, $\operatorname{pd}(A)=\operatorname{pd}(A Q)$ (resp. $\operatorname{pr}(A)=\operatorname{pr}(P A)$ ) for all $Q \in \Pi_{n}$ (resp. $P \in \Pi_{m}$ ).

The above property might be rephrased to emphasis on the strong relationships between packing of matrices and graph layout problems.
Property 2. Let $A$ be a $(0,1)$-matrix of order $n$. Then, $\operatorname{pd}(P A)=\operatorname{pd}\left(P A P^{T}\right)$ for all $P \in \Pi_{n}$.

We now introduce our main problem. The Matrix Packing Down problem asks to find a row permutation of a given $(0,1)$-matrix in such a way that the total sum of the first non-zero row indexes is maximized. Formally, this problem (in its natural decision version) is defined as follows :

## Matrix Packing Down

Instance: $A(0,1)$-matrix $A$ of size $m$ by $n$ and a positive integer $K$.
Question: Is there $P \in \Pi_{m}$ such that $\operatorname{pd}(P A) \geq K$ ?
The related Matrix Packing Right problem is defined analogously by replacing the above question by the question: "Is there $Q \in \Pi_{n}$ such that $\operatorname{pr}(A Q) \geq K$ ?". In the present paper we consider the problem of the existence of a polynomial-time algorithm for finding such a line permutation. Unfortunately, as we shall prove
in Sections 3 and 4, the Matrix Packing Down problem is NP-complete even when restricted to $(0,1)$-matrices with at most two 1 's per column or to zero trace symmetric $(0,1)$-matrices.

## 3. The packing down of $(0,1)$-matrix

We prove in this section that the Matrix Packing Down problem is NP-complete even when restricted to $(0,1)$-matrices with at most two 1 's per column. Combinatorial matrix theory includes a lot of graph theory [3]. It is therefore not surprising that our main tools to study the complexity of the Matrix Packing Down problem are graph layout problems. Let us start by considering the following problem.

## Max Min Edges

Instance: A graph $G=(V, E)$ and a positive integer $J$.
Question: Is there $\phi \in \Phi(G)$ such that $\sum_{e \in E} \min \{\phi(u) \mid u \in e\} \geq J$ ?
Most of the interest in the Max Min Edges problem stems from the following proposition.
Proposition 3.1. The Max Min Edges problem polynomially transforms to the Matrix Packing Down problem.

Proof. Let an arbitrary instance of the Max Min Edges problem be given by a graph $G=(G, E)$ and by a positive integer $J$. The corresponding instance of the Matrix Packing Down problem is given by the $n$ by $m$ vertex-edge incidence matrix $A$ of $G$. It is easily seen that there exists $\phi \in \Phi(G)$ such that $\sum_{e \in E} \min \{\phi(u) \mid u \in e\} \geq J$ if and only if there exists $P \in \Pi_{n}$ such that $\operatorname{pd}(P A) \geq J$.

It remains to prove that the Max Min Edges problem is NP-complete. This will be divided into two steps. Let us first consider a new related graph layout problem, namely the Max Max Edges problem, defined as follows.

```
Max Max Edges
Instance: A graph G = (V,E) and a positive integer J.
Problem: Is there }\phi\in\Phi(G)\mathrm{ such that }\mp@subsup{\sum}{e\inE}{}\operatorname{max}{\phi(u)|u\ine}\geqJ\mathrm{ ?
```

Observe that the Max Max Edges problem only differs from the Max Min Edges problem in the vertex taking into account for each edge. We prove that this problem is NP-complete even when restricted to planar graphs with maximum degree 3 by presenting a polynomial-time reduction from the VERTEX COVER problem restricted to cubic planar graphs, which is known to be NP-complete [9].

Proposition 3.2. The Max Max Edges problem is NP-complete even when restricted to planar graphs with maximum degree 3 .

Proof. We shall transform the Vertex Cover problem to the Max Max Edges problem. We will use the proof technique of Garey et al. [10] who showed that the Simple Optimal Linear Arrangement problem is NP-complete by exhibing a polynomial-time reduction from the Max Cut problem. Here, the basic idea is to show that there exists a mapping which maps the elements of an optimal vertex cover into a set of consecutive integers.

The Max Max Edges problem is obviously in NP. Let an arbitrary instance of the VERTEX COVER problem be given by a planar cubic graph $G_{1}=\left(V_{1}, E_{1}\right)$ of order $n$ and by a positive integer $K$. Of course, there is no loss of generality if we assume $K>1$. For convenience, write $p=6 n(K-1)$. Let $G_{2}=\left(V_{2}, E_{2}\right)$ be a new graph defined as follows:

$$
\begin{aligned}
V_{2} & =\left\{u_{1}, u_{2}, \ldots, u_{p}\right\} \\
E_{2} & =\left\{\left\{u_{i}, u_{j}\right\} \mid i+j=p+1\right\} .
\end{aligned}
$$

The corresponding instance for the Max Max Edges problem is given by a new graph $H=(W, F)$ defined by $W=V_{1} \cup V_{2}$ and $F=E_{1} \cup E_{2}$. Observe that $H$ is a planar graph with maximum degree 3 . Our construction is completed by setting $J=\frac{1}{4} p\left(\frac{3}{2} p+1\right)+\frac{1}{2} p(n-K)+\frac{3}{2} n(n-K+p+1)$. It is easily seen that $|W|=n+p$ and $|F|=\frac{1}{2}(3 n+p)$, and that our construction can be carried on in polynomial-time.

We claim that $G_{1}$ has a vertex cover of size $K$ if and only if there exists $\phi \in \Phi(H)$ such that $\sum_{e \in F} \max \{\phi(u) \mid u \in e\} \geq J$. For the sake of clarity, let us introduce the temporary notations

$$
\omega_{i}(\phi)=\sum_{e \in E_{i}} \max \{\phi(u) \mid u \in e\}
$$

for $1 \leq i \leq 2$, and $\omega(\phi)=\omega_{1}(\phi)+\omega_{2}(\phi)$.
Suppose that $G_{1}$ has a vertex cover $V_{1}^{\prime} \subseteq V_{1}$ of size $K$. Consider a one-to-one mapping $\phi \in \Phi(H)$ satisfying the following conditions:

$$
\begin{cases}1 \leq \phi(u) \leq n-K & \text { for } u \in V_{1}-V_{1}^{\prime} \\ n-K+1 \leq \phi\left(u_{i}\right) \leq n-K+\frac{1}{2} p & \text { for } u_{i} \in V_{2} \text { and } 1 \leq i \leq \frac{1}{2} p \\ n-K+\frac{1}{2} p+1 \leq \phi\left(u_{i}\right) \leq n-K+p & \text { for } u_{i} \in V_{2} \text { and } \frac{1}{2} p+1 \leq i \leq p \\ n-K+p+1 \leq \phi(u) \leq n+p & \text { for } u \in V_{1}^{\prime}\end{cases}
$$

Since $V_{1}^{\prime}$ is a vertex cover in $G_{1}$, we have $\max \{\phi(u) \mid u \in e\} \geq n-K+p+1$ for all $e \in E_{1}$. This observation coupled with the fact that a cubic graph of order $n$ has $\frac{3}{2} n$ edges yields

$$
\omega_{1}(\phi) \geq \frac{3}{2} n(n-K+p+1)
$$

Moreover, an easy computation shows that

$$
\omega_{2}(\phi)=\frac{1}{4} p\left(\frac{3}{2} p+1\right)+\frac{1}{2} p(n-K)
$$

and hence

$$
\begin{aligned}
\omega(\phi)=\omega_{1}(\phi)+\omega_{2}(\phi) & \\
& \geq \frac{1}{4} p\left(\frac{3}{2} p+1\right)+\frac{1}{2} p(n-K)+\frac{3}{2} n(n-K+p+1) \\
& =J
\end{aligned}
$$

Conversely, suppose that there exists a one-to-one mapping $\phi \in \Phi(H)$ such that $\sum_{e \in F} \max \{\phi(u) \mid u \in e\} \geq J$. Define

$$
W^{*}=\max _{\phi \in \Phi(H)} \omega(\phi)
$$

and

$$
\Phi^{*}(H)=\left\{\phi \in \Phi(H) \mid \omega(\phi)=W^{*}\right\}
$$

Clearly, $W^{*} \geq J$ and $\Phi^{*}(H)$ is non empty. We claim that there is at least one mapping $\phi \in \bar{\Phi}^{*}(H)$ which maps the vertices of $V_{2}$ to a set of consecutive integers, thereby partitioning $V_{1}$ into those vertices that go before and those that come afterwards. For each $\phi \in \Phi^{*}(H)$, define the set

$$
S(\phi)=\left\{u \in V_{1} \mid \exists u_{i}, u_{j} \in V_{2}, \quad \phi\left(u_{i}\right)<\phi(u)<\phi\left(u_{j}\right)\right\} .
$$

Then, there exists a one-to-one mapping $\phi \in \Phi^{*}(H)$ such that $|S(\phi)| \leq\left|S\left(\phi^{\prime}\right)\right|$ for all $\phi^{\prime} \in \Phi^{*}(H)$. We show that $|S(\phi)|=0$, and hence that $\phi$ is our desired mapping. Suppose, for the sake of contradiction, that $|S(\phi)|>0$. Let $u_{\max } \in S(\phi)$ be such that $\phi\left(u_{\max }\right) \geq \phi(u)$ for all $u \in S(\phi)$. Consider the following cases; cf. Figures 1a and b :
(1) If $\phi\left(u_{\max }\right)<\phi(u)$ for all $u \in N\left(u_{\max }\right)$. Let $v_{\min } \in V_{2}$ be such that $\phi\left(v_{\text {min }}\right) \leq \phi(v)$ for all $v \in V_{2}$. Consider the new mapping $\phi^{\prime} \in \Phi(H)$ which is identical to $\phi$ except that $\phi^{\prime}\left(u_{\max }\right)=\phi\left(v_{\min }\right)$ and $\phi^{\prime}\left(v_{\min }\right)=\phi\left(u_{\max }\right)$. It is easy to check that $\omega\left(\phi^{\prime}\right) \geq \omega(\phi)$ and $\left|S\left(\phi^{\prime}\right)\right|<|S(\phi)|$, which contradict the choice of $\phi$.
(2) If $\phi\left(u_{\max }\right)>\phi(u)$ for at least one $u \in N\left(u_{\max }\right)$. Let $v_{\max } \in V_{2}$ be such that $\phi\left(v_{\max }\right) \geq \phi(v)$ for all $v \in V_{2}$. Consider the new mapping $\phi^{\prime} \in \Phi(H)$ which is identical to $\phi$ except that $\phi^{\prime}\left(u_{\max }\right)=\phi\left(v_{\max }\right)$ and $\phi^{\prime}\left(v_{\max }\right)=\phi\left(u_{\max }\right)$. We check at once that $\omega\left(\phi^{\prime}\right) \geq \omega(\phi)$ and $\left|S\left(\phi^{\prime}\right)\right|<|S(\phi)|$, which contradict, once again, the choice of $\phi$.


Figure 1. The Max Max Edges problem: the vertices of $V_{2}$ are mapped by $\phi$ to a set of consecutive integers in Proposition 3.2. (a) $N\left(u_{\max }\right) \subseteq A$, (b) there exists $u \in N\left(u_{\max }\right)$ such that $u \in B$ and (c) the partition $\left(V_{1}^{\prime}, V_{1}^{\prime \prime}\right)$ defined by the mapping $\phi$.

Therefore, we must have $|S(\phi)|=0$, and hence the vertices of $V_{2}$ are mapped by $\phi$ to a set of consecutive integers. Define a partition $V_{1}=V_{1}^{\prime} \cup V_{1}^{\prime \prime}$ by

$$
\begin{array}{rlrl}
V_{1}^{\prime} & =\left\{u \in V_{1} \mid \forall u_{i} \in V_{2},\right. & & \left.\phi(u)>\phi\left(u_{i}\right)\right\} \\
V_{1}^{\prime \prime} & =\left\{u \in V_{1} \mid \forall u_{i} \in V_{2},\right. & \left.\phi(u)<\phi\left(u_{i}\right)\right\}
\end{array}
$$

and let $\left|V_{1}^{\prime}\right|=q ; c f$. Figure 1c. We claim that $q \leq K$. First, a trivial verification shows that

$$
\begin{equation*}
\omega_{2}(\phi) \leq \frac{1}{4} p\left(\frac{3}{2} p+1\right)+\frac{1}{2} p(n-q) . \tag{1}
\end{equation*}
$$

Furthermore, observe now that there is no loss of generality in assuming that

$$
\begin{cases}n-q+1 \leq \phi\left(u_{i}\right) \leq n-K+\frac{1}{2} p & \text { for all } u \in V_{2} \text { and } 1 \leq i \leq \frac{1}{2} p \\ n-q+\frac{1}{2} p+1 \leq \phi\left(u_{i}\right) \leq n-q+p & \text { for all } u \in V_{2} \text { and } \frac{1}{2} p+1 \leq i \leq p\end{cases}
$$

with the result that equality holds in (1). Then it follows that

$$
\begin{aligned}
\omega(\phi) & =\omega_{1}(\phi)+\omega_{2}(\phi) \\
& =\omega_{1}(\phi)+\frac{1}{4} p\left(\frac{3}{2} p+1\right)+\frac{1}{2} p(n-q) \\
& \geq J \\
& =\frac{1}{4} p\left(\frac{3}{2} p+1\right)+\frac{1}{2} p(n-K)+\frac{3}{2} n(n-K+p+1) .
\end{aligned}
$$

Therefore

$$
\omega_{1}(\phi) \geq \frac{1}{2} p(q-K)+\frac{3}{2} n(n-K+p+1)
$$

But

$$
\omega_{1}(\phi) \leq \frac{3}{2} n(n+p)
$$

which follows from the fact $G_{1}$ is a cubic graph. As a result, we are left with $3 n(K-1) \geq p(q-K)$. Replacing $p$ by $6 n(K-1)$ yields $\frac{1}{2} \geq q-K$, and we deduce that $q \leq K$.

It remains to prove that $V_{1}^{\prime}$ is indeed a vertex cover in $G_{1}$. Suppose, for the sake of contradiction, that there exists an edge $\{u, v\} \in E_{1}$ such that both $u$ and $v$ belong to $V_{1}^{\prime \prime}$. On the one hand,

$$
\omega(\phi) \leq(n-q)+\frac{1}{4} p\left(\frac{3}{2} p+1\right)+\frac{1}{2} p(n-q)+\left(\frac{3}{2} n-1\right)(n+p)
$$

On the other hand,

$$
\omega(\phi) \geq \frac{1}{4} p\left(\frac{3}{2} p+1\right)+\frac{1}{2} p(n-K)+\frac{3}{2} n(n-K+p+1) .
$$

Therefore,

$$
(n-q)+\frac{1}{2} p(n-q)-(n+p) \geq \frac{1}{2} p(n-K)-\frac{3}{2} n(K-1)
$$

and hence

$$
p \leq \frac{3 n(K-1)-2 q}{q-K+2} \leq \frac{3 n(K-1)}{q-K+2}
$$

Combining this with $q \leq K$ yields $p \leq 3 n(K-1)$. This is the desired contradiction since $p=6 n(K-1)>3 n(K-1)$, where the inequality follows from $K>1$. Therefore we must have $u \in V_{1}^{\prime}$ or $v \in V_{1}^{\prime}$ (possibly both) for all $\{u, v\} \in E_{1}$. Then it follows that $V_{1}^{\prime}$ is a vertex cover of size at most $K$ in $G_{1}$. The reduction is proved.

We proceed to show that the Max Min Edges problem is NP-complete. We need some additional notations and easy lemmas. Let $g$ be the function defined
by $g(n)=\frac{1}{3} n\left(n^{2}-1\right)$ for any positive integer $n$. Observe that, for any graph $G$ of order $n$ and any mapping $\phi \in \Phi(G)$, the following inequality holds

$$
\begin{equation*}
\sum_{e \in E} \max \{\phi(u) \mid u \in e\} \leq \frac{1}{3} n\left(n^{2}-1\right) \tag{2}
\end{equation*}
$$

Furthermore, $G$ and $K_{n}$ are isomorphic if and only if (2) holds as equality. This observation coupled with the fact that $G=(V, E \cup \bar{E})$ is isomorphic to $K_{n}$ yields the following lemma.
Lemma 3.3. Let $G=(V, E)$ be a graph of order $n$. Then

$$
\sum_{e \in E} \max \{\phi(u) \mid u \in e\}+\sum_{e \in \bar{E}} \max \{\phi(u) \mid u \in e\}=g(n)
$$

for all $\phi \in \Phi(G)$.
Lemma 3.4. Let $G=(V, E)$ be a graph, $|V|=n$ and $|E|=m$, and $J$ be a positive integer. There exists $\phi \in \Phi(G)$ such that $\sum_{e \in E} \max \{\phi(u) \mid u \in e\} \leq J$ if and only if there exists $\phi \in \Phi(G)$ such that $\sum_{e \in E} \min \{\phi(u) \mid u \in e\} \geq m(n+1)-J$.
Proposition 3.5. The Max Min Edges problem is NP-complete.
Proof. It is easily seen that the Max Min Edges problem is in NP. Let an arbitrary instance of the Max Max Edges problem be given by a graph $G=$ $(G, E)$ with $n$ vertices and $m$ edges and by a positive integer $K$. The corresponding instance for the Max Min Edges problem is given by the graph $\bar{G}=(V, \bar{E})$. Our construction can be carried on in polynomial-time.

Now, according to Lemmas 3.3 and 3.4, it is a simple matter to check that there exists $\phi \in \Phi(G)$ such that $\sum_{e \in E} \max \{\phi(u) \mid u \in e\} \geq J$ if and only if there exists $\phi^{\prime} \in \Phi(\bar{G})$ such that $\sum_{e \in \bar{E}} \min \left\{\phi^{\prime}(u) \mid u \in e\right\} \geq m(n+1)-g(n)+J$ which proves the proposition.

We can now formulate our main result concerning the Matrix Packing Down problem for sparse $(0,1)$-matrices.
Corollary 3.6. The Matrix Packing Down problem is NP-complete even when restricted to $(0,1)$-matrices with at most two 1's per column.

Proof. The NP-completeness of the Matrix Packing Down problem follows from Propositions 3.1 and 3.5. Moreover, the vertex-edge incidence matrix of a graph has exactly two 1's per column.

## 4. The packing down of a symmetric $(0,1)$-matrix

### 4.1. Introduction

We have shown in the preceding section that the Matrix Packing Down problem is NP-complete. We now prove a strengthening of this result, namely
that the Matrix Packing Down problem remains NP-complete for zero trace symmetric $(0,1)$-matrices. Furthermore, we exhibit strong relationships between this problem and a new simple graph layout problem. Indeed, according to Property 2, if $A$ is a square matrix of order $n$ then $\operatorname{pd}(P A)=\operatorname{pd}\left(P A P^{T}\right)$ for all $P \in \Pi_{n}$. As an immediate result, restricted to zero trace symmetric matrices, the Matrix Packing Down is nothing but the following graph layout problem:

```
Max Min Neighbor
Instance: A graph G=(V,E) and a positive integer K.
Question: Is there }\phi\in\Phi(G)\mathrm{ such that }\mp@subsup{\sum}{u\inV}{}\operatorname{min}{\phi(v)|v\inN(u)}\geqK\mathrm{ ?
```

In the sequel, we adhere to the convention that $\min \{\phi(v) \mid v \in N(u)\}=n+1$ if $u$ is an isolated vertex in $G$ for all $\phi \in \Phi(G)$.

Proposition 4.1. The Max Min Neighbor problem polynomially transforms to the symmetric Matrix Packing Down problem.

Proof. Immediate from the above discussion.
We are thus reduced to studying the computational complexity of the Max Min Neighbor problem. Unfortunately, as we shall prove in this section, the Max Min Neighbor problem is NP-complete. We will prove more, namely that this problem is NP-complete even when restricted to split graphs, bipartite graphs and co-bipartite graphs, and hence to chordal graphs, comparability graphs and co-comparability graphs.

### 4.2. Hardness results

In order to prove the NP-completeness of the Max Min Neighbor problem, we need some technical results that may be of independent interest.

Lemma 4.2. Let $A$ be a $(0,1)$-matrix of size $m$ by $n$. Then

$$
\begin{equation*}
\max _{P \in \Pi_{m}} \operatorname{pd}(P A)+m=\max _{Q \in \Pi_{n}} \operatorname{pr}(A Q)+n \tag{3}
\end{equation*}
$$

Proof. We begin by proving that $\max _{P \in \Pi_{m}} \operatorname{pd}(P A)-\max _{Q \in \Pi_{n}} \operatorname{pr}(A Q) \leq n-m$. Let $P_{\text {opt }} \in \Pi_{m}$ be such that $\operatorname{pd}\left(P_{\text {opt }} A\right)=\max _{P \in \Pi_{m}} \operatorname{pd}(P A)$. Now, let $Q_{\text {std }} \in \Pi_{n}$ be such that $P_{\mathrm{opt}} A Q_{\mathrm{std}}$ is in column standard form. According to Property 1, $\operatorname{pd}\left(P_{\text {opt }} A\right)=\operatorname{pd}\left(P_{\text {opt }} A Q_{\text {std }}\right)$. Suppose that $A$ contains $m^{\prime}$ rows and $n^{\prime}$ columns with at least one 1. Since $\operatorname{pd}\left(P_{\text {opt }} A\right)$ is maximal and $P_{\mathrm{opt}} A Q_{\text {std }}$ is in column standard form, we know that $P_{\mathrm{opt}} A Q_{\text {std }}$ is of the form

$$
P_{\mathrm{opt}} A Q_{\mathrm{std}}=\left[\begin{array}{cc}
0 & 0 \\
0 & A^{\prime}
\end{array}\right]
$$

where $A^{\prime}$ is a sub-matrix of size $m^{\prime}$ by $n^{\prime}$ with no row or column of 0 's. Moreover, there exist an integer $t \geq 1$ such that $A^{\prime}$ is of the form

$$
A^{\prime}=\left[\begin{array}{cccc}
0 & 0 & \cdots & A_{t} \\
\vdots & \vdots & . & \vdots \\
0 & A_{2} & \cdots & A_{2, t} \\
A_{1} & A_{1,2} & \cdots & A_{1, t}
\end{array}\right]
$$

where each $A_{i}, 1 \leq i \leq t$, is a sub-matrix of size $m_{i}$ by $n_{i}$ whose first row contains no 0 . Of course, $m_{1}+m_{2}+\ldots+m_{t}=m^{\prime}$ and $n_{1}+n_{2}+\ldots+n_{t}=n^{\prime}$. We claim that $A_{i}=J_{m_{i}, n_{i}}$ for all $1 \leq i \leq t$, i.e., each $A_{i}$ is the all 1 's matrix of size $m_{i}$ by $n_{i}$. Indeed, suppose, for the sake of contradiction, that there exists an integer $i, 1 \leq i \leq t$, such that the equality $A_{i}=J_{m_{i}, n_{i}}$ does not hold. Since the first row of $A_{i}$ contains no 0 , then it follows that $m_{i}>1$ and that there exists a row in $A_{i}$, say $k(k>1)$, which contains a 0 . Consider the permutation matrix $P \in \Pi_{m}$ which is identical to $P_{\text {opt }}$ except that rows 1 and $k$ in $A_{i}$ are permuted. A careful examination of $P A$ shows that $\operatorname{pd}(P A)>\operatorname{pd}\left(P_{\mathrm{opt}} A\right)$ which contradict the fact that $\operatorname{pd}\left(P_{\mathrm{opt}} A\right)$ is maximal. Therefore we must have $A_{i}=J_{m_{i}, n_{i}}$ for all $1 \leq i \leq t$. From this, observe that there is no loss of generality in assuming that $P_{\mathrm{opt}}$ and $Q_{\text {std }}$ form a lexical numbering of $A$, that is, for any pair of rows $i_{1}$, $i_{2}$ such that row $i_{1}$ comes before row $i_{2}$, the last column in which the entries differ has a 0 in row $i_{1}$ and a 1 in row $i_{2}$.

In order to establish the desired inequality, it is convenient to use a geometric argument on the matrix $P_{\text {opt }} A Q_{\text {std }}$. Indeed, according to the above, $P_{\mathrm{opt}} A Q_{\text {std }}$ is of the form

where $S$ is an polygon of 0's and every black rectangle is composed of 1's. Our objective is to evaluate the surface of $S$. We see at once that

$$
\operatorname{pd}\left(P_{\mathrm{opt}} A Q_{\mathrm{std}}\right)-n=S=\operatorname{pr}\left(P_{\mathrm{opt}} A Q_{\mathrm{std}}\right)-m
$$

On the one hand, $\operatorname{pd}\left(P_{\text {opt }} A Q_{\text {std }}\right)=\max _{P \in \Pi_{m}} \operatorname{pd}\left(P A Q_{\text {std }}\right)=\max _{P \in \Pi_{m}} \operatorname{pd}(P A)$. On the other hand, $\operatorname{pr}\left(P_{\mathrm{opt}} A Q_{\mathrm{std}}\right) \leq \max _{Q \in \Pi_{n}} \operatorname{pr}\left(P_{\mathrm{opt}} A Q\right)=\max _{Q \in \Pi_{n}} \operatorname{pr}(A Q)$. Then it follows that $\max _{P \in \Pi_{m}} \operatorname{pd}(P A)-\max _{Q \in \Pi_{n}} \operatorname{pr}(A Q) \leq n-m$.

We now apply this construction again, with $P_{\text {opt }} \in \Pi_{m}$ replaced by $Q_{\text {opt }} \in \Pi_{n}$ such that $\operatorname{pr}\left(A Q_{\text {opt }}\right)$ is maximized to obtain $\max _{P \in \Pi_{m}} \operatorname{pd}(P A)-$ $\max _{Q \in \Pi_{n}} \operatorname{pr}(A Q) \geq n-m$, and the proof is complete.

Let us now briefly mention some important consequences of Lemma 4.2.
Corollary 4.3. Let $A$ be a (0,1)-matrix of order $n$. Then

$$
\max _{P \in \Pi_{n}} \operatorname{pd}(P A)=\max _{Q \in \Pi_{n}} \operatorname{pd}\left(Q A^{T}\right)
$$

Proof. Restricted to $(0,1)$-matrices of order $n$, Lemma 4.2 is nothing but the statement that $\max _{P \in \Pi_{n}} \operatorname{pd}(P A)=\max _{Q \in \Pi_{n}} \operatorname{pr}(A Q)$. The result follows if we note that $\operatorname{pd}(P A)=\operatorname{pr}\left(A^{T} P^{T}\right)$ for all $P \in \Pi_{n}$.

Corollary 4.4. The Matrix Packing Down problem is NP-complete even when restricted to $(0,1)$-matrices with at most two 1 's per row.

Proof. Let $A=\left[a_{i, j}\right]$ be a $(0,1)$-matrix of size $m$ by $n$ with at most two 1 's per column. Consider the ( 0,1 )-matrix $B=\left[b_{i, j}\right]$ of size $n$ by $m$ with at most two 1 's per row defined by $B=A^{T}$. Then

$$
\begin{align*}
\max _{P \in \Pi_{n}} \operatorname{pd}(P B) & =n-m+\max _{Q \in \Pi_{m}} \operatorname{pr}(B Q)  \tag{4}\\
& =n-m+\max _{P \in \Pi_{m}} \operatorname{pd}(P A) \tag{5}
\end{align*}
$$

where (4) follows from Lemma 4.2; to deduce (5) from (4), consider a quarter-turn of $B$. According to Proposition 3.2, computing the right-hand side of (5) is an NP-complete problem, which proves the corollary.

It remains open however whether the Matrix Packing Down problem is NP-hard when restricted to $(0,1)$-matrices with fixed number of 1's per row and per column. The connections of the Matrix Packing down problem for zero trace symmetric matrices with graph classes are strengthened by the following three lemmas.
Lemma 4.5. Let $A$ be a (0,1)-matrix of order $n>1$ and $B$ be a zero trace symmetric $(0,1)$-matrix of order $3 n$ defined by

$$
B=\left[\begin{array}{ccc}
0 & 0 & A \\
0 & K_{n} & J_{n} \\
A^{T} & J_{n} & K_{n}
\end{array}\right]
$$

Then, there exists $P \in \Pi_{n}$ such that $\operatorname{pd}(P A) \geq k$ if and only there exists $S \in \Pi_{3 n}$ such that $\operatorname{pd}(S B) \geq 3 n^{2}+n+1+2 k$.

Proof. Suppose that there exists $P \in \Pi_{n}$ such that $\operatorname{pd}(P A) \geq k$. Let $Q \in$ $\Pi_{n}$ be such that $\operatorname{pd}\left(Q A^{T}\right)=\max _{R \in \Pi_{n}} \operatorname{pd}\left(R A^{T}\right)$. According to Corollary 4.3,
$\operatorname{pd}\left(Q A^{T}\right)=\max _{R \in \Pi_{n}} \operatorname{pd}(R A) \geq \operatorname{pd}(P A)$, and hence $\operatorname{pd}\left(Q A^{T}\right) \geq k$. Consider the permutation matrix $S \in \Pi_{3 n}$ defined as follow:

$$
S=\left[\begin{array}{ccc}
P & 0 & 0 \\
0 & I_{n} & 0 \\
0 & 0 & Q
\end{array}\right]
$$

Then

$$
S B S^{T}=\left[\begin{array}{ccc}
0 & 0 & P A Q^{T} \\
0 & K_{n} & J_{n} \\
Q A^{T} P^{T} & J_{n} & K_{n}
\end{array}\right]
$$

and hence

$$
\begin{aligned}
\operatorname{pd}(S B) & =\operatorname{pd}\left(S B S^{T}\right) \\
& =\left(2 n^{2}+\operatorname{pd}\left(Q A^{T} P^{T}\right)\right)+\left(n^{2}+\operatorname{pd}\left(K_{n}\right)\right)+\left(\operatorname{pd}\left(P A Q^{T}\right)\right) \\
& =3 n^{2}+n+1+\operatorname{pd}\left(Q A^{T}\right)+\operatorname{pd}(P A) \\
& \geq 3 n^{2}+n+1+2 k
\end{aligned}
$$

Conversely, suppose that there exists $S \in \Pi_{3 n}$ such that $\operatorname{pd}(S A) \geq 3 n^{2}+n+1+2 k$. Observe that there is no loss of generality in assuming that $S$ is of the form:

$$
S=\left[\begin{array}{ccc}
P & 0 & 0 \\
0 & I_{n} & 0 \\
0 & 0 & Q
\end{array}\right]
$$

for some $P, Q \in \Pi_{n}$ since for otherwise a permutation matrix $S^{\prime} \in \Pi_{3 n}$ of the desired form satisfying $\operatorname{pd}\left(S^{\prime} A\right) \geq \operatorname{pd}(S A)$ might be easily obtained by row permutation of $S A$. Then it follows that $\operatorname{pd}(S A)=\operatorname{pd}\left(S A S^{T}\right)=3 n^{2}+n+1+$ $\operatorname{pd}\left(Q A^{T}\right)+\operatorname{pd}(P A)$, and hence $\operatorname{pd}\left(Q A^{T}\right)+\operatorname{pd}(P A) \geq 2 k$. If $\operatorname{pd}(P A) \geq k$ we are done. If $\operatorname{pd}(P A)<k$ then $\operatorname{pd}\left(Q A^{T}\right)>k$. Let $P^{\prime} \in \Pi_{n}$ be such that $\operatorname{pd}\left(P^{\prime} A\right)=\max _{R \in \Pi_{n}} \operatorname{pd}(R A)$. According to Corollary 4.3, we have $\operatorname{pd}\left(P^{\prime} A\right)=$ $\max _{R \in \Pi_{n}} \operatorname{pd}\left(R A^{T}\right) \geq \operatorname{pd}\left(Q A^{T}\right)>k$.

The following two lemma can easily be established in much the same way as Lemma 4.5.
Lemma 4.6. Let $A$ be a (0,1)-matrix of order $n>1$ and $B$ be a zero trace symmetric $(0,1)$-matrix of order $4 n$ defined by

$$
B=\left[\begin{array}{cccc}
0 & 0 & 0 & A \\
0 & 0 & J_{n} & J_{n} \\
0 & J_{n} & 0 & 0 \\
A^{T} & J_{n} & 0 & 0
\end{array}\right]
$$

Then, there exists $P \in \Pi_{n}$ such that $\operatorname{pd}(P A) \geq k$ if and only there exists $S \in \Pi_{3 n}$ such that $\operatorname{pd}(S B) \geq 6 n^{2}+2 n+2 k$.

Lemma 4.7. Let $A$ be a (0,1)-matrix of order $n>1$ and $B$ be a zero trace symmetric $(0,1)$-matrix of order $3 n$ defined by

$$
B=\left[\begin{array}{ccc}
K_{n} & 0 & A \\
0 & K_{n} & J_{n} \\
A^{T} & J_{n} & K_{n}
\end{array}\right] .
$$

Then, there exists $P \in \Pi_{n}$ such that $\operatorname{pd}(P A) \geq k$ if and only there exists $S \in \Pi_{3 n}$ such that $\operatorname{pd}(S B) \geq n^{2}+2 n+2+k$.

Having disposed of these preliminary steps, we can now state our main result concerning Max Min Neighbor problem.

Corollary 4.8. The Max Min Neighbor problem is NP-complete even when restricted to split graphs, bipartite graphs and co-bipartite graphs.

## 5. Conclusions

We considered the problem of maximizing the packing down of a $(0,1)$-matrix by row permutation. We proved that this problem is NP-complete even when restricted to $(0,1)$-matrices with at most two 1 's per row, to $(0,1)$-matrices with at most two 1's per column or to zero trace symmetric $(0,1)$-matrices. Also, as intermediate results, several new simple layout problems were proven to be NP-complete: (1) finding $\phi \in \Phi(G)$ such that $\sum_{e \in E} \max \{\phi(u) \mid u \in e\}$ is maximized (even when restricted to at most cubic planar graphs), (2) finding $\phi \in \Phi(G)$ such that $\sum_{e \in E} \min \{\phi(u) \mid u \in e\}$ is maximized and (3) finding $\phi \in \Phi(G)$ such that $\sum_{u \in V} \min \{\phi(v) \mid v \in N(u)\}$ is maximized (even when restricted to split graphs, bipartite graphs and co-bipartite graphs).

There are many interesting related problems arising in the above context in a natural way. Here we mention just two of them: (1) What is the complexity of the Matrix Packing Down problem restricted to $(0,1)$-matrices with fixed number of 1's per row and per column and (2) How approximable the Matrix Packing Down problem is?

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[^1]:    ${ }^{1}$ They actually considered a similar problem on $(0,1)$-submatrices: given a connected bipartite graph $G=(U, V, E)$ with $|U|=|V|=n$ and a positive integer $k$, does there exists $U^{\prime} \subseteq U$, $V^{\prime} \subseteq V,\left|U^{\prime}\right|=\left|V^{\prime}\right|=k$ such that for some permutation $u_{1}^{\prime}, u_{2}^{\prime}, \ldots, u_{k}^{\prime}$ of the vertices in $U^{\prime}$ and for some permutation $v_{1}^{\prime}, v_{2}^{\prime}, \ldots, v_{k}^{\prime}$ of the vertices in $V^{\prime},\left\{u_{i}^{\prime}, v_{j}^{\prime}\right\} \notin E$ for any $1 \leq i \leq k$ and $i>j$ ?

