RAIRO-Inf. Theor. Appl. **39** (2005) 455-509
DOI: 10.1051/ita:2005027

DISTANCE DESERT AUTOMATA AND THE STAR HEIGHT PROBLEM *, **

Daniel Kirsten[1]

Abstract. We introduce the notion of nested distance desert automata as a joint generalization of distance automata and desert automata. We show that limitedness of nested distance desert automata is PSPACE-complete. As an application, we show that it is decidable in $2^{2^{\mathcal{O}(n)}}$ space whether the language accepted by an n-state non-deterministic automaton is of a star height less than a given integer h (concerning rational expressions with union, concatenation and iteration), which is the first ever complexity bound for the star height problem.

Mathematics Subject Classification. 20M35, 68Q17, 68Q70.

1. Introduction

The *star height problem* was raised by Eggan in 1963 [6]: is there an algorithm which computes the star height of recognizable languages? As Eggan, we consider star height concerning rational expressions with union, concatenation, and iteration in contrast to extended star height which also allows intersection and complement. For several years, in particular after Cohen refuted some promising ideas in 1970 [3], the star height problem was considered as the most difficult problem in the theory of recognizable languages, and it took 25 years until Hashiguchi

Keywords and phrases. Recognizable languages, star height, distance automata.

* *Short version was awarded by the European Association for Theoretical Computer Science (EATCS) as "Best Theoretical Paper presented at ETAPS 2004"* [23].

** *The main results of this paper were achieved during a 12 month stay at the Laboratoire d'Informatique Algorithmique, Fondements et Applications (LIAFA, Université Paris 7) which was funded by the grant KI 822/1–1 of the German Research Community (Deutsche Forschungsgemeinschaft).*

[1] Dresden University of Technology, Institute of Algebra, 01062 Dresden, Germany;
kirsten@math.tu-dresden.de

© EDP Sciences 2005

Article published by EDP Sciences and available at http://www.edpsciences.org/ita or http://dx.doi.org/10.1051/ita:2005027

showed the existence of such an algorithm which is one of the most important results in the theory of recognizable languages [13]. However, [13] is very difficult to read, *e.g.*, Pin commented “Hashiguchi’s solution for arbitrary star height relies on a complicated induction, which makes the proof very difficult to follow.” [45]. The entire proof stretches over [10–13], and Simon mentioned that it “takes more than a hundred pages of very heavy combinatorial reasoning” to present Hashiguchi’s solution in a self contained fashion [47]. Perrin wrote “the proof is very difficult to understand and a lot remains to be done to make it a tutorial presentation” [40].

Hashiguchi’s solution to the star height problem yields an algorithm of non-elementary complexity, and it remains open to deduce any upper complexity bound from Hashiguchi’s approach (*cf.* [34], Annexe B).

Motivated by his research on the star height problem, Hashiguchi introduced the notion of distance automata in 1982 [10,11]. *Distance automata* are nondeterministic finite automata with a set of marked transitions. The weight of a path is defined as the number of marked transitions in the path. The weight of a word is the minimum of the weights of all successful paths of the word. Thus, distance automata compute mappings from the free monoid to the positive integers. Hashiguchi showed that it is decidable whether a distance automaton is limited, *i.e.*, whether the range of the computed mapping is finite [10].

Distance automata and the more general weighted automata over the tropical semiring became a fruitful concept in theoretical computer science with many applications beyond their impact for the decidability of the star height hierarchy [13], *e.g.*, they have been of crucial importance in the research on the star problem in trace monoids [24, 37], but they are also of interest in industrial applications as speech recognition [38], database theory [8], and image compression [4, 20]. Consequently, distance automata and related concepts have been studied by many researchers beside Hashiguchi, *e.g.*, [14, 16, 25, 29, 33, 47, 49–51].

Bala and the author introduced independently the notion of desert automata in [1, 21, 22]. *Desert automata* are nondeterministic finite automata with a set of marked transitions. The weight of a path is defined as the length of a longest subpath which does not contain a marked transition. The weight of a word is the minimum of the weights of all successful paths of the word. The author showed that limitedness of desert automata is decidable [21, 22]. As an application, Bala and the author solved the so-called finite substitution problem which was open for more than 10 years: given recognizable languages K and L, it is decidable whether there exists a finite substitution σ such that $\sigma(K) = L$ [1, 21, 22].

Here, we introduce a joint generalization of distance automata and desert automata, the *nested distance desert automata*. By a generalization and further development of approaches from [14, 21–23, 27, 28, 32, 46, 47, 49], we show that limitedness of nested distance desert automata is PSPACE-complete. To achieve the decidability of limitedness in polynomial space we positively answer a question from Leung’s Ph.D. Thesis [27] from 1987.

As an application of nested distance desert automata, we give a new proof and the first ever complexity bound for the star height problem: given an integer h and an n-state nondeterministic automaton $\mathcal{A}$, it is decidable in $2^{2^{\mathcal{O}(n)}}$ space whether the star height of the language of $\mathcal{A}$ is less than h. The complexity bound does not depend on h because the star height of the language of an n-state nondeterministic automaton cannot exceed n.

The paper is organized as follows. In Section 2, we state preliminary notions and introduce nested distance desert automata. We present our main results in Section 2.4. In Section 3, we get familiar with some algebraic and technical foundations. In particular, we recall classic notions from ideal theory and the notion of a consistent mapping and we develop a finite semiring to describe nested distance desert automata in an algebraic fashion.

In Section 4, we develop two characterizations of unlimited nested distance desert automata which generalize classic results for distance automata by Hashiguchi, Leung, and Simon. One of these characterization utilizes Hashiguchi's so-called $\sharp$-expressions [14]. The other characterization relies on a solution of a so-called Burnside type problem which is similar to Leung's and Simon's approaches to the limitedness of distance automata. It gives immediately the decidability of limitedness of nested distance desert automata. To show the decidability of limitedness in PSPACE, we develop some more ideas in Section 5.

In Section 6, we reduce the star height problem to the limitedness of nested distance desert automata. Section 6 can be read independently of Sections 3–5. In Section 7, we discuss our approach and point out some open questions.

The present paper is self-contained.

2. Overview

2.1. Preliminaries

Let $\mathbb{N} = \{0, 1, \ldots\}$. For finite sets M, we denote by $|M|$ the number of elements of M. If p belongs to some set M, then we denote by p both the element p and the singleton set consisting of p. For sets M, we denote by $\mathcal{P}(M)$ the power set of M, and we denote by $\mathcal{P}_{ne}(M)$ the set of all non-empty subsets of M. We denote the union of disjoint sets by $\cup\!\!\!\cdot\;$.

A *semigroup* $(S, \cdot)$ consists of a set S and a binary associative operation "$\cdot$". Usually, we denote $(S, \cdot)$ for short by S, and we denote the operation $\cdot$ by juxtaposition.

Let S be a semigroup. We call S *commutative*, if $ab = ba$ for every $a, b \in S$. We call S *idempotent*, if $aa = a$ for every $a \in S$. We call an element $1 \in S$ an *identity*, if we have for every $a \in S$, $1a = a1 = a$. If S has an identity, then we call S a *monoid*. We call an element $0 \in S$ a *zero*, if we have for every $a \in S$, $a0 = 0a = 0$. There are at most one identity and at most one zero in a semigroup. We extend the operation of S to subsets of S in the usual way.

For subsets $T \subseteq S$, we call the closure of T under the operation of S the *subsemigroup generated by* T and denote it by $\langle T \rangle$. If $\langle T \rangle = T$, then we call T a *subsemigroup* of S.

Let $\leq$ be a binary relation over some semigroup S. We call $\leq$ *left stable* (resp. *right stable*) if for every $a, b, c \in S$ with $a \leq b$ we have $ca \leq cb$ (resp. $ac \leq bc$). We call $\leq$ *stable* if it is both left stable and right stable.

A *semiring* $(K, +, \cdot)$ consists of a set K and two binary operations $+$ and $\cdot$ whereas $(K, +)$ is a commutative monoid with an identity 0, $(K, \cdot)$ is a semigroup with zero 0, and the distributivity laws hold, *i.e.*, for every $a, b, c \in K$, we have $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$. Note that we do not require that a semiring has an identity for "$\cdot$".

During the main part of the paper, we fix some $n \geq 1$ which is used as the dimension of matrices. Whenever we do not explicitly state the range of a variable, then we assume that it ranges over the set $\{1, \ldots, n\}$. For example, phrases like "for every i, j" or "there is some l, such that" are understood as "for every $i, j \in \{1, \ldots, n\}$" resp. "there is some $l \in \{1, \ldots, n\}$, such that".

If $(K, +, \cdot)$ is a semiring, then we denote by $K_{n \times n}$ the semiring of all $n \times n$-matrices over K equipped with matrix multiplication (defined by $\cdot$ and $+$ as usual) and componentwise operation $+$.

Let Σ be a finite set of symbols within the entire paper. We denote by Σ^* the free monoid over Σ, *i.e.*, Σ^* consists of all *words* over Σ with concatenation as operation. We denote the empty word by ε. We denote by Σ^+ the free semigroup over Σ, *i.e.*, $\Sigma^+ := \Sigma^* \setminus \varepsilon$. For every $w \in \Sigma^*$, we denote by $|w|$ the length of w. We call subsets of Σ^* *languages*. We call a word u a *factor* of a word w if $w \in \Sigma^* u \Sigma^*$. Let $u, v \in \Sigma^*$, every factor of uv is the concatenation of a factor of u and a factor of v. For instance, if $u = aaab$ and $v = bb$, then the factor aa of $uv = aaabbb$ is the concatenation of aa and ε which are factors of u resp. v.

For $L \subseteq \Sigma^*$, let $L^* := L^0 \cup L^1 \cup \cdots = \cup_{i \in \mathbb{N}} L^i$ and $L^+ := L^1 \cup L^2 \cup \cdots = \cup_{i \geq 1} L^i$. Note that regardless of L, we have $L^0 = \{\varepsilon\}$. We call L^* the *iteration of* L.

Note that M^* is defined in two ways, depending on whether M is a set of symbols or M is a language. However, we will use the notation M^* in a way that no confusion arises.

2.2. Classic automata

We recall some standard terminology in automata theory.

A (nondeterministic) *automaton* is a tuple $\mathcal{A} = [Q, E, I, F]$ where

(1) Q is a finite set of *states*,
(2) $E \subseteq Q \times \Sigma \times Q$ is a set of *transitions*, and
(3) $I \subseteq Q$, $F \subseteq Q$ are sets called *initial* resp. *accepting states*.

Let $k \geq 1$. A *path* π in $\mathcal{A}$ of length k is a sequence (q_0, a_1, q_1) $(q_1, a_2, q_2) \ldots$ (q_{k-1}, a_k, q_k) of transitions in E. We say that π starts at q_0 and ends at q_k. We call the word $a_1 \ldots a_k$ the *label of* π. We denote $|\pi| := k$. As usual, we assume for every $q \in Q$ a path which starts and ends at q and is labeled with ε.

We call π *successful* if $q_0 \in I$ and $q_n \in F$. For every $0 \leq i \leq j \leq k$, we denote $\pi(i, j) := (q_i, a_i, q_{i+1}) \ldots (q_{j-1}, a_{i-1}, q_j)$ and call $\pi(i, j)$ a *factor of* π. For every

$p, q \in Q$ and every $w \in \Sigma^*$, we denote by $p \overset{w}{\rightsquigarrow} q$ the set of all paths with the label w which start at p and end at q.

We denote the *language of* $\mathcal{A}$ by $L(\mathcal{A})$ and define it as the set of all words in Σ^* which are labels of successful paths. We call some $L \subseteq \Sigma^*$ *recognizable*, if L is the language of some automaton. See, *e.g.,* [2, 7, 52] for a survey on recognizable languages.

The notion of a $\sharp$-expression was introduced by Hashiguchi in 1990 [14]. Intuitively, $\sharp$-expression provide a nested pumping technique. Every letter $a \in \Sigma$ is a *$\sharp$-expression.* For $\sharp$-expressions r and s, rs and $r^\sharp$ are *$\sharp$-expressions.*

Let r be a $\sharp$-expression. For every $k \geq 1$, r defines a word $r(k)$ as follows: If r is just a letter, then $r(k) := r$. For $\sharp$-expressions r and s, we set $rs(k) := r(k) \cdot s(k)$. Moreover, for every $\sharp$-expression r, $r^\sharp(k)$ yields the k-th power of $r(k)$, *i.e.*, we define $r^\sharp(k) := \big(r(k)\big)^k$.

The $\sharp$-height of $\sharp$-expressions is defined inductively. Letters are of $\sharp$-height 0, the $\sharp$-height of rs is the maximum of the $\sharp$-heights of r and s, and the $\sharp$-height of $r^\sharp$ is the $\sharp$-height of r plus 1.

2.3. Nested distance desert automata

Let $h \geq 0$ be arbitrary and $V := \{\angle_0, \curlyvee_0, \angle_1, \curlyvee_1, \ldots, \curlyvee_{h-1}, \angle_h\}$. We define a mapping $\Delta : V^* \to \mathbb{N}$ in a tricky way. Before we define Δ formally, we give an intuitive explanation. We regard the numbers $0, \ldots, h$ as colors. For every $0 \leq g \leq h$, there are coins of color g which are called g-coins. We have some bag to carry coins. The bag has exactly $h + 1$ partitions which are colored like the coins. For every $0 \leq g \leq h$, we can store g-coins in partition g, but we cannot store g-coins in any other partition. The *size of the bag* is an integer d whereas we can carry at most d 0-coins, d 1-coins, ..., and d h-coins at the same time. Hence, we can carry at most $d(h + 1)$ coins at the same time, but we cannot carry more than d coins of one and the same color at the same time.

Imagine that we plan to walk along some word[1] $\pi \in V^*$, and we have a bag of size $d \in \mathbb{N}$. Initially, the bag is completely filled, *i.e.*, there are d 0-coins, d 1-coins, ..., and d h-coins in the bag. Let $0 \leq g \leq h$ be arbitrary. If we walk along the letter $\angle_g$, then we have to pay a g-coin but we can obtain coins which are colored by a color less than g, *i.e.*, we can fill up our bag with 0-coins, 1-coins, ..., $(g - 1)$-coins. If we do not carry a g-coin in our bag, then we cannot walk along the letter $\angle_g$. We pronounce the letter $\angle_g$ as "péage g". If we walk along $\curlyvee_g$, then we need not to pay any coin but we can fill up our bag with 0-coins, 1-coins, ..., g-coins. We pronounce $\curlyvee_g$ as "water g". This notion arose from earlier variants of these automata in which $\curlyvee$ was considered as a source of water.

[1]Note that we utilize the letter π both to denote words over V but also to denote paths in automata.

Whenever we can obtain g-coins (at $\curlyvee_g, \angle_{g+1}, \ldots, \curlyvee_{h-1}, \angle_h$), then we can also obtain $0, \ldots, (g-1)$-coins. However, at $\curlyvee_{g-1}$ and $\angle_g$, we can obtain $0, \ldots, (g-1)$-coins but we cannot obtain g-coins. Thus, g-coins are considered as more valuable than $(g-1)$-coins.

It depends on the size of the bag (and of course on the word) whether we can walk along the entire word. We imagine $\Delta(\pi)$ as the least integer d such that we can walk along the word π with a bag of size d.

We define Δ formally. For every $0 \leq g \leq h$, we consider every factor π' of π in which we cannot obtain g-coins. More precisely, we consider factors π' of π with $\pi' \in \{\angle_0, \curlyvee_0, \ldots, \angle_g\}^*$ and count the number of occurrences of $\angle_g$. This is the number of g-coins which we need to walk along π'.

For $0 \leq g \leq h$ and $\pi \in V^*$, let $|\pi|_g$ be the number of occurrences of the letter $\angle_g$ in π. Let

(1) $\Delta_g(\pi) := \max_{\substack{\pi' \in \{\angle_0, \curlyvee_0, \ldots, \angle_g\}^* \\ \pi' \text{ is a factor of } \pi}} |\pi'|_g$ and

(2) $\Delta(\pi) := \max_{0 \leq g \leq h} \Delta_g(\pi)$.

It is easy to see that $\Delta(\pi) \leq |\pi|$.

An *h-nested distance desert automaton* is a tuple $\mathcal{A} = [Q, E, I, F, \theta]$ where $[Q, E, I, F]$ is an automaton and $\theta : E \to V$.

Let $\mathcal{A} = [Q, E, I, F, \theta]$ be an h-nested distance desert automaton. The notions of a path, a successful path, the language of $\mathcal{A}$, ... are understood w.r.t. $[Q, E, I, F]$. For every transition $e \in E$, we say that e is *marked by* $\theta(e)$. We extend θ to a homomorphism $\theta : E^* \to V^*$. We define the semantics of $\mathcal{A}$. For $w \in \Sigma^*$, let

$$\Delta_{\mathcal{A}}(w) := \min_{p \in I,\, q \in F,\, \pi \in p \overset{w}{\rightsquigarrow} q} \Delta(\theta(\pi)).$$

We have $\Delta_{\mathcal{A}}(w) = \infty$ iff $w \notin L(\mathcal{A})$. Hence, $\Delta_{\mathcal{A}}$ is a mapping $\Delta_{\mathcal{A}} : \Sigma^* \to \mathbb{N} \cup \{\infty\}$.

If there is a bound $d \in \mathbb{N}$ such that $\Delta_{\mathcal{A}}(w) \leq d$ for every $w \in L(\mathcal{A})$, then we say that $\mathcal{A}$ is *limited by d* or for short $\mathcal{A}$ is *limited*. Otherwise, we call $\mathcal{A}$ *unlimited*.

Clearly, h-nested distance desert automata are a particular case of $(h+1)$-nested distance desert automata.

For every 0-nested distance desert automaton $\mathcal{A}$, we have $\Delta_{\mathcal{A}}(w) = |w|$ for every $w \in L(\mathcal{A})$. Hence, 0-nested distance desert automaton $\mathcal{A}$ is limited iff $L(\mathcal{A})$ is finite.

The subclass of 1-nested distance desert automata for which $\theta : E \to \{\curlyvee_0, \angle_1\}$ are exactly Hashiguchi's distance automata [10]. If we consider the subclass of 1-nested distance desert automata with the restriction $\theta : E \to \{\angle_0, \curlyvee_0\}$, then we recover the definition of desert automata due to Bala and the author [1, 21, 22].

2.4. Main results

A main result of the present paper is a two-fold characterization of unlimited nested distance desert automata shown in Theorem 2.1, below. It generalizes results and ideas on distance and desert automata by Hashiguchi, Leung, Simon, and the author [14,21,22,27,28,32,46,47,49]. Our first characterization is algebraic.

It generalizes corresponding characterizations of unlimited distance automata due to Leung and Simon [27, 28, 32, 49] and unlimited desert automata due to the author [21, 22].

Our second characterization generalizes another well-known characterization of unlimited distance automata [14, 29, 32, 49] in terms of $\sharp$-expressions which were introduced by Hashiguchi in 1990 [14].

Theorem 2.1. *Let $h \in \mathbb{N}$. Let $\mathcal{A} = [Q, E, I, F, \theta]$ be a h-nested distance desert automaton. The following assertions are equivalent:*

(1) *$\mathcal{A}$ is unlimited.*
(2) *Let $T := \Psi(\Sigma)$. There is a matrix $a \in \langle T \rangle^\sharp$ such that $I \cdot a \cdot F = \omega$.*
(3) *There is a $\sharp$-expression r of a $\sharp$-height of at most $(h+1)|Q|$ such that for every $k \geq 1$, we have $r(k) \in L(\mathcal{A})$, and for increasing integers k, the weight $\Delta_{\mathcal{A}}\big(r(k)\big)$ is unbounded.*

The algebraic concepts involved in assertion (2) in Theorem 2.1 will be explained in Sections 3 and 4. At this point, it is not necessary to understand assertion (2).

Note that (3)⇒(1) in Theorem 2.1 is obvious. We prove (2)⇒(3) in Section 4.2 up to the bound on the $\sharp$-height of r which is considered in Section 5.3. The most difficult part in the proof of Theorem 2.1 is to show (1)⇒(2). It leads to an intriguing Burnside type problem and is shown in Section 4.6.

From Theorem 2.1, we derive the following result:

Theorem 2.2. *For $h \geq 1$, limitedness of h-nested distance desert automata is PSPACE-complete.*

Theorem 2.2 generalizes recent results due to Leung and Podolskiy [33] resp. Bala and the author [1, 21, 22] for PSPACE-completeness for limitedness of distance resp. desert automata. However, the proof of the decidability of limitedness of nested distance desert automata in PSPACE is not a generalization of these two particular cases, it is an new approach which is based on an analysis of the structure of the semigroup $\langle T \rangle^\sharp$ in assertion (2) of Theorem 2.1. In particular, we will positively answer a question from Leung's PhD thesis from 1987 [27] (see Cor. 5.6(2)).

Theorem 2.2 will be proved in Section 5. We show a nondeterministic PSPACE-algorithm which decides limitedness of nested distance desert automata in Section 5.4. PSPACE-hardness for $h \geq 1$ in Theorem 2.2 follows immediately from PSPACE-hardness of limitedness of distance automata [27, 28] and of desert automata [21, 22]. However, we use Leung's idea [27, 28] show PSPACE-hardness of limitedness of some more particular cases of nested distance desert automata in Section 5.5.

Limitedness of 0-nested distance desert automata is essentially the question whether $L(\mathcal{A})$ is finite which is decidable in polynomial time.

The equivalence problem for distance automata is undecidable [25], and hence, the equivalence problem for h-nested distance desert automata is undecidable for $h \geq 1$. The equivalence problem for 0-nested distance desert is essentially the question whether $L(\mathcal{A}) = L(\mathcal{A}')$ which is PSPACE-complete. The equivalence problem for desert automata is open [21, 22].

As an application of Theorem 2.2, we show the following result in Section 6:

Theorem 2.3. *Let $h \in \mathbb{N}$ and L be the language accepted by an n-state nondeterministic automaton. It is decidable in $2^{2^{\mathcal{O}(n)}}$ space whether L is of star height h.*

We prove Theorem 2.3 by a reduction of the star height h problem to the limitedness of h-nested distance desert automata. Note that this reduction is immediate for $h = 0$, because a language L is of star height 0 iff L is finite, and the finiteness problem of a language is exactly the limitedness of 0-nested distance desert automata.

3. Some algebraic and technical foundations

We develop some algebraic and technical foundations which are required in Sections 4 and 5.

In Section 3.1, we get familiar with classic ideas from ideal theory. In Section 3.2, we introduce the notion of a consistent mapping as an abstraction of several particular mappings used by Simon and Leung. In Section 3.3, we develop a semiring to describe nested distance desert automata in an algebraic fashion. In Section 3.4, we show some technical lemmas about connections between the concatenation and iteration of words in V^+ and their weights.

In order to understand Sections 4 and 5, it suffices to read Section 3.1 very briefly, and Sections 3.2 and 3.4 briefly. However, a good understanding of the ideas and constructions in Section 3.3 is necessary.

3.1. Ideal theory of finite semigroups

We introduce some concepts from ideal theory. This section is far away from being a comprehensive overview, for a deeper understanding, the author recommends teaching books, *e.g.*, [9, 26, 41]. We just state the notions and results which we need in the rest of the paper.

As already mentioned, a semigroup S is a set with a binary associative operation which we denote by juxtaposition. Let S be a semigroup within this section.

If there is no identity in S, then we denote by S^1 the semigroup consisting of the set $S \uplus 1$, on which the operation of S is extended in a way that 1 is the identity of S^1. If S has an identity, then we define S^1 to be S.

We call an $e \in S$ an *idempotent* if $e^2 = e$. We denote the set of all idempotents of S by $\mathsf{E}(S)$.

The following relations are called Green's relation. We show several equivalent definitions. Let $a, b \in S$.

(1) $a \leq_{\mathscr{J}} b \quad :\Longleftrightarrow \quad a \in S^1 b S^1 \quad \Longleftrightarrow \quad S^1 a S^1 \subseteq S^1 b S^1$
(2) $a \leq_{\mathscr{J}} b \quad :\Longleftrightarrow \quad a \in S^1 b \quad \Longleftrightarrow \quad S^1 a \subseteq S^1 b$
(3) $a \leq_{\mathscr{R}} b \quad :\Longleftrightarrow \quad a \in b S^1 \quad \Longleftrightarrow \quad a S^1 \subseteq b S^1$.

We allow to denote $a \leq_{\mathscr{J}} b$ by $b \geq_{\mathscr{J}} a$, and similarly for the other relations. The relation $\leq_{\mathscr{J}}$ is right stable, and similarly, $\leq_{\mathscr{R}}$ is left stable. However, we do not have a similar property for $\leq_{\mathscr{J}}$.

Again, let $a, b \in S$. We define:

(1) $a =_{\mathscr{L}} b \quad :\Longleftrightarrow \quad a \leq_{\mathscr{J}} b$ and $a \geq_{\mathscr{J}} b \quad \Longleftrightarrow \quad S^1aS^1 = S^1bS^1$
(2) $a =_{\mathscr{L}} b \quad :\Longleftrightarrow \quad a \leq_{\mathscr{J}} b$ and $a \geq_{\mathscr{L}} b \quad \Longleftrightarrow \quad S^1a = S^1b$
(3) $a =_{\mathscr{R}} b \quad :\Longleftrightarrow \quad a \leq_{\mathscr{R}} b$ and $a \geq_{\mathscr{R}} b \quad \Longleftrightarrow \quad aS^1 = bS^1$.

It is easy to see that $=_{\mathscr{L}}$, $=_{\mathscr{L}}$, and $=_{\mathscr{R}}$ are equivalence relations. We call their equivalence classes $\mathscr{J}$-classes (resp. $\mathscr{L}$-, $\mathscr{R}$-classes). For every $a \in S$, we denote by $\mathscr{J}(a)$, $\mathscr{L}(a)$, resp. $\mathscr{R}(a)$ the $\mathscr{J}$-, $\mathscr{L}$-, resp. $\mathscr{R}$-class of a. As above, $=_{\mathscr{L}}$ resp. $=_{\mathscr{R}}$ are right stable resp. left stable.

Remark 3.1.

(1) Let $e \in \mathsf{E}(S)$ and $a \leq_{\mathscr{J}} e$. There is some $p \in S^1$ such that $a = pe$. Hence, $ae = pee = pe = a$. Similarly, if $b \leq_{\mathscr{R}} e$, then $eb = b$.
(2) Let $e, f \in \mathsf{E}(S)$ with $e \leq_{\mathscr{J}} f$ and $e \geq_{\mathscr{R}} f$. Then, $ef = e$ and $ef = f$, *i.e.*, $e = f$.

On the set of idempotents $\mathsf{E}(S)$, one defines a natural ordering $\leq$ such that for every $e, f \in \mathsf{E}(S)$, we have $e \leq f$ iff $e = ef = fe$. By Remark 3.1, we have $e \leq f$ iff $e \leq_{\mathscr{J}} f$ and $e \leq_{\mathscr{R}} f$.

For every $a \in S^1$, let $a\cdot$ resp. $\cdot a$ be the left resp. right multiplication by a.

The following lemma due to Green is of crucial importance to understand the relations between $\mathscr{J}$-, $\mathscr{L}$-, and $\mathscr{R}$-classes.

Lemma 3.2. (Green's lemma). *Let S be a semigroup, $a, b \in S$ and $p, q \in S^1$.*

(1) *If $b = ap$ and $a = bq$, then $\cdot p$ and $\cdot q$ are mutually inverse, $\mathscr{R}$-class preserving bijections between $\mathscr{L}(a)$ and $\mathscr{L}(b)$.*
(2) *If $b = pa$ and $a = qb$, then $p\cdot$ and $q\cdot$ are mutually inverse, $\mathscr{L}$-class preserving bijections between $\mathscr{R}(a)$ and $\mathscr{R}(b)$.*

The notion *$\mathscr{R}$-class preserving* (and similarly *$\mathscr{L}$-class preserving*) means that we have $c =_{\mathscr{R}} cp$ for every $c \in \mathscr{L}(a)$ and $d =_{\mathscr{R}} dq$ for every $d \in \mathscr{L}(b)$.

Proof. We show (1). Let $c \in \mathscr{L}(a)$. We have $b = ap =_{\mathscr{L}} cp$, because $=_{\mathscr{L}}$ is right stable. Thus, $cp \in \mathscr{L}(b)$. There is an $x \in S^1$ such that $c = xa$. By $apq = a$, we have $xapq = xa$, *i.e.*, $cpq = c$.

By $cp \in \mathscr{L}(b)$ and $cpq = c$ for every $c \in \mathscr{L}(a)$, we know that $\cdot pq$ is the identity on $\mathscr{L}(a)$, and moreover, $\cdot p : \mathscr{L}(a) \to \mathscr{L}(b)$ is injective and $\cdot q : \mathscr{L}(b) \to \mathscr{L}(a)$ is surjective. In a symmetric way, we can show that $\cdot qp$ is the identity on $\mathscr{L}(b)$ and that $\cdot q : \mathscr{L}(b) \to \mathscr{L}(a)$ and $\cdot p : \mathscr{L}(a) \to \mathscr{L}(b)$ are injective resp. surjective. This completes (1). We can show (2) in a symmetric way. $\square$

There are several connections between Green's relations and multiplication.

We assume from now that S is finite. Let $a \in S$. There $l, m \geq 1$ such that $a^l = a^{l+m}$. Then, $a^{2lm} = a^{lm} \in \mathsf{E}(S)$. Thus, for every $a \in S$ there is some $k \geq 1$ such that $a^k \in \mathsf{E}(S)$.

Lemma 3.3. *Let S be a finite semigroup. Let $a, b \in S$ and $p, q \in S^1$ be arbitrary.*

(1) *If $a =_{\mathscr{J}} paq$, then $pa =_{\mathscr{L}} a =_{\mathscr{R}} aq$.*
(2) *If $a =_{\mathscr{J}} ab$, then $a =_{\mathscr{R}} ab$.*
(3) *If $b =_{\mathscr{J}} ab$, then $b =_{\mathscr{L}} ab$.*
(4) *If $a =_{\mathscr{J}} b$, we have $\mathscr{L}(a) \cap \mathscr{R}(b) \neq \emptyset$.*

Proof.

(1) By $a =_{\mathscr{J}} paq$, there are $r, s \in S^1$ such that $a = rpaqs$. Then, we have for every $k \geq 1$, $a = (rp)^k a(qs)^k$. Let $k \geq 1$ such that we have $(rp)^k \in \mathsf{E}(S)$. Then, $a = (rp)^k a(qs)^k = (rp)^k (rp)^k a(qs)^k = (rp)^k a$, and $a = (rp)^k a \leq_{\mathscr{J}} pa \leq_{\mathscr{J}} a$., *i.e.*, $pa =_{\mathscr{L}} a$, and by symmetry, $a =_{\mathscr{R}} aq$.
(2) By $a =_{\mathscr{J}} 1ab$ and (1), we have $a =_{\mathscr{R}} ab$.
(3) By $b =_{\mathscr{J}} ab1$ and (1), we have $ab =_{\mathscr{L}} b$.
(4) There are $p, q \in S^1$ such that $b = paq =_{\mathscr{J}} a$. By (1), we have $a =_{\mathscr{L}} pa$. By $paq =_{\mathscr{J}} a$, we have $paq =_{\mathscr{J}} pa$, and by (2), $paq =_{\mathscr{R}} pa$, *i.e.*, $b =_{\mathscr{R}} pa$. Thus, $pa \in \mathscr{L}(a) \cap \mathscr{R}(b)$. □

Lemma 3.3 cannot be generalized to infinite semigroups, although it is a rather challenging task to give a counter example without consulting a teaching book [41].

There is another Green's relation. For every $a, b \in S$ let

(1) $a =_{\mathscr{H}} b \quad :\Longleftrightarrow \quad a =_{\mathscr{L}} b$ and $a =_{\mathscr{R}} b$.

Clearly, $=_{\mathscr{H}}$ is the intersection of $=_{\mathscr{L}}$ and $=_{\mathscr{R}}$. Hence, $=_{\mathscr{H}}$ is an equivalence relation and its equivalence classes ($\mathscr{H}$-classes) are the non-empty intersections of $\mathscr{L}$- and $\mathscr{R}$-classes.

Lemma 3.4. *Let S be a finite semigroup, H be a $\mathscr{H}$-class, and J be the $\mathscr{J}$-class with $H \subseteq J$. We have $HH \cap J \neq \emptyset$ iff H is a group.*

Proof. If H is a group, then we have $HH = H \subseteq J$, *i.e.*, $HH \cap J = H \neq \emptyset$.

Conversely, assume $HH \cap J \neq \emptyset$. Let $p, q \in H$ satisfying $pq \in J$. Let $a, b \in H$ be arbitrary. By $a =_{\mathscr{L}} p$, we have $ab =_{\mathscr{L}} pb$. By $b =_{\mathscr{R}} q$, we have $pb =_{\mathscr{R}} pq$. Thus, $ab =_{\mathscr{J}} pq$, and by $a, b \in H$, we have $a =_{\mathscr{J}} b =_{\mathscr{J}} ab$. By Lemma 3.3(2,3), we have $ab \in \mathscr{R}(a) \cap \mathscr{L}(b)$, and by $a, b, p, q \in H$, we have $\mathscr{R}(a) = \mathscr{R}(p)$ and $\mathscr{L}(b) = \mathscr{L}(q)$. Consequently, $ab \in \mathscr{R}(p) \cap \mathscr{L}(q) = H$, *i.e.*, H is closed under multiplication.

By $a =_{\mathscr{R}} ab$, there is some $x \in S^1$ such that $abx = a$. By Lemma 3.2 $\cdot b : H \to H$ and $\cdot x : H \to H$ are mutually inverse bijections. Similarly, there is some $y \in S^1$ such that $yab = b$, and $a\cdot : H \to H$ and $y\cdot : H \to H$ are mutually inverse bijections. Hence, multiplication in H is cancelative, and thus, H is a group. □

The following lemma will be very useful.

Lemma 3.5. *Let S be a finite semigroup, let $a, b \in S$ satisfying $a =_{\mathscr{J}} b$.*
We have $a =_{\mathscr{J}} b =_{\mathscr{J}} ab$ iff there is an idempotent $e \in \mathsf{E}(S)$ such that $a =_{\mathscr{L}} e =_{\mathscr{R}} b$.

Proof. Let $e \in \mathsf{E}(S)$ such that $a =_{\mathscr{L}} e =_{\mathscr{R}} b$. There are $x, y \in S^1$ satisfying $xa = e = by$, *i.e.*, $ab \geq_{\mathscr{J}} xaby = ee = e =_{\mathscr{J}} a$. Clearly, $ab \leq_{\mathscr{J}} a$. To sum up, $ab =_{\mathscr{J}} a$.

Conversely, assume $a =_{\mathscr{J}} b =_{\mathscr{J}} ab$. Let $H := \mathscr{L}(a) \cap \mathscr{R}(b)$ and $J := \mathscr{J}(a) = \mathscr{J}(b) = \mathscr{J}(ab)$. By Lemma 3.3(4), choose a $p \in H$. There are $x, y \in S^1$ satisfying $a = xp$ and $b = py$. Hence, $pp \geq_{\mathscr{J}} xppy = ab$. Moreover, $pp \leq_{\mathscr{J}} p =_{\mathscr{J}} a$. Thus, $pp \in J$. By Lemma 3.4, H is a group. Let e be the identity of H. □

Usually, one visualizes a $\mathscr{J}$-class by an "egg-box picture" in which the columns are $\mathscr{L}$-classes and the rows are $\mathscr{R}$-classes. We can combine Lemma 3.3(2, 3, 4) and Lemma 3.5: if $a =_{\mathscr{J}} b =_{\mathscr{J}} ab$, then $ab \in \mathscr{R}(a) \cap \mathscr{L}(b)$ and there is an idempotent $e \in \mathscr{L}(a) \cap \mathscr{R}(b)$ as shown in the following table:

a		ab	
e		b	

One distinguishes two kinds of $\mathscr{J}$-classes. If some $\mathscr{J}$-class J satisfies the three equivalent conditions in Lemma 3.6, then we call J a *regular* $\mathscr{J}$-class, otherwise we call J *non-regular*. We call some element $a \in S$ *regular*, if $\mathscr{J}(a)$ is a regular $\mathscr{J}$-class. We denote the set of all regular elements of S by $\mathsf{Reg}(S)$.

Lemma 3.6. *Let J be a $\mathscr{J}$-class of a finite semigroup S. The following assertions are equivalent:*

(1) $JJ \cap J \neq \emptyset$.
(2) *There is at least one idempotent in J.*
(3) *In every $\mathscr{L}$-class of J and in every $\mathscr{R}$-class of J there is at least one idempotent.*

Proof. (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) are obvious, and (1) $\Rightarrow$ (2) is an immediate consequence of Lemma 3.5.

We show (2) $\Rightarrow$ (3). Let $e \in J$ be an idempotent, and let $a \in J$ be arbitrary. We show that there is an idempotent in $\mathscr{L}(a)$. There are $p, q \in S^1$ such that $e = paq = (paq)^3 = pa(qpa)^2q$. We have

$$a \geq_{\mathscr{J}} qpa \geq_{\mathscr{J}} (qpa)^2 \geq_{\mathscr{J}} pa(qpa)^2q = (paq)^3 = e =_{\mathscr{J}} a,$$

i.e., $qpa =_{\mathscr{J}} (qpa)^2 \in J$. By Lemma 3.5, there is an idempotent in $\mathscr{L}(qpa) \cap \mathscr{R}(qpa)$. By Lemma 3.3(3). We have $\mathscr{L}(qpa) = \mathscr{L}(a)$, *i.e.*, there is an idempotent in $\mathscr{L}(a)$.

By examining aqp, we can show in a symmetric way that there is an idempotent in $\mathscr{R}(a)$, and (3) follows from the arbitrary choice of a. □

Let T be a subsemigroup of S. We have $\mathsf{E}(T) = \mathsf{E}(S) \cap T$ and $\mathsf{Reg}(T) \subseteq \mathsf{Reg}(S)$. However, we do not necessarily have $\mathsf{Reg}(T) = \mathsf{Reg}(S) \cap T$.

The reader should be aware that in contrast to Lemma 3.4, a regular $\mathscr{J}$-class in not necessarily closed under multiplication. Even if a regular $\mathscr{J}$-class is closed under multiplication, then it is not necessarily a group.

The following property will be very useful.

Lemma 3.7. *Let S be a finite semigroup and let $a, b \in S$ such that $a =_{\mathscr{L}} b$. If $ab = a$, then $b \in \mathsf{E}(S)$. If $ab = b$, then $a \in \mathsf{E}(S)$.*

Proof. Assume $ab = a$. Then, $ab = a =_{\mathscr{L}} b$, and by Lemma 3.3(3), we have $ab =_{\mathscr{L}} b$, *i.e.*, $a =_{\mathscr{L}} b$. Hence, there is some $p \in S^1$ such that $pa = b$. Thus, $pab = pa$, *i.e.*, $b^2 = b \in \mathsf{E}(S)$. The other assertion follows by symmetry. □

The assumption $a =_{\mathscr{L}} b$ in Lemma 3.7 is crucial. Just assume that S has a zero and consider the case $a = 0$.

The next lemma is well-known in semigroup theory and of importance in the theory of recognizable languages [41].

Lemma 3.8. *Let S be a finite semigroup. Let $e, f \in \mathsf{E}(S)$ satisfying $e =_{\mathscr{L}} f$. For every $a \in \mathscr{R}(e) \cap \mathscr{L}(f)$, there is exactly one $b \in \mathscr{R}(f) \cap \mathscr{L}(e)$ satisfying both $ab = e$ and $ba = f$.*

We can visualize the relations between a, b, e, f in Lemma 3.7 by the following egg-box picture:

a		e	
f		b	

Proof. Let a, e, f as in the lemma. There are $p, q \in S^1$ such that $ap = e$ and $apq = eq = a$. Moreover, there are $x, y \in S^1$ such that $xa = f$ and $yxa = yf = a$.

By Lemma 3.2, $\cdot p$ and $\cdot q$ are mutually inverse bijections between $\mathscr{L}(a) = \mathscr{L}(f)$ and $\mathscr{L}(e)$. Similarly, $x\cdot$ and $y\cdot$ are mutually inverse bijections between $\mathscr{R}(a) = \mathscr{R}(e)$ and $\mathscr{R}(f)$.

The crucial fact is that $xap = xe$ but also $xap = fp$. We set $b := xe = fp$. By Lemma 3.2, we have $b \in \mathscr{R}(f) \cap \mathscr{L}(e)$.

By $a = xf$ and $f \in \mathsf{E}(S)$, we have $af = a$, and by symmetry $ea = a$.

We have $ab = afp = ap = e$ and $ba = xea = xa = f$.

Let $b' \in \mathscr{R}(f) \cap \mathscr{L}(e)$ such that $ab' = e$ and $b'a = f$. By $ab' = e$, we have $xab' = xe$, and thus, $fb' = b$. By $f =_{\mathscr{R}} b'$, we have $fb' = b'$, *i.e.* $b' = b$. □

For every $k > 0$ and $a_1, \ldots, a_k \in S$, we call $a_1, \ldots, a_k$ a *smooth product* if we have $a_1 =_{\mathscr{L}} a_2 =_{\mathscr{L}} \cdots =_{\mathscr{L}} a_k =_{\mathscr{L}} (a_1 \ldots a_k) \in \mathsf{Reg}(S)$. Note that this is not a classic notion.

Let J_1 and J_2 be two $\mathscr{J}$-classes. There are $a \in J_1$ and $b \in J_2$ satisfying $a \leq_{\mathscr{J}} b$ iff we have $a \leq_{\mathscr{J}} b$ for every $a \in J_1$ and $b \in J_2$. Hence, $\leq_{\mathscr{J}}$ extends to a partial ordering of the $\mathscr{J}$-classes.

In a finite semigroup, there is always a maximal $\mathscr{J}$-class, but it is not necessarily unique.

We call some subset of $I \subseteq S$ an *ideal* if $S^1 I S^1 \subseteq I$. Obviously, some subset $I \subseteq S$ is an ideal iff I is closed under $\leq_{\mathscr{J}}$, *i.e.*, iff for every $a \in S$, $b \in I$ with $a \leq_{\mathscr{J}} b$ we have $a \in I$. Every ideal of S is saturated by the $\mathscr{J}$-classes of S.

If S is finite, then there are some $z \geq 1$ and ideals $I_1, \ldots, I_{z+1}$ of S satisfying

$$S = I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_z \supsetneq I_{z+1} = \emptyset$$

such that for every $l \in \{1, \ldots, z\}$, the set $I_l \setminus I_{l+1}$ is a $\mathscr{J}$-class. Moreover, z is the number of $\mathscr{J}$-classes of S. It is easy to construct such a chain of ideals: you simply start by $I_1 := S$, then we set $I_2 := I_1 \setminus J_1$ where J_2 is a maximal $\mathscr{J}$-class and so on.

This closes our expedition to the realms of ideal theory. The reader should be aware that ideal theory is just an initial part of the huge field of the structure theory of semigroups. Moreover, the notions and results in this section are just the beginning of ideal theory, and there are many important aspects which are not covered here. For example, there is a deep theorem by Rees and Sushkevich which describes the inner structure of regular $\mathscr{J}$-classes of finite semigroups up to isomorphism.

3.2. Consistent mappings

We develop the notion of a consistent mapping as an abstraction from certain transformations (stabilization, perforation) of matrices over various semirings which play a key role in many articles by Simon and Leung [27–30, 32, 47–49]. Let S be a finite semigroup.

We call a mapping $\sharp : \mathsf{E}(S) \to \mathsf{E}(S)$ *consistent*, if for every $a, b \in S^1$ and $e, f \in \mathsf{E}(S)$ with $e =_{\mathscr{L}} f$ and $f = aeb$, we have $f^\sharp = ae^\sharp b$.

If $\sharp$ is a consistent mapping and $e \in \mathsf{E}(S)$, then $e = 1ee = ee1 = eee$, and thus, $e^\sharp = e^\sharp e = ee^\sharp = ee^\sharp e$, *i.e.*, $e^\sharp \leq_{\mathscr{J}} e$, and $e^\sharp \leq_{\mathscr{R}} e$. Thus, $e^\sharp \leq e$ in the natural ordering $\leq$ of the idempotents.

We use some results from finite semigroup theory to show that every consistent mapping admits a unique extension to regular elements. Lemma 3.9 was already shown by Leung in a more particular framework [27, 28, 32].

Lemma 3.9. *Let $\sharp$ be a consistent mapping. Let $e, f \in \mathsf{E}(S)$ and $a, b, c, d \in S^1$ satisfying $aeb = cfd =_{\mathscr{L}} e =_{\mathscr{L}} f$. We have $ae^\sharp b = cf^\sharp d$.*

Proof. Let J be the $\mathscr{J}$-class with $aeb = cfd =_{\mathscr{L}} e =_{\mathscr{L}} f \in J$. We have $ae, eb, rf, fd \in J$. As seen in Section 3.1, $ae =_{\mathscr{R}} aeb = cfd =_{\mathscr{R}} cf$ and $eb =_{\mathscr{L}} aeb = cfd =_{\mathscr{L}} fd$.

ae		cf		$aeb = cfd$
		f		fd
e				eb

There are $p, q \in S^1$ such that $pcf = f$ and $qf = cf$. By Green's lemma (Lem. 3.2), $p\cdot$ and $q\cdot$ are mutually inverse bijections between $\mathscr{R}(cf)$ and $\mathscr{R}(f)$. By $ae \in \mathscr{R}(cf)$, we have $ae = qpae$.

Similarly, there are $r, s \in S^1$ such that $fdr = f$ and $fs = fd$, and moreover, $eb = ebrs$.

By $cfd = aeb$, we have $pcfdr = paebr$, and thus, $f = paebr$. We have $f^\sharp = pae^\sharp br$. Then, we have $ff^\sharp f = paee^\sharp ebr$, and $qff^\sharp fs = qpaee^\sharp ebrs$, *i.e.*, $cff^\sharp fd = aee^\sharp eb$, and finally, $cf^\sharp d = ae^\sharp b$. $\square$

Lemma 3.9 allows us to extend consistent mappings to regular elements of S.

Corollary 3.10. *Let $\sharp : \mathsf{E}(S) \to \mathsf{E}(S)$ be a consistent mapping. By setting $(aeb)^\sharp := ae^\sharp b$ for every $a, b \in S^1$, $e \in \mathsf{E}(S)$ satisfying $e =_{\mathscr{J}} aeb$, we define a mapping $\sharp : \mathsf{Reg}(S) \to \mathsf{Reg}(S)$.*

Proof. By Lemma 3.9, it remains to show $ae^\sharp b \in \mathsf{Reg}(S)$. By $aeb =_{\mathscr{J}} e$, we have $ae =_{\mathscr{L}} e =_{\mathscr{R}} eb$. There are $c, d \in S^1$ such that $cae = e = ebd$. We obtain $cae^\sharp bd = caee^\sharp ebd = ee^\sharp e = e^\sharp$, and thus, $ae^\sharp b =_{\mathscr{J}} e^\sharp$, *i.e.*, $e^\sharp$ is an idempotent in $\mathscr{J}(ae^\sharp b)$. $\square$

Remark 3.11. Let $a \in S$ be arbitrary and $e, f \in S$ satisfying $e =_{\mathscr{R}} a =_{\mathscr{L}} f$. Then, $ea = af = a$ and $e^\sharp a = af^\sharp = a^\sharp$. Consequently, $a^\sharp \leq_{\mathscr{J}} a$ and $a^\sharp \leq_{\mathscr{R}} a$.

The next lemma allows to deal with consistent mappings in a very convenient way.

Lemma 3.12. *Let $a, b, c \in S^1$.*

(1) *If $abc =_{\mathscr{J}} b \in \mathsf{Reg}(S)$, then we have $(abc)^\sharp = ab^\sharp c$.*
(2) *If $a =_{\mathscr{J}} b =_{\mathscr{J}} ab \in \mathsf{Reg}(S)$, then we have $(ab)^\sharp = a^\sharp b = ab^\sharp = a^\sharp b^\sharp$.*

Proof.

(1) Because $b \in \mathsf{Reg}(S)$, there is some $e \in \mathsf{E}(S)$ with $e =_{\mathscr{L}} b$, *i.e.*, $be = b$. By the extension of $\sharp$, we have $b^\sharp = be^\sharp$ and $ab^\sharp c = abe^\sharp c$. For $(abc)^\sharp$, we obtain $(abc)^\sharp = (abec)^\sharp = abe^\sharp c$.
(2) There is some $e \in \mathsf{E}(S)$ such that $a =_{\mathscr{L}} e =_{\mathscr{R}} b$, *i.e.*, $ae = a$ and $eb = b$. We have $(ab)^\sharp = ae^\sharp b = (ae)^\sharp b = a^\sharp b$, $(ab)^\sharp = ae^\sharp b = a(eb)^\sharp = ab^\sharp$, and $(ab)^\sharp = ae^\sharp b = ae^\sharp e^\sharp b = (ae)^\sharp (eb)^\sharp = a^\sharp b^\sharp$.

$\square$

From (1) with $c = 1$, we get $(ab)^\sharp = ab^\sharp$. Similarly, $(bc)^\sharp = b^\sharp c$, if $a = 1$.

If $a, b, c \in S$ are a smooth product, then we can play with a consistent mapping:

$$(abc)^\sharp = a^\sharp bc = ab^\sharp c = abc^\sharp = a^\sharp b^\sharp c = a^\sharp bc^\sharp = ab^\sharp c^\sharp = a^\sharp b^\sharp c^\sharp = (ab)^\sharp c^\sharp.$$

For the consistent mappings (stabilizations) used by Simon and Leung we have $e^\sharp = (e^\sharp)^\sharp$ for every $e \in \mathsf{E}(S)$. However, this property does not hold for every consistent mapping, as the following example shows.

Example 3.13. Consider the monoid over $M = \{1, \ldots, 9\}$ with the maximum operation defined by the usual ordering of the integers. It is easy to verify that the mapping defined by $x^\sharp := x + 1$ for $x \in \{1, \ldots, 8\}$ and $9^\sharp = 9$ is consistent. However, we have, *e.g.*, $2^\sharp = 3 \neq 4 = (2^\sharp)^\sharp$. $\square$

There is a characterization of consistent mappings: A mapping $\sharp : \mathsf{E}(S) \to \mathsf{E}(S)$ is consistent iff for every $a, b \in S^1$ with $ab, ba \in \mathsf{E}(S)$, we have $(ab)^\sharp = a(ba)^\sharp b$ [21].

3.3. The nested distance desert semiring

In this section, we develop a semiring $\mathcal{V}$ to describe nested distance desert automata in an algebraic way. In particular, we use matrices over $\mathcal{V}$ as transformation matrices. Recall that we defined $V := \{\angle_0, \Upsilon_0, \angle_1, \Upsilon_1, \ldots, \Upsilon_{h-1}, \angle_h\}$.

Let $h \in \mathbb{N}$. Let $\mathcal{V} = V \cup \{\omega, \infty\}$ and consider the ordering

$$\angle_0 \sqsubseteq \Upsilon_0 \sqsubseteq \angle_1 \sqsubseteq \Upsilon_1 \sqsubseteq \ldots \sqsubseteq \Upsilon_{h-1} \sqsubseteq \angle_h \sqsubseteq \omega \sqsubseteq \infty$$

on $\mathcal{V}$. We define a multiplication $\cdot$ on $\mathcal{V}$ as the maximum for $\sqsubseteq$. Let $\psi : V^+ \to \mathcal{V}$ be the canonical homomorphism.

Let $\pi \in V^+$. We say that we can *walk along* π *in a cycle* iff there is some $d \in \mathbb{N}$ such that for every $k \in \mathbb{N}$, we have $\Delta(\pi^k) \leq d$. We show that we can walk along π in a cycle iff $\psi(\pi) \in \{\Upsilon_0, \ldots, \Upsilon_{h-1}\}$. This is a key property of ψ. Indeed, assume that $\psi(\pi) = \angle_g$ for some $0 \leq g \leq h$. Then, π contains the letter $\angle_g$, *i.e.*, we have to pay an g-coin when we walk along π. By the definition of ψ, π does not contain $\Upsilon_g, \angle_{g+1}, \ldots, \Upsilon_{h-1}, \angle_h$. Thus, we cannot obtain g-coins in π. Hence, for every $k \in \mathbb{N}$, $\Delta(\pi^k) \geq k$, *i.e.*, we cannot walk along π in a cycle. Conversely, if $\psi(\pi) = \Upsilon_g$ for some $0 \leq g < h$, then can walk along π in a cycle, because we can obtain 0-coins, …, g-coins and we do not have to pay $(g+1)$-coins, …, h-coins along π. As a conclusion, the set of all words $\pi \in V^+$ along which we can walk in a cycle is a recognizable language of V^+ and $\psi : V^+ \to \mathcal{V} \setminus \{\omega, \infty\}$ is its syntactic homomorphism.

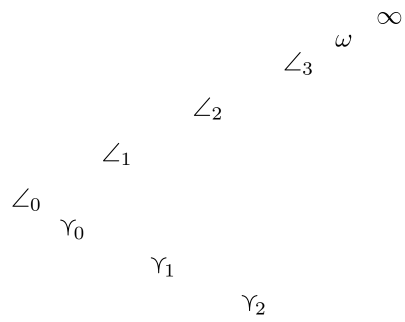
An extension of ψ to V^* is only possible by setting $\psi(\varepsilon) = \angle_0$, since otherwise, ψ is no longer a homomorphism. However, for every $k \in \mathbb{N}$, we have $\Delta(\varepsilon^k) = 0$. Hence, we do not have any longer the key property that we can walk along some path π in a cycle iff $\psi(\pi) \in \{\Upsilon_0, \ldots, \Upsilon_{h-1}\}$. Consequently, we rather leave $\psi(\varepsilon)$ undefined.

Now, consider the following ordering on $\leq$ on $\mathcal{V}$, which differs from $\sqsubseteq$:

$$\Upsilon_{h-1} \leq \Upsilon_{h-2} \leq \ldots \leq \Upsilon_0 \leq \angle_0 \leq \ldots \leq \angle_h \leq \omega \leq \infty. \tag{1}$$

Intuitively, $\leq$ reflects which transitions we prefer. Given the choice between two transitions marked resp. by Υ_g and Υ_{g-1} (for some $0 < g < h$), then we choose the transition marked by Υ_g, because $0, \ldots, g$-coins can be obtained at Υ_g, but just $0, \ldots, (g-1)$-coins can be obtained at Υ_{g-1}. Given the choice between two transitions marked resp. by $\angle_g$ and $\angle_{g+1}$ (for some $0 \leq g < h$), then we choose the transition marked by $\angle_g$, because $(g+1)$-coins are considered as more valuable than g-coins. We define an operation min on $\mathcal{V}$ as the minimum for $\leq$.

The following figure shows the relations $\sqsubseteq$ and $\leq$ for $h = 3$, where $\sqsubseteq$ corresponds to "left of" and $\leq$ corresponds to "below".



Let $z, z' \in \mathcal{V}$. If $z \sqsubseteq z'$ and $z \neq z'$, then we write $z \sqsubset z'$. We write $z < z'$ if $z \leq z'$ and $z \neq z'$.

Remark 3.14. Let $0 \leq g \leq h$. For every $z \sqsubseteq \angle_g$, we have $z \in \{\curlyvee_0, \ldots, \curlyvee_{g-1}, \angle_0, \ldots, \angle_g\}$, and thus, $z \leq \angle_g$. Similarly, $z \sqsubset \angle_g$ implies $z < \angle_g$.

Next, we show that the ordering $\leq$ on $\mathcal{V}$ is stable w.r.t. multiplication. We multiply the entire chain (1) by every member of $\mathcal{V}$. If we multiply (1) by ω (resp. ∞), then we obtain $\omega \leq \cdots \leq \omega \leq \infty$ (resp. $\infty \leq \cdots \leq \infty$), which is true. It is easy to see that (1) remains true if we multiply every element by $\curlyvee_g$ for $0 \leq g < h$ or by $\angle_g$ for $0 \leq g \leq h$.

As a consequence, for every $x, y, x', y' \in \mathcal{V}$ with $x \leq x'$ and $y \leq y'$, we have $xy \leq x'y \leq x'y'$.

Consequently, multiplication $\cdot$ on $\mathcal{V}$ distributes over min. Obviously, min and $\cdot$ are associative and commutative. Moreover, ∞ is a zero for $\cdot$ and an identity for min. Finally, $\angle_0$ is an identity for $\cdot$. Consequently, $(\mathcal{V}, \mathsf{min}, \cdot)$ is a commutative semiring which we call the *h-nested distance desert semiring.* We denote by $\mathcal{V}_{n\times n}$ the semiring of $n \times n$-matrices over $\mathcal{V}$.

For every $a, b \in \mathcal{V}_{n\times n}$ and every i, l, j, we have $(ab)[i,j] = \mathsf{min}_{1\leq k\leq n} a[i,k] \cdot b[k,j] \leq a[i,l] \cdot b[l,j]$.

Let $a, b \in \mathcal{V}_{n\times n}$. We denote $a \approx b$ if for every i, j we have $a[i,j] = \infty$ iff $b[i,j] = \infty$. It is straightforward to verify that $\approx$ is a congruence relation on $\mathcal{V}_{n\times n}$ and $\mathcal{V}_{n\times n}/\!\approx$ is isomorphic to the semiring of $n \times n$-matrices over the boolean semiring.

We close this section by a useful lemma for idempotent matrices.

Lemma 3.15. *Let $e \in \mathsf{E}(\mathcal{V}_{n\times n})$ and i, j be arbitrary. There is some l such that $e[i,j] = e[i,l] \cdot e[l,l] \cdot e[l,j]$.*

Proof. For every l, we have

$$e[i,j] = e^3[i,j] = \mathsf{min}_{1\leq k,k'\leq n} \left(e[i,k] \cdot e[k,k'] \cdot e[k',j]\right) \leq e[i,l] \cdot e[l,l] \cdot e[l,j].$$

Since $e = e^{n+2}$, there are $i = i_0, \ldots, i_{n+2} = j$ such that we have $e[i,j] = e[i_0, i_1] \cdots e[i_{n+1}, i_{n+2}]$. By a counting argument, there are $1 \leq p < q \leq (n+1)$ such that $i_p = i_q$. Let $l := i_p$. We have $e[i,l] = e^p[i,l] \leq e[i_0, i_1] \cdots e[i_{p-1}, i_p]$,

$e[l,l] = e^{q-p}[l,l] \le e[i_p, i_{p+1}] \cdots e[i_{q-1}, i_q]$, and $e[l,j] = e^{n+2-q}[l,j] \le e[i_q, i_{n+2}] \cdots e[i_{n+1}, i_{n+2}]$. Hence,

$$e[i,l] \cdot e[l,l] \cdot e[l,j] \;\le\; e[i_0, i_1] \cdots e[i_{n+1}, i_{n+2}] \;=\; e[i,j]$$

and the claim follows. □

Let $\mathcal{A} = [Q, E, I, F, \theta]$ be an h-nested distance desert automaton. Let $n := |Q|$ and assume $Q = \{1, \ldots, n\}$. We define a mapping $\Psi : \Sigma^+ \to \mathcal{V}_{n \times n}$ by setting for every $w \in \Sigma^+$, i, j

$$\Psi(w)[i,j] \;:=\; \mathsf{min}_{\pi \in i \overset{w}{\rightsquigarrow} j} \psi(\theta(\pi)).$$

It is well-known in the theory of weighted automata that Ψ is a homomorphism. It will be of crucial importance for the decidability of limitedness.

Let us mention that the semiring of $\mathcal{V}$ over the set $\mathcal{R} = \{\curlyvee_0, \angle_1, \omega, \infty\}$ was used by Simon and Leung to show the decidability of limitedness of distance automata [27, 28, 30, 32, 47, 49]. Similarly, the semiring of $\mathcal{V}$ over the set $\mathcal{D} = \{\angle_0, \curlyvee_0, \omega, \infty\}$ was used by the author to show the decidability of limitedness of desert automata [21, 22].

3.4. On the weights of words

We show some lemmas about the effects of the concatenation of words over V and their weights.

Lemma 3.16. *For every $\pi_1, \pi_2 \in V^+$, we have*

$$\mathsf{max}\big\{\Delta(\pi_1),\, \Delta(\pi_2)\big\} \;\le\; \Delta(\pi_1\pi_2) \;\le\; \Delta(\pi_1) + \Delta(\pi_2).$$

Proof. We have $\Delta(\pi_1) \le \Delta(\pi_1\pi_2)$ and $\Delta(\pi_2) \le \Delta(\pi_1\pi_2)$, because every factor of π_1 resp. π_2 is a factor of $\pi_1\pi_2$. We can show $\Delta(\pi_1\pi_2) \le \Delta(\pi_1) + \Delta(\pi_2)$, because every factor of $\pi_1\pi_2$ is a concatenation of a factor of π_1 and a factor of π_2. □

The bounds in Lemma 3.16 are sharp, just consider $\pi_1 := \pi_2 := \angle_0 \curlyvee_0$ (resp. $\pi_1 := \pi_2 := \angle_0\angle_0$).

Lemma 3.17.

(1) *Let $\pi \in V^+$ with $\psi(\pi) \in \{\angle_0, \ldots, \angle_h\}$. For every $k \ge 1$, we have $\Delta(\pi^k) \ge k$.*

(2) *Let $\pi \in V^+$ with $\psi(\pi) \in \{\curlyvee_0, \ldots, \curlyvee_{h-1}\}$. For every $k \ge 1$, we have $\Delta(\pi^k) \le 2\Delta(\pi)$ and $\Delta(\pi^k) < |\pi|$.*

Proof. (1) Let $0 \le g \le h$ such that $\psi(\pi) = \angle_g$. We have $\pi^k \in \{\angle_0, \curlyvee_0, \ldots, \angle_g\}^+$, and $|\pi^k|_g \ge k$, and thus, $|\Delta(\pi^k)| \ge k$.

(2) Let $0 \le g < h$ such that $\psi(\pi) = \curlyvee_g$. For every $g < g' \le h$, we have $|\pi^k|_{g'} = 0$. Now, let $0 \le g' \le g$, and let π' be a factor of π^k with $\pi' \in \{\angle_0, \curlyvee_0, \ldots, \curlyvee_{g'-1}, \angle_{g'}\}^*$. Because $\curlyvee_g$ occurs in π but not in π', we can factorize π' as $\pi' = \pi_1\pi_2$ for factors π_1, π_2 of π. We have

$$|\pi'|_{g'} \;=\; |\pi_1|_{g'} + |\pi_2|_{g'} \;\le\; 2\Delta(\pi).$$

Thus, $\Delta(\pi^k) \leq 2\Delta(\pi)$. We easily see $|\pi'| < |\pi|$, *i.e.*, $|\pi'|_{g'} < |\pi|$, and thus, $\Delta(\pi^k) < |\pi|$. □

The bounds in Lemma 3.17 are sharp: for (1), let $\pi = \curlyvee_0 \angle_1 \curlyvee_0$, and for (2), let $\pi = \angle_0 \angle_0 \curlyvee_0 \angle_0 \angle_0$.

Let $\mathbb{R}_+$ be the positive real numbers. For every $g \in \mathbb{N}$, we define a mapping $f_g : \mathbb{R}_+ \to \mathbb{R}_+$ by $f_g(x) := \sqrt[g+1]{x+1} - 1$ for $x \in \mathbb{R}_+$.

Lemma 3.18. *For every $g \in \mathbb{N}$, $x \in \mathbb{R}_+$, we have* $f_{g+1}(x) = f_g\left(\frac{x - f_{g+1}(x)}{f_{g+1}(x)+1}\right)$.

Proof. We have $\big(f_{g+1}(x)+1\big)^{g+2} = x+1$, *i.e.*, $\big(f_{g+1}(x)+1\big)^{g+1} = \dfrac{x+1}{f_{g+1}(x)+1}$, and further, $f_{g+1}(x) = \ldots$

$$= \sqrt[g+1]{\frac{x+1}{f_{g+1}(x)+1}} - 1 = \sqrt[g+1]{\frac{x - f_{g+1}(x)}{f_{g+1}(x)+1} + 1} - 1 = f_g\left(\frac{x - f_{g+1}(x)}{f_{g+1}(x)+1}\right). \quad \square$$

Lemma 3.19. *Let $0 \leq g \leq h$ be arbitrary. For every $k \geq 1$ and $\pi_1, \ldots, \pi_k \in V^+$ with $\psi(\pi_1 \ldots \pi_k) = \angle_g$ and $\psi(\pi_l) \in \{\angle_0, \ldots, \angle_h\}$ for every $0 \leq l \leq k$, we have $\Delta(\pi_1 \ldots \pi_k) \geq f_g(k) = \sqrt[g+1]{k+1} - 1$.*

At first, we sketch the idea to prove Lemma 3.19. For example, let $g = 2$ and $k = 124$, and $\pi_1, \ldots, \pi_{124} \in V^+$ as in the lemma. We denote $\pi := \pi_1, \ldots, \pi_{124}$. We have to show $\Delta(\pi) \geq 4$.

Let $P := \{\, l \,|\, 1 \leq l \leq 124,\ \psi(\pi_l) = \angle_2 \}$. We have $\Delta(\pi) \geq |P|$. If $|P| \geq 4$, then we are done.

Now, assume, *e.g.*, $P = \{10, 47, 93\}$. We consider the largest dense subset of $\{1, \ldots, 124\} \setminus P$, *i.e.*, we consider the set $\{48, \ldots, 92\}$ and examine the word $\pi' := \pi_{48}, \ldots, \pi_{92}$. We have $\psi(\pi') \in \{\angle_0, \angle_1\}$ by the definition of P and the assumptions on $\pi_1, \ldots, \pi_{124}$. By induction on g, we assume that the lemma is true for π'. Thus, we have $\Delta(\pi') \geq \sqrt[1]{46} - 1 = 45$ or $\Delta(\pi') \geq \sqrt[2]{46} - 1 > 5$.

There are $\pi_1, \ldots, \pi_{124} \in V^+$ with the above properties and $\Delta(\pi_1 \ldots \pi_{124}) = 4$. Just let $\pi_1' := {\angle_0}^4$, $\pi_2' := (\pi_1' \angle_1)^4 \pi_1'$, and $\pi := (\pi_2' \angle_2)^4 \pi_2'$, and let $\pi_1, \ldots, \pi_{124}$ be the letters of π. However, for $g = 2$ and $k = 125$, Lemma 3.19 shows $\Delta(\pi_1 \ldots \pi_{125}) \geq 5$.

Proof of Lemma 3.19. We show the lemma by an induction on g. At first, assume $g = 0$. We have $\pi_1 \ldots \pi_k \in {\angle_0}^+$ and $|\pi_1 \ldots \pi_k| \geq k$. Thus, $\Delta(\pi) \geq k = \sqrt[1]{k+1} - 1$.

Let $0 \leq g < h$. By induction, we assume that the claim is true for $0, \ldots, g$, and we show the claim for $g+1$. Choose some $k \geq 1$ and $\pi_1, \ldots, \pi_k \in V^+$ as in the lemma. Denote $\pi := \pi_1 \ldots \pi_k$. Let $P := \{\, l \,|\, 1 \leq l \leq k,\ \psi(\pi_l) = \angle_{g+1} \}$. If $|P| \geq f_{g+1}(k)$, then we have $\Delta(\pi) \geq f_{g+1}(k)$, because $\pi \in \{\angle_0, \curlyvee_0, \ldots, \angle_{g+1}\}^*$. We assume $|P| < f_{g+1}(k)$ in the rest of the proof.

We estimate the average cardinality of the maximal consecutive[2] subsets of $\{1,\ldots,k\}\setminus P$. There are at least $k-|P| > k-f_{g+1}(k)$ members in $\{1,\ldots,k\}\setminus P$. On the other hand, there are at most $|P|+1$, *i.e.*, at most $f_{g+1}(k)$ maximal consecutive subsets in $\{1,\ldots,k\}\setminus P$. Thus, the average cardinality of the maximal consecutive subsets of $\{1,\ldots,k\}\setminus P$ is at least

$$\frac{k-f_{g+1}(k)}{f_{g+1}(k)} =: k'.$$

Hence, there are $r \le s$ such that $\{r, r+1, \ldots, s\}$ is a subset of $\{1,\ldots,k\}\setminus P$ with a cardinality of at least k'. By the definition of P and the assumptions on $\pi, \pi_1, \ldots, \pi_k$, we have $\psi(\pi_r \ldots \pi_s) \in \{\angle_0, \ldots, \angle_g\}$. Let $g' \le g$ such that $\psi(\pi_r \ldots \pi_s) = \angle_{g'}$. We have $\Delta(\pi) \ge \ldots$

$$\Delta(\pi_r \ldots \pi_s) \ge f_{g'}(k') \ge f_g(k') = f_g\left(\frac{k-f_{g+1}(k)}{f_{g+1}(k)}\right) \ge f_g\left(\frac{k-f_{g+1}(k)}{f_{g+1}(k)+1}\right)$$

which yields $f_{g+1}(k)$ by Lemma 3.18. □

Lemma 3.20. *Let $k \ge 1$ and $\pi_1, \ldots, \pi_k \in V^+$ such that $\psi(\pi_1), \ldots, \psi(\pi_k) \in \{\curlyvee_0, \ldots, \curlyvee_{h-1}\}$. We have $\Delta(\pi_1 \ldots \pi_k) \le 2\max\big\{\Delta(\pi_1), \ldots, \Delta(\pi_k)\big\}$.*

Proof. Let $0 \le g \le h$ be arbitrary. Let π' be some factor of $\pi_1 \ldots \pi_k$ such that $\psi(\pi') = \angle_g$. We show $|\pi'|_g \le 2\max\big\{\Delta(\pi_1), \ldots, \Delta(\pi_k)\big\}$.

Case 1. There is some $1 \le l \le k$ such that π' is a factor of π_l.
We have $|\pi'|_g \le \Delta(\pi')$ (by $\psi(\pi') = \angle_g$) and $\Delta(\pi') \le \Delta(\pi_l)$ by Lemma 3.16.

Case 2. There are $1 \le l < l' \le k$ such that $\pi' = \tilde{\pi}_l \pi_{l+1} \ldots \pi_{l'-1} \tilde{\pi}_{l'}$, where $\tilde{\pi}_l$ (resp. $\tilde{\pi}_{l'}$) is a suffix of π_l (resp. prefix of $\pi_{l'}$).

By contradiction, assume that there is some $\angle_g$ in $\pi_{l+1} \ldots \pi_{l'-1}$, *i.e.*, $\angle_g \sqsubseteq \psi(\pi_{l+1} \ldots \pi_{l'-1})$. However, $\psi(\pi_{l+1} \ldots \pi_{l'-1}) \sqsubseteq \psi(\pi') = \angle_g$, *i.e.*, $\psi(\pi_{l+1} \ldots \pi_{l'-1}) = \angle_g$ which contradicts the assumption of the lemma. Hence, there is no $\angle_g$ in $\pi_{l+1} \ldots \pi_{l'-1}$. Thus, $|\pi'|_g = |\tilde{\pi}_l|_g + |\tilde{\pi}_{l'}|_g$.

We show $|\tilde{\pi}_l|_g \le \Delta(\pi_l)$. We have $\psi(\tilde{\pi}_l) \sqsubseteq \psi(\pi') = \angle_g$. If $\psi(\tilde{\pi}_l) \sqsubset \angle_g$, then $|\tilde{\pi}_l|_g = 0$. If $\psi(\tilde{\pi}_l) = \angle_g$, then $|\tilde{\pi}_l|_g \le \Delta(\tilde{\pi}_l) \le \Delta(\pi_l)$.

Similarly, we obtain $|\tilde{\pi}_{l'}|_g \le \Delta(\pi_{l'})$. Thus, $|\pi'|_g \le \Delta(\pi_l) + \Delta(\pi_{l'})$.

□

Lemma 3.21. *Let $k \ge 1$, $\pi_1, \ldots, \pi_{2k} \in V^*$, and $0 \le g < h$ such that for every $1 \le l \le k$:*

(1) *$\pi_{2l-1} = \varepsilon$ or $\psi(\pi_{2l-1}) \le \angle_g$ and*
(2) *$\pi_{2l} \ne \varepsilon$ and $\psi(\pi_{2l}) \le \curlyvee_g$.*

Then, we have $\Delta(\pi_1 \ldots \pi_{2k}) \le 4\max\big\{\Delta(\pi_1), \ldots, \Delta(\pi_{2k})\big\}$.

[2]We call some set $M \subseteq \{1,\ldots,k\}\setminus P$ *maximal consecutive*, if there are $r \le s$ such that $M = \{r, r+1, \ldots, s\}$ and $r-1 \notin M$, $s+1 \notin M$.

Proof. For every $1 \leq l \leq k$, we have $\psi(\pi_{2l-1}\pi_{2l}) \in \{\curlyvee_0, \ldots, \curlyvee_{h-1}\}$. Hence, we can apply Lemma 3.20 on $(\pi_1\pi_2), (\pi_3\pi_4), \ldots, (\pi_{2k-1}\pi_{2k})$. $\square$

Note that by Lemma 3.16, it follows that $\Delta(\pi_1 \ldots \pi_{2k-1})$, $\Delta(\pi_2 \ldots \pi_{2k})$, and $\Delta(\pi_2 \ldots \pi_{2k-1})$ are at most $4 \max\big\{\Delta(\pi_1), \ldots, \Delta(\pi_{2k})\big\}$.

4. The decidability of limitedness

In this section, we almost prove Theorem 2.1. Our solution is essentially a fusion and a further development of ideas from Hashiguchi, Leung, Simon, and the author [10,14,21,27,28,32,46,47,49]. We will prove (2)⇒(3) in Section 4.2 up to the bound on the $\sharp$-height of r which will be considered in Section 5.3. From Sections 4.3 to 4.5, we develop some tools to prove (1)⇒(2) in Theorem 2.1 in Section 4.6.

For the entire Section 4, let $h \in \mathbb{N}$ and $\mathcal{A} = [Q, E, I, F, \theta]$ be an h-nested distance desert automaton. Let $n := |Q|$ and assume $Q = \{1, \ldots, n\}$. We denote by T the transformation matrices of letters, *i.e.*, $T := \Psi(\Sigma)$. Clearly, $\langle T \rangle = \Psi(\Sigma^+)$.

4.1. Stabilization

We define a mapping $\sharp : \mathcal{V} \to \mathcal{V}$ which we call stabilization. For every $z \in \mathcal{V}$ let

$$z^\sharp := \begin{cases} z & \text{if } z \in \{\curlyvee_0, \ldots, \curlyvee_{h-1}\} \\ \omega & \text{if } z \in \{\angle_0, \ldots, \angle_h, \omega\} \\ \infty & \text{if } z = \infty. \end{cases}$$

We have $z \leq z^\sharp$ for every $z \in \mathcal{V}$.

If $z \in \{\curlyvee_0, \ldots, \curlyvee_{h-1}, \omega, \infty\}$, then we have $z = z^\sharp$, and thus, $zz^\sharp = z^\sharp z^\sharp = z^\sharp$. If $z \in \{\angle_0, \ldots, \angle_h\}$, then $z^\sharp = \omega$, and consequently, $zz^\sharp = z^\sharp$. To sum up, we have $zz^\sharp = z^\sharp z = z^\sharp$ for every $z \in \mathcal{V}$.

We define $\sharp : \mathsf{E}(\mathcal{V}_{n\times n}) \to \mathcal{V}_{n\times n}$. For every $e \in \mathsf{E}(\mathcal{V}_{n\times n})$ and i, j let

$$e^\sharp[i,j] := \mathsf{min}_{1\leq l\leq n} \big(e[i,l] \cdot (e[l,l])^\sharp \cdot e[l,j]\big).$$

This mapping is a joint generalization of Simon's and Leung's stabilization for idempotent matrices over $\mathcal{R}$ [27,28,32,47,49] and the author's stabilization for idempotent matrices over $\mathcal{D}$ [21,22].

We show a remark to get familiar with stabilization.

Remark 4.1. Let $e \in \mathsf{E}(\mathcal{V}_{n\times n})$ and i, j be arbitrary.

(1) Let $0 \leq g \leq g' \leq h$, and assume $e[i,j] = \angle_g$ but $e[j,j] = \curlyvee_{g'}$. It is easy to see that $e[i,j] = e^2[i,j] \leq e[i,j] \cdot e[j,j] = \curlyvee_{g'}$, which is a contradiction. Hence, i, j with these properties cannot exist. Similarly, it is impossible that for some $0 \leq g < g' \leq h$, we have $e[i,j] = \curlyvee_g$ and $e[j,j] = \curlyvee_{g'}$.

(2) We have $e^\sharp[i,j] \neq \angle_0$ by the definition of $e^\sharp$.

(3) We have $e[i,j] = e^3[i,j] \leq e^\sharp[i,j]$.
(4) Assume $e[i,j] \neq \infty$. By Lemma 3.15, there is some l such that $e[i,l] \cdot e[l,l] \cdot e[l,j] = e[i,j]$. Consequently, $\infty \neq e[i,l] \cdot (e[l,l])^\sharp \cdot e[l,j] \geq e^\sharp[i,j]$, *i.e.*, $e^\sharp[i,j] \neq \infty$.
Together with (3), we obtain $e \approx e^\sharp$.
(5) If $e[i,j] = \omega$, then (3) and (4) imply $e^\sharp[i,j] = \omega$.
(6) If $e[i,i] \in \{\curlyvee_0, \ldots, \curlyvee_{h-1}\}$, then $e^\sharp[i,i] \leq e[i,i] \cdot (e[i,i])^\sharp \cdot e[i,i] = e[i,i]$ by the definition of stabilization. In combination with (3), we obtain $e^\sharp[i,i] = e[i,i]$. $\square$

For subsets $T \subseteq \mathcal{V}_{n\times n}$ we define $\langle T\rangle^\sharp$ as the least subset of $\mathcal{V}_{n\times n}$ which contains T and is closed both under matrix multiplication and stabilization $\sharp$ of idempotent matrices. It is easy to see that $\langle T\rangle^\sharp$ can be effectively computed.

4.2. On $\sharp$-expressions

Recall that we defined the notion of a $\sharp$-expression already in Section 2.2.

We associate a type in $\mathcal{V}_{n\times n}$ to some $\sharp$-expressions. Every $a \in \Sigma$ is a $\sharp$-expression *of type* $\tau(a) := \Psi(a)$. If r and s are $\sharp$-expressions and $\tau(r)$ and $\tau(s)$ are defined, then rs is *of type* $\tau(rs) := \tau(r)\tau(s)$. If r is a $\sharp$-expression, $\tau(r)$ is defined, and $\tau(r) \in \mathsf{E}(\mathcal{V}_{n\times n})$, then $r^\sharp$ is *of type* $\tau(r^\sharp) := \tau(r)^\sharp$. If for a $\sharp$-expression r, $\tau(r)$ is defined, then r is called a *typed $\sharp$-expression*.

For every typed $\sharp$-expression r, we have $\tau(r) \in \langle T\rangle^\sharp$. Moreover, for every $a \in \langle T\rangle^\sharp$, there is a $\sharp$-expression r such that $\tau(r) = a$.

Lemma 4.2. *Let r be a typed $\sharp$-expression and $k \geq 1$.*

(1) *Let i,j be arbitrary. There is some path $i \overset{r(k)}{\rightsquigarrow} j$ iff $\tau(r)[i,j] < \infty$.*
(2) *We have $r(k) \in L(\mathcal{A})$ iff $I \cdot \tau(r) \cdot F < \infty$.*

Proof. We show (1). If r is a letter, then we have $\tau(r)[i,j] = \Psi(r)[i,j]$, and the claim is obvious.

Let r and s be typed $\sharp$-expressions and assume by induction that (1) is true for r and s.

Assume that there is some path in $i \overset{rs(k)}{\rightsquigarrow} j$. Hence, there is some l such that there are paths in $i \overset{r(k)}{\rightsquigarrow} l$ and $l \overset{s(k)}{\rightsquigarrow} j$. Thus, $\tau(r)[i,l] \neq \infty$ and $\tau(s)[l,j] \neq \infty$. Consequently, $\tau(rs)[i,j] = \big(\tau(r)\tau(s)\big)[i,j] \leq \tau(r)[i,l] \cdot \tau(s)[l,j] < \infty$.

Conversely, assume $\tau(rs)[i,j] < \infty$. Hence, there is some l such that $\tau(r)[i,l] \cdot \tau(s)[l,j] < \infty$, *i.e.*, $\tau(r)[i,l] < \infty$ and $\tau(s)[l,j] < \infty$. Thus, there are paths in $i \overset{r(k)}{\rightsquigarrow} l$ and $l \overset{s(k)}{\rightsquigarrow} j$. Since $r(k) \cdot s(k) = rs(k)$, there is some path in $i \overset{rs(k)}{\rightsquigarrow} j$.

Let r be a typed $\sharp$-expression such that $\tau(r) \in \mathsf{E}(\mathcal{V}_{n\times n})$ and assume that r satisfies (1).

Assume that there is some path in $i \overset{r^\sharp(k)}{\rightsquigarrow} j$. Since $r^\sharp(k) = (r(k))^k$, there are $i = i_0, \ldots, i_k = j$ such that for every $1 \leq l \leq k$, there is some path $i_{l-1} \overset{r(k)}{\rightsquigarrow} i_l$. Hence, for every $1 \leq l \leq k$, we have $\tau(r)[i_{l-1}, i_l] \leq \infty$. Thus, $\tau(r)^k[i,j] < \infty$, *i.e.*, $\tau(r) \leq \infty$, and by Remark 4.1(4), we have $\tau(r^\sharp)[i,j] = \tau(r)^\sharp[i,j] \leq \infty$.

Conversely, assume $\tau(r)[i,j] < \infty$. Since $\tau(r)[i,j] = \tau(r)^k[i,j]$ there are $i = i_0, \ldots, i_k = j$ such that for every $1 \leq l \leq k$, we have $\tau(r)[i_{l-1}, i_l] < \infty$, and hence, $\tau(r^\sharp)[i_{l-1}, i_l] < \infty$. Thus, for every $1 \leq l \leq k$, there is some path $i_{l-1} \overset{r(k)}{\rightsquigarrow} i_l$. Consequently, there is some path $i \overset{r^\sharp(k)}{\rightsquigarrow} j$.

Assertion (2) is an immediate consequence of (1). $\square$

Proposition 4.3. *Let r be a typed $\sharp$-expression.*
For every bound $d \geq 0$, there is some $K \geq 1$ such that for every $k \geq K$, we have: For every i, j and every path $\pi \in i \overset{r(k)}{\rightsquigarrow} j$ such that $\psi(\theta(\pi)) < \tau(r)[i,j]$, we have $\Delta(\theta(\pi)) \geq d$.

Proof. We proceed by an induction on typed $\sharp$-expressions.

If r is just a letter, then $\tau(r)[i,j] = \Psi(r)[i,j]$. Paths $\pi \in i \overset{r}{\rightsquigarrow} j$ such that $\psi(\theta(\pi)) < \tau(r)[i,j] = \Psi(r)[i,j]$ cannot exist, and we are done.

Let r and s be typed $\sharp$-expressions and assume that the claim is true for r and s. Let $d \geq 0$ be arbitrary, and let K be the maximum of the corresponding integers K for r and s, and let $k \geq K$.

Let i, j be arbitrary, and let $\pi \in i \overset{rs(k)}{\rightsquigarrow} j$ such that $\psi(\theta(\pi)) < \tau(r)[i,j]$. There is some l, $\pi_1 \in i \overset{r(k)}{\rightsquigarrow} l$, and $\pi_2 \in l \overset{s(k)}{\rightsquigarrow} j$ such that $\pi = \pi_1\pi_2$.

If $\psi(\theta(\pi_1)) < \tau(r)[i,l]$, then we have by induction $\Delta(\pi_1) \geq d$, *i.e.*, $\Delta(\pi_1\pi_2) \geq d$. If $\psi(\theta(\pi_2)) < \tau(s)[l,j]$, then we have $\Delta(\pi_1\pi_2) \geq d$ in the same way. It remains to consider the case that $\psi(\theta(\pi_1)) \geq \tau(r)[i,l]$ and $\psi(\theta(\pi_2)) \geq \tau(s)[l,j]$. We obtain

$$\psi(\theta(\pi)) \;=\; \psi(\theta(\pi_1))\psi(\theta(\pi_2)) \;\geq\; \tau(r)[i,l] \cdot \tau(s)[l,j] \;\geq\; \tau(rs)[i,j],$$

and we are done.

Finally, let r be a typed $\sharp$-expression such that $\tau(r) \in \mathsf{E}(\mathcal{V}_{n \times n})$ and assume that the claim is true for r. We show the claim for $r^\sharp$. Let $e := \tau(r)$. Let $d \geq 0$ be arbitrary, and let K be an integer which satisfies the condition

$$\sqrt[h+1]{\frac{K-1}{n}} - 1 \;\geq\; d.$$

Moreover, we assume that K is not smaller than the corresponding integer for r. Let $k \geq K$ and $\pi \in i \overset{r^\sharp(k)}{\rightsquigarrow} j$. There are $i = i_0, \ldots, i_k = j$ and for every $1 \leq p \leq k$, some $\pi_p \in i_{p-1} \overset{r(k)}{\rightsquigarrow} i_p$ such that $\pi = \pi_1 \ldots \pi_k$.

Let $1 \leq p \leq k$ be arbitrary. If $\psi(\theta(\pi_p)) < e[i_{p-1}, i_p]$, then we have by the inductive hypothesis $\Delta(\pi_p) \geq d$, and thus, $\Delta(\pi) \geq d$, and we are done. Hence, we assume $\psi(\theta(\pi_p)) \geq \tau(r)[i_{p-1}, i_p]$ in the rest of the proof.

Let $1 \leq p < q \leq k$ be arbitrary. We have

$$\psi(\theta(\pi_{p+1} \ldots \pi_q)) = \psi(\theta(\pi_{p+1}) \cdots \psi(\theta(\pi_q)) \;\geq\; \ldots$$
$$\cdots \geq\; e[i_p, i_{p+1}] \cdots e[i_{q-1}, i_q] \;\geq\; e^{q-p}[i_p, i_q] \;=\; e[i_p, i_q]. \qquad (2)$$

There is some l such that the set $I := \{p \mid 1 \leq p < k,\, i_p = l\}$ contains at least $\frac{k-1}{n}$ members.

Case 1. There are $p < q \in I$ such that $\psi(\theta(\pi_{p+1} \dots \pi_q)) \in \{\Upsilon_0, \dots, \Upsilon_{h-1}\}$.

By 2, we have $e[l,l] = e[i_p, i_q] \leq \psi(\theta(\pi_{p+1} \dots \pi_q))$, *i.e.*, we have $e[l,l] \in \{\Upsilon_0, \dots, \Upsilon_{h-1}\}$, and in particular, $(e[l,l])^\sharp = e[l,l]$. In the same way, we obtain $e[i,l] \leq \psi(\theta(\pi_1 \dots \pi_p))$ and $e[l,j] \leq \psi(\theta(\pi_{q+1} \dots \pi_k))$. To sum up,

$$\psi(\theta(\pi)) = \psi(\theta(\pi_1 \dots \pi_k)) \;\geq\; \dots$$

$$\dots \;\geq\; e[i,l] \cdot e[l,l] \cdot e[l,j] \;=\; e[i,l] \cdot (e[l,l])^\sharp \cdot e[l,j] \;\geq\; e^\sharp[i,j].$$

Hence, $\psi(\theta(\pi)) \geq \tau(r^\sharp)[i,j]$, and we are done.

Case 2. For every $p < q \in I$, we have $\psi(\theta(\pi_{p+1} \dots \pi_q)) \in \{\angle_0, \dots, \angle_h\}$.

There are at least $|I| - 1$ consecutive factors in π on which the image under $\psi \circ \theta$ belongs to $\{\angle_0, \dots, \angle_h\}$. By Lemma 3.19, we have $\Delta(\pi) \geq \sqrt[h+1]{|I|} - 1$. By $|I| \geq \frac{k-1}{n}$ and $k \geq K$, we have $|I| \geq \frac{K-1}{n}$. By the choice of K, we obtain $\Delta(\pi) \geq \sqrt[h+1]{\frac{K-1}{n}} - 1 \geq d$. □

Proposition 4.4. *Let r be a typed $\sharp$-expression such that $I \cdot \tau(r) \cdot F = \omega$. For every $k \geq 1$, we have $r(k) \in L(\mathcal{A})$, and $\Delta_{\mathcal{A}}\big(r(k)\big)$ is unbounded for increasing integers k.*

Proof. Let r be a typed $\sharp$-expression such that $I \cdot \tau(r) \cdot F = \omega$. For every $k \geq 1$, we have $r(k) \in L(\mathcal{A})$ by Lemma 4.2(2).

Let $d \geq 0$ be arbitrary, and let K be the integer provided by Proposition 4.3. To prove the assertion, we show that for every $k \geq K$, we have $\Delta(r(k)) \geq d$. Let $k \geq K$ and let π be a successful path for $r(k)$. Let $i \in I$ and $j \in F$ be the first (resp. last state of π). By Lemma 4.2(1), we have $\tau(r)[i,j] \neq \infty$, and since $I \cdot \tau(r) \cdot F = \omega$, we have $\tau(r)[i,j] \geq \omega$, *i.e.*, $\tau(r)[i,j] = \omega$. Moreover, $\psi(\theta(\pi)) < \omega = \tau(r)[i,j]$. By Proposition 4.3, we have $\Delta(\theta(\pi)) \geq d$. From the arbitrary choice of π, it follows $\Delta(r(k)) \geq d$. □

Proposition 4.4 almost proves (2)⇒(3) in Theorem 2.1. However, we have to invest some more ideas to construct a desired $\sharp$-expression r which is of $\sharp$-height of at most $(h+1)n$.

4.3. Stabilization is a consistent mapping

The aim of this section is to show that stabilization is a consistent mapping. At first, we show a preliminary lemma:

Lemma 4.5. *Let $e \in \mathsf{E}(\mathcal{V}_{n \times n})$. We have $e^\sharp = ee^\sharp = e^\sharp e = ee^\sharp e = e^\sharp e^\sharp$.*

Proof. Let i, j be arbitrary.

At first, we show $(ee^\sharp)[i,j] \geq e^\sharp[i,j]$. Let k such that $(ee^\sharp)[i,j] = e[i,k] \cdot e^\sharp[k,j]$, and let l such that $e^\sharp[k,j] = e[k,l] \cdot (e[l,l])^\sharp \cdot e[l,j]$. We obtain

$$(ee^\sharp)[i,j] \;=\; e[i,k] \cdot e[k,l] \cdot (e[l,l])^\sharp \cdot e[l,j] \;\geq\; e[i,l] \cdot (e[l,l])^\sharp \cdot e[l,j] \;\geq\; e^\sharp[i,j].$$

Now, we show $e^\sharp[i,j] \geq (ee^\sharp)[i,j]$. Let l such that $e^\sharp[i,j] = e[i,l] \cdot (e[l,l])^\sharp \cdot e[l,j]$, and let k such that $e[i,l] = e[i,k] \cdot e[k,l]$. We obtain

$$e^\sharp[i,j] \;=\; e[i,k] \cdot e[k,l] \cdot (e[l,l])^\sharp \cdot e[l,j] \;\geq\; e[i,k] \cdot e^\sharp[k,j] \;\geq\; (ee^\sharp)[i,j].$$

To sum up, $e^\sharp[i,j] = (ee^\sharp)[i,j]$, *i.e.*, $e^\sharp = ee^\sharp$. We can show $e^\sharp = e^\sharp e$ in a symmetric way, and from $e^\sharp = ee^\sharp = e^\sharp e$, we obtain immediately $e^\sharp = ee^\sharp e$.

It remains to show $e^\sharp = e^\sharp e^\sharp$. By Remark 4.1(3), we have $e^\sharp \geq e$, and hence, $e^\sharp e^\sharp \geq ee^\sharp = e^\sharp$. Let i, j be arbitrary. We show $(e^\sharp e^\sharp)[i,j] \leq e^\sharp[i,j]$. Let l such that $e^\sharp[i,j] = e[i,l] \cdot (e[l,l])^\sharp \cdot e[l,j]$. Since for every $z \in \mathcal{V}$, we have $z^\sharp = z^\sharp z^\sharp = z^\sharp z z z^\sharp$, we obtain

$$e^\sharp[i,j] \;=\; e[i,l] \cdot (e[l,l])^\sharp \cdot e[l,j] \;=\; e[i,l] \cdot (e[l,l])^\sharp \cdot e[l,l]) \cdot e[l,l] \cdot (e[l,l])^\sharp \cdot e[l,j] \;\geq$$

$$\ldots \geq e^\sharp[i,l] \cdot e^\sharp[l,j] \;\geq\; (e^\sharp e^\sharp)[i,j].$$

To sum up, $e^\sharp \geq e^\sharp e^\sharp$. □

Proposition 4.6. *Stabilization $\sharp$ on $\mathcal{V}_{n\times n}$ is a consistent mapping.*

Proof. By Lemma 4.5, we have $e^\sharp \in \mathsf{E}(\mathcal{V}_{n\times n})$ for every $e \in \mathsf{E}(\mathcal{V}_{n\times n})$. Hence, $\sharp$ is indeed a mapping from $\mathsf{E}(\mathcal{V}_{n\times n})$ to $\mathsf{E}(\mathcal{V}_{n\times n})$.

Let $a, b \in \mathcal{V}_{n\times n}$ and let $e, f \in \mathsf{E}(\mathcal{V}_{n\times n})$ such that $e =_{\mathscr{J}} f$, and in particular, $f = aeb$. To show that $\sharp$ is a consistent mapping, we have to show $f^\sharp = ae^\sharp b$.

We have $f = (ae)(eb)$ and $(ae) =_{\mathscr{L}} (eb) =_{\mathscr{L}} e$. We show $e = ebae$. We denote Green's relations between e, f, ae, and eb in the following egg-box picture:

eb		e	
f		ae	

Because the idempotent f belongs to $\mathscr{L}(eb) \cap \mathscr{R}(ae)$, we have by Lemma 3.5, $(eb)(ae) =_{\mathscr{L}} e$. Thus, $ebae =_{\mathscr{L}} e$, and $ebae =_{\mathscr{R}} e$, *i.e.*, $ebae =_{\mathscr{H}} e$. Moreover, we have $(ebae)(ebae) = ebfae = ebae \in \mathsf{E}(S)$. By Lemma 3.4, $\mathscr{H}(e)$ is a group. Thus, there is exactly one idempotent in $\mathscr{H}(e)$, and hence, $ebae = e$.

We show $f^\sharp \leq ae^\sharp b$. Let i, j be arbitrary.

Let r, s such that $aee^\sharp eb[i,j] = (ae)[i,r] \cdot e^\sharp[r,s] \cdot (eb)[s,j]$.

Let l such that $e^\sharp[r,s] = e[r,l] \cdot \big(e[l,l]\big)^\sharp \cdot e[l,s]$.

Let l' such that $e[l,l] = (eb)[l,l'] \cdot (ae)[l',l]$. By setting $x = (eb)[l,l']$ and $y = (ae)[l',l]$, we obtain

$$\big(e[l,l]\big)^\sharp = (xy)^\sharp = x(yx)^\sharp y = (eb)[l,l'] \cdot \Big((ae)[l',l] \cdot (eb)[l,l']\Big)^\sharp \cdot (ae)[l',l].$$

We have

$$aee^\sharp eb[i,j] = (ae)[i,r] \cdot \underbrace{e[r,l] \cdot \big(e[l,l]\big)^\sharp \cdot e[l,s]}_{=e^\sharp[r,s]} \cdot (eb)[s,j] = \ldots$$

$$(ae)[i,r] \cdot e[r,l] \cdot \underbrace{(eb)[l,l'] \cdot \Big((ae)[l',l] \cdot (eb)[l,l']\Big)^\sharp \cdot (ae)[l',l]}_{=(e[l,l])^\sharp} \cdot e[l,s] \cdot (eb)[s,j] \geq \ldots$$

$$(aeb)[i,l'] \cdot \Big((aeb)[l',l']\Big)^\sharp \cdot (aeb)[l',j] = f[i,l'] \cdot (f[l',l'])^\sharp \cdot f[l',j] \geq f^\sharp[i,j].$$

Hence, $(aee^\sharp eb)[i,j] \geq f^\sharp[i,j]$. By Lemma 4.5, we have $(ae^\sharp b)[i,j] \geq f^\sharp[i,j]$, *i.e.*, $(ae^\sharp b) \geq f^\sharp$.

We have seen $ebfae = e$. As above, we can show $ebf^\sharp ae \geq e^\sharp$. Hence, we have $aebf^\sharp aeb \geq ae^\sharp b$, *i.e.*, $ff^\sharp f \geq ae^\sharp b$, and by Lemma 4.5, $f^\sharp \geq ae^\sharp b$.

To sum up, $f^\sharp = ae^\sharp b$. □

By Lemma 4.6 and Corollary 3.10 we have a natural extension of stabilization to $\mathsf{Reg}(\mathcal{V}_{n\times n})$, and we can use Lemma 3.12 as a very convenient tool whenever we prove some assertion concerning stabilization.

At this point, we have to be very careful with the definition of $\langle T\rangle^\sharp$. Let $a \in \mathsf{Reg}(\langle T\rangle^\sharp)$. There is some $e \in \mathsf{E}(\langle T\rangle^\sharp)$ with $e =_{\mathscr{L}} a$. Then, $ae = a$ and $a^\sharp = ae^\sharp$, and thus, $a^\sharp \in \langle T\rangle^\sharp$, or more precisely, $a^\sharp \in \mathsf{Reg}(\langle T\rangle^\sharp)$. Consequently, $\langle T\rangle^\sharp$ is closed under stabilization of matrices in $\mathsf{Reg}(\langle T\rangle^\sharp)$.

However, for $b \in \mathsf{Reg}(\mathcal{V}_{n\times n})$, it is possible that $b \notin \mathsf{Reg}(\langle T\rangle^\sharp)$ and $b^\sharp \notin \langle T\rangle^\sharp$.

In the definition of $\langle T\rangle^\sharp$ we demand closure under stabilization of idempotents. *After* the definition of $\langle T\rangle^\sharp$ is given, we proved closure under stabilization of matrices which are regular in $\langle T\rangle^\sharp$.

If one defines $\langle T\rangle^\sharp$ in a way that $\langle T\rangle^\sharp$ has to be closed under stabilization of matrices which are regular in $\langle T\rangle^\sharp$, then the definition becomes a mess, because the term "regular matrix in $\langle T\rangle^\sharp$" does not have a meaning unless $\langle T\rangle^\sharp$ is defined.

4.4. Stabilization of regular matrices

We show two crucial lemmas about stabilization of regular matrices in $\mathcal{V}_{n\times n}$. Since $\mathsf{Reg}(\langle T\rangle^\sharp) \subseteq \mathsf{Reg}(\mathcal{V}_{n\times n})$, we can apply both lemmas for matrices in $\mathsf{Reg}(\langle T\rangle^\sharp)$.

Lemma 4.7. *For every $a \in \mathsf{Reg}(\mathcal{V}_{n\times n})$, we have*

(1) $a \leq a^\sharp$, $a \approx a^\sharp$, *and*
(2) *for every i,j, $a^\sharp[i,j] \neq \angle_0$.*

Proof. Let $e \in \mathsf{E}(\mathcal{V}_{n\times n})$ with $e =_{\mathscr{L}} a$, *i.e.*, $a = ae$, and $a^\sharp = ae^\sharp$. (1) is an immediate conclusion from Remark 4.1(3)(4), and the stability of $\leq$ and $\approx$ under matrix multiplication.

We have (2), because $\angle_0$ cannot occur in $e^\sharp$ by Remark 4.1(2). $\square$

Lemma 4.8. *Let $a, b, c \in \mathcal{V}_{n\times n}$ be a smooth product in $\mathcal{V}_{n\times n}$ and let i, j such that we have $(abc)^\sharp[i,j] \in \{\angle_1, \ldots, \angle_h\}$. Then, there are p, q such that*

(1) $a[i,p] \cdot b^\sharp[p,q] \cdot c[q,j] \;=\; (abc)^\sharp[i,j]$;
(2) $b^\sharp[p,q] \;<\; (abc)^\sharp[i,j]$, *and* $b^\sharp[p,q] \;\sqsubset\; (abc)^\sharp[i,j]$.

The reader should be aware that we state and prove Lemma 4.8 for smooth products in $\mathcal{V}_{n\times n}$, *i.e.*, for $a, b, c \in \mathcal{V}_{n\times n}$ satisfying $a =_{\mathscr{L}} b =_{\mathscr{L}} c =_{\mathscr{L}} abc$ in $\mathcal{V}_{n\times n}$.

Now, let $a, b, c \in \langle T\rangle^\sharp$ and assume that $a =_{\mathscr{L}} b =_{\mathscr{L}} c =_{\mathscr{L}} abc$ holds in $\langle T\rangle^\sharp$. Hence, we have $a, b, c, abc \in \mathsf{Reg}(\langle T\rangle^\sharp) \subseteq \mathsf{Reg}(\mathcal{V}_{n\times n})$ and it holds $a =_{\mathscr{L}} b =_{\mathscr{L}} c =_{\mathscr{L}} abc$ holds in $\mathcal{V}_{n\times n}$. Consequently, a, b, c satisfy the assumptions of Lemma 4.8, *i.e.*, we can apply Lemma 4.8 on a, b, c and we have $(abc)^\sharp \in \mathsf{Reg}(\langle T\rangle^\sharp)$.

Proof of Lemma 4.8. Let $0 \leq g \leq h$ such that $(abc)^\sharp[i,j] = \angle_g$. We denote Green's relations between a, b, c and their products in the following egg-box picture:

a		ab		abc
e		b		bc
d		f		c

By $a =_{\mathscr{L}} b =_{\mathscr{L}} ab$, there is an idempotent $e \in \mathscr{L}(a) \cap \mathscr{R}(b)$, and similarly, there is an idempotent $f \in \mathscr{L}(b) \cap \mathscr{R}(c)$. By Lemma 3.8, there is a $d \in \mathscr{L}(e) \cap \mathscr{R}(f)$ such that $bd = e$ and $db = f$.

We have $a = ae$, $abc = aebc$, and $(abc)^\sharp = ae^\sharp bc$.

Because $(abc)^\sharp[i,j] = ae^\sharp bc[i,j] = \angle_g$, there are r, s such that $a[i,r] \cdot e^\sharp[r,s] \cdot (bc)[s,j] = \angle_g$, and in particular $e^\sharp[r,s] \sqsubseteq \angle_g$. By the definition of stabilization, there is some p such that we have $e[r,p] \cdot (e[p,p])^\sharp \cdot e[p,s] \;=\; e^\sharp[r,s] \;\sqsubseteq\; \angle_g$, and hence, $(e[p,p])^\sharp \sqsubseteq \angle_g$. In combination with $(e[p,p])^\sharp \in \{\Upsilon_0, \ldots, \Upsilon_{h-1}, \omega, \infty\}$, we obtain $(e[p,p])^\sharp \in \{\Upsilon_0, \ldots, \Upsilon_{g-1}\}$, *i.e.*, $e[p,p] \in \{\Upsilon_0, \ldots, \Upsilon_{g-1}\}$. By Remark 4.1(6), we get $e^\sharp[p,p] = e[p,p]$. By $(e[p,p])^\sharp \in \{\Upsilon_0, \ldots, \Upsilon_{g-1}\}$, we have in particular $e^\sharp[p,p] = e[p,p] \sqsubset \angle_g$.

We have $(bc)[s,j] \sqsubseteq \angle_g$ and $e[p,s] \sqsubseteq \angle_g$, and by Remark 3.14, $(bc)[s,j] \leq \angle_g$ and $e[p,s] \leq \angle_g$.

By $e^\sharp = b^\sharp d$, there is some q such that $b^\sharp[p,q] \cdot d[q,p] = e^\sharp[p,p] \sqsubset \angle_g$, *i.e.*, $b^\sharp[p,q] \sqsubset \angle_g$ and $d[q,p] \sqsubset \angle_g$. By Remark 3.14, we have $b^\sharp[p,q] < \angle_g$ and $d[q,p] < \angle_g$ which proves (2).

We have $a[i,r] \sqsubseteq \angle_g$ and $e[r,p] \sqsubseteq \angle_g$, and thus, $a[i,r] \cdot e[r,p] \sqsubseteq \angle_g$, and by Remark 3.14, $a[i,r] \cdot e[r,p] \leq \angle_g$. Thus, $ae[i,p] = a[i,p] \leq \angle_g$.

We have $c = ffc = dbdbc = debc$, *i.e.*, $c[q,j] = (debc)[q,j] \;\leq\; d[q,p] \cdot e[p,s] \cdot (bc)[s,j] \;\leq\; \angle_g \cdot \angle_g \cdot \angle_g = \angle_g$, *i.e.*, $c[q,j] \leq \angle_g$.

Thus, $a[i,p]\cdot b^\sharp[p,q]\cdot c[q,j] \le \angle_g\cdot\angle_g\cdot\angle_g \le \angle_g$. On the other hand, $a[i,p]\cdot b^\sharp[p,q]\cdot c[q,j] \ge (ab^\sharp c)[i,j] = (abc)^\sharp[i,j] = \angle_g$. Consequently, $a[i,p]\cdot b^\sharp[p,q]\cdot c[q,j] = \angle_g$ which proves (1). □

A generalization of Lemma 4.8 to the case $(abc)^\sharp[i,j] = \angle_0$ is vacuously true, because we have $(abc)^\sharp[i,j] \ne \angle_0$ by Lemma 4.7(2). A generalization for $(abc)^\sharp[i,j] \in \{\Upsilon_0,\dots,\Upsilon_{h-1}\}$ is not possible, just let $a = b = c$ be the matrix in which every entry is Υ_0.

4.5. On the growth of entries

We consider pairs in $\mathcal{V}_{n\times n}\times\Sigma^+$. For every pair $(a,w)\in\mathcal{V}_{n\times n}\times\Sigma^+$, let $\Delta'(a,w)$ be the least non-negative integer such that for every i,j with $a[i,j]\notin\{\omega,\infty\}$ there is some $\pi\in i\overset{w}{\rightsquigarrow} j$ such that $\psi(\theta(\pi))\le a[i,j]$ and $\Delta(\theta(\pi))\le\Delta'(a,w)$. If such an integer does not exist, then we set $\Delta'(a,w):=\infty$. More precisely, we set

$$\Delta'(a,w) \;:=\; \mathsf{max}_{i,j,\,a[i,j]\notin\{\omega,\infty\}}\ \mathsf{min}\Big\{\Delta(\theta(\pi))\,\Big|\,\pi\in i\overset{w}{\rightsquigarrow} j,\ \psi(\theta(\pi))\le a[i,j]\Big\}.$$

The cartesian product $\mathcal{V}_{n\times n}\times\Sigma^+$ is a semigroup in a natural way whereas the operation is componentwise multiplication in $\mathcal{V}_{n\times n}$ and concatenation of words.

Proposition 4.9. *Let $k\ge 1$ and $(a_1,w_1),\dots,(a_k,w_k)\in\mathcal{V}_{n\times n}\times\Sigma^+$.*

(1) *We have $\Delta'\big(a_1\dots a_k,\, w_1\dots w_k\big) \le k\cdot\mathsf{max}_{1\le l\le k}\Delta'(a_l,w_l)$.*

(2) *If $a_1,\dots,a_k$ are a smooth product in $\mathcal{V}_{n\times n}$, then*

$$\Delta'\big((a_1\dots a_k)^\sharp,\, w_1\dots w_k\big) \;\le\; 2^{3h-1}\cdot\mathsf{max}_{1\le l\le k}\Delta'(a_l,w_l).$$

The most important fact in Proposition 4.9 is that the bound 2^{3h-1} in (2) does not depend on k. Although the bound 2^{3h-1} seems to be very large, this bound holds in contrast to (1) for arbitrarily large k.

Proof of Proposition 4.9. We denote $d:=\mathsf{max}_{1\le l\le k}\Delta'(a_l,w_l)$, $a:=a_1\cdots a_k$, and $w:=w_1\dots w_k$.

(1) Let i,j such that $a[i,j]\notin\{\omega,\infty\}$. There are $i=i_0,\dots,i_k=j$ such that $a[i,j]=a_1[i_0,i_1]\cdots a_k[i_{k-1},i_k]$. For every $1\le l\le k$, there is some $\pi_l\in i_{l-1}\overset{w_l}{\rightsquigarrow} i_l$ such that $\psi(\theta(\pi_l))\le a_l[i_{l-1},i_l]$ and $\Delta(\theta(\pi_l))\le d$. We have $\pi_1\dots\pi_k\in i\overset{w}{\rightsquigarrow} j$, $\Delta(\theta(\pi_1\dots\pi_k))\le kd$, and

$$\psi(\theta(\pi_1\dots\pi_k)) = \psi(\theta(\pi_1))\cdots\psi(\theta(\pi_k)) \;\le\; a_1[i_0,i_1]\cdots a_k[i_{k-1},i_k] \;=\; a[i,j],$$

and (1) follows.

To show (2), we show the following two claims (a),(b). Let i,j, and $0\le g\le h$ be arbitrary.

(a) If $a^\sharp[i,j]=\angle_g$, then there is a path $\pi\in i\overset{w}{\rightsquigarrow} j$ such that $\psi(\theta(\pi))\le\angle_g$ and $\Delta(\theta(\pi))\le 2^{3g-1}d$.

(b) If $a^\sharp[i,j] = \Upsilon_g$, then there is a path $\pi \in i \overset{w}{\rightsquigarrow} j$ such that $\psi(\theta(\pi)) \leq \Upsilon_g$ and $\Delta(\theta(\pi)) \leq 2^{3g+1}d$.

We show both claims by an induction over the ordering $\sqsubseteq$ of $\mathcal{V}$.

For $\angle_0$, claim (a) is vacuously true, because $a^\sharp[i,j] \neq \angle_0$ by Lemma 4.7(2).

For $k=1$, both claims (a) and (b) are obvious. For $k=2$, both claims (a) and (b) follow from assertion (1) since $a \leq a^\sharp$ and $2 = k \leq 2^{3g-1}$ for $g \geq 1$ (resp. $2 = k \leq 2^{3g+1}$ for $g \geq 0$). However, for $k=3$ we cannot simply use assertion (1) since for $g=0$ in (b) we have $k = 3 \not\leq 2^{3\cdot 0+1} = 2$.

In the rest of the proof we assume $k \geq 3$.

We show (b) for Υ_0. By Lemma 3.12(2), $(a_1 \ldots a_k)^\sharp = a_1^\sharp \ldots a_k^\sharp$. Thus, we have $(a_1^\sharp \ldots a_k^\sharp)[i,j] = \Upsilon_0$, *i.e.*, there are $i = i_0, \ldots, i_k = j$ such that for every $1 \leq l \leq k$, we have $a_l^\sharp[i_{l-1}, i_l] \in \{\Upsilon_0, \angle_0\}$, and by Lemma 4.7(2), $a_l^\sharp[i_{l-1}, i_l] = \Upsilon_0$.

Let $1 \leq l \leq k$. By Lemma 4.7(1), we have $a_l[i_{l-1}, i_l] \in \{\Upsilon_0, \ldots, \Upsilon_{h-1}\}$. Let $\pi_l \in i_{l-1} \overset{w_l}{\rightsquigarrow} i_l$ with $\psi(\theta(\pi_l)) \leq a_l[i_{l-1}, i_l]$ and $\Delta(\theta(\pi_l)) \leq d$. Then, $\psi(\theta(\pi_1 \ldots \pi_k)) \in \{\Upsilon_0, \ldots, \Upsilon_{h-1}\}$, *i.e.*, $\psi(\theta(\pi_1 \ldots \pi_k)) \leq \Upsilon_0$. By Lemma 3.20, $\Delta(\theta(\pi_1 \ldots \pi_k)) \leq 2d$. Hence, $\pi := \pi_1 \ldots \pi_k$ proves the claim.

Next, we show (a) for some $1 \leq g \leq h$, *i.e.*, we assume $(a_1 \ldots a_k)^\sharp[i,j] = \angle_g$. By induction, we assume that both (a) and (b) are true for $0 \leq g' < g$.

Since $k \geq 3$, we can apply Lemma 4.8 on $a_1(a_2 \ldots a_{k-1})a_k$. Let p, q be from Lemma 4.8. Let $\pi_1 \in i \overset{w_1}{\rightsquigarrow} p$ with $\psi(\theta(\pi_1)) \leq a_1[i,p]$ and $\Delta(\theta(\pi_1)) \leq d$. Similarly, let $\pi_k \in q \overset{w_k}{\rightsquigarrow} j$ with $\psi(\theta(\pi_{|w|})) \leq a_k[p,j]$ and $\Delta(\theta(\pi_k)) \leq d$.

Let $z := (a_2 \ldots a_{k-1})^\sharp[p,q]$. By Lemma 4.8(2), we have $z \sqsubset (a_1 \ldots a_k)[i,j] = \angle_g$, *i.e.*, we can apply the inductive hypothesis on $(a_2 \ldots a_{k-1})[p,q]$. Thus, there is a $\tilde{\pi} \in p \overset{w_2 \ldots w_{k-1}}{\rightsquigarrow} q$ such that $\psi(\theta(\tilde{\pi})) \leq (a_2 \ldots a_{k-1})^\sharp[p,q]$ and $\Delta(\theta(\tilde{\pi})) \leq 2^{3g-2}d$. We have by Lemma 4.8(1)

$$\psi(\theta(\pi_1)) \cdot \psi(\theta(\tilde{\pi})) \cdot \psi(\theta(\pi_k)) \;\leq\; a_1[i,p] \cdot (a_2 \ldots a_{k-1})^\sharp[p,q] \cdot a_k[q,j] \;=\; a^\sharp[i,j].$$

Moreover, we have $\Delta(\theta(\pi_1 \tilde{\pi} \pi_k)) \leq 2d + 2^{3g-2}d \leq (2 + 2^{3g-2})d \leq 2^{3g-1}d$, *i.e.*, claim (a) is true for $\pi := \pi_1 \tilde{\pi} \pi_k$. The estimation $(2 + 2^{3g-2}) \leq 2^{3g-1}$ is rough, but we rather want to avoid technical overhead.

We show (b) for some $1 \leq g < h$, *i.e.*, we assume $(a_1 \ldots a_k)^\sharp[i,j] = \Upsilon_g$. By induction, we assume that (a) and (b) are true for $\angle_0, \ldots, \angle_g$ (resp. $\Upsilon_0, \ldots, \Upsilon_{g-1}$).

By Lemma 3.12(2), we have $(a_1 \ldots a_k)^\sharp[i,j] = (a_1^\sharp \ldots a_k^\sharp)[i,j] = \Upsilon_g$. There is at least one sequence $i = i_0, \ldots, i_k = j$, such that

$$a_1^\sharp[i_0, i_1] \cdots a_k^\sharp[i_{k-1}, i_k] \;=\; \Upsilon_g.$$

We choose some sequence $i_0, \ldots, i_k$ with this property such that $a_l^\sharp[i_{l-1}, i_l] = \Upsilon_g$ for as many $1 \leq l \leq k$ as possible. There is at least one l with $a_l^\sharp[i_{l-1}, i_l] = \Upsilon_g$.

Now, let $1 \leq r \leq s \leq k$ such that $z_2 := a_r^\sharp[i_{r-1}, i_r] \ldots a_s^\sharp[i_{s-1}, i_s] \sqsubset \Upsilon_g$. Moreover, we assume that either $r = 1$ or $a_{r-1}^\sharp[i_{r-2}, i_{r-1}] = \Upsilon_g$. Similarly, we assume that $s = k$ or $a_{s+1}^\sharp[i_s, i_{s+1}] = \Upsilon_g$. Let us mention that it is not clear whether such r and s exist. However, the existence of r, s is not really important, we just want to develop some arguments which are required *if* there are r, s with these properties.

Let $z := (a_r^\sharp \dots a_s^\sharp)[i_{r-1}, i_s]$. We derive information on z.

- We have $z \le z_2 \sqsubset \curlyvee_g$. We can write $z_2 \sqsubset \curlyvee_g$ as $z_2 \in \{\angle_0, \curlyvee_0, \dots, \angle_g\}$. Hence, $z \le z_2$ simply means that z is less than or equal to (by $\le$) some element among $\{\angle_0, \curlyvee_0, \dots, \angle_g\}$. Consequently, z is less than or equal to (by $\le$) the biggest element (by $\le$) in $\{\angle_0, \curlyvee_0, \dots, \angle_g\}$, *i.e.*, $z \le \angle_g$.
- By contradiction, assume $z = \curlyvee_g$. If $r = s$, then we have $z_2 = \curlyvee_g$ which contradicts the choice of r and s. If $r < s$, then we can replace in $i_0, \dots, i_k$ the indices $i_r, \dots, i_{s-1}$ by $i'_r, \dots, i'_{s-1}$ such that $a_r^\sharp[i_{r-1}, i'_r] \cdots a_s^\sharp[i'_{s-1}, i_s] = \curlyvee_g$. Then, we have a contradiction to the choice of $i_0, \dots, i_{|w|}$, or more precisely, to the condition that $a_l^\sharp[i_{l-1}, i_l] = \curlyvee_g$ for as many l as possible.
- By contradiction, assume $z \in \{\curlyvee_{g+1}, \dots, \curlyvee_{h-1}\}$. We conclude

$$\curlyvee_g \;=\; (a_1^\sharp \dots a_k^\sharp)[i, j] \;\le\; \dots$$

$$\underbrace{a_1^\sharp[i_0, i_1] \cdots a_{r-1}^\sharp[i_{r-2}, i_{r-1}]}_{z_1 :=} \cdot \underbrace{\Big((a_r^\sharp \dots a_s^\sharp)[i_{r-1}, i_s]\Big)}_{= z} \cdot \underbrace{a_{s+1}^\sharp[i_s, i_{s+1}] \cdots a_k^\sharp[i_{k-1}, i_k]}_{z_3 :=}$$

  We have $z_1 z_2 z_3 = \curlyvee_g$, and thus, $z_1 \sqsubseteq \curlyvee_g$ and $z_3 \sqsubseteq \curlyvee_g$. By $\curlyvee_g \sqsubset z$, we have $z_1 z z_3 = z$. Thus, $\curlyvee_g \le z$ which contradicts the assumption $z \in \{\curlyvee_{g+1}, \dots, \curlyvee_{h-1}\}$.

By combining $z \le \angle_g$ and $z \notin \{\curlyvee_g, \dots, \curlyvee_{h-1}\}$, we obtain $(a_r^\sharp \cdots a_s^\sharp)[i_{r-1}, i_s] = z \sqsubset \curlyvee_g$. Thus, we can apply the inductive hypothesis on $(a_r \dots a_s)[i_{r-1}, i_s]$.

Consequently, there is some $\pi_{r,s} \in i_{r-1} \overset{w_r \dots w_s}{\rightsquigarrow} i_s$ such that we have $\Delta(\theta(\pi_{r,s})) \le 2^{3g-1}d$ and $\psi(\theta(\pi_{r,s})) \le (a_r \cdots a_s)^\sharp[i_{r-1}, i_s] = z \sqsubset \curlyvee_g$. We can denote $z \sqsubset \curlyvee_g$ as $z \in \{\angle_0, \curlyvee_0, \dots, \angle_g\}$. Hence, $\psi(\theta(\pi_{r,s})) \le z$ means than $\psi(\theta(\pi_{r,s}))$ is less (by $\le$) than the biggest element (by $\le$) of the set $\{\angle_0, \curlyvee_0, \dots, \angle_g\}$. Thus, $\psi(\theta(\pi_{r,s})) \le \angle_g$, *i.e.*, $\psi(\theta(\pi_{r,s})) \in \{\curlyvee_0, \dots, \curlyvee_{h-1}, \angle_0, \dots, \angle_g\}$. We assume such a path $\pi_{r,s}$ for every $1 \le r \le s \le k$ with the above properties.

For every $1 \le l \le k$ with $a_l^\sharp[i_{l-1}, i_l] = \curlyvee_g$, let $\pi_l \in i_{l-1} \overset{w_l}{\rightsquigarrow} i_l$ with $\Delta(\theta(\pi_l)) \le d$ and $\psi(\theta(\pi_l)) = a_l[i_{l-1}, i_l] \le a_l^\sharp[i_{l-1}, i_l] = \curlyvee_g$. Thus, $\psi(\theta(\pi_l)) \in \{\curlyvee_g, \dots, \curlyvee_{h-1}\}$. Note that there is at least one l with these properties.

Let π be the concatenation of all the paths $\pi_{r,s}$ and π_l "in correct order". Note that by the choice of r and s, there are no two consecutive $\pi_{r,s}$ factors in this concatenation. We have $\pi \in i \overset{w}{\rightsquigarrow} j$. By Lemma 3.21, we have $\Delta(\theta(\pi)) \le 4 \cdot 2^{3g-1}d = 2^{3g+1}d$.

For (b), it remains to show $\psi(\theta(\pi)) \le \curlyvee_g$. Since $\psi(\theta(\pi_l)) \in \{\curlyvee_g, \dots, \curlyvee_{h-1}\}$ and $\psi(\theta(\pi_{r,s})) \notin \{\angle_{g+1}, \dots, \angle_h\}$, we get $\psi(\theta(\pi)) \in \{\curlyvee_g, \dots, \curlyvee_{h-1}\}$, and hence, $\psi(\theta(\pi)) \le \curlyvee_g$. $\square$

4.6. The Proof of (1)⇒(2) in Theorem 2.1

In this section, we prove (1)⇒(2) in Theorem 2.1.

Proposition 4.10. *Let $I' \subsetneq I \subseteq \langle T\rangle^\sharp$ be ideals of $\langle T\rangle^\sharp$ such that $I \setminus I'$ is a $\mathscr{J}$-class of $\langle T\rangle^\sharp$.*
For every $k \geq 1$, $(a_1, w_1), \ldots, (a_k, w_k) \in \mathcal{V}_{n\times n} \times \Sigma^+$ satisfying

A1: $a_1, \ldots, a_k \in \langle T\rangle^\sharp$,
A2: *for every* $1 \leq l < k$, $a_l a_{l+1} \in I$,

there are $k' \geq 1$, $(a'_1, w'_1), \ldots, (a'_{k'}, w'_{k'}) \in \mathcal{V}_{n\times n} \times \Sigma^+$ such that

C1: $a'_1, \ldots, a'_{k'} \in \langle T\rangle^\sharp$,
C2: *for every* $1 \leq l < k'$, $a'_l a'_{l+1} \in I'$,
C3: $w'_1 \ldots w'_{k'} = w_1 \ldots w_k$, $\quad a_1 \cdots a_k \approx a'_1 \cdots a'_{k'}$ *, and*
C4: $\mathsf{max}_{1\leq l\leq k'} \Delta'\big(a'_l, w'_l\big) \leq 2^{3h+2} \cdot \mathsf{max}_{1\leq l\leq k} \Delta'\big(a_l, w_l\big)$.

Proof. We denote $d := \mathsf{max}_{1\leq l\leq k} \Delta'\big(a_l, w_l\big)$.

We factorize $(a_1, w_1), \ldots, (a_k, w_k)$. If $a_1 \in I'$, then let $v_1 := (a_1, w_1)$ and $l = 1$. If $a_1 \notin I'$, then let $1 \leq l \leq k$ be the largest integer such that $a_1 \cdots a_l \notin I'$ and set $v_1 = (a_1, w_1), \ldots, (a_l, w_l)$. If $l < k$, then we apply the same procedure to $(a_{l+1}, w_{l+1}), \ldots, (a_k, w_k)$ and obtain v_2. By repeating this procedure as many times as possible, we achieve sequences $v_1, \ldots, v_{k'}$ over $\mathcal{V}_{n\times n} \times \Sigma^+$ with $v_1 \ldots v_{k'} = (a_1, w_1), \ldots, (a_k, w_k)$. Note that k' is simply defined as the number of words which we obtain in the factorization.

Let $1 \leq l \leq k'$ be arbitrary. We define (a'_l, w'_l) from v_l. Let m be the number of pairs in v_l, and denote $v_l = (b_1, u_1), \ldots, (b_m, u_m)$. Let $w'_l := u_1 \ldots u_m$. Hence, we have $w'_1 \ldots w'_{k'} = w_1 \ldots w_k$. The definition of a'_l is more involved.

Case 1. $m \leq 3$.
We set $a'_l = b_1 \ldots b_m$. Then, $a'_l \in \langle T\rangle^\sharp$ (C1) and $b_1 \ldots b_m \approx a'_l$. By Lemma 4.9(1), we have (C4) for (a'_l, w'_l) since $m \leq 3 \leq 2^{3h+2}$.

Case 2. $m > 3$, m is even.
We set $a'_l := (b_1 \ldots b_m)^\sharp$. For this, we have to ensure that $b_1 \ldots b_m \in \mathsf{Reg}(\mathcal{V}_{n\times n})$. However, for (C1), we even have to show $b_1 \ldots b_m \in \mathsf{Reg}(\langle T\rangle^\sharp)$. Let $1 \leq p < m$ be arbitrary. By (A2), we have $b_p b_{p+1} \in I$ and $b_1 \ldots b_m \in I$. Since $m \geq 4$, we have by the definition of v_l and (A2), $b_1 \ldots b_m \notin I'$, and in particular $b_p b_{p+1} \notin I'$. Consequently, $b_1 b_2$, $b_3 b_4$, $\ldots$, $b_{m-1} b_m \in I \setminus I'$ and $b_1 \ldots b_m \in I \setminus I'$. Hence, $I \setminus I'$ is a regular $\mathscr{J}$-class of $\langle T\rangle^\sharp$, *i.e.*, $b_1 b_2$, $b_3 b_4$, $\ldots$, $b_{m-1} b_m$ are a smooth product in $\langle T\rangle^\sharp$. Thus, a'_l is defined and we have $a'_l \in \langle T\rangle^\sharp$ (C1) and $b_1 \ldots b_m \approx a'_l$.

We apply Lemma 4.9(2) on the sequence $(b_1 b_2, u_1 u_2)$, $(b_3 b_4, u_3 u_4)$, $\ldots$, $(b_{m-1} b_m, u_{m-1} u_m)$. For every odd $1 \leq p < m$, $\Delta'(b_p b_{p+1},\, u_p u_{p+1}) \leq 2d$. By Lemma 4.9(2), it follows

$$\Delta'(a'_l, u_1 \ldots u_m) \;\leq\; \Delta'\big((b_1 \ldots b_m)^\sharp,\, u_1 \ldots u_m\big) \;\leq\; 2^{3h} d,$$

i.e., $(a'_l, u_1 \ldots u_m)$ satisfies (C4).

Case 3. $m > 3$, m is odd.
We proceed as in case 2, but we consider the sequence $(b_1b_2b_3, u_1u_2u_3)$, $(b_4b_5, u_4u_5), \ldots, (b_{m-1}b_m, u_{m-1}u_m)$. We get $\Delta'(a'_l, u_1 \ldots u_m) \le 3 \cdot 2^{3h-1}d$, *i.e.*, $(a'_l, u_1 \ldots u_m)$ satisfies (C4).

Since $b_1 \cdots b_m \approx a'_l$ in each case, we have $a_1 \cdots a_k \approx a'_1 \cdots a'_{k'}$ which completes the proof of (C3).

It remains to show (C2). Let $1 \le l < k'$. We denote $v_l = (b_1, u_1), \ldots, (b_m, u_m)$. Moreover, we denote $v_{l+1} = (\hat{b}_1, \hat{u}_1), \ldots, (\hat{b}_{\hat{m}}, \hat{u}_{\hat{m}})$. We have $a'_l = b_1 \ldots b_m$ or $a'_l = (b_1 \ldots b_m)^\sharp$, and hence, $a'_l \le_{\mathscr{J}} b_1 \ldots b_m$, and similarly, $a'_{l+1} \le_{\mathscr{R}} \hat{b}_1 \ldots \hat{b}_{\hat{m}}$ (*cf.* Rem. 3.11). Consequently, $a'_l a'_{l+1} \le_{\mathscr{J}} b_1 \ldots b_m \hat{b}_1 \ldots \hat{b}_{\hat{m}}$. By our factorization method to obtain v_l and v_{l+1}, above, we have $b_1 \in I'$, $\hat{b}_1 \in I'$, or $b_1 \ldots b_m \hat{b}_1 \in I'$. To sum up, $a'_l a'_{l+1} \in I'$ (C2). □

Proof of (1)⇒(2) in Theorem 2.1. We assume that for every $a \in \langle T \rangle^\sharp$, we have $I \cdot a \cdot F \ne \omega$, and we show that $\mathcal{A}$ is limited. Let $w \in L(\mathcal{A})$ be an arbitrary, non-empty word. Let $k := |w|$ and denote $w = c_1 \ldots c_k$.

Let y be the number of $\mathscr{J}$-classes of $\langle T \rangle^\sharp$. Let $\langle T \rangle^\sharp = I_1 \supsetneq I_2 \supsetneq \cdots \supsetneq I_y \supsetneq I_{y+1} = \emptyset$ be ideals of $\langle T \rangle^\sharp$ such that for every $1 \le l \le y$ the set $I_l \setminus I_{l+1}$ is a $\mathscr{J}$-class of $\langle T \rangle^\sharp$.

Consider the sequence $\big(\Psi(c_1), c_1\big), \ldots, \big(\Psi(c_k), c_k\big)$. We apply Proposition 4.10 inductively y times for $I_0, \ldots, I_y$ on this sequence. Initially, $I = \langle T \rangle^\sharp$, and hence, (A2) is satisfied. Clearly, $\Psi(c_1), \ldots \Psi(c_k) \in T \subseteq \langle T \rangle^\sharp$, *i.e.*, (A1) is satisfied. In each application of Proposition 4.10, (C1) and (C2) provide (A1) and (A2) for the next application. In the last application, $I' = \emptyset$, and thus, (C2) implies $k' = 1$. Hence, we obtain a single pair (a, w), and we have by (C3, C4), $a \approx \Psi(c_1) \ldots \Psi(c_m) = \Psi(w)$ and $a \in \langle T \rangle^\sharp$. For every $1 \le l \le k$, we have $\Delta'\big(\Psi(c_l), c_l\big) \le 1$. By (C4) and since $y \le |\mathcal{V}_{n\times n}| = (2h+3)^{n^2}$, we get

$$\Delta'\big(a, w\big) \;\le\; 2^{(3h+2)(2h+3)^{n^2}}.$$

Since $w \in L(\mathcal{A})$, we have $I \cdot \Psi(w) \cdot F \ne \infty$, and since $a \approx \Psi(w)$, we get $I \cdot a \cdot F \ne \infty$. Since $a \in \langle T \rangle^\sharp$, we get $I \cdot a \cdot F \ne \omega$. Consequently, $I \cdot a \cdot F < \omega$. Hence, there is a successful path π in $\mathcal{A}$ with the label w and $\Delta(\pi) \le 2^{(3h+2)(2h+3)^{n^2}}$, *i.e.*, $\mathcal{A}$ is limited. □

5. On the complexity

5.1. The stabilization hierarchy

Let $T \subseteq \mathcal{V}_{n\times n}$ and set $T_0 := \langle T \rangle$. For every $p \in \mathbb{N}$, let

$$T_{p+1} := \Big\langle\; T_p \;\cup\; \big\{ e^\sharp \,\big|\, e \in \mathsf{E}(T_p) \big\} \;\Big\rangle.$$

We call $T_0 \subseteq T_1 \subseteq T_2 \ldots$ the *stabilization hierarchy of* T. Moreover, it is easy to see that $\langle T \rangle^\sharp = \bigcup_{p \geq 0} T_p$. Because for every $p \geq 0$, T_p is a subset of the finite set $\mathcal{V}_{n\times n}$, we have $T_{|\mathcal{V}_{n\times n}|} = T_{|\mathcal{V}_{n\times n}|+1}$, and hence, $\langle T \rangle^\sharp = T_{|\mathcal{V}_{n\times n}|}$.

A key question for the complexity of limitedness of nested distance desert automata is: at which level does the stabilization hierarchy collapse? This question was already raised by Leung in the framework of distance automata in 1987 [27].

Recall that $\mathcal{R} = \{\Upsilon_0, \angle_1, \omega, \infty\}$. For $T \subseteq \mathcal{R}_{n\times n}$, Leung conjectured $\langle T \rangle^\sharp = T_{n^2}$ [27], p. 38. In [47], p. 112 it was conjectured that there is a polynomial $B : \mathbb{N} \to \mathbb{N}$ such that $\langle T \rangle^\sharp = T_{B(n)}$ for every $T \subseteq \mathcal{R}_{n\times n}$. In [30], p. 522, the existence of such a polynomial B was again considered as an open question. This question was very important, because the existence of such a polynomial B implies that limitedness of distance automata is decidable in PSPACE [27,30].

However, in 1998, Leung suggested another strategy. He mentioned that limitedness of distance automata is decidable in PSPACE if there is some polynomial $C : \mathbb{N} \to \mathbb{N}$ such that every n-state distance automaton is either limited by $2^{C(n)}$ or unlimited [32]. Indeed, Hashiguchi showed that this assertion is true for $C(n) = 4n^3 + n\,\mathsf{ld}(n+2) + n \leq 4n^3 + n^2 + 2n$ [15–17]. Leung and Podolskiy improved this bound to $C(n) = 3n^3 + n\,\mathsf{ld}\,n + n - 1$ [33], and hence, limitedness of distance automata is decidable in PSPACE.

However, it remained open whether there is a polonium B for the collapse of the stabilization hierarchy. Let us mention that Leung showed for every $n \geq 2$ some set $T \subseteq \mathcal{R}_{n\times n}$ such that $T_{n-2} \subsetneq T_{n-1} = \langle T \rangle^\sharp$, *i.e.*, setting $B(n) := n-2$ is not sufficient [30].

Below, we will positively answer Leung's conjecture by showing $T_n = \langle T \rangle^\sharp$ in Corollary 5.6(2).

5.2. Index classes

Let $e \in \mathsf{E}(\mathcal{V}_{n\times n})$ and $0 \leq g \leq h$. We define a relation $\sim_{e,g}$ on $\{1,\ldots,n\}$ by setting

$$i \sim_{e,g} j \qquad :\Longleftrightarrow \qquad e[i,j] \leq \angle_g \text{ and } e[j,i] \leq \angle_g$$

for every i, j. Clearly, $\sim_{e,g}$ is symmetric, and since e is idempotent, $\sim_{e,g}$ is transitive. If for some i, there is a j such that $i \sim_{e,g} j$, then we have $i \sim_{e,g} i$. Consequently, the restriction of $\sim_{e,g}$ to the set

$$Z_{e,g} := \{ i \mid \text{there is some } j \text{ such that } i \sim_{e,g} j \}$$

is reflexive, *i.e.*, $\sim_{e,g}$ is an equivalence relation on $Z_{e,g}$. By *equivalence class of* $\sim_{e,g}$ we mean an equivalence class of $\sim_{e,g}$ on $Z_{e,g}$. For every $i \in Z_{e,g}$, we denote by $[i]_{e,g}$ the equivalence class of i. We denote by $\mathsf{Cl}(e,g)$ the set of equivalence classes of $\sim_{e,g}$.

Lemma 5.1. *Let* $e, f \in \mathsf{E}(\mathcal{V}_{n\times n})$ *such that* $e \geq_{\mathscr{J}} f$ *and* $0 \leq g \leq h$. *We have* $|\mathsf{Cl}(e,g)| \geq |\mathsf{Cl}(f,g)|$.

Proof. Let $a, b \in \mathcal{V}_{n\times n}$ such that $aeb = f$. We assume $ae = a$ and $eb = b$. If a and b do not satisfy these conditions, then we proceed the proof for $a' = ae$ and $b' = eb$.

We construct a partial surjective mapping $\beta : \mathsf{Cl}(e, g) \dashrightarrow \mathsf{Cl}(f, g)$. The mapping β depends on the choice of a and b. For every i, j with $i \sim_{e,g} i$ and $j \sim_{f,g} j$ satisfying $a[j, i] \cdot e[i, i] \cdot b[i, j] \le \angle_g$, we set $\beta([i]_{e,g}) := [j]_{f,g}$. To complete the proof, we have to show that β is well defined and that β is indeed surjective.

We show that β is well defined. Let i, i' such that $i \sim_{e,g} i$ and $i' \sim_{e,g} i'$. Moreover, let j, j' such that $j \sim_{f,g} j$ and $j' \sim_{f,g} j'$. Assume $a[j, i]\cdot e[i, i]\cdot b[i, j] \le \angle_g$ and $a[j', i'] \cdot e[i', i'] \cdot b[i', j'] \le \angle_g$. Thus, $\beta([i]_{e,g}) = [j]_{f,g}$ and $\beta([i']_{e,g}) = [j']_{f,g}$. To show that β is well defined, we have to show that if $[i]_{e,g} = [i']_{e,g}$, then we have $[j]_{f,g} = [j']_{f,g}$. Assume $[i]_{e,g} = [i']_{e,g}$, *i.e.*, $i \sim_{e,g} i'$. Hence, $e[i, i'] \le \angle_g$. Above, we assumed $a[j, i] \cdot e[i, i] \cdot b[i, j] \le \angle_g$, and thus, $a[j, i] \le \angle_g$. Similarly, $b[i', j'] \le \angle_g$. Consequently, $a[j, i] \cdot e[i, i'] \cdot b[i', j'] \le \angle_g$, *i.e.*, $f[j, j'] = (aeb)[j, j'] \le \angle_g$. By symmetry, we achieve $f[j', j] \le \angle_g$, and hence, $j \sim_{f,g} j'$.

We show that β is surjective. Let j such that $j \sim_{f,g} j$. We have to show some i such that $\beta([i]_{e,g}) = [j]_{f,g}$. Since $j \sim_{f,g} j$, we have $f[j, j] \le \angle_g$. Since $f = aeb$, there are k, l such that $a[j, k] \cdot e[k, l] \cdot b[l, j] \le \angle_g$, and in particular, $e[k, l] \le \angle_g$. By Lemma 3.15, there is some i such that $e[k, i] \cdot e[i, i] \cdot e[i, l] = e[k, l] \le \angle_g$, and in particular, $e[i, i] \le \angle_g$. We have $a[j, i] = (ae)[j, i] \le a[j, k] \cdot e[k, i] \le \angle_g$, and $b[i, j] = (eb)[i, j] = e[i, l] \cdot b[l, j] \le \angle_g$. To sum up, $a[j, i] \cdot e[i, i] \cdot b[i, j] \le \angle_g$, and hence, $\beta([i]_{e,g}) = [j]_{f,g}$. □

Lemma 5.2. *Let $e \in \mathsf{E}(\mathcal{V}_{n\times n})$ and $0 \le g \le h$. We have $\mathsf{Cl}(e, g) \supseteq \mathsf{Cl}(e^\sharp, g)$.*

Proof. Let i be such that $i \sim_{e^\sharp,g} i$. We show $[i]_{e^\sharp,g} = [i]_{e,g}$.

For every j with $i \sim_{e^\sharp,g} j$, we have by Remark 4.1(3), $i \sim_{e,g} j$, *i.e.*, $[i]_{e^\sharp,g} \subseteq [i]_{e,g}$.

Conversely, let $j \in [i]_{e,g}$. Hence, $e[i, j] \le \angle_g$. Since $i \sim_{e^\sharp,g} i$, we have $e^\sharp[i, i] \le \angle_g$. To sum up,

$$e^\sharp[i, j] = (e^\sharp e)[i, j] \le e^\sharp[i, i] \cdot e[i, j] \le \angle_g \cdot \angle_g = \angle_g,$$

and by symmetry, $e^\sharp[j, i] \le \angle_g$, *i.e.*, $i \sim_{e^\sharp,g} j$. Hence, $j \in [i]_{e^\sharp,g}$. □

Lemma 5.3. *Let $e \in \mathsf{E}(\mathcal{V}_{n\times n})$ and assume $e \ne e^\sharp$. There is some $0 \le g \le h$ such that we have $\mathsf{Cl}(e, g) \supsetneq \mathsf{Cl}(e^\sharp, g)$ and for some l, $e[l, l] = \angle_g$.*

Proof. Let i, j such that $e[i, j] \ne e^\sharp[i, j]$. By Lemma 3.15, there is some l such that $e[i, j] = e[i, l] \cdot e[l, l] \cdot e[l, j]$. By contradiction, assume $e[l, l] = e^\sharp[l, l]$. Hence,

$$e^\sharp[i, j] = (ee^\sharp e)[i, j] \le e[i, l] \cdot e^\sharp[l, l] \cdot e[l, j] = e[i, l] \cdot e[l, l] \cdot e[l, j] = e[i, j],$$

i.e., $e^\sharp[i, j] = e[i, j]$ which is a contradiction. Consequently, $e[l, l] < e^\sharp[l, l]$.

By Remark 4.1(4, 5, 6), we have $e[l, l] \notin \{\curlyvee_0, \ldots, \curlyvee_{h-1}, \omega, \infty\}$, and thus, $e[l, l] \in \{\angle_0, \ldots, \angle_h\}$. Let $0 \le g \le h$ such that $e[l, l] = \angle_g$. We have $e^\sharp[l, l] > \angle_g$. Consequently, $l \sim_{e,g} l$, but we do not have $l \sim_{e^\sharp,g} l$. Thus, $l \in Z_{e,g}$ but $l \notin Z_{e^\sharp,g}$. Hence, there is a class $[l]_{e,g}$ in $\mathsf{Cl}(e, g)$, but there is no class $[l]_{e^\sharp,g}$ in $\mathsf{Cl}(e^\sharp, g)$. In combination with Lemma 5.2, we obtain $\mathsf{Cl}(e, g) \supsetneq \mathsf{Cl}(e^\sharp, g)$. □

5.3. The collapse of the stabilization hierarchy

We fix some $T \subseteq \mathcal{V}_{n\times n}$ for Section 5.3. We define

$$\angle(T) := \bigl\{\angle_g \bigm| a[i,j] = \angle_g \text{ for some } a \in T \text{ and } i,j\bigr\}.$$

Note that $\angle(T) = \angle\bigl(\langle T\rangle^\sharp\bigr)$. For every $e \in \mathsf{E}(\mathcal{V}_{n\times n})$, we set

$$\mathsf{cls}(e) \;:=\; \sum_{\angle_g \,\in\, \angle(T)} \bigl|\mathsf{Cl}(e,g)\bigr|.$$

Note that $\mathsf{cls}(e)$ depends on the underlying set T. For example, let e be the matrix in which every entry is Υ_0. For every $0 \le g \le h$, the set $\{1,\ldots,n\}$ is the only equivalence class of $\sim_{e,g}$. Then, we have $|\mathsf{Cl}(e,g)| = 1$ and $\mathsf{cls}(e) = \angle(T)$.

For every $e \in \mathsf{E}(\mathcal{V}_{n\times n})$ and $0 \le g \le h$, we have $|\mathsf{Cl}(e,g)| \le n$, and hence, $\mathsf{cls}(e) \le \angle(T)n$. By Lemmas 5.2 and 5.3, we have $\mathsf{cls}(e) \le \mathsf{cls}(e^\sharp)$ for every $e \in \langle T\rangle^\sharp \cap \mathsf{E}(\mathcal{V}_{n\times n})$. If $e \neq e^\sharp$, then we even have $\mathsf{cls}(e) < \mathsf{cls}(e^\sharp)$. This observation allows us to show that the stabilization hierarchy of T collapses at level $\angle(T)n$.

Lemma 5.4. *Let $T \subseteq \mathcal{V}_{n\times n}$ and $p \ge 1$. For every $e \in T_p \setminus T_{p-1}$ with $e \in \mathsf{E}(\mathcal{V}_{n\times n})$, we have*

$$\mathsf{cls}(e) \;\le\; \angle(T)n - p.$$

In the particular case $p > \angle(T)n$, Lemma 5.4 implies that there is no idempotent in $T_p \setminus T_{p-1}$.

Proof. For a more lucid presentation of the proof, we set $T_{-1} := \emptyset$ and show the lemma for $p \ge 0$. We proceed by induction on p. For $p = 0$, the assertion is obvious.

Let $p \ge 0$. We show the claim for $p+1$. Let $e \in T_{p+1} \setminus T_p$ with $e \in \mathsf{E}(\mathcal{V}_{n\times n})$ be arbitrary. By the definition of T_{p+1}, there are some $k \ge 1$ and $a_1,\ldots,a_k \in \mathcal{V}_{n\times n}$ such that $e = a_1 \ldots a_k$ and for every $1 \le i \le k$, we have $a_i \in T_p$ or $a_i = e_i^\sharp$ for some $e_i \in \mathsf{E}(T_p)$. Since $e \in T_{p+1} \setminus T_p$, there is at least one $1 \le i \le k$ such that $a_i = e_i^\sharp$ for some $e_i \in \mathsf{E}(T_p)$ such that $e_i^\sharp \notin T_p$. By $e_i^\sharp \notin T_p$, we have $e_i \notin T_{p-1}$. Hence, $e_i \in T_p \setminus T_{p-1}$.

By induction, we have $\mathsf{cls}(e_i) \le \angle(T)n - p$. Since $e_i \neq e_i^\sharp$, we obtain by Lemmas 5.2 and 5.3 $\mathsf{cls}(e_i^\sharp) < \mathsf{cls}(e_i)$. Since $e \le_{\mathscr{J}} e_i^\sharp$, we obtain by Lemma 5.1, $\mathsf{cls}(e) \le \mathsf{cls}(e_i^\sharp)$. To sum up, we have $\mathsf{cls}(e) \le \angle(T)n - (p+1)$. ☐

Proposition 5.5. *Let $T \subseteq \mathcal{V}_{n\times n}$. We have $T_{\angle(T)n} = T_{\angle(T)n+1}$, i.e., the stabilization hierarchy of T collapses at level $\angle(T)n$, and in particular, $T_{\angle(T)n} = \langle T\rangle^\sharp$.*

Proof. Let $p := \angle(T)n$. By contradiction, let $e \in \mathsf{E}(T_p)$ such that $e^\sharp \notin T_p$. By the definition of T_{p+1}, we have $e^\sharp \in \mathsf{E}(T_{p+1})$. By Lemma 5.4, we have $\mathsf{cls}(e) \le \angle(T)n - (p+1) = -1$, which is a contradiction. ☐

For lucidity, we state the following corollary:

Corollary 5.6.

(1) *Let* $h \geq 1$ *and* $T \subseteq \mathcal{V}_{n\times n}$. *We have* $T_{(h+1)n} = \langle T \rangle^\sharp$.
(2) *For every subset* $T \subseteq \mathcal{R}_{n\times n}$, *we have* $T_n = \langle T \rangle^\sharp$.

Proof. Assertion (1) follows from Proposition 5.4 because $T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots$ and $\angle(T) \leq h+1$.

Since $\mathcal{R} = \{\Upsilon_0, \angle_1, \omega, \infty\}$, we have $|\angle(T)| \leq 1$. Hence, (2) follows from Proposition 5.4 □

As already mentioned, Leung's conjectured in 1987 [27] $T_{n^2} = \langle T \rangle^\sharp$ for every $T \subseteq \mathcal{R}_{n\times n}$. Corollary 5.6(2) is a positive answer to this conjecture, because $T_n \subseteq T_{n^2} \subseteq \langle T \rangle^\sharp$.

We complete the proof of Theorem 2.1 by showing (2)⇒(3).

Proof of (2)⇒(3) in Theorem 2.1. For every $a \in T_0$, we can construct a $\sharp$-expression r which does not contain any $\sharp$ such that $\tau(r) = a$. By induction, we can construct for every $p \geq 0$ and every $a \in T_p$ a typed $\sharp$-expression r such that $\tau(r) = a$ and the $\sharp$-height of r is at most p.

Let $a \in \langle T \rangle^\sharp$ such that $I \cdot a \cdot F = \omega$. By Corollary 5.6(1), $a \in T_{(h+1)|Q|}$, *i.e.*, there is a $\sharp$-expression r such that $\tau(r) = a$ and the $\sharp$-height of r is at most $(h+1)|Q|$. By Proposition 4.4, r proves (3). □

5.4. A nondeterministic PSPACE-algorithm

Again let $T \subseteq \mathcal{V}_{n\times n}$. We define

$$\mathsf{ent}(T) := \{a[i,j] \mid \text{ for some } a \in T \text{ and } i,j\}.$$

We have $\mathsf{ent}(T) \subseteq \mathsf{ent}(\langle T \rangle^\sharp) \subseteq \mathsf{ent}(T) \cup \{\omega\}$.

Lemma 5.7. *Let* $T \subseteq \mathcal{V}_{n\times n}$.

(1) *For every* $a \in T_0$, *there are* $1 \leq m \leq |\mathsf{ent}(T)|^{n^2}$ *and* $a_1, \ldots, a_m \in \mathcal{V}_{n\times n}$ *such that* $a = a_1 \ldots a_m$ *and for every* $1 \leq k \leq m$, $a_k \in T$.
(2) *Let* $p \geq 1$. *For every* $a \in T_p$, *there are* $1 \leq m \leq (|\mathsf{ent}(T)|+1)^{n^2}$ *and* $a_1, \ldots, a_m \in \mathcal{V}_{n\times n}$ *such that* $a = a_1 \ldots a_m$ *and for every* $1 \leq k \leq m$,
   (a) $a_k \in T_{p-1}$ *or*
   (b) *there is some* $e_k \in \mathsf{E}(T_{p-1})$ *such that* $a_k = e_k^\sharp$.

Proof. (1) Let $a \in T_0$. Since $T_0 = \langle T \rangle$, there are $m \geq 1$ and $a_1, \ldots, a_m \in T$ such that $a = a_1 \ldots a_m$. By a counting and cancellation argument, we can assume $m \leq |\mathsf{ent}(T)|^{n^2}$.

(2) We apply the definition of T_p. We can assume $m \leq (|\mathsf{ent}(T)|+1)^{n^2}$, because we have $a \in \langle T \rangle^\sharp \subseteq (\mathsf{ent}(T) \cup \{\omega\})_{n\times n}$. □

We give a nondeterministic algorithm to decide limitedness of nested distance desert automata. The key of the algorithm is the function `guess`, below. The input of `guess` are a non-empty set $T \subseteq \mathcal{V}_{n\times n}$ and an integer $p \geq 0$. Below, we will show that `guess` returns some matrix in $a \in \langle T\rangle^\sharp$.

We assume some function `guessbool` which returns nondeterministically `true` or `false`. We also assume some function `guessnum` whose argument is a integer. For every $k \geq 1$, `guessnum`(k) returns nondeterministically an integer between 1 and k. Moreover, we assume a function `choose`. The input of `choose` is a non-empty set of matrices in $\mathcal{V}_{n\times n}$ and it returns nondeterministically some matrix from the input set.

```
function guess(T, p)
a:= 1_{V_{n×n}}
for k:=1 to guessnum((|ent(T)| + 1)^{n^2}) do begin
  if p = 0 then a:= a · choose(T)
  else begin
    b:= guess(T, p − 1)
    if b · b = b and guessbool then b:= b^♯
    a:= a · b
  end
end
return a
```

Proposition 5.8. *Let $T \subseteq \mathcal{V}_{n\times n}$ be non-empty, $p \geq 0$, and $a \in \mathcal{V}_{n\times n}$. There is a run of* `guess`$(T, p)$ *which returns the matrix a iff $a \in T_p$.*

Proof. Let $p = 0$. Clearly, `guess`$(T, 0)$ returns a product of matrices in T. Conversely, by Lemma 5.7(1), there is for every $a \in T_0$ some run of `guess`$(T, 0)$ which returns a.

Let $p \geq 1$ and assume that the assertion is true for $p-1$. We prove the assertion for p.

$\ldots \Rightarrow \ldots$ Every run of `guess`(T, p) returns a product of matrices $a_1, \ldots, a_m \in \mathcal{V}_{n\times n}$ for some $1 \leq m \leq |\mathsf{ent}(T) + 1|^{n^2}$ whereas for every $1 \leq k \leq m$, a_k is a result of `guess`$(T, p-1)$ or $a_k = e_k^\sharp$ for some result e_k of `guess`$(T, p-1)$ with $e_k \in \mathsf{E}(\mathcal{V}_{n\times n})$. By induction, every result of `guess`$(T, p-1)$ belongs to T_{p-1}. By the definition of T_p, the result of `guess`(T, p) belongs to T_p.

$\ldots \Leftarrow \ldots$ Let $a \in T_p$. Let $1 \leq m \leq |\mathsf{ent}(T) + 1|^{n^2}$ and $a_1, \ldots, a_m \in \mathcal{V}_{n\times n}$ as in Lemma 5.7(2). We show a run of `guess`(T, p) which returns a. We assume that `guessnum`$(|\mathsf{ent}(T) + 1|^{n^2})$ returns m. Let $1 \leq k \leq m$ be arbitrary. We consider the k-th run of the loop. If $a_k \in T_{p-1}$, then we assume by the inductive hypothesis that `guess`$(T, p-1)$ returns a_k and `guessbool` returns false. If $a_k \notin T_{p-1}$, then $a_k = e_k^\sharp$ for some $e_k \in \mathsf{E}(T_{p-1})$, and we assume by the inductive hypothesis that `guess`$(T, p-1)$ returns b_k and `guessbool` returns true. By an induction on k, one can show that the value of a after the k-th run of the loop is $a_1 \cdots a_k$, and thus, `guess`(T, p) returns a. □

We show that limitedness of nested distance desert automata is decidable in PSPACE. We will show PSPACE-hardness in Section 5.5.

Proof of Theorem 2.2. We sketch a nondeterministic algorithm which decides limitedness of nested distance desert automata. Let $\mathcal{A} = [Q, E, I, F, \theta]$ be a nested distance desert automaton. At first, the algorithm constructs the set $\Psi(\Sigma) \in \mathcal{V}_{n\times n}$. Let us denote $n := |Q|$, $T := \Psi(\Sigma)$ and $p := |\angle(T)|n$. The algorithm computes $I \cdot \texttt{guess(}T\texttt{,}\, p\texttt{)} \cdot F$. If the result is ω, then the algorithm returns "$\mathcal{A}$ is not limited", otherwise, the computation fails.

If $\mathcal{A}$ is not limited, then there is a matrix $a \in \langle T\rangle^\sharp$ such that $I \cdot a \cdot F = \omega$. By Proposition 5.5, $a \in T_p$. By Proposition 5.8, there is a run of $\texttt{guess(}T\texttt{,}\, p\texttt{)}$ which returns a. Hence, there is a run on which the algorithm returns "$\mathcal{A}$ is not limited".

If there is a run on which the algorithm returns "$\mathcal{A}$ is not limited", then there is a run of $\texttt{guess(}T\texttt{,}\, p\texttt{)}$ which returns some matrix $a \in \mathcal{V}_{n\times n}$ for which $I \cdot a \cdot F = \omega$. By Proposition 5.8 and $T_p \subseteq \langle T\rangle^\sharp$, we have $a \in \langle T\rangle^\sharp$, *i.e.*, $\mathcal{A}$ is unlimited.

The algorithm requires $n^2 \mathsf{ld}(|\mathsf{ent}(T)|+1)$ bits to store the value $(\mathsf{ent}(T)+1)^{n^2}$ for `guessnum` calls. It is not necessary to store the set T explicitly. An implementation of $\texttt{choose(}T\texttt{)}$ can non-nondeterministically choose some letter from Σ and retrieve the corresponding matrix from the automaton.

For a run of $\texttt{guess(}T\texttt{,}\, p\texttt{)}$, one has to store the counter k, the matrices a and b, and a temporary matrix to compute matrix multiplication, stabilization, and the comparison $b \cdot b = b$. Moreover, the recursive call of $\texttt{guess(}T\texttt{,}\, p-1\texttt{)}$ requires space. One can store k in $\mathsf{ld}(|\mathsf{ent}(T)+1|)^{n^2} = n^2\mathsf{ld}(|\mathsf{ent}(T)+1|)$ bits, and one can store each matrix in $n^2\mathsf{ld}(|\mathsf{ent}(T)+1|)$ bits. Hence, a run of $\texttt{guess(}T\texttt{,}\, p\texttt{)}$ requires $4n^2\mathsf{ld}(|\mathsf{ent}(T)+1|)$ bits and additionally space for the recursive call of $\texttt{guess(}T\texttt{,}\, p-1\texttt{)}$. By an induction on p, we can show that every run of $\texttt{guess(}T\texttt{,}\, p\texttt{)}$ requires $(p+1)4n^2\mathsf{ld}(|\mathsf{ent}(T)+1|) = \big(|\angle(T)|n+1\big)4n^2\mathsf{ld}\big(|\mathsf{ent}(T)+1|\big) \in \mathcal{O}\big(|\angle(T)|n^3\mathsf{ld}\,\mathsf{ent}(T)\big)$ space.

We have $|\mathsf{ent}(T)| \le 2h+2$ and $|\angle(T)| \le h+1$. For fixed h, our nondeterministic algorithm requires $\mathcal{O}(n^3)$ space. If h is not fixed, we still have $|\mathsf{ent}(T)| \le |E|$ and $|\angle(T)| \le |E|$. We can assume $n \le 2|E|$. Thus, our nondeterministic algorithm requires $\mathcal{O}\big(|E|^3\mathsf{ld}|E|\big)$ space.

By Savitch's theorem, limitedness of nested distance desert automata is decidable in deterministic polynomial space. For fixed h, the space is polynomial in the number of states. For arbitrary h, the space is polynomial in the number of transitions. □

5.5. PSPACE-hardness of the limitedness problem

We show that limitedness is PSPACE-hard even for very restricted nested distance desert automata.

Proposition 5.9. *Let $g, h \in \mathbb{N}$ be arbitrary. Limitedness of nested distance desert automata in which each transition is marked by $\curlyvee_g$ or $\angle_h$ is PSPACE-hard.*

Proof. We follow the same idea as Leung's proof for PSPACE-hardness of limitedness of distance automata [27, 28]. Let $\mathcal{A} = [Q, E, I, F]$ be a nondeterministic automaton. The problem whether $L(\mathcal{A}) = \Sigma^*$ is known to be PSPACE-hard [19]. We construct a nested distance desert automaton which is limited iff $L(\mathcal{A}) = \Sigma^*$.

Let $c \notin \Sigma$ be a new letter. We can construct an automaton $\mathcal{A}'$ which accepts $L(\mathcal{A})c^+$ by adding just one state to $\mathcal{A}$. We mark every transition in $\mathcal{A}'$ by $\curlyvee_g$. For every $w \in L(\mathcal{A})$, $k \geq 1$, we have $\Delta_{\mathcal{A}'}(wc^k) = 0$.

By adding two more states to $\mathcal{A}'$, we can construct a nested distance desert automaton $\mathcal{A}''$ which accepts $\Sigma^* c^+$. We mark the new transitions by $\angle_h$. For every $w \in \Sigma^*$, $k \geq 1$, we have $\Delta_{\mathcal{A}''}(wc^k) = k$ if $w \notin L(\mathcal{A})$ but $\Delta_{\mathcal{A}''}(wc^k) = 0$ if $w \in L(\mathcal{A})$. Obviously, $\mathcal{A}''$ is limited iff $L(\mathcal{A}) = \Sigma^*$ and the size of $\mathcal{A}''$ is polynomial in the size of $\mathcal{A}$. □

6. On the star height problem

Let Σ be an alphabet. We denote the *star height* of a rational expression r by $\mathsf{sh}(r)$. Every word in $w \in \Sigma^*$ is a rational expression of star height 0, *i.e.*, $\mathsf{sh}(w) := 0$. Moreover, $\emptyset$ is a rational expression of *star height* 0. If r and s are rational expressions over Σ^*, then rs and $r \cup s$ are rational expressions of star height $\mathsf{max}\{\mathsf{sh}(r), \mathsf{sh}(s)\}$, but r^* is of star height $\mathsf{sh}(r) + 1$.

For every $k \in \mathbb{N}$, we define $\mathcal{L}_k := \{ L(r) \mid \mathsf{sh}(r) \leq k \}$. The class $\mathcal{L}_0$ consists of all finite languages. We denote the star height of a recognizable language L by $\mathsf{sh}(L)$ and define it as the least $k \in \mathbb{N}$ for which $L \in \mathcal{L}_k$. Already in 1963, Eggan showed $\mathcal{L}_k \subsetneq \mathcal{L}_{k+1}$ for every $k \in \mathbb{N}$, but he used an alphabet with $2^{k+1} - 1$ letters to construct a language in $\mathcal{L}_{k+1} \setminus \mathcal{L}_k$ [6]. In the same paper, he raised the *star height problem*:

(1) Is the inclusion $\mathcal{L}_k \subseteq \mathcal{L}_{k+1}$ strict for every $k \in \mathbb{N}$ for $\Sigma = \{a, b\}$?
(2) Is there an algorithm which computes the star height of recognizable languages?

During the recent 40 years, many papers have have dealt with the star height problem. For a detailed historical overview, the reader is referred to [42, 43, 47].

In 1966, Dejean and Schützenberger solved the first question by showing $\mathcal{L}_k \subsetneq \mathcal{L}_{k+1}$ for every $k \in \mathbb{N}$ for the alphabet $\Sigma = \{a, b\}$ [5]. In 1982, Hashiguchi showed that it is decidable whether a given recognizable language is of star height one [11, 12], and in 1988, he showed that the star height of recognizable languages is effectively computable [13]. Although this is a positive answer to the star height problem, there is still research for a better comprehension [34–36,39]. This research aims at a deeper understanding of the star height problem for particular classes of recognizable languages, *e.g.*, reversible languages.

Here, we give a new solution for the decidability of the star height problem by a reduction to limitedness of nested distance desert automata. This construction gives the first upper bound for the complexity of the star height problem.

6.1. Some preliminaries

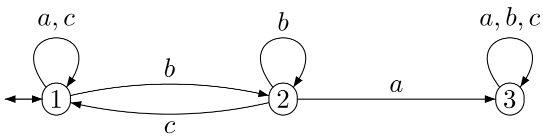
We recall some preliminary notions on automata.

For every automaton $\mathcal{A} = [Q, E, I, F]$, one can construct an automaton $\mathcal{A}' = \big[Q', E', \{q_I\}, \{q_F\}\big]$ such that $Q' = Q \,\dot\cup\, \{q_I, q_F\}$, $E' \subseteq (Q \setminus \{q_F\}) \times \Sigma \times (Q \setminus \{q_I\})$, and $L(\mathcal{A}) = L(\mathcal{A}')$.

An automaton $\mathcal{A} = [Q, E, I, F]$ is called *deterministic* if $|I| = 1$ and for every $p \in Q$, $a \in \Sigma$, there is at most one $q \in Q$ such that $(p, a, q) \in E$. If $\mathcal{A}$ is deterministic, then we can regard E as a partial mapping $\delta : Q \times \Sigma \dashrightarrow Q$. An automaton $\mathcal{A} = [Q, E, I, F]$ is called *total deterministic* if $|I| = 1$ and for every $p \in Q$, $a \in \Sigma$, there is exactly one $q \in Q$ such that $(p, a, q) \in E$. In this case, $\delta : Q \times \Sigma \to Q$ is total. For every automaton $\mathcal{A} = [Q, E, I, F]$, one can construct a total deterministic automaton $\mathcal{A}'$ with at most $2^{|Q|}$ states and $L(\mathcal{A}) = L(\mathcal{A}')$.

By Kleene's theorem, the union of the classes $\mathcal{L}_k$ for $k \in \mathbb{N}$ yields the class of all recognizable languages. From any proof of Kleene's theorem it follows that the star height of the language of an n-state nondeterministic automaton is at most n [52].

Example 6.1. We use an example to explain our constructions throughout this section. Let $L := (a^*b^*c)^*a^*$. Clearly, $\mathsf{sh}(L) \leq 2$. A total deterministic automaton $\mathcal{A} = [Q, \delta, 1, F]$ which accepts L is shown below whereas $I = F = \{1\}$.



We want to decide whether $\mathsf{sh}(L) \leq 1$. □

6.2. Normal forms of rational expressions

The following easy lemma allows to simplify some techniques, below.

Lemma 6.2. *Let $L \subseteq \Sigma^*$ be recognizable. We have $\mathsf{sh}(L) = \mathsf{sh}(L \setminus \varepsilon)$.*

Proof. We assume $\varepsilon \in L$, otherwise the claim is obvious.

If r is a rational expression with $L(r) = L \setminus \varepsilon$, then $r \cup \varepsilon$ is a rational expression of the same star height as r and $L(r \cup \varepsilon) = L$. Thus, $\mathsf{sh}(L) \leq \mathsf{sh}(L \setminus \varepsilon)$.

We show $\mathsf{sh}(L \setminus \varepsilon) \leq \mathsf{sh}(L)$. Let r be a rational expression with $L(r) = L$ and $\mathsf{sh}(r) = \mathsf{sh}(L)$. We transform r into a rational expression r' with $\mathsf{sh}(r') = \mathsf{sh}(r)$ and $L(r') = L(r) \setminus \varepsilon$.

If $r = \varepsilon$, then we set $r' := \emptyset$.

If $r = r_1 \cup r_2$, then we transform by induction r_1 and r_2 into r_1' and r_2' and set $r' := r_1' \cup r_2'$.

Assume $r = r_1 r_2$. By $\varepsilon \in L(r)$, we have $\varepsilon \in L(r_1)$ and $\varepsilon \in L(r_2)$. We transform r_1 and r_2 into r_1' and r_2' and set $r' := r_1' \cup r_2' \cup r_1' r_2'$.

Finally, assume $r = s^*$. We transform s into s' and set $r' := s' s'^*$. □

Example 6.1 (continued)**.** $L \setminus \varepsilon = (a^*b^*c)(a^*b^*c)^* \cup aa^* \cup (a^*b^*c)(a^*b^*c)^*aa^*$. □

We recall the notion of a string expression from Cohen [3]. We define the notions of a string expression, a single string expression and the degree in a simultaneous induction.

Every word $w \in \Sigma^*$ is a *single string expression* of star height $\mathsf{sh}(w) = 0$ and *degree* $\mathsf{dg}(w) := |w|$. Let $n \geq 1$ and $r_1, \ldots, r_n$ be single string expressions. We call $r := r_1 \cup \cdots \cup r_n$ a *string expression* of star height $\mathsf{sh}(r) = \mathsf{max}\{\mathsf{sh}(r_i) \,|\, 1 \leq i \leq n\}$ and *degree* $\mathsf{dg}(r) := \mathsf{max}\{\mathsf{dg}(r_i) \,|\, 1 \leq i \leq n\}$. The empty set $\emptyset$ is a *string expression* of star height $\mathsf{sh}(\emptyset) = 0$ and *degree* $\mathsf{dg}(\emptyset) := 0$.

Let $n \geq 2$, $a_1, \ldots, a_n \in \Sigma$, and $s_1, \ldots, s_{n-1}$ be string expressions. We call $s := a_1 s_1^* a_2 s_2^* \ldots s_{n-1}^* a_n$ a *single string expression* of star height $\mathsf{sh}(s) = 1 + \mathsf{max}\{\mathsf{sh}(s_i) \,|\, 1 \leq i < n\}$ and *degree* $\mathsf{dg}(s) := \mathsf{max}\big(\{n\} \cup \{\mathsf{dg}(s_i) \,|\, 1 \leq i < n\}\big)$.

String expressions define languages because they are rational expressions.

Let r and s be single string expressions. We can construct a single string expression t with $L(t) = L(r)L(s)$ and $\mathsf{sh}(t) = \mathsf{max}\{\mathsf{sh}(r), \mathsf{sh}(s)\}$ as follows: if $\mathsf{sh}(r) \geq 1$ and $\mathsf{sh}(s) \geq 1$, then we set $t := r\,\emptyset^* s$. If $\mathsf{sh}(r) = \mathsf{sh}(t) = 0$, then we simply set $t := rs$. Assume $\mathsf{sh}(r) = 0$ and $\mathsf{sh}(s) \geq 1$. If $r = \varepsilon$, then we set $t := s$. If $r \in \Sigma^+$, we can denote $r = a_1 \ldots a_{|r|}$ and set $t := a_1 \emptyset^* a_2 \emptyset^* \ldots \emptyset^* a_{|r|} \emptyset^* s$. If $\mathsf{sh}(r) \geq 1$ and $\mathsf{sh}(s) = 0$, then we proceed in a symmetric way.

Let r, t be single string expressions with $r \neq \varepsilon$ and $t \neq \varepsilon$, and let s be a string expression. We can regard rs^*t as a string expression with $\mathsf{sh}(rs^*t) = \mathsf{max}\big\{\mathsf{sh}(r),\, 1 + \mathsf{sh}(s),\, \mathsf{sh}(t)\big\}$. If r (or similarly t) is just a word $a_1 \ldots a_{|r|}$, then we understand rs^*t as $a_1 \emptyset^* a_2 \emptyset^* \ldots \emptyset^* a_{|r|} s^* t$.

The following lemma is due to Cohen [3].

Lemma 6.3. *Let $L \subseteq \Sigma^*$ be a recognizable language. There is a string expression s such that we have $L = L(s)$ and $\mathsf{sh}(s) = \mathsf{sh}(L)$.*

Proof. The lemma is an immediate conclusion from the following claim: for every rational expression r, there is a string expression s such that $L(s) = L(r)$ and $\mathsf{sh}(s) \leq \mathsf{sh}(r)$. To prove this claim, let r be a rational expression.

If r is just a word or $r = \emptyset$, then we let $s := r$.

Assume $r = r'r''$. If $L(r') = \emptyset$ or $L(r'') = \emptyset$, then let $s := \emptyset$. Otherwise, we assume by induction string expressions s' and s'' with $L(s') = L(r')$, $\mathsf{sh}(s') \leq \mathsf{sh}(r') \leq \mathsf{sh}(r)$ and similarly for s''. We denote $s' = s'_1 \cup \cdots \cup s'_{n'}$ and $s'' = s''_1 \cup \cdots \cup s''_{n''}$ for suitable n', n'', and single string expressions $s'_1, \ldots, s'_{n'}, s''_1, \ldots, s''_{n''}$. As seen above, we can concatenate every s'_i and s''_j to a single string expression $s'_i s''_j$ such that $L(s'_i)L(s''_j) = L(s'_i s''_j)$. We set

$$s := \bigcup_{1 \leq i \leq n',\ 1 \leq j \leq n''} s'_i s''_j.$$

Clearly, $L(s) = L(r)$ and $\mathsf{sh}(s) \leq \mathsf{sh}(r)$.

The case $r = r' \cup r''$ is similar but simpler.

Assume $r = r'^*$. If $L(r') = \emptyset$, then we set $s := \varepsilon$. Assume $L(r') \neq \emptyset$. By induction, let s' be a string expression with $L(s') = L(r')$ and $\mathsf{sh}(s') \leq \mathsf{sh}(r') = \mathsf{sh}(r) - 1$. We can denote $s' = s'_1 \cup \cdots \cup s'_{n'}$ for suitable n', $s'_1, \ldots, s'_{n'}$. We set

$$s \;:=\; \varepsilon \;\cup\; s' \;\cup \bigcup_{1 \leq i,j \leq n',\; s'_i \neq \varepsilon,\; s'_j \neq \varepsilon} s'_i\, s'^*\, s'_j.$$

Clearly, $L(s) = L(r)$ and $\mathsf{sh}(s) \leq \mathsf{sh}(r)$. □

Example 6.1 (continued)**.** Let $r_1 := aa^*bb^*c$, $r_2 := aa^*c$, $r_3 := bb^*c$, and $r_4 := c$. Then, $s := r_1 \cup r_2 \cup r_3 \cup r_4$ is a string expression for a^*b^*c and $\mathsf{sh}(s) = 1$ and $\mathsf{dg}(s) = \mathsf{dg}(r_1) = 3$. The language $(a^*b^*c)^* = s^*$ is generated by the string expression $\varepsilon \cup s \cup \bigcup_{1 \leq i,j \leq 4} r_i s^* r_j$ which is a union of 21 expressions. This expression is of star height two and its degree is $\mathsf{dg}(r_1 s^* r_1) = 6$.

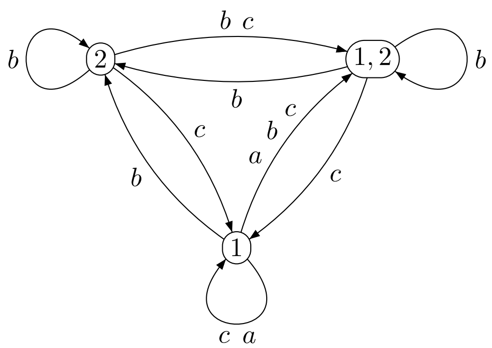
The language a^* is generated by the string expression $\varepsilon \cup a \cup aa^*a$ which is of star height 1 and degree 2. Then, we can construct a string expression for $L = (a^*b^*c)^*a^*$ which is a union of $3 \cdot 21 = 63$ subexpressions. It is of star height 2 and of degree 8 due to the subexpression $r_1 s^* r_1 \emptyset^* aa^*a$. □

6.3. The $T_{d,h}(P, R)$-hierarchy

For the rest of this section, let $L \subseteq \Sigma^*$ be a recognizable language. Let $\mathcal{A} = [Q, \delta, q_I, F]$ be a total deterministic automaton which accepts L. We extend δ to $\delta : \mathcal{P}(Q) \times \Sigma^* \to \mathcal{P}(Q)$ as usual. The totality of δ is crucial for our constructions, below. Otherwise, there are possibly languages $K \subseteq \Sigma^*$ with $\delta(q_I, K) = \emptyset$, and thus, $\delta(q_I, K) \subseteq F$ does not imply $K \subseteq L$.

Let $P, R \in \mathcal{P}_{ne}(Q)$. We define $\mathcal{T}(P, R) := \{ w \in \Sigma^+ \,|\, \delta(P, w) \subseteq R \}$.

Example 6.1 (continued)**.** In the following picture, we denote subsets of $\{1, 2\}$ as vertices of a graph. For every subsets $P, R \subseteq \{1, 2\}$, $a \in \Sigma$, we draw an edge (P, a, R) if $\delta(P, a) \subseteq R$. By considering the vertices P (resp. R) as initial (resp. accepting) state, we obtain an automaton which accepts $\mathcal{T}(P, R)$.



We omitted subsets $P \subseteq Q$ with $3 \in P$. □

Let $d \geq 1$, $P, R \in \mathcal{P}_{ne}(Q)$ and set $T_{d,0}(P,R) := \{w \mid \delta(P,w) \subseteq R,\ 1 \leq |w| \leq d\}$. We have

$$T_{d,0}(P,R) = \bigcup_{\substack{1 \leq d' \leq d,\\ P_0,\dots,P_{d'} \in \mathcal{P}_{ne}(Q),\\ P = P_0,\ P_{d'} \subseteq R}} T_{1,0}(P_0,P_1)\, T_{1,0}(P_1,P_2)\, \dots\, T_{1,0}(P_{d'-1},P_{d'}).$$

It is easy to see that $\mathcal{T}(P,R) = \bigcup_{d \geq 1} T_{d,0}(P,R)$.

Now, let $h \in \mathbb{N}$, and assume by induction that for every $P, R \in \mathcal{P}_{ne}(Q)$, $T_{d,h}(P,R)$ is already defined. We define $T_{d,\,h+1}(P,R) :=$

$$\bigcup_{(*)} T_{1,0}(P_0,P_1)\Big(T_{d,h}(P_1,P_1)\Big)^* T_{1,0}(P_1,P_2)\Big(T_{d,h}(P_2,P_2)\Big)^* \dots T_{1,0}(P_{d'-1},P_{d'})$$

whereas $(*)$ denotes the same conditions as in the definition of $T_{d,0}(P,R)$, above, which are $1 \leq d' \leq d$, $P_0, \dots, P_{d'} \in \mathcal{P}_{ne}(Q)$, and $P = P_0$, $P_{d'} \subseteq R$.

Let $d \geq 1$, $h \in \mathbb{N}$, and $P, R \in \mathcal{P}_{ne}(Q)$ be arbitrary. We have $\varepsilon \notin T_{d,h}(P,R)$.

Lemma 6.4. *Let $d \geq 1$, $h \in \mathbb{N}$, and $P, R \in \mathcal{P}_{ne}(Q)$. We have*

$$\big(T_{d,h}(P,P)\big)^* T_{1,0}(P,P)\big(T_{d,h}(P,P)\big)^* \ \subseteq\ \big(T_{d,h}(P,P)\big)^*.$$

Proof. The assertion follows, because $T_{1,0}(P,P) \subseteq T_{d,h}(P,P)$ and $\big(T_{d,h}(P,P)\big)^*$ is closed under concatenation. □

From the definition, it follows immediately for every $R \subseteq R' \in \mathcal{P}_{ne}(Q)$, $T_{d,h}(P,R) \subseteq T_{d,h}(P,R')$.

It is easy to show by an induction on h that for every $d' \geq d$, $T_{d,h}(P,R) \subseteq T_{d',h}(P,R)$. Moreover, we have $T_{d,0}(P,R) \subseteq T_{d,1}(P,R)$, and by an induction on h, we can show $T_{d,h}(P,R) \subseteq T_{d,\,h+1}(P,R)$. To sum up, for every $d' \geq d$ and $h' \geq h$, $T_{d,h}(P,R) \subseteq T_{d',h'}(P,R)$. For fixed $P, R \in \mathcal{P}_{ne}(Q)$, the sets $T_{d,h}(P,R)$ form a two-dimensional hierarchy. Whenever we use the notion $T_{d,h}(P,R)$*-hierarchy*, we regard $P, R \in \mathcal{P}_{ne}(Q)$ and $h \in \mathbb{N}$ as fixed, *i.e.*, it is a one-dimensional hierarchy w.r.t. the parameter $d \geq 1$.

By induction, we can construct a string expression r with $L(r) = T_{d,h}(P,R)$ such that $\mathsf{sh}(r) \leq h$ and $\mathsf{dg}(r) \leq d$, and hence, $\mathsf{sh}\big(T_{d,h}(P,R)\big) \leq h$. However, we cannot assume that there is a string expression r with $L(r) = T_{d,h}(P,R)$ such that $\mathsf{sh}(r) = h$ and $\mathsf{dg}(r) = d$. In the inductive construction of r, several sets $T_{1,0}(P_{i-1},P_i)$ may be empty, and then, the star-height (resp. degree) of r is possibly smaller than h (resp. d). Just consider the case $T_{d,h}(P,R) = \{a\}$ but $h > 1$, $d > 1$.

Lemma 6.5. *Let $d \geq 1$, $h \in \mathbb{N}$, $P, R \in \mathcal{P}_{ne}(Q)$. We have $T_{d,h}(P,R) \subseteq \mathcal{T}(P,R)$.*

Proof. We fix some arbitrary $d \geq 1$ for the entire proof.

For $h = 0$, the claim follows from the definitions of $T_{d,0}(P,R)$ and $\mathcal{T}(P,R)$.

Let $h \in \mathbb{N}$ and assume by induction that the claim is true for h. Consequently, for every $P' \in \mathcal{P}_{ne}(Q)$ and $u \in \big(T_{d,h}(P',P')\big)^*$, the inclusion $\delta(P',u) \subseteq P'$ holds.

Let $P, R \in \mathcal{P}_{ne}(Q)$ and $w \in T_{d,h+1}(P,R)$ be arbitrary. We show $\delta(P,w) \subseteq R$. According to the definition of $T_{d,\,h+1}(P,R)$ there are some $1 \le d' \le d$ and $P = P_0, \ldots, P_{d'} \subseteq R$ with the following property: there are $a_1, \ldots, a_{d'} \in \Sigma$ and $w_1, \ldots, w_{d'-1} \in \Sigma^*$ such that $w = a_1 w_1 a_2 w_2 \ldots w_{d'-1} a_d$ and

(1) for every $1 \le i \le d'$, we have $a_i \in T_{1,0}(P_{i-1}, P_i)$; and
(2) for every $1 \le i < d'$, we have $w_i \in \big(T_{d,h}(P_i,P_i)\big)^*$.

By the definition of $T_{1,0}$, we have for every $1 \le i \le d'$, $\delta(P_{i-1}, a_i) \subseteq P_i$. As seen above, we have for every $1 \le i < d'$, $\delta(P_i, w_i) \subseteq P_i$. Consequently, $\delta(P_0, w) \subseteq P_{d'}$, *i.e.*, $\delta(P,w) \subseteq R$. □

We have for every $h \in \mathbb{N}$ and $P, R \in \mathcal{P}_{ne}(Q)$:

$$\mathcal{T}(P,R) \;=\; \bigcup_{d \ge 1} T_{d,0}(P,R) \;\subseteq\; \bigcup_{d \ge 1} T_{d,h}(P,R) \;\subseteq\; \mathcal{T}(P,R).$$

Let $h \in \mathbb{N}$. We say that the $T_{d,h}(P,R)$-hierarchy collapses for h if there is some $d \ge 1$ such that $T_{d,h}(P,R) = \mathcal{T}(P,R)$. The key question is: for which $h \in \mathbb{N}$ does the $T_{d,h}(P,R)$-hierarchy collapse? If the $T_{d,h}(P,R)$-hierarchy collapses for some h, then it collapses for every $h' \ge h$. Hence, we can raise the key question as follows: given $P, R \in \mathcal{P}_{ne}(Q)$, what is the least h for which the $T_{d,h}(P,R)$-hierarchy collapses?

Let us consider the particular case $h = 0$. For every $d \ge 1$, the set $T_{d,0}(P,R)$ is finite. Thus, the $T_{d,0}(P,R)$-hierarchy collapses iff $\mathcal{T}(P,R)$ is finite. Consequently, the $T_{d,0}(P,R)$-hierarchy collapses iff $\mathcal{T}(P,R)$ is of star height 0. This observation leads us to the guess that the $T_{d,h}(P,R)$-hierarchy collapses for some $h \in \mathbb{N}$ iff $h \ge \mathsf{sh}\big(\mathcal{T}(P,R)\big)$. Below, Lemma 6.6 allows us to prove that our guess is right.

Example 6.1 (continued)**.** We have $L \setminus \varepsilon = \mathcal{T}\big(\{1\},\{1\}\big)$. The $T_{d,0}\big(\{1\},\{1\}\big)$-hierarchy does not collapse since $L \setminus \varepsilon$ is infinite. By contradiction, assume that the $T_{d,1}\big(\{1\},\{1\}\big)$-hierarchy collapses. Let d be an integer such that $T_{d,1}\big(\{1\},\{1\}\big) = \mathcal{T}\big(\{1\},\{1\}\big) = L \setminus \varepsilon$. Consider $w := \big(a^{d+1}b^{d+1}c\big)^{d+1} \in L \setminus \varepsilon$. We decompose w according to the definition of $T_{d,1}\big(\{1\},\{1\}\big)$, *i.e.*, there are some $1 \le d' \le d$, $P_0 \ldots, P_{d'} \in \mathcal{P}_{ne}(Q)$, $P_0 = P_{d'} = \{1\}$ such that

$$w \;\in\; T_{1,0}(P_0,P_1)\Big(T_{d,0}(P_1,P_1)\Big)^* T_{1,0}(P_1,P_2)\Big(T_{d,0}(P_2,P_2)\Big)^* \ldots T_{1,0}(P_{d'-1},P_{d'}).$$

Now, we show that there is some $0 < i < d'$ such that $\big(T_{d,0}(P_i,P_i)\big)^*$ contains some word w'' of the form a^+b^+. By a counting argument, there is some $0 < i < d'$ such that $\big(T_{d,0}(P_i,P_i)\big)^*$ contains some factor w' of w with $|w'| > 2d+2$. Hence, w' contains the letter c. If w' contains the letter c exactly once, then w' has either a prefix of the form a^+b^{d+1} or a suffix of the form $a^{d+1}b^+$ from which we can obtain w''. If w' contains the letter c at least twice, then $a^{d+1}b^{d+1}$ is a factor of w' and we can obtain w'' as a factor of $a^{d+1}b^{d+1}$.

Consequently, some word of the form a^+b^+ belongs to $\big(T_{d,0}(P_i, P_i)\big)^*$. Thus, there is also some word of the form $a^+b^+a^+b^+$ in $\big(T_{d,0}(P_i, P_i)\big)^*$. Hence, $3 \in P_i$. Since $3 \notin P_{d'}$, there is some $i \le j < d'$ such that $3 \in P_j$ and $3 \notin P_{j+1}$. Hence, $T_{1,0}(P_j, P_{j+1}) = \emptyset$, *i.e.*, $w \in \emptyset$. Consequently, the $T_{d,1}\big(\{1\}, \{1\}\big)$-hierarchy does not collapse.

We defined w by the $\sharp$-expression $(a^\sharp b^\sharp c)^\sharp$ which is of sharp height 2. We cannot use the $\sharp$-expression $r = a^\sharp b^\sharp c$, since for $k \ge 1$, we have $r(k) \in T_{3,1}\big(\{1\}, \{1\}\big)$. $\square$

Lemma 6.6. *Let r be a string expression, $d \ge \mathsf{dg}(r)$, $h \ge \mathsf{sh}(r)$. Let $P, R \in \mathcal{P}_{ne}(Q)$ such that $L(r) \subseteq \mathcal{T}(P, R)$. We have $L(r) \subseteq T_{d,h}(P, R)$.*

Proof. We assume $L(r) \ne \emptyset$. By $L(r) \subseteq \mathcal{T}(P, R)$, we have $\varepsilon \notin L(r)$.

Assume $\mathsf{sh}(r) = 0$. There are some $k \ge 1$ and $w_1, \ldots, w_k \in \Sigma^+$ such that $r = w_1 \cup \cdots \cup w_k$ and for every $1 \le i \le k$, we have $1 \le |w_i| \le d$, and moreover, $\delta(P, w_i) \subseteq R$. By the definition of $T_{d,0}(P, R)$, we have $w_i \in T_{d,0}(P, R)$, *i.e.*, $L(r) \subseteq T_{d,0}(P, R) \subseteq T_{d,h}(P, R)$.

Now, let $\mathsf{sh}(r) \ge 1$, and assume that the claim is true for every string expression r' with $\mathsf{sh}(r') < \mathsf{sh}(r)$.

Clearly, it suffices to consider the case that r is a single string expression. Let $d' \ge 2$ and $a_1, \ldots, a_{d'} \in \Sigma$ and $r_1, \ldots, r_{d'-1}$ be string expressions of a star height less that $\mathsf{sh}(r)$ such that $r = a_1 r_1^* a_2 r_2^* \ldots r_{d'-1}^* a_{d'}$. Let $d \ge \mathsf{dg}(r)$ and $h \ge \mathsf{sh}(r)$. Let $P, R \in \mathcal{P}_{ne}(Q)$ such that $L(r) \subseteq \mathcal{T}(P, R)$.

Let $P_0 := P$, and for $1 \le i < d'$, let $P_i := \delta\big(P_{i-1}, a_i L(r_i^*)\big)$. Finally, let $P_{d'} := \delta(P_{d'-1}, a_{d'})$. To show $L(r) \subseteq T_{d,h}(P, R)$, we apply the definition of $T_{d,h}(P, R)$ with $P_0, \ldots, P_{d'}$. We defined $P_0 = P$, and we can easily show $P_{d'} = \delta\big(P_0, L(r)\big) \subseteq R$. Clearly, $d' \le d$. To complete the proof, we show the following two assertions:

(1) for every $1 \le i \le d'$, we have $a_i \in T_{1,0}(P_{i-1}, P_i)$, and
(2) for every $1 \le i < d'$, we have $L(r_i) \subseteq T_{d,h-1}(P_i, P_i)$.

(1) Clearly, $\delta(P_{i-1}, a_i) \subseteq \delta\big(P_{i-1}, a_i L(r_i)^*\big) = P_i$. Hence, $a_i \in T_{1,0}(P_{i-1}, P_i)$ follows from the definition of $T_{1,0}(P_{i-1}, P_i)$.

(2) We have $\mathsf{sh}(r_i) < h$ and $\mathsf{dg}(r_i) \le d$. In order to apply the inductive hypothesis, we still have to show $\delta(P_i, L(r_i)) \subseteq P_i$. We have $a_i L(r_i)^* L(r_i) \subseteq a_i L(r_i)^*$. Thus, we obtain

$$\delta(P_i, L(r_i)) \;=\; \delta\Big(\delta\big(P_{i-1},\, a_i L(r_i)^*\big),\, L(r_i)\Big) \;=\; \ldots$$

$$\ldots \;=\; \delta\big(P_{i-1},\, a_i L(r_i)^* L(r_i)\big) \;\subseteq\; \delta\big(P_{i-1},\, a_i L(r_i)^*\big) \;=\; P_i.$$

$\square$

Example 6.1 (continued)**.** We have seen that L (and similarly $L \setminus \varepsilon$) is the language of a string expression of star height 2 and degree 8. Since $L \setminus \varepsilon = \mathcal{T}\big(\{1\}, \{1\}\big)$, we have by Lemma 6.6

$$L \setminus \varepsilon \;\subseteq\; T_{2,8}\big(\{1\}, \{1\}\big) \;\subseteq\; \mathcal{T}\big(\{1\}, \{1\}\big) \;=\; L \setminus \varepsilon,$$

i.e., $L \setminus \varepsilon = T_{2,8}\big(\{1\},\{1\}\big)$. It is tedious to verify this equation by hand, since by the definition of $T_{2,8}\big(\{1\},\{1\}\big)$ we have to consider for every $1 \leq d' \leq 8$ every sequence $P_0,\dots,P_{d'} \in \mathcal{P}_{ne}(Q)$ with $P_0 = P_{d'} = \{1\}$. Even if we omit sets which contain the state 3, we have to consider $\sum_{0\leq d'\leq 7} 3^{d'} = 3280$ sequences of states. □

Proposition 6.7. *Let $h \in \mathbb{N}$ and $P,R \in \mathcal{P}_{ne}(Q)$. There is a $d \geq 1$ such that $T_{d,h}(P,R) = \mathcal{T}(P,R)$ iff $\mathsf{sh}\big(\mathcal{T}(P,R)\big) \leq h$.*

Proof. Assume that there is some $d \geq 1$ such that $T_{d,h}(P,R) = \mathcal{T}(P,R)$. Hence, $\mathsf{sh}(\mathcal{T}(P,R)) = \mathsf{sh}\big(T_{d,h}(P,R)\big) \leq h$.

Assume $\mathsf{sh}\big(\mathcal{T}(P,R)\big) \leq h$. By Lemma 6.3, there is a string expression r with $L(r) = \mathcal{T}(P,R)$ and $\mathsf{sh}(r) \leq h$. Let $d := \mathsf{dg}(r)$. By Lemmas 6.5 and 6.6, we obtain $\mathcal{T}(P,R) = L(r) \subseteq T_{d,h}(R,P) \subseteq \mathcal{T}(P,R)$. □

Example 6.1 (continued)**.** We have seen that the $T_{d,1}\big(\{1\},\{1\}\big)$-hierarchy does not collapse, and hence, $\mathcal{T}\big(\{1\},\{1\}\big) = L \setminus \varepsilon$ is of a star height larger than 1. By Lemma 6.2, we have $\mathsf{sh}(L) > 1$, *i.e.*, $\mathsf{sh}(L) = 2$. □

6.4. A reduction to limitedness

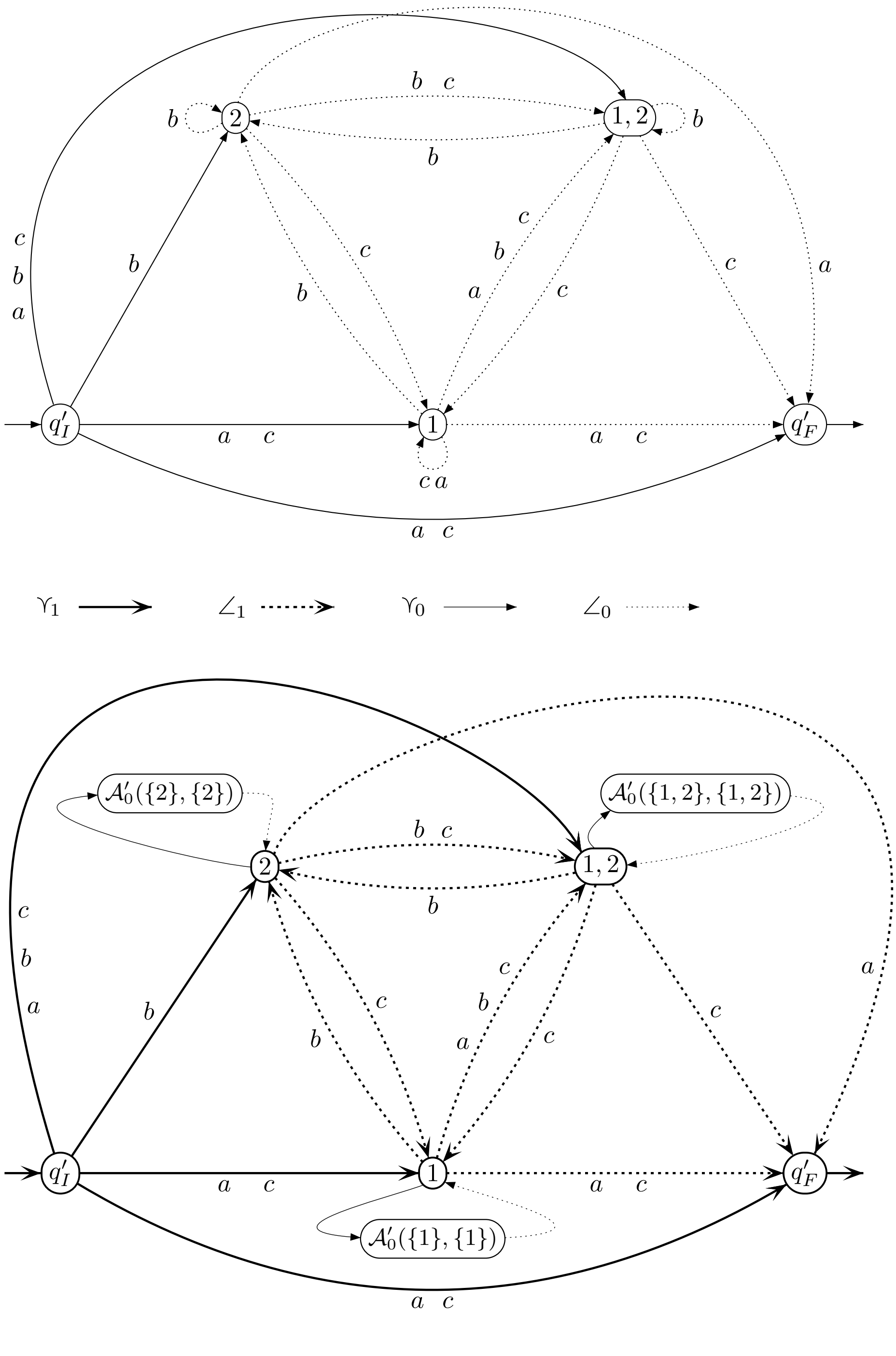
In this section, we construct for given $h \in \mathbb{N}$ and $P,R \in \mathcal{P}_{ne}(Q)$ a h-nested distance desert automaton which is limited iff $T_{d,h}(P,R) = \mathcal{T}(P,R)$ for some $d \geq 1$. In combination with Proposition 6.7 and the decidability of the limitedness of nested distance desert automata, this construction allows to decide whether the star height of the languages $\mathcal{T}(P,R)$ is less than h.

Proposition 6.8. *Let $h \in \mathbb{N}$ and $P,R \in \mathcal{P}_{ne}(Q)$. We can construct a $(h+1)$-nested distance desert automaton $\mathcal{A}'_h(P,R) = [Q',E',q'_I,q'_F,\theta']$ with the following properties:*

1. $E' \subseteq (Q' \setminus q'_F) \times \Sigma \times (Q' \setminus q'_I)$;
2. $|Q'| \leq 2^{|Q|(h+1)} + 1$;
3. *for every $(p,a,q) \in E'$, we have $\theta'((p,a,q)) = \curlyvee_h$ if $p = q'_I$, and $\theta'((p,a,q)) \in \{\curlyvee_0,\dots,\curlyvee_{h-1},\angle_0,\dots,\angle_h\}$ if $p \neq q'_I$;*
4. *for every $w \in \Sigma^*$, $\Delta(w) + 1 = \min\big\{d \geq 1 \,\big|\, w \in T_{d,h}(P,R)\big\}$.*

It follows from (4) that $\mathcal{A}'_h(P,R)$ computes on every word $w \in \Sigma$ the least integer d for which we have $w \in T_{d+1,h}(P,R)$, but it computes ∞ if $w \notin T_{d+1,h}(P,R)$ for every $d \in \mathbb{N}$. Hence, we have $L\big(\mathcal{A}'_h(P,R)\big) = \mathcal{T}(P,R)$. Moreover, $\mathcal{A}'_h(P,R)$ is limited iff the $T_{d,h}(P,R)$-hierarchy collapses. If so, then we have $T_{d+1,h}(P,R) = \mathcal{T}(P,R)$ whereas d is the biggest number which $\mathcal{A}'_h(P,R)$ computes.

Example 6.1 (continued)**.** The pictures on the next page show $\mathcal{A}'_0(\{1\},\{1\})$ (above) and $\mathcal{A}'_1(\{1\},\{1\})$ (below) to illustrate the constructions in the proof of Proposition 6.8. We omitted sets which contain the state 3.



q'_I
q'_F
1
2
$1,2$
$\curlyvee_1$
$\angle_1$
$\curlyvee_0$
$\angle_0$
$\mathcal{A}'_0(\{2\},\{2\})$
$\mathcal{A}'_0(\{1,2\},\{1,2\})$
$\mathcal{A}'_0(\{1\},\{1\})$

Proof of Proposition 6.8. We proceed by induction on h. Let $P, R \in \mathcal{P}_{ne}(Q)$ be arbitrary.

Let $h = 0$. At first, we construct an automaton which accepts every word w with $\delta(P, w) \subseteq R$. We use $\mathcal{P}_{ne}(Q)$ as states. For every $S, T \in \mathcal{P}_{ne}(Q)$, $a \in \Sigma$, we set a transition (S, a, T) iff $\delta(S, a) \subseteq T$. The initial state is P, every non-empty subset of R is an accepting state. We apply to this automation a standard construction to get an automation $[Q', E', q'_I, q'_F]$ which satisfies (1) whereas $Q' = \mathcal{P}_{ne}(Q) \cup \{q'_I, q'_F\}$. Hence, $|Q'| = |\mathcal{P}_{ne}(Q)| + 2 = 2^{|Q|} + 1$, *i.e.*, (2) is satisfied. For every transition $(q'_I, a, q) \in E'$, we set $\theta'((q'_I, a, q)) = \curlyvee_0$. For every transition $(p, a, q) \in E'$ with $p \neq q'_I$, we set $\theta'((p, a, q)) = \angle_0$. This completes the construction of $\mathcal{A}'_0(P, R) = [Q', E', q'_I, q'_F, \theta']$, and (3) is satisfied.

We show (4). For every $w \in \Sigma^*$ with $w \notin \mathcal{T}(P, R)$, the equation in (4) comes up to $\infty = \infty$ by the construction of $\mathcal{A}'_0(P, R)$ and Lemma 6.5. For $w \in \mathcal{T}(P, R)$, the equation in (4) comes up to $|w| = |w|$ by the construction of $\mathcal{A}'_0(P, R)$ and the definition of $T_{d,0}(P, R)$.

Now, let $h \in \mathbb{N}$. We assume that the claim is true for h and show the claim for $h + 1$. At first, we construct an automaton $\mathcal{A}'' := [Q'', E'', q'_I, q'_F]$. Let $Q'' := \mathcal{P}_{ne}(Q) \cup \{q'_I, q'_F\}$.

Let $a \in \Sigma$ and $S, T \in \mathcal{P}_{ne}(Q)$ be arbitrary. If $S \neq T$ and $\delta(S, a) \subseteq T$, then we put the transition (S, a, T) into E''. If $\delta(P, a) \subseteq T$, then we put the transition (q'_I, a, T) into E''. If $\delta(S, a) \subseteq R$, then we put the transition (S, a, q'_F) into E''. Finally, if $\delta(P, a) \subseteq R$, then we put the transition (q'_I, a, q'_F) into E''. For every word w which $[Q'', E'', q'_I, q'_F]$ accepts, we have $w \in \mathcal{T}(P, R)$.

We define $\theta'' : E'' \to \{\curlyvee_{h+1}, \angle_{h+1}\}$. For every transition $(q'_I, a, q) \in E''$, let $\theta''((q'_I, a, q)) = \curlyvee_{h+1}$. For every transition $(p, a, q) \in E''$ with $p \neq q'_I$, we set $\theta''((p, a, q)) = \angle_{h+1}$.

We construct $\mathcal{A}'_{h+1}(P, R)$. For every $S \in \mathcal{P}_{ne}(Q)$, we assume by induction an automaton $\mathcal{A}'_h(S, S)$ which satisfies (1, …, 4). We assume that the sets of states of the automata $\mathcal{A}'_h(S, S)$ are mutually disjoint. We construct $\mathcal{A}'_{h+1}(P, R) = [Q', E', q'_I, q'_F, \theta']$ as a disjoint union of $\mathcal{A}''$ and the automata $\mathcal{A}'_h(S, S)$ for every $S \in \mathcal{P}_{ne}(Q)$ and unifying both the initial and accepting state of $\mathcal{A}'_h(S, S)$ with the state S in $\mathcal{A}''$. Because we did not allow self loops in $\mathcal{A}''$, the union of the transitions is disjoint, and hence, θ' arises in a natural way as union of θ'' and the corresponding mappings of the automata $\mathcal{A}'_h(S, S)$. If $\theta'(t) \in \{\curlyvee_{h+1}, \angle_{h+1}\}$ for some $t \in E'$, then t stems from $\mathcal{A}''$. Conversely, if $\theta'(t) \in \{\curlyvee_0, \ldots, \curlyvee_h, \angle_0, \ldots, \angle_h\}$ for some $t \in E'$, then t stems from some automaton $\mathcal{A}'_h(S, S)$.

Let π be some path in $\mathcal{A}'_{h+1}(P, R)$ and assume that for every transition t in π, we have $\theta'(t) \in \{\curlyvee_0, \ldots, \curlyvee_h, \angle_0, \ldots, \angle_h\}$. Then, the entire path π stems from some automaton $\mathcal{A}'_h(S, S)$, *i.e.*, π cannot visit states in $\mathcal{P}_{ne}(Q) \setminus \{S\}$. Conversely, if π' is a path in $\mathcal{A}'_{h+1}(P, R)$, and two states $S, T \in \mathcal{P}_{ne}(Q)$ with $S \neq T$ occur in π', then π' contains some transition t with $\theta'(t) = \angle_{h+1}$.

Clearly, $\mathcal{A}'_{h+1}(P, R)$ satisfies (1) and (3). We show (2). Each automaton $\mathcal{A}'_h(S, S)$ has at most $2^{|Q|(h+1)} + 1$ states, but we lose one state by unifying the initial and accepting state of $\mathcal{A}'_h(S,S)$. Consequently, we obtain

$$|Q'| \leq 2^{|Q|(h+1)}\big(2^{|Q|}-1\big)+2 = 2^{|Q|(h+2)} - 2^{|Q|(h+1)} + 2 \leq 2^{|Q|(h+2)}.$$

To prove (4) for w, we show the following two claims:

4a) Let $d \geq 1$. For every $w \in T_{d,h+1}(P,R)$, there is a successful path π in $\mathcal{A}'_{h+1}(P,R)$ with the label w and $\Delta(\theta'(\pi)) + 1 \leq d$.

4b) Let π be a successful path in $\mathcal{A}'_{h+1}(P,R)$ with the label w. We have $w \in T_{\Delta(\theta'(\pi))+1,\, h+1}(P,R)$.

Claim (4a) (resp. 4b) proves "$\ldots \leq \ldots$" (resp. "$\ldots \geq \ldots$") in (4). Thus, (4) is a conclusion from (4a) and (4b).

We show (4a). We decompose w according to the definition of $T_{d,h+1}(P,R)$. There are some $1 \leq d' \leq d$ and $P_0, \ldots, P_{d'} \in \mathcal{P}_{ne}(Q)$ with $P_0 = P$ and $P_{d'} \subseteq R$. For every $1 \leq i \leq d'$, there is some $a_i \in T_{1,0}(P_{i-1}, P_i)$, and for every $1 \leq i < d'$ there is some $w_i \in \big(T_{d,h}(P_i,P_i)\big)^*$ such that $w = a_1 w_1 a_2 w_2 \ldots a_{d'}$. By Lemma 6.4, we can assume $P_{i-1} \neq P_i$ for every $2 \leq i < d'$.

If $d' = 1$, then w is a letter. We set $\pi := (q'_I, w, q'_F)$. Then, $\theta'(\pi) = \Upsilon_{h+1}$ and $\Delta(\theta'(\pi)) = 0$ which proves (4a). We assume $d' \geq 2$ in the rest of the proof of (4a).

Let $t_1 := (q'_I, a_1, P_1)$ and $t_{d'} := (P_{d'-1}, a_{d'}, q'_F)$. For every $2 \leq i < d'$, let $t_i := (P_{i-1}, a_i, P_i)$. Clearly, $t_1, \ldots, t_{d'}$ are transitions in $\mathcal{A}'_{h+1}(P,R)$, $\theta'(t_1) = \Upsilon_{h+1}$, and for $2 \leq i \leq d'$, $\theta'(t_i) = \angle_{h+1}$.

Let $1 \leq i < d'$. We decompose w_i. There is some $n_i \in \mathbb{N}$ and $w_{i,1}, \ldots, w_{i,n_i} \in T_{d,h}(P_i,P_i)$ such that $w_i = w_{i,1}, \ldots, w_{i,n_i}$.

Let $1 \leq i < d'$ and $1 \leq j \leq n_i$. Then, $w_{i,j} \in T_{d,h}(P_i,P_i)$. By the inductive hypothesis, there is a path $\tilde{\pi}_{i,j}$ in $\mathcal{A}'_h(P_i,P_i)$ with the label $w_{i,j}$ and $\Delta(\theta'(\tilde{\pi}_{i,j})) + 1 \leq d$. The first transition of this path is marked Υ_h, any other transition is marked by some member in $\{\Upsilon_0, \ldots, \Upsilon_{h-1}, \angle_0, \ldots, \angle_h\}$. We rename the first and the last state in $\tilde{\pi}_{i,j}$ to P_i and call the resulting path $\pi_{i,j}$. Since $\mathcal{A}'_{h+1}(P,R)$ contains $\mathcal{A}'_h(P_i,P_i)$, $\pi_{i,j}$ is a path in $\mathcal{A}'_{h+1}(P,R)$. Let $\pi_i := \pi_{i,1} \ldots \pi_{i,n_i}$. Clearly, π_i is a path in $\mathcal{A}'_{h+1}(P,R)$ from P_i to P_i with the label w_i. The transitions of π_i are marked by members in $\{\Upsilon_0, \ldots, \Upsilon_h, \angle_0, \ldots, \angle_h\}$. In the particular case $w_i = \varepsilon$, π_i is simply the empty path from P_i to P_i.

Clearly, $\pi := t_1 \pi_1 t_2 \pi_2 \ldots t_{d'}$ is a successful path in $\mathcal{A}'_{h+1}(P,R)$ with the label w. It remains to show $\Delta(\theta'(\pi))+1 \leq d$. We apply the definition of Δ from Section 2.3. Let π' be an arbitrary factor of $\theta'(\pi)$. We have $|\pi'|_{h+1} + 1 \leq |\theta'(\pi)|_{h+1} + 1 = d' \leq d$. Let $0 \leq g \leq h$, and assume $\pi' \in \{\Upsilon_0, \ldots, \Upsilon_{g-1}, \angle_0, \ldots, \angle_g\}^*$. Then, π' is a factor of $\theta'(\pi_{i,j})$ for some $1 \leq i < d'$, $1 \leq j \leq n_i$. Since $\Delta(\theta'(\tilde{\pi}_{i,j})) + 1 \leq d$, we have $|\pi'|_g + 1 \leq d$. Consequently, $\Delta(\theta'(\pi)) + 1 \leq d$.

We show (4b). Let π be a successful path in $\mathcal{A}'_{h+1}(P,R)$ with the label w. The first transition of π is marked Υ_{h+1}, any other transitions are marked by some member of $\{\Upsilon_0, \ldots, \Upsilon_h, \angle_0, \ldots, \angle_{h+1}\}$. Let $d' \geq 1$ and factorize π into $\pi = t_1 \pi_1 t_2 \pi_2 \ldots t_{d'}$ such that $t_2, \ldots, t_{d'}$ are the transitions in π which are marked by $\angle_{h+1}$. We have $\Delta(\theta'(\pi)) \geq d' - 1$, *i.e.*, $d' \leq \Delta(\theta'(\pi)) + 1$.

We denote the labels of $t_1, \ldots, t_{d'}$ and $\pi_1, \ldots, \pi_{d'-1}$ by $a_1, \ldots, a_{d'}$ and $w_1, \ldots, w_{d'-1}$, resp., *i.e.*, $w = a_1 w_1 a_2 w_2 \ldots a_{d'}$. Every transition $t_1, \ldots, t_{d'}$ starts and ends at some state in $\mathcal{P}_{ne}(Q)$ except t_1 which starts in q'_I and $t_{d'}$ which ends in q'_F.

Let $1 \leq i < d'$. Let P_i be the state in which π_i starts. Since the transitions of π_i are marked by members in $\{\Upsilon_0, \ldots, \Upsilon_h, \angle_0, \ldots, \angle_h\}$, π_i is a path inside $\mathcal{A}'_h(P_i, P_i)$. Clearly, π_i ends in the same state in which t_{i+1} starts, *i.e.*, π_i ends in some state in $\mathcal{P}_{ne}(Q)$. To sum up, π_i ends in P_i.

Let $P_0 := P$ and $P_{d'} := R$. By the construction of $\mathcal{A}'_{h+1}(P, R)$, (in particular by the definition of E''), we have for every $1 \leq i \leq d'$, $\delta(P_{i-1}, a_i) \subseteq P_i$, and thus, $a_i \in T_{1,0}(P_{i-1}, P_i)$.

To show $w \in T_{\Delta(\theta'(\pi))+1,\, h+1}(P, R)$, we show for every $1 \leq i < d'$, $w_i \in \big(T_{\Delta(\theta'(\pi))+1,h}(P_i, P_i)\big)^*$.

Let $1 \leq i < d'$. We decompose π_i into cycles. There are some $n_i \in \mathbb{N}$, and non-empty paths $\pi_{i,1}, \ldots, \pi_{i,n_i}$ such that $\pi_i = \pi_{i,1} \ldots \pi_{i,n_i}$ and every path among $\pi_{i,1}, \ldots, \pi_{i,n_i}$ starts and ends at P_i, but none of the paths $\pi_{i,1}, \ldots, \pi_{i,n_i}$ contains the state P_i inside.

Let $1 \leq j \leq n_i$. We denote the label of $\pi_{i,j}$ by $w_{i,j}$. In order to show $w_i \in \big(T_{\Delta(\theta'(\pi))+1,h}(P_i, P_i)\big)^*$, we show $w_{i,j} \in T_{\Delta(\theta'(\pi))+1,h}(P_i, P_i)$. We rename the first (resp. last) state of $\pi_{i,j}$ to q'_I (resp. q'_F) and obtain a path which we call $\tilde{\pi}_{i,j}$. Clearly, $\tilde{\pi}_{i,j}$ is an accepting path in $\mathcal{A}'_h(P_i, P_i)$ with the label $w_{i,j}$. Let d be the weight which $\mathcal{A}'_h(P_i, P_i)$ computes on $w_{i,j}$. We have $d \leq \Delta(\theta'(\tilde{\pi}_{i,j})) = \Delta(\theta'(\pi_{i,j})) \leq \Delta(\theta'(\pi))$. By induction, or more precisely, by (4) for $\mathcal{A}'_h(P_i, P_i)$, we have $w_{i,j} \in T_{d+1,h}(P_i, P_i)$, and thus, $w_{i,j} \in T_{\Delta(\theta'(\pi))+1,\, h}(P_i, P_i)$. $\square$

If $\mathsf{sh}(L) > h$, then we have $\mathsf{sh}\big(\mathcal{T}(\{q_I\}, F)\big) > h$, and by Proposition 6.7, the $T_{d,h}\big(\{q_I\}, F\big)$-hierarchy does not collapse. Thus, the automaton $\mathcal{A}'_h\big(\{q_I\}, F\big)$ from Proposition 6.8 is not limited. By Theorem 2.1, there is a $\sharp$-expression r such that $\mathcal{A}'_h\big(\{q_I\}, F\big)$ accepts $r(k)$ for every $k \geq 1$, but for increasing integers k the weight of $r(k)$ is unbounded. Hence, for every $d \geq 1$, there is some $k \geq 1$ such that $r(k) \notin T_{d,h}\big(\{q_I\}, F\big)$. Consequently, r is some kind of witness to show that the $T_{d,h}\big(\{q_I\}, F\big)$-hierarchy does not collapse.

For every $k \geq 1$, we have $r(k) \in L$.

Now, let $K \subseteq L$ be a recognizable language with $\mathsf{sh}(K) \leq h$. Let s be a string expression such that $\mathsf{sh}(s) = K \setminus \varepsilon$ and $\mathsf{sh}(s) \leq h$. By Lemma 6.6, we have $L(s) \subseteq T_{\mathsf{dg}(s),\, h}\big(\{q_I\}, F\big)$. Hence, there is some $k \geq 1$ such that $r(k) \notin L(s)$, *i.e.*, $r(k) \notin K$. Thus, every language $K \subseteq L$ with $\mathsf{sh}(K) \leq h$ cannot contain all the words generated by r. Consequently, r is some kind of witness to show that the star height of L is bigger than h.

Proposition 6.9. *Let $h \in \mathbb{N}$ and $P, R \in \mathcal{P}_{ne}(Q)$. We can construct a h-nested distance desert automaton $\mathcal{A}''_h(P, R)$ with at most $2^{|Q|(h+1)} + 1$ states such that $\mathcal{A}''_h(P, R)$ is limited iff there is some $d \geq 1$ such that $\mathcal{T}(P, R) = T_{d,h}(P, R)$.*

Proof. The automaton $\mathcal{A}'_h(P, R) = [Q', E', q'_I, q'_F, \theta']$ from Proposition 6.8 proves the assertion except that $\mathcal{A}_h(P, R)$ is a $(h+1)$-nested distance desert automaton. We construct $\mathcal{A}''_h(P, R)$ from $\mathcal{A}'_h(P, R)$ by modifying θ' and proving that $\mathcal{A}'_h(P, R)$

is limited iff $\mathcal{A}''_h(P,R)$ is limited. We change the mark of every transition which leaves the initial state. For every $(p,a,q) \in E'$, let

$$\theta''((p,a,q)) := \begin{cases} \angle_h & \text{if } p = q'_I \\ \theta'((p,a,q)) & \text{if } p \neq q'_I \end{cases}$$

and $\mathcal{A}''_h(P,R) = [Q', E', q'_I, q'_F, \theta'']$. Clearly, $\mathcal{A}''_h(P,R)$ is an h-nested distance desert automaton. Moreover, $\mathcal{A}'_h(P,R)$ and $\mathcal{A}''_h(P,R)$ have exactly the same accepting paths. Let π be an accepting path in $\mathcal{A}'_h(P,R)$ and $\mathcal{A}''_h(P,R)$. It is easy to see that either $\Delta(\theta''(\pi)) = \Delta(\theta'(\pi))$ or $\Delta(\theta''(\pi)) = \Delta(\theta'(\pi)) + 1$. Consequently, $\mathcal{A}'_h(P,R)$ is limited iff $\mathcal{A}''_h(P,R)$ is limited. $\square$

Proof of Theorem 2.3. Let $h \in \mathbb{N}$ and L be accepted by an n-state nondeterministic automaton. By any proof of Kleene's theorem, we can show $\mathsf{sh}(L) \leq n$.

An algorithm which decides whether $\mathsf{sh}(L) \leq h$ checks at first whether $n \leq h$. If so, then the algorithm answers "yes".

If $n > h$, then the algorithm proceeds as follows: it constructs a total deterministic automaton $\mathcal{A} = [Q, \delta, q_I, F]$ which recognizes L. Then, it constructs the automaton $\mathcal{A}''_h(q_I, F)$ by Proposition 6.9, and it decides by Theorem 2.2 whether $\mathcal{A}''_h(q_I, F)$ is limited. The algorithm answers "yes" if $\mathcal{A}''_h(q_I, F)$ is limited, otherwise it answers "no".

We have $L \setminus \varepsilon = \mathcal{T}(q_I, F)$. By Lemma 6.2, we have $\mathsf{sh}(L) = \mathsf{sh}\big(\mathcal{T}(q_I, F)\big)$, *i.e.*, $\mathsf{sh}(L) \leq h$ iff $\mathsf{sh}\big(\mathcal{T}(q_I, F)\big) \leq h$. By Proposition 6.7, we have $\mathsf{sh}\big(\mathcal{T}(q_I, F)\big) \leq h$ iff the $T_{d,h}(q_I, F)$-hierarchy collapses. By Proposition 6.9, the $T_{d,h}(q_I, F)$-hierarchy collapses iff $\mathcal{A}''_h(q_I, F)$ is limited.

Clearly, the initial test whether $n \leq h$ is not necessary for the correctness of the algorithm. However, this test increases the efficiency and it simplifies the analysis of the complexity.

We have $|Q| \leq 2^n$. The number of states of $\mathcal{A}''_h(q_I, F)$ is at most $2^{2^n(h+1)} + 1$. Since $n > h$, $\mathcal{A}''_h(q_I, F)$ has at most $2^{2^n n} + 1$ states. By Theorem 2.2, an algorithm requires $2^{2^{\mathcal{O}(n)}}$ space to decide whether $\mathcal{A}''_h(q_I, F)$ is limited. $\square$

6.5. PSPACE-hardness of the star height problem

In this section, we show that the star height problem over two letter alphabet is PSPACE-hard. Although this result seems to be well-known, the author did not find any proof in the literature. We reduce the star height problem to the universality problem of nondeterministic finite automata.

Lemma 6.10. *Let $\Sigma = \{a, b\}$ and $K, L \subseteq \Sigma^*$ be recognizable. Define $L' := \Sigma^* cL \cup Kc\Sigma^*$.*

(1) *If $L = \Sigma^*$, then $\mathsf{sh}(L') = 1$.*
(2) *If $L \subsetneq \Sigma^*$, then $\mathsf{sh}(L') \geq \mathsf{sh}(K)$.*

Proof. (1) If $L = \Sigma^*$, then $L' = \Sigma^* c\Sigma^*$, and hence, $\mathsf{sh}(L') = 1$.

(2) Let $h := \mathsf{sh}(L')$. For $K = \emptyset$, the claim is vacuously true. We assume $K \neq \emptyset$ in the rest of the proof. Hence, L' is infinite and $h \geq 1$.

We construct a rational expression r such that $L(r) = K$ and $\mathsf{sh}(r) \leq h$. By Lemma 6.3, there is a string expression s such that $L(s) = L'$ and $\mathsf{sh}(s) = h$. There are some $n \geq 1$ and single string expressions $s_1, \dots, s_n$ such that $s = s_1 \cup \dots \cup s_n$ and $\mathsf{sh}(s_i) \leq h$ for every $1 \leq i \leq n$.

Let $1 \leq i \leq n$ be arbitrary. There are some n_i, letters $a_1, \dots, a_{n_i} \in \Sigma \cup \{c\}$ and string expressions $t_1, \dots, t_{n_i - 1}$ such that $s_i = a_1 t_1^* a_2 t_2^* \dots t_{n_i-1}^* a_{n_i}$. Moreover, we have $\mathsf{sh}(t_j) < h$ for every $1 \leq j < n_i$. Note that the letters $a_1, \dots, a_{n_i}$ and $t_1, \dots, t_{n_i-1}$ depend on i.

By contradiction, assume that c occurs in an expression t_j for some $1 \leq j < n_i$. Then, $L(t_j^*)$ and $L(s)$ contain words with more than one occurrence of c. Hence, c cannot occur in t_j. Since every word in L' contains exactly one occurrence of c, exactly one of the letters $a_1, \dots, a_{n_j}$ is c. Let $1 \leq l \leq n_i$ such that $a_l = c$ and let $r_i := a_1 K_1^* \dots a_{l-1} K_{l-1}^*$. Note that r_i is not a string expression, because it ends by K_{l-1}^*. If $l = 1$, then $r_i = \varepsilon$. Similarly, let $r_i' := K_l^* a_{l+1} \dots K_{n_i-1}^* a_{n_i}$, *i.e.*, we have $s_i = r_i c r_i'$. We assume such expressions r_i and r_i' for every $1 \leq i \leq n$. Let

$$r := \bigcup_{1 \leq i \leq n,\ L(r_i) \subseteq K} r_i.$$

We have $\mathsf{sh}(r) \leq \mathsf{sh}(s) = h = \mathsf{sh}(L')$. It remains to show $L(r) = K$. From the definition of r, it follows immediately $L(r) \subseteq K$. Let $w \in K$ be arbitrary. We want to show $w \in L(r)$. Let $u \in \Sigma^* \setminus L$ be arbitrary. We have $wcu \in L'$. Hence, there is some $1 \leq i \leq n$ such that $wcu \in L(s_i)$, *i.e.*, $w \in L(r_i)$ and $u \in L(r_i')$. Thus, $L(r_i)cu \subseteq L'$. Since $u \notin L$, we have $L(r_i) \subseteq K$, and in particular, $w \in L(r_i) \subseteq L(r)$. □

Proposition 6.11. *Let $h \geq 1$. To decide whether for a nondeterministic automaton $\mathcal{A}$ over a three-letter alphabet, we have $\mathsf{sh}(L(\mathcal{A})) \leq h$ is PSPACE-hard.*

Proof. Let $h \geq 1$. Let $K \subseteq \{a, b\}^*$ be recognizable such that $\mathsf{sh}(K) = h + 1$. Such a language K exists due to [5].

Let $L \subseteq \{a, b\}^*$ be the language of some nondeterministic automaton. To decide whether $L = \Sigma^*$ is PSPACE-complete [19]. We construct an automaton $\mathcal{A}$ such that $L(\mathcal{A}) = \Sigma^* c L \cup K c \Sigma^*$, *i.e.*, $\mathcal{A}$ accepts L' from Lemma 6.10. Note that K does not depend on L, *i.e.*, $\mathcal{A}$ has just a bounded number of states more than the automaton for L. If $L = \Sigma^*$, then we know by Lemma 6.10(1), $\mathsf{sh}(L') = 1$, and in particular $\mathsf{sh}(L') \leq h$. Conversely, if $L \subsetneq \Sigma^*$, then we know by Lemma 6.10(2), $\mathsf{sh}(L') \geq \mathsf{sh}(K) > h$. To sum up, we have $L = \Sigma^*$ iff $\mathsf{sh}(L') \leq h$. Consequently, to decide whether $\mathsf{sh}(L') \leq h$ it is PSPACE-hard. □

In order to generalize Proposition 6.11, we apply a homomorphism which preserves star height. Let Γ and Σ be two alphabets and let $\alpha : \Gamma^* \to \Sigma^*$ be a homomorphism. For every recognizable language $L \subseteq \Gamma^*$, we have $\mathsf{sh}(L) \geq \mathsf{sh}(\alpha(L))$. We say that α *preserves star height* if for every recognizable language $L \subseteq \Gamma^*$, we have $\mathsf{sh}(L) = \mathsf{sh}(\alpha(L))$.

Assume that α is injective. By following [18], we say that α has the *tag property* if for every $u_1, u_2, v_1, v_2 \in \Sigma^*$ with $u_1v_1,\ u_1v_2,\ u_2v_1,\ u_2v_2\ \in \alpha(\Gamma^*)$, we have one of the following conditions:

(1) There are $x, u_1', u_2' \in \Sigma^*$ such that we have $u_1 = u_1'x,\ \ u_2 = u_2'x$ and $u_1',\ u_2',\ xv_1,\ xv_2 \in \alpha(\Gamma^*)$.
(2) There are $y, v_1', v_2' \in \Sigma^*$ such that we have $v_1 = yv_1',\ \ v_2 = yv_2'$ and $u_1y,\ u_2y,\ v_1',\ v_2' \in \alpha(\Gamma^*)$.

We use the following theorem by Hashiguchi and Honda from 1976 [18].

Theorem 6.12 [18]**.** *A homomorphism $\alpha : \Gamma^* \to \Sigma^*$ preserves star height iff α is injective and α has the tag property.*

Lemma 6.13. *Let $\Gamma = \{a, b, c\}$ and $\Sigma = \{a, b\}$. The homomorphism $\alpha : \Gamma^* \to \Sigma^*$ defined by $\alpha(a) := aa,\ \alpha(b) := ab,$ and $\alpha(c) := ba$ preserves star height.*

Proof. By Theorem 6.12, it suffices to show that α is injective and α has the tag property. Obviously, α is injective.

We show that h has the tag property. Let $u_1, u_2, v_1, v_2 \in \Sigma^*$ such that $u_1v_1, u_1v_2, u_2v_1, u_2v_2 \in \alpha(\Gamma^*)$. We show that one of the two conditions in the definition of the tag property holds.

If $|u_1|$ is even, then $|v_1|$, $|u_2|$, and $|v_2|$ are even, and both conditions hold for $x = \varepsilon$ (resp. $y = \varepsilon$).

From now, we assume that $|u_1|$ is odd. Hence, $|v_1|$, $|u_2|$, and $|v_2|$ are odd. We factorize $u_1 = u_1'x_1,\ u_2 = u_2'x_2,\ v_1 = y_1v_1',\ v_2 = y_2v_2'$ whereas $x_1, x_2, y_1, y_2 \in \Sigma$. Clearly, $|u_1'|$, $|u_2'|$, $|v_1'|$, and $|v_2'|$ are even and $u_1', u_2', v_1', v_2' \in h(\Gamma^*)$.

If $x_1 = x_2$, then condition (1) holds for $x := x_1 = x_2$. If $y_1 = y_2$, then condition (2) holds for $y := y_1 = y_2$. It remains to consider the case that $x_1 \neq x_2$ and $y_1 \neq y_2$. If so, then there are $1 \leq i, j \leq 2$ such that $x_i = y_j = b$, and thus, $u_iv_j = u_i'bbv_j' \notin \alpha(\Gamma^*)$, which is a contradiction. $\square$

Theorem 6.14. *Let $h \geq 1$. To decide whether for a nondeterministic automaton $\mathcal{A}$ over a two letter alphabet, we have $\mathsf{sh}(L(\mathcal{A})) \leq h$ is PSPACE-hard.*

Proof. Given a nondeterministic automaton $\mathcal{A}$ over the alphabet $\{a, b, c\}$, we can construct an automaton $\mathcal{A}'$ over $\{a, b\}$ such that $L(\mathcal{A}') = h(L(\mathcal{A}))$ whereas h is the homomorphism from Lemma 6.13, *i.e.*, $\mathsf{sh}(L(\mathcal{A})) = \mathsf{sh}(L(\mathcal{A}'))$. Moreover, we can construct $\mathcal{A}'$ in a way that $\mathcal{A}'$ has at most three times as many states as $\mathcal{A}$. The claim follows from Proposition 6.11. $\square$

7. Conclusions and challenges

In the author's opinion, there are two challenges concerning desert automata and the star height problem.

The first challenge is to determine the exact complexity of the star height problem. In particular, it is not clear whether its reduction to limitedness of nested distance desert automata can be achieved in a more efficient way.

The other challenge is an extension of our concepts to achieve decidability results for other hierarchies of classes of recognizable languages, *e.g.*, the Straubing-Thérien hierarchy, the dot-depth hierarchy, and the famous extended star height hierarchy [44]. It is not clear whether or how our principle of using nested distance desert automata to examine languages $T_{d,h}(P, R)$ can be generalized to decide language hierarchies which allow complement and intersection. Maybe, one needs to develop more involved automata concepts than nested distance desert automata.

Beside these two challenges, there are several other things to investigate.

As pointed out in Section 2.4, the decidability of the equivalence of two desert automata (1-nested distance desert automata in which transitions are marked by $\curlyvee_0$- and $\angle_0$) is an open question.

Another open question is to give a sharp bound on the range of the mappings of limited nested distance desert automata depending on the number of states. For limited n-state distance automata, the sharpest known upper bound on the range is $2^{3n^3+n\lg n+n-1}$ [33], but the worst known examples are limited by $2^n - 2$ [31,50].

The limitedness problem for distance automata was originally motivated by the star height problem, but it turned out to be useful in other areas, *e.g.*, [8, 24, 37]. At this point, there are two applications of desert and nested distance desert automata: the decidability of the finite substitution problem [21, 22] and a new proof for the decidability of the star height problem in the present paper. One should look for other applications and establish connections between nested distance desert automata and other concepts in theoretical computer science.

Acknowledgements. The author acknowledges the discussions with Jean-Éric Pin on the paper. The author thanks the referees of the present paper and the conference version [23] for their very careful work and their useful comments which resulted in many detail improvements. The author thanks Igor Walukiewicz for the invitation to publish in this special issue.

References

[1] S. Bala, Regular language matching and other decidable cases of the satisfiability problem for constraints between regular open terms, in *STACS'04 Proceedings*, edited by V. Diekert and M. Habib, Springer-Verlag, Berlin. *Lect. Notes Comput. Sci.* **2996** (2004) 596–607.

[2] J. Berstel, *Transductions and Context-Free Languages.* B.G. Teubner, Stuttgart (1979).

[3] R.S. Cohen, Star height of certain families of regular events. *J. Comput. Syst. Sci.* **4** (1970) 281–297.

[4] K. Culik II and J. Kari, Image compression using weighted finite automata. *Computer Graphics* **17** (1993) 305–313.

[5] F. Dejean and M. Schützenberger, On a question of Eggan. *Inform. Control* **9** (1966) 23–25.

[6] L.C. Eggan, Transition graphs and the star height of regular events. *Michigan Math. J.* **10** (1963) 385–397.

[7] S. Eilenberg, *Automata, Languages, and Machines,* Vol. A. Academic Press, New York (1974).

[8] G. Grahne and A. Thomo, Approximate reasoning in semi-structured databases, in *8th International Workshop on Knowledge Representation meets Databases (KRDB2001)*, edited by M. Lenzerini *et al.*, *CEUR Workshop Proceedings* **45** (2001).

[9] P.A. Grillet, *Semigroups: An Introduction to the Structure Theory*, Marcel Dekker, Inc., New York. *Monographs and Textbooks in Pure and Applied Mathematics* **193** (1995).
[10] K. Hashiguchi, Limitedness theorem on finite automata with distance functions. *J. Comput. Syst. Sci.* **24** (1982) 233–244.
[11] K. Hashiguchi, Regular languages of star height one. *Inform. Control* **53** (1982) 199–210.
[12] K. Hashiguchi, Representation theorems of regular languages. *J. Comput. Syst. Sci.* **27** (1983) 101–115.
[13] K. Hashiguchi, Algorithms for determining relative star height and star height. *Inform. Comput.* **78** (1988) 124–169.
[14] K. Hashiguchi, Improved limitedness theorems on finite automata with distance functions. *Theor. Comput. Sci.* **72** (1990) 27–38.
[15] K. Hashiguchi, New upper bounds to the limitedness of distance automata, in *ICALP'96 Proceedings*, edited by F. Meyer auf der Heide and B. Monien, Springer-Verlag, Berlin. *Lect. Notes Comput. Sci.* **1099** (1996) 324–335.
[16] K. Hashiguchi, New upper bounds to the limitedness of distance automata. *Theor. Comput. Sci.* **233** (2000) 19–32.
[17] K. Hashiguchi, Erratum to "New upper bounds to the limitedness of distance automata". *Theor. Comput. Sci.* **290** (2003) 2183–2184.
[18] K. Hashiguchi and N. Honda, Homomorphisms that preserve star height. *Inform. Control* **30** (1976) 247–266.
[19] J.E. Hopcroft and J.D. Ullman, *Introduction to Automata Theory Languages, and Computation*. Addison-Wesley, Reading (1979).
[20] F. Katritzke, *Refinements of Data Compression using Weighted Finite Automata*. Ph.D. Thesis, Universität Siegen (2001).
[21] D. Kirsten, A Burnside approach to the finite substitution problem. *Theory Comput. Syst.* (to appear).
[22] D. Kirsten, Desert automata and the finite substitution problem, in *STACS'04 Proceedings*, edited by V. Diekert and M. Habib, Springer-Verlag, Berlin. *Lect. Notes Comput. Sci.* **2996** (2004). 305–316.
[23] D. Kirsten, Distance desert automata and the star height one problem, in *FoSSaCS'04 Proceedings*, edited by I. Walukiewicz, Springer-Verlag, Berlin. *Lect. Notes Comput. Sci.* **2987** (2004) 257–272.
[24] D. Kirsten and J. Marcinkowski, Two techniques in the area of the star problem in trace monoids. *Theor. Comput. Sci.* **309** (2003) 381–412.
[25] D. Krob, The equality problem for rational series with multiplicities in the tropical semiring is undecidable. *Internat. J. Algebra Comput.* **4** (1994) 405–425.
[26] G. Lallement, *Semigroups and Combinatorial Applications*. John Wiley & Sons, New York (1979).
[27] H. Leung, *An Algebraic Method for Solving Decision Problems in Finite Automata Theory*. Ph.D. Thesis, Pennsylvania State University, Department of Computer Science (1987).
[28] H. Leung, On the topological structure of a finitely generated semigroup of matrices. *Semigroup Forum* **37** (1988) 273–287.
[29] H. Leung, Limitedness theorem on finite automata with distance functions: An algebraic proof. *Theor. Comput. Sci.* **81** (1991) 137–145.
[30] H. Leung, On some decision problems in finite automata, in *Monoids and Semigroups with Applications*, edited by J. Rhodes, World Scientific, Singapore (1991) 509–526.
[31] H. Leung, On finite automata with limited nondeterminism. *Acta Inform.* **35** (1998) 595–624.
[32] H. Leung, The topological approach to the limitedness problem on distance automata, in *Idempotency*, edited by J. Gunawardena, Cambridge University Press (1998) 88–111.
[33] H. Leung and V. Podolskiy, The limitedness problem on distance automata: Hashiguchi's method revisited. *Theor. Comput. Sci.* **310** (2004) 147–158.
[34] S. Lombardy, *Approche structurelle de quelques problèmes de la théorie des automates*. Ph.D. Thesis, École nationale supérieure des télécommunications, Paris (2001).

[35] S. Lombardy and J. Sakarovitch, On the star height of rational languages. A new proof for two old results, in *Proc. of the 3rd Int. Conf. on Languages, Words and Combinatorics, Kyoto'00* edited by M. Ito, World Scientific (2000).
[36] S. Lombardy and J. Sakarovitch, Star height of reversible languages and universal automata, in *LATIN'02 Proceedings*, Springer-Verlag, Berlin. *Lect. Notes Comput. Sci.* **2286** (2002) 76–89.
[37] Y. Métivier and G. Richomme. New results on the star problem in trace monoids. *Inform. Comput.* **119** (1995) 240–251.
[38] M. Mohri, Finite-state transducers in language and speech processing. *Comput. Linguistics* **23** (1997) 269–311.
[39] R. Montalbano and A. Restivo, On the star height of rational languages. *Internat. J. Algebra Comput.* **4** (1994) 427–441.
[40] D. Perrin, Finite automata, in *Handbook of Theoretical Computer Science*, edited by J. van Leeuwen, Elsevier Science Publishers **B** (1990) 1–57.
[41] J.-É. Pin, *Varieties of Formal Languages.* North Oxford Academic Publishers Ltd (1986).
[42] J.-É. Pin, Rational and recognizable langages, in *Lect. Appl. Math. Inform.* edited by Ricciardi, Manchester University Press (1990) 62–106.
[43] J.-É. Pin, Finite semigroups and recognizable languages: An introduction, in *NATO Advanced Study Institute, Semigroups, Formal Languages and Groups*, edited by J. Fountain, Kluwer Academic Publishers (1995) 1–32.
[44] J.-É. Pin, Syntactic semigroups, in *Handbook of Formal Languages, Vol. 1, Word, Language, Grammar*, edited by G. Rozenberg and A. Salomaa, Springer-Verlag, Berlin (1997) 679–746.
[45] J.-É. Pin, Tropical semirings, in *Idempotency*, edited by J. Gunawardena, Cambridge University Press (1998) 50–69.
[46] I. Simon, Limited subsets of a free monoid, in *Proc. of the 19th IEEE Annual Symposium on Foundations of Computer Science*, IEEE Computer Society Press, Long Beach, CA (1978) 143–150.
[47] I. Simon, Recognizable sets with multiplicities in the tropical semiring, in *MFCS'88 Proceedings*, edited by M.P. Chytil *et al.*, Springer-Verlag, Berlin. *Lect. Notes Comput. Sci.* **324** (1988) 107–120.
[48] I. Simon, The nondeterministic complexity of a finite automaton, in *Mots - mélanges offerts à M.P. Schützenberger*, edited by M. Lothaire, Hermes (1990) 384–400.
[49] I. Simon, On semigroups of matrices over the tropical semiring. *RAIRO-Inf. Theor. Appl.* **28** (1994) 277–294.
[50] A. Weber, Distance automata having large finite distance or finite ambiguity. *Math. Syst. Theor.* **26** (1993) 169–185.
[51] A. Weber, Finite valued distance automata. *Theor. Comput. Sci.* **134** (1994) 225–251.
[52] S. Yu, Regular Languages, in *Handbook of Formal Languages, Vol. 1, Word, Language, Grammar*, edited by G. Rozenberg and A. Salomaa, Springer-Verlag, Berlin (1997) 41–110.