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REGULAR LANGUAGES DEFINABLE BY LINDSTRÖM QUANTIFIERS *

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Abstract. In our main result, we establish a formal connection between Lindström quantifiers with respect to regular languages and the double semidirect product of finite monoids with a distinguished set of generators. We use this correspondence to characterize the expressive power of Lindström quantifiers associated with a class of regular languages.

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1. Introduction

By the classic result of Büchi [6], Elgot [12] and Trakhtenbrot [40], the regular languages are exactly those definable by the sentences of a certain monadic second-order logic over words. Moreover, Mc Naughton and Papert [22] proved that the first-order sentences of this logic define an important subclass of the regular languages, the star-free languages. By Schützenberger's theorem [29], the star-free languages are exactly those that can be recognized by the aperiodics, *i.e.*, by those finite monoids containing no nontrivial groups.

Because of the limited expressive power of first-order logic on words, and in the search for characterizations of other important subclasses of regular languages in

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terms of formal logic, Straubing, Therien and Thomas [34, 35] introduced generalized, or modular quantifiers $\exists^{(m,r)}$, where $m \geq 2$ and $r = 0, \ldots, m-1$, with the following meaning: A word u satisfies a sentence $\exists^{(m,r)}x.\varphi$ iff the number of assignments of positions in u to the variable x satisfying φ is congruent to r modulo m. They proved that a language is definable in the logic involving, in addition to first-order quantifiers, the above modular quantifiers iff its syntactic monoid is finite and solvable, *i.e.*, it contains only solvable groups. This class of regular languages first arose in [30, 36]. And if no first-order quantifiers are allowed, then a language is definable iff its syntactic monoid is a finite solvable group. In fact, Straubing, Therien and Thomas also studied the more general setting when the moduli of the generalized quantifiers in the logic are restricted to a finite set of (prime) numbers. See also Straubing and Therien [33] for a more recent account, and Baziramwabo, McKenzie and Therien [4] for a corresponding extension of linear temporal logic with modular counting.

In order to express regular languages having non-solvable syntactic monoids, Barrington, Immerman and Straubing [3] associated a family of quantifiers with each finite group containing a quantifier corresponding to each group element. When the group is cyclic of order m, the associated quantifiers are essentially the modular quantifiers $\exists^{(m,r)}$. They showed that a language is definable in first-order logic enriched with group quantifiers corresponding to the members of a subclass $\mathbf{G}$ of the finite groups iff the language is regular (so that its syntactic monoid is finite), and every simple group that divides the syntactic monoid of the language divides a group in $\mathbf{G}$. (See, *e.g.*, [11] for a definition of when a group divides a group or a monoid). Moreover, if only group quantifiers are allowed, then a language is definable iff in addition to the above conditions its syntactic monoid is a group. When $\mathbf{G}$ is empty, by Schützenberger's theorem one obtains the Mc Naughton–Papert characterization of first-order definable languages. The theorem of Barrington, Immerman and Straubing extends to quantifiers associated with finite monoids.

The quantifiers associated with finite monoids and groups, and thus the modular quantifiers, are all special cases of (simple) Lindström quantifiers associated with (regular) languages, defined in [7]. (For more general treatments of Lindström quantifiers the reader is referred to Lindström [19] and Ebbinghaus and Flum [10], Chap. 12. See also the generalized quantifiers of Immerman [16] and Väänänen [41]). By the results of Barrington, Immerman and Straubing [3], extended to monoids, it follows that when the Lindström quantifiers are associated with regular languages, then only regular languages can be defined. Moreover, when $\mathbf{G}$ is a class of finite groups (or monoids), and $\mathcal{L}_{\mathbf{G}}$ is the class of regular languages that can be recognized by the members of the class $\mathbf{G}$, then a language L is definable in first-order logic enriched with Lindström quantifiers associated with the languages in $\mathcal{L}_{\mathbf{G}}$ iff L is regular and every simple group divisor of the syntactic monoid of L divides a group (or monoid) in $\mathbf{G}$.

Our initial motivation for studying "regular Lindström quantifiers" was the question of characterizing those classes $\mathcal{L}$ of regular languages which are expressively complete in the sense that every regular language is definable by a sentence possibly involving, in addition to ordinary quantifiers, Lindström quantifiers with respect to the languages in $\mathcal{L}$. By the classic theorem of Büchi, Elgot and Trakhtenbrot, first-order logic, enriched with Lindström quantifiers with respect to the languages in an expressively complete class $\mathcal{L}$, has the same expressive power as monadic second-order logic. Moreover, by the above-mentioned results of Barrington, Immerman and Straubing, a necessary condition of the expressive completeness of a class $\mathcal{L}$ of regular languages is that $\mathcal{L}$ is group-complete, *i.e.*, every finite (non-abelian simple) group is a divisor of the syntactic monoid of a language in $\mathcal{L}$. We will show that this condition, together with a condition involving the existence of certain cycles in the syntactic monoids of the languages in $\mathcal{L}$, which is roughly equivalent to the expressibility of all of the one-letter languages $(a^n)^*$, $n \geq 2$, is necessary and sufficient as long as $\mathcal{L}$ satisfies certain natural assumptions. On the other hand, we show that neither condition is sufficient by itself. But when $\mathcal{L}$ is closed with respect to taking quotients and admits padding, then $\mathcal{L}$ is expressively complete iff it is group-complete and at least one of the one-letter languages $(a^n)^*$, $n \geq 2$ is definable.

Formal logic in connection with words and languages has several general techniques such as model theoretic games and deep algebraic techniques developed in the theory of finite semigroups and automata. A general account of these methods is given in Straubing [31]. In particular, the semidirect product and the wreath product, and their symmetric versions, the double semidirect product[1] and the block product, defined by Rhodes and Tilson [28] (or the triple product of Eilenberg [11]), and the Krohn–Rhodes theorem [17] have been the fundamental tools for several of the aforementioned results. The same holds for our investigation. In our main technical result, Theorem 9.5, we make a bridge between Lindström quantifiers and the double semidirect product, or the block product. Particular instances of this correspondence appear in above cited works, see, *e.g.*, the proofs of lemmas VI.1.2, VI.1.4, VII.2.2 and VII.2.3 in Straubing [31]. In fact, we will make use of a version of the double semidirect product and the block product that concerns finite monoids with a distinguished set of generators.

For the connection between circuit complexity and generalized quantifiers, we refer to Barrington, Immerman and Straubing [3], Barrington, Compton, Straubing and Therien [2], and the last two chapters of Straubing [31]. For second-order Lindström quantifiers and the relation of Lindström quantifiers to leaf language definability, see Burtschick and Vollmer [7], Peichl and Vollmer [23], Galota and Vollmer [15]. For results regarding the connection between the semidirect product and the expressive power of temporal logics, see Cohen, Perrin and Pin [8], Therien and Wilke [37] and Baziramwabo, McKenzie and Therien [4]. The last paragraph of the paper McKenzie, Schwentick, Therien and Vollmer [20] contains an indication of the possibility of handling nested monoidal quantifiers in the

[1] The double semidirect product is called the bilateral semidirect product in [31].

logical framework by series connections of 2-way automata. The texts Pin [25], Straubing [31] and Thomas [38, 39] are excellent surveys of the subject.

The paper is organized as follows. In Section 2, we associate a Lindström quantifier with any language and in Section 3, we establish some simple properties of Lindsröm quantifiers. In Section 4, we relate Lindsröm quantifiers to literal varieties of languages. Section 5 is devoted to mg-pairs, that is monoids equipped with a distinguished set of generators, and Section 6 to relativization. In Section 7, we define the operations of double semidirect product and block product on mg-pairs. Section 8 is devoted to varieties of finite mg-pairs, and to the operations of double semidirect product and block product on varieties. In Section 9, we establish a formal connection between Lindsröm quantifiers and the double semidirect product (block product, respectively) on varieties. In Section 10, we review the Krohn–Rhodes theorem and establish some of its consequences. In Section 11, we apply the results of Sections 9 and 10 to obtain additional characterizations of the expressive power of several concrete classes of Lindström quantifiers.

We have tried to make the paper accessible not only for the experts but also for a larger audience.

2. Lindström quantifiers, defined

For each alphabet (*i.e.*, finite nonempty set) Σ, the *formulas over* Σ are defined as follows. We assume that a fixed countable set of variables is given, and that each alphabet comes with a linear order defined on the letters of the alphabet.

- For each $a \in \Sigma$ and each variable x, $P_a(x)$ is an (atomic) formula. Moreover, when x, y are variables, $x < y$ is an (atomic) formula.
- For all formulas φ and ψ, both $\varphi \vee \psi$ and $\neg\varphi$ are formulas. Moreover, false is a formula.
- Suppose that $K \subseteq \Delta^*$, where $\Delta = \{b_1, \ldots, b_m\}$, $m \geq 1$ is some alphabet, ordered by $b_1 < \ldots < b_m$. Then for all variables x and formulas φ_{b_i}, $b_i \in \Delta$, $i < m$,

$$Q_K x.\langle \varphi_{b_1}, \ldots, \varphi_{b_{m-1}} \rangle \tag{1}$$

  is a formula.

We say that the variable x is *bound* in (1). The set of *free variables* of a formula is defined in the standard way. We identify any two formulas that differ only in the bound variables. Thus, we may assume that the bound variables of a formula are pairwise different, and different from any free variable. A formula with no free variables is called a *sentence*. The result of the *substitution* of a variable y for a (free) variable x in a formula φ, denoted $\varphi[y/x]$, is defined in the standard way.

Suppose that u is a word in Σ^* of length n, say $u = u_1 \ldots u_n$, where the u_i are letters. Moreover, suppose that φ is a formula over Σ whose free variables are contained in the finite set V. Given a function $\lambda : V \to [n]$, where $[n] = \{1, \ldots, n\}$, we say that (u, λ) *satisfies* φ, in notation $(u, \lambda) \models \varphi$, if

- φ is of the form $P_a(x)$ and $u_{\lambda(x)} = a$; or φ is of the form $x < y$ and $\lambda(x) < \lambda(y)$, or
- φ is of the form $\varphi_1 \vee \varphi_2$ and $(u, \lambda) \models \varphi_1$ or $(u, \lambda) \models \varphi_2$; or φ is of the form $\neg\psi$ and it is not the case that $(u, \lambda) \models \psi$, or
- φ is of the form (1) and the *characteristic word* [7] $\overline{u} = \overline{u}_1 \ldots \overline{u}_n$ determined by (u, λ) and the formula belongs to K, where for each $i \in [n]$, $\overline{u}_i$ is the least b_j, $j < m$ such that we have $(u, \kappa) \models \varphi_{b_j}$ for the function $\kappa : V \cup \{x\} \to [n]$ with $\kappa(y) = \lambda(y)$, for all $y \in V$, and $\kappa(x) = i$[2]. When no such b_j exists, we define $\overline{u}_i = b_m$. In particular, when $u = \epsilon$ is the empty word and $V = \emptyset$, then (u, λ) satisfies the formula (1) iff $\epsilon \in K$.

For all pairs (u, λ), relation $(u, \lambda) \models \mathsf{false}$ does not hold.

We will rely on the following lemma whose proof is standard.

Lemma 2.1. *Suppose that φ is a formula over Σ and X and Y both contain the free variables of φ. Then for any word $u = u_1 \ldots u_n \in \Sigma^*$ and functions $\kappa : X \to [n]$ and $\lambda : Y \to [n]$ that agree on all free variables of φ, $(u, \kappa) \models \varphi$ iff $(u, \lambda) \models \varphi$.*

Suppose that φ, ψ are formulas over Σ whose free variables are in X. We say that φ and ψ are *equivalent* if for all words $u = u_1 \ldots u_n \in \Sigma^*$ and functions $\lambda : X \to [n]$, $(u, \lambda) \models \varphi$ iff $(u, \lambda) \models \psi$.

Some notational conventions. In the sequel, in addition to the boolean connectives $\vee$ and $\neg$, we will also use the connectives $\wedge$ (conjunction), $\to$ (implication) and $\leftrightarrow$ (equivalence). These are treated as abbreviations. We use true to denote $\neg\mathsf{false}$. Moreover, we write $x \le y$ for $\neg(y < x)$, $x = y$ for $(x \le y) \wedge (y \le x)$, *etc.* In quantified formulas (1), we may assume that the sub-formulas φ_{b_i} are *pairwise inconsistent*, *i.e.*, no pair (u, λ) satisfies two or more φ_{b_i}. Then, we may define φ_{b_m} as $\neg(\bigvee_{i<m} \varphi_{b_i})$ and write (1) as $Q_K x.\langle \varphi_{b_i} \rangle_{b_i \in \Delta}$. Note that the ordering on Δ becomes irrelevant. *Below, when writing $Q_K x.\langle \varphi_{b_i} \rangle_{b_i \in \Delta}$, we will always assume that the φ_{b_i} form a deterministic family, i.e., the formulas φ_{b_i} are pairwise inconsistent and for any appropriate (u, λ) there is some b_i with $(u, \lambda) \models \varphi_{b_i}$.* When φ_b, $b \in \Delta$ are formulas over Σ with free variables in V, a *determinization* of the family φ_b, $b \in \Delta$ is a family φ'_b, $b \in \Delta$, where, given the linear order $b_1 < \ldots < b_n$ of Δ, it holds that $\varphi'_{b_i} = \varphi_{b_i} \wedge (\bigwedge_{j<i} \neg\varphi_{b_j})$, for all $i < n$, and $\varphi_{b_n} = \bigwedge_{j<n} \neg\varphi_{b_j}$. When φ is a sentence and V is empty, we will write $u \models \varphi$ whenever $(u, \lambda) \models \varphi$ for all, or for some $\lambda : V \to [n]$.

When writing a word u in the form $u = u_1 \ldots u_n$, we usually assume that the u_i are letters, so that u is a word of length n. By a *class of languages*, or *language class* $\mathcal{L}$ we mean a set $\mathcal{L}(\Sigma^*)$ of languages $L \subseteq \Sigma^*$, for each alphabet Σ. A *class of regular languages* contains only regular languages. When $\mathcal{L}$ is a class of languages, we let $\mathrm{Lin}(\mathcal{L})$ denote the logic whose formulas are those defined above, where

[2] When the φ_{b_j} contain no free variables other than x, then the function $u \mapsto \overline{u}$ is called a *translation* in [18] and [20].

Lindström quantification (1) is restricted to languages $K \in \mathcal{L}$. We also denote by $\mathrm{Lin}(\mathcal{L})$ the collection of formulas of this logic.

Example 2.2.

- Suppose that $K \subseteq \{b_1, b_2\}^*$ is the language $K_\exists = b_2^* b_1 (b_1 + b_2)^*$, where we assume $b_1 < b_2$. Then $(u, \lambda) \models Q_K x.\langle\varphi\rangle$ iff there is an extension $\kappa : V \cup \{x\} \to [n]$ of $\lambda : V \to [n]$ such that $(u, \kappa) \models \varphi$. Thus, the Lindtrsöm quantifier corresponding to $K_\exists$ is the ordinary existential quantifier. When $K_\forall \subseteq \{b_1, b_2\}^*$ is the language b_1^*, the corresponding Lindström quantifier is the ordinary universal quantifier.
- Suppose that M is a set of integers > 1. Let $\mathcal{C}_M$ consist of all languages $C_m^r \subseteq \{b_1, b_2\}^*$, $m \in M$, $r = 0, \ldots, m-1$, where C_m^r is the set of all words u in $\{b_1, b_2\}^*$ such that the number of b_1s in u is congruent to r modulo m. Then, assuming the order $b_1 < b_2$, $Q_{C_m^r}$ is the "modular quantifier" $\exists^{(m,r)}$ of Straubing, Therien and Thomas [35] and Straubing [31]. (Note that it is sufficient to allow modular quantifiers with respect to prime moduli as in [31, 35].)
- Let L_m^r, where $m \geq 1$ and $0 \leq r < m$, denote the language $(b_1^m)^* b_1^r$, considered as a subset of $\{b_1\}^*$. Then for every alphabet Σ, $Q_{L_m^r} x.\langle\rangle$ is a sentence over Σ, and for every word $u \in \Sigma^*$, $u \models Q_{L_m} x.\langle\rangle$ iff the length of u is congruent to r modulo m.
- In [2], Barrington, Compton, Straubing and Therien studied the extension of first-order logic with unary modular counting predicates $x \equiv r \bmod m$, where x is a variable, $m \geq 1$ and $r \in [m]$.[3] We show that this extension can be handled by Lindström quantifiers. For every m, r as above, let K_m^r denote the two-letter language $(b_2^m)^* b_2^{r-1} b_1 (b_1 + b_2)^*$, where we assume that $b_1 < b_2$. If u is word of length n and $\lambda : V \to [n]$, where V contains x, then (u, λ) satisfies $Q_K y.\langle y = x\rangle$ iff $\lambda(x)$ is congruent to r modulo m. Conversely, Lindsröm quantification with respect to the language K_m^r is expressible by the corresponding modular counting predicate and first-order quantification. In fact, $Q_{K_m^r} x.\langle\varphi\rangle$ is expressible as

$$\exists x[(x \equiv r \bmod m) \wedge \varphi \wedge \forall y(y < x \to \neg\varphi[y/x])].$$

- One can express temporal modalities by Lindström quantifiers. Recall from Pnueli [26], Cohen, Perrin and Pin [8] that the formulas of propositional linear temporal logic over an alphabet Σ are generated from atomic propositions p_a, $a \in \Sigma$, by the boolean connectives $\vee$ and $\neg$, and the next and until modalities denoted X and U. For more details and the definition of semantics we refer to [8, 26]. Let K_X denote the two-letter language $(b_1 + b_2) b_1 (b_1 + b_2)^*$, and let K_U denote the three-letter language $b_2^* b_1 (b_1 + b_2 + b_3)^*$, where we assume the orderings $b_1 < b_2$ and $b_1 < b_2 < b_3$, respectively. Using these notations, we can translate each

[3] In fact, the remainder r is taken between 0 and $m - 1$ in [2].

formula φ of propositional temporal logic over Σ into a sentence $\tau(\varphi)$ involving ordinary quantifiers and Lindström quantifiers with respect to the languages K_{X} and K_{U}. We define:

(1) $\tau(p_a) = \exists x.((\forall y.x \leq y) \wedge P_a(x))$, for all $a \in \Sigma$.

(2) $\tau(\varphi \vee \psi) = \tau(\varphi) \vee \tau(\psi)$ and $\tau(\neg\varphi) = \neg\tau(\varphi)$.

(3) $\tau(\mathsf{X}\varphi) = Q_{K_{\mathsf{X}}}x.\langle\tau(\varphi)[\geq x]\rangle$, if $\epsilon \not\models \varphi$, and $\tau(\mathsf{X}\varphi) = \psi \vee Q_{K_{\mathsf{X}}}x.\langle\tau(\varphi)[\geq x]\rangle$, if $\epsilon \models \varphi$, where ψ is a sentence only satisfied by the one-letter words.

(4) $\tau(\varphi\mathsf{U}\psi) = Q_{K_{\mathsf{U}}}x.\langle\tau(\psi)[\geq x], \tau(\varphi)[\geq x]\rangle$.

(Here, $\tau(\varphi)[\geq x]$ denotes a *relativization* of the formula $\tau(\varphi)$. See Straubing [31].) Then, for each word $u \in \Sigma^*$ and temporal logic formula φ, it holds that $u \models \varphi$ iff $u \models \tau(\varphi)$.

Given a sentence φ in Lin($\mathcal{L}$) over the alphabet Σ, we let $L_\varphi \subseteq \Sigma^*$ denote the *language defined by* φ:

$$L_\varphi = \{u \in \Sigma^* : u \models \varphi\}.$$

We write $\mathbf{Lin}(\mathcal{L})$ to denote the class of all languages definable by sentences of the logic Lin($\mathcal{L}$). Moreover, we define $\mathrm{FO}(\mathcal{L}) = \mathrm{Lin}(\mathcal{L} \cup \{K_\exists\})$ and $\mathbf{FO}(\mathcal{L}) = \mathbf{Lin}(\mathcal{L} \cup \{K_\exists\}) = \mathbf{Lin}(\mathcal{L} \cup \{K_\forall\})$. Thus, $\mathrm{FO} = \mathrm{FO}(\emptyset)$ is just ordinary first-order logic and $\mathbf{FO} = \mathbf{FO}(\emptyset)$ is the class of *first-order definable languages* of [22, 31].

Example 2.3.

- When $\mathcal{L} = \emptyset$, or when $\mathcal{L}(\Sigma^*) = \{\emptyset, \Sigma^*\}$, for each Σ, then for each alphabet Σ, $\mathbf{Lin}(\mathcal{L})$ consists of just the languages $\emptyset$ and Σ^*.
- Suppose that for a one-letter alphabet $\{a\}$, $\mathcal{L}$ consists of the language $\{\epsilon\}$, and $\mathcal{L}(\Sigma^*) = \emptyset$, for all other alphabets Σ. Then for each Σ, $\mathcal{L}' = \mathbf{Lin}(\mathcal{L})$ consists of the languages $\emptyset$, $\{\epsilon\}$, $\Sigma^+ = \Sigma^* - \{\epsilon\}$ and Σ^*. Moreover, $\mathbf{Lin}(\mathcal{L}') = \mathcal{L}'$.
- Let $\mathcal{L}$ consist of all finite languages. Then $\mathbf{Lin}(\mathcal{L})$ is the class of all finite or co-finite languages.
- Let $\mathcal{L}$ consist of the languages C_m^r defined in Example 2.2, where $m \geq 1$ and $0 \leq r < m$. Then, as shown in Straubing, Therien and Thomas [35], $\mathbf{Lin}(\mathcal{L})$ ($\mathbf{FO}(\mathcal{L})$, respectively) is the class of all regular languages whose syntactic monoid is a solvable group (monoid, respectively). See also Section 11.
- More generally, when M denotes a set of integers ≥ 1 and $\mathcal{C}_M = \{C_m^r : m \in M,\ 0 \leq r < m\}$ as above, then $\mathbf{Lin}(\mathcal{C}_M)$ consists of those regular languages L whose syntactic monoid is a solvable group of order n such that any prime divisor of n divides an integer in M. Moreover, $\mathbf{FO}(\mathcal{C}_M)$ consists of those regular languages L such that every subgroup of the syntactic monoid of L has this property. See Straubing, Therien and Thomas [35].
- Let $\mathcal{K}$ denote the collection of all languages K_m^r defined above. Then a language L belongs to $\mathbf{FO}(\mathcal{K})$ iff its syntactic monoid is quasi-aperiodic,

cf. Barrington, Compton, Straubing and Therien [2] and Straubing [31]. Moreover, $\mathbf{FO}(\mathcal{K}) = \mathbf{FO}(\{K_m^1 : m \geq 1\})$. More generally, when M is a subset of the positive integers and $\mathcal{K}_M = \{K_m^r : m \in M, r \in [m]\}$, then a characterization of the language class $\mathbf{FO}(\mathcal{K}_M)$ is given in Ésik and Ito [14].

In some of our results, we will also take into account the *quantification depth* $\mathrm{qd}(\varphi)$ of a formula φ, defined as the maximum number of nested quantifiers in φ. For a language class $\mathcal{L}$ and integer $n \geq 0$, we let $\mathrm{Lin}_n(\mathcal{L})$ ($\mathrm{FO}_n(\mathcal{L})$, respectively) denote the collection of all $\mathrm{Lin}(\mathcal{L})$ formulas ($\mathrm{FO}(\mathcal{L})$ formulas, respectively) whose quantification depth does not exceed n. Moreover, we denote by $\mathbf{Lin}_n(\mathcal{L})$ ($\mathbf{FO}_n(\mathcal{L})$, respectively) the class of all languages definable by sentences in $\mathrm{Lin}_n(\mathcal{L})$ ($\mathrm{FO}_n(\mathcal{L})$, respectively). Note that $\mathbf{Lin}(\mathcal{L}) = \bigcup_{n\geq 0} \mathbf{Lin}_n(\mathcal{L})$ and $\mathbf{FO}(\mathcal{L}) = \bigcup_{n\geq 0} \mathbf{FO}_n(\mathcal{L})$. Moreover, for every $n \geq 0$, the determinization of each family φ_b, $b \in \Delta$ of $\mathrm{Lin}_n(\mathcal{L})$ formulas consists of formulas in $\mathrm{Lin}_n(\mathcal{L})$, and similarly for families of $\mathrm{FO}_n(\mathcal{L})$ formulas.

For technical reasons, we also associate a language with formulas φ over Σ containing free variables. We follow the definitions in [31]. Let V denote a finite set of variables containing all of the free variables of φ. A *V-structure over* Σ is a word $u = u_1 \dots u_n$ in $(\Sigma \times P(V))^*$, where $P(V)$ denotes the power set of V, such that each variable in V appears exactly once in the right hand component of a letter $u_i = (a_i, X_i)$. Thus, the sets X_i are pairwise disjoint and their union is V. Note that the empty word ϵ is a V-structure iff $V = \emptyset$. Moreover, each letter $a \in \Sigma$ may be identified with the pair $(a, \emptyset)$. Given the V-structure u, the left hand components a_i determine a word $v = a_1 \dots a_n$ in Σ^*, and the right hand components determine a function $\lambda : V \to [n]$, defined by $\lambda(x) = i$ iff $x \in X_i$. Suppose that φ is a formula over Σ with free variables in V. We say that u satisfies φ, denoted $u \models \varphi$, if $(v, \lambda) \models \varphi$. The language L_φ defined by φ consists of all V-structures u over Σ with $u \models \varphi$.

3. Elementary properties of Lindström quantification

In this section we establish some basic properties of Lindström quantification.

Proposition 3.1. *For each class $\mathcal{L}$ of languages, it holds that $\mathcal{L} \subseteq \mathbf{Lin}_1(\mathcal{L})$. Moreover, when $\mathcal{L} \subseteq \mathcal{L}'$, then $\mathbf{Lin}_n(\mathcal{L}) \subseteq \mathbf{Lin}_n(\mathcal{L}')$, for all $n \geq 0$, so that $\mathbf{Lin}(\mathcal{L}) \subseteq \mathbf{Lin}(\mathcal{L}')$.*

Proof. Given $K \subseteq \Delta^*$ in $\mathcal{L}$, where $\Delta = \{b_1, \dots, b_k\}$, the language defined by the sentence $Q_K x.\langle P_{b_i}(x)\rangle_{b_i \in \Delta}$ is K. The second claim is obvious. □

Lemma 3.2. *Suppose that φ_b, $b \in B$ is a deterministic family of formulas over A with free variables in $\{x\} \cup Y$, and ψ is a formula over B whose free variables are in $\{x\} \cup Z$, where Y and Z are disjoint. Then the formula ψ' over the alphabet A with free variables in $\{x\} \cup Y \cup Z$ that results from ψ by replacing each sub-formula of the form $P_b(\xi)$, where ξ is any (free or bound) variable, by the formula $\varphi_b[\xi/x]$,* *is well-defined. Let* $(a_1, Y_1)\dots(a_n, Y_n)$ *be an* Y*-structure over* A *and for each* $j \in [n]$*, let* $b_j \in B$ *with*

$$(a_1, Y_1)\dots(a_j, \{x\} \cup Y_j)\dots(a_n, Y_n) \models \varphi_{b_j}. \tag{2}$$

Suppose that $Z_1, \dots, Z_n$ *are pairwise disjoint subsets of* Z *whose union is* Z*, so that* $(b_1, Z_1)\dots(b_n, Z_n)$ *is a* Z*-structure over* B*. Then for each* $i \in [n]$*,*

$$(a_1, Y_1 \cup Z_1)\dots(a_i, \{x\} \cup Y_i \cup Z_i)\dots(a_n, Y_n \cup Z_n) \models \psi' \tag{3}$$

iff

$$(b_1, Z_1)\dots(b_i, \{x\} \cup Z_i)\dots(b_n, Z_n) \models \psi. \tag{4}$$

Proof. We argue by induction on the structure of ψ. The basis case can be divided into four sub-cases. Let i denote an integer in $[n]$.

Case $\psi = \mathsf{false}$. In this case our claim is obvious.

Case $\psi = P_b(x)$*, for some* $b \in B$. Then $\psi' = \varphi_b$. Thus, using (2) and Lemma 2.1, we have that (3) holds iff

$$(a_1, Y_1)\dots(a_i, \{x_i\} \cup Y_i)\dots(a_n, Y_n) \models \varphi_b$$

iff $b = b_i$ iff (4) holds.

Case $\psi = P_b(z)$*, for some* $b \in B$ *and* $z \in Z$. Then $\psi' = \varphi_b[z/x]$. Thus, using (2) and Lemma 2.1 again, (3) holds iff

$$\exists j \in [n]\ z \in Z_j \wedge (a_1, Y_1)\dots(a_j, \{z\} \cup Y_j)\dots(a_n, Y_n) \models \varphi_b[z/x]$$

iff $\exists j \in [n]\ z \in Z_j \wedge b = b_j$ iff (4) holds.

Case $\psi = z_1 < z_2$*, where* $z_1, z_2 \in \{x\} \cup Z$. In this case $\psi' = \psi$, and (3) holds iff

$$(a_1, Z_1)\dots(a_i, \{x\} \cup Z_i)\dots(a_n, Z_n) \models z_1 < z_2$$

iff

$$(b_1, Z_1)\dots(b_i, \{x\} \cup Z_i)\dots(b_n, Z_n) \models z_1 < z_2,$$

i.e., when (4) holds.

The induction step can be divided into three sub-cases.

Case $\psi = \psi_1 \vee \psi_2$*, for some* ψ_1 *and* ψ_2. Then $\psi' = \psi_1' \vee \psi_2'$, where for $j = 1, 2$, ψ_j' is the formula that results from ψ_j by replacing each sub-formula of the form $P_b(\xi)$ with $\varphi_b[\xi/x]$. Since by the induction hypothesis

$$(a_1, Y_1 \cup Z_1)\dots(a_i, \{x\} \cup Y_i \cup Z_i)\dots(a_n, Y_n \cup Z_n) \models \psi_j'$$

iff

$$(b_1, Z_1) \dots (b_i, \{x\} \cup Z_i) \dots (b_n, Z_n) \models \psi_j,$$

for $j = 1, 2$, it follows that (3) holds iff (4) holds.

Case $\psi = \neg\psi_1$, for some ψ_1. This case is analogous to the previous one.

Case $\psi = Q_L z_0.\langle \tau_d \rangle_{d \in D}$ for some τ_d, $d \in D$, where each τ_d is a formula over B in the free variables $\{x\} \cup Z \cup \{z_0\}$. Now ψ' is $Q_L z_0.\langle \tau'_d \rangle_{d \in D}$, where each τ'_d is obtained from τ_d by replacing each sub-formula of the form $P_b(\xi)$ with $\varphi_b[\xi/x]$. By the induction hypothesis, each τ'_d is well-defined. To prove that ψ' is also well-defined, we have to show that the family τ'_d, $d \in D$, is deterministic, given that τ_d, $d \in D$ is a deterministic family. But by the induction hypothesis,

$$(a_1, Y_1 \cup Z'_1) \dots (a_j, Y_j \cup Z'_j \cup \{z_0\}) \dots (a_n, Y_n \cup Z'_n) \models \tau'_d$$

iff

$$(b_1, Z'_1) \dots (b_j, Z'_j \cup \{z_0\}) \dots (b_n, Z'_n) \models \tau_d,$$

for each $d \in D$ and $j \in [n]$, where $Z'_k = Z_k$ if $k \neq i$ and $Z'_k = Z_k \cup \{x\}$ if $k = i$. Since τ_d, $d \in D$ is deterministic, it follows that τ'_d, $d \in D$ is also deterministic. Moreover, the characteristic word determined by the structure $(a_1, Y_1 \cup Z_1) \dots (a_i, \{x\} \cup Y_i \cup Z_i) \dots (a_n, Y_n \cup Z_n)$ and ψ' is the same as that determined by $(b_1, Z_1) \dots (b_i, \{x\} \cup Z_i) \dots (b_n, Z_n)$ and ψ. It follows that (3) holds iff (4) does. □

Note the following special case of Lemma 3.2.

Corollary 3.3. *Suppose that φ_b, $b \in B$ is a deterministic family of formulas over A with free variables in $\{x\} \cup Y$ and ψ is formula over B with no free variable other than x. Let ψ' denote the formula over the alphabet A with free variables in $\{x\} \cup Y$ that results from ψ by replacing each sub-formula of the form $P_b(\xi)$, where ξ is any (free or bound) variable by the formula $\varphi_b[\xi/x]$. Let $(a_1, Y_1) \dots (a_n, Y_n)$ be an Y-structure over A and for each $j \in [n]$, let $b_j \in B$ with*

$$(a_1, Y_1) \dots (a_j, \{x\} \cup Y_j) \dots (a_n, Y_n) \models \varphi_{b_j}.$$

Then for each $i \in [n]$,

$$(a_1, Y_1) \dots (a_i, \{x\} \cup Y_i) \dots (a_n, Y_n) \models \psi'$$

iff

$$b_1 \dots (b_i, \{x\}) \dots b_n \models \psi.$$

Lemma 3.4. *Let $\varphi = Q_K x.\langle \varphi_b \rangle_{b \in B}$ be a formula over A with free variables in Y, and let $\psi = Q_L x.\langle \psi_c \rangle_{c \in C}$ be a sentence over B with $L_\psi = K$. For each $c \in C$,* *let ψ'_c denote the formula that results from ψ_c by replacing each sub-formula of the form $P_b(\xi)$, where ξ is a free or bound variable, by the formula $\varphi_b[\xi/x]$. Then the formula $\varphi' = Q_L x.\langle\psi'_c\rangle_{c\in C}$ of quantification depth less than or equal to* $\mathrm{qd}(\psi) + \mathrm{qd}(\varphi) - 1$ *is equivalent to φ, i.e., for all Y-structures $(a_1, Y_1)\dots(a_n, Y_n)$ over A,*

$$(a_1, Y_1)\dots(a_n, Y_n) \models \varphi \Leftrightarrow (a_1, Y_1)\dots(a_n, Y_n) \models \varphi'. \quad (5)$$

Proof. First note that by Lemma 3.2, φ' and each ψ'_c is well-defined, moreover, ψ'_c, $c \in C$ is a deterministic family. Let $(a_1, Y_1)\dots(a_n, Y_n)$ denote an Y-structure over A. For each $i \in [n]$, let b_i be the unique letter in B with

$$(a_1, Y_1)\dots(a_i, \{x\} \cup Y_i)\dots(a_n, Y_n) \models \varphi_{b_i}.$$

Moreover, let $c_i \in C$ with

$$b_1 \dots (b_i, \{x\}) \dots b_n \models \psi_{c_i}.$$

We have

$$\begin{aligned}(a_1, Y_1)\dots(a_n, Y_n) \models \varphi &\Leftrightarrow b_1 \dots b_n \in K\\ &\Leftrightarrow b_1 \dots b_n \models \psi\\ &\Leftrightarrow c_1 \dots c_n \in L.\end{aligned}$$

But by Corollary 3.3,

$$b_1 \dots (b_i, \{x\}) \dots b_n \models \psi_c \Leftrightarrow (a_1, Y_1)\dots(a_i, \{x\} \cup Y_i)\dots(a_n, Y_n) \models \psi'_c,$$

for each $i \in [n]$ and $c \in C$. Thus

$$(a_1, Y_1)\dots(a_n, Y_n) \models \varphi' \Leftrightarrow c_1 \dots c_n \in L,$$

proving (5). □

Theorem 3.5. *For any class $\mathcal{L}$ of languages,* $\mathbf{Lin}(\mathbf{Lin}(\mathcal{L})) \subseteq \mathbf{Lin}(\mathcal{L})$. *Moreover,* $\mathbf{Lin}_n(\mathbf{Lin}_1(\mathcal{L})) \subseteq \mathbf{Lin}_n(\mathcal{L})$, *for all $n \geq 0$.*

Proof. Given a formula φ in $\mathrm{Lin}(\mathbf{Lin}(\mathcal{L}))$ over some alphabet A, we show by induction on the structure of φ how to construct an equivalent formula $\tau(\varphi)$ in $\mathrm{Lin}(\mathcal{L})$. If φ is in $\mathrm{Lin}(\mathbf{Lin}_1(\mathcal{L}))$, we will also have $\mathrm{qd}(\tau(\varphi)) \leq \mathrm{qd}(\varphi)$. When φ is an atomic formula or false, we define $\tau(\varphi)$ to be the same formula. When φ is $\varphi_1 \vee \varphi_2$ or $\neg\varphi_1$, we define $\tau(\varphi)$ to be $\tau(\varphi_1) \vee \tau(\varphi_2)$ or $\neg(\tau(\varphi_1))$, respectively. Assume finally that φ is $Q_K x.\langle\varphi_b\rangle_{b\in B}$, where each φ_b is a formula over A in $\mathrm{Lin}(\mathbf{Lin}(\mathcal{L}))$ with free variables in $\{x\} \cup Y$ and K is a language in $\mathbf{Lin}(\mathcal{L})$ over the alphabet B. Now K is a boolean combination of languages $K_1, \dots, K_m$ definable by $\mathrm{Lin}(\mathcal{L})$ sentences of the form $Q_{L_i} x.\langle\psi^i_c\rangle_{c\in C_i}$, $i \in [m]$. Thus, φ is equivalent to a boolean combination of the formulas $Q_{K_i} x.\langle\varphi_b\rangle_{b\in B}$, $i \in [m]$, and by the induction hypothesis and

Lemma 3.4, each $Q_{K_i}x.\langle\varphi_b\rangle_{b\in B}$ is equivalent to the formula ψ_i of $\mathrm{Lin}(\mathcal{L})$ obtained from $Q_{L_i}x.\langle\psi^i_c\rangle_{c\in C_i}$ by replacing each occurrence of $P_b(\xi)$, where ξ is any variable, by $\varphi_b[\xi/x]$. Thus, when the quantification depth of $Q_{L_i}x.\langle\psi^i_c\rangle_{c\in C_i}$ is 1, so that the ψ^i_c contain no quantifiers, then $\mathrm{qd}(\psi_i)$ is at most the quantification depth of $Q_{K_i}x.\langle\varphi_b\rangle_{b\in B}$. It follows that φ is equivalent to a boolean combination $\tau(\varphi)$ of the formulas ψ_i, $i \in [m]$. Moreover, when $K \in \mathbf{Lin}_1(\mathcal{L})$, then $\mathrm{qd}(\varphi) \leq \mathrm{qd}(\tau(\varphi))$. □

Corollary 3.6. *Each of the assignments* $\mathcal{L} \mapsto \mathbf{Lin}(\mathcal{L})$ *and* $\mathcal{L} \mapsto \mathbf{Lin}_1(\mathcal{L})$ *defines a closure operator on language classes.*

Corollary 3.7. *The assignment* $\mathcal{L} \mapsto \mathbf{FO}(\mathcal{L})$ *defines a closure operator.*

Corollary 3.8. *For any language classes* $\mathcal{L}, \mathcal{L}'$*, we have* $\mathbf{Lin}(\mathcal{L}) \subseteq \mathbf{Lin}(\mathcal{L}')$ *iff for each formula of* $\mathrm{Lin}(\mathcal{L})$ *there is an equivalent formula of* $\mathrm{Lin}(\mathcal{L}')$*.*

Proof. The sufficiency part of the claim is obvious. Suppose now that $\mathbf{Lin}(\mathcal{L}) \subseteq \mathbf{Lin}(\mathcal{L}')$. Then any formula φ of $\mathrm{Lin}(\mathcal{L})$ that possibly contains free variables is also a formula of $\mathrm{Lin}(\mathbf{Lin}(\mathcal{L}'))$. Thus, since by the above proof of Theorem 3.5 every formula of $\mathrm{Lin}(\mathbf{Lin}(\mathcal{L}'))$ is equivalent to a formula of $\mathrm{Lin}(\mathcal{L}')$, there is a $\mathrm{Lin}(\mathcal{L}')$ formula equivalent to φ. □

By the same argument, we have:

Corollary 3.9. *For any language classes* $\mathcal{L}, \mathcal{L}'$*, we have* $\mathbf{Lin}_1(\mathcal{L}) \subseteq \mathbf{Lin}_1(\mathcal{L}')$ *iff for each formula* φ *of* $\mathrm{Lin}(\mathcal{L})$ *there is an equivalent formula* φ' *of* $\mathrm{Lin}(\mathcal{L}')$ *with* $\mathrm{qd}(\varphi') \leq \mathrm{qd}(\varphi)$*.*

Corollary 3.10. *For any class* $\mathcal{L}$ *of languages,* $\mathbf{Lin}(\mathcal{L}) = \mathbf{FO}(\mathcal{L})$ *iff* $K_\exists \in \mathbf{Lin}(\mathcal{L})$*.*

Of course, $K_\exists \in \mathbf{Lin}(\mathcal{L})$ iff $K_\forall \in \mathbf{Lin}(\mathcal{L})$.

Remark 3.11. When L is a regular language, one can use any finite automaton accepting L to express the Lindström quantifier Q_L in monadic second-order logic [31] and then use the theorem of Büchi [6], Elgot [12] and Trakhtenbrot [40] to establish that if each φ_{b_i} defines a regular language (of $(V \cup \{x\})$-structures), then the formula (1) defines a regular language (of V-structures). Thus, for any class $\mathcal{L}$ of regular languages, $\mathbf{Lin}(\mathcal{L})$ contains only regular languages. This fact may also be seen as an instance of a general property of Lindström quantifiers, *cf.* Exercise 12.1.1 in [10]. Moreover, this fact also follows from Theorem 9.5 below. See also Barrington, Immerman and Straubing [3], Lautemann, McKenzie and Schwentick [18], and Theorem 9.5.

Remark 3.12. All of the results of this section remain valid if instead of, or in addition to the predicate $<$, the language contains the successor predicate $y = x + 1$, or any other numerical predicate like the predicates $x \equiv r \bmod m$, where $m \geq 2$ and $r \in [m]$. In fact, Corollary 3.6 formulates a very general property of Lindström quantification that we have not been able to locate in the literature.

4. Literal varieties

We say that a class $\mathcal{L}$ of languages is *closed with respect to the boolean operations* if for each alphabet Σ, $\mathcal{L}(\Sigma^*)$ contains $\emptyset$ and Σ^* and forms a boolean algebra. Moreover, we say that $\mathcal{L}$ is *closed with respect to inverse literal (homo)morphisms* if for all alphabets Σ, Δ and letter preserving homomorphisms $h : \Sigma^* \to \Delta^*$ (*i.e.*, such that $h(\Sigma) \subseteq \Delta$), and for all languages $L \in \mathcal{L}(\Delta^*)$,

$$h^{-1}(L) = \{u \in \Sigma^* : h(u) \in L\}$$

is in $\mathcal{L}(\Sigma^*)$.

Suppose that $\mathcal{L}$ is a class of languages. We call $\mathcal{L}$ a *literal pre-variety* if it is closed with respect to the boolean operations and inverse literal homomorphisms. A *literal variety* is also closed with respect to left and right quotients. Thus, if $L, L_1, L_2 \subseteq \Sigma^*$ are in a literal variety $\mathcal{L}$ and $v \in \Sigma^*$, then $L_1 \cup L_2$, $\Sigma^* - L$ and the left and right *quotients* $v^{-1}L, Lv^{-1}$ are also in $\mathcal{L}$, where

$$\begin{aligned} v^{-1}L &= \{u \in \Sigma^* : vu \in L\} \\ Lv^{-1} &= \{u \in \Sigma^* : uv \in L\}. \end{aligned}$$

Moreover, if h is a literal morphism $\Sigma^* \to \Delta^*$ and $L \subseteq \Delta^*$ is in $\mathcal{L}$, then $h^{-1}(L)$ is also in $\mathcal{L}$. Note that every literal variety contains, for each alphabet Σ, the language Σ^* and the empty language. A *literal (pre)variety of regular languages* is a literal (pre)variety $\mathcal{L}$ such that every language in $\mathcal{L}$ is regular. Each class $\mathcal{L}$ of (regular) languages is contained in a least literal pre-variety $\mathcal{L}'$ and in a least literal variety $\mathcal{L}''$ of (regular) languages, respectively called the literal pre-variety and the literal variety generated by $\mathcal{L}$. Clearly, $\mathcal{L}' \subseteq \mathcal{L}''$. It is not difficult to see that a language belongs to $\mathcal{L}'$ iff it is an inverse image under a literal morphism of a boolean combination of languages in $\mathcal{L}$. Moreover, a language belongs to $\mathcal{L}''$ iff it is an inverse image under a literal morphism of a boolean combination of quotients of languages in $\mathcal{L}$. Thus, $\mathcal{L}' = \mathcal{L}''$ iff each quotient of a language in $\mathcal{L}$ belongs to $\mathcal{L}'$.

Literal varieties of regular languages are a generalization of the $*$-*varieties* of Eilenberg [11] and Pin [24] that are closed with respect to arbitrary inverse homomorphisms. Recently, Straubing [32] has defined the notion of C-*varieties*, where C is a category of morphisms between free monoids. Literal varieties of regular languages may be seen as that special case corresponding to the category C of literal morphisms.

Proposition 4.1. *For any language class $\mathcal{L}$ and integer $n \geq 0$, $\mathbf{Lin}_n(\mathcal{L})$ is a literal pre-variety. Thus, $\mathbf{Lin}(\mathcal{L})$ is also a literal pre-variety.*

Proof. It is obvious that for each n, $\mathbf{Lin}_n(\mathcal{L})$ is closed with respect to the boolean operations. To prove that $\mathbf{Lin}_n(\mathcal{L})$ is closed with respect to inverse literal homomorphisms, suppose that $L \subseteq \Delta^*$ is in $\mathbf{Lin}_n(\mathcal{L})$ and h is a literal homomorphism

$\Sigma^* \to \Delta^*$. We want to show $h^{-1}(L) \in \mathbf{Lin}_n(\mathcal{L})$. For any finite set V of variables, extend h to a literal homomorphism $(\Sigma \times P(V))^* \to (\Delta \times P(V))^*$ by defining

$$h((u_1, V_1) \dots (u_m, V_m)) = (h(u_1), V_1) \dots (h(u_m), V_m),$$

for all $(u_1, V_1) \dots (u_m, V_m) \in (\Sigma \times P(V))^*$. Then, for each formula φ in $\mathrm{Lin}(\mathcal{L})$ over Δ with free variables in V, let $\tau(\varphi)$ denote the formula obtained from φ by replacing each atomic sub-formula $P_b(x)$, where x is a variable and $b \in \Delta$, by the disjunction of all formulas $P_a(x)$ with $h(a) = b$. When b is not in the range of h, we replace $P_b(x)$ by the formula false. Note that $\mathrm{qd}(\tau(\varphi)) = \mathrm{qd}(\varphi)$ and $\tau(\varphi) \in \mathrm{Lin}(\mathcal{L})$. It follows by a straightforward induction argument that for all V-structures u over Σ,

$$u \models \tau(\varphi) \Leftrightarrow h(u) \models \varphi.$$

In particular, when φ is a sentence in $\mathrm{Lin}_n(\mathcal{L})$ defining L, we obtain $L_{\tau(\varphi)} = h^{-1}(L)$ and $\tau(\varphi) \in \mathrm{Lin}_n(\mathcal{L})$, proving our claim. □

By Corollary 3.6, $\mathbf{Lin}_1$ is a closure operator on language classes. The following result gives a precise description of this closure operator.

Proposition 4.2. *For each language class $\mathcal{L}$, $\mathbf{Lin}_1(\mathcal{L})$ is the literal pre-variety generated by $\mathcal{L}$.*

Proof. Let $\mathcal{L}'$ denote the literal pre-variety generated by $\mathcal{L}$. We already know from Proposition 3.1 and Proposition 4.1 that $\mathbf{Lin}_1(\mathcal{L})$ is a literal pre-variety containing $\mathcal{L}$, so that $\mathcal{L}' \subseteq \mathbf{Lin}_1(\mathcal{L})$. Suppose now that $L \subseteq \Sigma^*$ is defined by a sentence $\varphi = Q_K x.\langle \varphi_b \rangle_{b \in \Delta}$ in $\mathrm{Lin}(\mathcal{L})$ of quantification depth 1, so that each φ_b is quantifier-free and contains at most the variable x. It is easy to see that each φ_b is equivalent to a formula ψ_b which is a (possibly empty) disjunction of atomic formulas $P_a(x)$, $a \in \Sigma$. Moreover, since the family φ_b, $b \in \Delta$ is deterministic, for each letter $a \in \Sigma$ there is a unique $b \in \Delta$ such that $P_a(x)$ appears as a sub-formula of ψ_b. Now let $h : \Sigma^* \to \Delta^*$ denote the homomorphism that maps each letter $a \in \Sigma$ to the corresponding b. Then $L = h^{-1}(K)$, so that $L \in \mathcal{L}'$. Since each sentence in $\mathrm{Lin}(\mathcal{L})$ is equivalent to a boolean combination of sentences φ and since literal pre-varieties are closed with respect to the boolean operations, it follows now that $\mathbf{Lin}_1(\mathcal{L}) \subseteq \mathcal{L}'$. □

From Theorem 3.5 and Proposition 4.2 we immediately have:

Corollary 4.3. *For each language class $\mathcal{L}$ and integer $n \geq 0$, $\mathbf{Lin}_n(\mathcal{L}') = \mathbf{Lin}_n(\mathcal{L})$, where $\mathcal{L}'$ is the literal pre-variety generated by $\mathcal{L}$. Thus, $\mathbf{Lin}(\mathcal{L}') = \mathbf{Lin}(\mathcal{L})$.*

We say that *quotients are expressible in* $\mathrm{Lin}(\mathcal{L})$ if for each $L \in \mathcal{L}(\Delta^*)$ and $v \in \Delta^*$, and for any formulas $\varphi = Q_{v^{-1}L} x.\langle \varphi_b \rangle_{b \in \Delta^*}$ and $\varphi' = Q_{Lv^{-1}} x.\langle \varphi_b \rangle_{b \in \Delta^*}$, where φ_b, $b \in \Delta$ are $\mathrm{Lin}(\mathcal{L})$ formulas over some alphabet Σ with free variables in $\{x\} \cup Y$, there exist equivalent formulas in $\mathrm{Lin}(\mathcal{L})$, *i.e.*, $\mathrm{Lin}(\mathcal{L})$ formulas ψ and ψ' over Σ in the free variables Y such that for all Y-structures u over Σ,

$$u \models \varphi \Leftrightarrow u \models \psi \text{ and } u \models \varphi' \Leftrightarrow u \models \psi'.$$

If in addition $\mathrm{qd}(\psi) \leq \mathrm{qd}(\varphi)$ and $\mathrm{qd}(\psi') \leq \mathrm{qd}(\varphi')$, for all φ and φ', then we say that *quotients are strictly expressible in* $\mathrm{Lin}(\mathcal{L})$. We say that quotients are (strictly) expressible in $\mathrm{FO}(\mathcal{L})$ if quotients are strictly expressible in $\mathrm{Lin}(\mathcal{L} \cup \{K_\exists\})$.

Example 4.4.

- Let $\mathcal{L}$ contain only the language $K_\exists \subseteq \{b_1, b_2\}^*$. The only quotients of $K_\exists$ are $K_\exists$ and $\{b_1, b_2\}^*$. Since for any φ and x, $Q_{\{b_1,b_2\}^*}x.\langle\varphi\rangle$ is equivalent to true, it follows that quotients are strictly expressible in $\mathrm{Lin}(\mathcal{L})$, *i.e.*, in FO.
- Suppose that $m \geq 1$ and consider the language C_m^0 defined above. Its only quotients are the languages C_m^r, where $0 \leq r < m$. It is easy to see that quotients are expressible in $\mathrm{FO}(\{C_m^0\})$. Also, quotients are strictly expressible in $\mathrm{Lin}(\{C_2^0\})$, since $Q_{C_2^1}x.\langle\varphi\rangle$ is expressible as $\neg(Q_{C_2^0}x.\langle\varphi\rangle)$.
- For any set M of positive integers, quotients are strictly expressible in $\mathrm{Lin}(\mathcal{C}_M)$ and in $\mathrm{FO}(\mathcal{C}_M)$, where $\mathcal{C}_M = \{C_m^r : m \in M,\ 0 \leq r < m\}$ as above.
- For every $m \geq 1$ and $r \in [m]$, the quotients of $K_m^r = (b_2^m)^* b_2^{r-1} b_1 (b_1 + b_2)^*$ are the languages $\emptyset$, $(b_1 + b_2)^*$, $K_m^1, \ldots, K_m^m$ and $K_m^r \cup (b_2^m)^*, \ldots, K_m^r \cup (b_2^m)^* b_2^{m-1}$. Using this fact, it is easy to see that quotients are expressible in $\mathrm{FO}(\{K_m^r\})$. Moreover, it follows that quotients are expressible in $\mathrm{FO}(\mathcal{K}_M)$, where for a set M of positive integers, $\mathcal{K}_M = \{K_m^r : m \in M, r \in [m]\}$.

Proposition 4.5. *Suppose that quotients are expressible in* $\mathrm{Lin}(\mathcal{L})$. *Then* $\mathbf{Lin}(\mathcal{L})$ *is a literal variety. Moreover, if quotients are strictly expressible in* $\mathrm{Lin}(\mathcal{L})$, *then* $\mathbf{Lin}_n(\mathcal{L})$ *is a literal variety, for each* $n \geq 0$.

Proof. Suppose that φ is a formula of $\mathrm{Lin}(\mathcal{L})$ over the alphabet Σ possibly containing free variables from the finite set V. Let L_φ denote the set of all V-structures over Σ defined by φ. We argue by induction on the structure of φ to prove that for each letter $a \in \Sigma$, the set of V-structures $a^{-1}L_\varphi$ is definable by some formula ψ of $\mathrm{Lin}(\mathcal{L})$ with free variables in V. Moreover, for each letter $a \in \Sigma$ and variable $x \in V$, the set of $(V - \{x\})$-structures $(a, \{x\})^{-1}L_\varphi$ is definable by some formula ψ with free variables in $V - \{x\}$. When quotients are strictly expressible in $\mathrm{Lin}(\mathcal{L})$, then for each φ, the quantification depth of the formula ψ defining $a^{-1}L_\varphi$ or $(a, \{x\})^{-1}L_\varphi$ will be at most that of φ. The extension of the argument to left quotients $v^{-1}L_\varphi$, where v is any word in Σ^*, ore more generally, v is any V_1-structure over Σ, for some $V_1 \subseteq V$, is left to the reader. Right quotients can be dealt with symmetrically.

The basis case is obvious, including the case when φ is false, as are the cases when φ is of the form $\varphi_1 \vee \varphi_2$ or $\neg\psi$. One uses the fact that the operation of taking left quotients commutes with the boolean operations. Moreover, $a^{-1}L_\varphi = L_\varphi$, for all atomic formulas φ. Also, if φ is atomic, then $(a, \{x\})^{-1}L_\varphi$ is definable by a quantifier-free formula. Suppose finally that φ is of the form $Q_K x.\langle\psi_{b_j}\rangle_{b_j \in \Delta}$, where $K \subseteq \Delta^* = \{b_1, \ldots, b_m\}^*$ is a language in $\mathcal{L}$ and each ψ_{b_j} is a formula of $\mathrm{Lin}(\mathcal{L})$ over Σ with free variables in $V \cup \{x\}$, where $x \notin V$. For each b_j, let L_{b_j} denote the set of all $(V \cup \{x\})$-structures defined by ψ_{b_j}. It follows by the induction hypothesis that for each b_j there is a formula ρ_{b_j} of $\mathrm{Lin}(\mathcal{L})$ that defines the set $(a,\{x\})^{-1}L_{b_j}$, *i.e.*, the set of all V-structures u such that $(a,\{x\})u \models \psi_{b_j}$. Moreover, for each b_k there exists a formula ψ'_{b_k} of $\mathrm{Lin}(\mathcal{L})$ over Σ with free variables in $V \cup \{x\}$ defining the set $a^{-1}L_{b_k}$, *i.e.*, the set of all $(V \cup \{x\})$-structures u such that $au \models \psi_{b_k}$. For each b_j, let

$$\alpha_{b_j} = \rho_{b_j} \wedge Q_{b_j^{-1}K}x.\langle \psi'_{b_k} \rangle_{b_k \in \Delta}.$$

Given a V-structure u, we have $u \models \alpha_{b_j}$ iff $(a,\{x\})u \models \psi_{b_j}$ and $\overline{u} \in b_j^{-1}K$, where $\overline{u}$ is the characteristic word determined by u and the formula $Q_{b_j^{-1}K}x.\langle \psi'_{b_k} \rangle_{b_k \in \Delta}$. It follows that $u \models \alpha_{b_j}$ iff the characteristic word determined by au and the formula φ starts with b_j and belongs to K. Thus, letting

$$\alpha = \bigvee_{b_j \in \Delta} \alpha_{b_j},$$

it holds that $u \models \alpha$ iff $au \models \varphi$ iff $u \in a^{-1}L_\varphi$, showing that $a^{-1}L_\varphi$ is definable by α. Since by assumption there is a $\mathrm{Lin}(\mathcal{L})$ formula equivalent to α, it follows that $a^{-1}L_\varphi$ is in $\mathbf{Lin}(\mathcal{L})$. If quotients are strictly expressible in $\mathrm{Lin}(\mathcal{L})$, then by the induction hypothesis, $\mathrm{qd}(\rho_{b_j}), \mathrm{qd}(\psi'_{b_j}) \leq \mathrm{qd}(\psi_{b_j})$, for all $j \in [m]$. Thus, $\mathrm{qd}(\alpha) \leq \mathrm{qd}(\varphi)$, and since quotients are strictly expressible in $\mathrm{Lin}(\mathcal{L})$, it follows that there exists a $\mathrm{Lin}(\mathcal{L})$ formula of quantification depth at most $\mathrm{qd}(\varphi)$ which is equivalent to α.

The fact that $(a,\{x\})^{-1}L_\varphi$ is in $\mathbf{Lin}(\mathcal{L})$, for each variable $x \in V$ and letter $a \in \Sigma$, can be established by a similar argument. Again, it follows that if quotients are strictly expressible in $\mathrm{Lin}(\mathcal{L})$, then $(a,\{x\})^{-1}L_\varphi$ is definable by a formula of $\mathrm{Lin}(\mathcal{L})$ whose quantification depth is at most that of φ. □

Corollary 4.6. *The following conditions are equivalent for a class $\mathcal{L}$ of languages:*
1. *Each quotient of any language in $\mathcal{L}$ belongs to* $\mathbf{Lin}(\mathcal{L})$.
2. $\mathbf{Lin}(\mathcal{L})$ *is closed with respect to quotients.*
3. $\mathbf{Lin}(\mathcal{L})$ *is a literal variety.*
4. *Quotients are expressible in* $\mathrm{Lin}(\mathcal{L})$.

Proof. Let $\mathcal{L}_0$ denote the class of all quotients of languages in $\mathcal{L}$. If the first condition holds, then since $\mathbf{Lin}$ is a closure operator and $\mathcal{L} \subseteq \mathcal{L}_0 \subseteq \mathbf{Lin}(\mathcal{L})$, we have $\mathbf{Lin}(\mathcal{L}_0) = \mathbf{Lin}(\mathcal{L})$. Thus, by Corollary 3.8, for every $\mathrm{Lin}(\mathcal{L}_0)$ formula there is an equivalent $\mathrm{Lin}(\mathcal{L})$ formula. In particular, quotients are expressible in $\mathrm{Lin}(\mathcal{L})$. This proves that the first condition implies the last. That the last implies the third is the content of Proposition 4.5. Finally, it is clear that the third condition implies the second which in turn implies the first. □

Corollary 4.7. *The following conditions are equivalent for a language class $\mathcal{L}$.*
1. *Each quotient of any language in $\mathcal{L}$ belongs to the literal pre-variety generated by $\mathcal{L}$.*

(2) *The literal pre-variety generated by* $\mathcal{L}$ *is closed with respect to quotients and is thus a literal variety.*
(3) *Each quotient of any language in* $\mathcal{L}$ *is in* $\mathbf{Lin}_1(\mathcal{L})$.
(4) $\mathbf{Lin}_1(\mathcal{L})$ *is closed with respect to quotients.*
(5) $\mathbf{Lin}_1(\mathcal{L})$ *is a literal variety.*
(6) $\mathbf{Lin}_n(\mathcal{L})$ *is closed with respect to quotients, for all* $n \geq 0$.
(7) $\mathbf{Lin}_n(\mathcal{L})$ *is a literal variety, for all* $n \geq 0$.
(8) *Quotients are strictly expressible in* $\mathrm{Lin}(\mathcal{L})$.

Proof. By Propositions 4.1, 4.2, 4.5 and Corollary 3.9. □

Corollary 4.8. *For any language class* $\mathcal{L}$*, if quotients are (strictly) expressible in* $\mathrm{Lin}(\mathcal{L})$*, then the same holds for* $\mathrm{FO}(\mathcal{L})$.

Corollary 4.9. *If* $\mathbf{Lin}(\mathcal{L}) = \mathbf{Lin}(\mathcal{L}')$ *and quotients are expressible in* $\mathrm{Lin}(\mathcal{L})$*, then the same holds for* $\mathrm{Lin}(\mathcal{L}')$*. Similarly, if* $\mathbf{Lin}_1(\mathcal{L}) = \mathbf{Lin}_1(\mathcal{L}')$ *and quotients are strictly expressible in* $\mathrm{Lin}(\mathcal{L})$*, then the same holds for* $\mathrm{Lin}(\mathcal{L}')$.

Proof. Since quotients are expressible in $\mathrm{Lin}(\mathcal{L})$, by Corollary 4.6 we have that $\mathbf{Lin}(\mathcal{L}') = \mathbf{Lin}(\mathcal{L})$ is closed with respect to quotients. Thus, by Corollary 4.6 again, quotients are expressible in $\mathrm{Lin}(\mathcal{L}')$. The proof of the second claim uses Corollary 4.7. □

Corollary 4.10. *Assume that quotients are expressible in* $\mathrm{Lin}(\mathcal{L})$*. Then* $\mathbf{Lin}(\mathcal{L}) = \mathbf{Lin}(\mathcal{L}')$ *holds for the least literal variety* $\mathcal{L}'$ *containing* $\mathcal{L}$*. And if quotients are strictly expressible in* $\mathrm{Lin}(\mathcal{L})$*, then* $\mathbf{Lin}_n(\mathcal{L}) = \mathbf{Lin}_n(\mathcal{L}')$ *holds for all* $n \geq 0$*, where* $\mathcal{L}'$ *is the literal variety generated by* $\mathcal{L}$.

Proof. From Theorem 3.5 and Corollaries 4.3, 4.6 and 4.7. □

Remark 4.11. Suppose that $\mathbf{Lin}(\mathcal{L}) = \mathbf{Lin}(\mathcal{L}')$, where $\mathcal{L}'$ is the literal variety generated by $\mathcal{L}$, or any other literal variety. Then, by Proposition 4.5, $\mathbf{Lin}(\mathcal{L})$ is a literal variety. Thus, by Corollary 4.6, quotients are expressible in $\mathrm{Lin}(\mathcal{L})$. Also, if $\mathbf{Lin}_n(\mathcal{L}) = \mathbf{Lin}_n(\mathcal{L}')$ for all $n \geq 0$, or if $\mathbf{Lin}_1(\mathcal{L}) = \mathbf{Lin}_1(\mathcal{L}')$, for the literal variety $\mathcal{L}'$ generated by $\mathcal{L}$, or for any other literal variety, then quotients are strictly expressible in $\mathrm{Lin}(\mathcal{L})$.

Remark 4.12. We can show that for any class $\mathcal{L}$ of languages, $\mathbf{Lin}(\mathcal{L})$ is closed with respect to arbitrary inverse homomorphisms iff every inverse homomorphic image of a language in $\mathcal{L}$ belongs to $\mathbf{Lin}(\mathcal{L})$.

Remark 4.13. All of the results of this section remain valid if the $<$ predicate is replaced by the successor predicate. In fact, Proposition 4.2 remains valid whenever every atomic formula involving a numerical predicate and a *single* variable is equivalent to true or false, while in the proof of Proposition 4.5 we only used that when φ is atomic, then for any letter a and variable x, $a^{-1}L_\varphi$ and $(a, \{x\})^{-1}L_\varphi$ are definable by some quantifier-free formulas.

5. Monoids with a distinguished set of generators

A *monoid equipped with a set of distinguished generators*, or *mg-pair*, for short, consists of a monoid M and a nonempty set A of generators of M. When M is finite, we call (M, A) a *finite mg-pair*. A *morphism* $(M, A) \to (N, B)$ of mg-pairs is a monoid homomorphism $h : M \to N$ such that $h(A) \subseteq B$. It is clear that mg-pairs and their morphisms form a category with respect to function composition. When $\underline{1} = \{1\}$ denotes a trivial monoid, $(\underline{1}, \{1\})$ is a zero object of this category. We call these mg-pairs *trivial*. We call a morphism $h : (M, A) \to (N, B)$ *surjective* if $h(A) = B$, so that also $h(M) = N$, and *injective* if it is an injective function. Moreover, we call (N, B) a *quotient* of (M, A) if there is surjective morphism $(M, A) \to (N, B)$, and a *sub mg-pair* of (M, A) if $N \subseteq M$ and the inclusion $N \to M$ is a morphism (thus, $B = A \cap N$ and N is the submonoid of M generated by B). Finally, we say that (M, A) *covers* (N, B), or that (N, B) *divides* (M, A), if (N, B) is a quotient of a sub mg-pair of (M, A). We let $<$ denote the divisibility relation. It is clear that when $(N, B) < (M, A)$, the monoid N is a quotient of a submonoid of M, *i.e.*, $N < M$ as defined in [11]. Also, $<$ is reflexive and transitive both on mg-pairs and on monoids.

Remark 5.1. Let (M, A) and (N, B) be mg-pairs. Suppose that there are a *subsemigroup* S of M and a subset C of A that generates S such that there is a semigroup homomorphism $S \to N$ that maps C onto B. Then we have $(N, B) < (M, A)$. (Note that S may not contain the identity element of M.)

Remark 5.2. Suppose that M and N are monoids. We recall from Eilenberg [11] that a *covering* $N \to M$ is a relation $\varphi : N \to M$, viewed as a function $N \to P(M)$, such that

- $\varphi(n) \neq \emptyset$, for all $n \in N$,
- for all $n_1, n_2 \in N$, if $n_1 \neq n_2$ then $\varphi(n_1) \cap \varphi(n_2) = \emptyset$,
- $1 \in \varphi(1)$, and
- $\varphi(n_1)\varphi(n_2) \subseteq \varphi(n_1 n_2)$, for all $n_1, n_2 \in N$.

It is known that $N < M$ iff there is a covering $N \to M$. Suppose now that M and N are equipped with the nonempty sets of distinguished generators A and B, respectively. We define a covering $(N, B) \to (M, A)$ as a covering $\varphi : N \to M$ such that for each $b \in B$ there exists some $a \in A$ with $a \in \varphi(b)$. (The first condition above in the definition of covering then becomes redundant.) We will return to coverings in the Appendix.

Example 5.3.

- For every monoid M, the pair (M, M) is an mg-pair. Moreover, for monoids M, N, we have that $N < M$ iff $(N, N) < (M, M)$.
- When Σ is an alphabet, (Σ^*, Σ) is an mg-pair. Given any mg-pair (M, A) and function $h : \Sigma \to A$, there is a unique morphism $(\Sigma^*, \Sigma) \to (M, A)$ extending h. (We denote this morphism by h as well.) Thus, (Σ^*, Σ) is a *free* mg-pair.

- Each *automaton* $(Q, \Sigma, \cdot)$ with transition function $\cdot : Q \times \Sigma \to Q$ gives rise to an mg-pair $(M_Q, \overline{\Sigma})$. Its monoid component M_Q is the monoid of all state transformations $Q \to Q$ induced by the words in Σ^*, and the set of generators $\overline{\Sigma}$ consists of those transformations induced by the letters in Σ.
- Each mg-pair (M, A) may be regarded as an automaton $(M, A, \cdot)$ whose action is given by right multiplication $(m, a) \mapsto ma$, $m \in M$, $a \in A$. This automaton is freely generated by the identity element of M and is "input reduced": different input letters induce different state transformations. In fact, the category of mg-pairs is equivalent to the category of one-generated input reduced free automata whose morphisms preserve the free generator and the transitions (with a possible change in the alphabet).
- When (M, A) is an mg-pair, B is a nonempty subset of A, and if $Q \subseteq M$ is closed with respect to right multiplication with the elements of B, then Q and B determine an mg-pair $(N, \overline{B})$. Here, N is the quotient of the submonoid M' of M generated by the elements in B with respect to the congruence $\sim_Q$ defined by $x \sim_Q y$ iff $qx = qy$ for all $q \in Q$. Moreover, $\overline{B}$ consists of the congruence classes of the elements of B.

Below we will identify a monoid M with the mg-pair (M, M). Moreover, we call an mg-pair (M, A) an *mg-pair with identity*, or *mgi-pair*, if A contains the identity element of M.

Each mg-pair may be used as a recognizer. Let (M, A) denote a not necessarily finite mg-pair and let $h : (\Sigma^*, \Sigma) \to (M, A)$ be a morphism, so that h is a monoid homomorphism $\Sigma^* \to M$ with $h(\Sigma) \subseteq A$. Given a set $F \subseteq M$, the language *recognized*, or *accepted by* (M, A) *with* h *and* F is the set

$$h^{-1}(F) = \{u \in \Sigma^* : h(u) \in F\}.$$

It is clear that a language is regular iff it can be recognized by a finite mg-pair.

Any language can be recognized by an mg-pair. Given a language $L \subseteq \Sigma^*$, let M_L denote the syntactic monoid of L, and let $\eta_L : \Sigma^* \to M_L$ denote the syntactic homomorphism of L, *cf.* Eilenberg [11], Pin [24]. Then $Synt(L) = (M_L, \eta_L(\Sigma))$ is an mg-pair, called the *syntactic mg-pair* of L. Moreover, η_L is a morphism $(\Sigma^*, \Sigma) \to Synt(L)$, called the *syntactic morphism* of L.

The following fact is an adaptation of well-known results from Eilenberg [11] and Pin [24].

Proposition 5.4.

(1) *The language recognized by* $Synt(L) = (M_L, \eta_L(\Sigma))$ *with the syntactic morphism* η_L *and the set* $\eta_L(L)$ *is* L.

(2) *Suppose that* (M, A) *accepts* L *with* $h : (\Sigma^*, \Sigma) \to (M, A)$ *and* $F \subseteq M$. *Suppose that* h *is surjective. Then there is a (unique) morphism* $h' : (M, A) \to Synt(L)$ *such that*

$$\eta_L = (\Sigma^*, \Sigma) \xrightarrow{h} (M, A) \xrightarrow{h'} Synt(L).$$

(3) *A language* $L \subseteq \Sigma^*$ *can be recognized by an mg-pair* (M, A) *iff we have* $Synt(L) < (M, A)$.

The following fact is well-known.

Lemma 5.5. *Let* $L \subseteq A^*$ *be a regular language. Then every language in* A^* *recognizable by the syntactic morphism* η_L *is a boolean combination of quotients of* L.

Lemma 5.6. *Let* $L \subseteq A^*$ *be a regular language and let* B *denote an alphabet. Then every language in* B^* *recognizable by the syntactic mg-pair of* L *is the inverse image under a literal morphism* $B^* \to A^*$ *of a language in* A^* *which is a boolean combination of quotients of* L.

Proof. Let $h : (B^*, B) \to (M_L, \eta_L(A))$ be a morphism and let $K \subseteq B^*$ with $h^{-1}(h(K)) = K$. Since η_L is surjective and (B^*, B) is free, there exists a literal morphism $\varphi : B^* \to A^*$ such that

$$h = (B^*, B) \xrightarrow{\varphi} (A^*, A) \xrightarrow{\eta_L} (M_L, \eta_L(A)).$$

Thus, $K = \varphi^{-1}(K')$, where $K' = \eta_L^{-1}(h(K))$. By Lemma 5.5, K' is a boolean combination of quotients of L. □

Suppose that $\mathbf{K}$ is a class of mg-pairs. We define $\mathrm{Lin}(\mathbf{K}) = \mathrm{Lin}(\mathcal{L}_{\mathbf{K}})$ and $\mathbf{Lin}(\mathbf{K}) = \mathbf{Lin}(\mathcal{L}_{\mathbf{K}})$, where $\mathcal{L}_{\mathbf{K}}$ is the class of all languages recognizable by the members of $\mathbf{K}$. Moreover, we define $\mathrm{FO}(\mathbf{K}) = \mathrm{FO}(\mathcal{L}_{\mathbf{K}})$, $\mathbf{FO}(\mathbf{K}) = \mathbf{FO}(\mathcal{L}_{\mathbf{K}})$, and $\mathrm{Lin}_n(\mathbf{K}) = \mathrm{Lin}_n(\mathcal{L}_{\mathbf{K}})$, $\mathrm{FO}_n(\mathbf{K}) = \mathrm{FO}_n(\mathcal{L}_{\mathbf{K}})$, $\mathbf{Lin}_n(\mathbf{K}) = \mathbf{Lin}_n(\mathcal{L}_{\mathbf{K}})$, $\mathbf{FO}_n(\mathbf{K}) = \mathbf{FO}_n(\mathcal{L}_{\mathbf{K}})$, for each $n \geq 0$. Note that $\mathcal{L}_{\mathbf{K}}$ is closed with respect to quotients and inverse literal homomorphisms. Thus, by Corollary 4.7, quotients are strictly expressible in both $\mathrm{Lin}(\mathbf{K})$ and $\mathrm{FO}(\mathbf{K})$, for any $\mathbf{K}$.

Remark 5.7. For a class $\mathbf{K}$ of mg-pairs, let $\mathcal{L}'_{\mathbf{K}}$ denote those languages that can be recognized by an mg-pair $(M, A) \in \mathbf{K}$ with the homomorphism $(A^*, A) \to (M, A)$ which is the identity function on A. Then $\mathcal{L}'_{\mathbf{K}} \subseteq \mathcal{L}_{\mathbf{K}}$ and, moreover, every language in $\mathcal{L}_{\mathbf{K}}$ is the inverse image of a language in $\mathcal{L}'_{\mathbf{K}}$ under a literal homomorphism. It follows that $\mathbf{Lin}(\mathbf{K}) = \mathbf{Lin}(\mathcal{L}'_{\mathbf{K}})$ and $\mathbf{FO}(\mathbf{K}) = \mathbf{FO}(\mathcal{L}'_{\mathbf{K}})$. Moreover, for each $n \geq 0$, $\mathbf{Lin}_n(\mathbf{K}) = \mathbf{Lin}_n(\mathcal{L}'_{\mathbf{K}})$ and $\mathbf{FO}_n(\mathbf{K}) = \mathbf{FO}_n(\mathcal{L}'_{\mathbf{K}})$.

Corollary 5.8. *For each class* $\mathbf{K}$ *of mg-pairs and for each* $n \geq 0$, $\mathbf{Lin}_n(\mathbf{K})$ *is a literal variety. Moreover,* $\mathbf{Lin}(\mathbf{K})$ *is a literal variety. If* $\mathbf{K}$ *is a class of finite mg-pairs, then* $\mathbf{Lin}(\mathbf{K})$ *is a literal variety of regular languages.*

Proof. From Corollaries 4.6 and 4.7. When $\mathbf{K}$ consists of finite mg-pairs, then $\mathcal{L}_{\mathbf{K}}$ is a class of regular languages. Thus, by Remark 3.11, every language in $\mathbf{Lin}(\mathcal{L}_{\mathbf{K}})$ is regular. □

For a class $\mathcal{L}$ of languages, let $\mathbf{K}_{\mathcal{L}}$ consist of the mg-pairs $Synt(L)$, where $L \in \mathcal{L}$.

Proposition 5.9.

(1) *For any class* $\mathcal{L}$ *of languages,* $\mathbf{Lin}(\mathcal{L}) \subseteq \mathbf{Lin}(\mathbf{K}_{\mathcal{L}})$.

(2) *Suppose that* $\mathcal{L}$ *is a class of regular languages. Then:*
   - $\mathbf{Lin}(\mathbf{K}_{\mathcal{L}}) = \mathbf{Lin}(\mathcal{L}')$, *where* $\mathcal{L}'$ *is the literal variety generated by* $\mathcal{L}$.
   - $\mathbf{Lin}(\mathcal{L}) = \mathbf{Lin}(\mathbf{K}_{\mathcal{L}})$ *iff quotients are expressible in* $\mathrm{Lin}(\mathcal{L})$.

Proof. Since every language is recognizable by its syntactic mg-pair, the inclusion $\mathbf{Lin}(\mathcal{L}) \subseteq \mathbf{Lin}(\mathbf{K}_{\mathcal{L}})$ is obvious. Assume now that $\mathcal{L}$ consists of regular languages. By Lemma 5.6, every language recognizable by some mg-pair in $\mathbf{K}_{\mathcal{L}}$ is the inverse image with respect to a literal morphism of a boolean combination of quotients of a language in $\mathcal{L}$ and thus belongs to the least literal variety containing $\mathcal{L}$. Thus, $\mathbf{Lin}(\mathbf{K}_{\mathcal{L}}) \subseteq \mathbf{Lin}(\mathcal{L}')$. But since by Corollary 5.8 $\mathbf{Lin}(\mathbf{K}_{\mathcal{L}})$ is a literal variety containing $\mathcal{L}$, also $\mathcal{L}' \subseteq \mathbf{Lin}(\mathbf{K}_{\mathcal{L}})$ and thus $\mathbf{Lin}(\mathcal{L}') \subseteq \mathbf{Lin}(\mathbf{Lin}(\mathbf{K}_{\mathcal{L}})) = \mathbf{Lin}(\mathbf{K}_{\mathcal{L}})$ by Corollary 3.6. This proves $\mathbf{Lin}(\mathbf{K}_{\mathcal{L}}) = \mathbf{Lin}(\mathcal{L}')$. The last claim now follows from Corollary 4.10 and Remark 4.11. □

Remark 5.10. If $\mathbf{Lin}(\mathcal{L}) = \mathbf{Lin}(\mathbf{K})$ for any class $\mathbf{K}$ of mg-pairs, then quotients are expressible in $\mathrm{Lin}(\mathcal{L})$. See Remark 4.11.

Let U_1 denote a two-element semilattice. Note that U_1 is isomorphic to the syntactic monoid of the language $K_\exists$, defined in Example 2.2. Below we will write U_1 also for the mg-pair (U_1, U_1).

Proposition 5.11. *For each class* $\mathbf{K}$ *of mg-pairs,* $\mathbf{FO}(\mathbf{K}) = \mathbf{Lin}(\mathbf{K} \cup \{U_1\})$. *Moreover,* $\mathbf{FO}_n(\mathbf{K}) = \mathbf{Lin}_n(\mathbf{K} \cup \{U_1\})$, *for all* $n \geq 0$.

Proof. We only prove the first claim. By definition, $\mathbf{FO}(\mathbf{K}) = \mathbf{FO}(\mathcal{L}_{\mathbf{K}}) = \mathbf{Lin}(\mathcal{L}_{\mathbf{K}} \cup \{K_\exists\})$ and $\mathbf{Lin}(\mathbf{K} \cup \{U_1\}) = \mathbf{Lin}(\mathcal{L}_{\mathbf{K}} \cup \mathcal{L}_{\{U_1\}})$. Thus, since $K_\exists$ is in $\mathcal{L}_{\{U_1\}}$, we have $\mathbf{FO}(\mathbf{K}) \subseteq \mathbf{Lin}(\mathbf{K} \cup \{U_1\})$. But every language $L \subseteq \Delta^*$ recognizable by U_1 is either the empty language, or the language Δ^*, or the inverse image of $K_\exists$ or $K_\forall$ with respect to a literal morphism $\Delta^* \to \{b_1, b_2\}^*$. Using this fact and Corollary 4.3, we have $\mathbf{Lin}(\mathbf{K} \cup \{U_1\}) \subseteq \mathbf{FO}(\mathbf{K})$, as claimed. □

Below we will use the above fact without mention.

Example 5.12.

- Recall that for any integers $m \geq 1$ and $0 \leq r < m$, C_m^r denotes the language of all words over the two-letter alphabet $\{b_1, b_2\}$ such that the number of occurrences of b_1 is congruent to r modulo m. The syntactic mg-pair of each C_m^r is isomorphic to an mg-pair $(Z_m, \{a, 1\})$, where Z_m is a cyclic group of order m generated by the letter a and where 1 denotes the identity element of Z_m. Now when M is a set of positive integers and $\mathcal{C}_M = \{C_m^r : m \in M,\ 0 \leq r < m\}$, then $\mathcal{C}_M$ is closed with respect to quotients and thus $\mathbf{Lin}(\mathcal{C}_M) = \mathbf{Lin}(\{(Z_m, \{a, 1\}) : m \in M\})$. Moreover, since quotients are expressible in $\mathrm{FO}(\{\mathcal{C}_M\}) = \mathrm{Lin}(\mathcal{C}_M \cup \{K_\exists\})$, $\mathbf{FO}(\{\mathcal{C}_M\}) = \mathbf{FO}(\{(Z_m, \{a, 1\}) : m \in M\})$. In fact, $\mathbf{FO}(\{C_m^0 : m \in M\}) = \mathbf{FO}(\{(Z_m, \{a, 1\}) : m \in M\})$.

– Consider now a language $K_m^r = (b_2^m)^* b_2^{r-1} b_1 (b_1 + b_2)^*$, where $m \geq 1$ and $r \in [m]$. Its syntactic monoid has $2m$ elements and can be defined by the relations $b_2^m = 1$, $b_1 b_1 = b_1 b_2 = b_1$. Let M_m denote this monoid. Then $\mathit{Synt}(K_m^r) = (M_m, \{b_1, b_2\})$. Now for any set M of positive integers, quotients are expressible both in $\mathrm{FO}(\mathcal{K}_M)$ and in $\mathrm{FO}(\{K_m^1 : m \in M\})$. ($\mathcal{K}_M$ was defined in Ex. 2.3.) Thus, $\mathrm{FO}(\mathcal{K}_M) = \mathbf{FO}(\{K_m^1 : m \in M\}) = \mathbf{FO}(\{(M_m, \{b_1, b_2\}) : m \in M\})$. Moreover, when M is not empty, then $\mathbf{FO}(\{(M_m, \{b_1, b_2\}) : m \in M\}) = \mathbf{Lin}(\{(M_m, \{b_1, b_2\}) : m \in M\})$, since U_1 is a quotient of any $(M_m, \{b_1, b_2\})$.

We may refine the above results by taking into account the quantification depth of the formulas. Using Proposition 4.5 and Corollary 4.7, we have:

Proposition 5.13.

(1) *For any class $\mathcal{L}$ of languages and integer $n \geq 0$, $\mathbf{Lin}_n(\mathcal{L}) \subseteq \mathbf{Lin}_n(\mathbf{K}_{\mathcal{L}})$.*
(2) *Suppose that $\mathcal{L}$ is a class of regular languages. Then the following conditions are equivalent.*
  (a) *The quotient of each language in $\mathcal{L}$ belongs to the literal pre-variety generated by $\mathcal{L}$.*
  (b) *The literal pre-variety generated by $\mathcal{L}$ is a literal variety.*
  (c) *Quotients are strictly expressible in $\mathrm{Lin}(\mathcal{L})$.*
  (d) *$\mathbf{Lin}_1(\mathcal{L})$ is a literal variety.*
  (e) *For all $n \geq 0$, $\mathbf{Lin}_n(\mathcal{L})$ is a literal variety.*
  (f) *$\mathbf{Lin}_1(\mathcal{L}) = \mathbf{Lin}_1(\mathbf{K}_{\mathcal{L}})$.*
  (g) *For all $n \geq 0$, $\mathbf{Lin}_n(\mathcal{L}) = \mathbf{Lin}_n(\mathbf{K}_{\mathcal{L}})$.*

We also have:

Proposition 5.14. *For each class $\mathbf{K}$ of mg-pairs and $n \geq 0$, $\mathbf{FO}_n(\mathbf{K}) = \mathbf{Lin}_n(\mathbf{K} \cup \{U_1\})$.*

6. Relativization

We say that $\mathrm{Lin}(\mathcal{L})$ *admits relativization* if for each sentence φ in $\mathrm{Lin}(\mathcal{L})$, over any alphabet Σ, and for any variable x there exist $\mathrm{Lin}(\mathcal{L})$ formulas $\varphi[> x]$ and $\varphi[< x]$ over Σ in the free variable x such that for all $\{x\}$-structures $u = u_1 \dots (u_i, \{x\}) \dots u_n$ over Σ,

$$u \models \varphi[> x] \Leftrightarrow u_{i+1} \dots u_n \models \varphi$$

and

$$u \models \varphi[< x] \Leftrightarrow u_1 \dots u_{i-1} \models \varphi.$$

We say that $\mathrm{Lin}(\mathcal{L})$ *strictly admits relativization* if for each φ, the formulas $\varphi[> x]$ and $\varphi[< x]$ can be chosen so that their quantification depth does not exceed the quantification depth of φ. We say that FO($\mathcal{L}$) (strictly) admits relativization if Lin($\mathcal{L} \cup \{K_\exists\}$) does. Finally, when $\mathbf{K}$ is a class of mg-pairs, we say that Lin($\mathbf{K}$) (FO($\mathbf{K}$), respectively) admits, or strictly admits relativization, if Lin($\mathcal{L}_\mathbf{K}$) (FO($\mathcal{L}_\mathbf{K}$), respectively) does.

Example 6.1.

- It is clear that Lin($\emptyset$) and FO strictly admit relativization.
- Let $\mathcal{L}$ consist of the finite and co-finite languages. We show that Lin($\mathcal{L}$) does not admit relativization.

  Suppose it does. Then let $\mathsf{first}(x)$ denote the formula $(Q_{\{\epsilon\}} y.\langle\rangle)[< x]$ over the one-letter alphabet $\{a\}$, where $\{\epsilon\}$ is viewed as a one-letter language. Then an $\{x\}$-structure u over $\{a\}$ satisfies $\mathsf{first}(x)$ iff $u \neq \epsilon$ and its first letter is $(a, \{x\})$. But this leads to a contradiction, since for any formula φ of Lin($\mathcal{L}$) over $\{a\}$ in the free variable x, the set L_φ of $\{x\}$-structures over $\{a\}$ satisfying φ is either finite or co-finite in the sense that its complement with respect to the set of all $\{x\}$-structures is finite. Indeed, any such formula φ is a boolean combination of atomic formulas over $\{a\}$ in the variable x and of formulas of the sort $Q_K y.\langle\varphi_b\rangle_{b\in\Delta}$, where K is finite or co-finite and each φ_b is a formula over $\{a\}$ in Lin($\mathcal{L}$). Since the alphabet has only one letter, the set of $\{x\}$-structures satisfying an atomic formula in the variable x is either finite or co-finite. Moreover, the set of $\{x\}$-structures satisfying a formula of the sort $Q_K y.\langle\varphi_b\rangle_{b\in\Delta}$ is finite or co-finite depending on whether K is finite or co-finite. Since any boolean combination of finite and co-finite sets is finite or co-finite, it follows that the set of $\{x\}$-structures satisfying φ is also finite or co-finite.
- Let $\mathcal{L}$ consist of the one-letter language $(b_1^2)^*$. Then Lin($\mathcal{L}$) does not admit relativization. Indeed, otherwise there was a formula τ over the one-letter alphabet $\Sigma = \{a\}$ in the free variable x satisfied by those $\{x\}$-structures over Σ in which x appears at an odd position. But this results a contradiction, since it is easy to show that for any formula φ of Lin($\mathcal{L}$) over $\{a\}$ in the free variable x, an $\{x\}$-structure over $\{a\}$ satisfies φ iff all $\{x\}$-structures over $\{a\}$ of the same length satisfy φ. In a similar way, if $\mathcal{L}$ is any nonempty collection of languages $L_m^r = (b_1^m)^* b_1^r$, where $m \geq 2$ and $0 \leq r < m$, then Lin($\mathcal{L}$) does not admit relativization. In fact, if $\mathcal{L}$ is any set of languages over the one-letter alphabet $\{b_1\}$ containing at least one language different from $\emptyset$ and b_1^*, then Lin($\mathcal{L}$) does not admit relativization.

Proposition 6.2.

(1) *Suppose that* $\mathbf{Lin}(\mathcal{L}) = \mathbf{Lin}(\mathcal{L}')$. *Then* Lin($\mathcal{L}$) *admits relativization iff* Lin($\mathcal{L}'$) *does.*
(2) *Suppose that* $\mathbf{Lin}_1(\mathcal{L}) = \mathbf{Lin}_1(\mathcal{L}')$. *Then* Lin($\mathcal{L}$) *strictly admits relativization iff* Lin($\mathcal{L}'$) *does.*

Proof. The first claim is clear from the fact, shown in Corollary 3.8, that $\mathbf{Lin}(\mathcal{L}) = \mathbf{Lin}(\mathcal{L}')$ iff for each $\mathrm{Lin}(\mathcal{L})$ formula there exists an equivalent $\mathrm{Lin}(\mathcal{L}')$ formula and vice versa. The second claim follows from Corollary 3.9. □

Corollary 6.3. *Suppose that* $K_\exists \in \mathbf{Lin}(\mathcal{L})$. *Then* $\mathrm{Lin}(\mathcal{L})$ *admits relativization iff* $\mathrm{FO}(\mathcal{L})$ *does.*

Proof. Since $K_\exists \in \mathbf{Lin}(\mathcal{L})$ and thus $\mathbf{Lin}(\mathcal{L}) = \mathbf{Lin}(\mathcal{L} \cup \{K_\exists\})$, by Proposition 6.2 it holds that $\mathrm{Lin}(\mathcal{L})$ admits relativization iff $\mathrm{Lin}(\mathcal{L} \cup \{K_\exists\})$ does. But by definition, $\mathrm{Lin}(\mathcal{L} \cup \{K_\exists\})$ admits relativization iff $\mathrm{FO}(\mathcal{L})$ does. □

Below, for an alphabet Δ, we let Δ_0 denote the enlarged alphabet $\Delta \cup \{b_0\}$, where b_0 is a new symbol.

Lemma 6.4. *Suppose that for each* Δ *and* $K \subseteq \Delta^*$ *in* $\mathcal{L}$ *there exist languages* $K_<, K_> \subseteq \Delta_0^*$ *with* $b_0^*\Delta^* \cap K_> = b_0^* K$ *or* $b_0^*\Delta^* \cap K_> = b_0^+ K$, *and* $\Delta^* b_0^* \cap K_< = K b_0^*$ *or* $\Delta^* b_0^* \cap K_< = K b_0^+$, *and such that the Lindström quantifiers with respect to* $K_>$ *and* $K_<$ *are expressible in* $\mathrm{Lin}(\mathcal{L})$ *(or equivalently,* $K_>, K_<$ *are in* $\mathbf{Lin}(\mathcal{L})$*). Then* $\mathrm{Lin}(\mathcal{L})$ *admits relativization. And if* $K_>$ *and* $K_<$ *are in the literal pre-variety generated by* $\mathcal{L}$*, for each* K *in* $\mathcal{L}$*, then* $\mathrm{Lin}(\mathcal{L})$ *strictly admits relativization.*

Proof. Given a formula φ in $\mathrm{Lin}(\mathcal{L})$ over Σ with free variables in Y, where $x \notin Y$, we construct $\mathrm{Lin}(\mathcal{L})$ formulas $\varphi[< x]$ and $\varphi[> x]$ over Σ with free variables in $\{x\} \cup Y$ such that for all $(\{x\} \cup Y)$-structures over Σ of the form $v(a_0, \{x\})w$ and $w(a_0, \{x\})v$, where v is an Y-structure over Σ, $w \in \Sigma^*$ and $a_0 \in \Sigma$, we have

$$v(a_0, \{x\})w \models \varphi[< x] \Leftrightarrow v \models \varphi$$
$$w(a_0, \{x\})v \models \varphi[> x] \Leftrightarrow v \models \varphi.$$

We argue by induction on the structure of φ. When φ is an atomic formula or the formula false, then let $\varphi[< x] = \varphi[> x] = \varphi$. When $\varphi = \varphi_1 \vee \varphi_2$ or $\varphi = \neg\varphi_1$, for some φ_1, φ_2, define $\varphi[< x] = \varphi_1[< x] \vee \varphi_2[< x]$ or $\varphi[< x] = \neg(\varphi_1[< x])$, respectively, and define $\varphi[> x]$ symmetrically. Lastly, suppose that $\varphi = Q_K y.\langle\varphi_b\rangle_{b\in\Delta}$, for some (deterministic) family of formulas φ_b, $b \in \Delta$ in $\mathrm{Lin}(\mathcal{L})$. We only show how to define $\varphi[> x]$. For each $b \in \Delta$, let ψ'_b be the formula $x < y \wedge \varphi_b[> x]$, and let $\psi'_{b_0} = x \geq y$. Moreover, let ψ_b, $b \in \Delta_0$ be the determinization of the family ψ'_b, $b \in \Delta_0$ with respect to any linear order on Δ_0. Then, if $K_> \in \mathcal{L}$, we define $\varphi[> x] = Q_{K_>} y.\langle\psi_b\rangle_{b\in\Delta_0}$. Given $w(a_0, \{x\})v$ as above, the characteristic word determined by $w(a_0, \{x\})v$ and $\varphi[> x]$ can be written as $b_0^{|w|+1} p$, where p denotes the characteristic word determined by v and φ. Since $b_0^{|w|+1} p \in K_>$ iff $p \in K$, it follows that $w(a_0, \{x\})v \models \varphi[> x]$ iff $v \models \varphi$. When $K_>$ is not in $\mathcal{L}$, let $\varphi[> x]$ be a $\mathrm{Lin}(\mathcal{L})$ formula equivalent to $Q_{K_>} y.\langle\psi_b\rangle_{b\in\Delta_0}$. If $K_>$ belongs to the literal pre-variety generated by $\mathcal{L}$, *i.e.*, to the language class $\mathbf{Lin}_1(\mathcal{L})$, then by Corollary 3.9, there is such a formula $\varphi[> x]$ whose quantification depth is at most that of φ. □

Two languages $K_>$ satisfying the assumption $b_0^*\Delta^* \cap K_> = b_0^*K$ are b_0^*K and the *padding* of K, *i.e.*, the language $h^{-1}(K)$, where h is the homomorphism $\Delta_0^* \to \Delta$ which is the identity map on Δ and maps b_0 to ϵ.

Example 6.5. Let $\mathbf{K}$ be a class of mgi-pairs. Then for any $K \subseteq \Delta^*$ in $\mathcal{L}_{\mathbf{K}}$, the padding of K belongs to $\mathcal{L}_{\mathbf{K}}$.

Corollary 6.6. *Suppose that for each Δ and $K \subseteq \Delta^*$ in $\mathcal{L}$, Kb_0^* and b_0^*K, or Kb_0^+ and b_0^+K belong to* $\mathbf{Lin}(\mathcal{L})$. *Then* $\mathrm{Lin}(\mathcal{L})$ *admits relativization. Moreover, if for any language K in $\mathcal{L}$, Kb_0^* and b_0^*K, or Kb_0^+ and b_0^+K belong to the literal pre-variety generated by $\mathcal{L}$, then* $\mathrm{Lin}(\mathcal{L})$ *strictly admits relativization.*

Corollary 6.7. *Suppose that for each Δ and $K \subseteq \Delta^*$ in $\mathcal{L}$, the padding of K belongs to* $\mathbf{Lin}(\mathcal{L})$. *Then* $\mathrm{Lin}(\mathcal{L})$ *admits relativization. Moreover, if the padding of any language in $\mathcal{L}$ belongs to the literal pre-variety generated by $\mathcal{L}$, then* $\mathrm{Lin}(\mathcal{L})$ *strictly admits relativization.*

Corollary 6.8. *For any class* $\mathbf{K}$ *of mgi-pairs,* $\mathrm{Lin}(\mathbf{K})$ *strictly admits relativization.*

Lemma 6.9. *Suppose that for each Δ and $K \subseteq \Delta^*$ in $\mathcal{L}$ there exist languages $K_<, K_>$ with $\Delta^*b_0\Delta^* \cap K_> = \Delta^*b_0K$ and $\Delta^*b_0\Delta^* \cap K_< = Kb_0\Delta^*$ and such that the Lindström quantifiers with respect to $K_>$ and $K_<$ are expressible in* $\mathrm{Lin}(\mathcal{L})$ *(or equivalently, $K_>, K_<$ are in* $\mathbf{Lin}(\mathcal{L})$*). Then* $\mathrm{Lin}(\mathcal{L})$ *admits relativization. And if $K_>$ and $K_<$ are in the literal pre-variety generated by $\mathcal{L}$, for each K in $\mathcal{L}$, then* $\mathrm{Lin}(\mathcal{L})$ *strictly admits relativization.*

Proof. We argue as in the proof of Lemma 6.4 by constructing appropriate formulas $\varphi[< x], \varphi[> x]$ for each $\varphi \in \mathrm{Lin}(\mathcal{L})$. In the induction step, when $\varphi = Q_K y.\langle\varphi_b\rangle_{b\in\Delta}$, we define $\varphi[> x]$ as the formula $Q_{K_>} y.\langle\psi_b\rangle_{b\in\Delta_0}$, or as a formula equivalent to this formula, where ψ_b, $b \in \Delta_0$ is any determinization of the family ψ'_b, $b \in \Delta_0$ defined as follows. First, ψ'_{b_0} is the formula $x = y$, moreover, there is some $b \in \Delta$ with

$$\psi'_b = y < x \vee (y > x \wedge \varphi_b[> x]),$$

and for all $b' \in \Delta$ with $b' \neq b$,

$$\psi'_{b'} = y > x \wedge \varphi_b[> x].$$

We omit the details. □

Proposition 6.10. *The following conditions are equivalent for a class $\mathcal{L}$ of languages:*

1. (1) $\mathrm{FO}(\mathcal{L})$ *admits relativization.*
2. (2) *For each Δ and $K \in \mathcal{L}(\Delta^*)$, the languages b_0^*K and Kb_0^* (or b_0^+K and Kb_0^+) belong to* $\mathbf{FO}(\mathcal{L})$.
3. (3) *For each Δ and $K \in \mathcal{L}(\Delta^*)$ there exist languages $K_>$ and $K_<$ in* $\mathbf{FO}(\mathcal{L})$ *with $b_0^*\Delta^* \cap K_> = b_0^*K$ and $\Delta^*b_0^* \cap K_< = Kb_0^*$.*

Proof. We have already shown that the third condition implies the first. That the second implies the third is obvious. Thus, to complete the proof, we show that the first condition implies the second. So suppose that $\mathrm{FO}(\mathcal{L})$ admits relativization and $K \in \mathcal{L}(\Delta^*)$. Let K_1 denote the language $h_1^{-1}(K)$, where h_1 is a literal morphism $\Delta_0^* \to \Delta^*$ which is the identity function on Δ. We know that Lindström quantification with respect to K_1 is expressible in $\mathrm{FO}(\mathcal{L})$. Then b_0^+K is definable by

$$\exists y \left[\forall z (z \leq y \to P_{b_0}(z) \wedge z > y \to \bigvee_{b \in \Delta} P_b(z)) \wedge (Q_{K_1} z.\langle P_b(z) \rangle_{b \in \Delta_0})[> y] \right].$$

Moreover, b_0^*K is definable by the disjunction of the above formula with

$$\forall x (\bigvee_{b \in \Delta} P_b(x)) \wedge Q_{K_1} z.\langle P_b(z) \rangle_{b \in \Delta_0}.$$

The languages Kb_0^* and Kb_0^+ are definable in the same way. □

Proposition 6.11. *The following conditions are equivalent for a class $\mathcal{L}$ of languages:*

(1) $\mathrm{FO}(\mathcal{L})$ *admits relativization.*
(2) *For each Δ and $K \in \mathcal{L}(\Delta^*)$, the languages $\Delta^* b_0 K$ and $K b_0 \Delta^*$ belong to* $\mathbf{FO}(\mathcal{L})$.
(3) *For each Δ and $K \in \mathcal{L}(\Delta^*)$ there exist languages $K_>$ and $K_<$ in* $\mathbf{FO}(\mathcal{L})$ *with $\Delta^* b_0 \Delta^* \cap K_> = \Delta^* b_0 K$ and $\Delta^* b_0 \Delta^* \cap K_< = K b_0 \Delta^*$.*

Proof. We only need to show that the first condition implies the second. But given K, $\Delta^* b_0 K$ is defined by

$$\exists y \left[\forall z ((z \neq y \to \bigvee_{b \in \Delta} P_b(z)) \wedge (z = y \to P_{b_0}(z)) \wedge (Q_{K_1} z.\langle P_b(z) \rangle_{b \in \Delta_0})[> y] \right].$$

$K b_0 \Delta^*$ is definable in the same way. □

Problem 6.12. Characterize those classes of (regular) languages $\mathcal{L}$ such that $\mathrm{Lin}(\mathcal{L})$ admits relativization.

The same question arises when the language contains the successor predicate instead of the $<$ predicate.

7. Double semidirect product and block product

The double semidirect product and the block product of monoids were introduced in [28]. In this section, we extend these notions to mg-pairs.

Suppose that (S, A) and (T, B) are mg-pairs. We write the monoid operation of S additively without assuming that the operation is commutative. We denote by 0 the identity element of S and by 1 the identity element of T. A (monoidal) *left action* of T (or of (T, B)) on (S, A) is a function

$$\begin{aligned} T \times S &\to S \\ (t, s) &\mapsto ts \end{aligned}$$

subject to the following conditions for all $s, s' \in S$ and $t, t' \in T$:

$$\begin{aligned} (tt')s &= t(t's) \\ t(s + s') &= ts + ts' \\ 1s &= s \\ t0 &= 0. \end{aligned}$$

Moreover, it is required that

$$ta \in A, \quad \text{for all } t \in T, a \in A.$$

A *right action* $S \times T \to S$, $(s, t) \mapsto st$ is defined symmetrically. Actions $T \times S \to S$ and $S \times T \to S$ are *compatible* if

$$(ts)t' = t(st'),$$

for all $t, t' \in T$ and $s \in S$. Due to the above laws, we will write just tst' for $(ts)t' = t(st')$, $tt's$ for $t(t's) = (tt')s$, *etc.*

Remark 7.1. Any left action of (T, B) on (S, A) is uniquely determined by a function $B \times A \to A$ subject to certain conditions.

Given a compatible pair of left and right actions of T on (S, A), we define the *double semidirect product* $(S, A){\star\star}(T, B)$ as follows. First, we define

$$(s, t)(s', t') = (st' + ts', tt'),$$

for all $s, s' \in S$ and $t, t' \in T$. It is known, *cf.* [28, 31] that $S \times T$, equipped with this operation, is a monoid with identity element $(0, 1)$. This monoid $S{\star\star}T$ is called the double semidirect product of S and T determined by the actions. Let R denote the submonoid of $S{\star\star}T$ generated by the set $A \times B$. We define the double semidirect product $(S, A){\star\star}(T, B)$ to be the mg-pair $(R, A \times B)$. The monoid component R of $(S, A){\star\star}(T, B)$ is usually much smaller than the monoid $S{\star\star}T$. Given $(s, t) \in S \times T$, we have $(s, t) \in R$ iff there is some $n \geq 0$ such that (s, t) is an n-fold product over $A \times B$ in the monoid $S{\star\star}T$.

Two special cases are of particular interest. The notion of *semidirect product* $(S, A) \star (T, B)$ involves only a left action of T on (S, A) and corresponds to the double semidirect product $(S, A){\star\star}(T, B)$ determined by the same left action and the trivial right action: $st = s$, for all $s \in S$ and $t \in T$. When both actions are trivial, we obtain the *direct product* $(S, A) \times (T, B)$. This is the mg-pair $(R, A \times B)$, where R is the submonoid of the usual direct product $S \times T$ generated by $A \times B$. The direct product is the categorical product in the category of mg-pairs.

For later use we note:

Proposition 7.2. *For any finite mg-pairs* (S, A) *and* (T, B)*, any language recognizable by* $(S, A) \times (T, B)$ *is a boolean combination of languages recognizable by* (S, A) *and* (T, B).

Remark 7.3. The double semidirect product of monoids is closely related to the *triple product* of Eilenberg [11], vol. B. Any double semidirect product $S{\star\star}T$ of monoids S and T embeds in a triple product $[T, S, T]$ determined by the same actions. Moreover, as shown in Rhodes, Tilson [28], any triple product $[T_1, S, T_2]$ of monoids S, T_1, T_2 equipped with a monoidal right action $S \times T_1 \to S$, $(s, t_1) \mapsto st_1$, and a monoidal left action $T_2 \times S \to S$, $(t_2, s) \mapsto t_2 s$, is isomorphic to the double semidirect product $S{\star\star}(T_1 \times T_2)$ with actions $(t_1, t_2)s = t_2 s$ and $s(t_1, t_2) = st_1$, for all $s \in S$ and $t_i \in T_i$, $i = 1, 2$.

Suppose that (S, A) and (T, B) are mg-pairs. Then $(S, A)^{T\times T}$, the $(T \times T)$-fold direct product of (S, A) with itself is an mg-pair $(R, A^{T\times T})$, where R is the submonoid of $S^{T\times T}$ generated by the set $A^{T\times T}$. Thus, a function $f : T \times T \to S$ belongs to R iff there is some $n \geq 0$ such that for all t_1, t_2 in T, $f(t_1, t_2)$ is an n-fold product over A in the monoid S. The *block product* $(S, A)\Box(T, B)$ is the double semidirect product

$$(R, A^{T\times T}){\star\star}(T, B)$$

determined by the following compatible left and right actions of T on $(S, A)^{T\times T}$:

$$\begin{aligned}(tf)(t_1, t_2) &= f(t_1 t, t_2)\\ (ft)(t_1, t_2) &= f(t_1, tt_2),\end{aligned}$$

for all $f \in R$ and $t, t_1, t_2 \in T$. The reader should have no difficulty in verifying that $tf, ft \in R$ for all $f \in R$ and $t \in T$. Moreover, if $f \in A^{T\times T}$, then $tf, ft \in A^{T\times T}$, for all $t \in T$. The *wreath product* $(S, A) \circ (T, B)$ is defined in a similar way. It is the semidirect product $(R, A^T \times B) = (S, A)^T \star (T, B)$ determined by the left action

$$(tf)(t_1) = f(t_1 t), \quad t_1 \in T,$$

for all $f \in R$ and $t \in T$.

Proposition 7.4. *For any mg-pairs* (S, A) *and* (T, B)*, every double semidirect product* $(M, A \times B) = (S, A){\star\star}(T, B)$ *is isomorphic to a sub mg-pair of the block product* $(N, A^{T\times T} \times B) = (S, A)\Box(T, B)$.

Proof. We follow the proof of Proposition 7.1 in Rhodes and Tilson [28]. For each $s \in S$ let $f_s : T \times T \to S$ denote the function $(t_1, t_2) \mapsto t_1 s t_2$, $t_1 \in T_1, t_2 \in T_2$.

Note that when $s \in A$, then f_s maps $T \times T$ into A. It is shown in Rhodes [27] that the assignment

$$(s,t) \mapsto (f_s, t), \quad (s,t) \in S \times T$$

defines an injective morphism $S{\star\star}T \to S\Box T$. Moreover, if $(s,t) \in A \times B$, then $(f_s, t) \in A^{T\times T} \times B$. To complete the proof we still need to show that if $(s,t) \in M$, then $(f_s, t) \in N$. However, if (s,t) is an n-fold product over $A \times B$, for some $n \geq 0$, then (f_s, t) is an n-fold product over $A^{T\times T} \times B$. □

For later use we note:

Lemma 7.5. *Suppose that (M,A), (N,B) and (N',B') are mg-pairs such that (N',B') is a sub mg-pair of (N,B). If (S,C) is a sub mg-pair of $(M,A)\Box(N,B)$ such that $C \subseteq A^{N\times N} \times B'$, then (S,C) is isomorphic to a sub mg-pair of a double semidirect product $(M,A)^{N\times N}{\star\star}(N',B')$.*

Proof. Let T denote the monoid component of the direct power $(M,A)^{N\times N}$, so that $(M,A)^{N\times N} = (T, A^{N\times N})$. When $f \in T$ and $n \in N'$, define nf and fn in T by $nf(n_1,n_2) = f(n_1 n, n_2)$ and $fn(n_1,n_2) = f(n_1, nn_2)$, for all $n_1, n_2 \in N$. Since (N',B') is a sub mg-pair of (N,B), it follows that the double semidirect product $(M,A)^{N\times N}{\star\star}(N',B')$ determined by this compatible pair of actions is isomorphic to a sub mg-pair of $(M,A)\Box(N,B)$. Since this sub mg-pair contains (S,C), it follows that (S,C) is isomorphic to a sub mg-pair of a double semidirect product of $(M,A)^{N\times N}$ and (N',B'). □

8. Varieties of finite mg-pairs

In this section, other than free mg-pairs (Σ^*, Σ), we will only consider finite mg-pairs.

A *variety*, or *pseudo-variety of finite mg-pairs* is a class $\mathbf{V}$ of finite mg-pairs containing the trivial mg-pairs closed with respect to the direct product and division, *i.e.*, such that

- $(S,A), (T,B) \in \mathbf{V} \quad \Rightarrow \quad (S,A) \times (T,B) \in \mathbf{V}$, and
- $(S,A) < (T,B),\ (T,B) \in \mathbf{V} \quad \Rightarrow \quad (S,A) \in \mathbf{V}$.

A *closed class of finite mg-pairs* is a class of finite mg-pairs containing the trivial mg-pairs that is closed with respect to the double semidirect product and division. Since the direct product is a special case of the double semidirect product, any closed class of finite mg-pairs is a variety. Therefore we also call closed classes of finite mg-pairs as *closed varieties*. It is clear that each class $\mathbf{K}$ of finite mg-pairs is contained in a least variety $\mathbf{V}$ and in a least closed variety $\widehat{\mathbf{V}}$ (of finite mg-pairs).

Given varieties $\mathbf{V}$ and $\mathbf{W}$ of finite mg-pairs, we define

- $\mathbf{V}{\star\star}\mathbf{W}$ to be the variety generated by all double semidirect products $(M,A){\star\star}(T,B)$, where $(M,A) \in \mathbf{V}$ and $(T,B) \in \mathbf{W}$,
- $\mathbf{V}\Box\mathbf{W}$ to be the variety generated by all block products $(M,A)\Box(T,B)$, where $(M,A) \in \mathbf{V}$ and $(T,B) \in \mathbf{W}$.

Proposition 8.1. *For all varieties* $\mathbf{V}$ *and* $\mathbf{W}$ *of finite mg-pairs, it holds that* $\mathbf{V}{\star\star}\mathbf{W} = \mathbf{V}\Box\mathbf{W}$. *Moreover, an mg-pair is in* $\mathbf{V}{\star\star}\mathbf{W}$ *iff it is covered by a double semidirect product* $(S,A){\star\star}(T,B)$, *or, equivalently, by a block product* $(S,A)\Box(T,B)$, *where* $(S,A) \in \mathbf{V}$ *and* $(T,B) \in \mathbf{W}$.

Proof. Since a block product $(S,A)\Box(T,B)$ is a double semidirect product

$$(S,A)^{B\times B}{\star\star}(T,B),$$

and since varieties are closed with respect to the direct product, it follows that $\mathbf{V}\Box\mathbf{W} \subseteq \mathbf{V}{\star\star}\mathbf{W}$. The reverse inclusion follows from Proposition 7.4.

The proof of the second claim uses the fact that any direct product of double semidirect products is isomorphic to a double semidirect product of direct products. Moreover, if $(S_i,A_i) < (T_i,B_i)$, $i = 1,2$, then $(S_1,A_1)\Box(S_2,A_2) < (T_1,B_1)\Box(T_2,B_2)$. The argument is quite standard. All facts formulated in Proposition 8.1 are well-known for varieties of finite monoids. See, *e.g.*, Rhodes [27]. □

Corollary 8.2. *A class of finite mg-pairs is a closed variety iff it is not empty and is closed with respect to division and the block product.*

Proposition 8.3. *For all varieties* $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$ *of finite mg-pairs, it holds that*

$$(\mathbf{V}_1{\star\star}\mathbf{V}_2){\star\star}\mathbf{V}_3 \subseteq \mathbf{V}_1{\star\star}(\mathbf{V}_2{\star\star}\mathbf{V}_3).$$

This fact is known to hold for varieties of finite monoids, *cf.* Rhodes [27], p. 460. As communicated to the authors by John Rhodes, the proof uses the Kernel Theorem (Th. 7.4) of Rhodes and Tilson [28]. In the Appendix, we extend the kernel construction to mg-pairs.

Corollary 8.4. *For every variety* $\mathbf{V}$ *of finite mg-pairs, the least closed variety* $\widehat{\mathbf{V}}$ *containing* $\mathbf{V}$ *can be constructed as the class* $\bigcup_{n\geq 0} \mathbf{V}^{(n)}$, *where* $\mathbf{V}^{(0)}$ *is the class of all trivial mg-pairs and* $\mathbf{V}^{(n+1)} = \mathbf{V}{\star\star}\mathbf{V}^{(n)} = \mathbf{V}\Box\mathbf{V}^{(n)}$, *for all* $n \geq 0$.

A version of Eilenberg's Variety Theorem [11, 24] holds. The proof is standard.

Theorem 8.5. *The function that maps a variety* $\mathbf{V}$ *of finite mg-pairs to the class* $\mathcal{L}_{\mathbf{V}}$ *of (regular) languages recognizable by the members of* $\mathbf{V}$ *is an order isomorphism from the lattice of varieties of finite mg-pairs onto the lattice of literal varieties of regular languages. The inverse of this function takes a literal variety of regular languages* $\mathcal{V}$ *to the variety of finite mg-pairs that only accept languages in* $\mathcal{V}$.

It follows by Proposition 5.4 that the function $\mathbf{V} \mapsto \mathcal{L}_{\mathbf{V}}$ maps a variety of finite mg-pairs $\mathbf{V}$ to the class of all (regular) languages whose syntactic mg-pair is in $\mathbf{V}$. Moreover, the inverse assignment takes a literal variety $\mathcal{V}$ of regular languages to the variety generated by $\mathbf{K}_{\mathcal{V}}$.

Proposition 8.6.

(1) *For a class* $\mathbf{K}$ *of finite mg-pairs, let* $\mathbf{V}$ *denote the variety of finite mg-pairs generated by* $\mathbf{K}$. *Then* $\mathbf{Lin}(\mathbf{K}) = \mathbf{Lin}(\mathbf{V})$.

(2) *Suppose that* $\mathcal{L}$ *is a class of regular languages and* $\mathbf{V}$ *is the variety generated by* $\mathbf{K}_{\mathcal{L}}$. *Then* $\mathbf{Lin}(\mathcal{L}) \subseteq \mathbf{Lin}(\mathbf{V})$. *Moreover,* $\mathbf{Lin}(\mathcal{L}) = \mathbf{Lin}(\mathbf{V})$ *iff quotients are expressible in* $\mathrm{Lin}(\mathcal{L})$.

Proof. As for the first claim, since $\mathbf{K} \subseteq \mathbf{V}$, the inclusion $\mathbf{Lin}(\mathbf{K}) \subseteq \mathbf{Lin}(\mathbf{V})$ is obvious. For the reverse inclusion, note that any language that can be recognized by an mg-pair in $\mathbf{V}$ is a boolean combination of languages recognizable by the mg-pairs in $\mathbf{K}$ (use Proposition 7.2), and then apply Corollary 4.3. The second claim is immediate from the first and Proposition 5.9. □

Remark 8.7. If $\mathbf{Lin}(\mathcal{L}) = \mathbf{Lin}(\mathbf{V})$, where $\mathcal{L}$ is a class of regular languages and $\mathbf{V}$ is any variety, then quotients are expressible in $\mathrm{Lin}(\mathcal{L})$.

By our previous results we also have:

Proposition 8.8.

(1) *For any class* $\mathbf{K}$ *of finite mg-pairs and integer* $n \geq 0$, $\mathbf{Lin}_n(\mathbf{K}) = \mathbf{Lin}_n(\mathbf{V})$, *where* $\mathbf{V}$ *denotes the variety generated by* $\mathbf{K}$.

(2) *Suppose that* $\mathcal{L}$ *is a class of regular languages and* $\mathbf{V}$ *is the variety generated by* $\mathbf{K}_{\mathcal{L}}$. *Then* $\mathbf{Lin}_n(\mathcal{L}) \subseteq \mathbf{Lin}_n(\mathbf{V})$, *for each* $n \geq 0$. *Moreover, the following conditions are equivalent:*
   - (a) *The quotient of each language in* $\mathcal{L}$ *belongs to the literal pre-variety generated by* $\mathcal{L}$.
   - (b) *The literal pre-variety generated by* $\mathcal{L}$ *is a literal variety.*
   - (c) *Quotients are strictly expressible in* $\mathrm{Lin}(\mathcal{L})$.
   - (d) $\mathbf{Lin}_1(\mathcal{L})$ *is a literal variety.*
   - (e) *For all* $n \geq 0$, $\mathbf{Lin}_n(\mathcal{L})$ *is a literal variety.*
   - (f) $\mathbf{Lin}_1(\mathcal{L}) = \mathbf{Lin}_1(\mathbf{V})$.
   - (g) *For all* $n \geq 0$, $\mathbf{Lin}_n(\mathcal{L}) = \mathbf{Lin}_n(\mathbf{V})$.

Corollary 8.9. *Suppose that* $\mathbf{K}$ *is a class of finite mg-pairs and* $\mathbf{V}$ *denotes the variety generated by* $\mathbf{K}$. *Then* $\mathrm{Lin}(\mathbf{K})$ *(strictly) admits relativization iff* $\mathrm{Lin}(\mathbf{V})$ *does.*

Proof. This follows from Proposition 6.2, Proposition 8.6 and Proposition 8.8. □

9. Lindström quantifiers and the block product

In this section, we assume that $\mathbf{K}$ is a class of finite mg-pairs and let $\mathbf{V}$ denote the variety of finite mg-pairs generated by $\mathbf{K}$. Moreover, we denote by either $\widehat{\mathbf{K}}$ or $\widehat{\mathbf{V}}$ the least closed variety containing $\mathbf{K}$ (or $\mathbf{V}$).

Proposition 9.1. *Suppose that* $\mathrm{Lin}(\mathbf{K})$ *admits relativization. If all languages recognizable by the finite mg-pairs* (S, A) *and* (T, B) *belong to* $\mathbf{Lin}(\mathbf{K})$, *then any* *language recognizable by any double semidirect product* $(R, A\times B) = (S, A)\star\star(T, B)$ *belongs to* $\mathbf{Lin}(\mathbf{K})$.

Proof. Let Σ be an alphabet, and let

$$h : \Sigma^* \to R$$

denote a monoid homomorphism with $h(\Sigma) \subseteq A \times B$, so that h is a morphism $(\Sigma^*, \Sigma) \to (R, A \times B)$. It suffices to show that $L = h^{-1}(r_0)$ is in $\mathbf{Lin}(\mathbf{K})$, for each $r_0 = (s_0, t_0) \in R$.

For each $\sigma \in \Sigma$, let $s_\sigma \in A$ denote the left-hand component of $h(\sigma)$. We have

$$h(w) = \left(\sum_{w=w'\sigma w''} \pi(h(w'))s_\sigma\pi(h(w'')),\ \pi(h(w)) \right),$$

for all $w \in \Sigma^*$, where π denotes the projection $R \to T$, $\pi((s,t)) = t$, for all $(s,t) \in R$. Note that each $\pi(h(w'))s_\sigma\pi(h(w''))$ belongs to A. Since the composite of h and π is a homomorphism $\Sigma^* \to T$ with $\pi(h(\Sigma)) \subseteq B$, it follows from our assumptions that for each $t \in T$ there is a sentence α_t of $\mathrm{Lin}(\mathcal{L})$ such that for all words $w \in \Sigma^*$, $\pi(h(w)) = t$ iff $w \models \alpha_t$. For each $a \in A$ let $\varphi_a = \varphi_a(x)$ be the formula in the free variable x,

$$\bigvee_{t's_\sigma t''=a} (P_\sigma(x) \wedge \alpha_{t'}[< x] \wedge \alpha_{t''}[> x]),$$

where $\alpha_{t'}[< x]$ and $\alpha_{t''}[> x]$ denote relativizations of $\alpha_{t'}$ and $\alpha_{t''}$, respectively, which exist by assumption. Note that the family φ_a, $a \in A$ is deterministic. Then let ψ be the sentence

$$Q_K x.\langle\varphi_a\rangle_{a\in A}, \tag{6}$$

where $K \subseteq A^*$ denotes the regular language recognized by (S, A) with the element s_0 and the morphism $(A^*, A) \to (S, A)$ which is the identity function on A. It is clear from the construction that ψ defines the set of all words $w \in \Sigma^*$ such that the left-hand component of $h(w)$ is s_0. Thus, $\alpha_{t_0} \wedge \psi$ defines L. This shows that $L \in \mathbf{Lin}(\mathbf{Lin}(\mathbf{K}))$. Thus, by Corollary 3.6, $L \in \mathbf{Lin}(\mathbf{K})$. □

Proposition 9.2. *Suppose that* $\mathrm{Lin}(\mathbf{K})$ *strictly admits relativization. If all languages recognizable by the finite mg-pair* (T, B) *belong to* $\mathbf{Lin}_n(\mathbf{K})$ *and if* $(S, A) \in \mathbf{V}$*, then any language recognizable in any double semidirect product* $(R, A \times B) = (S, A)\star\star(T, B)$ *belongs to* $\mathbf{Lin}_{n+1}(\mathbf{K})$.

Proof. The argument is the same as above. The sentence (6) is in $\mathrm{Lin}_{n+1}(\mathbf{V})$. But by Proposition 8.8, $\mathbf{Lin}_{n+1}(\mathbf{V}) = \mathbf{Lin}_{n+1}(\mathbf{K})$. □

Suppose that Σ is an alphabet and V is a finite set of variables. Given a morphism $h : ((\Sigma \times P(V))^*, \Sigma \times P(V)) \to (M, A)$ and a set $F \subseteq M$, where (M, A)

is an mg-pair, the *language of V-structures (over Σ) recognized by* (M, A) *with* h *and* F consists of all V-structures $u \in (\Sigma \times P(V))^*$ such that $h(u) \in F$. In other words, it is the intersection of the language of all V-structures over Σ and the language $h^{-1}(F)$ recognized in the usual sense by (M, A) with h and F.

Suppose now that $K \subseteq \Delta^* = \{b_1, \ldots, b_k\}^*$, and consider a formula

$$\psi = Q_K x.\langle \varphi_{b_i} \rangle_{b_i \in \Delta},$$

where each $\varphi_{b_i} = \varphi_{b_i}(x, y_1, \ldots, y_m)$ is a formula of $\mathrm{Lin}(\mathbf{K})$ over the alphabet Σ whose free variables are among $x, y_1, \ldots, y_m$. Let (M, A) denote the syntactic mg-pair of K, or any mg-pair by which K can be recognized. For each $b_i \in \Delta$, let (N_i, B_i) denote an mg-pair recognizing the language of $(V \cup \{x\})$-structures $L_i = L_{\varphi_{b_i}} \subseteq (\Sigma \times P(V \cup \{x\}))^*$ defined by φ_{b_i}, where $V = \{y_1, \ldots, y_m\}$.

Proposition 9.3. *The language L_ψ of V-structures over Σ defined by ψ can be recognized by the block product*

$$(M, A)\Box[(N_1, B_1) \times \ldots \times (N_k, B_k)].$$

Proof. Let (N, B) denote the product $(N_1, B_1) \times \ldots \times (N_k, B_k)$, so that $B = B_1 \times \ldots \times B_k$ and N is the submonoid generated by B in the direct product $N_1 \times \ldots \times N_k$. For each $i \in [k]$, let η_i denote a morphism

$$((\Sigma \times P(V \cup \{x\}))^*, \Sigma \times P(V \cup \{x\})) \to (N_i, B_i),$$

recognizing L_i, and let η_K denote the syntactic morphism $(\Delta^*, \Delta) \to (M, A)$ of K, or any morphism recognizing K.

Let us order B by $b_1 < \ldots < b_k$. We define

$$\theta : ((\Sigma \times P(V))^*, \Sigma \times P(V)) \to (M, A)\Box(N, B)$$

by

$$\theta((a, X)) = (F_{(a,X)}, \eta_1((a, X)), \ldots, \eta_k((a, X))),$$

$(a, X) \in \Sigma \times P(V)$, where for all $n_1, n_1' \in N_1, \ldots, n_k, n_k' \in N_k$ such that $(n_1, \ldots, n_k)$ and $(n_1', \ldots, n_k')$ are in N,

$$F_{(a,X)}((n_1, \ldots, n_k), (n_1', \ldots, n_k')) = \eta_K(b_i)$$

for the least $b_i \in \Delta$ such that $n_i \eta_i((a, X \cup \{x\}))n_i' \in \eta_i(L_i)$, if there is such a letter b_i. Otherwise, we define $F_{(a,X)}((n_1, \ldots, n_k), (n_1', \ldots, n_k'))$ to be any element of A. Note that we indeed have that

$$F_{(a,X)} \in A^{N \times N}$$

and

$$\theta((a, X)) \in A^{N\times N} \times B_1 \times \ldots \times B_k.$$

Let $w = (a_1, X_1)\ldots(a_n, X_n) \in (A \times P(V))^*$ be a V-structure and write F_i for $F_{(a_i,X_i)}$, for all $i \in [n]$. Then we have

$$\begin{aligned}\theta(w) &= (F_1, \eta_1((a_1, X_1)), \ldots, \eta_k((a_1, X_1)))\ldots\\ &\quad\ldots(F_n, \eta_1((a_n, X_n)), \ldots, \eta_k((a_n, X_n)))\\ &= (F, \eta_1(w), \ldots, \eta_k(w)),\end{aligned}$$

where

$$\begin{aligned}&F((1,\ldots,1),(1,\ldots,1)) =\\ &\quad\prod_{i=1}^{n} F_i((\eta_1((a_1, X_1)\ldots(a_{i-1}, X_{i-1})), \ldots, \eta_k((a_1, X_1)\ldots(a_{i-1}, X_{i-1}))),\\ &\qquad(\eta_1((a_{i+1}, X_{i+1})\ldots(a_n, X_n)), \ldots, \eta_k((a_{i+1}, X_{i+1})\ldots(a_n, X_n)))) = \prod_{i=1}^{n} G_i.\end{aligned}$$

Now, for each $i \in [n]$, since $(a_1, X_1)\ldots(a_i, X_i \cup \{x\})\ldots(a_n, X_n)$ is a $(V \cup \{x\})$-structure, $G_i = \eta_K(b_j)$ for the unique (and thus least) j such that

$$\eta_j((a_1, X_1)\ldots(a_i, X_i \cup \{x\})\ldots(a_n, X_n)) \in \eta_j(L_j),$$

i.e., the unique j with

$$(a_1, X_1)\ldots(a_i, X_i \cup \{x\})\ldots(a_n, X_n) \in L_j.$$

Thus, $F((1,\ldots,1),(1,\ldots,1)) = \eta_K(\overline{w})$, where $\overline{w}$ is the characteristic word determined by w and the formula ψ. It follows that L_ψ is exactly the language of V-structures recognized by θ with those elements $(F, n_1, \ldots, n_k)$ of the block product satisfying $F((1,\ldots,1),(1\ldots,1)) \in \eta_K(K)$. □

Remark 9.4. Let π denote the projection

$$(M, A)\Box[(N_1, B_1) \times \ldots \times (N_k, B_k)] \to (N_1, B_1) \times \ldots \times (N_k, B_k),$$

and for each $i \in [k]$, let π_i denote the projection

$$(N_1, B_1) \times \ldots \times (N_k, B_k) \to (N_i, B_i).$$

The morphism θ constructed above has the property that for each $i \in [k]$, the composite

$$\begin{aligned}(\Sigma \times P(V))^* &\xrightarrow{\theta} (M,A)\Box[(N_1,B_1)\times \ldots \times (N_k,B_k)]\\ &\xrightarrow{\pi} (N_1,B_1)\times \ldots \times (N_k,B_k)\\ &\xrightarrow{\pi_i} (N_i,B_i)\end{aligned}$$

is the morphism $(\Sigma \times P(V))^* \to (N_i, B_i)$ obtained by restricting the function $\eta_i : (\Sigma \times P(V \cup \{x\}))^* \to N_i$ to $(\Sigma \times P(V))^*$. In particular, for each $i \in [k]$, the restriction of the above composite morphism to Σ^* agrees with the restriction of η_i to Σ^*. (We regard Σ as a subset of $\Sigma \times P(V)$ which in turn is a subset of $\Sigma \times P(V \cup \{x\})$.) Thus, if each $(N_i', B_i') = (\eta_i(\Sigma^*), \eta_i(\Sigma))$ belongs to $\mathbf{W}$, for some variety $\mathbf{W}$, then $(\theta(\Sigma^*), \theta(\Sigma))$ belongs to $\mathbf{V}{\star\star}\mathbf{W}$, since by Lemma 7.5, $(\theta(\Sigma^*), \theta(\Sigma))$ embeds in a double semidirect product of a direct power of (M, A) and the mg-pair $(N_1', B_1') \times \ldots \times (N_k', B_k')$.

Recall that by Corollary 8.4, we have $\widehat{\mathbf{K}} = \widehat{\mathbf{V}} = \bigcup_{n\geq 0} \mathbf{V}^{(n)}$, where $\mathbf{V}^{(0)}$ is the class of trivial mg-pairs and $\mathbf{V}^{(n+1)} = \mathbf{V}{\star\star}\mathbf{V}^{(n)} = \mathbf{V}\Box\mathbf{V}^{(n)}$.

We are now ready to prove the main result of this section.

Theorem 9.5. *Suppose that* $\mathbf{K}$ *is a class of finite mg-pairs and* $\mathbf{V}$ *denotes the variety generated by* $\mathbf{K}$*. Then for each* $n \geq 0$*, every language in* $\mathbf{Lin}_n(\mathbf{K})$ *is recognizable by some mg-pair in* $\mathbf{V}^{(n)}$*, i.e.,* $\mathbf{Lin}_n(\mathbf{K}) \subseteq \mathcal{L}_{\mathbf{V}^{(n)}}$*. Thus,* $\mathbf{Lin}(\mathbf{K}) \subseteq \mathcal{L}_{\widehat{\mathbf{K}}}$*. Moreover, if* Lin($\mathbf{K}$) *admits relativization, then a language belongs to* $\mathbf{Lin}(\mathbf{K})$ *iff it can be recognized by an mg-pair in* $\widehat{\mathbf{K}}$*, i.e.,* $\mathbf{Lin}(\mathbf{K}) = \mathcal{L}_{\widehat{\mathbf{K}}}$*. And if* Lin($\mathbf{K}$) *strictly admits relativization, then for each* $n \geq 0$*,* $\mathbf{Lin}_n(\mathbf{K}) = \mathcal{L}_{\mathbf{V}^{(n)}}$*.*

Proof. In order to prove the first claim, suppose that φ is a formula of Lin($\mathbf{K}$) over the alphabet Σ with free variables included in the finite set V. Let $\mathrm{qd}(\varphi) = n$. We argue by induction on the structure of φ to show that L_φ can be recognized by a morphism $\theta : (\Sigma \times P(V))^* \to (M, A)$ such that $(\theta(\Sigma^*), \theta(\Sigma))$ belongs to $\mathbf{V}^{(n)}$. When φ is an atomic formula, or the formula false, then $n = 0$ and any two words in Σ^* are equivalent with respect to the syntactic congruence of L_φ. Let η_{L_φ} denote the syntactic morphism $((\Sigma \times P(V))^*, \Sigma \times P(V)) \to \mathit{Synt}(L_\varphi)$. Then $(\eta_{L_\varphi}(\Sigma^*), \eta_{L_\varphi}(\Sigma))$ is trivial and thus belongs to $\mathbf{V}^{(0)}$, proving the claim. Suppose now that φ is $\varphi_1 \vee \varphi_2$, and that L_{φ_i} can be recognized by the morphism $\theta_i : (\Sigma \times P(V))^* \to (M_i, A_i)$ such that $(\theta_i(\Sigma^*), \theta_i(\Sigma))$ belongs to $\mathbf{V}^{(n)}$, $i = 1, 2$. Then L_φ can be recognized by the target pairing

$$\begin{aligned}\theta = \langle\theta_1, \theta_2\rangle : (\Sigma \times P(V))^* &\to (M_1, A_1) \times (M_2, A_2)\\ (\sigma, X) &\mapsto (\theta_1((\sigma, X)),\, \theta_2((\sigma, X))).\end{aligned}$$

Since

$$(\theta(\Sigma^*), \theta(\Sigma)) < (\theta_1(\Sigma^*), \theta_1(\Sigma)) \times (\theta_2(\Sigma^*), \theta_2(\Sigma))$$

and varieties are closed with respect to direct product and division, and since $\mathrm{qd}(\varphi_1), \mathrm{qd}(\varphi_2) \le n$, it follows by the induction hypothesis that $(\theta(\Sigma^*), \theta(\Sigma))$ belongs to $\mathbf{V}^{(n)}$. When φ is of the form $\neg\psi$, the result follows by using that L_φ and L_ψ can be recognized by the same mg-pairs. Suppose finally that $\varphi = Q_K x.\langle\varphi_{b_i}\rangle_{b_i\in\Delta}$, where $B \subseteq \Delta^*$, $\Delta = \{b_1, \ldots, b_k\}$, is a language recognized by some mg-pair (M, A) in $\mathbf{K}$, and where each φ_{b_i} is a formula of $\mathrm{Lin}(\mathbf{K})$ over Σ with free variables in $V \cup \{x\}$ of quantifier depth at most $n-1$. By the induction hypothesis, each $L_{\varphi_{b_i}}$ can be recognized by a morphism

$$\theta_i : (A \times P(V \cup \{x\}))^* \to (N_i, B_i)$$

such that $(\theta_i(\Sigma^*), \theta_i(\Sigma)) \in \mathbf{V}^{(n-1)}$. But then, by Proposition 9.3 and Remark 9.4, L_φ can be recognized by a morphism

$$\theta : (\Sigma \times P(V))^* \to (M, A)\Box[(N_1, B_1) \times \ldots \times (N_k, B_k)]$$

such that $(\theta(\Sigma^*), \theta(\Sigma))$ is in $\mathbf{V}^{(n)}$.

Assume now that $\mathrm{Lin}(\mathbf{K})$ admits relativization. We show that if the syntactic mg-pair $\mathit{Synt}(L)$ of a language $L \subseteq \Sigma^*$ belongs to $\mathbf{V}^{(n)}$, for some $n \ge 0$, then $L \in \mathbf{Lin}(\mathbf{K})$. When $n = 0$, $\mathit{Synt}(L)$ is trivial and thus L is either the empty set or Σ^*. In either case, L can be defined by a sentence of $\mathrm{Lin}(\mathbf{K})$, namely by false or true, proving that $L \in \mathbf{Lin}(\mathbf{K})$. We proceed by induction on n. When $n > 0$ and $\mathit{Synt}(L) \in \mathbf{V}^{(n)}$, then L can be recognized by a double semidirect product

$$(S, A)\star\star(T, B),$$

where $(S, A) \in \mathbf{V}$ and $(T, B) \in \mathbf{V}^{(n-1)}$. By the induction hypothesis, every language recognizable by (T, B) is in $\mathbf{Lin}(\mathbf{K})$. Thus, by Proposition 9.1 , L belongs to $\mathbf{Lin}(\mathbf{K})$. The same argument using Proposition 9.2 proves that if $\mathrm{Lin}(\mathbf{K})$ strictly admits relativization, then $\mathbf{Lin}_n(\mathbf{K}) \subseteq \mathcal{L}_{\mathbf{V}^{(n)}}$, for all $n \ge 0$. □

Corollary 9.6. *For any class $\mathbf{K}$ of finite mg-pairs, $\mathbf{FO}(\mathbf{K}) \subseteq \mathcal{L}_{\widehat{\mathbf{K}}_1}$, where $\mathbf{K}_1 = \mathbf{K} \cup \{U_1\}$ and $\widehat{\mathbf{K}}_1$ is the closed variety generated by $\mathbf{K}_1$. Moreover, when $\mathrm{FO}(\mathbf{K})$ admits relativization, then $\mathbf{FO}(\mathbf{K}) = \mathcal{L}_{\widehat{\mathbf{K}}_1}$.*

Corollary 9.7. *Suppose that $\mathbf{K}$ is a class of finite mg-pairs such that $\mathrm{FO}(\mathbf{K})$ strictly admits relativization. Let $\mathbf{V}_1$ denote the variety generated by $\mathbf{K}_1 = \mathbf{K} \cup \{U_1\}$. Then for each n, $\mathbf{FO}_n(\mathbf{K}) = \mathcal{L}_{\mathbf{V}_1^{(n)}}$.*

Proof. By Proposition 5.11, $\mathbf{FO}(\mathbf{K}) = \mathbf{Lin}(\mathbf{K}_1)$ and $\mathbf{FO}_n(\mathbf{K}) = \mathbf{Lin}_n(\mathbf{K}_1)$, for each $n \ge 1$. Moreover, since $\mathrm{FO}(\mathbf{K})$ strictly admits relativization, so does $\mathrm{Lin}(\mathbf{K}_1)$. Thus the result follows from Theorem 9.5 applied to $\mathbf{K}_1$. □

Corollary 9.8. *For any class $\mathcal{L}$ of regular languages, $\mathbf{Lin}(\mathcal{L}) \subseteq \mathcal{L}_{\widehat{\mathbf{V}}}$, where $\widehat{\mathbf{V}} = \widehat{\mathbf{K}}_{\mathcal{L}}$, the least closed variety containing the syntactic mg-pairs of the languages in $\mathcal{L}$. Moreover, when quotients are expressible in $\mathrm{Lin}(\mathcal{L})$ and $\mathrm{Lin}(\mathcal{L})$ admits relativization, then $\mathbf{Lin}(\mathcal{L}) = \mathcal{L}_{\widehat{\mathbf{V}}}$.*

Proof. It is clear that $\mathbf{Lin}(\mathcal{L}) \subseteq \mathbf{Lin}(\mathbf{K}_{\mathcal{L}})$. Moreover, $\mathbf{Lin}(\mathbf{K}_{\mathcal{L}}) \subseteq \mathcal{L}_{\widehat{\mathbf{V}}}$, by Theorem 9.5. Assume now that quotients are expressible in $\mathrm{Lin}(\mathcal{L})$ and $\mathrm{Lin}(\mathcal{L})$ admits relativization. We know that $\mathbf{Lin}(\mathcal{L}) = \mathbf{Lin}(\mathbf{K}_{\mathcal{L}})$ (Proposition 5.9). Since $\mathrm{Lin}(\mathcal{L})$ admits relativization, so does $\mathrm{Lin}(\mathbf{K}_{\mathcal{L}})$, by Proposition 6.2. Thus, by Theorem 9.5, $\mathbf{Lin}(\mathcal{L}) = \mathcal{L}_{\widehat{\mathbf{V}}}$. □

Corollary 9.9. *For any class $\mathcal{L}$ of regular languages, $\mathbf{FO}(\mathcal{L}) \subseteq \mathcal{L}_{\widehat{\mathbf{V}}_1}$, where $\widehat{\mathbf{V}}_1$ denotes the least closed variety containing U_1 and the syntactic mg-pairs of the languages in $\mathcal{L}$. Moreover, when quotients are expressible in* $\mathrm{FO}(\mathcal{L})$ *and* $\mathrm{FO}(\mathcal{L})$ *admits relativization, then $\mathbf{FO}(\mathcal{L}) = \mathcal{L}_{\widehat{\mathbf{V}}_1}$.*

Corollary 9.10. *Suppose that $\mathcal{L}$ is a class of regular languages such that the quotient of any language in $\mathcal{L}$ belongs to the literal pre-variety generated by $\mathcal{L}$. Suppose that* $\mathrm{Lin}(\mathcal{L})$ *strictly admits relativization. Then for each $n \geq 0$, $\mathbf{Lin}_n(\mathcal{L}) = \mathcal{L}_{\mathbf{V}^{(n)}}$, where $\mathbf{V}$ denotes the variety generated by $\mathbf{K}_{\mathcal{L}}$.*

Proof. By Theorem 9.5, Proposition 8.8 and Proposition 6.2. □

Corollary 9.11. *Suppose that $\mathcal{L}$ is a class of regular languages such that the quotient of any language in $\mathcal{L}$ belongs to the literal pre-variety generated by $\mathcal{L}$. Suppose that* $\mathrm{FO}(\mathcal{L})$ *strictly admits relativization. Then for each $n \geq 0$, $\mathbf{FO}_n(\mathcal{L}) = \mathcal{L}_{\mathbf{V}_1^{(n)}}$, where $\mathbf{V}_1$ denotes the variety generated by $\mathbf{K}_1 = \mathbf{K}_{\mathcal{L}} \cup \{U_1\}$.*

Proof. This follows from Corollary 9.10 by noting that for each n, $\mathbf{FO}_n(\mathcal{L}) = \mathbf{Lin}_n(\mathcal{L} \cup \{K_\exists\})$. Moreover, every quotient of any language in $\mathcal{L} \cup \{K_\exists\}$ is either $K_\exists$, or it belongs to the literal pre-variety generated by $\mathcal{L}$. Since by assumption $\mathrm{FO}(\mathcal{L})$ strictly admits relativization, so does $\mathrm{Lin}(\mathcal{L} \cup \{K_\exists\})$. □

Example 9.12.

- Let $\mathcal{L}$ consist of the one-letter languages $L_m^r = (b_1^m)^* b_1^r$, $m \geq 1$, $0 \leq r < m$, so that $\mathcal{L}$ is closed with respect to quotients. Then a language $L \subseteq \Sigma^*$ belongs to $\mathbf{Lin}(\mathcal{L})$ iff there exists some m such that L is the union of some languages $h^{-1}(L_m^r)$, where h denotes the unique literal homomorphism $\Sigma^* \to \{a\}^*$. Also, $\mathbf{K}_{\mathcal{L}}$ is the class of all mg-pairs of the form $(Z_m, \{a\})$ where Z_m is a cyclic group of order m with cyclic generator a, and $\widehat{\mathbf{K}}_{\mathcal{L}}$ is the class of all mg-pairs of the form $(Z_n, \{a\})$ where n divides m. We have that a language belongs to $\mathbf{Lin}(\mathcal{L})$ iff its syntactic mg-pair is in $\widehat{\mathbf{K}}_{\mathcal{L}}$, yet $\mathrm{Lin}(\mathcal{L})$ does not admit relativization. See Example 6.1.
- Let $\mathcal{L}$ consist of the finite and co-finite languages, so that $\mathcal{L} = \mathbf{Lin}(\mathcal{L})$. Then $\mathcal{L}$ is closed with respect to quotients, moreover, $\widehat{\mathbf{K}}_{\mathcal{L}}$ is the class of all finite *nilpotent* mg-pairs, *i.e.*, those finite mg-pairs (M, A) such that for some $n \geq 1$, all m-fold products $a_1 \dots a_m$ of the distinguished generators with $m \geq n$ give the same monoid element. Again, $L \in \mathbf{Lin}(\mathcal{L})$ holds for a language L iff $\mathit{Synt}(L)$ is in $\widehat{\mathbf{K}}_{\mathcal{L}}$, but as shown in Example 6.1, $\mathrm{Lin}(\mathcal{L})$ does not admit relativization.

Example 9.13. We give an example of a class $\mathcal{L}$ of regular languages closed with respect to quotients such that the variety of finite mg-pairs corresponding to $\mathbf{FO}(\mathcal{L}) = \mathbf{Lin}(\mathcal{L} \cup \{K_\exists\})$ is properly included in the least closed variety $\widehat{\mathbf{V}}_1$ containing the syntactic mg-pairs of the languages in $\mathcal{L} \cup \{K_\exists\}$. Thus, the assumption that $\mathrm{Lin}(\mathcal{L})$ admits relativization is essential to have $\mathbf{Lin}(\mathcal{L}) = \mathcal{L}_{\widehat{\mathbf{V}}}$ in Corollary 9.8. Similarly, the equality $\mathbf{FO}(\mathcal{L}) = \mathcal{L}_{\widehat{\mathbf{V}}_1}$ of Corollary 9.9 fails in general if $\mathrm{FO}(\mathcal{L})$ does not admit relativization.

Let $\mathcal{L} = \{L_m^r : m \geq 1,\ 0 \leq r < m\}$, where for each m and r, L_m^r is the one-letter language $(b_1^m)^* b_1^r$. Note that $\mathcal{L}$ is closed with respect to quotients. One can argue by induction on the structure of the formula φ over Σ of $\mathrm{FO}(\mathcal{L})$ to show that there exists some m such that φ is equivalent to a formula of the form

$$(\varphi_0 \wedge Q_{L_m^0} x.\langle\rangle) \vee \ldots \vee (\varphi_{m-1} \wedge Q_{L_m^{m-1}} x.\langle\rangle),$$

where each φ_i is a first-order formula over Σ. Thus, a language $L \subseteq \Sigma^*$ belongs to $\mathbf{FO}(\mathcal{L})$ iff there exists some $m \geq 1$ such that for each r with $0 \leq r < m$, $L \cap (\Sigma^m)^* \Sigma^r$ is in $\mathbf{FO}$.

Now let $\widehat{\mathbf{V}}_1$ denote the closed variety generated by $\mathbf{K}_{\mathcal{L} \cup \{K_\exists\}}$, *i.e.*, the closed variety generated by the mg-pairs $\{U_1, (Z_m, \{a\}) : m \geq 1\}$. It is known, *cf.* [2], that an mg-pair (M, A) belongs to $\widehat{\mathbf{V}}_1$ iff it is *quasi-aperiodic*, *i.e.*, it is finite and for each integer k, every group in M which is contained in the set A^k of all k-fold products of the generators is trivial. Let L be the two-letter language $(abab)^*$. The reader can easily verify that $Synt(L)$ is quasi-aperiodic. Indeed, denoting $Synt(L)$ by $(M_L, \{a, b\})$ (*i.e.*, we identify the letters a and b with the corresponding generators of M_L), the only nontrivial groups contained in M_L are two groups of order 2. These are the groups $\{ab, abab\}$ and $\{ba, baba\}$. But the length of any word representing the element ab or ba is congruent to 2 modulo 4, while the length of any word representing $abab$ or $baba$ is congruent to 0 modulo 4. Thus, there exist no k such that both ab and $abab$ can be represented by length k words, and similarly for ba and $baba$, showing that $Synt(L)$ is quasi-aperiodic. On the other hand, L is not in $\mathbf{FO}(\mathcal{L})$. To see this, let m be any positive integer, and let n denote the l.c.m. of m and 4. We have $L \cap (\Sigma^m)^* = ((abab)^{n/4})^*$ which is not aperiodic, since its syntactic monoid contains a group of order 2. Thus, by the theorem of McNaughton and Papert [22], $L \cap (\Sigma^m)^*$ is not in $\mathbf{FO}$, hence L is not in $\mathbf{FO}(\mathcal{L})$. It follows that $\mathbf{FO}(\mathcal{L}) \subset \mathcal{L}_{\widehat{\mathbf{V}}_1}$, or equivalently, the variety $\mathbf{W}$ generated by $\mathbf{K}_{\mathbf{FO}(\mathcal{L})}$ is strictly included in $\widehat{\mathbf{V}}_1$. (In fact, $\mathbf{W}$ is the direct product of the variety of aperiodic mg-pairs and the variety of *counters*, *i.e.*, mg-pairs of the form $(Z_m, \{a\})$, $m \geq 1$.)

The following results can be derived by combining some facts from Section 6 with Theorem 9.5 and the above corollaries of Theorem 9.5.

Corollary 9.14. *Suppose that $\mathcal{L}$ is a class of regular languages such that quotients are expressible in* $\mathrm{Lin}(\mathcal{L})$ *and for each K in $\mathcal{L}$, the padding of K belongs to* $\mathbf{Lin}(\mathcal{L})$. *Then* $\mathbf{Lin}(\mathcal{L}) = \mathcal{L}_{\widehat{\mathbf{V}}}$, *where $\widehat{\mathbf{V}}$ denotes the closed variety generated by $\mathbf{K}_{\mathcal{L}}$. Moreover, if quotients are expressible in* $\mathrm{FO}(\mathcal{L})$ *and for each K in $\mathcal{L}$, the* *padding of K belongs to* $\mathbf{FO}(\mathcal{L})$*, then* $\mathbf{FO}(\mathcal{L}) = \mathcal{L}_{\widehat{\mathbf{V}}_1}$*, where* $\widehat{\mathbf{V}}_1$ *denotes the closed variety generated by* $\mathbf{K}_{\mathcal{L}} \cup \{U_1\}$*.*

Corollary 9.15. *Suppose that $\mathcal{L}$ is a class of regular languages such that the padding and each quotient of any language in $\mathcal{L}$ belong to the literal pre-variety generated by $\mathcal{L}$. Then for each n,* $\mathbf{Lin}_n(\mathcal{L}) = \mathbf{V}^{(n)}$ *and* $\mathbf{FO}_n(\mathcal{L}) = \mathbf{V}_1^{(n)}$*, where* $\mathbf{V}$ *denotes the variety generated by* $\mathbf{K}_{\mathcal{L}}$ *and* $\mathbf{V}_1$ *is the variety generated by* $\mathbf{K}_{\mathcal{L}} \cup \{U_1\}$*.*

Corollary 9.16. *For any class* $\mathbf{K}$ *of finite mgi-pairs and for every integer $n \geq 0$,* $\mathbf{Lin}_n(\mathbf{K}) = \mathcal{L}_{\mathbf{V}^{(n)}}$ *and* $\mathbf{FO}(\mathbf{K}) = \mathcal{L}_{\mathbf{V}_1^{(n)}}$*, where* $\mathbf{V}$ *denotes the variety generated by* $\mathbf{K}$ *and* $\mathbf{V}_1$ *is the variety generated by* $\mathbf{K} \cup \{U_1\}$*.*

Remark 9.17. The main result, Theorem 9.5 extends to the situation when the language contains the successor predicate instead of the $<$ predicate. In fact, the only property one needs to require from the numerical predicates is that for any atomic formula φ over an alphabet Σ involving a numerical predicate, and for any two words $u, v \in \Sigma^*$, it holds that u and v are collapsed by the syntactic congruence of L_φ iff $u = v$. However, the assumption in the second part of the Theorem that the logic admits relativization seems to be quite strong when the successor predicate is the only numerical predicate.

10. The Krohn–Rhodes theorem

In this section, we first review a version of the fundamental theorem of Krohn and Rhodes [1, 17] which involves the double semidirect product (block product). The original formulation involved the wreath product, and its automata theoretic equivalent, the cascade product. Our presentation follows Straubing [31], Appendix A. Then we review some results from Dömösi, Ésik [9] and Ésik [13] and apply them in conjunction with the Krohn–Rhodes theorem to obtain descriptions of certain closed varieties of finite mg-pairs. The results of this section will be applied in Section 11 in the characterization of the expressive power of Lindström quantifiers with respect to concrete classes of regular languages.

All monoids considered in this section are assumed to be finite. As before, we will identify a finite monoid M with the mg-pair (M, M). For a class $\mathbf{K}$ of finite monoids (mg-pairs), we let $\widehat{\mathbf{K}}$ denote the least class of finite monoids (mg-pairs) containing $\mathbf{K}$ which is closed with respect to the double semidirect product (block product) and division. It is a simple matter to show that when $\mathbf{K}$ is a class of finite monoids and $\mathbf{K}_1$ is the class of mg-pairs (M, M), where $M \in \mathbf{K}$, then an mg-pair (S, A) belongs to $\widehat{\mathbf{K}}_1$ iff S belongs to $\widehat{\mathbf{K}}$. If $\mathbf{K}$ denotes a class of finite mg-pairs, then we let $\overline{\mathbf{K}}$ denote the class of all monoid components of the mg-pairs in $\mathbf{K}$. When $\mathbf{K}$ is a class of finite monoids and M is a finite monoid, $M < \mathbf{K}$ means that there is a monoid $S \in \mathbf{K}$ with $M < S$.

Recall that U_1 denotes a two-element semilattice. Moreover, recall that a (finite) group G is called *simple* if it is nontrivial and has no nontrivial normal subgroup. It is a simple matter to show that a finite monoid M is group iff $U_1 < M$ does not hold. Indeed, if M is not a group, then it contains an idempotent e other than the identity element 1. Then $\{1, e\}$ is a submonoid of M which is isomorphic to U_1.

Theorem 10.1. Krohn–Rhodes Theorem

- *Part 1. The following two conditions are equivalent for a nontrivial finite monoid M.*
  - *M is a simple group or isomorphic to U_1.*
  - *For every class $\mathbf{K}$ of finite monoids, if $M \in \widehat{\mathbf{K}}$ then $M < \mathbf{K}$.*
- *Part 2. Suppose that M is a finite monoid and $\mathbf{K}$ is a class of finite monoids containing at least one monoid which covers U_1. Then $M \in \widehat{\mathbf{K}}$ iff for every finite simple group G, if $G < M$ then $G < \mathbf{K}$. Moreover, when $\mathbf{K}$ is a class of finite groups, then $M \in \widehat{\mathbf{K}}$ iff M is a finite group and for every finite simple group G, if $G < M$ then $G < \mathbf{K}$.*

Remark 10.2. If one defines $\widehat{\mathbf{K}}$ as the closure of $\mathbf{K}$ with respect to the semidirect product (or wreath product) and division, then the result remains true provided that in Part 1 both U_1 and the three element monoid U_2 with two right zero elements are allowed, and if U_1 is replaced by U_2 in Part 2. In fact, the original formulations of the Krohn–Rhodes theorem (Krohn and Rhodes [17], Arbib [1]) used the wreath product and/or the corresponding automata theoretic notion of cascade composition.

Recall that a finite monoid M is called *aperiodic*, *cf.* Eilenberg [11], Pin [24], if it contains no nontrivial group, or equivalently, if no nontrivial group (or simple group) divides M. Moreover, recall that M is *solvable*, *cf.* Pin [24], Straubing [31], if every group included in M is solvable. (Such a group does not necessarily contain the identity element of M.) We denote the class of all aperiodics and the class of all finite groups by $\mathbf{A}$ and $\mathbf{G}$, respectively. Moreover, we denote by $\mathbf{GSol}$ the class of finite solvable groups, and by $\mathbf{MSol}$ the class of finite solvable monoids. Moreover, when P is a set of prime numbers, we denote by $\mathbf{GSol}_P$ the subclass of $\mathbf{GSol}$ determined by those finite solvable groups whose order is a product of primes in P. The variety $\mathbf{MSol}_P$ is defined likewise. Note that when P is empty, $\mathbf{MSol}_P = \mathbf{A}$, and when P is the set of all prime numbers, then $\mathbf{GSol}_P = \mathbf{GSol}$ and $\mathbf{MSol}_P = \mathbf{MSol}$. More generally, when $\mathbf{S}$ denotes a class of finite simple groups closed with respect to division, we let $\mathbf{G_S}$ denote the class of finite groups all of whose simple group divisors lie in $\mathbf{S}$. Moreover, we let $\mathbf{M_S}$ denote the class of those finite monoids that only contain groups in $\mathbf{G_S}$. When $\mathbf{S}$ is empty, $\mathbf{M_S}$ is the class of all aperiodic monoids. Moreover, when $\mathbf{S}$ is the class of cyclic groups of prime order, then $\mathbf{G_S} = \mathbf{GSol}$ and $\mathbf{M_S} = \mathbf{MSol}$. And when $\mathbf{S}$ contains all finite simple groups, then $\mathbf{G_S}$ is the class $\mathbf{G}$ of all finite groups, and $\mathbf{M_S} = \mathbf{M}$ is the class of all finite monoids. By the Krohn–Rhodes theorem, the above classes are all closed varieties (of finite monoids), *i.e.*, they are closed with respect to the double semidirect product (block product) and division.

Corollary 10.3. *Let $\mathbf{K}$ denote a class of finite monoids and let $\mathbf{S}$ denote a class of finite simple groups closed with respect to division.*

– $\mathbf{G_S} \subseteq \widehat{\mathbf{K}}$ *iff* $G < \mathbf{K}$ *holds for all* $G \in \mathbf{S}$. *Moreover,* $\widehat{\mathbf{K}} = \mathbf{G_S}$ *iff* $\mathbf{K} \subseteq \mathbf{G_S}$ *and* $G < \mathbf{K}$ *holds for all* $G \in \mathbf{S}$, *iff* $U_1 \not< \mathbf{K}$ *and for all finite simple groups* G *it holds that* $G < \mathbf{K}$ *iff* $G \in \mathbf{S}$.
– $\mathbf{M_S} \subseteq \widehat{\mathbf{K}}$ *iff* $U_1 < \mathbf{K}$ *and* $G < \mathbf{K}$, *for all* $G \in \mathbf{S}$. *Moreover,* $\widehat{\mathbf{K}} = \mathbf{M_S}$ *iff* $\mathbf{K} \subseteq \mathbf{M_S}$ *and* $U_1 < \mathbf{K}$ *and* $G < \mathbf{K}$, *for all* $G \in \mathbf{S}$, *iff* $U_1 < \mathbf{K}$ *and for all finite simple groups* G *it holds that* $G < \mathbf{K}$ *iff* $G \in \mathbf{S}$.

In particular, we obtain:

– $\widehat{\mathbf{K}} = \mathbf{M}$ iff the monoid U_1 as well as each finite (non-abelian simple) group is covered by some monoid in $\mathbf{K}$.
– $\mathbf{G} \subseteq \widehat{\mathbf{K}}$ iff $G < \mathbf{K}$ holds for all finite simple groups G. Moreover, $\widehat{\mathbf{K}} = \mathbf{G}$ iff $\mathbf{K} \subseteq \mathbf{G}$ and $G < \mathbf{K}$ holds for all finite (non-abelian simple) groups G, iff $U_1 \not< \mathbf{K}$ and $G < \mathbf{K}$ holds for all finite (non-abelian simple) groups G.
– $\mathbf{A} \subseteq \widehat{\mathbf{K}}$ iff $U_1 < \mathbf{K}$. Moreover, $\widehat{\mathbf{K}} = \mathbf{A}$ iff $\mathbf{K} \subseteq \mathbf{A}$ and $U_1 < \mathbf{K}$, iff no nontrivial finite group divides $\mathbf{K}$ and $U_1 < \mathbf{K}$.
– $\mathbf{GSol}_P \subseteq \widehat{\mathbf{K}}$ iff $Z_p < \mathbf{K}$ holds for all cyclic groups Z_p of prime order $p \in P$. Moreover, $\widehat{\mathbf{K}} = \mathbf{GSol}_P$ iff $\mathbf{K} \subseteq \mathbf{GSol}_P$ and $Z_p < \mathbf{K}$ holds for all cyclic groups Z_p with $p \in P$, iff $U_1 \not< \mathbf{K}$ and for each simple group G we have $G < \mathbf{K}$ iff G is cyclic with order in P.
– $\mathbf{MSol}_P \subseteq \widehat{\mathbf{K}}$ iff $U_1 < \mathbf{K}$ and $Z_p < \mathbf{K}$ hold for all cyclic groups Z_p of prime order $p \in P$. Moreover, $\widehat{\mathbf{K}} = \mathbf{MSol}_P$ iff $\mathbf{K} \subseteq \mathbf{MSol}_P$ and $U_1 < K$ and $Z_p < \mathbf{K}$ hold for all cyclic groups Z_p with $p \in P$, iff $U_1 < \mathbf{K}$ and for each simple group G we have $G < \mathbf{K}$ iff G is cyclic with order in P.

We now turn our attention to mg-pairs. *Below we identify any class* $\mathbf{K}$ *of finite monoids with the class of all mg-pairs* (M, A) *such that* $M \in \mathbf{K}$. Thus, for example, $\mathbf{M}$ also denotes the class of all finite mg-pairs, $\mathbf{MSol}$ the class all finite mg-pairs whose monoid component is solvable, etc. It holds that $\mathbf{K}$ is a variety of finite monoids (*cf.* [24]) iff $\mathbf{K}$ is a variety as a class of finite mg-pairs. Moreover, $\mathbf{K}$, as a variety of finite monoids is closed with respect to the double semidirect product iff $\mathbf{K}$, as a class of finite mg-pairs, is a closed variety.

Suppose that M is a finite monoid, (S, A) is a finite mg-pair, and $n \geq 1$. Following Dömösi and Ésik [9], we say that M *divides* (S, A) *in length* $n \geq 1$, denoted $M|^{(n)}(S, A)$, if S contains a subsemigroup T that maps homomorphically onto M under a homomorphism $h : T \to M$ such that each set $h^{-1}(m)$, $m \in M$ contains an n-fold product of elements in A (*i.e.*, an element in A^n). We define $M|(S, A)$ iff there is some n with $M|^{(n)}(S, A)$. (This relation may be called *divisibility in equal lengths.*)

Proposition 10.4. *Let* T *denote the submonoid generated by* A^n *in* S. *Then* $M|^{(n)}(S, A)$ *iff* $M < (T, A^n)$.

Proof. If $(M, M) < (T, A^n)$, then a sub mg-pair (T', B) of (T, A^n) maps homomorhically onto (M, M). Let h denote a surjective homomorphism $(T', B) \to (M, M)$. Since h maps $B \subseteq A^n$ onto M, each element of M is the image of an n-fold product over A. Thus, $M|^{(n)}(S, A)$.

Suppose now that $M|^{(n)}(S,A)$. Then let T' be a subsemigroup of S and $h : T' \to M$ a surjective semigroup homomorphism such that $h^{-1}(m) \cap A^n$ is not empty, for all $m \in M$. For each $m \in M$, let $b_m = h^{-1}(m) \cap A^n$, and let T'' denote the submonoid of S generated by $B = \{b_m : m \in M\}$, so that (T'', B) is a sub mg-pair of (T, A^n). It is clear that (M, M) is a quotient of (T'', B), proving $(M, M) < (T, A^n)$. □

Proposition 10.5. *If $M|^{(n)}(S,A)$ then there is a multiple m of n and a subsemigroup T of S contained in A^m such that M is a homomorphic image of T.*

This is shown in Ésik [13], *cf.* Lemma 3.3. Since M is a monoid, T can be chosen to be a monoid as well. However, T may not contain the identity element of S. Also, when M is a group, T can be assumed to be a group as well.

Proposition 10.6. *Suppose that (S,A) is a finite mg-pair and M is a finite non-abelian simple group or the monoid U_1. If $M < S$ then $M|(S,A)$.*

This is a very particular case of Proposition 3.5 in Ésik [13]. See also Maurer and Rhodes [21]. The case $M = U_1$ is obvious.

Proposition 10.7. *Suppose that M is U_1 or a finite simple group, moreover, suppose that $M|(S,A)\star\star(T,B)$, where (S,A) and (T,B) are finite mg-pairs. Then either $M|(S,A)$ or $M|(T,B)$.*

Proof. When M is U_1, or a finite non-abelian simple group, this follows from the first part of the Krohn–Rhodes theorem and Proposition 10.6. Thus, to complete the proof, it suffices to establish the claim for (cyclic) groups of prime order. So suppose that G is a cyclic group with prime order p such that $G|(S,A)\star\star(T,B) = (M, A \times B)$. By Proposition 10.5, G is a homomorphic image of a group in M all of whose members are n-fold products of elements in $A \times B$, for some $n \geq 1$. It follows easily that this group in turn contains a cyclic subgroup H of order p. Let (f,e) denote the identity element of H and let (s,t) denote any element of H different from (f,e). If $t \neq e$ then clearly t generates a cyclic group of order p in T (whose identity element is e), all of whose elements are n-fold products over B. We conclude that $G|^{(n)}(T,B)$. So suppose now that $t = e$. Then the right-hand component of each element of H is e. It follows as in Straubing [31], p. 64, or Eilenberg [11], v. B, p. 143, that the function $(s,e) \mapsto ese$, $(s,e) \in H$ is an injective homomorphism $H \to S$. Since each $(s,e) \in H$ is an n-fold product over $A \times B$, it follows that each element ese, $(s,e) \in H$ is an n-fold product over A. Thus, we have $G|^{(n)}(S,A)$. □

Suppose that M is a finite monoid, $\mathbf{K}$ is a class of finite mg-pairs, and $n \geq 1$. Below we will write $M|^{(n)}\mathbf{K}$ ($M|\mathbf{K}$, respectively) to denote that there exists an mg-pair $(S,A) \in \mathbf{K}$ such that $M|^{(n)}(S,A)$ ($M|(S,A)$, respectively).

Corollary 10.8. *Let M be U_1 or a finite simple group, and let $\mathbf{K}$ denote a class of finite mg-pairs. If $M|\widehat{\mathbf{K}}$ then $M|\mathbf{K}$.*

Recall from Example 9.13 that a counter of length n is an mg-pair consisting of a cyclic group Z_n of order n and a singleton generating set. We let $(Z_n, \{a\})$ denote a counter of order n. A nontrivial counter is a counter of length > 1. Given a monoid M and an element $a \in M$, the *period* of a is the least positive integer p such that there exists some m with $a^m = a^{m+p}$, *i.e.*, the period of the cyclic semigroup generated by a.

The following fact is clear.

Lemma 10.9. *Suppose that (T, B) is a finite mg-pair and $n \geq 1$. Then $(Z_n, \{a\}) < (T, B)$ iff there is some $b \in B$ whose period is a multiple of n.*

We say that a sequence $s_0, s_1, s_2, \ldots$ is *ultimately periodic* if there exist some $k \geq 0$ and $p \geq 1$ such that $s_{i+p} = s_i$ for all $i \geq k$. It is clear that if the sequence $s_0, s_1, s_2, \ldots$ is ultimately periodic, then there exists a least $p \geq 1$ such that the above property holds for some k. This number p is called the *period* of the sequence.

Lemma 10.10. *Suppose that M is a finite monoid and $a \in M$ has period n. Then for each m, m' in M, the sequence $mm', mam', ma^2m', \ldots$ is ultimately periodic with a period p that divides n.*

Proof. It is clear that the sequence $mm', mam', ma^2m', \ldots$ is ultimately periodic, moreover, there is an integer k such that $ma^{i+n}m' = ma^im'$, for all $i \geq k$. But the period p is a divisor of any such n. □

Lemma 10.11. *Suppose that a nontrivial counter divides a double semidirect product $(S, A \times B) = (M, A)\star\star(N, B)$. Then there is a nontrivial counter which divides (M, A) or (N, B).*

Proof. By Lemma 10.9, there is some $(a, b) \in A \times B \subseteq S$ with period $n > 1$. If the period of b is > 1, then we are done. So suppose that the period of b is 1, *i.e.*, $b^k = b^{k+1}$, for some k. Consider the sequence

$$(a, b), (a, b)^2, \ldots$$

which is, by assumption, ultimately periodic with period n. But for all $\ell \geq 0$,

$$\begin{aligned}(a,b)^{2k+\ell} &= (ab^{2k+\ell-1} + bab^{2k+\ell-2} + \ldots + b^{2k+\ell-1}a,\ b^{2k+\ell})\\ &= (ab^k + bab^k + \ldots + \overbrace{b^kab^k + \ldots + b^kab^k}^{\ell \text{ times}} + b^kab^{k-1} + \ldots + b^ka,\ b^k).\end{aligned}$$

Thus, by Lemma 10.10, n divides the period of b^kab^k. It follows that a counter of length n divides (M, A). □

Corollary 10.12. *Given a class $\mathbf{K}$ of mg-pairs, $\widehat{\mathbf{K}}$ contains a nontrivial counter iff a nontrivial counter divides an mg-pair in $\mathbf{K}$.*

Remark 10.13. The proof of Lemma 10.11 can easily be modified to show the following fact: a counter of length $n > 1$ divides a double semidirect product

$(M,A)\star\star(N,B)$ iff there exist integers p,q such that n divides pq, moreover, a counter of length p divides (N,B), and a counter of length q divides (M',A^p), where M' is the submonoid of M generated by A^p.

It is shown in Dömösi and Ésik [9] that if $M|^n(S,A)$, then the monoid M, or more precisely, the mg-pair (M,M) divides a wreath product

$$(S,A)\circ(R,B)\circ(Z_n,\{a\}), \tag{7}$$

where R is aperiodic. (Actually this fact is shown in [9] for finite automata and the cascade product, moreover, only a special type of aperiodic automata, namely *definite automata* are needed in the construction. The wreath product is associative, this is why no parentheses appear in (7).) Thus, by the Krohn–Rhodes theorem and Proposition 8.3, we have:

Proposition 10.14. *Suppose that (S,A) is a finite mg-pair and M is a finite monoid with $M|^n(S,A)$. Then $M\in\widehat{\mathbf{K}}$, where $\mathbf{K}$ consists of U_1, a counter of length n, and the mg-pair (S,A).*

Corollary 10.15. *Suppose that $\mathbf{K}$ is a class of finite mg-pairs such that $U_1\in\widehat{\mathbf{K}}$ and for each finite simple group G with $G<\overline{\mathbf{K}}$ there exists some $n\geq 1$ with $G|^{(n)}\mathbf{K}$ and $(Z_n,\{a\})\in\widehat{\mathbf{K}}$. Then a finite mg-pair (M,A) belongs to $\widehat{\mathbf{K}}$ iff for every finite simple group G, if $G<M$ then $G<\overline{\mathbf{K}}$.*

Proof. One direction is immediate from the Krohn–Rhodes Theorem. The other direction follows from the Krohn–Rhodes Theorem, Theorem 10.1, and Proposition 10.14. Indeed, assume that every simple group divisor of M divides the underlying monoid of an mg-pair in $\mathbf{K}$. Let $G_1,\ldots,G_k$ denote, up to isomorphism, all of the simple group divisors of M. By assumption, for each i there exists n_i with $G_i|^{(n_i)}\mathbf{K}$ and $(Z_{n_i},\{a\})\in\widehat{\mathbf{K}}$. Since also $U_1\in\widehat{\mathbf{K}}$, it follows from Proposition 10.14 that $G_i\in\widehat{\mathbf{K}}$. Since this holds for all $i\in[k]$, thus, by the Krohn–Rhodes Theorem, $(M,A)\in\widehat{\mathbf{K}}$. □

Corollary 10.16. *Suppose that $\mathbf{K}$ is a class of finite mg-pairs such that $\widehat{\mathbf{K}}$ contains U_1 as well as all the counters. Moreover, suppose that for all finite simple groups G, if $G<\overline{\mathbf{K}}$ then $G|\mathbf{K}$. Then a finite mg-pair (M,A) belongs to $\widehat{\mathbf{K}}$ iff for every finite simple group G, if $G<M$ then $G|\mathbf{K}$.*

Call a class $\mathbf{K}$ of finite mg-pairs *group-complete* if every finite group divides some monoid in $\overline{\mathbf{K}}$. Since every finite group embeds in a finite (non-abelian) simple group, by Proposition 10.6 we have that $\mathbf{K}$ is group-complete iff every finite (non-abelian) simple group divides in equal lengths some mg-pair in $\mathbf{K}$.

Corollary 10.17. *Let $\mathbf{K}$ be a class of finite mg-pairs. Then $\widehat{\mathbf{K}}$ is the class of all finite mg-pairs iff the following hold:*

(1) *$\widehat{\mathbf{K}}$ contains U_1 and all counters.*
(2) *$\mathbf{K}$ is group-complete.*

Remark 10.18. It is clear that $\widehat{\mathbf{K}}$ contains all counters iff it contains all counters of prime power length.

Example 10.19. For each $n \geq 1$, let S_n denote the *symmetric group* of all permutations of the set $[n]$. If $n \geq 3$, S_n is generated by the cyclic permutation $\rho = (12\dots n)$ and the transposition $\pi = (12)$. Hence, $(S_n, \{\rho, \pi\})$ is an mg-pair. Let $\mathbf{K}$ consist of (U_1, U_1) and all the mg-pairs $(S_n, \{\rho, \pi\})$, $n \geq 3$. Then both conditions of Corollary 10.17 are satisfied, so that $\widehat{\mathbf{K}}$ is the class of all finite mg-pairs.

Note that $(S_n, \{\rho, \pi\})$ is just the mg-pair of the automaton (*cf.* Ex. 5.3) whose states are the integers in the set $[n]$ which has two input letters that induce the permutations ρ and π, respectively.

Example 10.20. We modify the previous example to show that there is a group-complete class $\mathbf{K}$ with $U_1 \in \mathbf{K}$ such that $\widehat{\mathbf{K}}$ contains no counter. So let $\mathbf{K}$ consist of U_1 and, for each $n \geq 3$, the mg-pair of the following automaton Q_n with $2n+1$ states. The state set of Q_n consists of the integers $1, 2, \dots, 2n$ and the state $2'$, and there are four input letters, a, b, c, d. For each state q and letter x, $qx = q$, except for the following cases.

$$
\begin{aligned}
(2i-1)a &= 2i, \quad i \in [n] \\
(2i)b &= 2i+1, \quad i \in [n-1] \\
(2n)b &= 1 \\
1c &= 2 \\
2d &= 3 \\
3c &= 2' \\
2'd &= 1.
\end{aligned}
$$

Thus, on the set of odd integers, the word ab induces the cyclic permutation $(13...(2n-1))$ and cd induces the transposition (13). Thus, $\mathbf{K}$ is group-complete and contains U_1. However, no non-trivial counter divides any mg-pair in $\mathbf{K}$, since for each Q_n, any letter $x \in \{a, b, c, d\}$ induces the same function as x^2, and similarly for U_1. (See Corollary 10.12).

This example can be modified to show that there is a class $\mathbf{K}$ of finite mg-pairs which is group-complete, contains U_1 as well as each counter whose length is not a multiple of a given prime number p, but such that no counter of length p belongs to $\widehat{\mathbf{K}}$.

Corollary 10.21. *Let* $\mathbf{K}$ *be a class of finite mg-pairs. Then* $\widehat{\mathbf{K}} \supseteq \mathbf{MSol}$ *iff the following hold:*

(1) $\widehat{\mathbf{K}}$ *contains* U_1 *and all counters.*
(2) *For each (cyclic) group* G *of prime order, it holds that* $G|\mathbf{K}$.

Moreover, $\widehat{\mathbf{K}} = \mathbf{MSol}$ *iff the above conditions hold and* $\mathbf{K} \subseteq \mathbf{MSol}$.

More generally, we have:

Corollary 10.22. *Let* $\mathbf{K}$ *be a class of finite mg-pairs and let* $\mathbf{S}$ *be a class of finite simple groups containing the cyclic groups of prime order and closed with respect to division. Then* $\widehat{\mathbf{K}} \supseteq \mathbf{M_S}$ *iff the following hold:*

(1) $\widehat{\mathbf{K}}$ *contains* U_1 *and all counters.*
(2) *For each* $G \in \mathbf{S}$ *it holds that* $G|\mathbf{K}$.

Moreover, $\widehat{\mathbf{K}} = \mathbf{M_S}$ *iff the above conditions hold and* $\mathbf{K} \subseteq \mathbf{M_S}$.

10.1. Mg-pairs with identity

Recall that an mg-pair (M, A) is termed an mgi-pair if the identity element of M belongs to A. For example, $(Z_n, \{a, 1\})$ is an mgi-pair, for each $n \geq 1$, where a is a cyclic generator of Z_n and 1 is the identity.

Proposition 10.23. *For each* n, *it holds that* Z_n *divides a direct power of* $(Z_n, \{a, 1\})$.

Proof. Map each $(n-1)$-tuple $(a^{k_1}, \ldots, a^{k_{n-1}})$ in the direct power

$$(Z_n, \{a, 1\})^{n-1} = (Z_n^{n-1}, \{a, 1\}^{n-1})$$

to the element

$$a^{k_1} a^{2k_2} \ldots a^{(n-1)k_{n-1}}$$

in Z_n. □

Proposition 10.24. *Suppose that* $\mathbf{K}$ *is a class of finite mg-pairs and* P *is a set of prime numbers. Then* $\widehat{\mathbf{K}} \supseteq \mathbf{GSol}_P$ *iff for each prime number* $p \in P$ *it holds that* $(Z_p, \{a, 1\}) \in \widehat{\mathbf{K}}$. *Moreover,* $\widehat{\mathbf{K}} = \mathbf{GSol}_P$ *iff the above condition holds and* $\mathbf{K} \subseteq \mathbf{GSol}_P$.

Proof. This follows from Proposition 10.23 and Corollary 10.3. □

Lemma 10.25. *Suppose that* S *is a monoid and* (M, A) *is a finite mgi-pair. If* $S < M$ *then there exists some* n_0 *such that* $S|^{(n)}(M, A)$ *holds for all* $n \geq n_0$. *In particular,* $S|(M, A)$.

Proof. Since A is a set of generators for M, each element of M can be written as a product of elements of A. Let n_0 be the maximum number of factors in such a representation for each element of M. Since the identity element is in A, it follows that each $m \in M$ is the product of n generators, for every $n \geq n_0$. Thus, $M|^{(n)}(M, A)$. It follows that $S|^{(n)}(M, A)$ for all monoids S with $S < M$ and for all $n \geq n_0$. □

Lemma 10.26. *Let* $\mathbf{K}$ *be a class of finite mgi-pairs. Then* $U_1 \in \widehat{\mathbf{K}}$ *iff* $\overline{\mathbf{K}}$ *contains a monoid which is not a group.*

Proof. If $U_1 \in \widehat{\mathbf{K}}$, then, by the Krohn–Rhodes theorem, it holds that $U_1 < \overline{\mathbf{K}}$. But this is possible only if $\overline{\mathbf{K}}$ contains a monoid which is not a group.

Suppose now that (M, A) is a finite mg-pair in $\mathbf{K}$ which is not a group. Then A contains the identity element 1. Moreover, since M is not a group and A generates M, there exists some $a \in A$ such that $a^k \neq 1$, for all $k \geq 1$. Thus, U_1 is a homomorphic image of the submonoid M' of M generated by a. It follows that (U_1, U_1) is a homomorphic image of $(M', \{a, 1\})$. This proves that $U_1 < \mathbf{K}$, so that $U_1 \in \widehat{\mathbf{K}}$. □

Lemma 10.27. *Let* $\mathbf{K}$ *denote a class of finite mgi-pairs. The following conditions are equivalent.*

(1) *There is a nontrivial counter* $(Z_n, \{a\})$ *with* $(Z_n, \{a\}) < \mathbf{K}$.
(2) $\widehat{\mathbf{K}}$ *contains a nontrivial counter.*
(3) $\widehat{\mathbf{K}}$ *contains an infinite number of non-isomorphic counters.*

Proof. We already know that the first and second conditions are equivalent (Cor. 10.12). The third condition clearly implies the second. To complete the proof we show that the first condition implies the third. Given that $(Z_n, \{a\}) < \mathbf{K}$, where $n > 1$, also $(Z_n, \{a, 1\}) < \mathbf{K}$, since $\mathbf{K}$ consists of mgi-pairs. Thus, by Proposition 10.24, $(Z_m, \{a\}) \in \widehat{\mathbf{K}}$ for all integers m such that every prime divisor of m divides n. □

Proposition 10.28. *Suppose that* $\mathbf{K}$ *is a class of finite mgi-pairs such that* $\overline{\mathbf{K}}$ *contains a monoid which is not a group and there is a nontrivial counter that divides an mg-pair in* $\mathbf{K}$. *Then* $\widehat{\mathbf{K}}$ *contains and mg-pair* (M, A) *iff every simple group divisor of* M *divides a monoid in* $\overline{\mathbf{K}}$.

Proof. By the Krohn–Rhodes theorem, $\widehat{\mathbf{K}}$ contains at most those finite mg-pairs (M, A) such that every simple group divisor of M divides a monoid $\overline{\mathbf{K}}$. In the rest of the proof, we show that every such mg-pair is in indeed in $\widehat{\mathbf{K}}$.

By Lemma 10.26 we have $U_1 \in \widehat{\mathbf{K}}$. Consider now an arbitrary finite monoid M such that $M < \overline{\mathbf{K}}$. By Lemma 10.25 there exist some n_0 and $(S, B) \in \mathbf{K}$ such that $M|^{(n)}(S, B)$ for all $n \geq n_0$. Also, by Lemma 10.27, there exist an infinite number of counters in $\widehat{\mathbf{K}}$ of pairwise different length. We conclude that for some n, both $M|^{(n)}(S, B)$ and $(Z_n, \{a\}) \in \widehat{\mathbf{K}}$ hold. Thus, by Proposition 10.14 and Proposition 8.3, $(M, M) \in \widehat{\mathbf{K}}$. In particular, it follows that whenever G is a simple group with $G < \overline{\mathbf{K}}$, then $(G, G) \in \widehat{\mathbf{K}}$. Since also $U_1 \in \widehat{\mathbf{K}}$, it follows from the Krohn–Rhodes theorem that $\widehat{\mathbf{K}}$ contains every finite mg-pair (M, A) such that every simple group divisor of M divides a monoid in $\overline{\mathbf{K}}$. □

Corollary 10.29. *Suppose that* $\mathbf{K}$ *is a class of finite mgi-pairs. Then* $\widehat{\mathbf{K}}$ *is the class of all finite mg-pairs iff the following hold:*

(1) $\overline{\mathbf{K}}$ *contains a monoid which is not a group.*
(2) *There is a nontrivial counter which divides an mg-pair in* $\mathbf{K}$.
(3) $\mathbf{K}$ *is group-complete.*

Again, the above three conditions are independent.

Corollary 10.30. *Suppose that* $\mathbf{S}$ *is a nonempty class of finite simple groups closed with respect to division. Let* $\mathbf{K}$ *be a class of finite mgi-pairs. Then* $\widehat{\mathbf{K}} \supseteq \mathbf{M_S}$ *iff the following hold:*

(1) $\overline{\mathbf{K}}$ *contains a monoid which is not a group.*
(2) *There is a nontrivial counter which divides an mg-pair in* $\mathbf{K}$*.*
(3) $G < \mathbf{K}$ *holds for each* $G \in \mathbf{S}$*.*

Moreover, $\widehat{\mathbf{K}} = \mathbf{M_S}$ *iff the above conditions hold and* $\mathbf{K} \subseteq \mathbf{M_S}$*.*

Corollary 10.31. *For a class* $\mathbf{K}$ *of finite mgi-pairs,* $\widehat{\mathbf{K}} \supseteq \mathbf{A}$ *iff* $\mathbf{K}$ *contains a finite mgi-pair whose underlying monoid is not a group. Moreover,* $\widehat{\mathbf{K}} = \mathbf{A}$ *if this condition holds and* $\mathbf{K} \subseteq \mathbf{A}$*.*

11. Completeness

Call a class $\mathcal{L}$ of regular languages *Lindström-complete* if $\mathbf{Lin}(\mathcal{L})$ is the class of all regular languages, and *expressively complete* if $\mathbf{FO}(\mathcal{L})$ is the class of all regular languages. Similarly, call a class $\mathbf{K}$ of finite mg-pairs *Lindström-complete* if $\mathbf{Lin}(\mathbf{K})$ is the class of all regular languages, and *expressively complete* if $\mathbf{FO}(\mathbf{K})$ is the class of all regular languages. In this section, we combine results from Section 9 and Section 10 to obtain characterizations of Lindström-complete and expressively complete classes. We will also include relative expressive completeness results.

In the following propositions, $\mathcal{L}$ denotes a class of regular languages and $\mathbf{K}$ a class of finite mg-pairs.

Lemma 11.1. *If* $\mathcal{L}$ *is Lindström-complete then quotients are expressible in* $\mathrm{Lin}(\mathcal{L})$ *and* $\mathrm{Lin}(\mathcal{L})$ *admits relativization. If* $\mathbf{K}$ *is Lindsröm-complete, then* $\mathrm{Lin}(\mathbf{K})$ *admits relativization.*

Proof. Since the second claim follows from the first one, we only prove the first. So assume that $\mathcal{L}$ is Lindström-complete. Then $\mathbf{Lin}(\mathcal{L})$ is the class of all regular languages which is closed with respect to quotients. Thus, by Corollary 4.6, quotients are expressible in $\mathrm{Lin}(\mathcal{L})$. Since $K_\exists \in \mathbf{Lin}(\mathcal{L})$, by Corollary 6.3 $\mathrm{Lin}(\mathcal{L})$ admits relativization iff $\mathrm{FO}(\mathcal{L})$ does. But by Proposition 6.10, $\mathrm{FO}(\mathcal{L})$ admits relativization. □

Corollary 11.2. *If* $\mathcal{L}$ *is expressively complete then quotients are expressible in* $\mathrm{FO}(\mathcal{L})$ *and* $\mathrm{FO}(\mathcal{L})$ *admits relativization. If* $\mathbf{K}$ *is expressively complete, then* $\mathrm{FO}(\mathbf{K})$ *admits relativization.*

Lemma 11.3. *Every finite mg-pair* (M, A) *divides a direct product of the syntactic mg-pairs of some regular languages recognizable by* (M, A)*.*

Proof. Consider the morphism $h : (A^*, A) \to (M, A)$ which is the identity on A. For each $m \in M$, let (N_m, A_m) denote the syntactic mg-pair of $h^{-1}(m)$. Then let (N, B) denote the image of (A^*, A) under the target tupling η of the syntactic morphisms $(A^*, A) \to (N_m, A_m)$, $m \in M$. Now (M, A) is a quotient of (N, B) under the map $\eta(u) \mapsto m$ iff $u \in h^{-1}(m)$, for all $u \in A^*$ and $m \in M$. □

Theorem 11.4. *A class* $\mathbf{K}$ *of finite mg-pairs is Lindström-complete iff* $\mathrm{Lin}(\mathbf{K})$ *admits relativization,* $\mathbf{K}$ *is group-complete and* $\widehat{\mathbf{K}}$ *contains* U_1 *and all counters. Moreover,* $\mathbf{K}$ *is expressively complete iff* $\mathrm{FO}(\mathbf{K})$ *admits relativization,* $\mathbf{K}$ *is group-complete and* $\widehat{\mathbf{K}}$ *contains all counters.*

Proof. Suppose that $\mathbf{K}$ is Lindström-complete. Then by Lemma 11.1, $\mathrm{Lin}(\mathbf{K})$ admits relativization. Thus, by Theorem 9.5, $\widehat{\mathbf{K}}$ contains the syntactic mg-pair of every regular language. Thus, by Lemma 11.3, $\widehat{\mathbf{K}}$ is the class of all finite mg-pairs. It follows now from the Krohn–Rhodes theorem that $\mathbf{K}$ is group-complete.

Suppose now that $\mathrm{Lin}(\mathbf{K})$ admits relativization, moreover, $\mathbf{K}$ is group-complete and $\widehat{\mathbf{K}}$ contains U_1 and the counters. Then, by Corollary 10.17, $\widehat{\mathbf{K}}$ is the class of all finite mg-pairs, and thus by Theorem 9.5, $\mathbf{Lin}(\mathbf{K})$ is the class of all regular languages.

Suppose next that $\mathbf{K}$ is expressively complete. By Corollary 11.2, $\mathrm{FO}(\mathbf{K})$ admits relativization. By Proposition 5.11, $\mathbf{K}_1 = \mathbf{K} \cup \{U_1\}$ is Lindström-complete. Hence, $\mathbf{K}_1$ is group-complete and $\widehat{\mathbf{K}}_1$ contains the counters. Since U_1 is aperiodic, it follows that $\mathbf{K}$ is group-complete. Since $\widehat{\mathbf{K}}_1$ contains the counters, it is not difficult to show that $\widehat{\mathbf{K}}$ also contains the counters. (See also Rem. 10.13.)

Finally, suppose that $\mathrm{FO}(\mathbf{K})$ admits relativization, $\mathbf{K}$ is group-complete, and $\widehat{\mathbf{K}}$ contains the counters. Let $\mathbf{K}_1$ be defined as above. Then $\mathrm{Lin}(\mathbf{K}_1)$ admits relativization, $\mathbf{K}_1$ is group-complete, and $\widehat{\mathbf{K}}_1$ contains U_1 and the counters. Thus, $\mathbf{FO}(\mathbf{K}) = \mathbf{Lin}(\mathbf{K}_1)$ is the class of all regular languages. □

Corollary 11.5. *A class* $\mathcal{L}$ *of regular languages is Lindström-complete iff the following hold.*

(1) *Quotients are expressible in* $\mathrm{Lin}(\mathcal{L})$.
(2) $\mathrm{Lin}(\mathcal{L})$ *admits relativization.*
(3) *Every finite group divides the syntactic mg-pair of a language in* $\mathcal{L}$.
(4) $\widehat{\mathbf{K}}_{\mathcal{L}}$ *contains* U_1 *and the counters.*

Proof. If $\mathcal{L}$ is Lindström-complete, then by Lemma 11.1, quotients are expressible in $\mathrm{Lin}(\mathcal{L})$. Moreover, if quotients are expressible in $\mathrm{Lin}(\mathcal{L})$, then, by Proposition 5.9, $\mathbf{Lin}(\mathcal{L}) = \mathbf{Lin}(\mathbf{K}_{\mathcal{L}})$ and thus $\mathrm{Lin}(\mathcal{L})$ admits relativization iff $\mathrm{Lin}(\mathbf{K}_{\mathcal{L}})$ does (*cf.* Prop. 6.2). Thus, our claim follows from Theorem 11.4. □

Corollary 11.6. *A class* $\mathcal{L}$ *of regular languages is Lindström-complete iff the following hold.*

(1) *Quotients are expressible in* $\mathrm{Lin}(\mathcal{L})$.
(2) $\mathrm{Lin}(\mathcal{L})$ *admits relativization.*
(3) *Every finite group divides the syntactic mg-pair of a language in* $\mathcal{L}$.
(4) $K_\exists \in \mathbf{Lin}(\mathcal{L})$, *and for each* n, *the one-letter language* $(a^n)^*$ *belongs to* $\mathbf{Lin}(\mathcal{L})$.

Proof. This follows from Corollary 11.5 and Corollary 9.8 by noting that (U_1, U_1) is the syntactic mg-pair of $K_\exists$ and for each n, the syntactic mg-pair of $(a^n)^*$ is a counter of length n. □

Corollary 11.7. *A class $\mathcal{L}$ of regular languages is expressively complete iff the following hold.*

(1) *Quotients are expressible in* $\mathrm{FO}(\mathcal{L})$.
(2) $\mathrm{FO}(\mathcal{L})$ *admits relativization.*
(3) *Every finite group divides the syntactic mg-pair of a language in* $\mathcal{L}$.
(4) $\widehat{\mathbf{K}}_{\mathcal{L}}$ *(or* $\widehat{\mathbf{K}}_1$*, where* $\mathbf{K}_1 = \mathbf{K}_{\mathcal{L}} \cup \{U_1\}$*) contains the counters.*

Corollary 11.8. *A class $\mathcal{L}$ of regular languages is expressively complete iff the following hold.*

(1) *Quotients are expressible in* $\mathrm{FO}(\mathcal{L})$.
(2) $\mathrm{FO}(\mathcal{L})$ *admits relativization.*
(3) *Every finite group divides the syntactic mg-pair of a language in* $\mathcal{L}$.
(4) *For each n, the one-letter language $(a^n)^*$ belongs to* $\mathbf{FO}(\mathcal{L})$ *(or to* $\mathbf{Lin}(\mathcal{L})$*).*

Example 11.9. The class $\mathbf{K}$ presented in Example 10.19 is Lindström-complete. Thus there exists a Lindström-complete class of finite mg-pairs with two generators. Also, there exists a Lindström-complete class of two-letter regular languages. On the other hand, no class of mg-pairs with a single generator is group-complete, hence no such class is Lindström-complete, or expressively complete.

Corollary 11.10. *There exists no finite Lindström-complete class of finite mg-pairs, or regular languages. Each Lindström-complete class of finite mg-pairs contains an infinite number of mg-pairs with 2 or more generators. Also, any Lindström-complete class of regular languages contains an infinite number of languages over an alphabet with two or more letters.*

This corollary is related to the main result of Beauquier and Rabinovitch [5].

Proposition 11.11. *Suppose that $\mathbf{S}$ is a class of finite simple groups closed with respect to division and suppose that $\mathbf{K}$ is a class of finite mg-pairs.*

(1) *Suppose that* $\mathrm{Lin}(\mathbf{K})$ *admits relativization. Then* $\mathbf{Lin}(\mathbf{K})$ *contains all regular languages whose syntactic monoid is in* $\mathbf{M_S}$ *iff* $\widehat{\mathbf{K}}$ *contains (the mg-pairs corresponding to)* U_1 *and the simple groups in* $\mathbf{S}$.
(2) $\mathbf{Lin}(\mathbf{K})$ *is the class of all regular languages whose syntactic monoid is in* $\mathbf{M_S}$ *iff* $\mathrm{Lin}(\mathbf{K})$ *admits relativization, each simple group divisor of the monoid component of any mg-pair in* $\mathbf{K}$ *is in* $\mathbf{S}$*, and* $\widehat{\mathbf{K}}$ *contains (the mg-pairs corresponding to)* U_1 *and the simple groups in* $\mathbf{S}$.
(3) *If* $\mathrm{FO}(\mathbf{K})$ *admits relativization, then* $\mathbf{FO}(\mathbf{K})$ *contains all regular languages whose syntactic monoid is in* $\mathbf{M_S}$ *iff* $\widehat{\mathbf{K}}_1$ *contains (the mg-pairs corresponding to) the simple groups in* $\mathbf{S}$*, where* $\mathbf{K}_1 = \mathbf{K} \cup \{U_1\}$.
(4) $\mathbf{FO}(\mathbf{K})$ *is the class of all regular languages whose syntactic monoid is in* $\mathbf{M_S}$ *iff* $\mathrm{FO}(\mathbf{K})$ *admits relativization and each simple group divisor of the monoid component of any mg-pair in* $\mathbf{K}$ *is in* $\mathbf{S}$ *and* $\widehat{\mathbf{K}}_1$ *contains (the mg-pairs corresponding to) the simple groups in* $\mathbf{S}$*, where* $\mathbf{K}_1 = \mathbf{K} \cup \{U_1\}$.

Proof. We only prove the first two claims. Assume that Lin($\mathbf{K}$) admits relativization. Then, by Theorem 9.5, a language belongs to $\mathbf{Lin}(\mathbf{K})$ iff its syntactic mg-pair is in $\widehat{\mathbf{K}}$. Given a finite group G, let h denote the morphism $h : (G^*, G) \to (G, G)$ which is the identity mapping on G. It is not difficult to show that the syntactic mg-pair of the language $h^{-1}(1)$ is isomorphic to (G, G). Thus, if $\mathbf{Lin}(\mathbf{K})$ contains all languages whose syntactic monoid is in $\mathbf{M_S}$, and if $G \in \mathbf{S}$, then $\widehat{\mathbf{K}}$ contains (G, G). Since the syntactic mg-pair of the language $K_\exists$ is isomorphic to (U_1, U_1) and thus aperiodic, it follows that (U_1, U_1) is also in $\widehat{\mathbf{K}}$. Conversely, if $\widehat{\mathbf{K}}$ contains $\mathbf{S}$ and U_1, then, by the Krohn–Rhodes theorem, $\mathbf{M_S} \subseteq \widehat{\mathbf{K}}$. Thus, by Theorem 9.5, $\mathbf{Lin}(\mathbf{K})$ contains all regular languages whose syntactic monoid is in $\mathbf{M_S}$.

As for the second claim, assume that $\mathbf{Lin}(\mathbf{K})$ is the class of all (regular) languages whose syntactic mg-pair is in $\mathbf{M_S}$. Then, by Theorem 3.5, $\mathbf{Lin}(\mathbf{K}) = \mathbf{Lin}(\mathbf{M_S})$. Now it follows using Corollary 6.8 that Lin($\mathbf{M_S}$) admits relativization. Thus, by Proposition 6.2, Lin($\mathbf{K}$) also admits relativization. Thus, by the first claim, $\widehat{\mathbf{K}}$ contains U_1 and the mg-pair corresponding to each group $G \in \mathbf{S}$. Moreover, by Theorem 9.5, every simple group divisor of $\overline{\mathbf{K}}$ belongs to $\mathbf{S}$. Conversely, if these hold and if Lin($\mathbf{K}$) admits relativization, then again by the first claim, $\mathbf{Lin}(\mathbf{K})$ contains every language whose syntactic mg-pair is in $\mathbf{M_S}$. Moreover, by Theorem 9.5 and the Krohn–Rhodes Theorem, it does not contain any other language. □

Proposition 11.12. *Suppose that* $\widehat{\mathbf{K}}$ *contains* U_1 *and the counters, and has the following property: for every simple group* G*, if* $G < \overline{\mathbf{K}}$ *then* $G|\mathbf{K}$*. Suppose that* Lin($\mathbf{K}$) *admits relativization. Then a language* L *is in* $\mathbf{Lin}(\mathbf{K})$ *iff every simple group divisor of the syntactic monoid of* L *divides* $\mathbf{K}$*.*

Proof. By Corollary 10.16, $\widehat{\mathbf{K}} = \mathbf{M_S}$, where $\mathbf{S}$ is the class of all simple groups G with $G < \mathbf{K}$. Thus, the result follows from Theorem 9.5. □

Proposition 11.13. *Suppose that* $\mathbf{S}$ *is a class of finite simple groups closed with respect to division containing all cyclic groups of prime order. Moreover, suppose that* $\mathbf{K}$ *is a class of finite mg-pairs.*

(1) *If* Lin($\mathbf{K}$) *admits relativization then* $\mathbf{Lin}(\mathbf{K})$ *contains all regular languages whose syntactic monoid is in* $\mathbf{M_S}$ *iff* $\widehat{\mathbf{K}}$ *contains* U_1 *and the counters, and for each finite simple group* G *in* $\mathbf{S}$ *it holds that* $G|\mathbf{K}$*.*
(2) $\mathbf{Lin}(\mathbf{K})$ *is the class of all regular languages whose syntactic monoid is in* $\mathbf{M_S}$ *iff* Lin($\mathbf{K}$) *admits relativization,* $\widehat{\mathbf{K}}$ *contains* U_1 *and the counters, and for every finite simple group* G *it holds that* $G \in \mathbf{S}$ *iff* $G|\mathbf{K}$*.*
(3) *If* FO($\mathbf{K}$) *admits relativization then* $\mathbf{FO}(\mathbf{K})$ *contains all regular languages whose syntactic monoid is in* $\mathbf{M_S}$ *iff* $\widehat{\mathbf{K}}$ *contains the counters and for each simple group* G *in* $\mathbf{S}$ *it holds that* $G|\mathbf{K}$*.*
(4) $\mathbf{FO}(\mathbf{K})$ *is the class of all regular languages whose syntactic monoid is in* $\mathbf{M_S}$ *iff* FO($\mathbf{K}$) *admits relativization,* $\widehat{\mathbf{K}}$ *contains the counters, and for every finite simple group* G *it holds that* $G \in \mathbf{S}$ *iff* $G|\mathbf{K}$*.*

Proof. From Proposition 11.11 and Corollary 10.22. □

Without proof we mention:

Proposition 11.14. *Suppose that* $\mathbf{S}$ *is a class of finite simple groups closed with respect to division. If* $\mathrm{Lin}(\mathbf{K})$ *admits relativization then* $\mathbf{Lin}(\mathbf{K})$ *contains all regular languages whose syntactic monoid is in* $\mathbf{G_S}$ *iff* $\widehat{\mathbf{K}}$ *contains (the mg-pairs corresponding to) the simple groups in* $\mathbf{S}$*. Moreover,* $\mathbf{Lin}(\mathbf{K})$ *is the class of all regular languages whose syntactic monoid is in* $\mathbf{G_S}$ *iff* $\mathbf{Lin}(\mathbf{K})$ *admits relativization,* $\widehat{\mathbf{K}}$ *contains (the mg-pairs corresponding to) the simple groups in* $\mathbf{S}$*, and* $\mathbf{K} \subseteq \mathbf{G_S}$*.*

By taking $\mathbf{S}$ to be the class of all cyclic groups of prime order, from Proposition 11.13 we obtain:

Proposition 11.15.

(1) *If* $\mathrm{Lin}(\mathbf{K})$ *admits relativization then* $\mathbf{Lin}(\mathbf{K})$ *contains all regular languages whose syntactic monoid is in* $\mathbf{MSol}$ *iff* $\widehat{\mathbf{K}}$ *contains* U_1 *and the counters, and for every prime number* p *it holds that* $Z_p|\mathbf{K}$*. Moreover,* $\mathbf{Lin}(\mathbf{K})$ *is the class of all regular languages whose syntactic monoid is in* $\mathbf{MSol}$ *iff* $\mathbf{Lin}(\mathbf{K})$ *admits relativization,* $\mathbf{K} \subseteq \mathbf{MSol}$*,* $\widehat{\mathbf{K}}$ *contains* U_1 *and the counters, and for every prime number* p *it holds that* $Z_p|\mathbf{K}$*.*

(2) *If* $\mathrm{FO}(\mathbf{K})$ *admits relativization then* $\mathbf{FO}(\mathbf{K})$ *contains all regular languages whose syntactic monoid is in* $\mathbf{MSol}$ *iff* $\widehat{\mathbf{K}}$ *contains the counters, and for every prime number* p *it holds that* $Z_p|\mathbf{K}$*. Moreover,* $\mathbf{FO}(\mathbf{K})$ *is the class of all regular languages whose syntactic monoid is solvable iff* $\mathrm{FO}(\mathbf{K})$ *admits relativization,* $\mathbf{K} \subseteq \mathbf{MSol}$*,* $\widehat{\mathbf{K}}$ *contains the counters, and for every prime number* p *it holds that* $Z_p|\mathbf{K}$*.*

Proposition 11.16. *Let* P *denote a set of prime numbers. If* $\mathrm{Lin}(\mathbf{K})$ *admits relativization then* $\mathbf{Lin}(\mathbf{K})$ *contains all regular languages whose syntactic monoid is in* $\mathbf{GSol}_P$ *iff for every prime number* $p \in P$ *it holds that* $(Z_p, \{a,1\}) \in \widehat{\mathbf{K}}$*. Moreover,* $\mathbf{Lin}(\mathbf{K})$ *is the class of all regular languages whose syntactic monoid is in* $\mathbf{GSol}_P$ *iff* $\mathrm{Lin}(\mathbf{K})$ *admits relativization,* $\mathbf{K} \subseteq \mathbf{GSol}_P$*, and for every prime number* $p \in P$ *it holds that* $(Z_p, \{a,1\}) \in \widehat{\mathbf{K}}$*.*

Proof. This follows from Proposition 11.14 and Proposition 10.24. □

We also have:

Proposition 11.17. *Suppose that* P *is a set of prime numbers.*

(1) *If* $\mathrm{Lin}(\mathbf{K})$ *admits relativization then* $\mathbf{Lin}(\mathbf{K})$ *contains all regular languages whose syntactic monoid is in* $\mathbf{MSol}_P$ *iff* $U_1 \in \widehat{\mathbf{K}}$ *and for every prime number* $p \in P$ *it holds that* $(Z_p, \{a,1\}) \in \widehat{\mathbf{K}}$*. Moreover,* $\mathbf{Lin}(\mathbf{K})$ *is the class of all regular languages whose syntactic monoid is in* $\mathbf{MSol}_P$ *iff* $\mathrm{Lin}(\mathbf{K})$ *admits relativization,* $\mathbf{K} \subseteq \mathbf{MSol}_P$*,* $U_1 \in \widehat{\mathbf{K}}$*, and for every prime number* $p \in P$ *it holds that* $(Z_p, \{a,1\}) \in \widehat{\mathbf{K}}$*.*

(2) $\mathbf{FO}(\mathbf{K})$ *is the class of all regular languages whose syntactic monoid is in* $\mathbf{MSol}_P$ *iff* $\mathrm{FO}(\mathbf{K})$ *admits relativization,* $\mathbf{K} \subseteq \mathbf{MSol}_P$*, moreover, we have* $(Z_p, \{a,1\}) \in \widehat{\mathbf{K}}_1$*, for all* $p \in P$*, where* $\mathbf{K}_1 = \mathbf{K} \cup \{U_1\}$*.*

In particular, when P is empty, we have:

Proposition 11.18. $\mathbf{Lin}(\mathbf{K}) = \mathbf{FO}$ *iff* $\mathrm{Lin}(\mathbf{K})$ *admits relativization,* $U_1 \in \widehat{\mathbf{K}}$*, and* $\mathbf{K} \subseteq \mathbf{A}$.

We now translate some of the above results to classes $\mathcal{L}$ of regular languages.

Corollary 11.19. *Suppose that* $\mathbf{S}$ *is a class of finite simple groups closed with respect to division and suppose that* $\mathcal{L}$ *is a class of regular languages.*

(1) *Suppose that* $\mathrm{Lin}(\mathcal{L})$ *admits relativization and quotients are expressible in* $\mathrm{Lin}(\mathcal{L})$*. Then* $\mathbf{Lin}(\mathcal{L})$ *contains all regular languages whose syntactic monoid is in* $\mathbf{M_S}$ *iff* $\widehat{\mathbf{K}}_{\mathcal{L}}$ *contains (the mg-pairs corresponding to)* U_1 *and the simple groups in* $\mathbf{S}$.

(2) $\mathbf{Lin}(\mathcal{L})$ *is the class of all regular languages whose syntactic monoid is in* $\mathbf{M_S}$ *iff* $\mathrm{Lin}(\mathcal{L})$ *admits relativization, quotients are expressible in* $\mathrm{Lin}(\mathcal{L})$*, each simple group divisor of the monoid component of any mg-pair in* $\mathbf{K}_{\mathcal{L}}$ *is in* $\mathbf{S}$*, and* $\widehat{\mathbf{K}}_{\mathcal{L}}$ *contains (the mg-pairs corresponding to)* U_1 *and the simple groups in* $\mathbf{S}$.

(3) *If* $\mathrm{FO}(\mathcal{L})$ *admits relativization and quotients are expressible in* $\mathrm{FO}(\mathcal{L})$*, then* $\mathbf{FO}(\mathcal{L})$ *contains all regular languages whose syntactic monoid is in* $\mathbf{M_S}$ *iff* $\widehat{\mathbf{K}}_1$ *contains (the mg-pairs corresponding to) the simple groups in* $\mathbf{S}$*, where* $\mathbf{K}_1 = \mathbf{K}_{\mathcal{L}} \cup \{U_1\}$.

(4) $\mathbf{FO}(\mathcal{L})$ *is the class of all regular languages whose syntactic monoid is in* $\mathbf{M_S}$ *iff* $\mathrm{FO}(\mathcal{L})$ *admits relativization, quotients are expressible in* $\mathrm{FO}(\mathcal{L})$*, each simple group divisor of the monoid component of any mg-pair in* $\mathbf{K}_{\mathcal{L}}$ *is in* $\mathbf{S}$*, and* $\widehat{\mathbf{K}}_1$ *contains (the mg-pairs corresponding to) the simple groups in* $\mathbf{S}$*, where* $\mathbf{K}_1 = \mathbf{K}_{\mathcal{L}} \cup \{U_1\}$.

Corollary 11.20. *Suppose that* $\mathbf{S}$ *is a class of finite simple groups closed with respect to division containing all cyclic groups of prime order. Moreover, suppose that* $\mathcal{L}$ *is a class of regular languages.*

(1) *If* $\mathrm{Lin}(\mathcal{L})$ *admits relativization and quotients are expressible in* $\mathrm{Lin}(\mathcal{L})$ *then* $\mathbf{Lin}(\mathcal{L})$ *contains all regular languages whose syntactic monoid is in* $\mathbf{M_S}$ *iff* $\widehat{\mathbf{K}}_{\mathcal{L}}$ *contains* U_1 *and the counters, and for each finite simple group* G *in* $\mathbf{S}$ *it holds that* $G|\mathbf{K}_{\mathcal{L}}$.

(2) $\mathbf{Lin}(\mathcal{L})$ *is the class of all regular languages whose syntactic monoid is in* $\mathbf{M_S}$ *iff* $\mathrm{Lin}(\mathbf{K})$ *admits relativization, quotients are expressible in* $\mathrm{Lin}(\mathcal{L})$*,* $\widehat{\mathbf{K}}_{\mathcal{L}}$ *contains* U_1 *and the counters, and for every finite simple group* G *it holds that* $G \in \mathbf{S}$ *iff* $G|\mathbf{K}_{\mathcal{L}}$.

(3) *If* $\mathrm{FO}(\mathcal{L})$ *admits relativization and quotients are expressible in* $\mathrm{FO}(\mathcal{L})$ *then* $\mathbf{FO}(\mathcal{L})$ *contains all regular languages whose syntactic monoid is in* $\mathbf{M_S}$ *iff* $\widehat{\mathbf{K}}_{\mathcal{L}}$ *contains the counters and for each simple group* G *in* $\mathbf{S}$ *it holds that* $G|\mathbf{K}_{\mathcal{L}}$.

(4) $\mathbf{FO}(\mathcal{L})$ *is the class of all regular languages whose syntactic monoid is in* $\mathbf{M_S}$ *iff* $\mathrm{FO}(\mathbf{K})$ *admits relativization, quotients are expressible in* $\mathrm{FO}(\mathcal{L})$*,*

$\widehat{\mathbf{K}}$ contains the counters, and for every finite simple group G it holds that $G \in \mathbf{S}$ iff $G|\mathbf{K}_{\mathcal{L}}$.

Corollary 11.21.

(1) *If $\mathrm{Lin}(\mathcal{L})$ admits relativization and quotients are expressible in $\mathrm{Lin}(\mathcal{L})$ then $\mathbf{Lin}(\mathcal{L})$ contains all regular languages whose syntactic monoid is in $\mathbf{MSol}$ iff $\widehat{\mathbf{K}}_{\mathcal{L}}$ contains U_1 and the counters, and for every prime number p it holds that $Z_p|\mathbf{K}_{\mathcal{L}}$. Moreover, $\mathbf{Lin}(\mathcal{L})$ is the class of all regular languages whose syntactic monoid is in $\mathbf{MSol}$ iff $\mathbf{Lin}(\mathcal{L})$ admits relativization, quotients are expressible in $\mathrm{Lin}(\mathcal{L})$, $\mathbf{K}_{\mathcal{L}} \subseteq \mathbf{MSol}$, $\widehat{\mathbf{K}}_{\mathcal{L}}$ contains U_1 and the counters, and for every prime number p it holds that $Z_p|\mathbf{K}_{\mathcal{L}}$.*

(2) *If $\mathrm{FO}(\mathcal{L})$ admits relativization and quotients are expressible in $\mathrm{FO}(\mathcal{L})$ then $\mathbf{FO}(\mathcal{L})$ contains all regular languages whose syntactic monoid is in $\mathbf{MSol}$ iff $\widehat{\mathbf{K}}_{\mathcal{L}}$ contains the counters, and for every prime number p it holds that $Z_p|\mathbf{K}_{\mathcal{L}}$. Moreover, $\mathbf{FO}(\mathcal{L})$ is the class of all regular languages whose syntactic monoid is solvable iff $\mathrm{FO}(\mathcal{L})$ admits relativization, quotients are expressible in $\mathrm{FO}(\mathcal{L})$, $\mathbf{K}_{\mathcal{L}} \subseteq \mathbf{MSol}$, $\widehat{\mathbf{K}}_{\mathcal{L}}$ contains the counters, and for every prime number p it holds that $Z_p|\mathbf{K}_{\mathcal{L}}$.*

Corollary 11.22. *Suppose that P is a set of prime numbers.*

(1) *If $\mathrm{Lin}(\mathcal{L})$ admits relativization and quotients are expressible in $\mathrm{Lin}(\mathcal{L})$, then $\mathbf{Lin}(\mathcal{L})$ contains all regular languages whose syntactic monoid is in $\mathbf{MSol}_P$ iff $\widehat{\mathbf{K}}_{\mathcal{L}}$ contains U_1 and the mgi-pairs $Z_p^1 = (Z_p, \{a, 1\})$, where p is any prime in P. Moreover, $\mathbf{Lin}(\mathcal{L})$ is the class of all regular languages whose syntactic monoid is in $\mathbf{MSol}_P$ iff the syntactic monoid of each language in $\mathcal{L}$ belongs to $\mathbf{MSol}_P$, $\mathrm{Lin}(\mathcal{L})$ admits relativization, quotients are expressible in $\mathrm{Lin}(\mathcal{L})$, and the above conditions hold.*

(2) *If $\mathrm{FO}(\mathcal{L})$ admits relativization and quotients are expressible in $\mathrm{FO}(\mathcal{L})$, then $\mathbf{FO}(\mathcal{L})$ contains all regular languages whose syntactic monoid is in $\mathbf{MSol}_P$ iff $\widehat{\mathbf{K}}_1$ contains the mgi-pairs $Z_p^1 = (Z_p, \{a, 1\})$, where p is any prime in P and $\mathbf{K}_1 = \mathbf{K}_{\mathcal{L}} \cup \{U_1\}$. Moreover, $\mathbf{FO}(\mathcal{L})$ is the class of all regular languages whose syntactic monoid is in $\mathbf{MSol}_P$ iff the syntactic monoid of each language in $\mathcal{L}$ belongs to $\mathbf{MSol}_P$, $\mathrm{FO}(\mathcal{L})$ admits relativization, quotients are expressible in $\mathrm{FO}(\mathcal{L})$, and the above condition holds.*

Corollary 11.23. *Suppose that P is a set of prime numbers. If $\mathrm{Lin}(\mathcal{L})$ admits relativization and quotients are expressible in $\mathrm{Lin}(\mathcal{L})$, then $\mathbf{Lin}(\mathcal{L})$ contains all regular languages whose syntactic monoid is in $\mathbf{GSol}_P$ iff $\widehat{\mathbf{K}}_{\mathcal{L}}$ contains the mgi-pairs $Z_p^1 = (Z_p, \{a, 1\})$, where p is any prime in P. Moreover, $\mathbf{Lin}(\mathcal{L})$ is the class of all regular languages whose syntactic monoid is in $\mathbf{GSol}_P$ iff the syntactic monoid of each language in $\mathcal{L}$ belongs to $\mathbf{GSol}_P$, $\mathrm{Lin}(\mathcal{L})$ admits relativization and quotients are expressible in $\mathrm{Lin}(\mathcal{L})$, and the above condition holds.*

Corollary 11.24. $\mathbf{Lin}(\mathcal{L}) = \mathbf{FO}$ *iff $\mathrm{Lin}(\mathcal{L})$ admits relativization, quotients are expressible in $\mathrm{Lin}(\mathcal{L})$, $U_1 \in \widehat{\mathbf{K}}_{\mathcal{L}}$ and the syntactic monoid of each language in $\mathcal{L}$ is aperiodic.*

11.1. Completeness and padding

Our characterizations become simpler when $\mathbf{K}$ is a class of mgi-pairs that we assume in the rest of this section. We only present three results and skip the proofs that use Corollaries 10.29, 10.30 and 10.31. So let $\mathbf{K}$ denote a class of mgi-pairs.

Proposition 11.25.

(1) $\mathbf{K}$ *is Lindström-complete iff* $\mathbf{K}$ *is group-complete, contains an mgi-pair whose underlying monoid is not a group, moreover, there exists some* $n > 1$ *with* $(Z_n, \{a\}) < \mathbf{K}$.

(2) $\mathbf{K}$ *is expressively complete iff* $\mathbf{K}$ *is group-complete and there exists some* $n > 1$ *with* $(Z_n, \{a\}) < \mathbf{K}$.

Proposition 11.26. *Suppose that* $\mathbf{S}$ *is a nonempty class of simple groups closed with respect to division.*

(1) $\mathbf{Lin}(\mathbf{K})$ *contains the regular languages whose syntactic monoids are in* $\mathbf{M_S}$ *iff for each* $G \in \mathbf{S}$ *it holds that* $G < \mathbf{K}$*, moreover,* $\mathbf{K}$ *contains an mgi-pair whose underlying monoid is not a group and there exists some* $n > 1$ *with* $(Z_n, \{a\}) < \mathbf{K}$*. Further,* $\mathbf{Lin}(\mathbf{K})$ *is the class of all regular languages whose syntactic monoid is in* $\mathbf{M_S}$ *iff the above conditions hold and* $\overline{\mathbf{K}} \subseteq \mathbf{M_S}$.

(2) $\mathbf{FO}(\mathbf{K})$ *contains the regular languages whose syntactic monoids are in* $\mathbf{M_S}$ *iff for each* $G \in \mathbf{S}$ *it holds that* $G < \mathbf{K}$ *and there exists some* $n > 1$ *with* $(Z_n, \{a\}) < \mathbf{K}$*. Further,* $\mathbf{Lin}(\mathbf{K})$ *is the class of all regular languages whose syntactic monoid is in* $\mathbf{M_S}$ *iff the above conditions hold and* $\overline{\mathbf{K}} \subseteq \mathbf{M_S}$.

Proposition 11.27. $\mathbf{Lin}(\mathbf{K}) \supseteq \mathbf{FO}$ *iff* $\mathbf{K}$ *contains an mgi-pair whose monoid component is not a group.*

In the next three corollaries, we assume that $\mathcal{L}$ *is closed with respect to quotients and padding.*

Corollary 11.28.

(1) $\mathcal{L}$ *is Lindström-complete iff every (non-abelian simple) group divides the syntactic monoid of some language in* $\mathcal{L}$*, moreover,* $\mathcal{L}$ *contains a language whose syntactic monoid is not a group, and there is some* $n > 1$ *such that the one-letter language* $(a^n)^*$ *is the inverse image of a language in* $\mathcal{L}$ *under a literal homomorphism.*

(2) $\mathcal{L}$ *is expressively complete iff every (non-abelian simple) group divides the syntactic monoid of some language in* $\mathcal{L}$*, moreover, there is some* $n > 1$ *such that the one-letter language* $(a^n)^*$ *is the inverse image of a language in* $\mathcal{L}$ *under a literal homomorphism.*

Corollary 11.29. *Suppose that* $\mathbf{S}$ *is a nonempty class of simple groups closed with respect to division.*

(1) $\mathbf{Lin}(\mathcal{L})$ *contains the regular languages whose syntactic monoids are in* $\mathbf{M_S}$ *iff for each* $G \in \mathbf{S}$ *it holds that* $G < \mathbf{K}_\mathcal{L}$*, moreover,* $\mathbf{K}_\mathcal{L}$ *contains an mgi-pair whose underlying monoid is not a group and there exists some* $n > 1$ *such that* $(a^n)^*$ *is the inverse image of a language in* $\mathcal{L}$ *under a literal*

homomorphism. Further, $\mathbf{Lin}(\mathcal{L})$ *is the class of all regular languages whose syntactic monoid is in* $\mathbf{M_S}$ *iff the above conditions hold and* $\overline{\mathbf{K}}_{\mathcal{L}} \subseteq \mathbf{M_S}$.

(2) $\mathbf{FO}(\mathcal{L})$ *contains the regular languages whose syntactic monoids are in* $\mathbf{M_S}$ *iff for each* $G \in \mathbf{S}$ *it holds that* $G < \mathbf{K}_{\mathcal{L}}$ *and there exists some* $n > 1$ *such that* $(a^n)^*$ *is the inverse image of a language in* $\mathcal{L}$ *under a literal homomorphism. Further,* $\mathbf{Lin}(\mathcal{L})$ *is the class of all regular languages whose syntactic monoid is in* $\mathbf{M_S}$ *iff the above conditions hold and* $\overline{\mathbf{K}}_{\mathcal{L}} \subseteq \mathbf{M_S}$.

Corollary 11.30. $\mathbf{Lin}(\mathcal{L}) \supseteq \mathbf{FO}$ *iff* $\mathcal{L}$ *contains a language whose syntactic monoid is not a group.*

12. Further work

Extensions of the main result to other structures including ω-words, ordinal words, words over discrete linear orders and trees will be considered in subsequent papers.

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Appendix

In this appendix, we prove Proposition 8.3. The argument is an adaptation of the proof of the corresponding fact for monoid varieties, communicated to the authors by John Rhodes. In this section, by a monoid or category we will always mean a finite monoid, or category, respectively.

First, we recall from Rhodes and Tilson [28] the notion of the *kernel* K_φ of a monoid morphism $\varphi : M \to N$. It is a category constructed as a quotient of category W_φ defined as follows. The *objects* of W_φ are all ordered pairs $\mathbf{n} = (n_L, n_R)$ of elements n_L, n_R in the image $\varphi(M)$ of M. (We will follow the convention of [28] that if a boldface letter $\mathbf{x}$ denotes a pair of elements of a monoid, then x_L and x_R are the left and right hand components of this pair.) An *arrow* $\mathbf{n} \to \mathbf{n}'$ of W_φ takes the form $(n_L, (m, n), n'_R)$, where $m \in M$, $n \in N$ with $n = \varphi(m)$ are such that $n_L n = n'_L$ and $n n'_R = n_R$. (Thus, we could as well just write (n_L, m, n'_R), but we want to keep the notation consistent with that of [28]. The reason for the more complex notation of [28] is due to the fact that the kernel construction also applies to relational morphisms φ of monoids, whereas in this paper we only consider the particular case when φ is a function.) Note that n_R and n'_L and thus $\mathbf{n}$ and $\mathbf{n}'$ can be recovered from the notation $(n_L, (m, n), n'_R)$. Below we will sometimes just write (m,n) for $(n_L,(m,n),n'_R)$ when there is no danger of confusion. The composite of consecutive arrows $(n_L,(m,n),n'_R):\mathbf{n}\to\mathbf{n}'$ and $(n'_L,(m',n'),n''_R):\mathbf{n}'\to\mathbf{n}''$ is defined as $(n_L,(mm',nn'),n''_R)$. Note that the identity arrows of W_φ take the form $(n_L,(1,1),n_R)$. Each arrow $(n_L,(m,n),n'_R)$ induces a function

$$[n_L,(m,n),n'_R]:\varphi^{-1}(n_L)\times\varphi^{-1}(n'_R)\to M$$
$$(m_L,m_R)\mapsto m_L m m_R.$$

The relation that identifies any two parallel arrows inducing the same function is shown to be a (category) congruence in [28]. The kernel K_φ is then defined as the quotient of W_φ with respect to this congruence. Following [28], we will denote a morphism of K_φ as $[n_L,(m,n),n'_R]$, or just $[m,n]$.

Suppose now that (M,A) and (N,B) are mg-pairs and φ is a morphism $(M,A)\to(N,B)$, so that φ is also a monoid homomorphism $M\to N$. Then we define the kernel of φ to be the pair (K_φ,A_φ), where K_φ is the category constructed above, and where A_φ is a distinguished collection of morphisms of K_φ: it consists of those morphisms $[a,b]$ of K_φ with $a\in A$. Since $b=\varphi(a)$, and since φ is a morphism of mg-pairs, it follows that $b\in B$. Since A is a generating set of M and φ preserves the generators, it follows that (K_φ,A_φ) is a *category equipped with a distinguished set of generators*, or *cg-pair*: each arrow of K_φ is either an identity arrow or the composite of some arrows in A_φ.

Suppose that K is a category, N is a monoid, and φ is a relation from the arrows of K to N, viewed as a function from the arrows of K to the set of all subsets of N. We say that φ is a *covering* $K\to N$ if the following conditions hold:

- $\varphi(m)\varphi(m')\subseteq\varphi(mm')$, for all composable arrows m,m'.
- For all identity arrows e it holds that $1\in\varphi(e)$.
- For all arrows m it holds that $\varphi(m)\neq\emptyset$.
- For all m,m', if $m\neq m'$ then $\varphi(m)\cap\varphi(m')=\emptyset$.

When K and N are equipped with distinguished sets of generators, *i.e.*, when (K,A) is a cg-pair and (N,B) is an mg-pair, then a covering $\varphi:(K,A)\to(N,B)$ also satisfies that for each $a\in A$ there is some $b\in B$ with $b\in\varphi(a)$. Since A is a set of generators, the third condition above becomes redundant. Note that each covering $\varphi:(K,A)\to(N,B)$ contains a covering φ' such that whenever $n\in\varphi'(m)$, it holds that either m is an identity arrow and $n=1$, or there exist $a_1,\dots,a_k\in A$, $b_1,\dots,b_k\in B$, $k\geq 1$ with $b_1\in\varphi(a_1),\dots,b_k\in\varphi(a_k)$ such that m is the composite $a_1\dots a_k$ and n is $b_1\dots b_k$. The above definition also applies to one object categories K which may conveniently be identified with their homsets. In that case the concept reduces to the notion of covering defined earlier in Section 5. Note that each injective morphism $(M,A)\to(N,B)$ is a covering $(M,A)\to(N,B)$, moreover, the relational inverse of each surjective morphism $(M,A)\to(N,B)$ is a covering $(N,B)\to(M,A)$, *i.e.*, in the opposite direction.

The notion of covering can be generalized to a pair of categories, and in fact to cg-pairs. Given categories K and K', a covering $\varphi:K\to K'$ assigns an object to each object of K, a set $\varphi(m)$ of morphisms of K' to each morphism m of K, compatible with the object map, such that the obvious analogies of the above conditions hold. A covering $\varphi : (K, A) \to (K', B)$ between cg-pairs (K, A) and (K', B) also satisfies that for each arrow $a \in A$ there is an arrow $b \in B$ with $b \in \varphi(a)$. The composite of two coverings is defined in the expected way.

Lemma 12.1. *The composite of coverings $\psi : K \to K'$ and $\psi' : K' \to K''$ is a covering $K \to K''$. Similarly, if $\psi : (K, A) \to (K', A')$ and $\psi' : (K', A') \to (K'', A'')$ are coverings, then the composite of ψ with ψ' is a covering $K \to K''$.*

The notion of covering is related to the double semidirect product by the Kernel Theorem of Rhodes and Tilson [28], also known as the Covering Lemma. We need a version of this result.

Theorem 12.2. *Let $\varphi : (M, A) \to (N, B)$ be a morphism of mg-pairs, and let (V, C) be an mg-pair satisfying $(K_\varphi, A_\varphi) < (V, C)$. Then $(M, A) < (V, C)\Box(N, B)$.*

Proof. We follow the argument given in the proof of the Kernel Theorem (Th. 7.4) in Rhodes and Tilson [28]. Let $\psi : (K_\varphi, A_\varphi) \to (V, C)$ be a covering. For each pair $m \in M,\ n \in N$ with $n \in \varphi(m)$, define

$$F(m, n) = \{f \in V^{N\times N} : f(n_1, n_2) \in \psi([n_1, (m, n), n_2]),\ n_1, n_2 \in \varphi(M)\}.$$

Then let the relation $\theta : M \to V\Box N$ be defined by

$$\theta(m) = \{(f, n) : n \in \varphi(m),\ f \in F(m, n)\}.$$

It is shown in [28] that θ is a covering $M \to V\Box N$. For each $m \in M$ let $\theta'(m) = \theta(m)\cap W$, where W denotes the monoid component of $(V, C)\Box(N, B)$, *i.e.*, $(V, C)\Box(N, B) = (W, C^{N\times N}\times B)$. If we can show that for each $a \in A$ there is some $b \in B$ and $f \in C^{N\times N}$ with $(f, b) \in \theta(a)$, then, using the fact that W is a submonoid of $V\Box N$, it follows that θ' is a covering $(M, A) \to (V, C)\Box(N, B)$. Given a, let $b = \varphi(a)$. Since ψ is a covering $(K_\varphi, A_\varphi) \to (V, C)$, for each $n_1, n_2 \in \varphi(M)$ there is some $c \in C$ with $c \in \psi([n_1, (a, b), n_2])$. So let f map each pair $(n_1, n_2) \in \varphi(M)^2$ to such a c, and let $f(n_1, n_2)$ be an arbitrary element of C if n_1 or n_2 is not in $\varphi(M)$. $\Box$

Suppose now that $(M, A)\star\star(T, C)$ and $(N, B)\star\star(T, C)$ are double semidirect products so that T acts on M and on N on the left and on the right. Following Rhodes and Tilson [28], we say that these actions are compatible with a morphism $\varphi : (M, A) \to (N, B)$ if for all $m \in M,\ n \in N$ and $t \in T$, if $\varphi(m) = n$ then $\varphi(tm) = tn$ and $\varphi(mt) = nt$. In this case we define a morphism

$$\begin{aligned}\varphi\star\star(T, C) : (M, A)\star\star(T, C) &\to (N, B)\star\star(T, C)\\ (m, t) &\mapsto (\varphi(m), t).\end{aligned}$$

The reader should have no difficulty to check that $\varphi\star\star(T, C)$ is indeed a morphism. In the same way, we define

$$\begin{aligned}\varphi\star\star T : M\star\star T &\to N\star\star T\\ (m, t) &\mapsto (\varphi(m), t),\end{aligned}$$

where $M{\star\star}T$ and $N{\star\star}T$ are respectively the double semidirect products of M and N with T determined by the actions.

Proposition 12.3. *Under the previous assumptions, if the actions of T on M and on N are compatible with φ, then*

$$(K_{\varphi\star\star(T,C)}, A_{\varphi\star\star(T,C)}) < (K_\varphi, A_\varphi).$$

Proof. In the proof of Rhodes and Tilson [28], Theorem 6.2, it is shown that $K_{\varphi\star\star T} < K_\varphi$. This is achieved by mapping each object $(\mathbf{n}, \mathbf{t})$ of $K_{\varphi\star\star T}$ to the object $(n_L t_R, t_L n_R)$ of K_φ, and by relating each arrow

$$[(m,t),(n,t)] : (\mathbf{n}, \mathbf{t}) \to (\mathbf{n}', \mathbf{t}')$$

in $K_{\varphi\star\star T}$ to

$$[t_L m t'_R, t_L n t'_R] : (n_L t_R, t_L n_R) \to (n'_L t'_R, t'_L n'_R).$$

Note that when $m \in A$ (and thus by $\varphi(m) = n$, also $n \in B$), then $t_L m t'_R \in A$ and $t_L n t'_R \in B$. Let ψ denote this covering. By Lemma 12.4 and its proof, there is a covering

$$\rho : K_{\varphi\star\star(T,C)} \to K_{\varphi\star\star T}$$

which is the identity on objects and relates a morphism $[(a,c),(b,c)]$ in $K_{\varphi\star\star(T,C)}$, where $a \in A$, $b \in B$ and $c \in C$, with the corresponding morphism $[(a,c),(b,c)]$ in $K_{\varphi\star\star T}$. The composite of the two coverings ρ and ψ is the required covering. $\square$

Lemma 12.4. *Suppose that φ is a homomorphism $M \to N$, M' is a submonoid of M, and N' is a submonoid of N such that the restriction of φ to M is a homomorphism $M' \to N'$. Then $K_{\varphi'} < K_\varphi$.*

This is proved in Rhodes and Tilson [28], Corollary 5.4. It is clear that every object of $K_{\varphi'}$ is an object of K_φ. It is shown in [28] that the relation that is the identity function on objects and relates each morphism $[(m,n)]$ in $K_{\varphi'}$ with the morphism $[(m,n)]$ in K_φ is a covering.

We let $\underline{1}$ denote a trivial monoid $\underline{1} = \{1\}$. Thus, $(\underline{1}, \{1\})$ is a trivial mg-pair.

Proposition 12.5. *Let (M,A) denote an mg-pair and let φ denote the unique (collapsing) morphism $(M,A) \to (\underline{1}, \{1\})$. Then*

$$(K_\varphi, A_\varphi) < (M, A).$$

Proof. The relation that relates each arrow $[1,(m,1),1]$ with m is a covering. $\square$

Corollary 12.6. *Suppose that $(M,A){\star\star}(N,B)$ is a double semidirect product. Let π denote the projection $(M,A){\star\star}(N,B) \to (N,B)$, $(m,n) \mapsto n$. Then $(K_\pi, A_\pi) < (M,A)$.*

Proof. The projection π is (essentially) $\varphi{\star\star}(N,C)$, where φ denotes the collapsing morphism $(M,A) \to (\underline{1},\{1\})$. (Note that φ is compatible with any actions.) Thus, the result follows from Propositions 12.5 and Proposition 12.3. □

Given a double semidirect product $((M,A){\star\star}(N,B)){\star\star}(T,C)$, where we denote $(M,A){\star\star}(N,B) = (V, A\times B)$, we say that the actions of T (on V) are *point-wise* if there exist left and right actions of T on M and on N such that

$$
\begin{aligned}
t(m,n) &= (tm,tn)\\
(m,n)t &= (mt,nt),
\end{aligned}
$$

for all $(m,n)\in V$ and $t\in T$. It follows that the left and right actions of T on N are compatible and determine a double semidirect product $(N,B){\star\star}(T,C)$.

Lemma 12.7. *Suppose that $((M,A){\star\star}(N,B)){\star\star}(T,C)$ is a double semidirect product of finite mg-pairs such that the actions of T are point-wise and thus determine a double semidirect product $(N,B){\star\star}(T,C)$. Then the actions of T on $(M,A){\star\star}(N,B)$ and on (N,B) are compatible with the projection morphism π : $(M,A){\star\star}(N,B)\to(N,B)$.*

Proof. Immediate from the definitions. □

Proposition 12.8. *Suppose that $\mathbf{V}_1,\mathbf{V}_2$ and $\mathbf{V}_3$ are varieties of finite mg-pairs. Then an mg-pair is in $(\mathbf{V}_1{\star\star}\mathbf{V}_2){\star\star}\mathbf{V}_3$ iff it divides a semidirect product*

$$((M,A){\star\star}(N,B)){\star\star}(T,C)$$

such that $(M,A)\in\mathbf{V}_1$, $(N,B)\in\mathbf{V}_2$, $(T,C)\in\mathbf{V}_3$ and the actions of T are point-wise.

Proof. One direction is trivial. Suppose now that (S,D) is in $(\mathbf{V}_1{\star\star}\mathbf{V}_2){\star\star}\mathbf{V}_3$. Then, by Proposition 8.1, (S,D) divides a block product

$$((M,A)\Box(N,B))\Box(T,C)$$

which is a double semidirect product

$$((M,A)^{N\times N}{\star\star}(N,B))^{T\times T}{\star\star}(T,C)$$

with suitable actions. This double semidirect product is in turn isomorphic to a double semidirect product

$$((M,A)^{N\times N\times T\times T}{\star\star}(N,B)^{T\times T}){\star\star}(T,C),$$

where the actions of T are given by

$$
\begin{aligned}
t(f,g) &= (f',g')\\
(f,g)t &= (f'',g''),
\end{aligned}
$$

where

$$f'(n_1, n_2, t_1, t_2) = f(n_1, n_2, t_1 t, t_2)$$
$$g'(t_1, t_2) = g(t_1 t, t_2),$$

and similarly for f'' and g''. Since f' does not depend on g and g' does not depend on f, the left action is point-wise. Likewise the right action. Now let

$$(M', A') = (M, A)^{N \times N \times T \times T}$$
$$(N', B') = (N, B)^{T \times T}.$$

We have that (S, D) divides a double semidirect product

$$((M', A') \star\star (N', B')) \star\star (T, C)$$

such that the actions of T are point-wise. Since varieties are closed with respect to the direct product, we have $(M', A') \in \mathbf{V}_1$ and $(N', B') \in \mathbf{V}_2$. □

We now complete the proof of Proposition 8.3. We want to prove that for all varieties of finite mg-pairs $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$,

$$(\mathbf{V}_1 \star\star \mathbf{V}_2) \star\star \mathbf{V}_3 \subseteq \mathbf{V}_1 \star\star (\mathbf{V}_2 \star\star \mathbf{V}_3).$$

By Proposition 8.1, we only need to show that each mg-pair

$$((M, A) \star\star (N, B)) \star\star (T, C)$$

such that $(M, A) \in \mathbf{V}_1$, $(N, B) \in \mathbf{V}_2$ and $(T, C) \in \mathbf{V}_3$ is in $\mathbf{V}_1 \star\star (\mathbf{V}_2 \star\star \mathbf{V}_3)$. Moreover, by Proposition 12.8, we may assume that the actions of T are point-wise. But then, by Lemma 12.7, the projection $\pi : (M, A) \star\star (N, B) \to (N, B)$ is compatible with the actions of T, and moreover, by Corollary 12.6, it holds that

$$(K_{\pi \star\star (T,C)}, A_{\pi \star\star (T,C)}) < (M, A).$$

Thus, by Theorem 12.2 applied to the morphism

$$\pi \star\star (T, C) : ((M, A) \star\star (N, B)) \star\star (T, C) \to (N, B) \star\star (T, C),$$

we have

$$((M, A) \star\star (N, B)) \star\star (T, C) < (M, A) \square ((N, B) \star\star (T, C)),$$

proving that $((M, A) \star\star (N, B)) \star\star (T, C)$ is in $\mathbf{V}_1 \star\star (\mathbf{V}_2 \star\star \mathbf{V}_3)$.

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